

Comparison of several methods for analysis of simple undamped pendulum

Input parameters

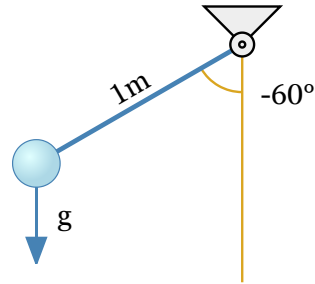
Gravitational acceleration (m/s^2) - $g = 9.81 \frac{\text{m}}{\text{s}^2}$

Pendulum length - $l = 1 \text{ m}$

Pendulum mass - $m = 1 \text{ kg}$

Initial angle - $\theta_0 = -60^\circ = -1.05 \text{ rad}$

Maximum simulation time - $t_{\text{max}} = 10 \text{ s}$



Analytical solution for small rotations

$\theta \ll 1$ or $\sin(\theta) \approx \theta$

Differential equation - $\theta'' + \frac{g}{l} \cdot \theta = 0$

Angular frequency - $\omega = \sqrt{\frac{g}{l}} \cdot \text{rad} = 3.13 \text{ rad/s}$

Cyclic frequency - $f = \frac{\omega}{2 \cdot \pi \cdot \text{rad}} = 0.498 \text{ Hz}$

Period - $T = \frac{1}{f} = 2.01 \text{ s}$

Equation of motion - $\theta(t) = \theta_0 \cdot \cos(\omega \cdot t)$

Analytical solution for large rotations (exact)

Differential equation - $\theta'' + \frac{g}{l} \cdot \sin(\theta) = 0$

Incomplete elliptic integral of the first kind

$$F(\varphi; k) = \int_0^\varphi \frac{1}{\sqrt{1 - k^2 \cdot \sin^2(\theta)}} d\theta$$

Jacobi elliptic functions

Modulus - $k = \sin\left(\frac{\theta_0}{2}\right) = 0.5$

$\text{am}(u; k) = \text{Root}\{F(\varphi; k) = u; \varphi \in [0; 10 \cdot \pi]\}$

$\text{sn}(u; k) = \sin(\text{am}(u; k))$, $\text{cn}(u; k) = \cos(\text{am}(u; k))$

$\text{dn}(u; k) = \sqrt{1 - k^2 \cdot \text{sn}^2(u; k)}$, $\text{cd}(u; k) = \frac{\text{cn}(u; k)}{\text{dn}(u; k)}$

Period - $T_e = 4 \cdot \sqrt{\frac{l}{g}} \cdot F\left(\frac{\pi}{2}; k\right) = 2.15 \text{ s}$

$$\text{Error} - \delta_T = \frac{|T - T_e|}{T_e} = 6.82\%$$

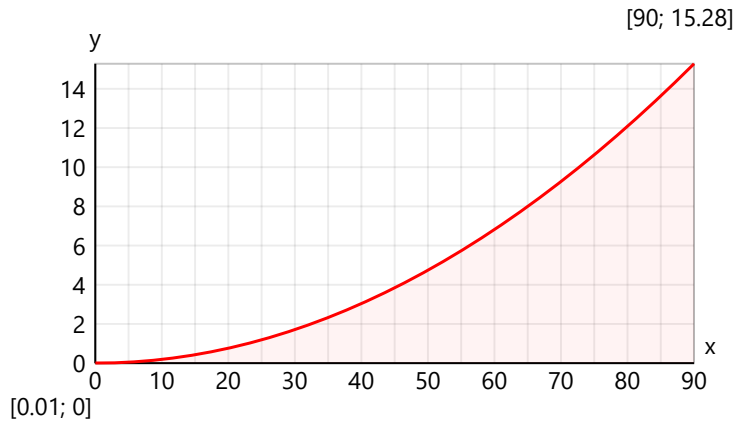
Cyclic frequency - $f_e = \frac{1}{T_e} = 0.464 \text{ Hz}$

Angular frequency - $\omega_e = 2 \cdot \pi \cdot \text{rad} \cdot f_e = 2.92 \text{ rad/s}$

Equation of motion - $\theta_e(t) = 2 \cdot \sin\left(k \cdot cd\left(\sqrt{\frac{g}{l}} \cdot t; k\right)\right)$

Energy - $E_0 = m \cdot l \cdot g \cdot (1 - \cos(\theta_0)) = 4.9 \text{ J}$

Relative error δ_T [%] of small displacements period versus initial angle θ_0 [°] plot



Solution by forward Euler method (explicit)

For that purpose, the II order equation is reduced the following system of I order equations

$$\theta' = \omega \text{ and } \omega' = \frac{g}{l} \cdot \sin(-\theta)$$

The solution is performed iteratively

Step size - $h = 0.05s$

$$\text{Number of steps} - n = \frac{t_{\max}}{h} = 200$$

For each time step $n = 200$ the values for the next step will be obtained by using the following equations

$$\begin{aligned}\theta_{n+1} &= \theta_n + h \cdot \omega_n \\ \omega_{n+1} &= \omega_n + h \cdot \frac{g}{l} \cdot \sin \theta_n\end{aligned}$$

Allocate vectors

[illegible]

[illegible]

[illegible]

Set initial conditions

$$\vec{\theta}_{\text{fwE.1}} = \frac{\theta_0}{1 \text{ rad}} = -1.05, \quad \vec{\omega}_{\text{fwE.1}} = \frac{0}{\text{s}}$$

Perform Euler steps

Rotation - $\bar{\theta}_{CN.2} = \text{Root}\{f(\theta) = 0; \theta \in [-2\pi; 2\pi]\} = -1.04$

Angular velocity - $\bar{\omega}_{CN.2} = \bar{\omega}_{CN.1} + \frac{h \cdot g}{2 \cdot l} \cdot (\sin(-\bar{\theta}_{CN.1}) + \sin(-\bar{\theta}_{CN.2})) = 0.423 \text{ s}^{-1}$

Energy - $\bar{E}_{CN.1} = m \cdot l^2 \cdot \left(\frac{1}{2} \cdot \bar{\omega}_{CN.1}^2 + \frac{g}{l} \cdot (1 - \cos(\bar{\theta}_{CN.1})) \right) = 4.9 \text{ J}$

Solution by Runge-Kutta RK4 method (explicit)

The following iterative procedure is applied:

First step (k1) - $k_{1,\theta} = \omega_i$, $k_{1,\omega} = \frac{g}{l} \cdot \sin(-\theta_i)$

Second step (k2) - $k_{2,\theta} = \omega_i + 0.5 \cdot h \cdot k_{1,\omega}$, $k_{2,\omega} = \frac{g}{l} \cdot \sin(-(\theta_i + 0.5 \cdot h \cdot k_{1,\theta}))$

Third step (k3) - $k_{3,\theta} = \omega_i + 0.5 \cdot h \cdot k_{2,\omega}$, $k_{3,\omega} = \frac{g}{l} \cdot \sin(-(\theta_i + 0.5 \cdot h \cdot k_{2,\theta}))$

Fourth step (k4) - $k_{4,\theta} = \omega_i + h \cdot k_{3,\omega}$, $k_{4,\omega} = \frac{g}{l} \cdot \sin(-(\theta_i + h \cdot k_{3,\theta}))$

Update values using weighted averages

$$\begin{cases} \theta_{n+1} = \theta_i + \frac{h}{6} \cdot (k_{1,\theta} + 2 \cdot k_{2,\theta} + 2 \cdot k_{3,\theta} + k_{4,\theta}) \\ \omega_{n+1} = \omega_i + \frac{h}{6} \cdot (k_{1,\omega} + 2 \cdot k_{2,\omega} + 2 \cdot k_{3,\omega} + k_{4,\omega}) \end{cases}$$

Allocate vectors

$$\bar{\theta}_{RK4} = \text{vector}(n) = [0 \ 0 \dots 0]$$

$$\bar{\omega}_{RK4} = \text{vector}(n) = [0 \ 0 \dots 0]$$

$$\bar{E}_{RK4} = \text{vector}(n) = [0 \ 0 \dots 0]$$

Set initial conditions

$$\bar{\theta}_{RK4.1} = \frac{\theta_0}{1 \text{ rad}} = -1.05, \quad \bar{\omega}_{RK4.1} = \frac{0}{s}$$

Perform Runge-Kutta 4 steps

RK4 factors

$$k_{1,\theta} = \bar{\omega}_{RK4.1} = 0 \text{ s}^{-1}, \quad k_{1,\omega} = \frac{g}{l} \cdot \sin(-\bar{\theta}_{RK4.1}) = 8.49 \text{ s}^{-2}$$

$$k_{2,\theta} = \bar{\omega}_{RK4.1} + 0.5 \cdot h \cdot k_{1,\omega} = 0.212 \text{ s}^{-1}$$

$$k_{2,\omega} = \frac{g}{l} \cdot \sin(-(\bar{\theta}_{RK4.1} + 0.5 \cdot h \cdot k_{1,\theta})) = 8.49 \text{ s}^{-2}$$

$$k_{3,\theta} = \bar{\omega}_{RK4.1} + 0.5 \cdot h \cdot k_{2,\omega} = 0.212 \text{ s}^{-1}$$

$$k_{3,\omega} = \frac{g}{l} \cdot \sin(-(\bar{\theta}_{RK4.1} + 0.5 \cdot h \cdot k_{2,\theta})) = 8.47 \text{ s}^{-2}$$

$$k_{4,\theta} = \bar{\omega}_{RK4.1} + h \cdot k_{3,\omega} = 0.423 \text{ s}^{-1}$$

$$k_{4,\omega} = \frac{g}{l} \cdot \sin(-(\bar{\theta}_{RK4.1} + h \cdot k_{3,\theta})) = 8.44 \text{ s}^{-2}$$

Update values using weighted averages

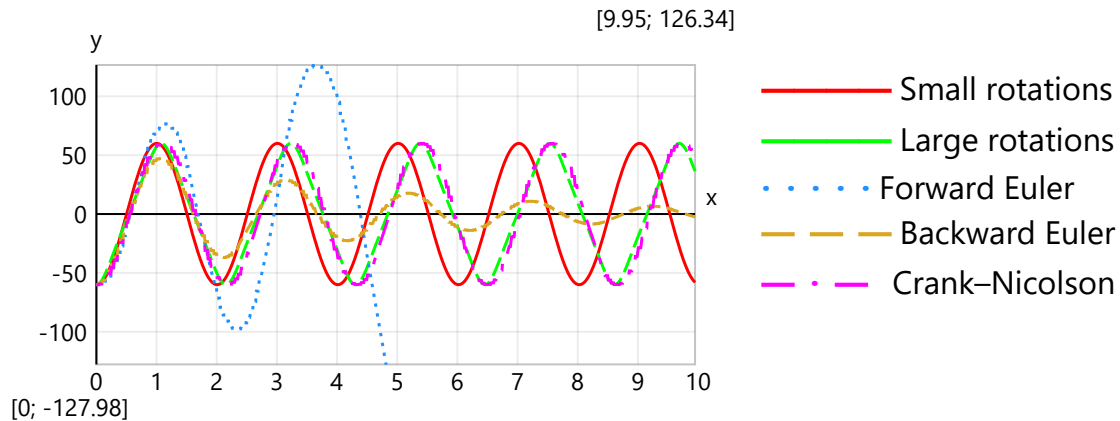
$$\text{Rotation} - \vec{\theta}_{\text{RK4.2}} = \vec{\theta}_{\text{RK4.1}} + \frac{h}{6} \cdot (k_{1,\theta} + 2 \cdot k_{2,\theta} + 2 \cdot k_{3,\theta} + k_{4,\theta}) = -1.04$$

$$\text{Angular velocity} - \vec{\omega}_{\text{RK4.2}} = \vec{\omega}_{\text{RK4.1}} + \frac{h}{6} \cdot (k_{1,\omega} + 2 \cdot k_{2,\omega} + 2 \cdot k_{3,\omega} + k_{4,\omega}) = 0.424 \text{ s}^{-1}$$

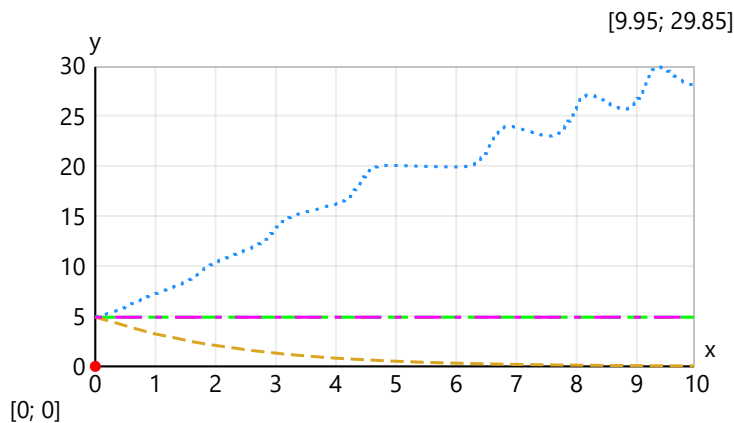
$$\text{Energy} - \vec{E}_{\text{RK4.1}} = m \cdot l^2 \cdot \left(\frac{1}{2} \cdot \vec{\omega}_{\text{RK4.1}}^2 + \frac{g}{l} \cdot (1 - \cos(\vec{\theta}_{\text{RK4.1}})) \right) = 4.9 \text{ J}$$

Plot results

Rotation θ [deg] versus time t [s] plot

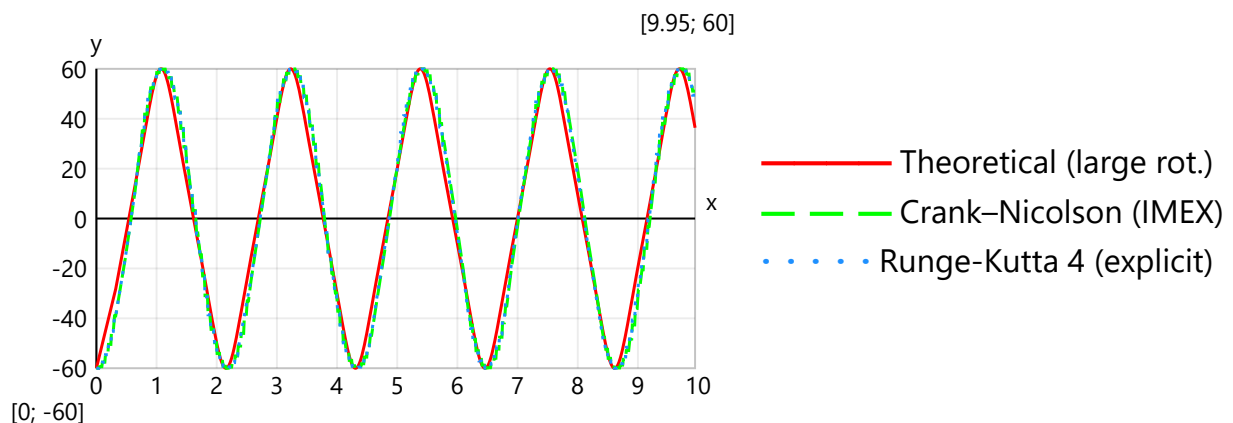


Energy E [J] versus time t [s] plot

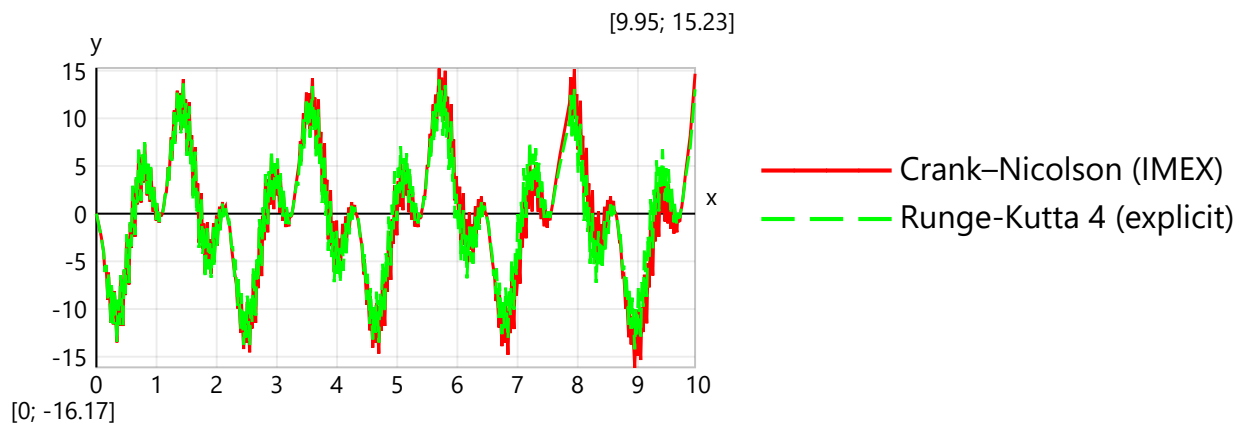


Comparision of Crank–Nicolson and Runge-Kutta 4 methods

Rotation θ [deg] versus time t [s] plot



Absolute error $\Delta\theta$ [°] versus time t [s] plot



Energy E [J] versus time t [s] plot

