PLSC 503 – Spring 2024
Variances, Collinearity, etc.

February 12, 2024

Variances: Why We Care

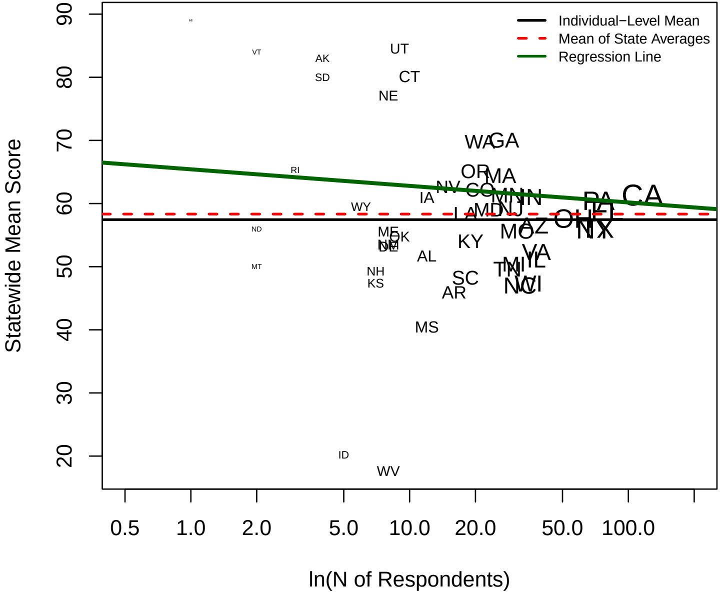
2016 ANES pilot study "feeling thermometer" toward gays and lesbians ($N = 1200$):

```
> summary(ANES$ftgay)
   Min. 1st Qu.  Median    Mean 3rd Qu.    Max.    NA's
    0.0    40.5    54.0    57.4    88.5   100.0       1
> summary(ANES$presjob)
   Min. 1st Qu.  Median    Mean 3rd Qu.    Max.
   1.00    2.00    4.00    4.19    7.00    7.00
```

Suppose we wanted to create aggregate measures, by state ($N = 51$). We would get:

```
> summary(StateFT)
    State               Nresp          meantherm      meanpresapp
 Length:50          Min.   :  1.0   Min.   :17.6   Min.   :2.00
 Class :character   1st Qu.:  8.0   1st Qu.:51.3   1st Qu.:3.75
 Mode  :character   Median : 18.0   Median :57.1   Median :4.24
                    Mean   : 24.0   Mean   :58.3   Mean   :4.15
                    3rd Qu.: 30.8   3rd Qu.:62.5   3rd Qu.:4.61
                    Max.   :116.0   Max.   :89.0   Max.   :5.80
```

Variances: Why We Care



Start with:

$$Y_i = \mathbf{X}_i\boldsymbol{\beta} + u_i$$

with:

$$\text{Var}(u_i) = \sigma^2/w_i$$

with w_i known.

WLS now minimizes:

$$\text{RSS} = \sum_{i=1}^{N} w_i(Y_i - \mathbf{X}_i\boldsymbol{\beta}).$$

which gives:

$$\begin{aligned}\hat{\boldsymbol{\beta}}_{WLS} &= [\mathbf{X}'(\sigma^2\boldsymbol{\Omega})^{-1}\mathbf{X}]^{-1}\mathbf{X}'(\sigma^2\boldsymbol{\Omega})^{-1}\mathbf{Y} \\ &= [\mathbf{X}'\mathbf{W}^{-1}\mathbf{X}]^{-1}\mathbf{X}'\mathbf{W}^{-1}\mathbf{Y}\end{aligned}$$

where:

$$\mathbf{W} = \begin{bmatrix} \frac{\sigma^2}{w_1} & 0 & \cdots & 0 \\ 0 & \frac{\sigma^2}{w_2} & \cdots & \vdots \\ \vdots & 0 & \ddots & 0 \\ 0 & \cdots & 0 & \frac{\sigma^2}{w_N} \end{bmatrix}$$

The variance-covariance matrix is:

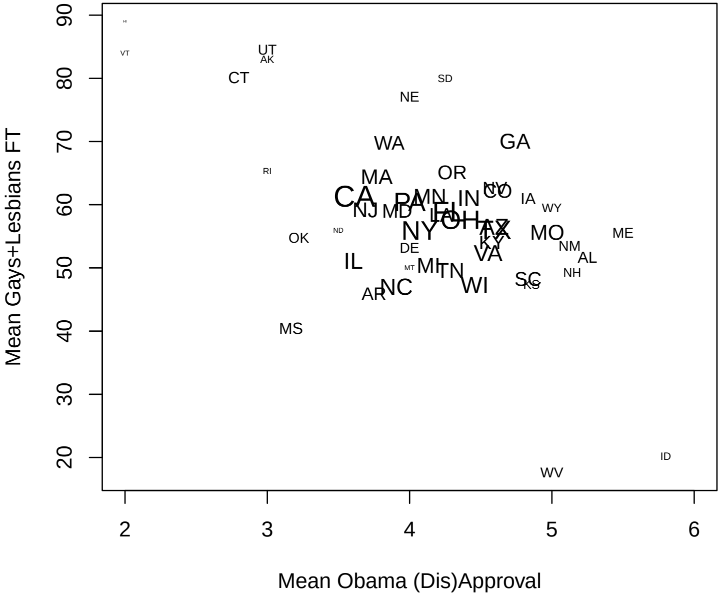
$$\begin{aligned}\text{Var}(\hat{\beta}_{WLS}) &= \sigma^2(\mathbf{X}'\boldsymbol{\Omega}^{-1}\mathbf{X})^{-1} \\ &\equiv (\mathbf{X}'\mathbf{W}^{-1}\mathbf{X})^{-1}\end{aligned}$$

A common case is:

$$\text{Var}(u_i) = \sigma^2 \frac{1}{N_i}$$

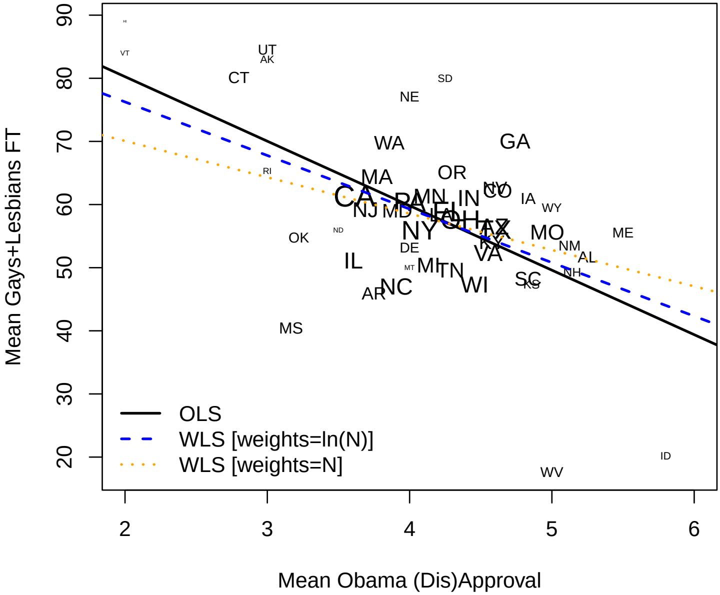
where N_i is the number of observations upon which (aggregate) observation i is based.

Feeling Thermometer Example



	Mean Gay/Lesbian FTs		
	OLS	WLS [1/ln(N)]	WLS [1/N]
Mean Pres. Disapproval	-10.200^{***}	-8.480^{***}	-5.760^{**}
	(1.980)	(2.200)	(2.190)
Constant	101.000^{***}	93.200^{***}	81.600^{***}
	(8.340)	(9.380)	(9.240)
Observations	50	50	50
R^2	0.358	0.237	0.126
Adjusted R^2	0.344	0.221	0.108
Residual Std. Error (df = 48)	11.100	17.100	37.900
F Statistic (df = 1; 48)	26.700^{***}	14.900^{***}	6.930^{**}
Note:			$^{*}p<0.1$; $^{**}p<0.05$; $^{***}p<0.01$

Regressions, Plotted



Recall that, if $\sigma_i^2 \neq \sigma_j^2 \; \forall \; i \neq j$,

$$\begin{aligned}\text{Var}(\beta_{\text{Het.}}) &= (\mathbf{X}'\mathbf{X})^{-1}(\mathbf{X}'\mathbf{W}^{-1}\mathbf{X})(\mathbf{X}'\mathbf{X})^{-1} \\ &= (\mathbf{X}'\mathbf{X})^{-1}\,\mathbf{Q}\,(\mathbf{X}'\mathbf{X})^{-1}\end{aligned}$$

where $\mathbf{Q} = (\mathbf{X}'\mathbf{W}^{-1}\mathbf{X})$ and $\mathbf{W} = \sigma^2\boldsymbol{\Omega}$.

We can rewrite $\mathbf{Q}$ as

$$\begin{aligned}\mathbf{Q} &= \sigma^2(\mathbf{X}'\boldsymbol{\Omega}^{-1}\mathbf{X}) \\ &= \sum_{i=1}^{N} \sigma_i^2 \mathbf{X}_i \mathbf{X}_i'\end{aligned}$$

Estimate $\hat{\mathbf{Q}}$ as:

$$\widehat{\mathbf{Q}} = \sum_{i=1}^{N} \hat{u}_i^2 \mathbf{X}_i \mathbf{X}_i'$$

Yields:

$$\begin{aligned}
\widehat{\text{Var}(\beta)}_{\text{Robust}} &= (\mathbf{X}'\mathbf{X})^{-1}(\mathbf{X}'\widehat{\mathbf{Q}}^{-1}\mathbf{X})(\mathbf{X}'\mathbf{X})^{-1} \\
&= (\mathbf{X}'\mathbf{X})^{-1}\left[\mathbf{X}'\left(\sum_{i=1}^{N} \hat{u}_i^2 \mathbf{X}_i \mathbf{X}_i'\right)^{-1}\mathbf{X}\right](\mathbf{X}'\mathbf{X})^{-1}
\end{aligned}$$

"Robust" standard error estimates:

- are heteroscedasticity-consistent, but
- are biased in small samples, and
- are less efficient than "naive" estimates when $\text{Var}(u) = \sigma^2 \mathbf{I}$.
- Come in various "versions"
  - Called "HC0," "HC1," "HC2," "HC3," etc.
  - See the Long and Ervin (2000) paper for details...

"Clustering"

Huber / White

?????????

WLS / GLS

I know very little about my error variances...

I know a great deal about my error variances...

A common case:

$$Y_{ij} = \mathbf{X}_{ij}\boldsymbol{\beta} + u_{ij}$$

with

$$\sigma_{ij}^2 = \sigma_{ik}^2.$$

"Robust, clustered" estimator:

$$\widehat{\text{Var}(\beta)}_{\text{Clustered}} = (\mathbf{X}'\mathbf{X})^{-1}\left\{\mathbf{X}'\left[\sum_{i=1}^{N}\left(\sum_{j=1}^{n_j}\hat{u}_{ij}^2\mathbf{X}_{ij}\mathbf{X}_{ij}'\right)\right]^{-1}\mathbf{X}\right\}(\mathbf{X}'\mathbf{X})^{-1}$$

Robust / Clustered SEs: A Simulation

```
> set.seed(7222009)
> X <- rnorm(10)
> Y <- 1 + X + rnorm(10)
> df10 <- data.frame(ID=seq(1:10),X=X,Y=Y)
>
> fit10 <- lm(Y~X,data=df10)
> summary(fit10)

Residuals:
     Min       1Q   Median       3Q      Max
-1.12328 -0.65321 -0.05073  0.43937  1.81661

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)   0.8438     0.3020   2.794   0.0234 *
X             0.3834     0.3938   0.974   0.3588
---
Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1   1

Residual standard error: 0.9313 on 8 degrees of freedom
Multiple R-squared:  0.1059,Adjusted R-squared:  -0.005832
F-statistic: 0.9478 on 1 and 8 DF,  p-value: 0.3588

> rob10 <- vcovHC(fit10,type="HC1")
> sqrt(diag(rob10))
(Intercept)           X
  0.2932735   0.2859552
```

Robust / Clustered SEs: A Simulation (continued)

```
> # "Clone" each observation 100 times:
>
> df1K <- df10[rep(seq_len(nrow(df10)), each=100),]
> df1K <- pdata.frame(df1K, index="ID")
> fit1K <- lm(Y~X,data=df1K)
> summary(fit1K)

Residuals:
    Min      1Q  Median      3Q     Max
-1.1233 -0.6755 -0.0507  0.4840  1.8166

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)   0.8438     0.0270    31.2   <2e-16 ***
X             0.3834     0.0353    10.9   <2e-16 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 0.834 on 998 degrees of freedom
Multiple R-squared:  0.106,Adjusted R-squared:  0.105
F-statistic:  118 on 1 and 998 DF,  p-value: <2e-16


> # With clustered SEs (HC1):
>
> clustSE<-sqrt(diag(vcovCL(fit1K,cluster=df1K$ID)))
> clustOLS<-coeftest(fit1K,vcov.=vcovCL,cluster=~df1K$ID)
> clustOLS

t test of coefficients:

            Estimate Std. Error t value Pr(>|t|)
(Intercept)    0.844      0.277    3.05   0.0023 **
X              0.383      0.270    1.42   0.1555
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

Regressions, Again

	Mean Gay/Lesbian FTs			
	OLS	OLS (robust)	WLS [1/ln(N)]	WLS [1/N]
Mean Pres. Disapproval	-10.200^{***}	-10.200^{***}	-8.480^{***}	-5.760^{**}
	(1.980)	(2.340)	(2.200)	(2.190)
Constant	101.000^{***}	101.000^{***}	93.200^{***}	81.600^{***}
	(8.340)	(9.720)	(9.380)	(9.240)
Observations	50		50	50
R^2	0.358		0.237	0.126
Adjusted R^2	0.344		0.221	0.108
Residual Std. Error (df = 48)	11.100		17.100	37.900
F Statistic (df = 1; 48)	26.700^{***}		14.900^{***}	6.930^{**}

Note: $^{*}p<0.1$; $^{**}p<0.05$; $^{***}p<0.01$

Expanded State-Level ANES Example

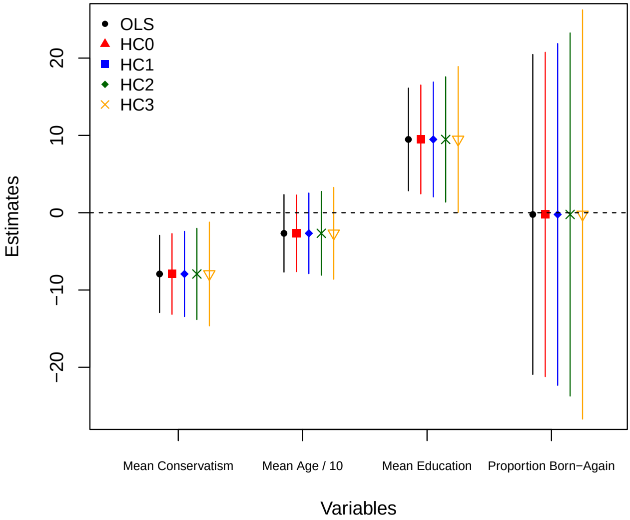
```
> psych::describe(StateData)
             vars  n  mean    sd median trimmed   mad   min    max  range  skew kurtosis   se
State*          1 50 25.50 14.58  25.50   25.50 18.53  1.00  50.00  49.00  0.00    -1.27 2.06
NResp           2 50 24.00 23.74  18.00   19.48 16.31  1.00 116.00 115.00  1.79     3.34 3.36
LGBTTherm       3 50 58.33 13.74  57.11   58.11  8.51 17.62  89.00  71.38 -0.22     1.40 1.94
MeanCons        4 50  3.97  0.77   4.00    3.98  0.55  1.50   5.60   4.10 -0.47     1.28 0.11
MeanAge         5 50  4.74  0.64   4.78    4.74  0.43  3.10   6.50   3.40  0.11     1.10 0.09
MeanEducation   6 50  3.25  0.52   3.22    3.22  0.41  2.33   5.00   2.67  0.84     1.44 0.07
BornAgainProp   7 50  0.28  0.18   0.25    0.28  0.19  0.00   0.72   0.72  0.11    -0.62 0.02
```

Basic regression:

```
> OLS<-lm(LGBTTherm~MeanCons+MeanAge+MeanEducation+BornAgainProp,data=StateData)
> summary(OLS)

Coefficients:
              Estimate Std. Error t value Pr(>|t|)
(Intercept)     71.646     16.577    4.32 0.000084 ***
MeanCons        -7.926      2.544   -3.12   0.0032 **
MeanAge         -2.669      2.557   -1.04   0.3022
MeanEducation    9.477      3.384    2.80   0.0075 **
BornAgainProp   -0.227     10.560   -0.02   0.9829
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Residual standard error: 10.6 on 45 degrees of freedom
Multiple R-squared:  0.459,Adjusted R-squared:  0.411
F-statistic: 9.54 on 4 and 45 DF,  p-value: 0.0000113
```

```
> hccm(OLS,type="hc3") # "HC3" var-cov matrix
              (Intercept) MeanCons MeanAge MeanEducation BornAgainProp
(Intercept)         605.4   -43.05 -37.251       -89.915        122.75
MeanCons            -43.0    11.71  -1.234         4.969        -38.74
MeanAge             -37.3    -1.23   9.170        -0.645         -3.44
MeanEducation       -89.9     4.97  -0.645        23.148         -4.41
BornAgainProp       122.7   -38.74  -3.439        -4.406        182.30

> sqrt(diag(hccm(OLS,type="hc3"))) # "HC3" robust SEs
  (Intercept)      MeanCons       MeanAge MeanEducation BornAgainProp
       24.60          3.42          3.03          4.81         13.50

> coeftest(OLS,vcov.=vcovHC)

t test of coefficients:

              Estimate Std. Error t value Pr(>|t|)
(Intercept)     71.646     24.604    2.91   0.0056 **
MeanCons        -7.926      3.422   -2.32   0.0251 *
MeanAge         -2.669      3.028   -0.88   0.3828
MeanEducation    9.477      4.811    1.97   0.0550 .
BornAgainProp   -0.227     13.502   -0.02   0.9866
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1
```

$\hat{\beta}$s and 95% CIs: Various Types of Robust SEs



Cases, Variables, and Collinearity

OLS (and regression methods more generally) requires:

- **X** is full column rank.
- $N > K$.
- "Sufficient" variability in **X**.

Formally: There cannot be any set of λs such that:

$$\lambda_0 \mathbf{1} + \lambda_1 \mathbf{X}_1 + ... + \lambda_K \mathbf{X}_K = \mathbf{0}$$

If there was, it would imply

$$\mathbf{X}_j = \frac{-\lambda_0}{\lambda_j}\mathbf{1} + \frac{-\lambda_1}{\lambda_j}\mathbf{X}_1 + ... + \frac{-\lambda_K}{\lambda_j}\mathbf{X}_K$$

which means

$$\begin{aligned}
Y &= \beta_0 \mathbf{1} + \beta_1 \mathbf{X}_1 + ... + \beta_j \mathbf{X}_j + ... + \beta_K \mathbf{X}_K + \mathbf{u} \\
&= \beta_0 \mathbf{1} + \beta_1 \mathbf{X}_1 + ... + \beta_j \left(\frac{-\lambda_0}{\lambda_j}\mathbf{1} + \frac{-\lambda_1}{\lambda_j}\mathbf{X}_1 + ... + \frac{-\lambda_K}{\lambda_j}\mathbf{X}_K \right) + ... + \beta_K \mathbf{X}_K + \mathbf{u} \\
&= \left[\beta_0 + \beta_j \left(\frac{-\lambda_0}{\lambda_j} \right) \right] \mathbf{1} + \left[\beta_1 + \beta_j \left(\frac{-\lambda_1}{\lambda_j} \right) \right] \mathbf{X}_1 + ... + \left[\beta_K + \beta_j \left(\frac{-\lambda_K}{\lambda_j} \right) \right] \mathbf{X}_K + \mathbf{u} \\
&= \left(\beta_0 + \frac{\gamma_0}{\lambda_j} \right) \mathbf{1} + \left(\beta_1 + \frac{\gamma_1}{\lambda_j} \right) \mathbf{X}_1 + ... + \left(\beta_K + \frac{\gamma_K}{\lambda_j} \right) \mathbf{X}_K + \mathbf{u}
\end{aligned}$$

```
> Africa$newgdp<-(Africa$gdppppd-mean(Africa$gdppppd))*1000

> fit<-with(Africa, lm(adrate~gdppppd+newgdp+healthexp+subsaharan+
+                         muslperc+literacy))
> summary(fit)

Call:
lm(formula = adrate ~ gdppppd + newgdp + healthexp + subsaharan +
    muslperc + literacy)

Residuals:
    Min      1Q  Median      3Q     Max
-15.291  -4.329  -1.412   2.723  20.682

Coefficients: (1 not defined because of singularities)
                      Estimate Std. Error t value Pr(>|t|)
(Intercept)           -7.78020   10.33872  -0.753   0.4565
gdppppd                0.36142    0.58214   0.621   0.5385
newgdp                      NA         NA      NA       NA
healthexp              1.87001    0.75667   2.471   0.0182 *
subsaharanSub-Saharan  3.64354    4.54163   0.802   0.4275
muslperc              -0.07908    0.05967  -1.325   0.1932
literacy               0.12445    0.09867   1.261   0.2151
---
Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1   1

Residual standard error: 7.665 on 37 degrees of freedom
Multiple R-squared:  0.4782,Adjusted R-squared:  0.4077
F-statistic: 6.782 on 5 and 37 DF,  p-value: 0.0001407
```

So...

- Perfect multicollinearity is terrible, but

- Perfect multicollinearity not a problem at all.

$N > K$...

Statistically,

- we lack sufficient degrees of freedom to identify $\hat{\boldsymbol{\beta}}$.
- $\hat{\boldsymbol{\beta}}$ is "overdetermined."

Conceptually:

- Variables $>$ Cases means
- ...no unique conclusion about explanatory / causal factors.

$N = K$ in Practice

```
> smallAfrica<-subset(Africa,subsaharan=="Not Sub-Saharan")
> fit2<-with(smallAfrica,lm(adrate~gdppppd+healthexp+muslperc+
+                             literacy+war))
> summary(fit2)

Call:
lm(formula = adrate ~ gdppppd + healthexp + muslperc + literacy +
    war)

Residuals:
ALL 6 residuals are 0: no residual degrees of freedom!

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept) -0.12430         NA      NA       NA
gdppppd     -0.97906         NA      NA       NA
healthexp   -0.45166         NA      NA       NA
muslperc     0.01413         NA      NA       NA
literacy     0.09512         NA      NA       NA
war         -0.96429         NA      NA       NA

Residual standard error: NaN on 0 degrees of freedom
Multiple R-squared:      1,Adjusted R-squared:    NaN
F-statistic:   NaN on 5 and 0 DF,  p-value: NA
```

Recall that

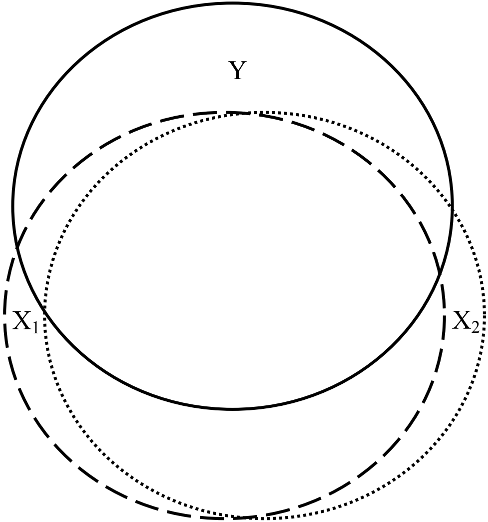
$$\widehat{\text{Var}(\hat{\boldsymbol{\beta}})} = \hat{\sigma}^2(\mathbf{X}'\mathbf{X})^{-1}$$

We can write the kth diagonal element of $(\mathbf{X}'\mathbf{X})^{-1}$ as:

$$\frac{1}{(\mathbf{X}_k'\mathbf{X}_k)(1 - \hat{R}_k^2)}$$

where $\hat{R}_k^2$ is the R^2 from the regression of $\mathbf{X}_k$ on all the other variables in $\mathbf{X}$.

The Obligatory Venn Diagram



High (Non-Perfect) Multicollinearity

Things to understand:

1. Multicollinearity is a *sample problem*.
2. Multicollinearity is a matter of *degree*.

Near-Perfect Collinearity: An Example

Consider:

$$\text{HIV}_i = \beta_0 + \beta_1(\text{Civil War}_i) + \beta_2(\text{Intensity}_i) + u_i$$

```
> with(Africa, table(internalwar,intensity))

           intensity
internalwar  0  1  2  3
          0 30  0  0  0
          1  0  6  2  5
```

Table: Three Models

	Dependent variable:		
	adrate		
	(1)	(2)	(3)
internalwar	−4.459		−2.849
	(3.274)		(6.682)
intensity		−1.955	−0.837
		(1.481)	(3.018)
Constant	10.713^{***}	10.502^{***}	10.713^{***}
	(1.800)	(1.734)	(1.821)
Observations	43	43	43
R^2	0.043	0.041	0.045
Adjusted R^2	0.020	0.017	−0.003
Residual Std. Error	9.860 (df = 41)	9.873 (df = 41)	9.973 (df = 40)
F Statistic	1.855 (df = 1; 41)	1.743 (df = 1; 41)	0.945 (df = 2; 40)
Note:			*p<0.1; **p<0.05; ***p<0.01

(Near-Perfect) Multicollinearity: Detection

Symptoms:

1. *High R^2, but nonsignificant coefficients.*
2. *High pairwise correlations among independent variables.*
3. *High partial correlations among the* $\mathbf{X}$*s.*
4. *VIF and Tolerance.*

If $\hat{R}_k^2 = 0$, then

$$\widehat{\text{Var}(\hat{\beta}_k)} = \frac{\hat{\sigma}^2}{\mathbf{X}_k'\mathbf{X}_k};$$

So:

$$\text{VIF}_k = \frac{1}{1 - \hat{R}_k^2}$$

$$\text{Tolerance} = \frac{1}{\text{VIF}_k}$$

Rule of Thumb: VIF > 10 is a problem...

What To Do?

Don't:

- **Blindly drop covariates!!!**
- Restrict $\boldsymbol{\beta}$s...

Do:

- **Add data**.
- **Transform the covariates**
    - Data reduction
    - First differences
    - Orthogonalize
- **Shrinkage / Regularization Methods**

What To Do? Shrinkage Methods

OLS is:

$$\begin{aligned}
\text{MSE} &= \text{E}\{[\mathbf{Y} - \text{E}(\mathbf{Y})]^2\} \\
&= \text{E}[(Y_i - \mathbf{X}_i\hat{\boldsymbol{\beta}})^2] \\
&= [Y_i - \text{E}(\mathbf{X}_i\hat{\boldsymbol{\beta}})]^2 + \{\text{E}[(\mathbf{X}_i\hat{\boldsymbol{\beta}}) - \text{E}(\mathbf{X}_i\hat{\boldsymbol{\beta}})]\}^2 \\
&= (\text{Bias})^2 + \text{Variance}
\end{aligned}$$

"Ridge regression":

$$\hat{\boldsymbol{\beta}}^R = (\mathbf{X}'\mathbf{X} + \lambda\mathbf{I})^{-1}\mathbf{X}'\mathbf{Y}$$

- <u>Biases</u> $\hat{\boldsymbol{\beta}}$, but
- Increases the (perceived) independent variability in $\mathbf{X}$
- Yields:

$$\widehat{\text{Var}(\hat{\boldsymbol{\beta}}_\ell^R)} = \frac{\hat{\sigma}^2}{(\mathbf{X}_\ell'\mathbf{X}_\ell + \lambda)(1 - R_\ell^2)}$$

"LASSO" $=$ "Least Absolute Shrinkage and Selection Operator."

- Formally:

$$\min_{\boldsymbol{\beta}} \left\{ \frac{1}{N} \sum_{i=1}^{N} (Y_i - \mathbf{X}_i \boldsymbol{\beta})^2 \right\} \text{ subject to } \sum_{j=1}^{p} |\beta_j| \leq t.$$

- Combines variable selection and shrinkage...
- Like ridge regression, but with some $\hat{\beta}$s set to zero
- Reduces overfitting $+$ makes the model more interpretable

OLS, Ridge Regression, Lasso, & Elastic Net

Objective / Loss Functions:

$$
\begin{aligned}
\text{OLS} &= \sum_{i=1}^{N}\left(Y_i - \sum_{k=1}^{k}\mathbf{X}_{ik}\beta_k\right)^2 \\
\text{LASSO} &= \sum_{i=1}^{N}\left(Y_i - \sum_{k=1}^{k}\mathbf{X}_{ij}\beta_k\right)^2 + \lambda\sum_{k=1}^{k}|\beta_k| \\
\text{Ridge Regression} &= \sum_{i=1}^{N}\left(Y_i - \sum_{k=1}^{k}\mathbf{X}_{ik}\beta_k\right)^2 + \lambda\sum_{k=1}^{k}\beta_k^2 \\
\text{Elastic Net} &= \sum_{i=1}^{N}\left(Y_i - \sum_{k=1}^{k}\mathbf{X}_{ik}\beta_k\right)^2 + \lambda\left[(1-\alpha)\sum_{k=1}^{k}\beta_k^2 + \alpha\sum_{k=1}^{k}|\beta_k|\right]
\end{aligned}
$$

Shrinkage Methods Using `glmnet`

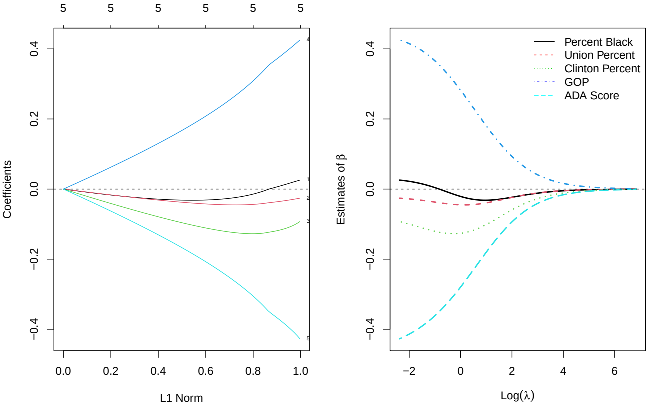
The `glmnet` package fits (generalized) linear models with regularization.

- Model controlled by α:
  - $\alpha = 0 \rightarrow$ ridge regression
  - $\alpha = 1 \rightarrow$ lasso
  - $0 < \alpha < 1 \rightarrow$ elastic net
- Fits multiple models over a range of values of λ, and
- Allows for selection of λ via *k*-fold cross-validation
- Plots, diagnostics, etc.

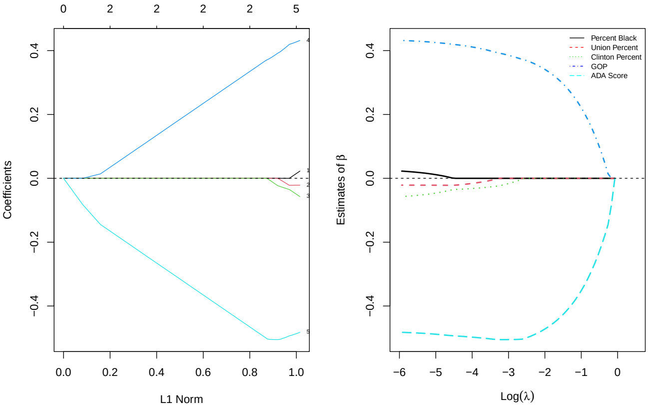
```
> summary(impeachment)
     name               state              district     votesum
 Length:433         Length:433         Min.   : 1   Min.   :0.00
 Class :character   Class :character   1st Qu.: 3   1st Qu.:0.00
 Mode  :character   Mode  :character   Median : 6   Median :2.00
                                       Mean   :10   Mean   :1.85
                                       3rd Qu.:13   3rd Qu.:4.00
                                       Max.   :52   Max.   :4.00
    pctbl96        unionpct          clint96       GOPmember         ADA98
 Min.   : 0.0   Min.   :0.0257   Min.   :26.0   Min.   :0.000   Min.   :  0.0
 1st Qu.: 2.0   1st Qu.:0.0930   1st Qu.:42.0   1st Qu.:0.000   1st Qu.:  5.0
 Median : 5.4   Median :0.1690   Median :48.0   Median :1.000   Median : 30.0
 Mean   :11.9   Mean   :0.1636   Mean   :50.3   Mean   :0.527   Mean   : 46.3
 3rd Qu.:14.0   3rd Qu.:0.2150   3rd Qu.:57.0   3rd Qu.:1.000   3rd Qu.: 90.0
 Max.   :74.0   Max.   :0.3733   Max.   :94.0   Max.   :1.000   Max.   :100.0
```

```
> # Standardize all the variables:
>
> ImpStd<-data.frame(scale(impeachment[,4:9]))
> cor(ImpStd)
          votesum  pctbl96 unionpct clint96 GOPmember   ADA98
votesum    1.0000 -0.28765 -0.26199 -0.6408    0.9198 -0.9280
pctbl96   -0.2876  1.00000 -0.09394  0.6165   -0.3091  0.3029
unionpct  -0.2620 -0.09394  1.00000  0.3331   -0.1941  0.2756
clint96   -0.6408  0.61651  0.33305  1.0000   -0.6120  0.6703
GOPmember  0.9198 -0.30911 -0.19406 -0.6120    1.0000 -0.9392
ADA98     -0.9280  0.30288  0.27563  0.6703   -0.9392  1.0000

> # OLS w/o intercept:
>
> fit<-with(ImpStd,lm(votesum~pctbl96+unionpct+clint96+GOPmember+ADA98-1))
> summary(fit)

Coefficients:
          Estimate Std. Error t value Pr(>|t|)
pctbl96     0.0301     0.0233    1.29    0.199
unionpct   -0.0212     0.0193   -1.09    0.274
clint96    -0.0650     0.0301   -2.16    0.031 *
GOPmember   0.4367     0.0492    8.88   <2e-16 ***
ADA98      -0.4775     0.0530   -9.01   <2e-16 ***
---
Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1   1

Residual standard error: 0.343 on 428 degrees of freedom
Multiple R-squared:  0.883,Adjusted R-squared:  0.882
F-statistic:  648 on 5 and 428 DF,  p-value: <2e-16

> vif(fit)
  pctbl96  unionpct   clint96 GOPmember     ADA98
    1.998     1.371     3.313     8.878    10.292
```

```
> # Ridge regression:
>
> X<-ImpStd[,2:6] # Predictors
> Y<-ImpStd[,1]   # Response
>
> ridge.fit<-glmnet(X,Y,alpha=0)
```



```
> # Lasso regression:
>
> lasso.fit<-glmnet(X,Y,alpha=1)
```



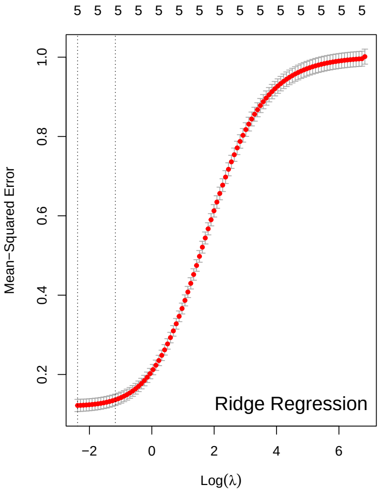
Getting λ Via Cross-Validation: Ridge Regression

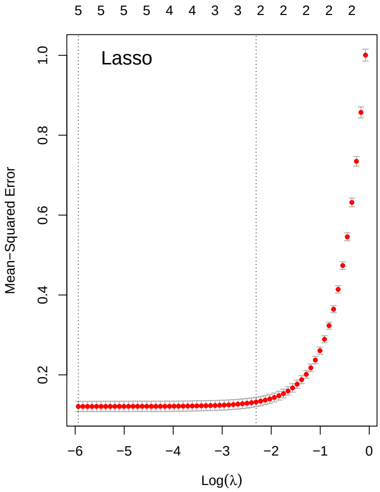
```
> # Ridge regression:
>
> ridge.cv<-cv.glmnet(as.matrix(X),as.matrix(Y),alpha=0,intercept=FALSE)
> ridge.cv

Call:  cv.glmnet(x = as.matrix(X), y = as.matrix(Y), alpha = 0, intercept = FALSE)

Measure: Mean-Squared Error

    Lambda Index Measure     SE Nonzero
min 0.0927   100   0.122 0.0161       5
1se 0.3107    87   0.136 0.0150       5

> coef(ridge.cv,s="lambda.min")
6 x 1 sparse Matrix of class "dgCMatrix"
                  s1
(Intercept)  .
pctbl96      0.02561
unionpct    -0.02598
clint96     -0.09265
GOPmember    0.42533
ADA98       -0.42733

> coef(ridge.cv,s="lambda.1se")
6 x 1 sparse Matrix of class "dgCMatrix"
                   s1
(Intercept)  .
pctbl96      0.008112
unionpct    -0.035391
clint96     -0.117373
GOPmember    0.376793
ADA98       -0.372307
```

Getting λ Via Cross-Validation: Lasso

```
> # Lasso:
>
> lasso.cv<-cv.glmnet(as.matrix(X),as.matrix(Y),alpha=1,intercept=FALSE)
> lasso.cv

Call:  cv.glmnet(x = as.matrix(X), y = as.matrix(Y), alpha = 1, intercept = FALSE)

Measure: Mean-Squared Error

    Lambda Index Measure      SE Nonzero
min 0.0026    64   0.119 0.00906       5
1se 0.0825    27   0.127 0.00812       2

> coef(lasso.cv,s="lambda.min")
6 x 1 sparse Matrix of class "dgCMatrix"
                  s1
(Intercept)  .
pctbl96      0.02255
unionpct    -0.02141
clint96     -0.05732
GOPmember    0.43174
ADA98       -0.48234

> coef(lasso.cv,s="lambda.1se")
6 x 1 sparse Matrix of class "dgCMatrix"
                 s1
(Intercept)  .
pctbl96      .
unionpct     .
clint96      .
GOPmember    0.3686
ADA98       -0.4992
```

Cross-Validation Plots





Other Things To Know

On regularization / shrinkage methods...

- Other useful R packages:
  - `caret` (will do all this, and more)
  - `grplasso`, `elasticnet`, etc.
  - More generally, see the CRAN Task View on machine learning
- Ridge regression / lasso / etc. are widely used for model selection in machine learning / prediction contexts, because
- ...they are automated ways of reducing model complexity and/or overfitting

On multicollinearity in general:

- Can often be ignored without issue
- Consider combining predictors when you can, or
- ...analyzing subsets of the data ($\rightarrow$ interactions)