PLSC 503 – Spring 2024
Regression Models for Nominal and Ordinal Outcomes

April 8, 2024

Outcome variable has $J > 2$ *unordered* categories:

$$Y_i \in \{1, 2, ...J\}$$

Write:

$$\Pr(Y_i = j) = P_{ij}$$

Means that:

$$\sum_{j=1}^{J} P_{ij} = 1$$

And set:

$$P_{ij} = \exp(\mathbf{X}_i \boldsymbol{\beta}_j)$$

Rescale:

$$\Pr(Y_i = j) \equiv P_{ij} = \frac{\exp(\mathbf{X}_i \boldsymbol{\beta}_j)}{\sum_{j=1}^{J} \exp(\mathbf{X}_i \boldsymbol{\beta}_j)}$$

Ensures

- $\Pr(Y_i = j) \in (0, 1)$
- $\sum_{j=1}^{J} \Pr(Y_i = j) = 1.0$

Constrain $\boldsymbol{\beta}_1 = \mathbf{0}$; then:

$$\Pr(Y_i = 1) = \frac{1}{1 + \sum_{j=2}^{J} \exp(\mathbf{X}_i \boldsymbol{\beta}'_j)}$$

$$\Pr(Y_i = j) = \frac{\exp(\mathbf{X}_i \boldsymbol{\beta}'_j)}{1 + \sum_{j=2}^{J} \exp(\mathbf{X}_i \boldsymbol{\beta}'_j)}$$

where $\boldsymbol{\beta}'_j = \boldsymbol{\beta}_j - \boldsymbol{\beta}_1$.

Alternative Motivation: Discrete *Choice*

Utility:

$$U_{ij} = \mu_i + \epsilon_{ij}$$

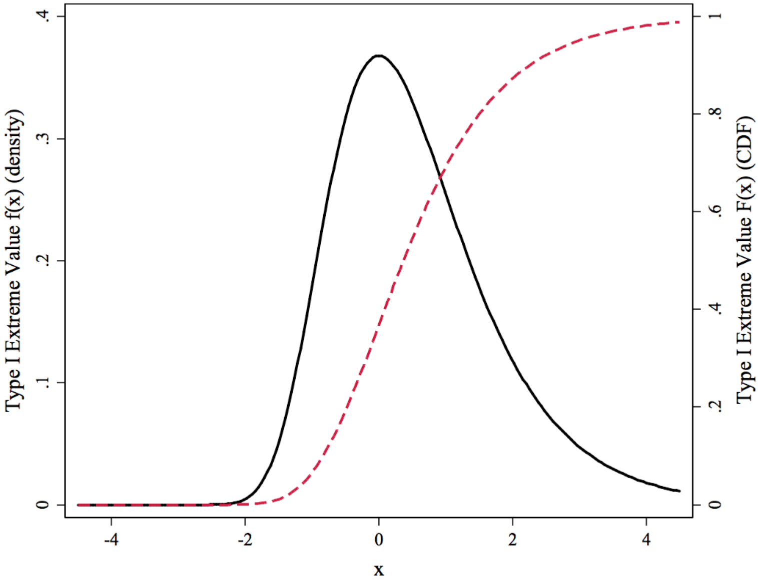
$$\mu_i = \mathbf{X}_i \boldsymbol{\beta}_j$$

$$\begin{aligned}
\Pr(Y_i = j) &= \Pr(U_{ij} > U_{i\ell} \,\forall\, \ell \neq j \in J) \\
&= \Pr(\mu_i + \epsilon_{ij} > \mu_i + \epsilon_{i\ell} \,\forall\, \ell \neq j \in J) \\
&= \Pr(\mathbf{X}_i \boldsymbol{\beta}_j + \epsilon_{ij} > \mathbf{X}_i \boldsymbol{\beta}_\ell + \epsilon_{i\ell} \,\forall\, \ell \neq j \in J) \\
&= \Pr(\epsilon_{ij} - \epsilon_{i\ell} > \mathbf{X}_i \boldsymbol{\beta}_\ell - \mathbf{X}_i \boldsymbol{\beta}_j \,\forall\, \ell \neq j \in J)
\end{aligned}$$

$\epsilon \sim$???

- *Type I Extreme Value*
- Density: $f(\epsilon) = \exp[-\epsilon - \exp(-\epsilon)]$
- CDF: $\int f(\epsilon) \equiv F(\epsilon) = \exp[-\exp(-\epsilon)]$

Type I Extreme Value



The probability of choosing choice j is:

$$\begin{aligned}
\Pr(Y_i = j) &= \Pr(U_j > U_1, U_j > U_2, ...U_j > U_J) \\
&= \int f(\epsilon_j) \left[\int_{-\infty}^{\epsilon_{ij} + \mathbf{X}_i \boldsymbol{\beta}_j - \mathbf{X}_i \boldsymbol{\beta}_1} f(\epsilon_1) d\epsilon_1 \times \int_{-\infty}^{\epsilon_{ij} + \mathbf{X}_i \boldsymbol{\beta}_j - \mathbf{X}_i \boldsymbol{\beta}_2} f(\epsilon_2) d\epsilon_2 \times ... \right] d\epsilon_j \\
&= \int f(\epsilon_j) \times \exp[-\exp(\epsilon_{ij} + \mathbf{X}_i \boldsymbol{\beta}_j - \mathbf{X}_i \boldsymbol{\beta}_1)] \times \\
&\quad \exp[-\exp(\epsilon_{ij} + \mathbf{X}_i \boldsymbol{\beta}_j - \mathbf{X}_i \boldsymbol{\beta}_2)] \times ... d\epsilon_j \\
&= \frac{\exp(\mathbf{X}_i \boldsymbol{\beta}_j)}{\sum_{j=1}^{J} \exp(\mathbf{X}_i \boldsymbol{\beta}_j)}
\end{aligned}$$

Define:

$$\begin{aligned} \delta_{ij} &= 1 \text{ if } Y_i = j, \\ &= 0 \text{ otherwise.} \end{aligned}$$

Then:

$$\begin{aligned} L_i &= \prod_{j=1}^{J} [\Pr(Y_i = j)]^{\delta_{ij}} \\ &= \prod_{j=1}^{J} \left[\frac{\exp(\mathbf{X}_i \boldsymbol{\beta}_j)}{\sum_{j=1}^{J} \exp(\mathbf{X}_i \boldsymbol{\beta}_j)} \right]^{\delta_{ij}} \end{aligned}$$

So:

$$L = \prod_{i=1}^{N} \prod_{j=1}^{J} \left[\frac{\exp(\mathbf{X}_i \boldsymbol{\beta}_j)}{\sum_{j=1}^{J} \exp(\mathbf{X}_i \boldsymbol{\beta}_j)} \right]^{\delta_{ij}}$$

and (of course):

$$\ln L = \sum_{i=1}^{N} \sum_{j=1}^{J} \delta_{ij} \ln \left[\frac{\exp(\mathbf{X}_i \boldsymbol{\beta}_j)}{\sum_{j=1}^{J} \exp(\mathbf{X}_i \boldsymbol{\beta}_j)} \right]$$

Example: The 1992 U.S. Presidential Election

Data:

- Y (`PresVote`) $\in \{\text{Bush}(=1), \text{Clinton}(=2), \text{Perot}(=3)\}$
- $\mathbf{X}$ = political demographic characteristics + "feeling thermometers"

```
> describe(NES92)
           vars    n    mean      sd median trimmed     mad  min  max range  skew kurtosis    se
ID            1 1473 4671.15 1104.02   5113 4681.28 1546.35 3001 6251  3250 -0.11    -1.66 28.77
VotedFor*     2 1473    1.85    0.71      2    1.82    1.48    1    3     2  0.22    -1.03  0.02
PresVote      3 1473    1.85    0.71      2    1.82    1.48    1    3     2  0.22    -1.03  0.02
PartyID       4 1473    3.75    2.11      3    3.69    2.97    1    7     6  0.15    -1.39  0.06
Age           5 1473   45.89   16.67     43   44.85   17.79   18   91    73  0.50    -0.72  0.43
FamIncome     6 1473   15.53    5.76     16   16.10    5.93    1   24    23 -0.78    -0.17  0.15
Female        7 1473    0.51    0.50      1    0.52    0.00    0    1     1 -0.06    -2.00  0.01
White*        8 1473    1.88    0.33      2    1.97    0.00    1    2     1 -2.31     3.36  0.01
FT.Bush       9 1473   51.75   27.26     60   52.99   29.65    0  100   100 -0.30    -0.72  0.71
FT.Clinton   10 1473   55.77   25.08     60   57.35   29.65    0  100   100 -0.45    -0.37  0.65
FT.Perot     11 1473   44.85   26.51     50   44.90   29.65    0  100   100 -0.16    -0.68  0.69
```

Model:

$$\texttt{PresVote}_i = f(\beta_0 + \beta_1 \times \texttt{PartyID}_i + \beta_2 \times \texttt{Age}_i + \beta_3 \times \texttt{White}_i + \beta_4 \times \texttt{Female}_i)$$

MNL #1, using `vglm` ("Baseline" = Perot)

```
> NES92.mlogit<-vglm(PresVote~PartyID+Age+White+Female,multinomial,data=NES92)
> summary(NES92.mlogit)

Call:
vglm(formula = PresVote ~ PartyID + Age + White + Female, family = multinomial,
    data = NES92)

Coefficients:
              Estimate Std. Error z value  Pr(>|z|)
(Intercept):1 -1.98008    0.52454   -3.77   0.00016 ***
(Intercept):2  3.82657    0.46402    8.25   < 2e-16 ***
PartyID:1      0.50132    0.04870   10.29   < 2e-16 ***
PartyID:2     -0.63429    0.04918  -12.90   < 2e-16 ***
Age:1          0.01556    0.00504    3.09   0.00203 **
Age:2          0.01296    0.00510    2.54   0.01096 *
WhiteWhite:1  -0.87918    0.43605   -2.02   0.04377 *
WhiteWhite:2  -1.86826    0.38611   -4.84 0.0000013 ***
Female:1       0.50928    0.16266    3.13   0.00174 **
Female:2       0.38427    0.16267    2.36   0.01816 *
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Names of linear predictors: log(mu[,1]/mu[,3]), log(mu[,2]/mu[,3])
Residual deviance: 2107 on 2936 degrees of freedom
Log-likelihood: -1054 on 2936 degrees of freedom
Number of Fisher scoring iterations: 5
No Hauck-Donner effect found in any of the estimates
Reference group is level  3  of the response
```

MNL #2, using `multinom` ("Baseline" = Perot)

```
> NES92$PresVote2<-factor(NES92$PresVote,
+                         levels = c("3", "1", "2"),
+                         labels = c("Perot", "Bush", "Clinton"))

> NES92.mlogit2<-multinom(PresVote2~PartyID+Age+White+Female,data=NES92)
# weights:  18 (10 variable)
initial  value 1618.255901
iter  10 value 1080.908630
final  value 1053.650588
converged

> summary(NES92.mlogit2)
Call:
multinom(formula = PresVote2 ~ PartyID + Age + White + Female,
    data = NES92)

Coefficients:
        (Intercept) PartyID    Age WhiteWhite Female
Bush          -1.98   0.501 0.0156     -0.879  0.509
Clinton        3.83  -0.634 0.0130     -1.868  0.384

Std. Errors:
        (Intercept) PartyID     Age WhiteWhite Female
Bush          0.525  0.0487 0.00504      0.436  0.163
Clinton       0.464  0.0492 0.00510      0.386  0.163

Residual Deviance: 2107
AIC: 2127
```

First, we have to "reshape" the data:

```
> head(NES92)
    ID VotedFor PresVote PartyID Age FamIncome Female     White FT.Bush FT.Clinton FT.Perot PresVote2
1 3001     Bush        1       6  31        20      0     White      85         30        0      Bush
2 3002     Bush        1       7  89         9      1     White     100          0        0      Bush
3 3003     Bush        1       7  35        17      1     White      85         30       60      Bush
4 3005  Clinton        2       6  27         3      1 Non-White      40         60       60   Clinton
5 3006  Clinton        2       2  54        15      1     White      30         70       50   Clinton
6 3007  Clinton        2       1  45         2      1 Non-White      15         70       50   Clinton

> AltNES92<-dfidx(NES92,varying=9:11,shape="wide",choice="VotedFor")

> head(AltNES92)
~~~~~~~
 first 10 observations out of 4419
~~~~~~~
     ID VotedFor PresVote PartyID Age FamIncome Female     White PresVote2  FT    idx
1  3001     TRUE        1       6  31        20      0     White      Bush  85 1:Bush
2  3001    FALSE        1       6  31        20      0     White      Bush  30 1:nton
3  3001    FALSE        1       6  31        20      0     White      Bush   0 1:erot
4  3002     TRUE        1       7  89         9      1     White      Bush 100 2:Bush
5  3002    FALSE        1       7  89         9      1     White      Bush   0 2:nton
6  3002    FALSE        1       7  89         9      1     White      Bush   0 2:erot
7  3003     TRUE        1       7  35        17      1     White      Bush  85 3:Bush
8  3003    FALSE        1       7  35        17      1     White      Bush  30 3:nton
9  3003    FALSE        1       7  35        17      1     White      Bush  60 3:erot
```

MNL #3, using `mlogit` (continued)

Now, fit the model:

```
> NES92.mlogit3<-mlogit(VotedFor~0|PartyID+Age+White+Female,data=AltNES92,reflevel="Perot")
> summary(NES92.mlogit3)

Call:
mlogit(formula = VotedFor ~ 0 | PartyID + Age + White + Female,
    data = AltNES92, reflevel = "Perot", method = "nr")

Frequencies of alternatives:choice
  Perot    Bush Clinton
  0.191   0.339   0.469

nr method
5 iterations, 0h:0m:0s
g'(-H)^-1g = 4.94E-08
gradient close to zero

Coefficients :
                    Estimate Std. Error z-value Pr(>|z|)
(Intercept):Bush    -1.98008    0.52454   -3.77  0.00016 ***
(Intercept):Clinton  3.82657    0.46403    8.25  2.2e-16 ***
PartyID:Bush         0.50132    0.04870   10.29  < 2e-16 ***
PartyID:Clinton     -0.63429    0.04918  -12.90  < 2e-16 ***
Age:Bush             0.01556    0.00504    3.09  0.00203 **
Age:Clinton          0.01296    0.00510    2.54  0.01096 *
WhiteWhite:Bush     -0.87918    0.43606   -2.02  0.04378 *
WhiteWhite:Clinton  -1.86826    0.38612   -4.84  1.3e-06 ***
Female:Bush          0.50928    0.16266    3.13  0.00174 **
Female:Clinton       0.38427    0.16267    2.36  0.01816 *
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Log-Likelihood: -1050
McFadden R^2:  0.311
Likelihood ratio test : chisq = 952 (p.value = <2e-16)
```

MNL: 1992 Election ("Baseline" = Bush)

```
> Bush.nes92.mlogit<-vglm(PresVote~PartyID+Age+White+Female,
+                         data=NES92,family=multinomial(refLevel=1))
> summary(Bush.nes92.mlogit)

Call:
vglm(formula = PresVote ~ PartyID + Age + White + Female, family = multinomial(refLevel = 1),
    data = NES92)

Coefficients:
              Estimate Std. Error z value Pr(>|z|)
(Intercept):1  5.80665    0.44301   13.11  < 2e-16 ***
(Intercept):2  1.98008    0.52454    3.77  0.00016 ***
PartyID:1     -1.13561    0.05486  -20.70  < 2e-16 ***
PartyID:2     -0.50132    0.04870  -10.29  < 2e-16 ***
Age:1         -0.00260    0.00514   -0.51  0.61276
Age:2         -0.01556    0.00504   -3.09  0.00203 **
WhiteWhite:1  -0.98908    0.31346   -3.16  0.00160 **
WhiteWhite:2   0.87918    0.43605    2.02  0.04377 *
Female:1      -0.12500    0.16895   -0.74  0.45936
Female:2      -0.50928    0.16266   -3.13  0.00174 **
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Names of linear predictors: log(mu[,2]/mu[,1]), log(mu[,3]/mu[,1])
Residual deviance: 2107 on 2936 degrees of freedom
Log-likelihood: -1054 on 2936 degrees of freedom
Number of Fisher scoring iterations: 5
No Hauck-Donner effect found in any of the estimates
Reference group is level  1  of the response
```

MNL: 1992 Election (“Baseline” = Clinton)

```
> Clinton.nes92.mlogit<-vglm(PresVote~PartyID+Age+White+Female,
+                          data=NES92,family=multinomial(refLevel=2))
> summary(Clinton.nes92.mlogit)

Call:
vglm(formula = PresVote ~ PartyID + Age + White + Female, family = multinomial(refLevel = 2),
    data = NES92)

Coefficients:
              Estimate Std. Error z value  Pr(>|z|)
(Intercept):1 -5.80665    0.44301  -13.11   < 2e-16 ***
(Intercept):2 -3.82657    0.46402   -8.25   < 2e-16 ***
PartyID:1      1.13561    0.05486   20.70   < 2e-16 ***
PartyID:2      0.63429    0.04918   12.90   < 2e-16 ***
Age:1          0.00260    0.00514    0.51    0.6128
Age:2         -0.01296    0.00510   -2.54    0.0110 *
WhiteWhite:1   0.98908    0.31346    3.16    0.0016 **
WhiteWhite:2   1.86826    0.38611    4.84 0.0000013 ***
Female:1       0.12500    0.16895    0.74    0.4594
Female:2      -0.38427    0.16267   -2.36    0.0182 *
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Names of linear predictors: log(mu[,1]/mu[,2]), log(mu[,3]/mu[,2])
Residual deviance: 2107 on 2936 degrees of freedom
Log-likelihood: -1054 on 2936 degrees of freedom
Number of Fisher scoring iterations: 5
Warning: Hauck-Donner effect detected in the following estimate(s):
'(Intercept):2'
Reference group is level  2  of the response
```

PartyID Coefficient Estimates and "Baselines"

Note: PartyID is 1 (strong Republican) $\rightarrow$ 7 (strong Democrat)

		"Baseline" category		
		Clinton	Perot	Bush
Comparison	Clinton	–	-0.63	-1.14
Category	Perot	0.63	–	-0.50
	Bush	1.14	0.50	–

Consider the choice of Bush vs. Perot:

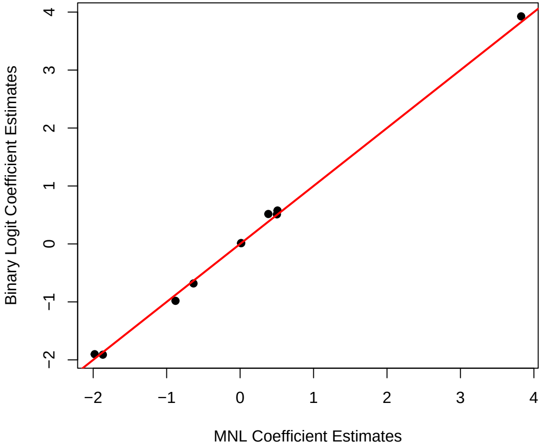
```
> NES92$PickBush<-NA
> NES92$PickBush<-ifelse(NES92$VotedFor=="Bush",1,NES92$PickBush)
> NES92$PickBush<-ifelse(NES92$VotedFor=="Perot",0,NES92$PickBush)
> BushBinary<-glm(PickBush~PartyID+Age+White+Female,data=NES92,family="binomial")
> summary(BushBinary)

Call:
glm(formula = PickBush ~ PartyID + Age + White + Female, family = "binomial",
    data = NES92)

Coefficients:
            Estimate Std. Error z value Pr(>|z|)
(Intercept)  -1.9024     0.5372   -3.54  0.00040 ***
PartyID       0.5106     0.0505   10.12  < 2e-16 ***
Age           0.0143     0.0052    2.75  0.00595 **
WhiteWhite   -0.9817     0.4586   -2.14  0.03230 *
Female        0.5768     0.1683    3.43  0.00061 ***
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

(Dispersion parameter for binomial family taken to be 1)

    Null deviance: 1022.50  on 781  degrees of freedom
Residual deviance:  880.28  on 777  degrees of freedom
  (691 observations deleted due to missingness)
AIC: 890.3

Number of Fisher Scoring iterations: 4
```

What about Clinton vs. Perot?:

```
> NES92$PickClinton<-NA
> NES92$PickClinton<-ifelse(NES92$VotedFor=="Clinton",1,NES92$PickClinton)
> NES92$PickClinton<-ifelse(NES92$VotedFor=="Perot",0,NES92$PickClinton)
> ClintonBinary<-glm(PickClinton~PartyID+Age+White+Female,data=NES92,family="binomial")
> summary(ClintonBinary)

Call:
glm(formula = PickClinton ~ PartyID + Age + White + Female, family = "binomial",
    data = NES92)

Coefficients:
            Estimate Std. Error z value Pr(>|z|)
(Intercept)  3.92614    0.48490    8.10  5.6e-16 ***
PartyID     -0.68125    0.05301  -12.85  < 2e-16 ***
Age          0.01381    0.00537    2.57   0.0101 *
WhiteWhite  -1.91056    0.39879   -4.79  1.7e-06 ***
Female       0.51690    0.17024    3.04   0.0024 **
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

(Dispersion parameter for binomial family taken to be 1)

    Null deviance: 1171.48  on 972  degrees of freedom
Residual deviance:  861.57  on 968  degrees of freedom
  (500 observations deleted due to missingness)
AIC: 871.6

Number of Fisher Scoring iterations: 5
```

MNL and Binary Logit (continued)

Are the $\hat{\beta}$s the same? (A: Yes, basically...)



It is exactly the same as the multinomial logit model. Period.

Choice-Specific Covariates: Data Structure

```
> head(AltNES92)

      ID VotedFor PresVote PartyID   Age FamIncome Female White     PresVote2    FT idx$id1 $id2
 * <dbl> <lgl>       <dbl>   <dbl> <dbl>     <dbl>  <dbl> <chr>     <fct>     <dbl>   <int> <fct>
 1  3001 TRUE            1       6    31        20      0 White     Bush         85       1 Bush
 2  3001 FALSE           1       6    31        20      0 White     Bush         30       1 Clinton
 3  3001 FALSE           1       6    31        20      0 White     Bush          0       1 Perot
 4  3002 TRUE            1       7    89         9      1 White     Bush        100       2 Bush
 5  3002 FALSE           1       7    89         9      1 White     Bush          0       2 Clinton
 6  3002 FALSE           1       7    89         9      1 White     Bush          0       2 Perot
 7  3003 TRUE            1       7    35        17      1 White     Bush         85       3 Bush
 8  3003 FALSE           1       7    35        17      1 White     Bush         30       3 Clinton
 9  3003 FALSE           1       7    35        17      1 White     Bush         60       3 Perot
10  3005 FALSE           2       6    27         3      1 Non-White Clinton      40       4 Bush
# 4,409 more rows
# Use 'print(n = ...)' to see more rows
```

Note that:

$$\Pr(Y_{ij} = 1) = \frac{\exp(\mathbf{Z}_{ij}\gamma)}{\sum_{j=1}^{J} \exp(\mathbf{Z}_{ij}\gamma)}$$

Combinations: $\mathbf{X}_i\boldsymbol{\beta}$ and $\mathbf{Z}_{ij}\gamma$:

- "Fixed effects" (choice-specific intercepts), plus
- Observation-specific **X**s, plus
- Interactions...

CL in R (Feeling Thermometers only)

```
> NES92.clogit<-mlogit(VotedFor~FT,data=AltNES92,reflevel="Perot")
> summary(NES92.clogit)

Call:
mlogit(formula = VotedFor ~ FT, data = AltNES92, reflevel = "Perot",
    method = "nr")

Frequencies of alternatives:choice
  Perot    Bush Clinton
  0.191   0.339   0.469

nr method
6 iterations, 0h:0m:0s
g'(-H)^-1g = 0.00219
successive function values within tolerance limits

Coefficients :
                    Estimate Std. Error z-value   Pr(>|z|)
(Intercept):Bush     0.03307    0.10039    0.33       0.74
(Intercept):Clinton  0.45841    0.09253    4.95 0.00000073 ***
FT                   0.07512    0.00314   23.89    < 2e-16 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Log-Likelihood: -801
McFadden R^2:  0.476
Likelihood ratio test : chisq = 1460 (p.value = <2e-16)
```

CL in R ("Full" Model)

```
> NES92.clogit2<-mlogit(VotedFor~FT|PartyID+Age+White+Female,data=AltNES92,reflevel="Perot")
> summary(NES92.clogit2)

Frequencies of alternatives:choice
  Perot    Bush Clinton
  0.191   0.339   0.469

nr method
6 iterations, 0h:0m:0s
g'(-H)^-1g = 3.06E-08
gradient close to zero

Coefficients :
                    Estimate Std. Error z-value Pr(>|z|)
(Intercept):Bush    -0.39416    0.58730   -0.67  0.50214
(Intercept):Clinton  2.69235    0.50403    5.34  9.2e-08 ***
FT                   0.06231    0.00325   19.17  < 2e-16 ***
PartyID:Bush         0.22298    0.05842    3.82  0.00014 ***
PartyID:Clinton     -0.40620    0.05798   -7.01  2.4e-12 ***
Age:Bush             0.00868    0.00612    1.42  0.15639
Age:Clinton          0.00839    0.00598    1.40  0.16040
WhiteWhite:Bush     -1.31961    0.47725   -2.77  0.00569 **
WhiteWhite:Clinton  -1.57156    0.41404   -3.80  0.00015 ***
Female:Bush          0.39271    0.19995    1.96  0.04953 *
Female:Clinton       0.28585    0.19474    1.47  0.14213
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Log-Likelihood: -723
McFadden R^2:  0.527
Likelihood ratio test : chisq = 1610 (p.value = <2e-16)
```

Interpretation: Baseline MNL Results

```
> NES.MNL<-vglm(PresVote~PartyID+Age+White+Female,data=NES92,
+               multinomial(refLevel=1)) # Bush is comparison category
> summaryvglm(NES.MNL)

Call:
vglm(formula = PresVote ~ PartyID + Age + White + Female, family = multinomial(refLevel = 1),
    data = NES92)

Coefficients:
              Estimate Std. Error z value Pr(>|z|)
(Intercept):1  5.80665    0.44301   13.11  < 2e-16 ***
(Intercept):2  1.98008    0.52454    3.77  0.00016 ***
PartyID:1     -1.13561    0.05486  -20.70  < 2e-16 ***
PartyID:2     -0.50132    0.04870  -10.29  < 2e-16 ***
Age:1         -0.00260    0.00514   -0.51  0.61276
Age:2         -0.01556    0.00504   -3.09  0.00203 **
WhiteWhite:1  -0.98908    0.31346   -3.16  0.00160 **
WhiteWhite:2   0.87918    0.43605    2.02  0.04377 *
Female:1      -0.12500    0.16895   -0.74  0.45936
Female:2      -0.50928    0.16266   -3.13  0.00174 **
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Names of linear predictors: log(mu[,2]/mu[,1]), log(mu[,3]/mu[,1])
Residual deviance: 2107 on 2936 degrees of freedom
Log-likelihood: -1054 on 2936 degrees of freedom
Number of Fisher scoring iterations: 5
No Hauck-Donner effect found in any of the estimates
Reference group is level  1  of the response
```

Global $\ln LR$ statistic Q tests:

$$\hat{\boldsymbol{\beta}} = \mathbf{0} \, \forall \, j, k$$

$$Q \sim \chi^2_{(J-1)(k-1)}$$

Test H: No Effect of Age

Is the effect of Age across the three candidates equal to zero?

```
> library(aod)
> wald.test(b=c(t(coef(NES.MNL))),Sigma=vcov(NES.MNL),Terms=c(5,6))

Wald test:
----------

Chi-squared test:
X2 = 11.0, df = 2, P(> X2) = 0.0042
```

Test H: No Difference – Clinton vs. Bush

Are the estimated coefficients for Clinton (vs. Bush) jointly equal to zero?

```
> wald.test(b=c(t(coef(NES.MNL))),Sigma=vcov(NES.MNL),Terms=c(1,3,5,7,9))

Wald test:
----------

Chi-squared test:
X2 = 444.6, df = 5, P(> X2) = 0.0
```

Interpretation: Marginal Effects

$$\frac{\partial \text{Pr}(Y_i = j)}{\partial X_k} = \text{Pr}(Y_i = j|\mathbf{X}) \left[\hat{\boldsymbol{\beta}}_{jk} - \sum_{j=1}^{J} \hat{\boldsymbol{\beta}}_{jk} \times \text{Pr}(Y_i = j|\mathbf{X}) \right]$$

Depends on:

- $\text{Pr}\widehat{(Y_i = j)}$
- $\hat{\beta}_{jk}$
- $\sum_{j=1}^{J} \hat{\boldsymbol{\beta}}_{jk}$

Available for `-multinom-` (in the `-nnet-` package) via the `-margins-` package...

Marginal Effects: Illustrated

```
> MNL.alt<-multinom(PresVote2~PartyID+Age+White+Female,data=NES92,Hess=TRUE)
# weights:  18 (10 variable)
initial  value 1618.255901
iter  10 value 1080.908630
final  value 1053.650588
converged

> summary(marginal_effects(MNL.alt))
  dydx_PartyID         dydx_Age            dydx_Female        dydx_WhiteWhite
 Min.   :-0.0908   Min.   :-0.00362   Min.   :-0.1158   Min.   :0.0416
 1st Qu.:-0.0439   1st Qu.:-0.00282   1st Qu.:-0.0892   1st Qu.:0.0943
 Median : 0.0185   Median :-0.00215   Median :-0.0674   Median :0.1352
 Mean   : 0.0083   Mean   :-0.00207   Mean   :-0.0648   Mean   :0.1435
 3rd Qu.: 0.0618   3rd Qu.:-0.00138   3rd Qu.:-0.0420   3rd Qu.:0.1848
 Max.   : 0.1070   Max.   :-0.00011   Max.   :-0.0034   Max.   :0.2926
```

Odds ("Relative Risk") Ratios

MNL has:

$$\ln\left[\frac{\Pr(Y_i = j|\mathbf{X})}{\Pr(Y_i = j'|\mathbf{X})}\right] = \mathbf{X}(\hat{\boldsymbol{\beta}}_j - \hat{\boldsymbol{\beta}}_{j'})$$

Setting $\hat{\boldsymbol{\beta}}_{j'} = \mathbf{0}$:

$$\ln\left[\frac{\Pr(Y_i = j|\mathbf{X})}{\Pr(Y_i = j'|\mathbf{X})}\right] = \mathbf{X}\hat{\boldsymbol{\beta}}_j$$

One-Unit Change in X_k:

$$RRR_{jk} = \exp(\boldsymbol{\beta}_{jk})$$

δ-Unit Change in X_k:

$$RRR_{jk} = \exp(\boldsymbol{\beta}_{jk} \times \delta)$$

Odds ("Relative Risk") Ratios

```
> mnl.or <- function(model) {
    coeffs <- c(t(coef(NES.MNL)))
    lci <- exp(coeffs - 1.96 * diag(vcov(NES.MNL))^0.5)
    or <- exp(coeffs)
    uci <- exp(coeffs + 1.96* diag(vcov(NES.MNL))^0.5)
    lreg.or <- cbind(lci, or, uci)
    lreg.or
  }

> mnl.or(NES.MNL)
                    lci      or     uci
(Intercept):1 139.540 332.504 792.309
(Intercept):2   2.591   7.243  20.250
PartyID:1       0.288   0.321   0.358
PartyID:2       0.551   0.606   0.666
Age:1           0.987   0.997   1.008
Age:2           0.975   0.985   0.994
WhiteWhite:1    0.201   0.372   0.688
WhiteWhite:2    1.025   2.409   5.662
Female:1        0.634   0.882   1.229
Female:2        0.437   0.601   0.827
```

Odds Ratios: Interpretation

Odds ratio interpretations:

- A one unit increase in `partyid` corresponds to:
  - A decrease in the odds of a Clinton vote, versus a vote for Bush, of $\exp(-1.136) = 0.321$ (or about 68 percent), and
  - A decrease in the odds of a Perot vote, versus a vote for Bush, of $\exp(-0.501) = 0.606$ (or about 40 percent).
  - These are *large* decreases in the odds – not surprisingly, more Republican voters are *much* more likely to vote for Bush than for Perot or Clinton.
- Similarly, `female` voters are:
  - No more or less likely to vote for Clinton vs. Bush (OR=0.88), but
  - Roughly 40 percent less likely to have voted for Perot (OR=0.60).

$$
\begin{aligned}
\Pr(\widehat{\texttt{presvote}_i} = \text{Bush}) &= \frac{\exp(\mathbf{X}_i \hat{\boldsymbol{\beta}}_{\text{Bush}})}{\sum_{j=1}^{J} \exp(\mathbf{X}_i \hat{\boldsymbol{\beta}}_j)} \\
&= \frac{1}{1 + \sum_{j=2}^{J} \exp(\mathbf{X}_i \hat{\boldsymbol{\beta}}_j)}
\end{aligned}
$$

In-Sample Predicted Outcomes

Generate predicted vote choices:

```
> NES92$Predictions<-" "
> NES92$Predictions<-ifelse(fitted.values(NES.MNL)[,1]>fitted.values(NES.MNL)[,2]
+                  & fitted.values(NES.MNL)[,1]>fitted.values(NES.MNL)[,3],
+                  paste("Bush"),NES92$Predictions) # Bush
> NES92$Predictions<-ifelse(fitted.values(NES.MNL)[,2]>fitted.values(NES.MNL)[,1]
+                  & fitted.values(NES.MNL)[,2]>fitted.values(NES.MNL)[,3],
+                  paste("Clinton"),NES92$Predictions) # Clinton
> NES92$Predictions<-ifelse(fitted.values(NES.MNL)[,3]>fitted.values(NES.MNL)[,1]
+                  & fitted.values(NES.MNL)[,3]>fitted.values(NES.MNL)[,2],
+                  paste("Perot"),NES92$Predictions) # Perot)

> # "Confusion Table":
>
> table(NES92$VotedFor,NES92$Predictions)

         Bush Clinton Perot
  Bush     415      77     8
  Clinton   56     619    16
  Perot    135     133    14
```

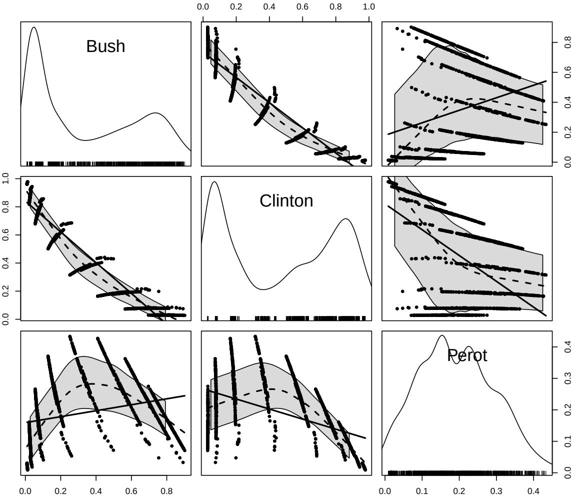
<u>Model fit</u>:

- "Null" Model: $\left(\frac{691}{1473}\right) = 46.9\%$ correct.
- Estimated model: $\frac{(415+619+14)}{1473} = \frac{1048}{1473} = 71.2\%$ correct.
- PRE $= \frac{1048-691}{1473-691} = \frac{357}{782} = 45.7\%$.
- Correct predictions: 90% Clinton, 83% Bush, 5% Perot.

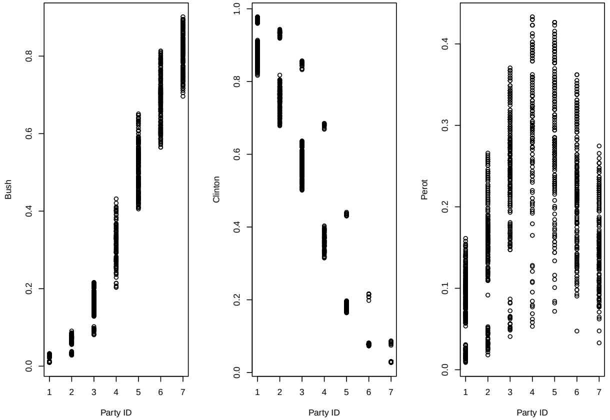
In-Sample Predicted Probabilities

```
> hats<-as.data.frame(fitted.values(NES.MNL))
> names(hats)[3]<-"Perot" # nice names...
> names(hats)[2]<-"Clinton"
> names(hats)[1]<-"Bush"
> attach(hats)

> library(car)
> scatterplot.matrix(~Bush+Clinton+Perot,
  diagonal="histogram",col=c("black","grey"))
```

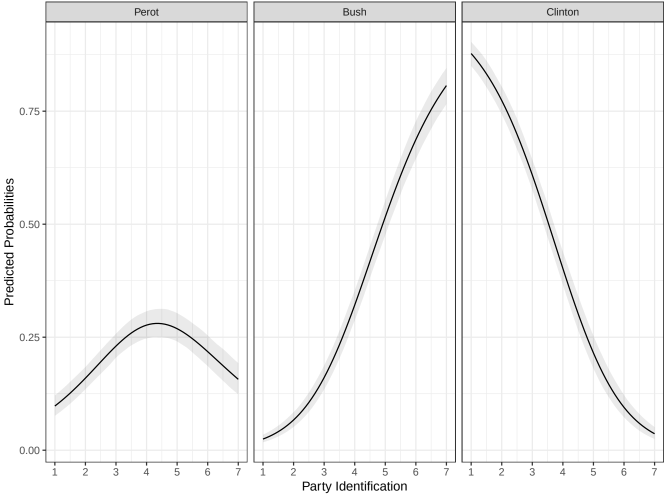
In-Sample $\widehat{\Pr}$s



In-Sample $\widehat{\text{Pr}}$s vs. `partyid`

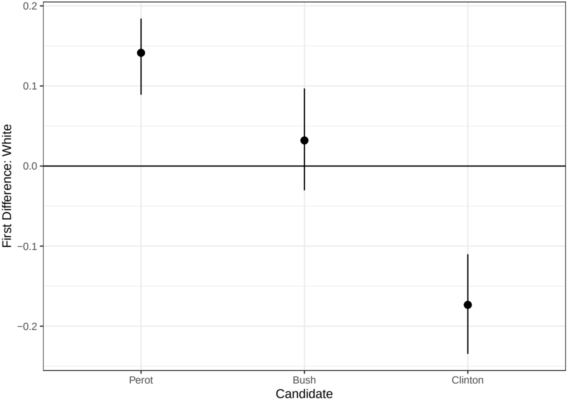


Out-Of-Sample Predictions (using `MNLpred`)



OOS First Differences (using `MNLpred`)

First differences in probabilities associated with `White`:

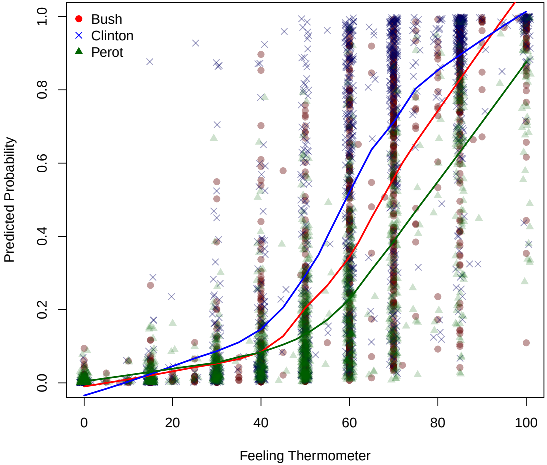


Conditional Logit: Example

```
> nes92.clogit<-mlogit(PVote~FT|partyid,data=nes92CL)
> summary(nes92.clogit)

Call:
mlogit(formula = PVote ~ FT | partyid, data = nes92CL, method = "nr")

Frequencies of alternatives:choice
   Bush Clinton   Perot
  0.339   0.469   0.191

nr method
6 iterations, 0h:0m:0s
g'(-H)^-1g = 0.00293
successive function values within tolerance limits

Coefficients :
                    Estimate Std. Error z-value Pr(>|z|)
(Intercept):Clinton  2.81272    0.26880   10.46  < 2e-16 ***
(Intercept):Perot    0.94353    0.28563    3.30  0.00096 ***
FT                   0.06299    0.00322   19.58  < 2e-16 ***
partyid:Clinton     -0.63187    0.06225  -10.15  < 2e-16 ***
partyid:Perot       -0.19212    0.05703   -3.37  0.00076 ***
---
Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1   1

Log-Likelihood: -736
McFadden R^2:  0.519
Likelihood ratio test : chisq = 1590 (p.value = <2e-16)
```

Conditional Logit: In-Sample Predicted Probabilities

```
> CLhats<-predict(NES92.clogit2,AltNES92)
```



Other Topics (for PLSC 504)

- "Independence of Irrelevant Alternatives"
- $\rightarrow$ Multinomial Probit
- $\rightarrow$ Heteroscedastic Extreme Value model
- "Mixed" Logit
- Nested Logit

Models for Ordinal Outcomes

Ordinal data are:

- Discrete: $Y \in \{1, 2, ...\}$
- *Grouped Continuous* Data
- *Assessed Ordered* Data

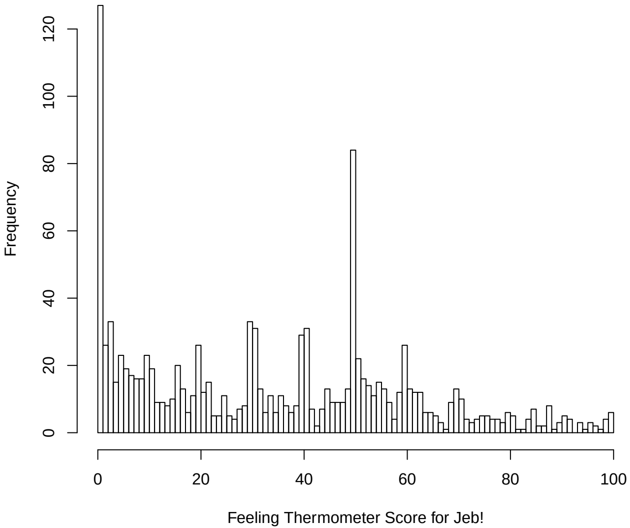
In general:

- Some things can be ordered, but shouldn't be
- Some things are ordered in some circumstances but not others
- Orderings can differ across applications

Ordinal vs. Continuous Response Models

"I'd like to get your feelings toward some of our political leaders and other people who are in the news these days. I'll read the name of a person and I'd like you to rate that person using something we call the feeling thermometer. Ratings between 50 and 100 degrees mean that you feel favorably and warm toward the person; ratings between 0 and 50 degrees mean that you don't feel favorably toward the person and that you don't care too much for that person. You would rate the person at the 50 degree mark if you don't feel particularly warm or cold toward the person."

Thermometer Scores for Jeb! (2016)



A Fake-Data Example

$$Y_i^* = 0 + 1.0X_i + u_i,$$

$$X_i \sim U[0, 10]$$

$$u_i \sim N(0, 1)$$

$$\begin{aligned} Y_{1i} &= 1 \quad \text{if} \quad Y_i^* < 2.5 \\ &= 2 \quad \text{if} \quad 2.5 \leq Y_i^* < 5 \\ &= 3 \quad \text{if} \quad 5 \leq Y_i^* < 7.5 \\ &= 4 \quad \text{if} \quad Y_i^* > 7.5 \end{aligned}$$

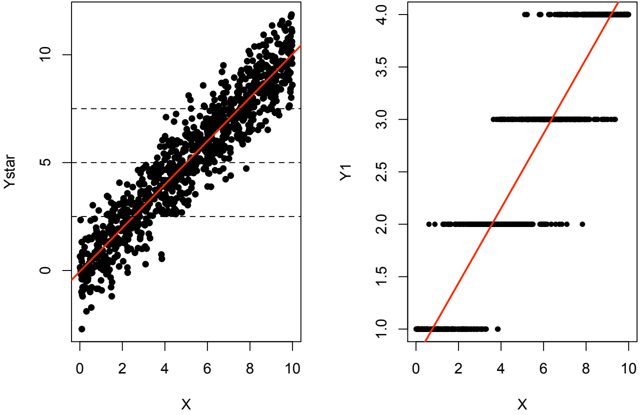
$$\begin{aligned} Y_{2i} &= 1 \quad \text{if} \quad Y_i^* < 2 \\ &= 2 \quad \text{if} \quad 2 \leq Y_i^* < 8 \\ &= 3 \quad \text{if} \quad 8 \leq Y_i^* < 9 \\ &= 4 \quad \text{if} \quad Y_i^* > 9 \end{aligned}$$

```
> summary(lm(Ystar~X))

Call:
lm(formula = Ystar ~ X)

Residuals:
   Min     1Q Median     3Q    Max 
-3.006 -0.654 -0.049  0.643  3.298 

Coefficients:
            Estimate Std. Error t value             Pr(>|t|)    
(Intercept)  -0.0830     0.0609   -1.36                 0.17    
X             1.0110     0.0106   95.48 <0.0000000000000002 ***
---
Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1   1

Residual standard error: 0.988 on 998 degrees of freedom
Multiple R-squared:  0.901,Adjusted R-squared:  0.901 
F-statistic: 9.12e+03 on 1 and 998 DF,  p-value: <0.0000000000000002
```

Also A Pretty Good Regression

```
> summary(lm(Y1~X))

Call:
lm(formula = Y1 ~ X)

Residuals:
    Min      1Q  Median      3Q     Max
-1.2889 -0.2439  0.0158  0.2592  1.3968

Coefficients:
            Estimate Std. Error t value            Pr(>|t|)
(Intercept)  0.69979    0.02639    26.5 <0.0000000000000002 ***
X            0.35825    0.00459    78.0 <0.0000000000000002 ***
---
Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1   1

Residual standard error: 0.428 on 998 degrees of freedom
Multiple R-squared:  0.859,Adjusted R-squared:  0.859
F-statistic: 6.09e+03 on 1 and 998 DF,  p-value: <0.0000000000000002
```

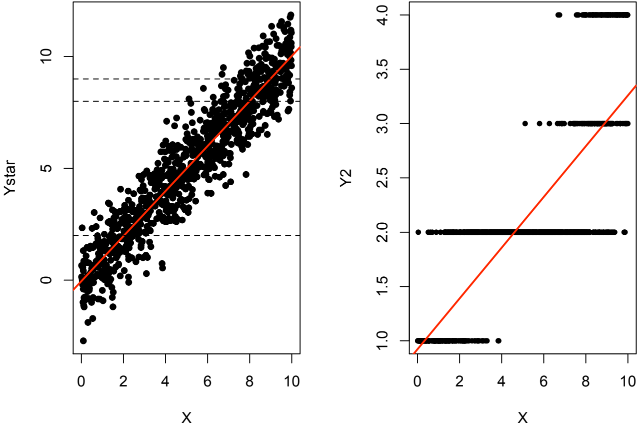
What That Looks Like



A Not-So-Good Regression

```
> summary(lm(Y2~X))

Call:
lm(formula = Y2 ~ X)

Residuals:
    Min      1Q  Median      3Q     Max
-1.3115 -0.3205 -0.0405  0.2914  1.4876

Coefficients:
            Estimate Std. Error t value             Pr(>|t|)
(Intercept)  0.88919    0.03069    29.0 <0.0000000000000002 ***
X            0.24383    0.00534    45.7 <0.0000000000000002 ***
---
Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1   1

Residual standard error: 0.498 on 998 degrees of freedom
Multiple R-squared:  0.676,Adjusted R-squared:  0.676
F-statistic: 2.09e+03 on 1 and 998 DF,  p-value: <0.0000000000000002
```

What That Looks Like



$$Y_i^* = \mu + u_i$$

$$Y_i = j \text{ if } \tau_{j-1} \leq Y_i^* < \tau_j, \; j \in \{1, ...J\}$$

$$\begin{aligned}
Y_i &= 1 \;\text{ if }\; -\infty \leq Y_i^* < \tau_1 \\
&= 2 \;\text{ if }\; \tau_1 \leq Y_i^* < \tau_2 \\
&= 3 \;\text{ if }\; \tau_2 \leq Y_i^* < \tau_3 \\
&= 4 \;\text{ if }\; \tau_3 \leq Y_i^* < \infty
\end{aligned}$$

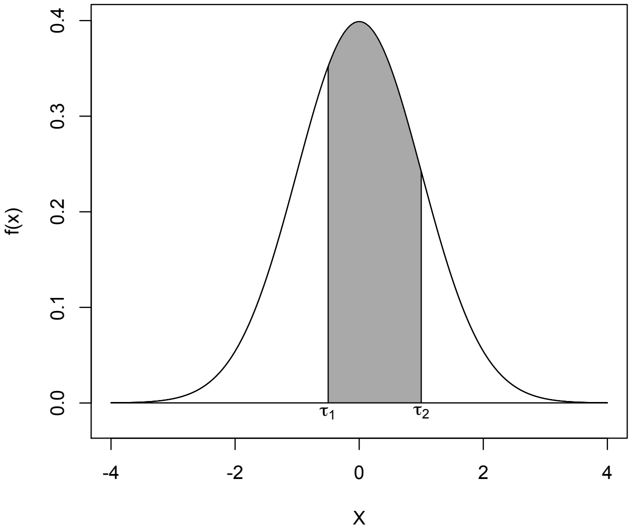
Ordinal Response Models: Probabilities

$$\begin{aligned} \Pr(Y_i = j) &= \Pr(\tau_{j-1} \leq Y^* < \tau_j) \\ &= \Pr(\tau_{j-1} \leq \mu_i + u_i < \tau_j) \end{aligned} \tag{1}$$

$$\mu_i = \mathbf{X}_i \boldsymbol{\beta}$$

$$\begin{aligned} \Pr(Y_i = j | \mathbf{X}, \boldsymbol{\beta}) &= \Pr(\tau_{j-1} \leq Y_i^* < \tau_j | \mathbf{X}) \\ &= \Pr(\tau_{j-1} \leq \mathbf{X}_i \boldsymbol{\beta} + u_i < \tau_j) \\ &= \Pr(\tau_{j-1} - \mathbf{X}_i \boldsymbol{\beta} \leq u_i < \tau_j - \mathbf{X}_i \boldsymbol{\beta}) \\ &= \int_{-\infty}^{\tau_j - \mathbf{X}_i \boldsymbol{\beta}} f(u_i) du - \int_{-\infty}^{\tau_{j-1} - \mathbf{X}_i \boldsymbol{\beta}} f(u_i) du \\ &= F(\tau_j - \mathbf{X}_i \boldsymbol{\beta}) - F(\tau_{j-1} - \mathbf{X}_i \boldsymbol{\beta}) \end{aligned}$$

What That Looks Like



Probabilities, etc.

$$
\begin{aligned}
\Pr(Y_i = 1) &= \Phi(\tau_1 - \mathbf{X}_i\boldsymbol{\beta}) - 0 \\
\Pr(Y_i = 2) &= \Phi(\tau_2 - \mathbf{X}_i\boldsymbol{\beta}) - \Phi(\tau_1 - \mathbf{X}_i\boldsymbol{\beta}) \\
\Pr(Y_i = 3) &= \Phi(\tau_3 - \mathbf{X}_i\boldsymbol{\beta}) - \Phi(\tau_2 - \mathbf{X}_i\boldsymbol{\beta}) \\
\Pr(Y_i = 4) &= 1 - \Phi(\tau_3 - \mathbf{X}_i\boldsymbol{\beta})
\end{aligned}
$$

Define:

$$\begin{aligned}\delta_{ij} &= 1 \text{ if } Y_i = j \\ &= 0 \text{ otherwise.}\end{aligned}$$

Likelihood:

$$L(Y|\mathbf{X}, \boldsymbol{\beta}, \tau) = \prod_{i=1}^{N} \prod_{j=1}^{J} [F(\tau_j - \mathbf{X}_i \boldsymbol{\beta}) - F(\tau_{j-1} - \mathbf{X}_i \boldsymbol{\beta})]^{\delta_{ij}}$$

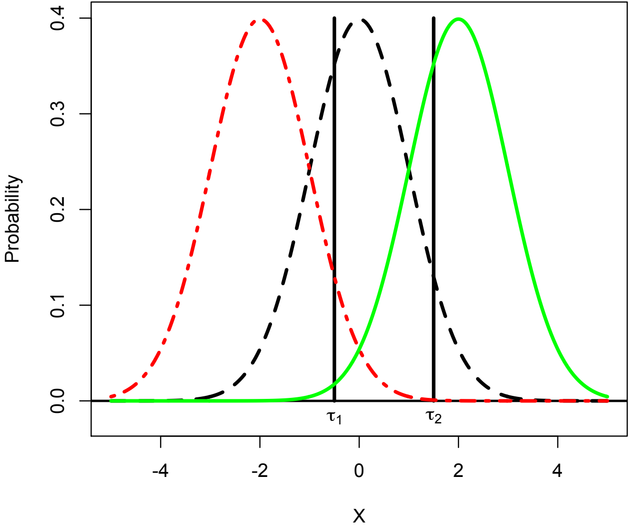
Log-Likelihood, probit:

$$\ln L(Y|\mathbf{X}, \boldsymbol{\beta}, \tau) = \sum_{i=1}^{N} \sum_{j=1}^{J} \delta_{ij} \ln[\Phi(\tau_j - \mathbf{X}_i \boldsymbol{\beta}) - \Phi(\tau_{j-1} - \mathbf{X}_i \boldsymbol{\beta})]$$

Log-Likelihood, logit:

$$\ln L(Y|\mathbf{X}, \boldsymbol{\beta}, \tau) = \sum_{i=1}^{N} \sum_{j=1}^{J} \delta_{ij} \ln[\Lambda(\tau_j - \mathbf{X}_i \boldsymbol{\beta}) - \Lambda(\tau_{j-1} - \mathbf{X}_i \boldsymbol{\beta})]$$

The Intuition



Identification

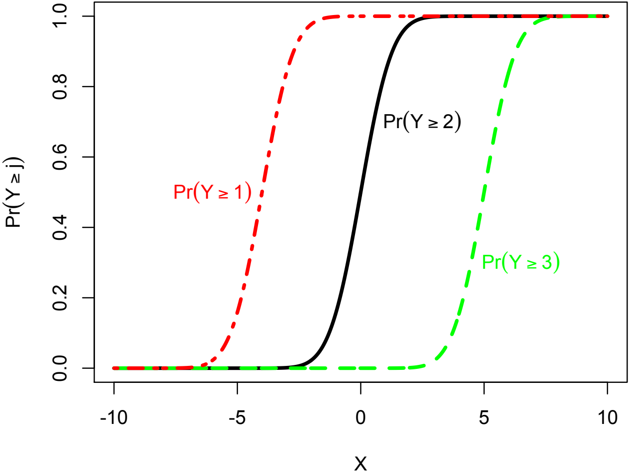
- (Usual) Assumption about $\sigma^2_{Y^*}$
- β_0 vs. the τs...
- Must either omit β_0 or drop one of the $J - 1$ τs
- In practice: Stata & R omit β_0

Parallel Regressions

$$\frac{\partial \text{Pr}(Y_i \geq j)}{\partial X} = \frac{\partial \text{Pr}(Y_i \geq j')}{\partial X} \; \forall \; j \neq j'$$

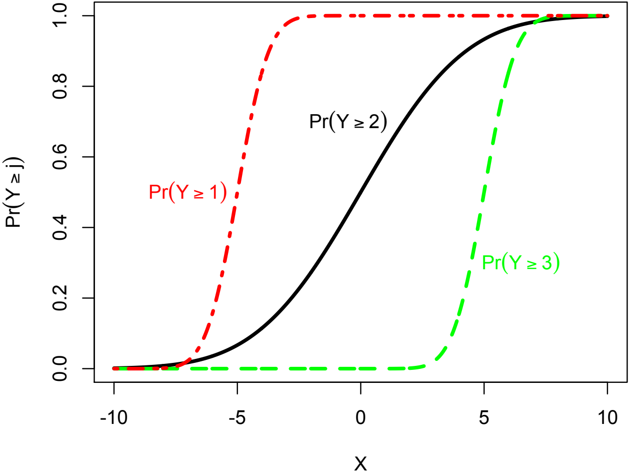
(aka "proportional odds"...)

Parallel Regressions Envisioned



$$\frac{\partial \Pr(Y_i \geq j)}{\partial X} \neq \frac{\partial \Pr(Y_i \geq j')}{\partial X} \,\forall\, j \neq j'$$

Nonparallel Regressions Envisioned



Estimation (in R)

- `polr` (in `MASS`)
- `ologit`/`oprobit` (in `Zelig`; calls `polr`)
- `vglm` (in `VGAM`)

1996 Consumer Reports Beer Survey:

```
> summary(beer)

    name              contqual        quality          price          calories
 Length:69          Min.   :24.00   Min.   :1.000   Min.   :2.360   Min.   : 58.0
 Class :character   1st Qu.:49.00   1st Qu.:2.000   1st Qu.:3.900   1st Qu.:142.0
 Mode  :character   Median :70.00   Median :3.000   Median :4.790   Median :148.0
                    Mean   :64.78   Mean   :2.536   Mean   :4.963   Mean   :142.3
                    3rd Qu.:80.00   3rd Qu.:4.000   3rd Qu.:6.240   3rd Qu.:160.0
                    Max.   :98.00   Max.   :4.000   Max.   :7.800   Max.   :201.0

    alcohol          craftbeer           bitter          malty                        class
 Min.   :0.500   Min.   :0.0000   Min.   : 8.00   Min.   : 5.00   Craft Lager        :13
 1st Qu.:4.400   1st Qu.:0.0000   1st Qu.:21.00   1st Qu.:12.00   Craft Ale          :17
 Median :4.900   Median :0.0000   Median :31.00   Median :23.00   Imported Lager     :10
 Mean   :4.471   Mean   :0.4348   Mean   :35.44   Mean   :33.13   Regular or Ice Beer:16
 3rd Qu.:5.100   3rd Qu.:1.0000   3rd Qu.:52.50   3rd Qu.:50.50   Light Beer         : 6
 Max.   :6.000   Max.   :1.0000   Max.   :80.50   Max.   :86.00   Nonalcoholic       : 7
```

```
> library(MASS)
> beer.logit<-polr(as.factor(quality)~price+calories+craftbeer+bitter
  +malty,data=beer)
> summary(beer.logit)

Call:
polr(formula = as.factor(quality) ~ price + calories + craftbeer +
    bitter + malty)

Coefficients:
           Value Std. Error t value
price     -0.451      0.293    -1.5
calories   0.047      0.012     3.8
craftbeer -1.705      0.942    -1.8
bitter    -0.030      0.042    -0.7
malty      0.051      0.025     2.1

Intercepts:
    Value  Std. Error t value
1|2  2.771  1.674      1.655
2|3  4.270  1.725      2.475
3|4  5.578  1.760      3.170
```

Ordered Probit

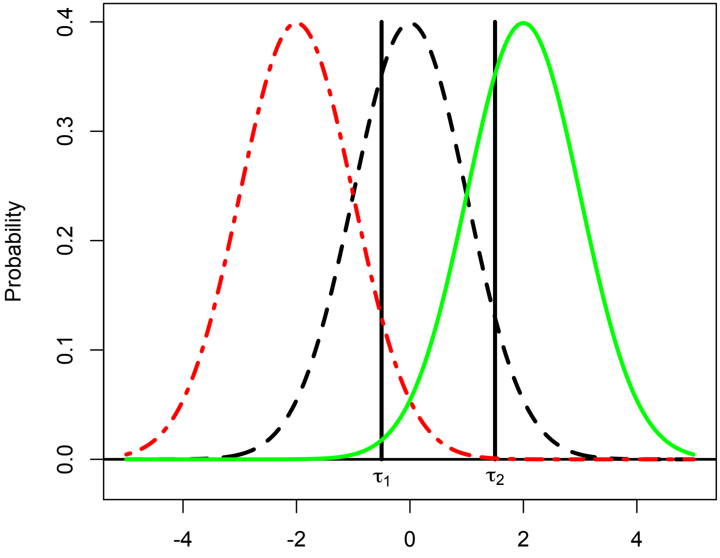
```
> beer.probit<-polr(as.factor(quality)~price+calories+craftbeer+bitter+malty,
+  data=beer,method="probit")
> summary(beer.probit)

Call:
polr(formula = as.factor(quality) ~ price + calories + craftbeer +
    bitter + malty, method = "probit")

Coefficients:
             Value Std. Error t value
price     -0.27914   0.172012 -1.6228
calories   0.02800   0.007184  3.8979
craftbeer -0.98427   0.559020 -1.7607
bitter    -0.01737   0.024719 -0.7025
malty      0.02855   0.014321  1.9937

Intercepts:
    Value  Std. Error t value
1|2  1.647  1.018      1.619
2|3  2.508  1.034      2.426
3|4  3.290  1.049      3.136
```

$$\begin{aligned}\frac{\partial \Pr(Y=j)}{\partial X_k} &= \frac{\partial F(\hat{\tau}_{j-1} - \bar{\mathbf{X}}\hat{\beta})}{\partial X_k} - \frac{\partial F(\hat{\tau}_j - \bar{\mathbf{X}}\hat{\beta})}{\partial X_k} \\ &= \hat{\beta}_k[f(\hat{\tau}_{j-1} - \bar{\mathbf{X}}\hat{\beta}) - f(\hat{\tau}_j - \bar{\mathbf{X}}\hat{\beta})]\end{aligned}$$

So:

- $\text{sign}\left(\frac{\partial \Pr(Y=1)}{\partial X_k}\right) =$ -sign$(\hat{\beta}_k)$
- $\text{sign}\left(\frac{\partial \Pr(Y=J)}{\partial X_k}\right) = \text{sign}(\hat{\beta}_k)$
- $\frac{\partial \Pr(Y=\ell)}{\partial X_k},\ \ell \in \{2, 3, ...J-1\}$ are non-monotonic

Marginal Effects, Illustrated



For a δ-unit change in X_k:

$$
\begin{aligned}
\mathrm{OR}_{X_k} &= \frac{\frac{\Pr(Y>j|\mathbf{X},X_k+\delta)}{\Pr(Y\leq j|\mathbf{X},X_k+\delta)}}{\frac{\Pr(Y>j|\mathbf{X},X_k)}{\Pr(Y\leq j|\mathbf{X},X_k)}} \\
&= \exp(\delta\hat{\beta}_k)
\end{aligned}
$$

Calculating Odds Ratios

```
> olreg.or <- function(model)
+  {
+  coeffs <- coef(summary(model))
+  lci <- exp(coeffs[ ,1] - 1.96 * coeffs[ ,2])
+  or <- exp(coeffs[ ,1])
+  uci <- exp(coeffs[ ,1] + 1.96 * coeffs[ ,2])
+  lreg.or <- cbind(lci, or, uci)
+  lreg.or
+  }

> olreg.or(beer.logit)

             lci       or      uci
price     0.3586   0.6373    1.133
calories  1.0231   1.0479    1.073
craftbeer 0.0287   0.1818    1.152
bitter    0.8933   0.9707    1.055
malty     1.0023   1.0518    1.104
1|2       0.6003  15.9748  425.133
2|3       2.4319  71.4963 2101.961
3|4       8.4053 264.4357 8319.319
```

Odds Ratios: Explication

- `craftbeer`:
  - $\exp(-1.705) = 0.18$
  - "The odds of being rated "Good" or better (versus "Fair") are more than 80 percent lower for a craft beer than for a regular beer."
  - "The odds of being rated "Very Good" or better (versus "Fair" or "Good") are more than 80 percent lower for a craft beer than for a regular beer."
- `calories`:
  - $\exp(0.047) = 1.05$
  - "A one-calorie increase raises the odds of being in a higher set of categories (versus all lower ones) by about five percent."
  - etc.

Predicted Probabilities: Basics

$$\Pr(\widehat{Y_i = j}|\mathbf{X}) = F(\hat{\tau}_j - \bar{\mathbf{X}}_i\hat{\beta}) - F(\hat{\tau}_{j-1} - \bar{\mathbf{X}}_i\hat{\beta})$$

Means:

- `price` $= 4.96$, `calories` $= 142$, `craftbeer` $= 0$, `bitter` $= 35.4$, `malty` $= 33.1$.
- Yields:

$$\begin{aligned}
\sum_{k=1}^{K} \bar{\mathbf{X}}_k\hat{\beta}_k &= -0.45 \times 4.96 + 0.047 \times 142 - 1.70 \times 0 - \\
& \quad 0.03 \times 35.4 + 0.05 \times 33.1 \\
&= -2.23 + 6.67 - 0 - 1.06 + 1.66 \\
&= \mathbf{5.04}.
\end{aligned}$$

Predicted Probabilities: "By Hand"

$$\begin{aligned}
\Pr(Y = 1) &= \Lambda(2.77 - 5.04) - 0 \\
&= \frac{\exp(-2.27)}{1 + \exp(-2.27)} \\
&= \mathbf{0.09}.
\end{aligned}$$

$$\begin{aligned}
\Pr(Y = 2) &= \Lambda(4.27 - 5.04) - \Lambda(2.77 - 5.04) \\
&= \Lambda(-0.77) - \Lambda(-2.27) \\
&= 0.32 - 0.09 \\
&= \mathbf{0.23}.
\end{aligned}$$

$$\begin{aligned}
\Pr(Y = 3) &= \Lambda(5.58 - 5.04) - \Lambda(4.27 - 5.04) \\
&= \Lambda(0.54) - \Lambda(-0.77) \\
&= 0.63 - 0.32 \\
&= \mathbf{0.31}.
\end{aligned}$$

$$\begin{aligned}
\Pr(Y = 4) &= 1 - \Lambda(5.58 - 5.04) \\
&= 1 - \Lambda(0.54) \\
&= 1 - 0.63 \\
&= \mathbf{0.37}.
\end{aligned}$$

Changes in Predicted Probabilities

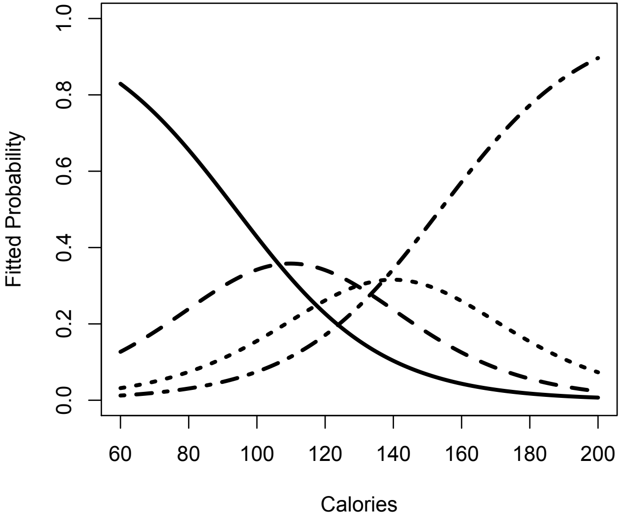
For `craftbeer`=1:

- $\Pr(Y = 1) = \Lambda(2.77 - 3.34) - 0 = \mathbf{0.36}$.
- $\Pr(Y = 2) = \Lambda(4.27 - 3.34) - \Lambda(2.77 - 3.34) = 0.72 - 0.36 = \mathbf{0.36}$.
- $\Pr(Y = 3) = \Lambda(5.58 - 3.34) - \Lambda(4.27 - 3.34) = 0.90 - 0.72 = \mathbf{0.18}$.
- $\Pr(Y = 4) = 1 - 0.90 = \mathbf{0.10}$.

Outcome	Change in Probability
ΔPr(Fair)	0.27
ΔPr(Good)	0.13
ΔPr(Very Good)	-0.13
ΔPr(Excellent)	-0.27

- Can be category-specific or "cumulative"
- In-sample in `$fitted.values`
- `polr` class supports `predict`, `confint`, etc.

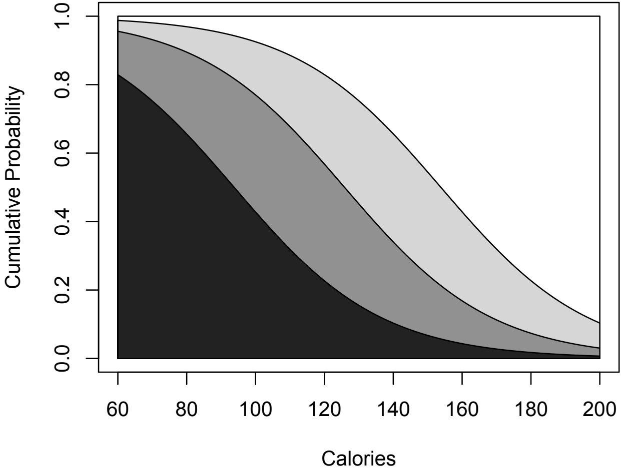
Plot by Outcome



(How'd He Do That?)

```
> calories<-seq(60,200,1)
> price<-mean(beer$price)
> craftbeer<-median(beer$craftbeer)
> bitter<-mean(beer$bitter)
> malty<-mean(beer$malty)
> beersim<-cbind(calories,price,craftbeer,bitter,malty)
> beer.hat<-predict(beer.logit,beersim,type='probs')
> plot(c(60,200), c(0,1), type='n', xlab="Calories", ylab='Fitted
  Probability')
> lines(60:200, beer.hat[1:141, 1], lty=1, lwd=3)
> lines(60:200, beer.hat[1:141, 2], lty=2, lwd=3)
> lines(60:200, beer.hat[1:141, 3], lty=3, lwd=3)
> lines(60:200, beer.hat[1:141, 4], lty=4, lwd=3)
```

Cumulative Predicted Probabilities



(code...)

```
> xaxis<-c(60,60:200,200)
> yaxis1<-c(0,beer.hat[,1],0)
> yaxis2<-c(0,beer.hat[,2]+beer.hat[,1],0)
> yaxis3<-c(0,beer.hat[,3]+beer.hat[,2]+beer.hat[,1],0)
> yaxis4<-c(0,beer.hat[,4]+beer.hat[,3]+beer.hat[,2]+beer.hat[,1],0)
>
> plot(c(60,200), c(0,1), type='n', xlab="Calories", ylab="Cumulative
  Probability")
> polygon(xaxis,yaxis4,col="white")
> polygon(xaxis,yaxis3,col="grey80")
> polygon(xaxis,yaxis2,col="grey50")
> polygon(xaxis,yaxis1,col="grey10")
```

Variants / Extensions (also for PLSC 504...)

- *Generalized* models (relax parallel regressions; Brant (1990))
- *Heteroscedastic* models
- Varying τs (Maddala, Terza, Sanders)
- Models for "balanced" scales (Jones & Sobel)
- Compound Ordered Hierarchical Probit ("`chopit`") (Wand & King)
- "Zero-Inflated" Ordered Models (Hill, Bagozzi, Moore & Mukherjee)
- Latent class/mixture models (Winkelmann, etc.)