

PLSC 503 – Spring 2024

Variable Selection, Specification Bias, and Multiplicative Interactions

February 26, 2024

Among our OLS / linear regression assumptions:

- $\text{Cov}(\mathbf{X}, \mathbf{u}) = 0$, and
- the distribution of \mathbf{X} does not depend on either β or σ^2 .

“Truth”:

$$Y_i = \beta_0 + \beta_1 X_{1i} + \beta_2 X_{2i} + u_i$$

Fitted model:

$$Y_i = \gamma_0 + \gamma_1 X_{1i} + e_i$$

Then:

$$e_i = \beta_2 X_{2i} + u_i$$

Note:

$$\begin{aligned}E(e) &= E(\beta_2 X_2 + u) \\&= X_2 E(\beta_2) + E(u) \\&\neq 0\end{aligned}$$

$$\begin{aligned}E(\gamma_1) &= \beta_1 + \frac{\sum_{i=1}^N (X_{1i} - \bar{X}_1)(X_{2i} - \bar{X}_2)}{\sum_{i=1}^N (X_{1i} - \bar{X}_1)^2} \beta_2 \\&= \beta_1 + b_{X_2 X_1} \beta_2\end{aligned}$$

where $b_{X_2 X_1}$ is the “slope” coefficient one obtains from regressing X_2 on X_1 .

Omitted Variable Bias, continued

If X_2 is omitted and $\text{Cov}(X_1, X_2) = 0$, then

- $E(\hat{\gamma}_1) = \beta_1$, but
- $E(\hat{\gamma}_0) \neq \beta_0$.

If $\text{Cov}(X_1, X_2) \neq 0$ then

- $E(\hat{\gamma}_1) \neq \beta_1$ and $E(\hat{\gamma}_0) \neq \beta_0$
- In the simple bivariate case,
 - if $\text{Cov}(X_1, X_2) > 0$ then $E(|\hat{\gamma}_1|) > |\beta_1|$,
 - if $\text{Cov}(X_1, X_2) < 0$ then $E(|\hat{\gamma}_1|) < |\beta_1|$.

Omitted Variables and Inference

Recall that for one X :

$$\widehat{\text{Var}}(\hat{\beta}_1) = \frac{\hat{\sigma}^2}{\sum_{i=1}^N (X_i - \bar{X})^2}.$$

and for two X s:

$$\widehat{\text{Var}}(\hat{\beta}_1) = \frac{\hat{\sigma}^2}{\sum_{i=1}^N (X_i - \bar{X})^2 (1 - R_{X_1 X_2}^2)}$$

Also, because $\hat{e}_i \neq \hat{u}_i$,

$$E(\sigma_e^2) = \sigma_u^2 + f(\beta_2, X_1) \leftarrow \text{Bias}$$

Multivariate Regression

For the “true” DGP

$$\mathbf{Y} = \mathbf{X}\boldsymbol{\beta} + \mathbf{u}$$

and fitted model

$$\mathbf{Y} = \mathbf{Z}\boldsymbol{\Gamma} + \mathbf{e}$$

where $\mathbf{Z} \subset \mathbf{X}$, we have

$$\begin{aligned}\boldsymbol{\Gamma} &= (\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'(\mathbf{X}\boldsymbol{\beta} + \mathbf{u}) \\ &= (\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'\mathbf{X}\boldsymbol{\beta} + (\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'\mathbf{u}\end{aligned}$$

and so

$$\begin{aligned}E(\boldsymbol{\Gamma}) &= (\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'\mathbf{X}\boldsymbol{\beta} \\ &= \mathbf{P}\boldsymbol{\beta}.\end{aligned}$$

Now assume a “true” model:

$$\mathbf{Y} = \mathbf{X}\beta + \mathbf{u}$$

and fitted model:

$$\mathbf{Y} = \mathbf{Z}\Gamma + \mathbf{e}$$

where $\mathbf{X} \subset \mathbf{Z}$. This means:

$$\mathbf{Y} = \mathbf{X}\beta + \mathbf{W}\Theta + \mathbf{u}$$

where $\mathbf{Z} = \mathbf{X} \cup \mathbf{W}$ and $\Theta = 0$.

Results:

- $E(\hat{\beta}) = \beta$ and $E(\hat{\sigma}^2) = \sigma^2$, but
- $\widehat{\text{Var}}(\beta) > \text{Var}(\beta) \leftarrow$ Inefficiency

Implication: *Pre-Test Bias*

Omitted Variable Bias: Simulated Example

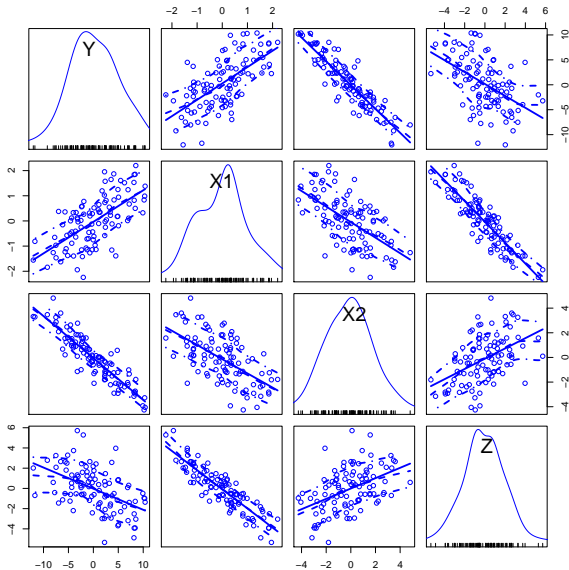
“True” model:

$$Y_i = 0 + 1.0X_{1i} - 2.0X_{2i} + u_i$$

Simulation:

```
> N <- 100
> X1<-rnorm(N)                # <- X1
> X2<-(-X1)+1.5*(rnorm(N))     # <- correlated w/X1
> Y<-X1-(2*X2)+(2*(rnorm(N))) # <- Y
> Z <- (-2*X1) + rnorm(N)      # <- correlated w/X1 but irrelevant
> data <- data.frame(Y=Y,X1=X1,X2=X2,Z=Z)
```

Scatterplot Matrix



Correctly Specified Model

```
> correct<-lm(Y~X1+X2)
> summary(correct)
```

Residuals:

Min	1Q	Median	3Q	Max
-5.721	-1.209	0.093	1.198	5.915

Coefficients:

	Estimate	Std. Error	t value	Pr(> t)
(Intercept)	-0.03311	0.21249	-0.156	0.87651
X1	0.81690	0.26718	3.057	0.00288 **
X2	-2.13652	0.13844	-15.433	< 2e-16 ***

Signif. codes: 0 *** 0.001 ** 0.01 * 0.05 . 0.1 1

Residual standard error: 2.116 on 97 degrees of freedom
Multiple R-squared: 0.8295, Adjusted R-squared: 0.826
F-statistic: 236 on 2 and 97 DF, p-value: < 2.2e-16

Overspecified Model

```
> overspec<-lm(Y~X1+X2+Z)
> summary(overspec)
```

Residuals:

Min	1Q	Median	3Q	Max
-5.9809	-1.0442	-0.0265	1.2609	6.0201

Coefficients:

	Estimate	Std. Error	t value	Pr(> t)
(Intercept)	-0.01570	0.21420	-0.073	0.94173
X1	0.82148	0.26785	3.067	0.00281 **
X2	-2.11735	0.14105	-15.011	< 2e-16 ***
Z	0.01662	0.02202	0.755	0.45220

Signif. codes: 0 *** 0.001 ** 0.01 * 0.05 . 0.1 1

Residual standard error: 2.12 on 96 degrees of freedom
Multiple R-squared: 0.8306, Adjusted R-squared: 0.8253
F-statistic: 156.8 on 3 and 96 DF, p-value: < 2.2e-16

Underspecified Model

```
> incorrect<-lm(Y~X1)
> summary(incorrect)
```

Residuals:

Min	1Q	Median	3Q	Max
-9.3297	-2.9762	-0.0672	2.4828	8.7787

Coefficients:

	Estimate	Std. Error	t value	Pr(> t)
(Intercept)	0.2704	0.3913	0.691	0.491
X1	3.2783	0.3964	8.270	6.71e-13 ***

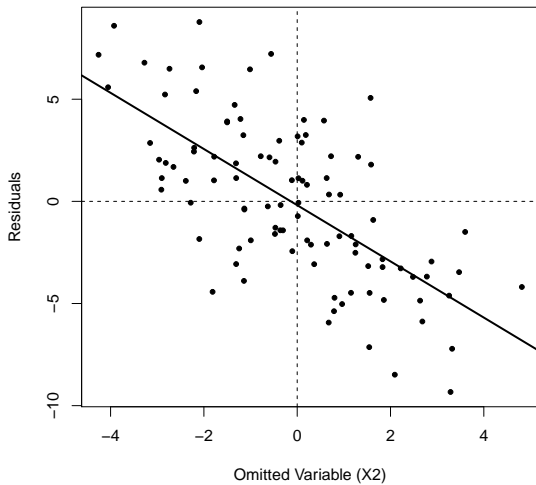
Signif. codes: 0 *** 0.001 ** 0.01 * 0.05 . 0.1 1

Residual standard error: 3.913 on 98 degrees of freedom

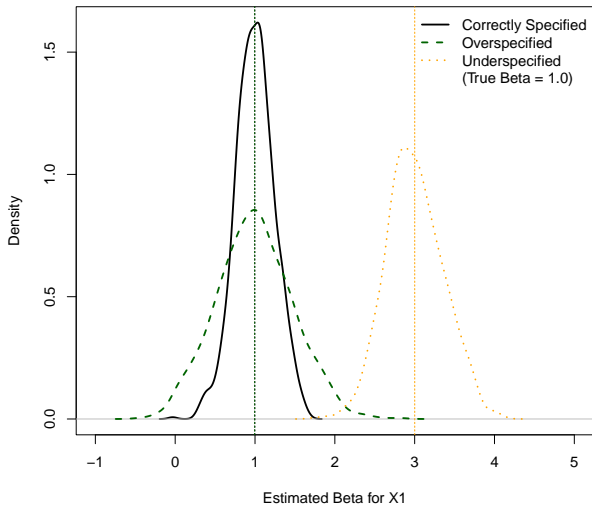
Multiple R-squared: 0.411, Adjusted R-squared: 0.405

F-statistic: 68.39 on 1 and 98 DF, p-value: 6.714e-13

Omitted Variable Plot



Repeated 1000 Times...



Nothing Beats a Good Theory. Period.

Also:

- “Model specification tests” \leftarrow meh
- Examine residuals
- Proxy variables...
- *Resist the urge to overspecify!*

Multiplicative Interactions

Our old friend:

$$\mathbf{Y} = \mathbf{X}\boldsymbol{\beta} + \mathbf{u}$$

$$Y_i = \beta_0 + \beta_1 D_{1i} + \beta_2 D_{2i} + u_i$$

means:

$$E(Y|D_1 = 1, D_2 = 0) - E(Y|D_1 = 0, D_2 = 0) = E(Y|D_1 = 1, D_2 = 1) - E(Y|D_1 = 0, D_2 = 1) [\equiv \beta_1]$$

and

$$E(Y|D_1 = 0, D_2 = 1) - E(Y|D_1 = 0, D_2 = 0) = E(Y|D_1 = 1, D_2 = 1) - E(Y|D_1 = 1, D_2 = 0) [\equiv \beta_2].$$

Consider:

$$Y_i = \beta_0 + \beta_1 X_{1i} + \beta_2 X_{2i} + \beta_3 X_{1i} X_{2i} + u_i$$

$$\begin{aligned} E(Y_i) &= \beta_0 + \beta_1 X_{1i} + \beta_2 X_{2i} + \beta_3 X_{1i} X_{2i} \\ &= \beta_0 + \beta_2 X_{2i} + (\beta_1 + \beta_3 X_{2i}) X_{1i} \\ &= \beta_0 + \beta_2 X_{2i} + \psi_1 X_{1i} \end{aligned}$$

where $\psi_1 = \beta_1 + \beta_3 X_{2i}$. This means:

$$\frac{\partial E(Y_i)}{\partial X_1} = \beta_1 + \beta_3 X_{2i}.$$

Similarly:

$$\begin{aligned} E(Y_i) &= \beta_0 + \beta_1 X_{1i} + (\beta_2 + \beta_3 X_{1i}) X_{2i} \\ &= \beta_0 + \beta_1 X_{1i} + \psi_2 X_{2i} \end{aligned}$$

which implies:

$$\frac{\partial E(Y_i)}{\partial X_2} = \beta_2 + \beta_3 X_{1i}.$$

If $X_2 = 0$, then:

$$\begin{aligned} E(Y_i) &= \beta_0 + \beta_1 X_{1i} + \beta_2(0) + \beta_3 X_{1i}(0) \\ &= \beta_0 + \beta_1 X_{1i}. \end{aligned}$$

Similarly, for $X_1 = 0$:

$$\begin{aligned} E(Y_i) &= \beta_0 + \beta_1(0) + \beta_2 X_{2i} + \beta_3(0)X_{2i} \\ &= \beta_0 + \beta_2 X_{2i} \end{aligned}$$

In most instances, the quantities we care about are not β_1 and β_2 , but rather ψ_1 and ψ_2 .

Point estimates:

$$\hat{\psi}_1 = \hat{\beta}_1 + \hat{\beta}_3 X_2$$

and

$$\hat{\psi}_2 = \hat{\beta}_2 + \hat{\beta}_3 X_1.$$

For variance, recall that:

$$\text{Var}(a + bZ) = \text{Var}(a) + Z^2 \text{Var}(b) + 2Z \text{Cov}(a, b)$$

Means that:

$$\widehat{\text{Var}}(\hat{\psi}_1) = \widehat{\text{Var}}(\hat{\beta}_1) + X_2^2 \widehat{\text{Var}}(\hat{\beta}_3) + 2X_2 \widehat{\text{Cov}}(\hat{\beta}_1, \hat{\beta}_3).$$

and

$$\widehat{\text{Var}}(\hat{\psi}_2) = \widehat{\text{Var}}(\hat{\beta}_2) + X_1^2 \widehat{\text{Var}}(\hat{\beta}_3) + 2X_1 \widehat{\text{Cov}}(\hat{\beta}_2, \hat{\beta}_3).$$

Types of Interactions: Dichotomous X s

For

$$Y_i = \beta_0 + \beta_1 D_{1i} + \beta_2 D_{2i} + \beta_3 D_{1i} D_{2i} + u_i$$

we have:

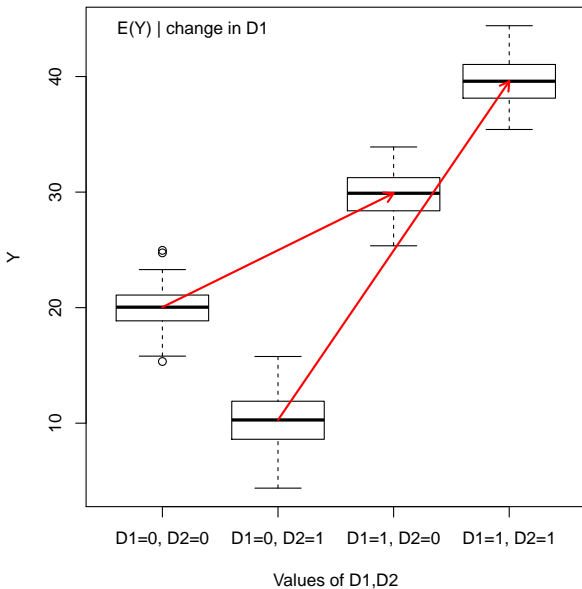
$$E(Y|D_1 = 0, D_2 = 0) = \beta_0$$

$$E(Y|D_1 = 1, D_2 = 0) = \beta_0 + \beta_1$$

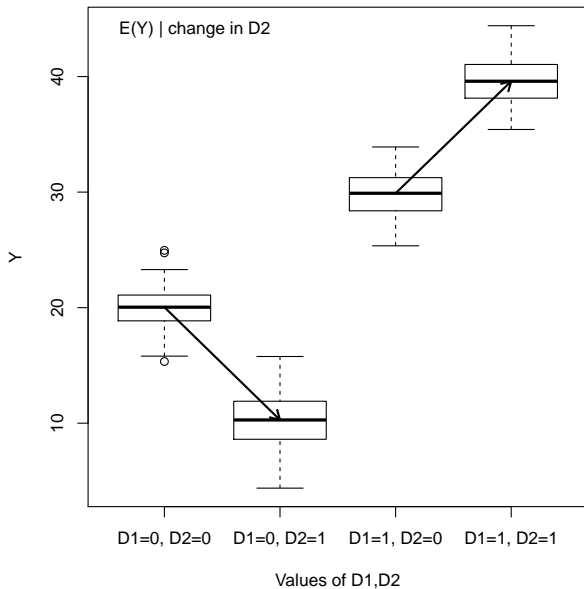
$$E(Y|D_1 = 0, D_2 = 1) = \beta_0 + \beta_2$$

$$E(Y|D_1 = 1, D_2 = 1) = \beta_0 + \beta_1 + \beta_2 + \beta_3$$

Values of $E(Y)$ for Changes in D_1



Values of $E(Y)$ for Changes in D_2



Dichotomous and Continuous X s

Writing:

$$Y_i = \beta_0 + \beta_1 X_i + \beta_2 D_i + \beta_3 X_i D_i + u_i$$

gives:

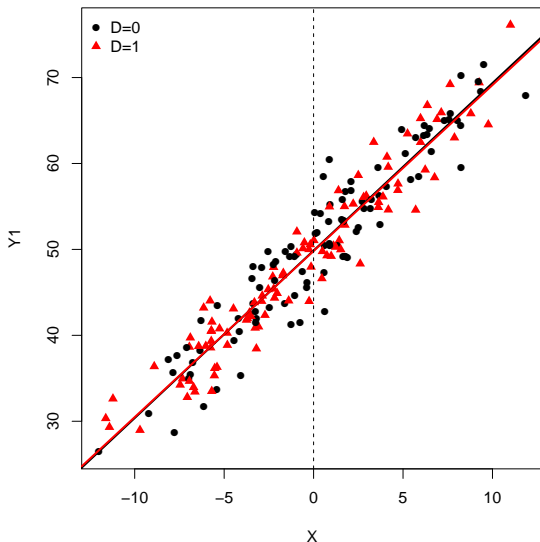
$$E(Y|X, D = 0) = \beta_0 + \beta_1 X$$

$$E(Y|X, D = 1) = (\beta_0 + \beta_2) + (\beta_1 + \beta_3)X$$

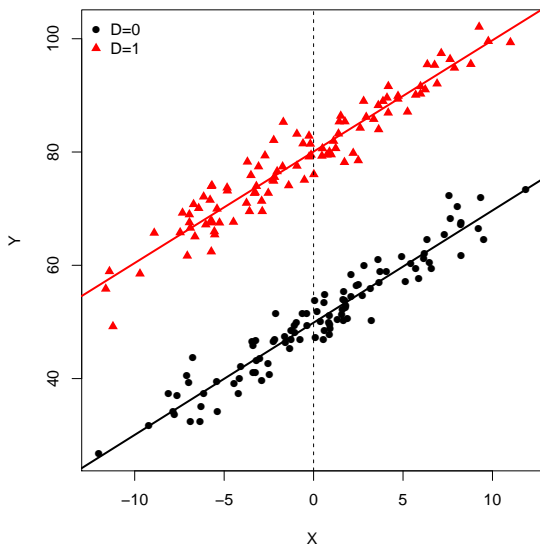
Four possibilities:

- $\beta_2 = \beta_3 = 0$
- $\beta_2 \neq 0$ and $\beta_3 = 0$
- $\beta_2 = 0$ and $\beta_3 \neq 0$
- $\beta_2 \neq 0$ and $\beta_3 \neq 0$

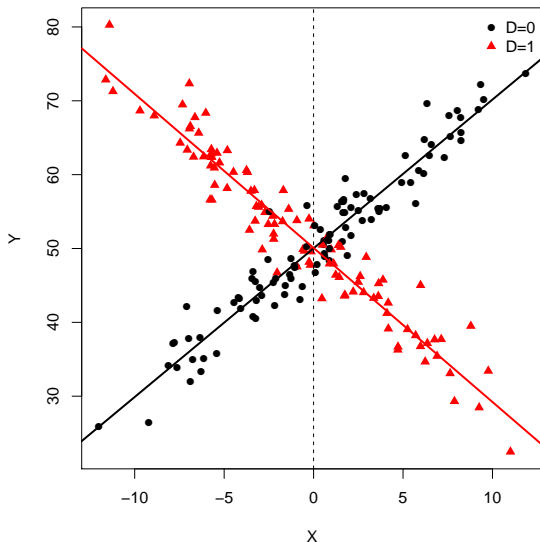
Scatterplot and Regression Lines of Y on X for $D = 0$ and $D = 1$: No Slope or Intercept Differences ($\beta_2 = \beta_3 = 0$)



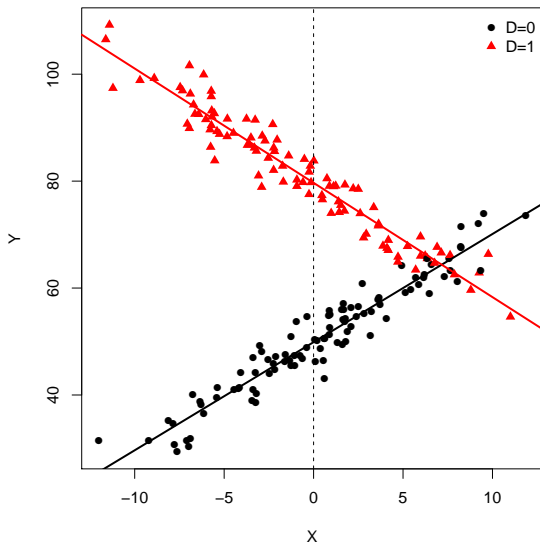
Scatterplot and Regression Lines of Y on X for $D = 0$ and $D = 1$: Intercept Shift ($\beta_2 \neq 0, \beta_3 = 0$)



Scatterplot and Regression Lines of Y on X for $D = 0$ and $D = 1$: Slope Change ($\beta_2 = 0$, $\beta_3 \neq 0$)



Scatterplot and Regression Lines of Y on X for $D = 0$ and $D = 1$: Slope and Intercept Change ($\beta_2 \neq 0$, $\beta_3 \neq 0$)



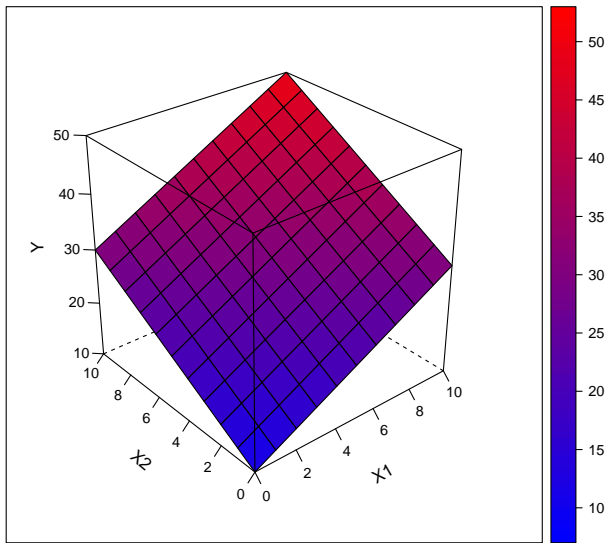
Two Continuous X s

$$Y_i = \beta_0 + \beta_1 X_{1i} + \beta_2 X_{2i} + \beta_3 X_{1i} X_{2i} + u_i.$$

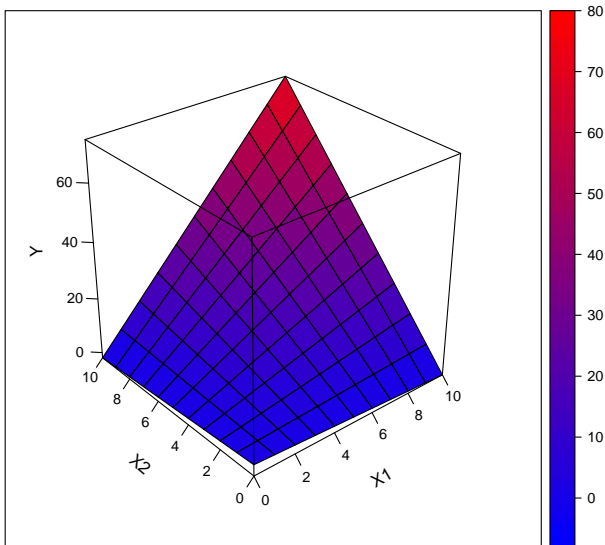
Implies

$$\beta_3 = 0 \rightarrow \frac{\partial E(Y)}{\partial X_1} = \beta_1 \forall X_2 \text{ and } \frac{\partial E(Y)}{\partial X_2} = \beta_2 \forall X_1$$

Two Continuous Variables: No Interactive Effects



Two Continuous Variables: Interaction Present



Quadratic, Cubic, and Other Polynomial Effects

$$Y_i = \beta_0 + \beta_1 X_i + \beta_2 X_i^2 + \beta_3 X_i^3 + \dots + \beta_j X_i^j + u_i.$$

In general:

$$\frac{\partial E(Y)}{\partial X} = \beta_1 + 2\beta_2 X + 3\beta_3 X^2 + \dots + j\beta_j X^{j-1}$$

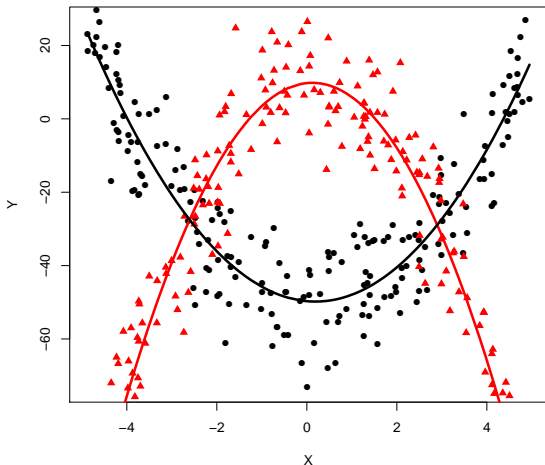
Quadratic case ($j = 2$):

$$Y_i = \beta_0 + \beta_1 X_i + \beta_2 X_i^2 + u_i.$$

implies

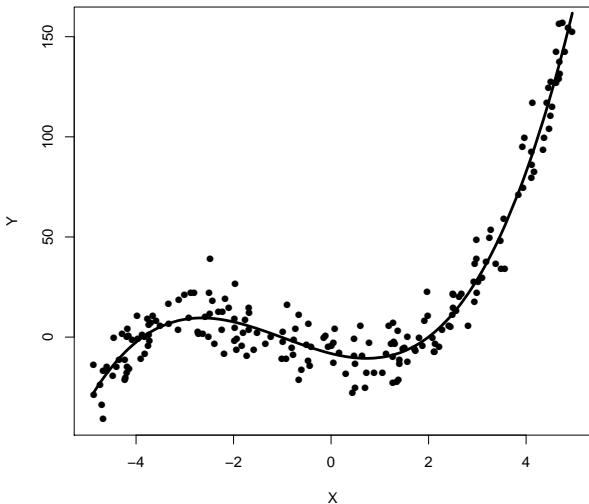
$$\frac{\partial E(Y)}{\partial X} = \beta_1 + 2\beta_2 X$$

Two Quadratic Relationships



Note: Red line is $Y_i = 10 + 1X_i - 5X_i^2 + u_i$; black line is $Y_i = -50 - 1X_i + 3X_i^2 + u_i$.

Example of a Cubic Relationship



Note: Solid line is $Y_i = -1 + 1X_i - 8X_i^2 + 5X_i^3 + u_i$.

Higher-Order Interactive Models

Three-way interaction:

$$Y_i = \beta_0 + \beta_1 X_{1i} + \beta_2 X_{2i} + \beta_3 X_{3i} + \\ \beta_4 X_{1i} X_{2i} + \beta_5 X_{1i} X_{3i} + \beta_6 X_{2i} X_{3i} + \beta_7 X_{1i} X_{2i} X_{3i} + u_i$$

Special case of dichotomous X_1, X_2 :

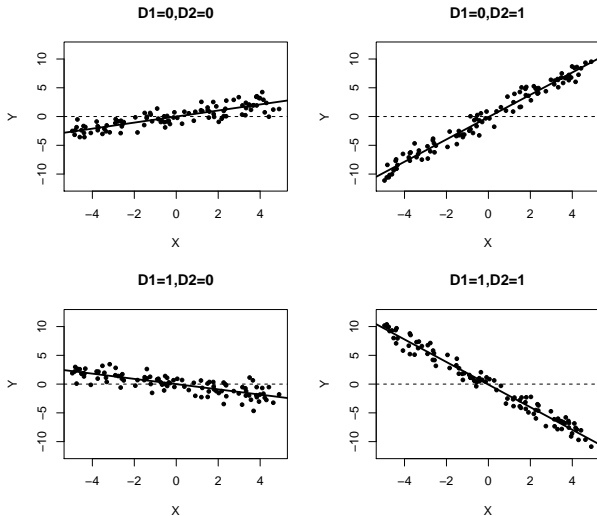
$$Y_i = \beta_0 + \beta_1 X_i + \beta_2 D_{1i} + \beta_3 D_{2i} + \\ \beta_4 X_i D_{1i} + \beta_5 X_i D_{2i} + \beta_6 D_{1i} D_{2i} + \beta_7 X_i D_{1i} D_{2i} + u_i$$

YO DAWG I HEARD YOU LIKE INTERACTIONS

**SO WE PUT SOME INTERACTIONS
IN YOUR INTERACTIONS**

imgflip.com

Three-Way Interaction: Two Dummy and One Continuous Covariates



Elderly 1996 NES Data

```
> ClintonTherm<-read.csv("ClintonTherm.csv")  
> summary(ClintonTherm)
```

caseid	ClintonTherm	RConserv	ClintonConserv
Min. :1001	Min. : 0	Min. :1.000	Min. :1.000
1st Qu.:1440	1st Qu.: 30	1st Qu.:3.000	1st Qu.:2.000
Median :1854	Median : 60	Median :4.000	Median :3.000
Mean :2001	Mean : 57	Mean :4.323	Mean :2.985
3rd Qu.:2262	3rd Qu.: 85	3rd Qu.:5.000	3rd Qu.:4.000
Max. :3403	Max. :100	Max. :7.000	Max. :7.000

PID	GOP
Min. :1.000	Min. :0.0000
1st Qu.:1.000	1st Qu.:0.0000
Median :2.000	Median :0.0000
Mean :2.059	Mean :0.3161
3rd Qu.:3.000	3rd Qu.:1.0000
Max. :5.000	Max. :1.0000

A Basic Regression

```
> summary(with(ClintonTherm, lm(ClintonTherm~RConserv+GOP)))
```

Call:

```
lm(formula = ClintonTherm ~ RConserv + GOP)
```

Coefficients:

	Estimate	Std. Error	t value	Pr(> t)
(Intercept)	93.4756	2.2278	41.96	<2e-16 ***
RConserv	-6.4866	0.5373	-12.07	<2e-16 ***
GOP	-26.6699	1.6056	-16.61	<2e-16 ***

Signif. codes: 0 *** 0.001 ** 0.01 * 0.05 . 0.1 1

Residual standard error: 23.65 on 1294 degrees of freedom

Multiple R-squared: 0.3795, Adjusted R-squared: 0.3786

F-statistic: 395.7 on 2 and 1294 DF, p-value: < 2.2e-16

An Interactive Model

```
> fit1<-with(ClintonTherm,  
             lm(ClintonTherm~RConserv+GOP+RConserv*GOP))
```

```
> summary(fit1)
```

Coefficients:

	Estimate	Std. Error	t value	Pr(> t)	
(Intercept)	89.927	2.487	36.16	<2e-16	***
RConserv	-5.571	0.609	-9.15	<2e-16	***
GOP	-6.484	6.569	-0.99	0.3238	
RConserv:GOP	-4.058	1.281	-3.17	0.0016	**

Signif. codes: 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 23.6 on 1293 degrees of freedom

Multiple R-squared: 0.384, Adjusted R-squared: 0.383

F-statistic: 269 on 3 and 1293 DF, p-value: <2e-16

Same Model, (Slightly) Different Formula

```
> fit1a<-with(ClintonTherm,  
              lm(ClintonTherm~(RConserv+GOP)^2))
```

```
> summary(fit1a)
```

Coefficients:

	Estimate	Std. Error	t value	Pr(> t)	
(Intercept)	89.927	2.487	36.16	<2e-16	***
RConserv	-5.571	0.609	-9.15	<2e-16	***
GOP	-6.484	6.569	-0.99	0.3238	
RConserv:GOP	-4.058	1.281	-3.17	0.0016	**

Signif. codes: 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 23.6 on 1293 degrees of freedom

Multiple R-squared: 0.384, Adjusted R-squared: 0.383

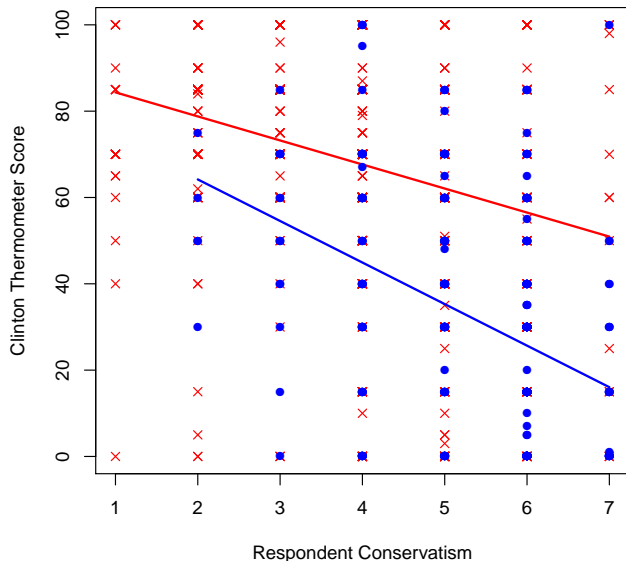
F-statistic: 269 on 3 and 1293 DF, p-value: <2e-16

Two Regressions, Sort Of

$$\begin{aligned} E(\text{Thermometer} \mid \text{Non-GOP})_i &= 89.9 - 6.5(0) - 5.6(\text{R's Conservatism}_i) \\ &\quad - 4.0(0 \times \text{R's Conservatism}_i) \\ &= \mathbf{89.9 - 5.6(\text{R's Conservatism}_i)} \end{aligned}$$

$$\begin{aligned} E(\text{Thermometer} \mid \text{GOP})_i &= [89.9 - 6.5(1)] + [-5.6 - 4.0(1 \times \text{R's Conservatism}_i)] \\ &= \mathbf{83.4 - 9.6(\text{R's Conservatism}_i)} \end{aligned}$$

Thermometer Scores by Conservatism, GOP and Non-GOP



Interactive Results are (Almost) Identical to Separate Regressions

```
> NonReps<-subset(ClintonTherm,GOP==0)
> summary(with(NonReps, lm(ClintonTherm~RConserv)))
```

Call:

```
lm(formula = ClintonTherm ~ RConserv, data = NonReps)
```

Coefficients:

	Estimate	Std. Error	t value	Pr(> t)
(Intercept)	89.9271	2.4695	36.416	<2e-16 ***
RConserv	-5.5705	0.6043	-9.217	<2e-16 ***

Signif. codes: 0 *** 0.001 ** 0.01 * 0.05 . 0.1 1

Residual standard error: 23.41 on 885 degrees of freedom

Multiple R-squared: 0.08759, Adjusted R-squared: 0.08656

F-statistic: 84.96 on 1 and 885 DF, p-value: < 2.2e-16

Interactive Results are (Almost) Identical to Separate Regressions

```
> Reps<-subset(ClintonTherm,GOP==1)
> summary(with(Reps, lm(ClintonTherm~RConserv)))
```

Call:

```
lm(formula = ClintonTherm ~ RConserv, data = Reps)
```

Coefficients:

	Estimate	Std. Error	t value	Pr(> t)
(Intercept)	83.443	6.170	13.524	< 2e-16 ***
RConserv	-9.629	1.144	-8.419	6.52e-16 ***

Signif. codes: 0 *** 0.001 ** 0.01 * 0.05 . 0.1 1

Residual standard error: 23.92 on 408 degrees of freedom

Multiple R-squared: 0.148, Adjusted R-squared: 0.1459

F-statistic: 70.88 on 1 and 408 DF, p-value: 6.518e-16

Discovering $\hat{\psi}_1$ and $\hat{\psi}_2$

For RConserv:

$$\begin{aligned}\text{Clinton Thermometer}_i &= \beta_0 + (\beta_1 + \beta_3 \text{GOP}_i) \text{R's Conservatism}_i + \\ &\quad \beta_2 \text{GOP}_i + u_i \\ &= \beta_0 + \psi_{1i} \text{R's Conservatism}_i + \beta_2 \text{GOP}_i + u_i.\end{aligned}$$

So:

$$\hat{\psi}_{1i} = \hat{\beta}_1 + \hat{\beta}_3 \times \text{GOP}_i$$

and

$$\hat{\sigma}_{\psi_1} = \sqrt{\widehat{\text{Var}}(\hat{\beta}_1) + (\text{GOP})^2 \widehat{\text{Var}}(\hat{\beta}_3) + 2(\text{GOP}) \widehat{\text{Cov}}(\hat{\beta}_1, \hat{\beta}_3)}.$$

Discovering $\hat{\psi}_1$ and $\hat{\psi}_2$

For GOP:

$$\begin{aligned}\text{Clinton Thermometer}_i &= \beta_0 + (\beta_2 + \beta_3 \times \text{R's Conservatism}_i)\text{GOP}_i + \\ &\quad \beta_1(\text{R's Conservatism}_i) + u_i \\ &= \beta_0 + \psi_{2i}\text{GOP}_i + \beta_1(\text{R's Conservatism}_i) + u_i.\end{aligned}$$

So:

$$\hat{\psi}_{2i} = \hat{\beta}_2 + \hat{\beta}_3 \times (\text{R's Conservatism}_i).$$

and

$$\hat{\sigma}_{\psi_2} = \sqrt{\widehat{\text{Var}}(\hat{\beta}_2) + (\text{R's Conservatism}_i)^2 \widehat{\text{Var}}(\hat{\beta}_3) + 2k \widehat{\text{Cov}}(\hat{\beta}_2, \hat{\beta}_3)}.$$

Discovering $\hat{\psi}_1$ and $\hat{\psi}_2$

```
> Psi1<-fit1$coeff[2]+fit1$coeff[4]

> Psi1
  RConserv
-9.628577

> SPsi1<-sqrt(vcov(fit1)[2,2] + (1)^2*vcov(fit1)[4,4] + 2*1*vcov(fit1)[2,4])
> SPsi1
[1] 1.127016

> Psi1 / SPsi1 # <-- t-statistic
  RConserv
-8.543422
```

Discovering $\hat{\psi}_1$ and $\hat{\psi}_2$

```
> # psi_2 | RConserv = 1
> fit1$coeff[3]+(1 * fit1$coeff[4])
      GOP
-10.54208

> sqrt(vcov(fit1)[3,3] + (1)^2*vcov(fit1)[4,4] + 2*1*vcov(fit1)[3,4])
[1] 5.335847

# Implies t is approximately 2

> # psi_2 | RConserv = 7
> fit1$coeff[3]+(7 * fit1$coeff[4])
      GOP
-34.89045

> sqrt(vcov(fit1)[3,3] + (7)^2*vcov(fit1)[4,4] + 2*7*vcov(fit1)[3,4])
[1] 3.048302

# t is approximately 11
```

An Easier Way: linearHypothesis()

```
> library(car)
> linearHypothesis(fit1,"RConserv+RConserv:GOP")
Linear hypothesis test
```

Hypothesis:

RConserv + RConserv:GOP = 0

Model 1: restricted model

Model 2: ClintonTherm ~ RConserv + GOP + RConserv * GOP

	Res.Df	RSS	Df	Sum of Sq	F	Pr(>F)
1	1294	758714				
2	1293	718173	1	40541	72.99	< 2.2e-16 ***

Signif. codes: 0 *** 0.001 ** 0.01 * 0.05 . 0.1 1

> # Note: Same as t-test:

```
> sqrt(72.99)
```

```
[1] 8.543419
```

An Easier Way: linearHypothesis()

```
> # psi_2 | RConserv = 7:  
> linearHypothesis(fit1,"GOP+7*RConserv:GOP")  
Linear hypothesis test
```

Hypothesis:

$\text{GOP} + 7 \text{ RConserv:GOP} = 0$

Model 1: restricted model

Model 2: ClintonTherm ~ RConserv + GOP + RConserv * GOP

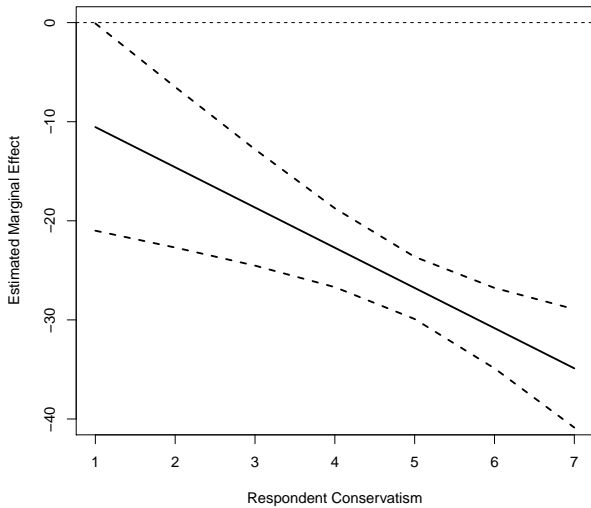
	Res.Df	RSS	Df	Sum of Sq	F	Pr(>F)
1	1294	790938				
2	1293	718173	1	72766	131.01	< 2.2e-16 ***

Signif. codes: 0 *** 0.001 ** 0.01 * 0.05 . 0.1 1

Marginal Effects Plots, I

```
> ConsSim<-seq(1,7,1)
> psis<-fit1$coeff[3]+(ConsSim * fit1$coeff[4])
> psis.ses<-sqrt(vcov(fit1)[3,3] +
  (ConsSim)^2*vcov(fit1)[4,4] + 2*ConsSim*vcov(fit1)[3,4])

> plot(ConsSim,psis,t="l",lwd=2,xlab="Respondent Conservatism",
  ylab="Estimated Marginal Effect",ylim=c(-40,0))
> lines(ConsSim,psis+(1.96*psis.ses),lty=2,lwd=2)
> lines(ConsSim,psis-(1.96*psis.ses),lty=2,lwd=2)
> abline(h=0,lwd=1,lty=2)
```



Interacting Two Continuous Covariates

```
> fit2<-with(ClintonTherm,  
+           lm(ClintonTherm~RConserv+ClintonConserv+RConserv*ClintonConserv))  
> summary(fit2)
```

Call:

```
lm(formula = ClintonTherm ~ RConserv + ClintonConserv + RConserv *  
    ClintonConserv)
```

Coefficients:

	Estimate	Std. Error	t value	Pr(> t)
(Intercept)	119.3515	5.1634	23.115	< 2e-16 ***
RConserv	-19.5673	1.0362	-18.884	< 2e-16 ***
ClintonConserv	-7.9311	1.6477	-4.813	1.66e-06 ***
RConserv:ClintonConserv	3.6293	0.3394	10.695	< 2e-16 ***

Signif. codes: 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 22.03 on 1293 degrees of freedom

Multiple R-squared: 0.4619, Adjusted R-squared: 0.4606

F-statistic: 370 on 3 and 1293 DF, p-value: < 2.2e-16

Hypothesis Tests

```
> fit2$coef[2]+(1*fit2$coef[4])
```

```
RConserv
```

```
-15.93803
```

```
> sqrt(vcov(fit2)[2,2] + (1)^2*vcov(fit2)[4,4] + 2*1*vcov(fit2)[2,4])
```

```
[1] 0.7439696
```

```
> linearHypothesis(fit2,"RConserv+1*RConserv:ClintonConserv")
```

```
Linear hypothesis test
```

```
Hypothesis:
```

```
RConserv + RConserv:ClintonConserv = 0
```

```
Model 1: restricted model
```

```
Model 2: ClintonTherm ~ RConserv + ClintonConserv + RConserv * ClintonConserv
```

	Res.Df	RSS	Df	Sum of Sq	F	Pr(>F)
1	1294	850442				
2	1293	627658	1	222784	458.94	< 2.2e-16 ***

```
---
```

```
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

More Hypothesis Tests

```
> # psi_1 | ClintonConserv = mean
> fit2$coef[2]+((mean(ClintonTherm$ClintonConserv))*fit2$coef[4])

RConserv
-8.735424

> sqrt(vcov(fit2)[2,2] + (mean(ClintonTherm$ClintonConserv)^2*vcov(fit2)[4,4] +
+ 2*(mean(ClintonTherm$ClintonConserv))*vcov(fit2)[2,4]))

[1] 0.4507971

> pt(((fit2$coef[2]+(2.985*fit2$coef[4])) / sqrt(vcov(fit2)[2,2] +
+ (2.985)^2*vcov(fit2)[4,4] + 2*2.985*vcov(fit2)[2,4])),df=1293)

RConserv
6.483788e-74

> # psi_2 | RConserv = 1
> fit2$coef[3]+(1*fit2$coef[4])

ClintonConserv
-4.301803

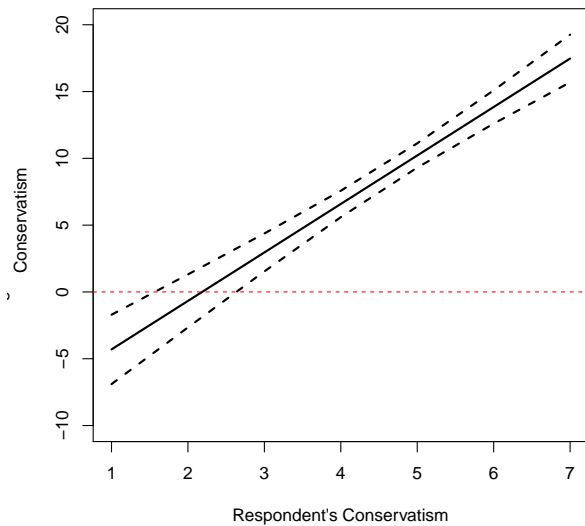
> # psi_2 | RConserv = 6
> fit2$coef[3]+(6*fit2$coef[4])

ClintonConserv
13.84463
```

Marginal Effect Plot, II

```
> psis2<-fit2$coef[3]+(ConsSim*fit2$coef[4])
> psis2.ses<-sqrt(vcov(fit2)[3,3] + (ConsSim)^2*vcov(fit2)[4,4]
+ 2*ConsSim*vcov(fit2)[3,4])

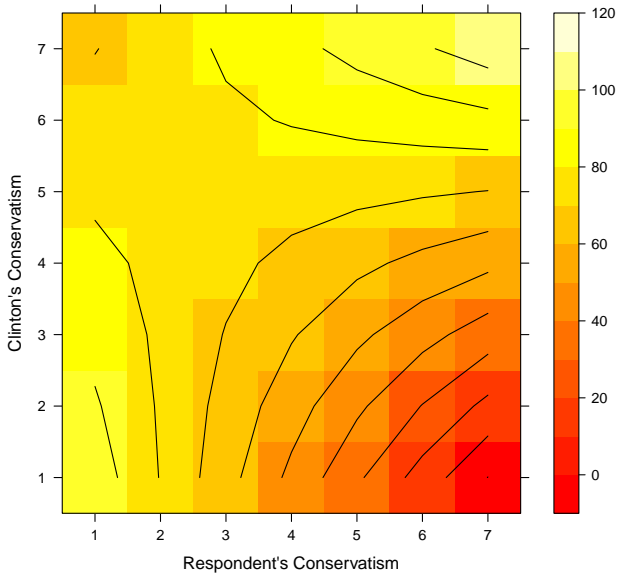
> plot(ConsSim,psis2,t="l",lwd=2,xlab="Respondent's
  Conservatism",ylab="Marginal Effect of Clinton's
  Conservatism",ylim=c(-10,20))
> lines(ConsSim,psis2+(1.96*psis2.ses),lty=2,lwd=2)
> lines(ConsSim,psis2-(1.96*psis2.ses),lty=2,lwd=2)
> abline(h=0,lty=2,lwd=1,col="red")
```



Predicted Values: A Contour Plot

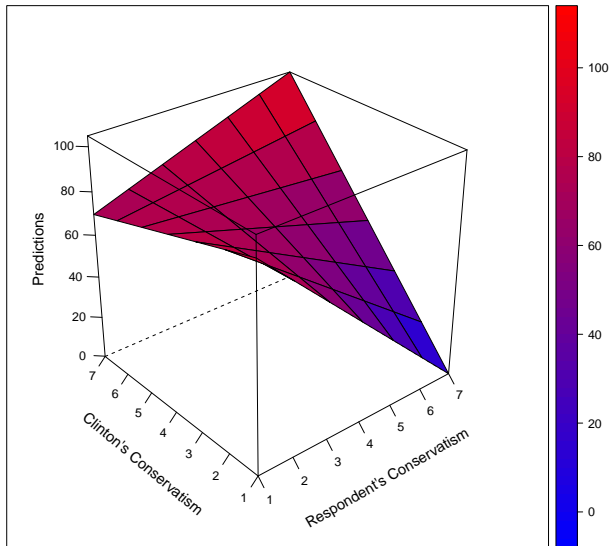
```
> library(lattice)
> grid<-expand.grid(RConserv=seq(1,7,1),
  ClintonConserv=seq(1,7,1))
> hats<-predict(fit2,newdata=grid)

> levelplot(hats~grid$RConserv*grid$ClintonConserv,
  contour=TRUE,
  cuts=12,pretty=TRUE,xlab="Respondent's Conservatism",
  ylab="Clinton's Conservatism",
  col.regions=heat.colors)
```

Predicted Values: A Wireframe Plot

```
> trellis.par.set("axis.line",list(col="transparent"))  
  
> wireframe(hats~grid$RConserv*grid$ClintonConserv,  
  drape=TRUE,  
  xlab=list("Respondent's Conservatism",rot=30),  
  ylab=list("Clinton's Conservatism",  
    rot=-40),zlab=list("Predictions",rot=90),  
  scales=list(arrows=FALSE,col="black"),  
  zoom=0.85,pretty=TRUE),  
  col.regions=colorRampPalette(c("blue","red"))(100))
```



Interpreting Interactions: Other Tools

A partial list of packages and things:

- `interactions` ([vignette](#))
- `emmeans` (see, e.g., [this vignette](#))
- `plot_model` in `sjPlot` ([tutorial](#))
- `interactionR`
- `interplot`
- `InteractionPowerR` (power analyses for interactive models)