Finite Element Analysis of Flat Slab with Calcpad

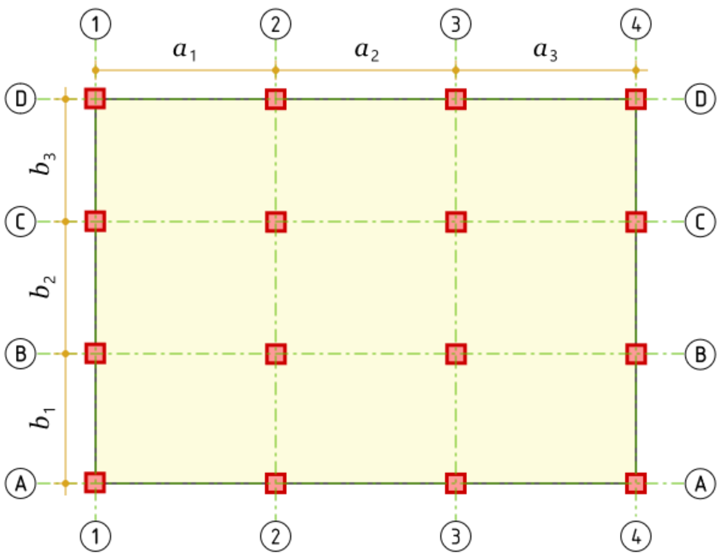
Using Bogner-Fox-Schmit (BFS) plate element [1]

Input data

Span lengths

$\vec{a} = \mathbf{hp}([3.6; 4.2; 4.2; 3.6]) = [3.6\ \ 4.2\ \ 4.2\ \ 3.6]$ m , $\vec{b} = \mathbf{hp}([3; 3.6; 3]) = [3\ \ 3.6\ \ 3]$ m

Number of axes - $n_{sa} = \mathbf{len}(\vec{a}) + 1 = 5$, $n_{sb} = \mathbf{len}(\vec{b}) + 1 = 4$



Axis coordinates - $\vec{x}_s = [0\ \ 3.6\ \ 7.8\ \ 12\ \ 15.6]$ m, $\vec{y}_s = [0\ \ 3\ \ 6.6\ \ 9.6]$ m

Slab dimensions - $l_a = \vec{x}_{s_.} = 15.6$ m, $l_b = \vec{y}_{s_.} = 9.6$ m

Thickness - $t = 0.2$ m

Load - $q = 10$ kN/m²

Modulus of elasticity - $E = 35000$ MPa

Poisson`s ratio - $\nu = 0.2$

Finite element mesh

We will use BFS rectangular finite element with $n_{DOFs} = 16$

Element dimensions - $a_1 = 0.6$ m, $b_1 = 0.6$ m

Number of elements and joints along *a* and *b*

$\vec{n}_a = \mathbf{ceiling}\left(\frac{\vec{a}}{a_1}\right) = [6\ \ 7\ \ 7\ \ 6]$, $n_{ea} = \mathbf{sum}(\vec{n}_a) = 26$, $n_{ja} = n_{ea} + 1 = 27$

$\vec{n}_b = \mathbf{ceiling}\left(\frac{\vec{b}}{b_1}\right) = [5\ \ 6\ \ 5]$, $n_{eb} = \mathbf{sum}(\vec{n}_b) = 16$, $n_{jb} = n_{eb} + 1 = 17$

Total number of elements - $n_e = n_{ea} \cdot n_{eb} = 416$

Total number of joints - $n_j = n_{ja} \cdot n_{jb} = 459$

Supported joints count - $n_s = n_{sa} \cdot n_{sb} = 20$

Joint coordinates

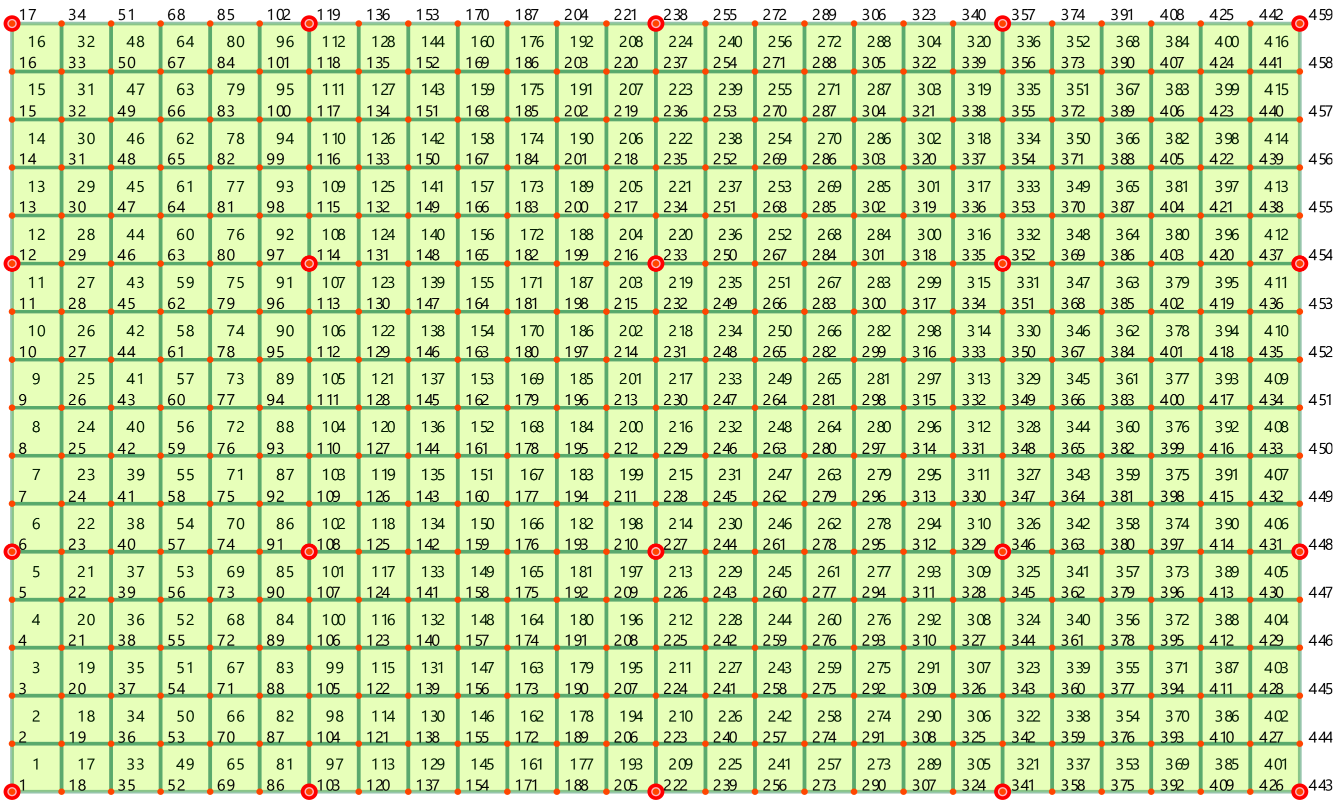
$\vec{x}_j = [0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; \ldots \;\; 15.6]$ m , $\vec{y}_j = [0 \;\; 0.6 \;\; 1.2 \;\; 1.8 \;\; 2.4 \;\; 3 \;\; 3.6 \;\; 4.2 \;\; 4.8 \;\; 5.4 \;\; \ldots \;\; 9.6]$ m

Numbers of joints at elements' corners

$$\mathbf{transp}(e_j) = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & \cdots & 441 \\ 18 & 19 & 20 & 21 & 22 & 23 & 24 & 25 & 26 & 27 & \cdots & 458 \\ 19 & 20 & 21 & 22 & 23 & 24 & 25 & 26 & 27 & 28 & \cdots & 459 \\ 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & \cdots & 442 \end{bmatrix}$$

Supported joints - $\vec{s}_j = [1 \;\; 6 \;\; 12 \;\; 17 \;\; 103 \;\; 108 \;\; 114 \;\; 119 \;\; 222 \;\; 227 \;\; \ldots \;\; 459]$

Joints for element e - $j_e(e) = \mathbf{row}(e_j; e)$



Finite element formulation

Shape functions

Along dimension *a*

Base functions

$$\Phi_{1a}(\xi) = 1 - \xi^2 \cdot (3 - 2 \cdot \xi)$$

$$\Phi_{2a}(\xi) = \xi \cdot a_1 \cdot \left(1 - \xi \cdot (2 - \xi)\right)$$

$$\Phi_{3a}(\xi) = \xi^2 \cdot (3 - 2 \cdot \xi)$$

$$\Phi_{4a}(\xi) = \xi^2 \cdot a_1 \cdot (-1 + \xi)$$

First derivatives

$$\Phi'_{1a}(\xi) = -6 \cdot \frac{\xi}{a_1} \cdot (1 - \xi)$$

$$\Phi'_{2a}(\xi) = 1 - \xi \cdot (4 - 3 \cdot \xi)$$

$$\Phi'_{3a}(\xi) = 6 \cdot \frac{\xi}{a_1} \cdot (1 - \xi)$$

$$\Phi'_{4a}(\xi) = -\xi \cdot (2 - 3 \cdot \xi)$$

Second derivatives

$$\Phi''_{1a}(\xi) = -\frac{6}{{a_1}^2} \cdot (1 - 2 \cdot \xi)$$

$$\Phi''_{2a}(\xi) = -\frac{2}{a_1} \cdot (2 - 3 \cdot \xi)$$

$$\Phi''_{3a}(\xi) = \frac{6}{{a_1}^2} \cdot (1 - 2 \cdot \xi)$$

$$\Phi''_{4a}(\xi) = -\frac{2}{a_1} \cdot (1 - 3 \cdot \xi)$$

Along dimension *b*

Base functions

$$\Phi_{1b}(\eta) = 1 - \eta^2 \cdot (3 - 2 \cdot \eta)$$

$$\Phi_{2b}(\eta) = \eta \cdot b_1 \cdot (1 - \eta \cdot (2 - \eta))$$

$$\Phi_{3b}(\eta) = \eta^2 \cdot (3 - 2 \cdot \eta)$$

$$\Phi_{4b}(\eta) = \eta^2 \cdot b_1 \cdot (-1 + \eta)$$

First derivatives

$$\Phi'_{1b}(\eta) = -6 \cdot \frac{\eta}{b_1} \cdot (1 - \eta)$$

$$\Phi'_{2b}(\eta) = 1 - \eta \cdot (4 - 3 \cdot \eta)$$

$$\Phi'_{3b}(\eta) = 6 \cdot \frac{\eta}{b_1} \cdot (1 - \eta)$$

$$\Phi'_{4b}(\eta) = -\eta \cdot (2 - 3 \cdot \eta)$$

Second derivatives

$$\Phi''_{1b}(\eta) = -\frac{6}{{b_1}^2} \cdot (1 - 2 \cdot \eta)$$

$$\Phi''_{2b}(\eta) = -\frac{2}{b_1} \cdot (2 - 3 \cdot \eta)$$

$$\Phi''_{3b}(\eta) = \frac{6}{{b_1}^2} \cdot (1 - 2 \cdot \eta)$$

$$\Phi''_{4b}(\eta) = -\frac{2}{b_1} \cdot (1 - 3 \cdot \eta)$$

For vertical displacements

$$N_{1,w}(\xi;\eta) = \Phi_{1a}(\xi) \cdot \Phi_{1b}(\eta)$$

$$N_{2,w}(\xi;\eta) = \Phi_{3a}(\xi) \cdot \Phi_{1b}(\eta)$$

$$N_{3,w}(\xi;\eta) = \Phi_{3a}(\xi) \cdot \Phi_{3b}(\eta)$$

$$N_{4,w}(\xi;\eta) = \Phi_{1a}(\xi) \cdot \Phi_{3b}(\eta)$$

For rotations ϑ_x

$$N_{1,\theta_x}(\xi;\eta) = \Phi_{2a}(\xi) \cdot \Phi_{1b}(\eta)$$

$$N_{2,\theta_x}(\xi;\eta) = \Phi_{4a}(\xi) \cdot \Phi_{1b}(\eta)$$

$$N_{3,\theta_x}(\xi;\eta) = \Phi_{4a}(\xi) \cdot \Phi_{3b}(\eta)$$

$$N_{4,\theta_x}(\xi;\eta) = \Phi_{2a}(\xi) \cdot \Phi_{3b}(\eta)$$

For rotations ϑ_y

$$N_{1,\theta_y}(\xi;\eta) = \Phi_{1a}(\xi) \cdot \Phi_{2b}(\eta)$$

$$N_{2,\theta_y}(\xi;\eta) = \Phi_{3a}(\xi) \cdot \Phi_{2b}(\eta)$$

$$N_{3,\theta_y}(\xi;\eta) = \Phi_{3a}(\xi) \cdot \Phi_{4b}(\eta)$$

$$N_{4,\theta_y}(\xi;\eta) = \Phi_{1a}(\xi) \cdot \Phi_{4b}(\eta)$$

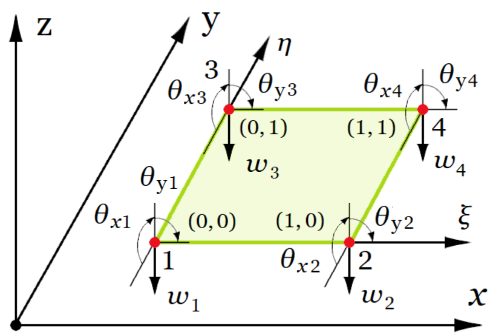
For twist ψ

$$N_{1,\psi}(\xi;\eta) = \Phi_{2a}(\xi) \cdot \Phi_{2b}(\eta)$$

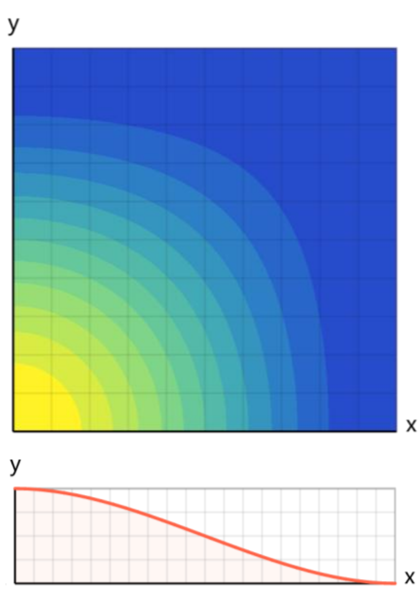
$$N_{2,\psi}(\xi;\eta) = \Phi_{4a}(\xi) \cdot \Phi_{2b}(\eta)$$

$$N_{3,\psi}(\xi;\eta) = \Phi_{4a}(\xi) \cdot \Phi_{4b}(\eta)$$

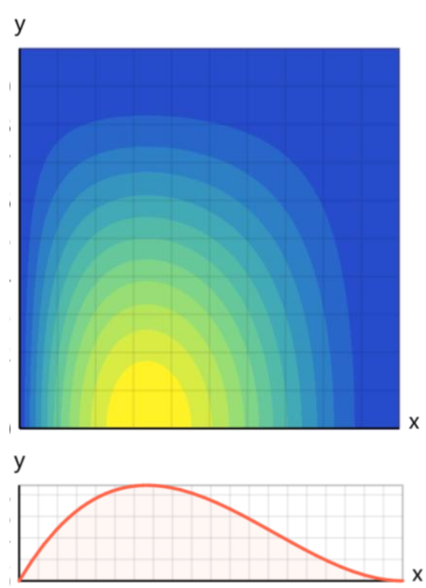
$$N_{4,\psi}(\xi;\eta) = \Phi_{2a}(\xi) \cdot \Phi_{4b}(\eta)$$



$N_{1,w}$ shape function plot



$N_{1,\theta x}$ shape function plot



Shape functions vector $N(i;\xi;\eta) = \textbf{take}\left(\begin{array}{l} i;\ N_{1,w}(\xi;\eta); N_{1,\theta_x}(\xi;\eta); N_{1,\theta_Y}(\xi;\eta); N_{1,\psi}(\xi;\eta); \\ N_{2,w}(\xi;\eta); N_{2,\theta_x}(\xi;\eta); N_{2,\theta_Y}(\xi;\eta); N_{2,\psi}(\xi;\eta); \\ N_{3,w}(\xi;\eta); N_{3,\theta_x}(\xi;\eta); N_{3,\theta_Y}(\xi;\eta); N_{3,\psi}(\xi;\eta); \\ N_{4,w}(\xi;\eta); N_{4,\theta_x}(\xi;\eta); N_{4,\theta_Y}(\xi;\eta); N_{4,\psi}(\xi;\eta) \end{array}\right)$

Constitutive matrix (stress - strain relationship)

$$D = \frac{E\cdot t^3}{12\cdot(1-\nu^2)}\cdot\textbf{hp}\left(\left[1;\nu;0\mid\nu;1;0\mid 0;0;\frac{1-\nu}{2}\right]\right) = \begin{bmatrix} 24.31 & 4.86 & 0 \\ 4.86 & 24.31 & 0 \\ 0 & 0 & 9.72 \end{bmatrix}\ \text{kNm}$$

Strain-displacement matrix

$$B_1(j;\xi;\eta) = \textbf{take}\big(j; \Phi''_{1a}(\xi)\cdot\Phi_{1b}(\eta); \Phi''_{2a}(\xi)\cdot\Phi_{1b}(\eta); \Phi''_{1a}(\xi)\cdot\Phi_{2b}(\eta); \Phi''_{2a}(\xi)\cdot\Phi_{2b}(\eta); \Phi''_{3a}(\xi)\cdot\Phi_{1b}(\eta); \Phi''_{4a}(\xi)\cdot\Phi_{1b}(\eta); \Phi''_{3a}(\xi)\cdot\Phi_{2b}(\eta); \Phi''_{4a}(\xi)\cdot\Phi_{2b}(\eta); \Phi''_{3a}(\xi)\cdot\Phi_{3b}(\eta); \Phi''_{4a}(\xi)\cdot\Phi_{3b}(\eta); \Phi''_{3a}(\xi)\cdot\Phi_{4b}(\eta); \Phi''_{4a}(\xi)\cdot\Phi_{4b}(\eta); \Phi''_{1a}(\xi)\cdot\Phi_{3b}(\eta); \Phi''_{2a}(\xi)\cdot\Phi_{3b}(\eta); \Phi''_{1a}(\xi)\cdot\Phi_{4b}(\eta); \Phi''_{2a}(\xi)\cdot\Phi_{4b}(\eta)\big)$$

$$B_2(j;\xi;\eta) = \textbf{take}\big(j; \Phi_{1a}(\xi)\cdot\Phi''_{1b}(\eta); \Phi_{2a}(\xi)\cdot\Phi''_{1b}(\eta); \Phi_{1a}(\xi)\cdot\Phi''_{2b}(\eta); \Phi_{2a}(\xi)\cdot\Phi''_{2b}(\eta); \Phi_{3a}(\xi)\cdot\Phi''_{1b}(\eta); \Phi_{4a}(\xi)\cdot\Phi''_{1b}(\eta); \Phi_{3a}(\xi)\cdot\Phi''_{2b}(\eta); \Phi_{4a}(\xi)\cdot\Phi''_{2b}(\eta); \Phi_{3a}(\xi)\cdot\Phi''_{3b}(\eta); \Phi_{4a}(\xi)\cdot\Phi''_{3b}(\eta); \Phi_{3a}(\xi)\cdot\Phi''_{4b}(\eta); \Phi_{4a}(\xi)\cdot\Phi''_{4b}(\eta); \Phi_{1a}(\xi)\cdot\Phi''_{3b}(\eta); \Phi_{2a}(\xi)\cdot\Phi''_{3b}(\eta); \Phi_{1a}(\xi)\cdot\Phi''_{4b}(\eta); \Phi_{2a}(\xi)\cdot\Phi''_{4b}(\eta)\big)$$

$$B_3(j;\xi;\eta) = 2\cdot\textbf{take}\big(j; \Phi'_{1a}(\xi)\cdot\Phi'_{1b}(\eta); \Phi'_{2a}(\xi)\cdot\Phi'_{1b}(\eta); \Phi'_{1a}(\xi)\cdot\Phi'_{2b}(\eta); \Phi'_{2a}(\xi)\cdot\Phi'_{2b}(\eta); \Phi'_{3a}(\xi)\cdot\Phi'_{1b}(\eta); \Phi'_{4a}(\xi)\cdot\Phi'_{1b}(\eta); \Phi'_{3a}(\xi)\cdot\Phi'_{2b}(\eta); \Phi'_{4a}(\xi)\cdot\Phi'_{2b}(\eta); \Phi'_{3a}(\xi)\cdot\Phi'_{3b}(\eta); \Phi'_{4a}(\xi)\cdot\Phi'_{3b}(\eta); \Phi'_{3a}(\xi)\cdot\Phi'_{4b}(\eta); \Phi'_{4a}(\xi)\cdot\Phi'_{4b}(\eta); \Phi'_{1a}(\xi)\cdot\Phi'_{3b}(\eta); \Phi'_{2a}(\xi)\cdot\Phi'_{3b}(\eta); \Phi'_{1a}(\xi)\cdot\Phi'_{4b}(\eta); \Phi'_{2a}(\xi)\cdot\Phi'_{4b}(\eta)\big)$$

$$B(j;\xi;\eta) = \textbf{hp}([B_1(j;\xi;\eta); B_2(j;\xi;\eta); B_3(j;\xi;\eta)])$$

The coefficients of the stiffness matrix will be calculated by using the equation

$$K_{e,ij} = a_1\cdot b_1\cdot\int_0^1\int_0^1 B_i(\xi;\eta)^T\cdot D\cdot B_j(\xi;\eta)\,d\xi\ d\eta$$

Element stiffness matrix – numerical evaluation

$$BTDB_e(i;j;\xi;\eta) = \textbf{transp}\big(B(i;\xi;\eta)\big)\cdot D\cdot B(j;\xi;\eta)$$

$$K_e(i;j) = a_1\cdot b_1\cdot\int_0^1\int_0^1 BTDB_e(i;j;\xi;\eta)\,d\eta\ d\xi$$

$$\$Repeat\left\{\$Repeat\left\{K_{e.i,j} = K_e(i;j); j = i\ldots n\right\}; i = 1\ldots n\right\}$$

$$K_e = \begin{bmatrix} 796.3 & 135.19 & 135.19 & 16.74 & -391.2 & 84.95 & -13.66 & 4.1 & -13.89 & 36.57 & \cdots & 4.1 \\ 0 & 46.67 & 21.6 & 4.67 & -84.95 & 14.03 & -4.09 & 0.708 & -36.57 & 10.28 & \cdots & -0.583 \\ 0 & 0 & 46.67 & 4.67 & -13.66 & 4.09 & 1.94 & -0.583 & -36.57 & 8.54 & \cdots & 0.708 \\ 0 & 0 & 0 & 0.978 & -4.1 & 0.708 & 0.583 & -0.161 & -8.54 & 1.62 & \cdots & -0.161 \\ 0 & 0 & 0 & 0 & 796.3 & -135.19 & 135.19 & -16.74 & -391.2 & 13.66 & \cdots & 8.54 \\ 0 & 0 & 0 & 0 & 0 & 46.67 & -21.6 & 4.67 & 13.66 & 1.94 & \cdots & -1.62 \\ 0 & 0 & 0 & 0 & 0 & 0 & 46.67 & -4.67 & -84.95 & 4.1 & \cdots & 1.62 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.978 & 4.1 & 0.583 & \cdots & -0.231 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 796.3 & -135.19 & \cdots & 4.1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 46.67 & \cdots & -0.708 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0.978 \end{bmatrix}$$

Element load vector

$$F_{e,i} = a_1 \cdot b_1 \cdot \int_0^1 \int_0^1 N_i(\xi;\eta)^T \cdot q \, d\xi \ d\eta$$

$\vec{F_e} = [0.9 \ \ 0.09 \ \ 0.09 \ \ 0.009 \ \ 0.9 \ \ \text{-}0.09 \ \ 0.09 \ \ \text{-}0.009 \ \ 0.9 \ \ \text{-}0.09 \ \ \ldots \ \ \text{-}0.009]$ kN

Solution

Global stiffness matrix

$$K = \begin{bmatrix} 10^{20} & 135.19 & 135.19 & 16.74 & -391.2 & -13.66 & 84.95 & 4.1 & 0 & 0 & \cdots & 0 \\ 135.19 & 46.67 & 21.6 & 4.67 & -13.66 & 1.94 & 4.1 & -0.583 & 0 & 0 & \cdots & 0 \\ 135.19 & 21.6 & 46.67 & 4.67 & -84.95 & -4.1 & 14.03 & 0.708 & 0 & 0 & \cdots & 0 \\ 16.74 & 4.67 & 4.67 & 0.978 & -4.1 & 0.583 & 0.708 & -0.161 & 0 & 0 & \cdots & 0 \\ -391.2 & -13.66 & -84.95 & -4.1 & 1592.59 & 270.37 & 0 & 0 & -391.2 & -13.66 & \cdots & 0 \\ -13.66 & 1.94 & -4.1 & 0.583 & 270.37 & 93.33 & 0 & 0 & -13.66 & 1.94 & \cdots & 0 \\ 84.95 & 4.1 & 14.03 & 0.708 & 0 & 0 & 93.33 & 9.33 & -84.95 & -4.1 & \cdots & 0 \\ 4.1 & -0.583 & 0.708 & -0.161 & 0 & 0 & 9.33 & 1.96 & -4.1 & 0.583 & \cdots & 0 \\ 0 & 0 & 0 & 0 & -391.2 & -13.66 & -84.95 & -4.1 & 1592.59 & 270.37 & \cdots & 0 \\ 0 & 0 & 0 & 0 & -13.66 & 1.94 & -4.1 & 0.583 & 270.37 & 93.33 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0.978 \end{bmatrix}$$

Global load vector

$\vec{F} = [0.9 \ \ 0.09 \ \ 0.09 \ \ 0.009 \ \ 1.8 \ \ 0.18 \ \ 0 \ \ 0 \ \ 1.8 \ \ 0.18 \ \ \ldots \ \ 0.009]$ kN

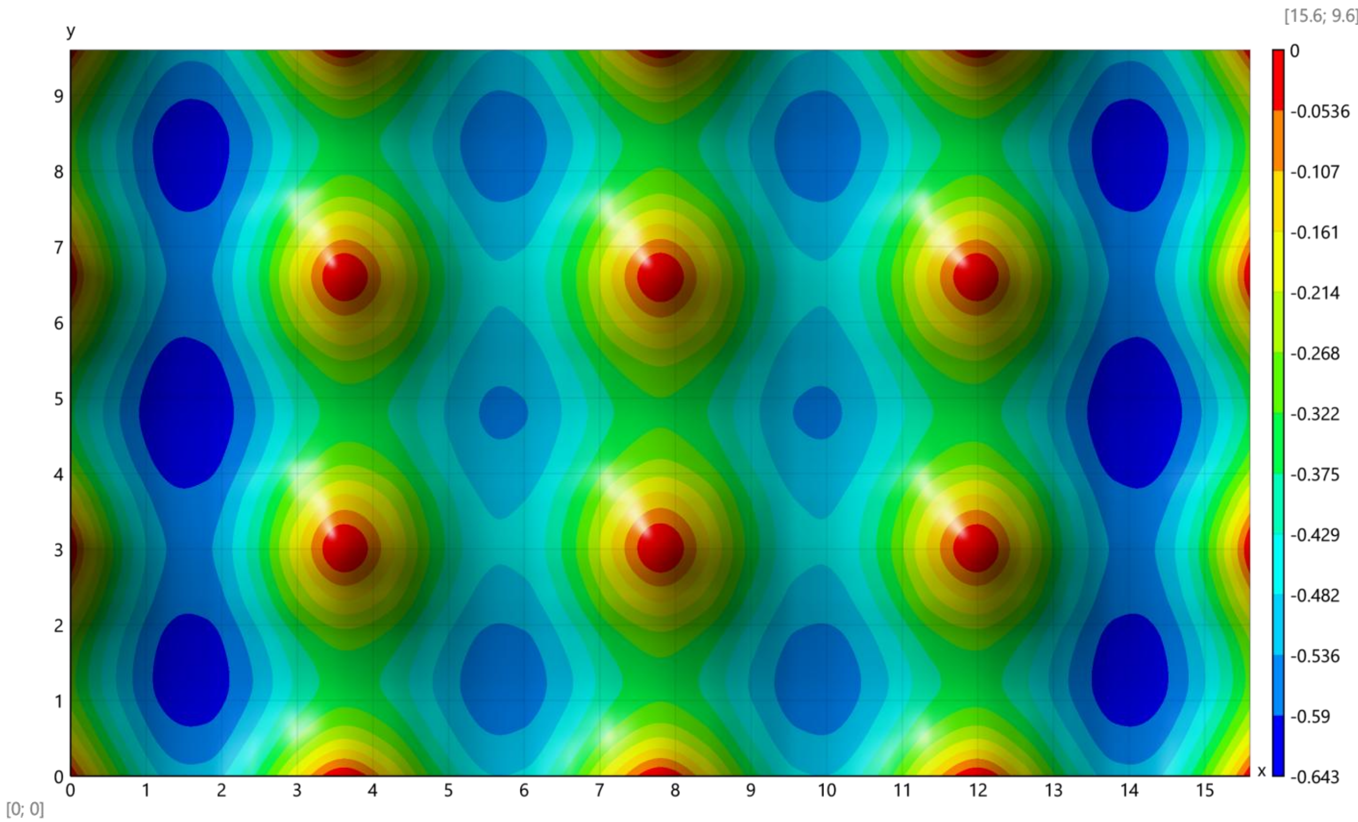
Solution of the system of equations

$\vec{Z} = \mathbf{slsolve}(K;\vec{F}) = [0 \ \ 0.552 \ \ 0.383 \ \ \text{-}0.416 \ \ 0.203 \ \ 0.373 \ \ 0.265 \ \ \text{-}0.194 \ \ 0.299 \ \ 0.309 \ \ \ldots \ \ \text{-}0.416]$ mm

Results

Joint displacements – $\mathbf{transp}(W_z) =$

$$\begin{bmatrix} 0 & 0.303 & 0.488 & 0.512 & 0.383 & 0.165 & 0 & 0.139 & 0.34 & 0.469 & \cdots & 0 \\ 0.203 & 0.419 & 0.562 & 0.581 & 0.485 & 0.337 & 0.25 & 0.31 & 0.438 & 0.531 & \cdots & 0.203 \\ 0.299 & 0.482 & 0.605 & 0.62 & 0.537 & 0.417 & 0.35 & 0.387 & 0.485 & 0.562 & \cdots & 0.299 \\ 0.261 & 0.461 & 0.593 & 0.608 & 0.513 & 0.375 & 0.299 & 0.342 & 0.455 & 0.542 & \cdots & 0.261 \\ 0.121 & 0.386 & 0.55 & 0.565 & 0.443 & 0.253 & 0.138 & 0.217 & 0.381 & 0.493 & \cdots & 0.121 \\ 0 & 0.34 & 0.527 & 0.542 & 0.404 & 0.173 & 0 & 0.135 & 0.336 & 0.463 & \cdots & 0 \\ 0.139 & 0.398 & 0.556 & 0.566 & 0.44 & 0.247 & 0.129 & 0.206 & 0.367 & 0.478 & \cdots & 0.139 \\ 0.299 & 0.487 & 0.608 & 0.612 & 0.511 & 0.369 & 0.287 & 0.325 & 0.433 & 0.516 & \cdots & 0.299 \\ 0.363 & 0.526 & 0.632 & 0.635 & 0.544 & 0.42 & 0.35 & 0.376 & 0.464 & 0.536 & \cdots & 0.363 \\ 0.299 & 0.487 & 0.608 & 0.612 & 0.511 & 0.369 & 0.287 & 0.325 & 0.433 & 0.516 & \cdots & 0.299 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0.303 & 0.488 & 0.512 & 0.383 & 0.165 & 0 & 0.139 & 0.34 & 0.469 & \cdots & 0 \end{bmatrix} \text{mm}$$



Bending moments

$$Z_j(j) = \mathbf{slice}(\vec{Z}; k_1 \cdot (j-1) + 1; k_1 \cdot j)$$

$$Z_e(e) = \mathbf{hp}\left(\left[Z_j\left(e_{j_{.e,1}}\right); Z_j\left(e_{j_{.e,2}}\right); Z_j\left(e_{j_{.e,3}}\right); Z_j\left(e_{j_{.e,4}}\right)\right]\right)$$

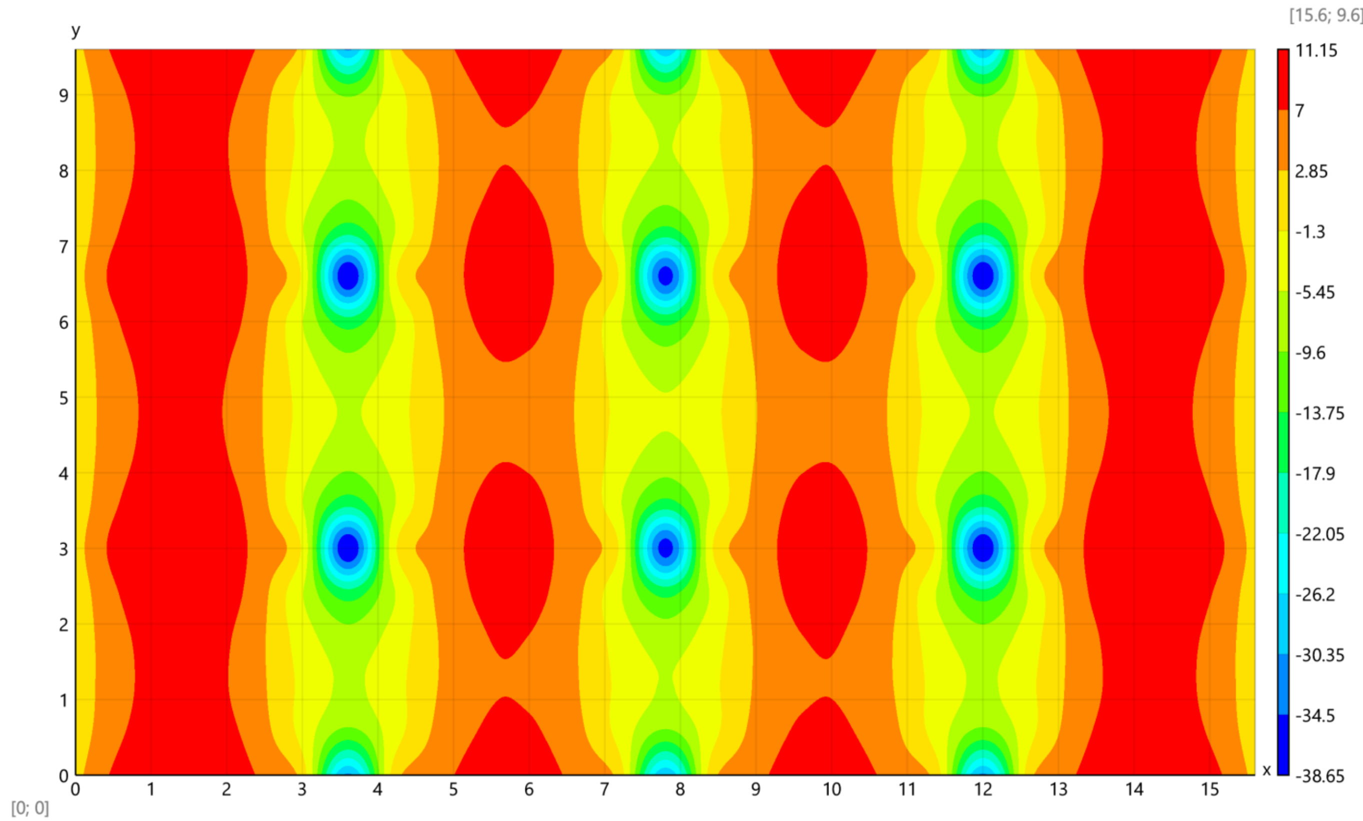
Average bending moments at joints, kNm/m

$$M_j = \begin{bmatrix} 1.5 & 0.31 & 0.22 & 0.156 & 0.157 & 0.998 & 0.156 & 0.152 & 0.194 & 0.152 & \cdots & 1.5 \\ 1.57 & 7.81 & 9.34 & 7.67 & 3.33 & -28.36 & 3.15 & 7.32 & 8.9 & 7.32 & \cdots & 1.57 \\ 8.09 & 3.77 & 0.486 & -2.48 & -4.46 & 0.155 & 4.78 & 2.84 & 1.19\times10^{-8} & -2.84 & \cdots & 8.09 \end{bmatrix}$$

Bending moments for the plate

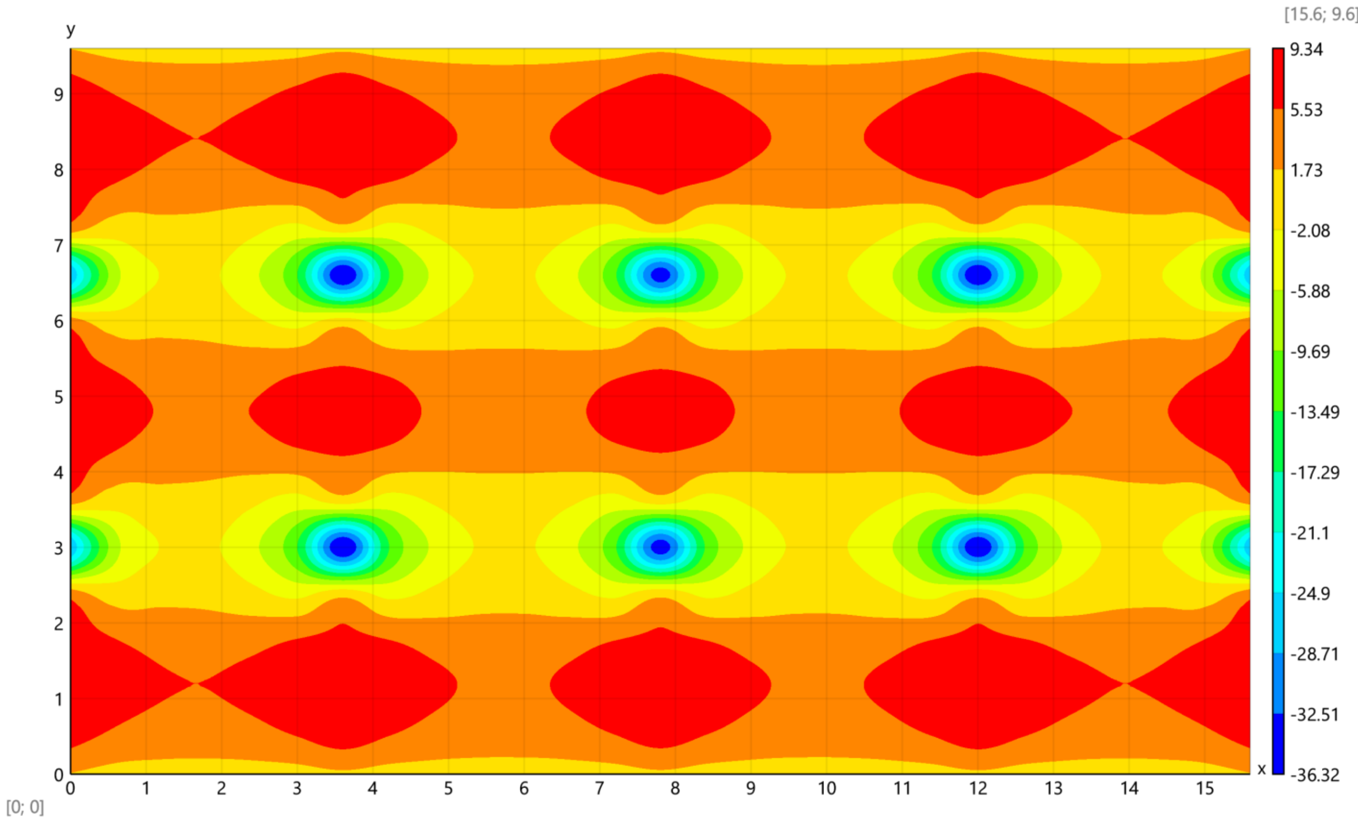
Bending moments - M_x - $\mathbf{transp}(Mx) =$

$$\begin{bmatrix} 1.5 & 8.5 & 11.07 & 10.48 & 6.8 & 0.884 & -31.24 & 0.302 & 5.64 & 8.78 & \cdots & 1.5 \\ 0.31 & 6.48 & 9.43 & 8.88 & 4.92 & -2.7 & -10.24 & -3.26 & 3.81 & 7.25 & \cdots & 0.31 \\ 0.22 & 5.78 & 8.72 & 8.14 & 4.04 & -2.26 & -6.06 & -2.82 & 2.93 & 6.52 & \cdots & 0.22 \\ 0.156 & 5.99 & 9.13 & 8.49 & 4.06 & -3.26 & -7.99 & -3.83 & 2.91 & 6.8 & \cdots & 0.156 \\ 0.157 & 7.27 & 10.37 & 9.56 & 4.97 & -5.27 & -16.12 & -5.87 & 3.77 & 7.71 & \cdots & 0.157 \\ 0.998 & 9.04 & 11.08 & 10.12 & 5.76 & -2.14 & -38.65 & -2.75 & 4.52 & 8.19 & \cdots & 0.998 \\ 0.156 & 7.22 & 10.28 & 9.44 & 4.85 & -5.36 & -16.18 & -5.96 & 3.62 & 7.56 & \cdots & 0.156 \\ 0.152 & 5.87 & 8.93 & 8.23 & 3.77 & -3.42 & -8.03 & -4.01 & 2.59 & 6.47 & \cdots & 0.152 \\ 0.194 & 5.51 & 8.38 & 7.69 & 3.47 & -2.49 & -5.74 & -3.07 & 2.31 & 5.98 & \cdots & 0.194 \\ 0.152 & 5.87 & 8.93 & 8.23 & 3.77 & -3.42 & -8.03 & -4.01 & 2.59 & 6.47 & \cdots & 0.152 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1.5 & 8.5 & 11.07 & 10.48 & 6.8 & 0.884 & -31.24 & 0.302 & 5.64 & 8.78 & \cdots & 1.5 \end{bmatrix} \text{kNm/m}$$



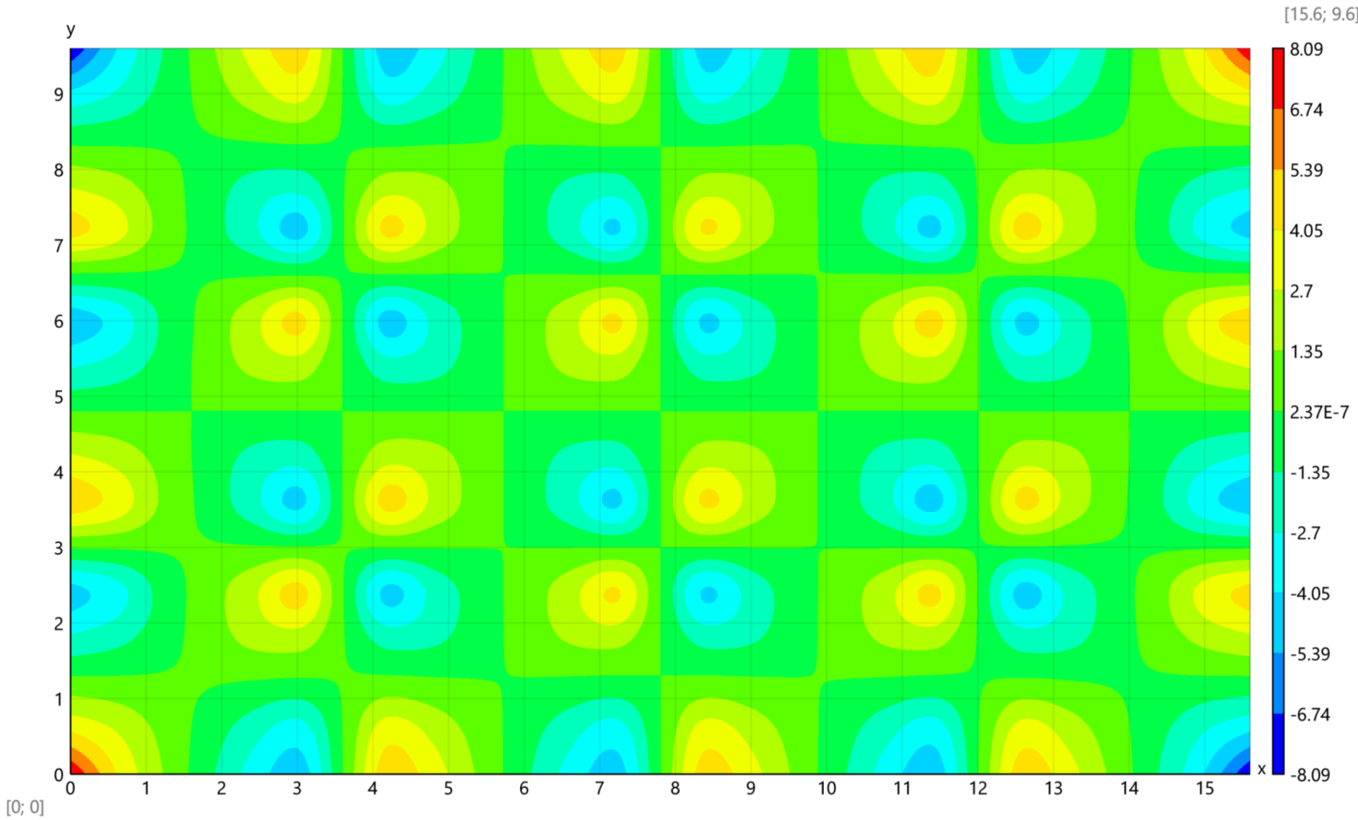
Bending moments M_y – $\mathbf{transp}(My) =$

$$\begin{bmatrix} 1.57 & 0.321 & 0.233 & 0.215 & 0.168 & 0.18 & 1.02 & 0.179 & 0.167 & 0.21 & \cdots & 1.57 \\ 7.81 & 5.38 & 4.31 & 4.13 & 4.74 & 6.34 & 8.23 & 6.28 & 4.61 & 3.89 & \cdots & 7.81 \\ 9.34 & 7.13 & 5.8 & 5.55 & 6.41 & 8.12 & 9.05 & 8.01 & 6.19 & 5.14 & \cdots & 9.34 \\ 7.67 & 5.42 & 4.26 & 3.92 & 4.23 & 5.42 & 6.34 & 5.3 & 3.98 & 3.48 & \cdots & 7.67 \\ 3.33 & -0.328 & 0.54 & 0.406 & -1.17 & -3.41 & -0.33 & -3.53 & -1.4 & 0.0996 & \cdots & 3.33 \\ -28.36 & -7.04 & -1.94 & -1.65 & -4.99 & -13.64 & -36.32 & -13.77 & -5.22 & -1.86 & \cdots & -28.36 \\ 3.15 & -0.507 & 0.349 & 0.18 & -1.46 & -3.76 & -0.723 & -3.92 & -1.77 & -0.252 & \cdots & 3.15 \\ 7.32 & 5.09 & 3.91 & 3.49 & 3.68 & 4.69 & 5.5 & 4.48 & 3.25 & 2.79 & \cdots & 7.32 \\ 8.9 & 6.74 & 5.35 & 4.96 & 5.61 & 6.91 & 7.57 & 6.69 & 5.14 & 4.18 & \cdots & 8.9 \\ 7.32 & 5.09 & 3.91 & 3.49 & 3.68 & 4.69 & 5.5 & 4.48 & 3.25 & 2.79 & \cdots & 7.32 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1.57 & 0.32 & 0.233 & 0.215 & 0.168 & 0.18 & 1.02 & 0.179 & 0.167 & 0.21 & \cdots & 1.57 \end{bmatrix} \text{kNm/m}$$



Bending moments M_{xy} – **transp**(Mxy) =

$$
\begin{bmatrix}
8.09 & 4.11 & 1.42 & -0.946 & -3.23 & -4.78 & 0.0218 & 4.84 & 3.34 & 1.14 & \cdots & -8.09 \\
3.77 & 2.57 & 0.983 & -0.516 & -2.06 & -3.23 & 0.0582 & 3.35 & 2.21 & 0.731 & \cdots & -3.77 \\
0.486 & 0.367 & 0.21 & 0.0857 & -0.12 & -0.314 & 0.0735 & 0.461 & 0.267 & 0.0568 & \cdots & -0.486 \\
-2.48 & -1.8 & -0.601 & 0.617 & 1.75 & 2.09 & 0.0763 & -1.95 & -1.63 & -0.581 & \cdots & 2.48 \\
-4.46 & -3.24 & -0.898 & 0.683 & 2.37 & 4.52 & 0.0758 & -4.38 & -2.26 & -0.682 & \cdots & 4.46 \\
0.155 & 0.161 & 0.149 & 0.125 & 0.0964 & 0.078 & 0.0739 & 0.0699 & 0.0513 & 0.0211 & \cdots & -0.155 \\
4.78 & 3.57 & 1.19 & -0.451 & -2.21 & -4.4 & 0.0635 & 4.54 & 2.38 & 0.725 & \cdots & -4.78 \\
2.84 & 2.13 & 0.868 & -0.438 & -1.68 & -2.08 & 0.0378 & 2.17 & 1.8 & 0.63 & \cdots & -2.84 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & -0 \\
-2.84 & -2.13 & -0.868 & 0.438 & 1.68 & 2.08 & -0.0378 & -2.17 & -1.8 & -0.63 & \cdots & 2.84 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
-8.09 & -4.11 & -1.42 & 0.946 & 3.23 & 4.78 & -0.0218 & -4.84 & -3.34 & -1.14 & \cdots & 8.09
\end{bmatrix}
\text{ kNm/m}
$$



Element stiffness matrix calculation by analytical expressions (faster)

$$\alpha = \frac{b_1}{a_1} = 1$$

$$K_{e_{1,1}} = \frac{156}{35} \cdot \left(\frac{\alpha}{{a_1}^2} + \frac{1}{\alpha \cdot {b_1}^2}\right) + \frac{72}{25 \cdot A_1} = 32.76$$

$K_{e_{5,5}} = K_{e_{1,1}}$, $K_{e_{9,9}} = K_{e_{1,1}}$, $K_{e_{13,13}} = K_{e_{1,1}}$

$$K_{e_{1,2}} = \frac{2}{35} \cdot \left(\frac{39 \cdot \alpha}{a_1} + \frac{11}{\alpha^2 \cdot b_1}\right) + \frac{30 \cdot \nu + 6}{25 \cdot b_1} = 5.56$$

$K_{e_{5,6}} = -K_{e_{1,2}}$, $K_{e_{9,10}} = -K_{e_{1,2}}$, $K_{e_{13,14}} = K_{e_{1,2}}$

$$K_{e_{1,3}} = \frac{2}{35} \cdot \left(\frac{11 \cdot \alpha^2}{a_1} + \frac{39}{\alpha \cdot b_1}\right) + \frac{30 \cdot \nu + 6}{25 \cdot a_1} = 5.56$$

$K_{e_{5,7}} = K_{e_{1,3}}$, $K_{e_{9,11}} = -K_{e_{1,3}}$, $K_{e_{13,15}} = -K_{e_{1,3}}$

$$K_{e_{1,4}} = \frac{11}{35} \cdot \left(\alpha^2 + \frac{1}{\alpha^2}\right) + \frac{10 \cdot \nu + 1}{50} = 0.689$$

$K_{e_{5,8}} = -K_{e_{1,4}}$, $K_{e_{9,12}} = K_{e_{1,4}}$, $K_{e_{13,16}} = -K_{e_{1,4}}$

$$K_{e_{1,5}} = \frac{2}{35} \cdot \left(\frac{27}{\alpha \cdot {b_1}^2} - \frac{78 \cdot \alpha}{{a_1}^2}\right) - \frac{72}{25 \cdot A_1} = -16.1$$

$K_{e_{9,13}} = K_{e_{1,5}}$

$$K_{e_{1,6}} = \frac{13}{35} \cdot \left(\frac{6 \cdot \alpha}{a_1} - \frac{1}{\alpha^2 \cdot b_1}\right) + \frac{6}{25 \cdot b_1} = 3.5$$

$K_{e_{2,5}} = -K_{e_{1,6}}$, $K_{e_{9,14}} = -K_{e_{1,6}}$, $K_{e_{10,13}} = K_{e_{1,6}}$

$$K_{e_{1,7}} = \frac{1}{35} \cdot \left(\frac{27}{\alpha \cdot b_1} - \frac{22 \cdot \alpha^2}{a_1}\right) - \frac{30 \cdot \nu + 6}{25 \cdot a_1} = -0.562$$

$K_{e_{3,5}} = K_{e_{1,7}}$, $K_{e_{9,15}} = -K_{e_{1,7}}$, $K_{e_{11,13}} = -K_{e_{1,7}}$

$$K_{e_{2,4}} = \frac{2}{35} \cdot \left(\frac{11 \cdot \alpha \cdot b_1}{3} + \frac{a_1}{\alpha^2}\right) + \frac{2 \cdot a_1 \cdot (5 \cdot \nu + 1)}{75} = 0.192$$

$K_{e_{6,8}} = K_{e_{2,4}}$, $K_{e_{10,12}} = -K_{e_{2,4}}$, $K_{e_{14,16}} = -K_{e_{2,4}}$

$$K_{e_{2,6}} = \frac{1}{35} \cdot \left(26 \cdot \alpha - \frac{3}{\alpha^3}\right) - \frac{2}{25 \cdot \alpha} = 0.577$$

$K_{e_{10,14}} = K_{e_{2,6}}$

$$K_{e_{2,8}} = \frac{1}{35} \cdot \left(\frac{11 \cdot \alpha \cdot b_1}{3} - \frac{3 \cdot a_1}{2 \cdot \alpha^2}\right) - \frac{a_1 \cdot (5 \cdot \nu + 1)}{150} = 0.0291$$

$K_{e_{4,6}} = K_{e_{2,8}}$, $K_{e_{10,16}} = -K_{e_{2,8}}$, $K_{e_{12,14}} = -K_{e_{2,8}}$

$$K_{e_{2,10}} = \frac{3}{35} \cdot \left(\frac{1}{\alpha^3} + 3 \cdot \alpha\right) + \frac{2}{25 \cdot \alpha} = 0.423$$

$K_{e_{6,14}} = K_{e_{2,10}}$

$$K_{e_{2,12}} = -\frac{1}{70} \cdot \left(\frac{3 \cdot a_1}{\alpha^2} + \frac{13 \cdot \alpha \cdot b_1}{3}\right) - \frac{a_1}{150} = -0.0669$$

$K_{e_{4,10}} = -K_{e_{2,12}}$, $K_{e_{6,16}} = K_{e_{2,12}}$, $K_{e_{8,14}} = -K_{e_{2,12}}$

$$K_{e_{2,14}} = \frac{2}{35} \cdot \left(9 \cdot \alpha - \frac{2}{\alpha^3}\right) - \frac{8}{25 \cdot \alpha} = 0.08$$

$K_{e_{6,10}} = K_{e_{2,14}}$

$$K_{e_{2,15}} = \frac{1}{35} \cdot \left(\frac{11}{\alpha^2} - \frac{13 \cdot \alpha^2}{2}\right) + \frac{5 \cdot \nu + 1}{50} = 0.169$$

$K_{e_{3,14}} = -K_{e_{2,15}}$, $K_{e_{6,11}} = -K_{e_{2,15}}$, $K_{e_{7,10}} = K_{e_{2,15}}$

$$K_{e_{1,14}} = \frac{1}{35} \cdot \left(\frac{27 \cdot \alpha}{a_1} - \frac{22}{\alpha^2 \cdot b_1}\right) - \frac{30 \cdot \nu + 6}{25 \cdot b_1} = -0.562$$

$K_{e_{2,13}} = K_{e_{1,14}}$, $K_{e_{5,10}} = -K_{e_{1,14}}$, $K_{e_{6,9}} = -K_{e_{1,14}}$

$$K_{e_{1,15}} = \frac{13}{35} \cdot \left(\frac{6}{\alpha \cdot b_1} - \frac{\alpha^2}{a_1}\right) + \frac{6}{25 \cdot a_1} = 3.5$$

$K_{e_{3,13}} = -K_{e_{1,15}}$, $K_{e_{5,11}} = K_{e_{1,15}}$, $K_{e_{7,9}} = -K_{e_{1,15}}$

$$K_{e_{1,16}} = \frac{1}{35} \cdot \left(\frac{11}{\alpha^2} - \frac{13 \cdot \alpha^2}{2} \right) + \frac{5 \cdot \nu + 1}{50} = 0.169$$

$K_{e_{4,13}} = \text{-}K_{e_{1,16}}$, $K_{e_{8,9}} = K_{e_{1,16}}$, $K_{e_{5,12}} = \text{-}K_{e_{1,16}}$

$$K_{e_{2,2}} = \frac{4}{35} \cdot \left(13 \cdot \alpha + \frac{1}{\alpha^3} \right) + \frac{8}{25 \cdot \alpha} = 1.92$$

$K_{e_{6,6}} = K_{e_{2,2}}$, $K_{e_{10,10}} = K_{e_{2,2}}$, $K_{e_{14,14}} = K_{e_{2,2}}$

$$K_{e_{2,3}} = \frac{11}{35} \cdot \left(\alpha^2 + \frac{1}{\alpha^2} \right) + \frac{60 \cdot \nu + 1}{50} = 0.889$$

$K_{e_{6,7}} = \text{-}K_{e_{2,3}}$, $K_{e_{10,11}} = K_{e_{2,3}}$, $K_{e_{14,15}} = \text{-}K_{e_{2,3}}$

$$K_{e_{3,16}} = \text{-}\frac{1}{35} \cdot \left(\frac{3 \cdot \alpha^2 \cdot b_1}{2} - \frac{11 \cdot a_1}{3 \cdot \alpha} \right) - \frac{b_1 \cdot (5 \cdot \nu + 1)}{150} = 0.0291$$

$K_{e_{4,15}} = K_{e_{3,16}}$, $K_{e_{7,12}} = \text{-}K_{e_{3,16}}$, $K_{e_{8,11}} = \text{-}K_{e_{3,16}}$

$$K_{e_{4,4}} = \frac{4}{105} \cdot \left(\alpha \cdot {b_1}^2 + \frac{{a_1}^2}{\alpha} \right) + \frac{8 \cdot A_1}{225} = 0.0402$$

$K_{e_{8,8}} = K_{e_{4,4}}$, $K_{e_{12,12}} = K_{e_{4,4}}$, $K_{e_{16,16}} = K_{e_{4,4}}$

$$K_{e_{4,8}} = \frac{1}{35} \cdot \left(\frac{2 \cdot \alpha \cdot {b_1}^2}{3} - \frac{{a_1}^2}{\alpha} \right) - \frac{2 \cdot A_1}{225} = \text{-}0.00663$$

$K_{e_{12,16}} = K_{e_{4,8}}$

$$K_{e_{4,12}} = \text{-}\frac{1}{70} \cdot \left(\frac{{a_1}^2}{\alpha} + \alpha \cdot {b_1}^2 \right) + \frac{A_1}{450} = \text{-}0.00949$$

$K_{e_{8,16}} = K_{e_{4,12}}$

$$K_{e_{4,16}} = \frac{1}{35} \cdot \left(\frac{2 \cdot {a_1}^2}{3 \cdot \alpha} - \alpha \cdot {b_1}^2 \right) - \frac{2 \cdot A_1}{225} = \text{-}0.00663$$

$K_{e_{8,12}} = K_{e_{4,16}}$

$$K_{e_{1,8}} = \frac{1}{35} \cdot \left(11 \cdot \alpha^2 - \frac{13}{2 \cdot \alpha^2} \right) + \frac{5 \cdot \nu + 1}{50} = 0.169$$

$K_{e_{2,7}} = \text{-}K_{e_{1,8}}$, $K_{e_{3,6}} = K_{e_{1,8}}$, $K_{e_{4,5}} = \text{-}K_{e_{1,8}}$, $K_{e_{9,16}} = K_{e_{1,8}}$

$K_{e_{10,15}} = \text{-}K_{e_{1,8}}$, $K_{e_{11,14}} = K_{e_{1,8}}$, $K_{e_{12,13}} = \text{-}K_{e_{1,8}}$

$$K_{e_{1,9}} = \text{-}\frac{54}{35} \cdot \left(\frac{\alpha}{{a_1}^2} + \frac{1}{\alpha \cdot {b_1}^2} \right) + \frac{72}{25 \cdot A_1} = \text{-}0.571$$

$K_{e_{5,13}} = K_{e_{1,9}}$

$$K_{e_{1,10}} = \frac{1}{35} \cdot \left(\frac{27 \cdot \alpha}{a_1} + \frac{13}{\alpha^2 \cdot b_1} \right) - \frac{6}{25 \cdot b_1} = 1.5$$

$K_{e_{2,9}} = -K_{e_{1,10}}$, $K_{e_{5,14}} = -K_{e_{1,10}}$, $K_{e_{6,13}} = K_{e_{1,10}}$

$$K_{e_{1,11}} = \frac{1}{35} \cdot \left(\frac{13 \cdot \alpha^2}{a_1} + \frac{27}{\alpha \cdot b_1}\right) - \frac{6}{25 \cdot a_1} = 1.5$$

$K_{e_{3,9}} = -K_{e_{1,11}}$, $K_{e_{5,15}} = K_{e_{1,11}}$, $K_{e_{7,13}} = -K_{e_{1,11}}$

$$K_{e_{1,12}} = -\frac{13}{70} \cdot \left(\alpha^2 + \frac{1}{\alpha^2}\right) + \frac{1}{50} = -0.351$$

$K_{e_{2,11}} = -K_{e_{1,12}}$, $K_{e_{3,10}} = -K_{e_{1,12}}$, $K_{e_{4,9}} = K_{e_{1,12}}$, $K_{e_{5,16}} = -K_{e_{1,12}}$

$K_{e_{6,15}} = K_{e_{1,12}}$, $K_{e_{7,14}} = K_{e_{1,12}}$, $K_{e_{8,13}} = -K_{e_{1,12}}$

$$K_{e_{1,13}} = \frac{2}{35} \cdot \left(\frac{27 \cdot \alpha}{{a_1}^2} - \frac{78}{\alpha \cdot {b_1}^2}\right) - \frac{72}{25 \cdot A_1} = -16.1$$

$K_{e_{5,9}} = K_{e_{1,13}}$

$$K_{e_{2,16}} = \frac{1}{35} \cdot \left(\frac{2 \cdot a_1}{\alpha^2} - \frac{13 \cdot \alpha \cdot b_1}{3}\right) + \frac{2 \cdot a_1}{75} = -0.024$$

$K_{e_{4,14}} = -K_{e_{2,16}}$, $K_{e_{6,12}} = K_{e_{2,16}}$, $K_{e_{8,10}} = -K_{e_{2,16}}$

$$K_{e_{3,3}} = \frac{4}{35} \cdot \left(\alpha^3 + \frac{13}{\alpha}\right) + \frac{8 \cdot \alpha}{25} = 1.92$$

$K_{e_{7,7}} = K_{e_{3,3}}$, $K_{e_{11,11}} = K_{e_{3,3}}$, $K_{e_{15,15}} = K_{e_{3,3}}$

$$K_{e_{3,4}} = \frac{2}{35} \cdot \left(\alpha^2 \cdot b_1 + \frac{11 \cdot a_1}{3 \cdot \alpha}\right) + \frac{2 \cdot b_1 \cdot (5 \cdot \nu + 1)}{75} = 0.192$$

$K_{e_{7,8}} = -K_{e_{3,4}}$, $K_{e_{11,12}} = -K_{e_{3,4}}$, $K_{e_{15,16}} = K_{e_{3,4}}$

$$K_{e_{3,7}} = \frac{2}{35} \cdot \left(-2 \cdot \alpha^3 + \frac{9}{\alpha}\right) - \frac{8 \cdot \alpha}{25} = 0.08$$

$K_{e_{11,15}} = K_{e_{3,7}}$

$$K_{e_{3,8}} = \frac{1}{35} \cdot \left(2 \cdot \alpha^2 \cdot b_1 - \frac{13 \cdot a_1}{3 \cdot \alpha}\right) + \frac{2 \cdot b_1}{75} = -0.024$$

$K_{e_{4,7}} = -K_{e_{3,8}}$, $K_{e_{11,16}} = -K_{e_{3,8}}$, $K_{e_{12,15}} = K_{e_{3,8}}$

$$K_{e_{3,11}} = \frac{3}{35} \cdot \left(\alpha^3 + \frac{3}{\alpha}\right) + \frac{2 \cdot \alpha}{25} = 0.423$$

$K_{e_{7,15}} = K_{e_{3,11}}$

$$K_{e_{3,12}} = -\frac{1}{70} \cdot \left(3 \cdot \alpha^2 \cdot b_1 + \frac{13 \cdot a_1}{3 \cdot \alpha}\right) - \frac{b_1}{150} = -0.0669$$

$K_{e_{4,11}} = -K_{e_{3,12}}$, $K_{e_{7,16}} = -K_{e_{3,12}}$, $K_{e_{8,15}} = K_{e_{3,12}}$

$$K_{e_{3,15}} = \frac{1}{35} \cdot \left(\frac{26}{\alpha} - 3 \cdot \alpha^3\right) - \frac{2 \cdot \alpha}{25} = 0.577$$

$K_{e_{7,11}} = K_{e_{3,15}}$

Element stiffness matrix coefficients (above the main diagonal only)

$$K_e = D_{1,1} \cdot K_e = \begin{bmatrix} 796.3 & 135.19 & 135.19 & 16.74 & -391.2 & 84.95 & -13.66 & 4.1 & -13.89 & 36.57 & \cdots & 4.1 \\ 0 & 46.67 & 21.6 & 4.67 & -84.95 & 14.03 & -4.1 & 0.708 & -36.57 & 10.28 & \cdots & -0.583 \\ 0 & 0 & 46.67 & 4.67 & -13.66 & 4.1 & 1.94 & -0.583 & -36.57 & 8.54 & \cdots & 0.708 \\ 0 & 0 & 0 & 0.978 & -4.1 & 0.708 & 0.583 & -0.161 & -8.54 & 1.63 & \cdots & -0.161 \\ 0 & 0 & 0 & 0 & 796.3 & -135.19 & 135.19 & -16.74 & -391.2 & 13.66 & \cdots & 8.54 \\ 0 & 0 & 0 & 0 & 0 & 46.67 & -21.6 & 4.67 & 13.66 & 1.94 & \cdots & -1.63 \\ 0 & 0 & 0 & 0 & 0 & 0 & 46.67 & -4.67 & -84.95 & 4.1 & \cdots & 1.63 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.978 & 4.1 & 0.583 & \cdots & -0.231 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 796.3 & -135.19 & \cdots & 4.1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 46.67 & \cdots & -0.708 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0.978 \end{bmatrix}$$

Element load vector

$$\vec{F}_e = \frac{q \cdot A_1}{24} \cdot \left[6; a_1; b_1; \frac{A_1}{6}; 6; -a_1; b_1; \frac{-A_1}{6}; 6; -a_1; -b_1; \frac{A_1}{6}; 6; a_1; -b_1; \frac{-A_1}{6}\right] =$$

$[0.9 \;\; 0.09 \;\; 0.09 \;\; 0.009 \;\; 0.9 \;\; -0.09 \;\; 0.09 \;\; -0.009 \;\; 0.9 \;\; -0.09 \;\; \ldots \;\; -0.009]$ kN

The obtained element stiffness matrix and load vector are identical to the numerical formulation.

[1] Bogner, F. K., Fox, R. L., and Schmit, L. A. The generation of interelement compatible stiffness and mass matrices by the use of interpolation formulae, *Proceedings of the Conference on Matrix Methods in Structural Mechanics*, 397–444, 1965