# bnormal

Calculate the total bnormal of all coils on plasma surface and the derivatives with respect to coil geometry and currents, including the first and second dirivatives. Calling *bnormal(0), bnormal(1), bnormal(2)* calculates the $0-order$, $1^{st}-order$ and $2^{nd}-order$ derivatives respectively.

[called by: costfun.] [calls: bfield.]

## contents



## 1.1 Bnormal on plasma surface (0-order)

1. The "energy" of normal magnetic field is defined as the "quadratic-flux" on a given "plasma boundary",

$$\begin{aligned} bnorm &= \int_{\mathcal{S}} \frac{1}{2} (B_n)^2 \, ds \\ &= \Delta\theta \, \Delta\zeta \sum_{j,k} \sqrt{g}_{j,k} \frac{1}{2} \left(B_{n,j,k}\right)^2 \end{aligned} \tag{1}$$

   where

$$B_{n,j,k} \equiv n^x_{j,k} \sum\nolimits_i B^{x,i}_{j,k} + n^y_{j,k} \sum\nolimits_i B^{y,i}_{j,k} + n^z_{j,k} \sum\nolimits_i B^{z,i}_{j,k}, \tag{2}$$

   where $B^{x,i}_{j,k}$, $B^{y,i}_{j,k}$ and $B^{z,i}_{j,k}$ are the Cartesian components of the magnetic field, which depend explicity on the geometry of and the current in the $i$-th coil. The normal vector to the plasma boundary at the angular location $(\theta_{j,k}, \zeta_{j,k})$ is $\mathbf{n}_{j,k} \equiv n^x_{j,k}\, \mathbf{i} + n^y_{j,k}\, \mathbf{j} + n^z_{j,k}\, \mathbf{k}$. (This is pre-computed in surface.)

2. The resolution of the discretized surface integral is given by `Nteta` and `Nzeta` (see global and surface).

3. The magnetic field at $\mathbf{x} \equiv x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ is given by the Biot-Savart integral,

$$\mathbf{B} \equiv I \int_{\mathcal{C}} \frac{d\mathbf{l} \times \mathbf{r}}{r^3} \tag{3}$$

   where $\mathbf{r} \equiv \mathbf{x} - \bar{\mathbf{x}}$, where $\bar{\mathbf{x}}$ is a point on the plasma boundary

4. In component form, Eq.(3) is

$$B^x \equiv I \int_0^{2\pi} \frac{\dot{\bar{y}}(z-\bar{z}) - \dot{\bar{z}}(y-\bar{y})}{r^3} dt, \tag{4}$$

$$B^y \equiv I \int_0^{2\pi} \frac{\dot{\bar{z}}(x-\bar{x}) - \dot{\bar{x}}(z-\bar{z})}{r^3} dt, \tag{5}$$

$$B^z \equiv I \int_0^{2\pi} \frac{\dot{\bar{x}}(y-\bar{y}) - \dot{\bar{y}}(x-\bar{x})}{r^3} dt, \tag{6}$$

   where $\dot{\bar{x}} \equiv d\bar{x}/dt$, etc. and the Fourier representation of the curves is given in iccoil.

## 1.2 First derivatives

1. The derivatives of $bnorm$ with respect to the coil parameters, e.g. $\alpha^i$, take the form:

$$\frac{\partial bnorm}{\partial \alpha^i} = \Delta\theta\,\Delta\zeta \sum_{j,k} \sqrt{g}_{j,k} \left( n^x_{j,k} \sum\nolimits_i B^{x,i}_{j,k} + n^y_{j,k} \sum\nolimits_i B^{y,i}_{j,k} + n^z_{j,k} \sum\nolimits_i B^{z,i}_{j,k} \right) \left( n^x_{j,k} \frac{\partial B^{x,i}_{j,k}}{\partial \alpha^i} + n^y_{j,k} \frac{\partial B^{y,i}_{j,k}}{\partial \alpha^i} + n^z_{j,k} \frac{\partial B^{z,i}_{j,k}}{\partial \alpha^i} \right) \tag{7}$$

2. The integrals over $t$ are provided by NAG:D01EAF. Only the integrands are required:

$$\begin{aligned}
B^x &\equiv \frac{\Delta y\ \dot{z} - \Delta z\ \dot{y}}{r^3} \\
\frac{\partial B^x}{\partial x_{c,m}} &\equiv -B^x 3\Delta x \cos(mt)/r^2 \\
\frac{\partial B^x}{\partial y_{c,m}} &\equiv \frac{\cos(mt)\ \dot{z} + \Delta z\ \sin(mt) m}{r^3} - B^x 3\Delta y \cos(mt)/r^2 \\
\frac{\partial B^x}{\partial z_{c,m}} &\equiv \frac{-\Delta y\ \sin(mt) m - \cos(mt)\ \dot{y}}{r^3} - B^x 3\Delta z \cos(mt)/r^2 \\
\frac{\partial B^x}{\partial x_{s,m}} &\equiv -B^x 3\Delta x \sin(mt)/r^2 \\
\frac{\partial B^x}{\partial y_{s,m}} &\equiv \frac{\sin(mt)\ \dot{z} - \Delta z\ \cos(mt) m}{r^3} - B^x 3\Delta y \sin(mt)/r^2 \\
\frac{\partial B^x}{\partial z_{s,m}} &\equiv \frac{\Delta y\ \cos(mt) m - \sin(mt)\ \dot{y}}{r^3} - B^x 3\Delta z \sin(mt)/r^2
\end{aligned} \tag{8}$$

## 1.3 The second derivatives

1. The integrands in $B^x$, $B^y$ and $B^z$ can be expressed in a concise way:

$$B^i \equiv g^i r^{-3} \tag{9}$$

where,

$$g^i \equiv \varepsilon^{ijk}\ \Delta l^j\ \dot{l}^k \tag{10}$$

Here $\varepsilon^{ijk}$ is Levi-Civita symbol and for simplification I will omit this symbol in later $i, j, k$ cases.

2. Therefore, the derivatives of $B^x$, $B^y$ and $B^z$ integrands (both the first and second derivatives) can be expressed as:

$$\begin{aligned}
\frac{\partial B^i}{\partial x_l} &\equiv \frac{\partial g^i}{\partial x_l} r^{-3} - \frac{3}{r^4}\frac{\partial r}{\partial x_l} g^i \\
\frac{\partial^2 B^i}{\partial x_l \partial x_m} &\equiv \frac{\partial^2 g^i}{\partial x_l \partial x_m} r^{-3} - \frac{3}{r^4}\frac{\partial r}{\partial x_m}\frac{\partial g^i}{\partial x_l} + \frac{12}{r^5}\frac{\partial r}{\partial x_l}\frac{\partial r}{\partial x_m} g^i - \frac{3}{r^{-4}}\frac{\partial^2 r}{\partial x_l \partial x_m} g^i - \frac{3}{r^4}\frac{\partial r}{\partial x_l}\frac{\partial g^i}{\partial x_m}
\end{aligned} \tag{11}$$

3. In that case, $\frac{\partial^2 B^i}{\partial x_l \partial x_m}$ is just related to the derivatives of $g^i$ and $r$.So we can also write out all the derivatives of $g^i$ and $r$.

$$\begin{aligned}
\frac{\partial g^i}{\partial x_l} &\equiv \frac{\partial \Delta l^j}{\partial x_l}\dot{l}^k + \Delta l^j \frac{\partial \dot{l}^k}{\partial x_l} - \frac{\partial \Delta l^k}{\partial x_l}\dot{l}^j - \Delta l^k \frac{\partial \dot{l}^j}{\partial x_l} \\
\frac{\partial^2 g^i}{\partial x_l \partial x_m} &\equiv \frac{\partial^2 \Delta l^j}{\partial x_l \partial x_m}\dot{l}^k + \frac{\partial \Delta l^j}{\partial x_l}\frac{\partial \dot{l}^k}{\partial x_m} + \frac{\partial \Delta l^j}{\partial x_m}\frac{\partial \dot{l}^k}{\partial x_l} + \Delta l^j \frac{\partial^2 \dot{l}^k}{\partial x_l \partial x_m} \\
&\quad - \frac{\partial^2 \Delta l^k}{\partial x_l \partial x_m}\dot{l}^j - \frac{\partial \Delta l^k}{\partial x_l}\frac{\partial \dot{l}^j}{\partial x_m} - \frac{\partial \Delta l^k}{\partial x_m}\frac{\partial \dot{l}^j}{\partial x_l} - \Delta l^k \frac{\partial^2 \dot{l}^j}{\partial x_l \partial x_m} \\
\frac{\partial r}{\partial x_l^i} &\equiv \frac{\Delta l^i}{r}\frac{\partial \Delta l^i}{\partial x_l} \\
\frac{\partial^2 r}{\partial x_l^i \partial x_m^j} &\equiv \delta_j^i \frac{1}{r}\frac{\partial \Delta l^i}{\partial x_l}\frac{\partial \Delta l^j}{\partial x_m} - \frac{\Delta l^i}{r^2}\frac{\partial \Delta r}{\partial x_m}\frac{1}{r}\frac{\partial \Delta l^i}{\partial x_l}
\end{aligned} \tag{12}$$

4. The derivatives of $bnorm$ in Eq.(1) can be written as,

$$
\begin{aligned}
\frac{\partial bnorm}{\partial I^i} &\equiv \int_{\mathcal{S}} \sum_j I^j (B_x^j n_x + B_y^j n_y + B_z^j n_z)\,(B_x^i n_x + B_y^i n_y + B_z^i n_z)\, ds && (13)\\
\frac{\partial bnorm}{\partial x_n^i} &\equiv \int_{\mathcal{S}} \sum_j I^j (B_x^j n_x + B_y^j n_y + B_z^j n_z)\,(\frac{\partial B_x^i}{\partial x_n^i} n_x + \frac{\partial B_y^i}{\partial x_n^i} n_y + \frac{\partial B_z^i}{\partial x_n^i} n_z) I^i\, ds && (14)\\
\frac{\partial^2 bnorm}{\partial I^i \partial I^j} &\equiv \int_{\mathcal{S}} (B_x^j n_x + B_y^j n_y + B_z^j n_z)(B_x^i n_x + B_y^i n_y + B_z^i n_z)\, ds && (15)\\
\frac{\partial^2 bnorm}{\partial I^i \partial x_n^j} &\equiv \int_{\mathcal{S}} (B_x^i n_x + B_y^i n_y + B_z^i n_z)(\frac{\partial B_x^j}{\partial x_n^j} n_x + \frac{\partial B_y^j}{\partial x_n^j} n_y + \frac{\partial B_z^j}{\partial x_n^j} n_z) I^i\, ds && (16)\\
\frac{\partial^2 bnorm}{\partial x_m^i \partial x_n^j} &\equiv \int_{\mathcal{S}} (\frac{\partial B_x^j}{\partial x_n^j} n_x + \frac{\partial B_y^j}{\partial x_n^j} n_y + \frac{\partial B_z^j}{\partial x_n^j} n_z)(\frac{\partial B_x^i}{\partial x_n^i} n_x + \frac{\partial B_y^i}{\partial x_n^i} n_y + \frac{\partial B_z^i}{\partial x_n^i} n_z) I^i I^j\, ds && (17)\\
& && (18)
\end{aligned}
$$

### 1.4 Normalization

1. It's recommended to normalize all the cost functions, even the weights may need to be normalized. While dealing with bnormal function, it's normalized as,

$$
Bnorm \equiv \int_s \frac{1}{2} \frac{(\vec{B} \cdot \vec{n})^2}{|B|^2} ds \tag{19}
$$

2. For simplification, we can denote $Bn$ for $\vec{B} \cdot \vec{n}$ and $Bm$ for $|B|^2$. And their derivatives can be written as,

$$
\begin{aligned}
\frac{\partial Bn}{\partial I^i} &= B_x^i n_x + B_y^i n_y + B_z^i n_z \\
\frac{\partial Bn}{\partial x_m^i} &= (\tfrac{\partial B_x^i}{\partial x_m^i} n_x + \tfrac{\partial B_y^i}{\partial x_m^i} n_y + \tfrac{\partial B_z^i}{\partial x_m^i} n_z)\, I^i \\
\frac{\partial Bm}{\partial I^i} &= 2(B_x^i B_x + B_y^i B_y + B_z^i B_z) \\
\frac{\partial Bm}{\partial x_m^i} &= 2(\tfrac{\partial B_x^i}{\partial x_m^i} B_x + \tfrac{\partial B_y^i}{\partial x_m^i} B_y + \tfrac{\partial B_z^i}{\partial x_m^i} B_z)\, I^i
\end{aligned} \tag{20}
$$

Here, the superscript $i$ is denoted as the $i^th$ coil's current ($I$) or geometric variables ($x$). And $B_x$ (or $B_y$ and $B_z$) means the total magnetic field at the surface point, while $B_x^i$ means the magnetic field generated by the $i^th$ coil without timing current and Biot-Savart constant $\frac{\mu}{4\pi}$.

3. Similarily, we can also write down the second derivatives for $Bn$ and $Bm$.

$$
\begin{aligned}
\frac{\partial^2 Bn}{\partial I^i \partial I^j} &= 0 \\
\frac{\partial^2 Bn}{\partial I^i \partial x_n^j} &= \begin{cases} 0 & \text{, if i} \neq \text{j;} \\ \frac{\partial B_x^i}{\partial x_m^i} n_x + \frac{\partial B_y^i}{\partial x_m^i} n_y + \frac{\partial B_z^i}{\partial x_m^i} n_z & \text{, if i = j} \end{cases} \\
\frac{\partial^2 Bn}{\partial x_m^i \partial x_n^j} &= \begin{cases} 0 & \text{, if i} \neq \text{j;} \\ (\frac{\partial^2 B_x^i}{\partial x_m^i \partial x_n^i} n_x + \frac{\partial^2 B_y^i}{\partial x_m^i \partial x_n^i} n_y + \frac{\partial^2 B_z^i}{\partial x_m^i \partial x_n^i} n_z) & \text{, if i = j} \end{cases}
\end{aligned} \tag{21}
$$

$$
\begin{aligned}
\frac{\partial^2 Bm}{\partial I^i \partial I^j} &= 2(B_x^i B_x^j + B_y^i B_y^j + B_z^i B_z^j) \\
\frac{\partial^2 Bm}{\partial I^i \partial x_m^j} &= \begin{cases} 2(B_x^i \frac{\partial B_x^j}{\partial x_m^j} + B_y^i \frac{\partial B_y^j}{\partial x_m^j} + B_z^i \frac{\partial B_z^j}{\partial x_m^j}) I^j & \text{, if i} \neq \text{j;} \\ 2(B_x^i \frac{\partial B_x^j}{\partial x_m^j} + B_y^i \frac{\partial B_y^j}{\partial x_m^j} + B_z^i \frac{\partial B_z^j}{\partial x_m^j}) I^j + 2(\frac{\partial B_x^i}{\partial x_m^i} B_x + \frac{\partial B_y^i}{\partial x_m^i} B_y + \frac{\partial B_z^i}{\partial x_m^i} B_z) & \text{, if i = j} \end{cases} \\
\frac{\partial^2 Bm}{\partial x_m^i \partial x_n^j} &= \begin{cases} 2(\frac{\partial B_x^i}{\partial x_n^i} \frac{\partial B_x^j}{\partial x_m^j} + \frac{\partial B_y^i}{\partial x_n^i} \frac{\partial B_y^j}{\partial x_m^j} + \frac{\partial B_z^i}{\partial x_n^i} \frac{\partial B_z^j}{\partial x_m^j}) I^i I^j & \text{, if i} \neq \text{j;} \\ 2(\frac{\partial B_x^i}{\partial x_n^i} \frac{\partial B_x^j}{\partial x_m^j} + \frac{\partial B_y^i}{\partial x_n^i} \frac{\partial B_y^j}{\partial x_m^j} + \frac{\partial B_z^i}{\partial x_n^i} \frac{\partial B_z^j}{\partial x_m^j}) I^i I^j + 2(\frac{\partial^2 B_x^i}{\partial x_m^i \partial x_n^i} B_x + \frac{\partial^2 B_y^i}{\partial x_m^i \partial x_n^i} B_y + \frac{\partial^2 B_z^i}{\partial x_m^i \partial x_n^i} B_z) I^i & \text{, if i = j} \end{cases}
\end{aligned} \tag{22}
$$

4. So the derivatives of bnormal cost function can be represented as,

$$\begin{aligned}
\frac{\partial bnorm}{\partial x} &= \frac{Bn}{Bm}\frac{\partial Bn}{\partial x} - \frac{1}{2}\frac{Bn^2}{Bm^2}\frac{\partial Bm}{\partial x} \\
\frac{\partial^2 bnorm}{\partial x_1 \partial x_2} &= \frac{1}{Bm}\frac{\partial Bn}{\partial x_1}\frac{\partial Bn}{\partial x_2} - \frac{Bn}{Bm^2}\frac{\partial Bn}{\partial x_1}\frac{\partial Bm}{\partial x_2} + \frac{Bn}{Bm}\frac{\partial^2 Bn}{\partial x_1 \partial x_2} \\
& \frac{Bn^2}{Bm^3}\frac{\partial Bm}{\partial x_1}\frac{\partial Bm}{\partial x_2} - \frac{Bn}{Bm^2}\frac{\partial Bm}{\partial x_1}\frac{\partial Bn}{\partial x_2} - \frac{1}{2}\frac{Bn^2}{Bm^2}\frac{\partial^2 Bm}{\partial x_1 \partial x_2}
\end{aligned} \tag{23}$$

Here,$x, x_1, x_2$ can be both currents and geometric variables.

5. **Note:** *This normalization method works for 0-order cost function. But derivatives on current are actually divided by corresponding currents additionally.*