specwid

Condense the Fourier spectra of X, Y and Z to eliminate the nonuniqueness of the poloidal angel in the parametric representation of space curves. The spectral width can be represented as $M = \frac{\sum_{m=1,M} m^{p+q}({X_{c,m}}^2)+{X_{s,m}}^2)+{Y_{c,m}}^2)+{Y_{s,m}}^2)+{Z_{c,m}}^2)+{Z_{s,m}}^2)}{\sum_{m=1,M} m^{p}({X_{c,m}}^2)+{X_{s,m}}^2)+{Y_{c,m}}^2)+{Y_{s,m}}^2)+{Z_{c,m}}^2)+{Z_{s,m}}^2)}$. We have mupltiple approaches for deriving the minimum of M.

[called by: denergy.]

contents

1.1 Introduction

1. For $p=0, q=2$ and ignoring the normalizing denominator, the spectral width can be written as, $M = \sum_{m=1,M} m^2({X_{c,m}}^2 + {X_{s,m}}^2 + {Y_{c,m}}^2 + {Y_{s,m}}^2 + {Z_{c,m}}^2 + {Z_{s,m}}^2)$ (*Here, I'm not quite sure if this simplification is appropriate.*) And we do not want this term changes our physics properties. So it's implemented in tangential variation.

$$\begin{aligned}
\delta x &= \dot{x}\delta u && (1)\\
\delta y &= \dot{y}\delta u && (2)\\
\delta z &= \dot{z}\delta u && (3)\\
\delta X_{c,m} &= \int_0^{2\pi} \delta x \cos(m\theta)\, \mathrm{d}\theta && (4)\\
\delta M &= \sum_{m=1,M} 2m^2 \left(X_{c,m}\delta X_{c,m} + X_{s,m}\delta X_{s,m} + Y_{c,m}\delta Y_{c,m} + Y_{s,m}\delta Y_{s,m} + X_{c,m}\delta X_{c,m} + X_{s,m}\delta X_{s,m}\right)\\
&= \sum_{m=1,M} 2m^2 \left(X_{c,m}\int_0^{2\pi} \dot{x}\delta u \cos(m\theta)\, \mathrm{d}\theta + \ldots + Z_{s,m}\int_0^{2\pi} \dot{x}\delta u \cos(m\theta)\, \mathrm{d}\theta\right)\\
&= 2\int_0^{2\pi}\left(\sum_{m=1,M} m^2 X_{c,m}\cos(m\theta)\dot{x} + \ldots + \sum_{m=1,M} m^2 Z_{c,m}\cos(m\theta)\dot{z}\right)\delta u\, \mathrm{d}\theta\\
&= -2\int_0^{2\pi}\left(\ddot{x}\dot{x} + \ldots + \ddot{z}\dot{z}\right)\delta u\, \mathrm{d}\theta && (5)
\end{aligned}$$

2. Thus, if we want to minimize M, make $\delta M = 0$, which equals making $\ddot{x}\dot{x} + \ddot{y}\dot{y} + \ddot{z}\dot{z} = 0$ or $\dot{x}^2 + \dot{y}^2 + \dot{z}^2 = const.$. That's so-called equal-arclength.

3. In later constructions, we will use the equal-arclength constraint, rather than minimize spectral width directly.

1.2 Differential flow A

1. For differential flow, we can just include the equal-arclength condition and to avoid the length term, we use $\ddot{x}\dot{x} + \ddot{y}\dot{y} + \ddot{z}\dot{z} = 0$. So the equal-arclength constraint can be written as,

$$A = \frac{1}{2}\sum_i \int_0^{2\pi} (\ddot{x}\dot{x} + \ddot{y}\dot{y} + \ddot{z}\dot{z})^2\, \mathrm{d}\theta \tag{6}$$

2. The equal-arclength contraint's first derivatives can be written as,

$$\begin{aligned}
\frac{\partial A}{\partial x^i_m} &= \int_0^{2\pi}\left(\frac{\partial \ddot{x}}{\partial x^i_m}\dot{x} + \frac{\partial \dot{x}}{\partial x^i_m}\ddot{x}\right)(\ddot{x}\dot{x} + \ddot{y}\dot{y} + \ddot{z}\dot{z})\, \mathrm{d}\theta && (7)\\
x &= \sum_n X_{c,n}\cos(n\theta) + X_{s,n}\sin n\theta && (8)\\
\dot{x} &= \sum_n -nX_{c,n}\sin(n\theta) + nX_{s,n}\cos n\theta && (9)\\
\ddot{x} &= \sum_n -n^2X_{c,n}\cos(n\theta) - n^2X_{s,n}\sin n\theta && (10)
\end{aligned}$$

3. And the second derivatives can be written as, (*Actually, there is no need constructing the second derivatives for differential flow method.*)

$$\begin{aligned}
\frac{\partial^2 A}{\partial x_m^i \partial x_n^i} &= \int_0^{2\pi} \left(\frac{\partial \ddot{x}}{\partial x_m^i} \frac{\partial \dot{x}}{\partial x_n^i} + \frac{\partial \dot{x}}{\partial x_m^i} \frac{\partial \ddot{x}}{\partial x_n^i} \right) (\ddot{x}\dot{x} + \ddot{y}\dot{y} + \ddot{z}\dot{z}) + \left(\frac{\partial \ddot{x}}{\partial x_m^i} \dot{x} + \frac{\partial \dot{x}}{\partial x_m^i} \ddot{x} \right) \left(\frac{\partial \ddot{x}}{\partial x_n^i} \dot{x} + \frac{\partial \dot{x}}{\partial x_n^i} \ddot{x} \right) \mathrm{d}\theta \\
\frac{\partial^2 A}{\partial x_m^i \partial y_n^i} &= \int_0^{2\pi} \left(\frac{\partial \ddot{x}}{\partial x_m^i} \dot{x} + \frac{\partial \dot{x}}{\partial x_m^i} \ddot{x} \right) \left(\frac{\partial \ddot{y}}{\partial y_n^i} \dot{y} + \frac{\partial \dot{y}}{\partial y_n^i} \ddot{y} \right) \mathrm{d}\theta
\end{aligned} \tag{11}$$

1.3 Differential flow B

1. In the above method, we are trying to minimize the line integral of equal arclength constraint, which will be zero only when the integrand equals zero in the whole domain. That is hard to accomplish. Thus, similar as what we did with the Langrange multipiler, we discretize the distribution function of the integrand using Fourier transformation and try to minimize the square sum of the Fourier coefficients to zero.

$$\begin{aligned}
f(\theta) &= \ddot{x}\dot{x} + \ddot{y}\dot{y} + \ddot{z}\dot{z} \\
&= \sum_{n=0,N} f_{c,n} \cos(n\theta) + f_{s,n} \sin(n\theta) \qquad (12) \\
f_{c,n} &= \frac{1}{\pi} \int_0^{2\pi} f(\theta) \cos(n\theta) \, \mathrm{d}\theta \qquad (13) \\
f_{s,n} &= \frac{1}{\pi} \int_0^{2\pi} f(\theta) \sin(n\theta) \, \mathrm{d}\theta \qquad (14) \\
A &= \frac{1}{2} \sum_i \sum_n ({f_{c,n}}^2 + {f_{s,n}}^2) \qquad (15)
\end{aligned}$$

2. If the equal arclength constraint is written as above equation, then its derivatives can be derived as,

$$\frac{\partial A}{\partial x_m^i} = \sum_n \left(f_{c,n}^i \frac{\partial f_{c,n}}{\partial x_m^i} + f_{s,n}^i \frac{\partial f_{s,n}}{\partial x_m^i} \right) \tag{16}$$

$$\frac{\partial^2 A}{\partial x_p^i \partial x_q^i} = \sum_n \left(\frac{\partial f_{c,n}}{\partial x_p^i} \frac{\partial f_{c,n}}{\partial x_q^i} + f_{c,n}^i \frac{\partial^2 f_{c,n}}{\partial x_p^i \partial x_q^i} + \frac{\partial f_{s,n}}{\partial x_p^i} \frac{\partial f_{s,n}}{\partial x_q^i} + f_{s,n}^i \frac{\partial^2 f_{s,n}}{\partial x_p^i \partial x_q^i} \right) \tag{17}$$

1.4 Newton method

1. For minimizing problems subjecting to some conditions, the most effective method is Langrange multipiler. In this case, we can include :

$$E_{total} = E_{physics} + \sum_i \lambda_i(\theta)(\ddot{x}\dot{x} + \ddot{y}\dot{y} + \ddot{z}\dot{z}) \tag{18}$$

Here, $\lambda_i(\theta)$ is a function of θ. In the code, we do a FFT on the constraint function as

$$\begin{aligned}
f(\theta) &= \ddot{x}\dot{x} + \ddot{y}\dot{y} + \ddot{z}\dot{z} \\
&= \sum_{n=0,N} f_{c,n} \cos(n\theta) + f_{s,n} \sin(n\theta) \qquad (19) \\
f_{c,n} &= \frac{1}{\pi} \int_0^{2\pi} f(\theta) \cos(n\theta) \, \mathrm{d}\theta \qquad (20) \\
f_{s,n} &= \frac{1}{\pi} \int_0^{2\pi} f(\theta) \sin(n\theta) \, \mathrm{d}\theta \qquad (21)
\end{aligned}$$

And the new Langrange multipiler is,

$$E_{total} = E_{physics} + \sum_i \sum_n (\lambda_{i,c} f_{c,n} + \lambda_{i,s} f_{s,n}) \tag{22}$$

Here,i is denoted as coil label and n for fourier modes number.

2. First derivatives of the total "energy" function are,

$$\frac{\partial E_{total}}{\partial x_m^i} = \frac{\partial E_{physics}}{\partial x_m^i} + \sum_n \left(\lambda_{i,c} \frac{\partial f_{c,n}^i}{\partial x_m^i} + \lambda_{i,s} \frac{\partial f_{s,n}^i}{\partial x_m^i} \right) \tag{23}$$

$$\frac{\partial E_{total}}{\partial \lambda_{i,c}} = f_{c,n}^i \tag{24}$$

$$\frac{\partial E_{total}}{\partial \lambda_{i,s}} = f_{s,n}^i \tag{25}$$

And we have

$$\frac{\partial f_{c,n}}{\partial x_m^i} = \frac{1}{\pi} \int_0^{2\pi} \frac{\partial f(\theta)}{\partial x_m^i} \cos(n\theta)\, \mathrm{d}\theta \tag{26}$$

$$\frac{\partial f_{s,n}}{\partial x_m^i} = \frac{1}{\pi} \int_0^{2\pi} \frac{\partial f(\theta)}{\partial x_m^i} \sin(n\theta)\, \mathrm{d}\theta \tag{27}$$

$$\frac{\partial f(\theta)}{\partial x_m^i} = \frac{\partial \ddot{x}}{\partial x_m^i} \dot{x} + \frac{\partial \dot{x}}{\partial x_m^i} \ddot{x} \tag{28}$$

3. The second derivatives are written as,

$$\frac{\partial^2 E_{total}}{\partial \lambda_{i,c} \partial x_m^i} = \frac{\partial f_{c,n}^i}{\partial x_m^i} \tag{29}$$

$$\frac{\partial^2 E_{total}}{\partial x_m^i \partial x_n^i} = \frac{\partial^2 E_{phsyics}}{\partial x_m^i \partial x_n^i} + \sum_n \left(\lambda_{i,c} \frac{\partial^2 f_{c,n}^i}{\partial x_m^i \partial x_n^i} + \lambda_{i,s} \frac{\partial^2 f_{s,n}^i}{\partial x_m^i \partial x_n^i} \right) \tag{30}$$

And we have

$$\frac{\partial^2 f_{c,n}}{\partial x_m^i \partial x_n^i} = \frac{1}{\pi} \int_0^{2\pi} \frac{\partial^2 f(\theta)}{\partial x_m^i \partial x_n^i} \cos(n\theta)\, \mathrm{d}\theta \tag{31}$$

$$\frac{\partial^2 f_{s,n}}{\partial x_m^i \partial x_n^i} = \frac{1}{\pi} \int_0^{2\pi} \frac{\partial^2 f(\theta)}{\partial x_m^i \partial x_n^i} \sin(n\theta)\, \mathrm{d}\theta \tag{32}$$

$$\frac{\partial^2 f(\theta)}{\partial x_m^i \partial x_n^i} = \frac{\partial \ddot{x}}{\partial x_m^i} \frac{\partial \dot{x}}{\partial x_n^i} + \frac{\partial \dot{x}}{\partial x_m^i} \frac{\partial \ddot{x}}{\partial x_n^i} \tag{33}$$

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