

Free Vibrations of Simply Supported Beam with Distributed Mass

AI generated calculations (as continuous mass system)

Dynamic analysis based on Euler-Bernoulli beam theory

1. Input Data

Beam geometry

Span length - $L = 6 \text{ m}$

Cross-section (rectangular)

Width - $b = 0.3 \text{ m}$

Height - $h = 0.5 \text{ m}$

Cross-section area - $A = b \cdot h = 0.3 \text{ m} \cdot 0.5 \text{ m} = 0.15 \text{ m}^2$

Second moment of area - $I = \frac{b \cdot h^3}{12} = \frac{0.3 \text{ m} \cdot (0.5 \text{ m})^3}{12} = 0.00313 \text{ m}^4$

Material properties

Modulus of elasticity - $E = 30 \text{ GPa}$

Mass density - $\rho = 2500 \frac{\text{kg}}{\text{m}^3}$

Derived quantities

Bending stiffness - $EI = E \cdot I = 30 \text{ GPa} \cdot 0.00313 \text{ m}^4 = 0.0938 \text{ Mt} \cdot \text{m}^3/\text{s}^2$

Distributed mass per unit length - $m = \rho \cdot A = 2500 \text{ kg/m}^3 \cdot 0.15 \text{ m}^2 = 375 \text{ kg/m}$

Total mass of the beam - $M_{tot} = m \cdot L = 375 \text{ kg/m} \cdot 6 \text{ m} = 2250 \text{ kg}$

2. Analytical Solution

Theory

For a simply supported beam with uniformly distributed mass, the equation of free vibrations is:

$$EI \cdot \frac{\partial^4 w}{\partial x^4} + m \cdot \frac{\partial^2 w}{\partial t^2} = 0$$

The solution is obtained by separation of variables:

$$w(x, t) = \phi(x) \cdot q(t)$$

Mode shapes: $\phi_n(x) = \sin(n\pi x/L)$

Natural circular frequencies:

$$\omega_n = (n\pi/L)^2 \cdot \sqrt{EI/m}$$

Number of modes to compute

$$n = 5$$

3. Natural Frequencies and Periods

Circular frequencies ω_n

$$\omega(k) = \left(\frac{k \cdot \pi}{L}\right)^2 \cdot \sqrt{\frac{EI}{m}}$$

Natural frequencies f_n

$$f(k) = \frac{\omega(k)}{2 \cdot \pi}$$

Vibration periods T_n

$$T(k) = \frac{1}{f(k)}$$

Results for the first $n = 5$ modes:

Mode	ω_n [rad/s]	f_n [Hz]	T_n [s]
1	137.08	21.82	0.05
2	548.31	87.27	0.01
3	1233.7	196.35	0.01
4	2193.25	349.07	0
5	3426.95	545.42	0

Fundamental frequency (1st mode)

Circular frequency - $\omega_1 = \omega(1) = 137.08$ Hz

Natural frequency - $f_1 = f(1) = 21.82$ Hz

Vibration period - $T_1 = T(1) = 0.0458$ s

4. Mode Shapes

The mode shapes are sinusoidal functions:

$$\phi_n(x) = \sin(n\pi x/L)$$

$$\varphi(\mathbf{k}; \mathbf{x}) = \sin\left(\frac{\mathbf{k} \cdot \pi \cdot \mathbf{x}}{L}\right)$$

PlotHeight = 120



5. Modal Properties

Modal mass

For sinusoidal mode shapes of a simply supported beam:

$$M_n = \int_0^L m \cdot \phi_n^2(x) dx = m \cdot L / 2$$

$$M_n = \frac{m \cdot L}{2} = \frac{375 \text{ kg/m} \cdot 6 \text{ m}}{2} = 1125 \text{ kg}$$

Modal stiffness

$$K_n = \omega_n^2 \cdot M_n$$

Mode	M_n [kg]	K_n [kN/m]
1	1125	21139.13
2	1125	338226.01
3	1125	1712269.18
4	1125	5411616.17
5	1125	13211953.54

6. Free Vibration Response

Initial conditions

Initial midspan displacement - $w_0 = 5 \text{ mm}$

Initial velocity - $v_0 = 0 \frac{\text{m}}{\text{s}}$

Response at midspan

For initial displacement w_0 at midspan with zero initial velocity,

the modal initial conditions are:

$$q_n(0) = (2/L) \cdot \int_0^L w_0 \cdot \delta(x-L/2) \cdot \sin(n\pi x/L) dx$$

For concentrated initial displacement at midspan:

$$q_n(0) = w_0 \cdot \sin(n\pi/2) \cdot (2/(m \cdot L))$$

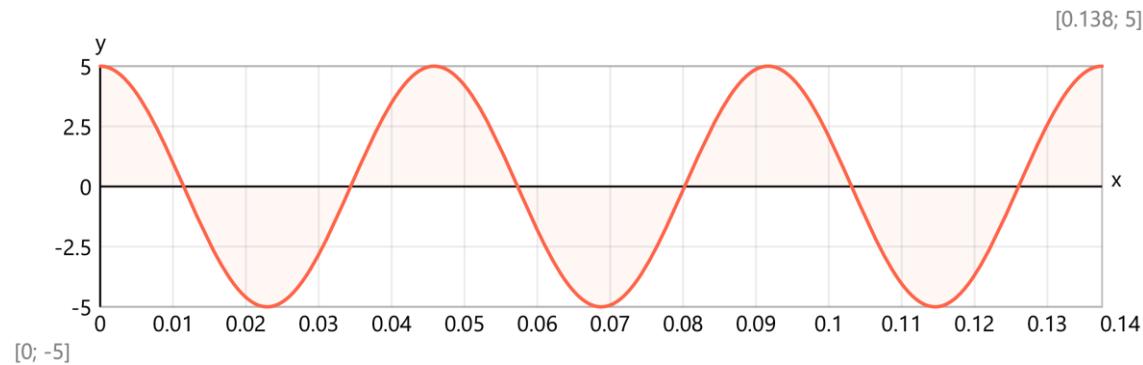
Only odd modes are excited (modes 1, 3, 5,...).

Time history of midspan displacement (first mode approximation)

Amplitude of first mode - $A_1 = w_0 = 5 \text{ mm}$

$$\text{Response} - w(t) = A_1 \cdot \sin\left(\frac{\pi \cdot L}{2} \cdot t\right) \cdot \cos(\omega(1) \cdot t)$$

$$t_{max} = 3 \cdot T(1) = 0.138 \text{ s}$$



7. Verification

Comparison with approximate formulas

Static deflection under self-weight:

$$\delta_{st} = 5 \cdot m \cdot g \cdot L^4 / (384 \cdot EI)$$

$$g = 9.81 \frac{\text{m}}{\text{s}^2}$$

$$\delta_{st} = \frac{5 \cdot m \cdot g \cdot L^4}{384 \cdot EI} = \frac{5 \cdot 375 \text{ kg/m} \cdot 9.81 \text{ m/s}^2 \cdot (6 \text{ m})^4}{384 \cdot 0.0938 \text{ Mt} \cdot \text{m}^3/\text{s}^2} = 0.662 \text{ mm}$$

Approximate fundamental frequency (Rayleigh method):

$$f_{1R} = \frac{1}{2 \cdot \pi} \cdot \sqrt{\frac{g}{\delta_{st}}} = \frac{1}{2 \cdot 3.14} \cdot \sqrt{\frac{9.81 \text{ m/s}^2}{0.662 \text{ mm}}} = 19.37 \text{ s}^{-1}$$

Exact fundamental frequency: $f_1 = \pi / (2L^2) \cdot \sqrt{EI/m}$

$$\text{Ratio (exact/Rayleigh)} - r = \frac{f_1}{f_{1R}} = \frac{21.82 \text{ Hz}}{19.37 \text{ s}^{-1}} = 1.13$$

The Rayleigh method gives a slightly different result because it uses a static deflection shape (parabolic) instead of the exact sinusoidal mode shape.

8. Frequency Ratios

Relationship between higher modes and fundamental frequency

For a simply supported beam: $f_n/f_1 = n^2$

Mode n	f_n/f_1	n^2
1	1	1
2	4	4
3	9	9
4	16	16
5	25	25

This confirms the characteristic property of simply supported beams:

natural frequencies increase proportionally to the square of the mode number.

Manual verification (as equivalent discrete mass SDOF or MDOF systems)

Structure type - simply supported beam

Beam length - $L = 6 \text{ m}$

Cross section - rectangular with dimensions:

Width - $b = 300 \text{ mm}$

Height - $h = 500 \text{ mm}$

Area - $A = b \cdot h = 300 \text{ mm} \cdot 500 \text{ mm} = 1500 \text{ cm}^2$

Second moment of area - $I = \frac{b \cdot h^3}{12} = \frac{300 \text{ mm} \cdot (500 \text{ mm})^3}{12} = 3125000000 \text{ mm}^4$

Shear area - $A_Q = \frac{5}{6} \cdot A = \frac{5}{6} \cdot 1500 \text{ cm}^2 = 1250 \text{ cm}^2$

Modulus of elasticity - $E = 30 \text{ GPa}$

Poisson's ratio - $\nu = 0.2$

Shear modulus - $G = \frac{E}{2 \cdot (1 + \nu)} = \frac{30 \text{ GPa}}{2 \cdot (1 + 0.2)} = 12.5 \text{ GPa}$

Mass density - $\rho = 2500 \frac{\text{kg}}{\text{m}^3}$

Distributed mass per unit length - $m = \rho \cdot A = 2500 \text{ kg/m}^3 \cdot 1500 \text{ cm}^2 = 375 \text{ kg/m}$

Simple solution as equivalent SDOF system

The structure is reduced to a SDOF system for simplicity

$$\text{Dynamically equivalent mass} - M_{tot} = \frac{2 \cdot L}{\pi} \cdot m = \frac{2 \cdot 6 \text{ m}}{3.14} \cdot 375 \text{ kg/m} = 1432.39 \text{ kg}$$

Structural stiffness for a vertical force applied at the middle point of the span

$$K = \frac{48 \cdot E \cdot I}{L^3} = \frac{48 \cdot 30 \text{ GPa} \cdot 3125000000 \text{ mm}^4}{(6 \text{ m})^3} = 20833.3 \text{ kN/m}$$

$$\text{Natural circular frequency} - \omega_1 = \sqrt{\frac{K}{M_{tot}}} = \sqrt{\frac{20833.3 \text{ kN/m}}{1432.39 \text{ kg}}} = 120.6 \text{ s}^{-1}$$

$$\text{Vibration frequency} - f_1 = \frac{\omega_1}{2 \cdot \pi} = \frac{120.6 \text{ s}^{-1}}{2 \cdot 3.14} = 19.19 \text{ s}^{-1}$$

$$\text{Vibration period} - T_1 = \frac{2 \cdot \pi}{\omega_1} = \frac{2 \cdot 3.14}{120.6 \text{ s}^{-1}} = 0.0521 \text{ s}$$

Solution as an MDOF system

Number of intermediate joints - $n_J = 11$

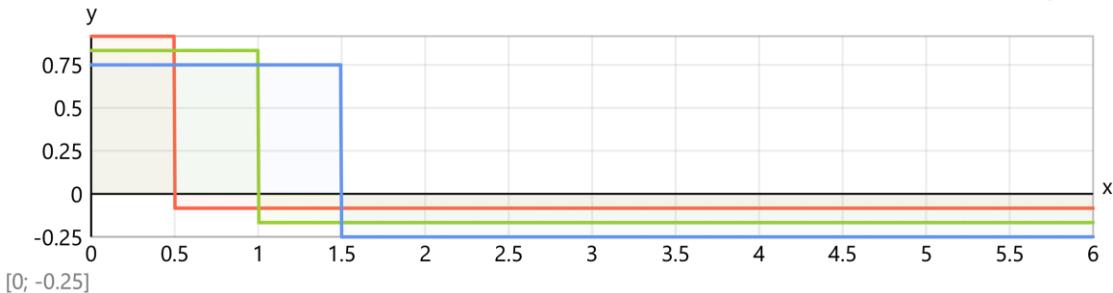
$$\text{Length of one segment} - \Delta x = \frac{L}{n_J + 1} = \frac{6 \text{ m}}{11 + 1} = 0.5 \text{ m}$$

Coordinate of joint j - $x_j(j) = \Delta x \cdot j$

Shear forces due to unit vertical load $F_j = 1$ at joint j

$$V_1(x; j) = \begin{cases} \text{if } x < x_j(j): & 1 - \frac{x_j(j)}{L} \\ \text{else:} & \frac{-x_j(j)}{L} \end{cases}$$

[6; 0.917]

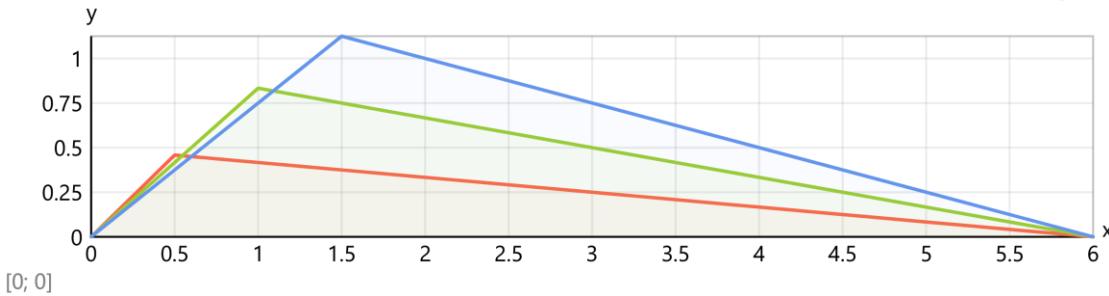


Bending moments due to unit vertical load $F_j = 1$ at joint j

$$M_{1,max}(j) = \left(\frac{x_j(j)}{L} - 1 \right) \cdot x_j(j)$$

$$M_1(x; j) = M_{1,max}(j) \cdot \begin{cases} \text{if } x < x_j(j): & \frac{x}{x_j(j)} \\ \text{else:} & \frac{L-x}{L-x_j(j)} \end{cases}$$

[6; 1.12]



Flexibility matrix

$$D(i; j) = \left(\int_{0 \text{ m}}^L M_1(x; i) \cdot M_1(x; j) dx \right) \cdot \frac{1}{E \cdot I} + \$\text{Area}\{V_1(x; i)\} * V_1(x; j) @ x = 0 \text{ m} : L * (1 / (G * A_Q))$$

(skipped for compatibility)

$$D = \begin{bmatrix} 0.00448 & 0.00796 & 0.0103 & 0.0117 & 0.0122 & 0.0119 & 0.0109 & 0.00941 & 0.00744 & 0.00515 & 0.00263 \\ 0.00796 & 0.0148 & 0.0197 & 0.0225 & 0.0236 & 0.0231 & 0.0213 & 0.0184 & 0.0146 & 0.0101 & 0.00515 \\ 0.0103 & 0.0197 & 0.027 & 0.0316 & 0.0334 & 0.033 & 0.0306 & 0.0264 & 0.021 & 0.0146 & 0.00744 \\ 0.0117 & 0.0225 & 0.0316 & 0.0379 & 0.041 & 0.0409 & 0.0381 & 0.0332 & 0.0264 & 0.0184 & 0.00941 \\ 0.0122 & 0.0236 & 0.0334 & 0.041 & 0.0454 & 0.0461 & 0.0435 & 0.0381 & 0.0306 & 0.0213 & 0.0109 \\ 0.0119 & 0.0231 & 0.033 & 0.0409 & 0.0461 & 0.048 & 0.0461 & 0.0409 & 0.033 & 0.0231 & 0.0119 \\ 0.0109 & 0.0213 & 0.0306 & 0.0381 & 0.0435 & 0.0461 & 0.0454 & 0.041 & 0.0334 & 0.0236 & 0.0122 \\ 0.00941 & 0.0184 & 0.0264 & 0.0332 & 0.0381 & 0.0409 & 0.041 & 0.0379 & 0.0316 & 0.0225 & 0.0117 \\ 0.00744 & 0.0146 & 0.021 & 0.0264 & 0.0306 & 0.033 & 0.0334 & 0.0316 & 0.027 & 0.0197 & 0.0103 \\ 0.00515 & 0.0101 & 0.0146 & 0.0184 & 0.0213 & 0.0231 & 0.0236 & 0.0225 & 0.0197 & 0.0148 & 0.00796 \\ 0.00263 & 0.00515 & 0.00744 & 0.00941 & 0.0109 & 0.0119 & 0.0122 & 0.0117 & 0.0103 & 0.00796 & 0.00448 \end{bmatrix}$$

mm/kN

Mass matrix

$$d_{M,j} = \frac{m \cdot \Delta x}{t} = \frac{375 \text{ kg/m} \cdot 0.5 \text{ m}}{t} = 0.188$$

$$M = \begin{bmatrix} 0.188 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.188 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.188 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.188 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.188 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.188 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.188 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.188 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.188 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.188 & 0 \end{bmatrix}$$

Total mass of the structure - $\text{sum}(\vec{d}_M) = 2.06 \text{ t}$

Eigenvalues

$$\mathbf{M}_{sq} = \sqrt{\mathbf{M}} = \begin{bmatrix} 0.433 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.433 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.433 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.433 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.433 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.433 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.433 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.433 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.433 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.433 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.433 \end{bmatrix}$$

$$\mathcal{C} = \text{copy}(\mathbf{M}_{sq} \cdot \mathbf{D} \cdot \mathbf{M}_{sq}; \text{symmetric}(n_J); 1; 1) = \text{copy}(\mathbf{M}_{sq} \cdot \mathbf{D} \cdot \mathbf{M}_{sq}; \text{symmetric}(11); 1; 1)$$

$$= \begin{bmatrix} 0.00084 & 0.00149 & 0.00194 & 0.00219 & 0.00228 & 0.00223 & 0.00205 & 0.00176 & 0.0014 & 0.000965 & 0.000493 \\ 0.00149 & 0.00278 & 0.00369 & 0.00422 & 0.00442 & 0.00433 & 0.00399 & 0.00344 & 0.00273 & 0.00189 & 0.000965 \\ 0.00194 & 0.00369 & 0.00506 & 0.00592 & 0.00627 & 0.00619 & 0.00573 & 0.00496 & 0.00394 & 0.00273 & 0.0014 \\ 0.00219 & 0.00422 & 0.00592 & 0.00711 & 0.00768 & 0.00767 & 0.00715 & 0.00622 & 0.00496 & 0.00344 & 0.00176 \\ 0.00228 & 0.00442 & 0.00627 & 0.00768 & 0.00851 & 0.00865 & 0.00816 & 0.00715 & 0.00573 & 0.00399 & 0.00205 \\ 0.00223 & 0.00433 & 0.00619 & 0.00767 & 0.00865 & 0.009 & 0.00865 & 0.00767 & 0.00619 & 0.00433 & 0.00223 \\ 0.00205 & 0.00399 & 0.00573 & 0.00715 & 0.00816 & 0.00865 & 0.00851 & 0.00768 & 0.00627 & 0.00442 & 0.00228 \\ 0.00176 & 0.00344 & 0.00496 & 0.00622 & 0.00715 & 0.00767 & 0.00768 & 0.00711 & 0.00592 & 0.00422 & 0.00219 \\ 0.0014 & 0.00273 & 0.00394 & 0.00496 & 0.00573 & 0.00619 & 0.00627 & 0.00592 & 0.00506 & 0.00369 & 0.00194 \\ 0.000965 & 0.00189 & 0.00273 & 0.00344 & 0.00399 & 0.00433 & 0.00442 & 0.00422 & 0.00369 & 0.00278 & 0.00149 \\ -0.000493 & 0.000965 & 0.0014 & 0.00176 & 0.00205 & 0.00223 & 0.00228 & 0.00219 & 0.00194 & 0.00084 & 0.000493 \end{bmatrix}$$

$$\vec{\lambda} = \text{eigvals}(\mathcal{C} \cdot 10^{-3}; -7)$$

$$= [5.32 \times 10^{-5} \ 3.33 \times 10^{-6} \ 6.57 \times 10^{-7} \ 2.08 \times 10^{-7} \ 8.57 \times 10^{-8} \ 4.17 \times 10^{-8} \ 2.29 \times 10^{-8}]$$

Natural circular frequencies - $\vec{\omega} = \sqrt{\frac{1}{\vec{\lambda}}} =$
 $[137.08 \ 548.28 \ 1233.32 \ 2190.89 \ 3416.68 \ 4898.98 \ 6609.4] \text{ s}^{-1}$

Natural vibration frequencies - $\vec{f} = \frac{\vec{\omega}}{2 \cdot \pi} \cdot \text{Hz} = \frac{\vec{\omega}}{2 \cdot 3.14} \cdot \text{Hz} =$
 $[21.82 \text{Hz} \ 87.26 \text{Hz} \ 196.29 \text{Hz} \ 348.69 \text{Hz} \ 543.78 \text{Hz} \ 779.7 \text{Hz} \ 1051.92 \text{Hz}]$

Natural vibration periods - $\vec{T} = \frac{1}{\vec{f}} =$
 $[0.0458 \text{s} \ 0.0115 \text{s} \ 0.00509 \text{s} \ 0.00287 \text{s} \ 0.00184 \text{s} \ 0.00128 \text{s} \ 0.000951 \text{s}]$

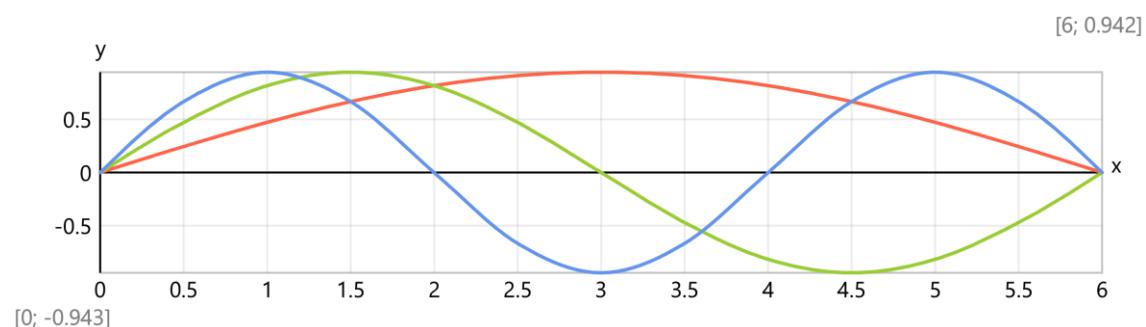
Eigenvectors

$$V = \text{transp}(\text{eigenvecs}(C \cdot 10^{-3}; -7)) = \begin{bmatrix} 0.106 & 0.204 & 0.289 & 0.354 & 0.394 & 0.408 & 0.394 \\ 0.204 & 0.354 & 0.408 & 0.354 & 0.204 & -6 \times 10^{-14} & -0.204 \\ 0.289 & 0.408 & 0.289 & 0 & -0.289 & -0.408 & -0.289 \\ 0.354 & 0.354 & 0 & -0.354 & -0.354 & 9.54 \times 10^{-14} & 0.354 \\ 0.394 & 0.204 & -0.289 & -0.354 & 0.106 & 0.408 & 0.106 \\ 0.408 & 0 & -0.408 & 0 & 0.408 & -1.04 \times 10^{-14} & -0.408 \\ 0.394 & -0.204 & -0.289 & 0.354 & 0.106 & -0.408 & 0.106 \\ 0.354 & -0.354 & 0 & 0.354 & -0.354 & 8.73 \times 10^{-15} & 0.354 \\ 0.289 & -0.408 & 0.289 & -1.48 \times 10^{-14} & -0.289 & 0.408 & -0.289 \\ 0.204 & -0.354 & 0.408 & -0.354 & 0.204 & -5.96 \times 10^{-14} & -0.204 \\ 0.106 & -0.204 & 0.289 & -0.354 & 0.394 & -0.408 & 0.394 \end{bmatrix}$$

$$\Phi = \text{inverse}(M_{sq}) \cdot V = \begin{bmatrix} 0.244 & 0.471 & 0.667 & 0.816 & 0.911 & 0.943 & 0.911 \\ 0.471 & 0.816 & 0.943 & 0.816 & 0.471 & -1.39 \times 10^{-13} & -0.471 \\ 0.667 & 0.943 & 0.667 & 0 & -0.667 & -0.943 & -0.667 \\ 0.816 & 0.816 & 0 & -0.816 & -0.816 & 2.2 \times 10^{-13} & 0.816 \\ 0.911 & 0.471 & -0.667 & -0.816 & 0.244 & 0.943 & 0.244 \\ 0.943 & 0 & -0.943 & 0 & 0.943 & -2.39 \times 10^{-14} & -0.943 \\ 0.911 & -0.471 & -0.667 & 0.816 & 0.244 & -0.943 & 0.244 \\ 0.816 & -0.816 & 0 & 0.816 & -0.816 & 2.02 \times 10^{-14} & 0.816 \\ 0.667 & -0.943 & 0.667 & -3.41 \times 10^{-14} & -0.667 & 0.943 & -0.667 \\ 0.471 & -0.816 & 0.943 & -0.816 & 0.471 & -1.38 \times 10^{-13} & -0.471 \\ 0.244 & -0.471 & 0.667 & -0.816 & 0.911 & -0.943 & 0.911 \end{bmatrix}$$

$$X = \text{stack}(\text{matrix}(1; 3); \Phi; \text{matrix}(1; 3))$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.244 & 0.471 & 0.667 & 0.816 & 0.911 & 0.943 & 0.911 \\ 0.471 & 0.816 & 0.943 & 0.816 & 0.471 & -1.39 \times 10^{-13} & -0.471 \\ 0.667 & 0.943 & 0.667 & 0 & -0.667 & -0.943 & -0.667 \\ 0.816 & 0.816 & 0 & -0.816 & -0.816 & 2.2 \times 10^{-13} & 0.816 \\ 0.911 & 0.471 & -0.667 & -0.816 & 0.244 & 0.943 & 0.244 \\ 0.943 & 0 & -0.943 & 0 & 0.943 & -2.39 \times 10^{-14} & -0.943 \\ 0.911 & -0.471 & -0.667 & 0.816 & 0.244 & -0.943 & 0.244 \\ 0.816 & -0.816 & 0 & 0.816 & -0.816 & 2.02 \times 10^{-14} & 0.816 \\ 0.667 & -0.943 & 0.667 & -3.41 \times 10^{-14} & -0.667 & 0.943 & -0.667 \\ 0.471 & -0.816 & 0.943 & -0.816 & 0.471 & -1.38 \times 10^{-13} & -0.471 \\ 0.244 & -0.471 & 0.667 & -0.816 & 0.911 & -0.943 & 0.911 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$



Comparison of the results

AI gen, Hz	Manual, Hz	Difference
21.82	21.82	0.00%
87.27	87.26	0.01%
196.35	196.29	0.03%
349.07	348.69	0.11%
545.42	543.78	0.30%