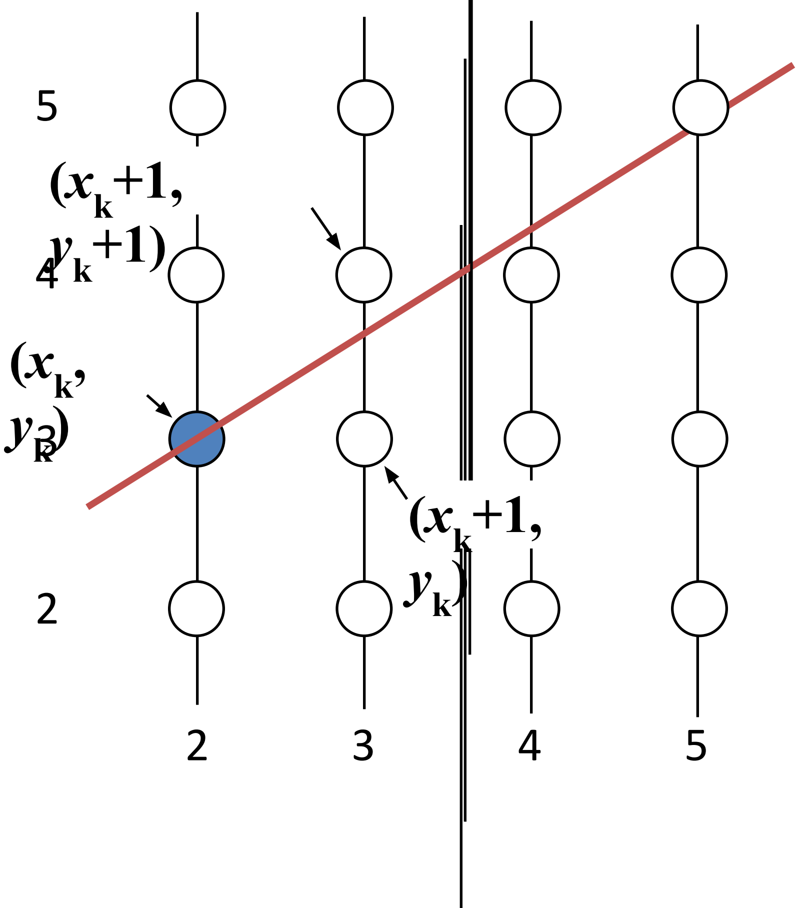
Computer Graphics (CSE-321)
Lecture-4

The Bresenham Line Algorithm

- The Bresenham algorithm is another incremental scan conversion algorithm
- The big advantage of this algorithm is that it uses only integer calculations: integer addition, subtraction and multiplication by 2, which can be accomplished by a simple arithmetic shift operation.

- Move across the x axis in unit intervals and at each step choose between two different y coordinates

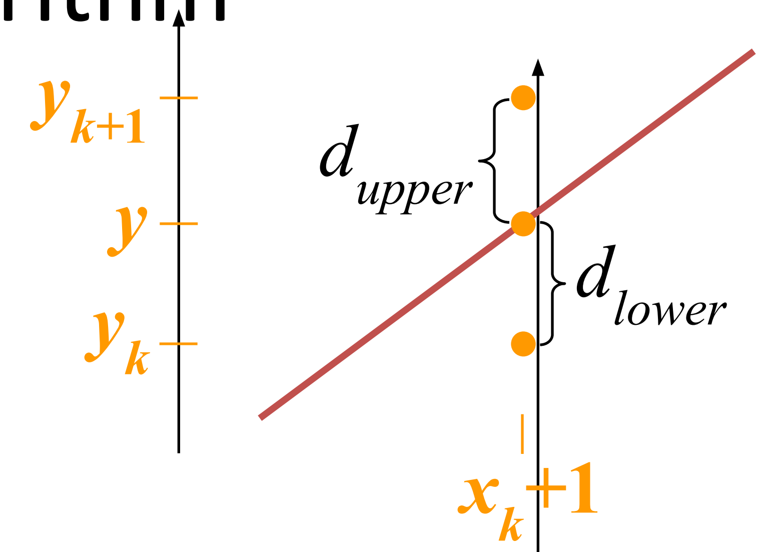


For example, from position (2, 3) we have to choose between (3, 3) and (3, 4)

We would like the point that is closer to the original line

Deriving The Bresenham Line Algorithm

- At sample position $x_k + 1$ the vertical separations from the mathematical line are labelled d_{upper} and d_{lower}



The y coordinate on the mathematical line at $x_k + 1$ is:

$$y = m(x_k + 1) + b$$

Deriving The Bresenham Line Algorithm…

- So, d_{upper} and d_{lower} are given as follows:

$$
\begin{aligned}
d_{lower} &= y - y_k \\
&= m(x_k + 1) + b - y_k
\end{aligned}
$$

- and:

$$
\begin{aligned}
d_{upper} &= (y_k + 1) - y \\
&= y_k + 1 - m(x_k + 1) - b
\end{aligned}
$$

- We can use these to make a simple decision about which pixel is closer to the mathematical line

Deriving The Bresenham Line Algorithm…

- This simple decision is based on the difference between the two pixel positions:

$$d_{lower} - d_{upper} = 2m(x_k + 1) - 2y_k + 2b - 1$$

- Let's substitute m with $\Delta y / \Delta x$ where Δx and Δy are the differences between the end-points:

$$\Delta x(d_{lower} - d_{upper}) = \Delta x(2\frac{\Delta y}{\Delta x}(x_k + 1) - 2y_k + 2b - 1)$$

$$= 2\Delta y \cdot x_k - 2\Delta x \cdot y_k + 2\Delta y + \Delta x(2b - 1)$$

$$= 2\Delta y \cdot x_k - 2\Delta x \cdot y_k + c$$

Deriving The Bresenham Line Algorithm…

- So, a decision parameter p_k for the kth step along a line is given by:

$$p_k = \Delta x(d_{lower} - d_{upper})$$
$$= 2\Delta y \cdot x_k - 2\Delta x \cdot y_k + c$$

- The sign of the decision parameter p_k is the same as that of $d_{lower} - d_{upper}$
- If p_k is negative, then we choose the lower pixel, otherwise we choose the upper pixel

Deriving The Bresenham Line Algorithm…

- Remember coordinate changes occur along the x axis in unit steps so we can do everything with integer calculations
- At step k+1 the decision parameter is given as:

$$p_{k+1} = 2\Delta y \cdot x_{k+1} - 2\Delta x \cdot y_{k+1} + c$$

- Subtracting p_k from this we get:

$$p_{k+1} - p_k = 2\Delta y(x_{k+1} - x_k) - 2\Delta x(y_{k+1} - y_k)$$

Deriving The Bresenham Line Algorithm…

- But, x_{k+1} is the same as x_k+1 so:

$$p_{k+1} = p_k + 2\Delta y - 2\Delta x(y_{k+1} - y_k)$$

- where y_{k+1} - y_k is either 0 or 1 depending on the sign of p_k
- The first decision parameter p0 is evaluated at (x0, y0) is given as:

$$p_0 = 2\Delta y - \Delta x$$

The Bresenham Line Algorithm…

BRESENHAM'S LINE DRAWING ALGORITHM

(for $|m| < 1.0$)

1. Input the two line end-points, storing the left end-point in (x_0, y_0)
2. Plot the point (x_0, y_0)
3. Calculate the constants Δx, Δy, $2\Delta y$, and $(2\Delta y - 2\Delta x)$ and get the first value for the decision parameter as:

$$p_0 = 2\Delta y - \Delta x$$

1. At each x_k along the line, starting at $k = 0$, perform the following test. If $p_k < 0$, the next point to plot is (x_k+1, y_k) and:

$$p_{k+1} = p_k + 2\Delta y$$

The Bresenham Line Algorithm…

Otherwise, the next point to plot is (x_k+1, y_k+1) and:

$$p_{k+1} = p_k + 2\Delta y - 2\Delta x$$

5. Repeat step 4 $(\Delta x - 1)$ times

- The algorithm and derivation above assumes slopes are less than 1. for other slopes we need to adjust the algorithm slightly

Bresenham's Line Algorithm (Example)

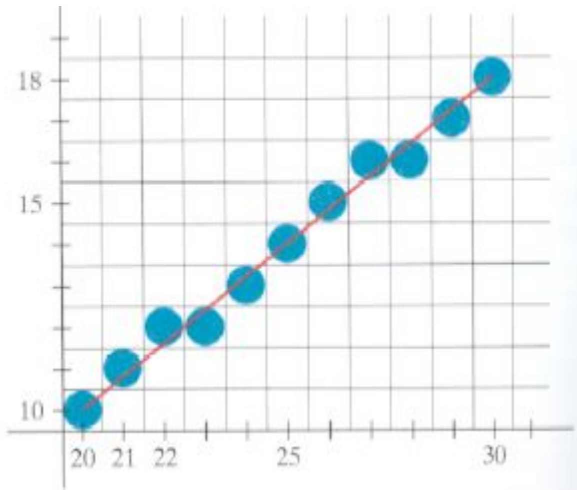
- using Bresenham's Line-Drawing Algorithm, Digitize the line with endpoints (20,10) and (30,18).

- $\Delta y = 18 – 10 = 8$,
- $\Delta x = 30 – 20 = 10$
- $m = \Delta y / \Delta x = 0.8$
- $2*\Delta y = 16$
- $2*\Delta y – 2* \Delta x = -4$
- plot the first point $(x0, y0) = (20, 10)$
- $p0 = 2 * \Delta y – \Delta x = 2 * 8 – 10 = 6$, so the next point is (21, 11)

Example (cont.)

K	P_k	(x_{k+1}, y_{k+1})	K	P_k	(x_{k+1}, y_{k+1})
0	6	(21,11)	5	6	(26,15)
1	2	(22,12)	6	2	(27,16)
2	-2	(23,12)	7	-2	(28,16)
3	14	(24,13)	8	14	(29,17)
4	10	(25,14)	9	10	(30,18)

Example (cont.)



Bresenham’s Line Algorithm (cont.)

- Notice that bresenham’s algorithm works on lines with slope in range $0 < m < 1$.
- We draw from left to right.
- To draw lines with slope > 1, interchange the roles of x and y directions.

Code (0 < slope < 1)

```
Bresenham ( int xA, yA, xB, yB) {
    int d, dx, dy, xi, yi;
    int incE, incNE;

    dx = xB - xA;
    dy = yB - yA;
    incE = dy << 1;                    // Q
    incNE = incE - dx << 1;            // Q + R
    d = incE - dx;            // initial d = Q + R/2
    xi = xA; yi = yA;
    writePixel(xi, yi);
    while(xi < xB) {
         xi++;
         if(d < 0)                // choose E
       d += incE;
         else {          // choose NE
       d += incNE;
       yi++;
     }
          writePixel(xi, yi);
     }}
```

Bresenham Line Algorithm Summary

- The Bresenham line algorithm has the following advantages:
  - An fast incremental algorithm
  - Uses only integer calculations
- Comparing this to the DDA algorithm, DDA has the following problems:
  - Accumulation of round-off errors can make the pixelated line drift away from what was intended
  - The rounding operations and floating point arithmetic involved are time consuming

A Simple Circle Drawing Algorithm

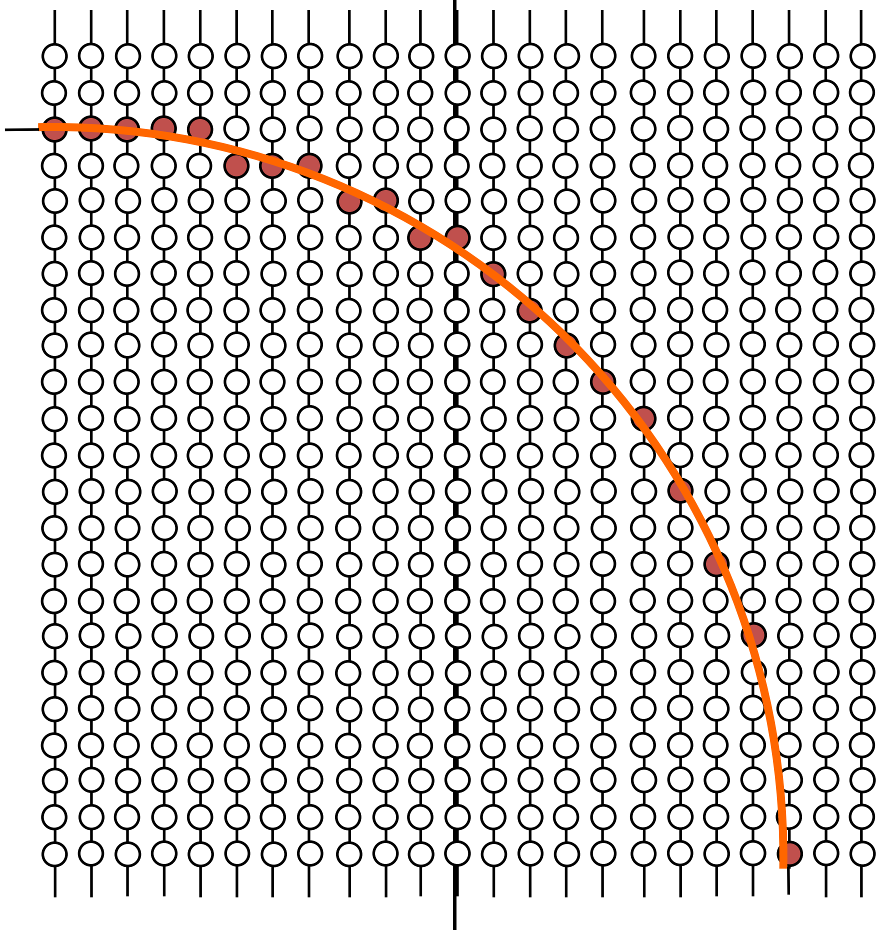
- The equation for a circle is:

$$x^2 + y^2 = r^2$$

- where r is the radius of the circle
- So, we can write a simple circle drawing algorithm by solving the equation for y at unit x intervals using:

$$y = \pm\sqrt{r^2 - x^2}$$

A Simple Circle Drawing Algorithm (cont…)



$$y_0 = \sqrt{20^2 - 0^2} \approx 20$$

$$y_1 = \sqrt{20^2 - 1^2} \approx 20$$

$$y_2 = \sqrt{20^2 - 2^2} \approx 20$$

$$\vdots$$

$$y_{19} = \sqrt{20^2 - 19^2} \approx 6$$

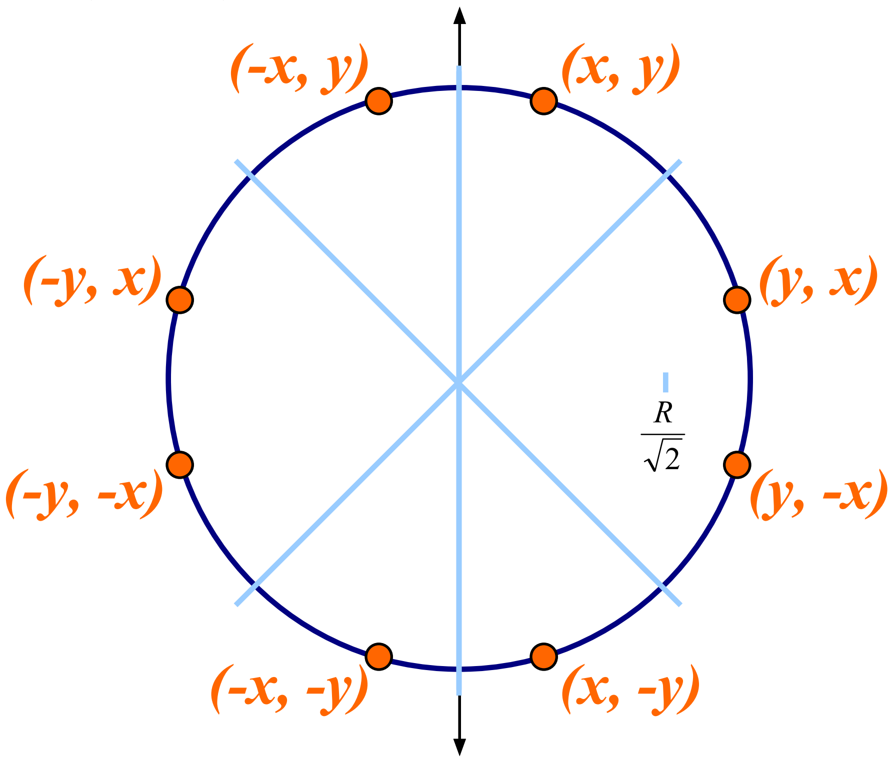
$$y_{20} = \sqrt{20^2 - 20^2} \approx 0$$

A Simple Circle Drawing Algorithm (cont…)

- However, unsurprisingly this is not a brilliant solution!
- Firstly, the resulting circle has large gaps where the slope approaches the vertical
- Secondly, the calculations are not very efficient
  - The square (multiply) operations
  - The square root operation – try really hard to avoid these!
- We need a more efficient, more accurate solution

Eight-Way Symmetry

- The first thing we can notice to make our circle drawing algorithm more efficient is that circles centred at $(0, 0)$ have *eight-way symmetry*

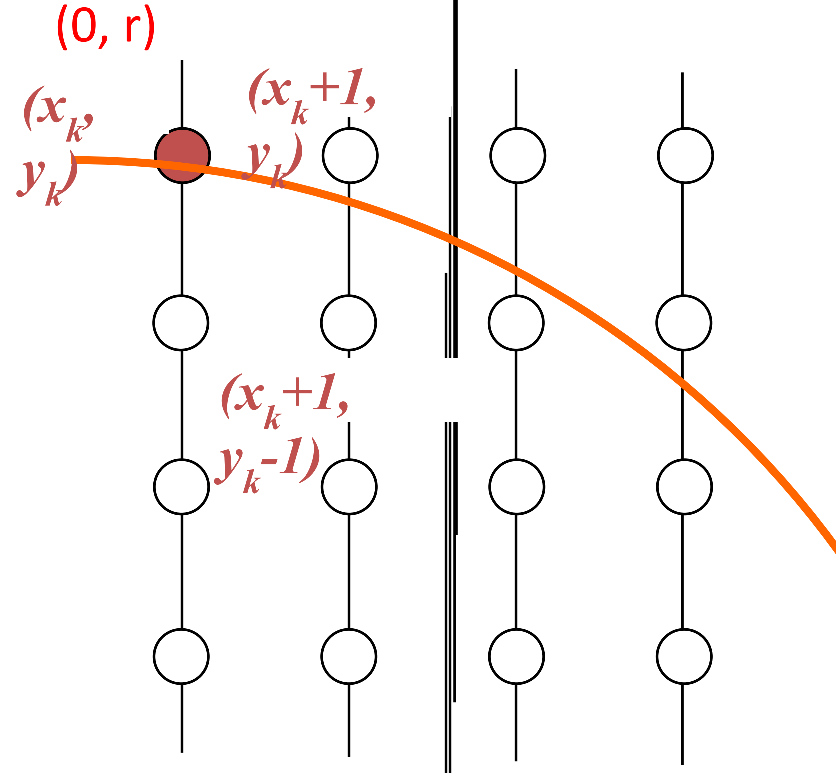


Mid-Point Circle Algorithm

- Similarly to the case with lines, there is an incremental algorithm for drawing circles – the *mid-point circle algorithm*
- In the mid-point circle algorithm we use eight-way symmetry so only ever calculate the points for the top right eighth of a circle, and then use symmetry to get the rest of the points

Mid-Point Circle Algorithm (cont…)

- Assume that we have



Mid-Point Circle Algorithm (cont…)

- Let's re-jig the equation of the circle slightly to give us:

$$f_{circ}(x, y) = x^2 + y^2 - r^2$$

- The equation evaluates as follows:

$$f_{circ}(x, y) \begin{cases} < 0, \text{ if } (x, y) \text{ is inside the circle boundary} \\ = 0, \text{ if } (x, y) \text{ is on the circle boundary} \\ > 0, \text{ if } (x, y) \text{ is outside the circle boundary} \end{cases}$$

- By evaluating this function at the midpoint between the candidate pixels we can make our decision

Mid-Point Circle Algorithm (cont…)

- Assuming we have just plotted the pixel at (x_k, y_k) so we need to choose between (x_k+1, y_k) and (x_k+1, y_k-1)
- Our decision variable can be defined as:

$$p_k = f_{circ}(x_k + 1, y_k - \tfrac{1}{2})$$

$$= (x_k + 1)^2 + (y_k - \tfrac{1}{2})^2 - r^2$$

- If $p_k < 0$ the midpoint is inside the circle and and the pixel at y_k is closer to the circle
- Otherwise the midpoint is outside and y_k-1 is closer

Mid-Point Circle Algorithm (cont…)

- To ensure things are as efficient as possible we can do all of our calculations incrementally
- First consider: $p_{k+1} = f_{circ}\left(x_{k+1}+1, y_{k+1}-\frac{1}{2}\right)$

$$= [(x_k+1)+1]^2 + \left(y_{k+1}-\frac{1}{2}\right)^2 - r^2$$

- or:

$$p_{k+1} = p_k + 2(x_k+1) + (y_{k+1}^2 - y_k^2) - (y_{k+1} - y_k) + 1$$

- where y_{k+1} is either y_k or y_k-1 depending on the sign of p_k

Mid-Point Circle Algorithm (cont…)

- The first decision variable is given as:

$$p_0 = f_{circ}(1, r - \tfrac{1}{2})$$
$$= 1 + (r - \tfrac{1}{2})^2 - r^2$$
$$= \tfrac{5}{4} - r$$

- Then if $p_k < 0$ then the next decision variable is given as: $p_{k+1} = p_k + 2x_{k+1} + 1$

- If $p_k > 0$ then the decision variable is:

$$p_{k+1} = p_k + 2x_{k+1} + 1 - 2y_k + 1$$

Mid-point Circle Algorithm - Steps

1. Input radius **r** and circle center **(x_c, y_c).** set the first point **(x_0 , y_0) = (0, *r*).**

1. Calculate the initial value of the decision parameter as **$p_0 = 1 – r$.**
   ($p_0 = 5/4 – r \cong 1 – r$)

3. If **$p_k < 0$**,
   plot **($x_k + 1$, y_k)** and **$p_{k+1} = p_k + 2x_{k+1} + 1$**,

   Otherwise,

   plot **($x_k + 1$, $y_k – 1$)** and **$p_{k+1} = p_k + 2x_{k+1} + 1 – 2y_{k+1}$**,

   where **$2x_{k+1} = 2x_k + 2$** and **$2y_{k+1} = 2y_k – 2$**.

Mid-point Circle Algorithm - Steps

4. Determine symmetry points on the other seven octants.

4. Move each calculated pixel position (x, y) onto the circular path centered on (x_c, y_c) and plot the coordinate values: $\boldsymbol{x = x + x_c}$ **,** $\boldsymbol{y = y + y_c}$

4. Repeat steps 3 though 5 until $\boldsymbol{x \geq y}$.

4. For all points, add the center point (x_c, y_c)

Mid-point Circle Algorithm - Steps

- Now we drew a part from circle, to draw a complete circle, we must plot the other points.
- We have $(x_c + x , y_c + y)$, the other points are:
  - $(x_c - x , y_c + y)$
  - $(x_c + x , y_c - y)$
  - $(x_c - x , y_c - y)$
  - $(x_c + y , y_c + x)$
  - $(x_c - y , y_c + x)$
  - $(x_c + y , y_c - x)$
  - $(x_c - y , y_c - x)$

Mid-point circle algorithm (Example)

- Given a circle radius $r = 10$, demonstrate the midpoint circle algorithm by determining positions along the circle octant in the first quadrant from $x = 0$ to $x = y$.

<u>Solution:</u>

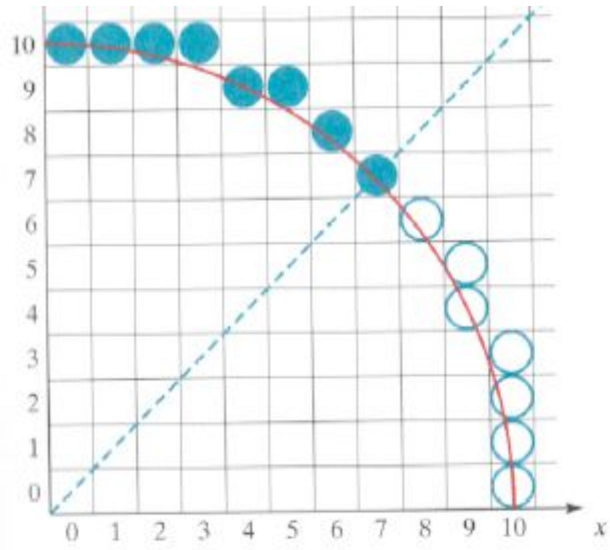
- $p_0 = 1 - r = -9$
- Plot the initial point $(x_0, y_0) = (0, 10)$,

- $2x_0 = 0$ and $2y_0 = 20$.
- Successive decision parameter values and positions along the circle path are calculated using the midpoint method as appear in the next table:

Mid-point circle algorithm (Example)

K	P_k	(x_{k+1}, y_{k+1})	x_{k+1} 2	y_{k+1} 2
0	9 –	(10 ,1)	2	20
1	6 –	(10 ,2)	4	20
2	1 –	(10 ,3)	6	20
3	6	(9 ,4)	8	18
4	3 –	(9 ,5)	10	18
5	8	(6,8)	12	16
6	5	(7,7)	14	14

Mid-point circle algorithm (Example)



Mid-point Circle Algorithm – Example (2)

- Given a circle radius $r = 15$, demonstrate the midpoint circle algorithm by determining positions along the circle octant in the first quadrant from $x = 0$ to $x = y$.

Solution:

- $p_0 = 1 - r = -14$
- plot the initial point $(x_0, y_0) = (0, 15)$,
- $2x_0 = 0$ and $2y_0 = 30$.
- Successive decision parameter values and positions along the circle path are calculated using the midpoint method as:

Mid-point Circle Algorithm – Example (2)

K	P_k	(x_{k+1}, y_{k+1})	$2x_{k+1}$	$2y_{k+1}$
0	– 14	(1, 15)	2	30
1	– 11	(2, 15)	4	30
2	– 6	(3, 15)	6	30
3	1	(4, 14)	8	28
4	– 18	(5, 14)	10	28

Mid-point Circle Algorithm – Example (2)

K	P_k	(x_{k+1}, y_{k+1})	x_{k+1} 2	y_{k+1} 2
5	7 –	(6,14)	12	28
6	6	(7,13)	14	26
7	5 –	(8,13)	16	26
8	12	(9,12)	18	24
9	7	(10,11)	20	22
10	6	(11,10)	22	20