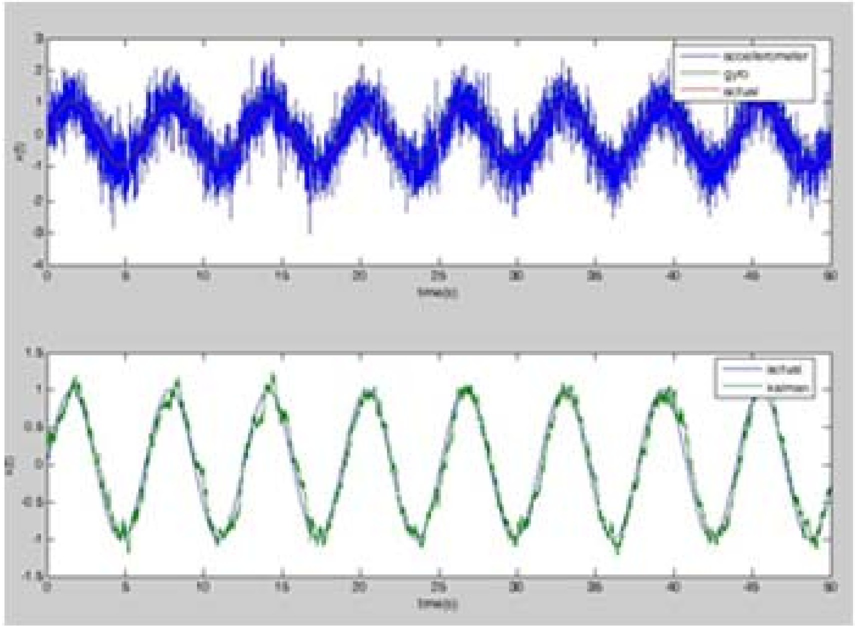
Continuous-Time Filters

생체공학과 임창환

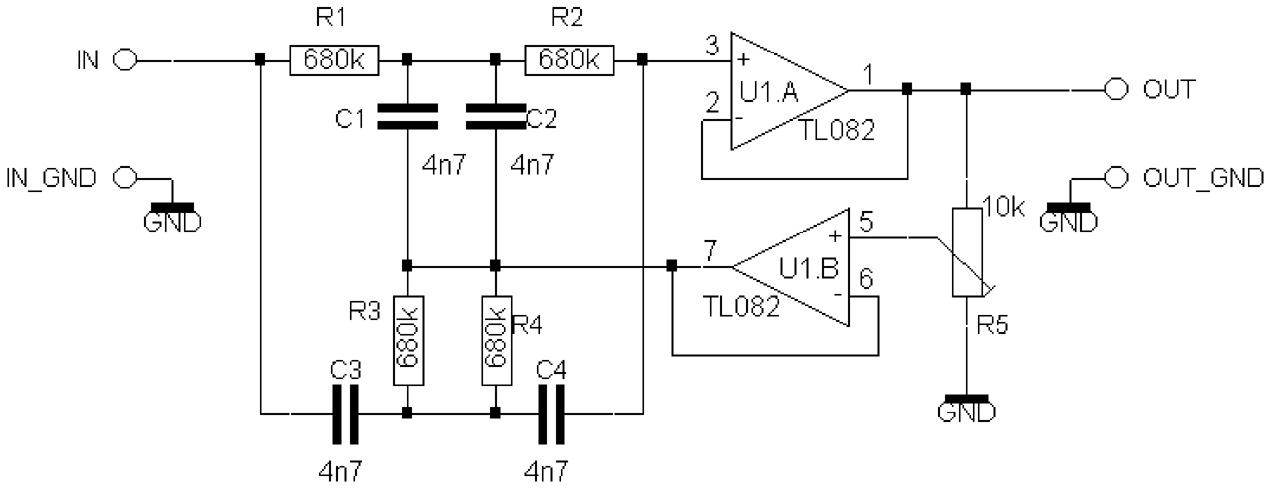
Filter

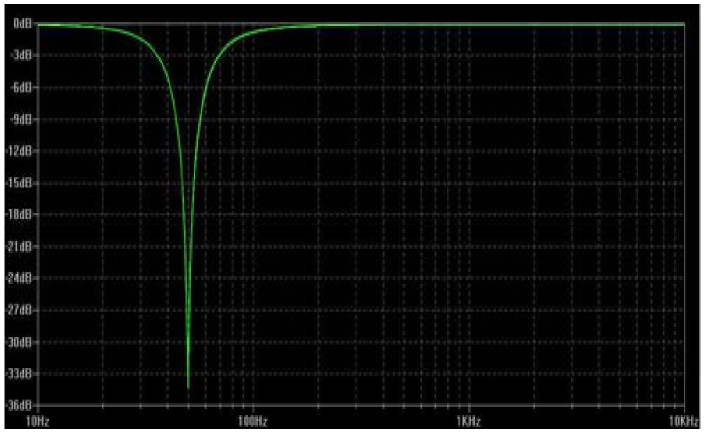
실 생활에 많은 편의와 정보를 주는 라디오의 원리는 위와 같이 매우 단순합니다. 여기서 우리가 짚고 넘어가야 할 부분은 **filtering** 입니다. Filter 의 사전적 의미는 "무엇인가를 걸러내다"라고 인지하고 있으실 것입니다. 그럼 filtering 이란 무엇인가요? 흔히 신호처리 분야에서 filter 는 주파수(frequency)영역에서 특정 주파수 대역만을 제거(혹은 통과)하는 것을 말합니다.

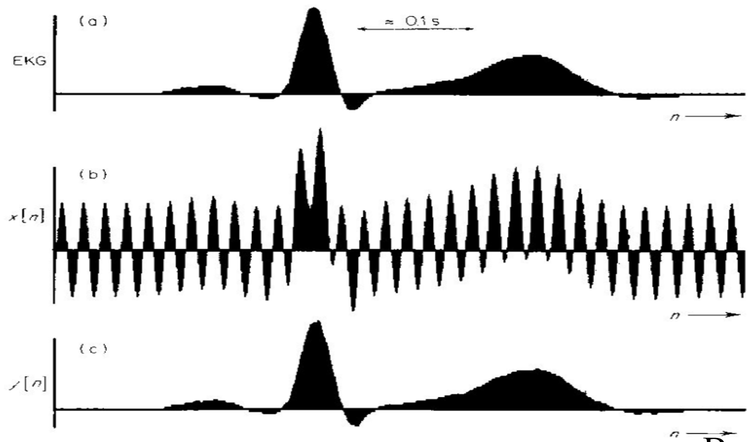


- Analog implementation of signal processing methods

  - Analog circuit for 60 Hz notch filter







ECG

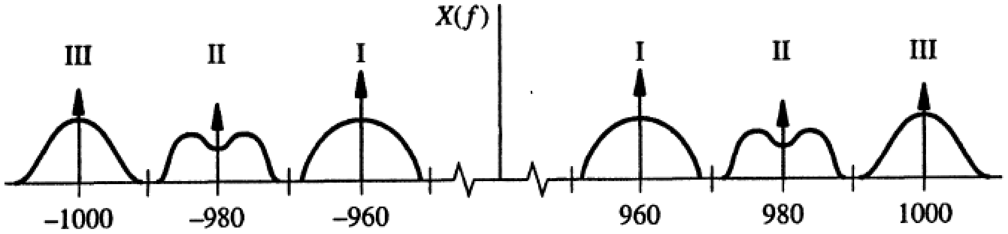
ECG + 60Hz 잡음

대역저지 출력

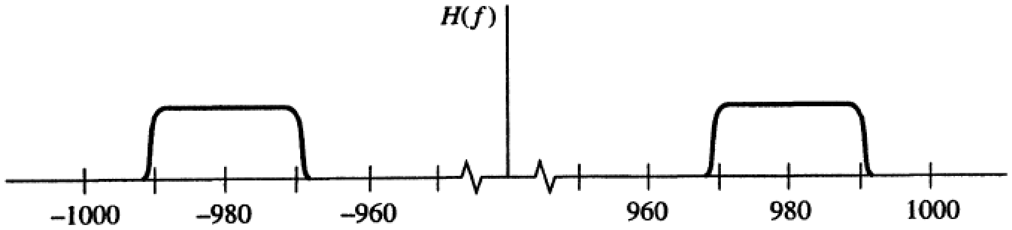
Band-stopped output

Continuous-Time Filters (Analog Filters)

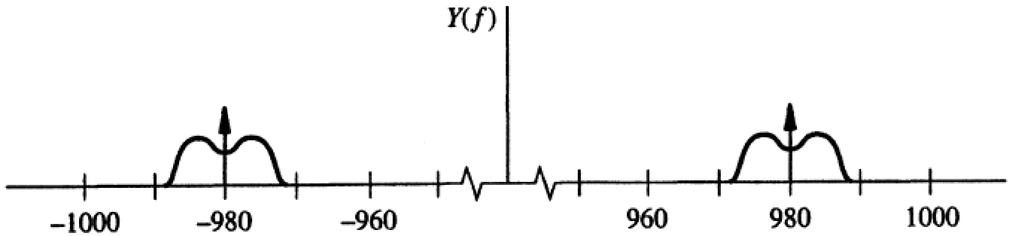
Figure 8.1
Illustration of Signal Selection Filtering in an AM Radio Receiver



(a) Receiver Input Signal Spectrum



(b) Station-Selection Filter Frequency Response



(c) Station-Selection Filter Output-Signal Spectrum

Distortionless Transmission System

Definition

A *distortionless transmission system* passes any signal with no change, except possibly amplification and time delay.

The output signal from a distortionless transmission system that has gain K and time delay τ is

$$y(t) = Kx(t - \tau) \tag{8.1}$$

when the input signal is $x(t)$. Computing the Fourier transform of eq. (8.1) we obtain the output-signal spectrum

$$Y(f) = KX(f)e^{-j2\pi f\tau} \tag{8.2}$$

We then compute the frequency response of the distortionless transmission system as

$$H(f) = Y(f)/X(f) = Ke^{-j2\pi f\tau} \tag{8.3}$$

Distortionless vs Physical Transmission System

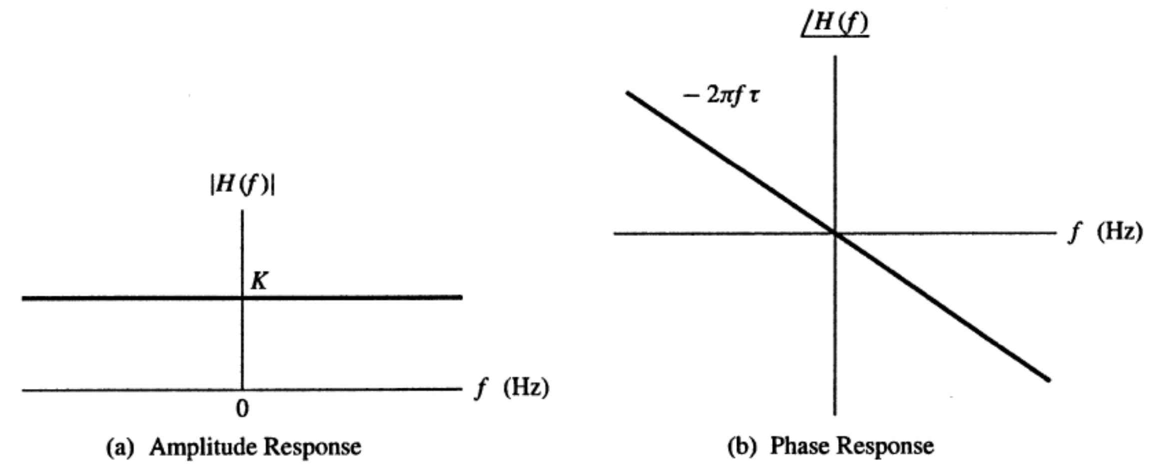


Figure 8.2 Frequency Response for a Distortionless Transmission System

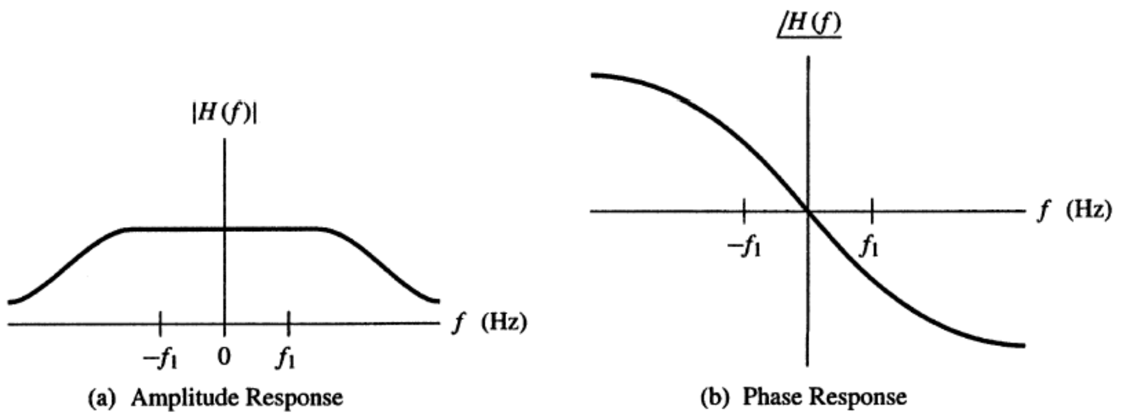


Figure 8.3 Frequency Response for a Physical Transmission System

Ideal vs Approximated Low-Pass Filter

Figure 8.4
Frequency Response and Impulse Response for an Ideal Low-Pass Filter

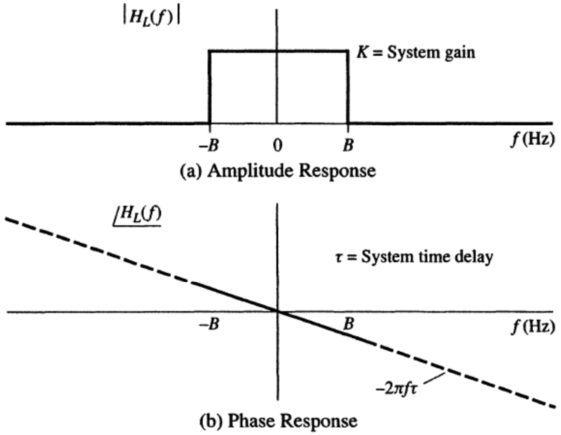
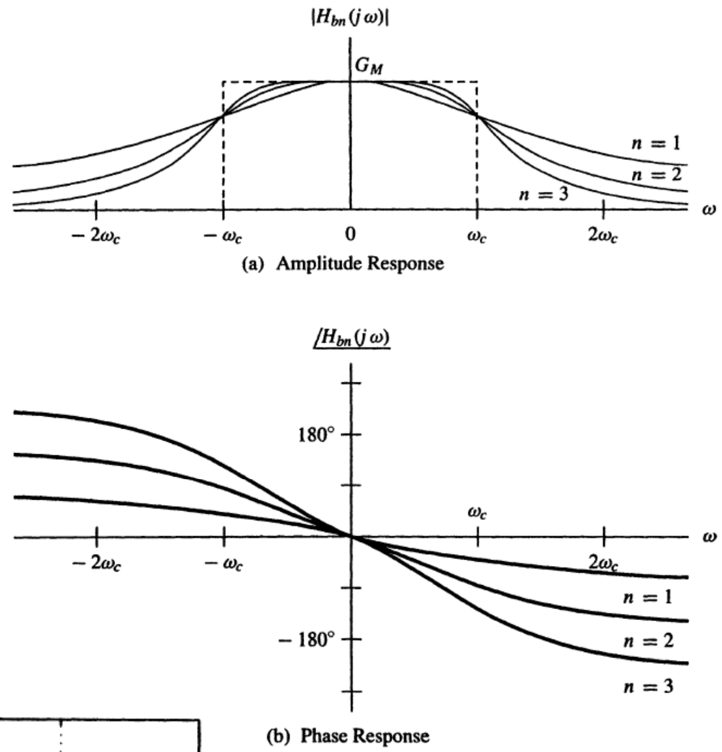


Figure 8.14
Amplitude and Phase Responses for First-, Second-, and Third-Order Low-Pass Butterworth Filters



Butterworth Filter
Chebyshev Filter

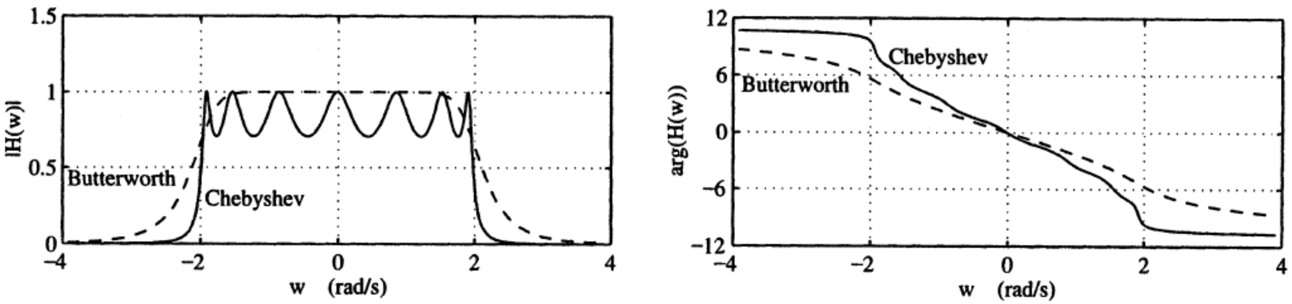
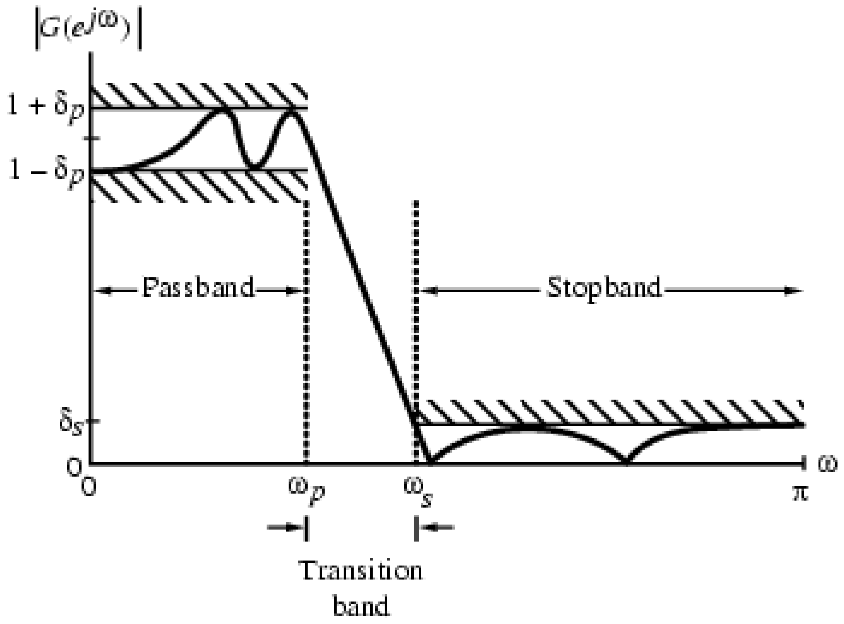


Figure 8.18 Amplitude and Phase Response for Seventh-Order Butterworth and Chebyshev Filters

회로로 구현
C, R, OpAmp

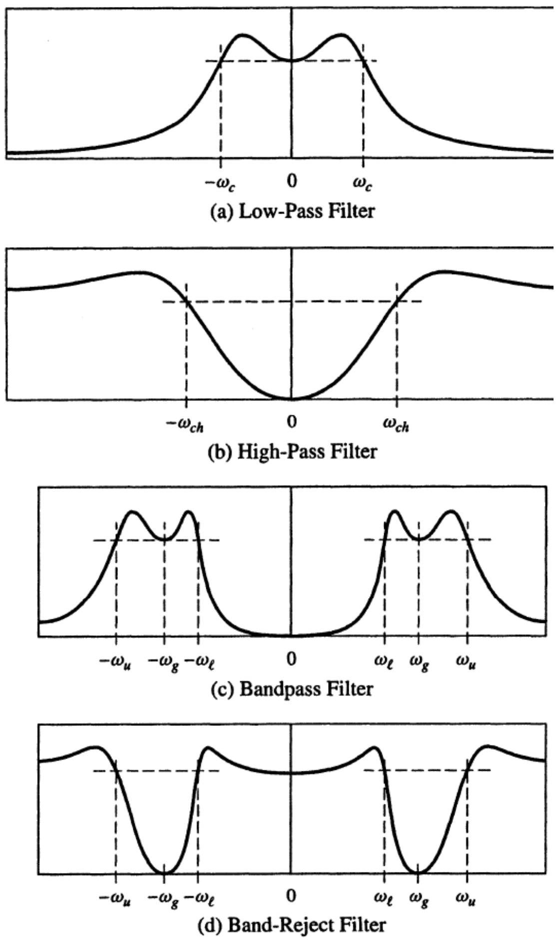
Filter



- ω_p - **passband edge frequency**
- ω_s - **stopband edge frequency**
- δ_p - **peak ripple value** in the passband
- δ_s - **peak ripple value** in the stopband

Practical Analog Filters

Figure 8.19
Filter Generation from a Low-Pass Filter Using the Frequency Transformations Given in Eqs. (8.33), (8.34), and (8.35)



Low-pass filter (LPF)

High-pass filter (HPF)

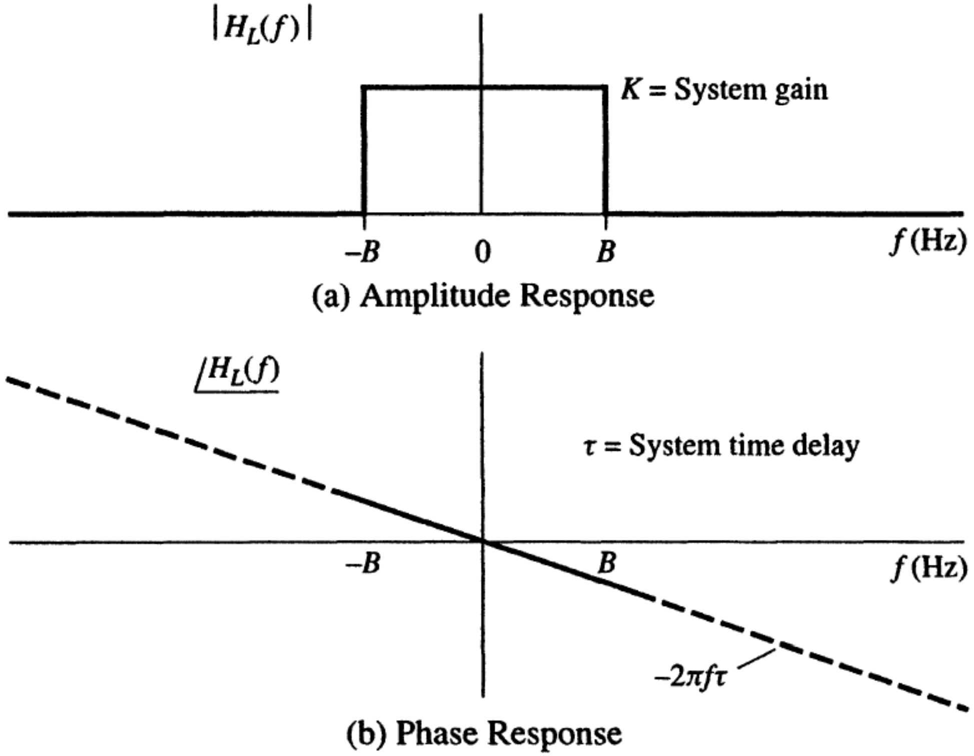
Bandpass filter (BPF)

Band-reject filter (BRF)

Analog Filters

Butterworth, Chebyshev, and elliptic filters: approximation of ideal filters

Figure 8.4
Frequency Response and Impulse Response for an Ideal Low-Pass Filter



Analog Filters – Butterworth filter

Definition

A low-pass Butterworth filter has the amplitude response

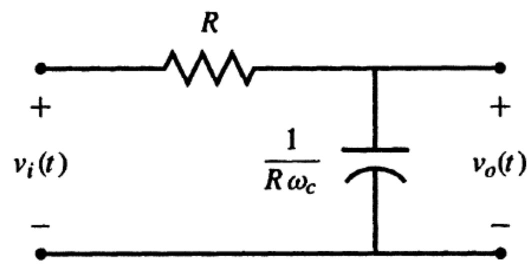
$$|H_{bn}(j\omega)| = G_M/\sqrt{1+(\omega/\omega_c)^{2n}} \tag{8.19}$$

where $n \geq 1$ is the filter order and the subscript b denotes a Butterworth filter.

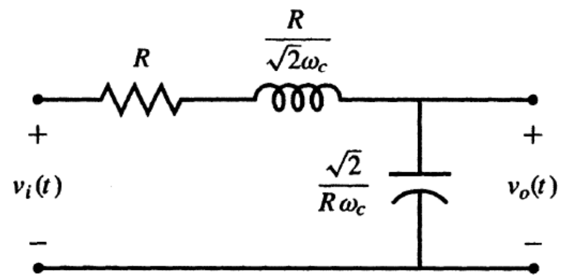
Filter cutoff frequency: half-power cutoff frequency or 3-dB cutoff frequency

$$H_{b1}(s) = \frac{G_M\omega_c}{\omega_c + s}$$

$$H_{b2}(s) = \frac{G_M\omega_c^2}{\omega_c^2 + \sqrt{2}\omega_c s + s^2}$$



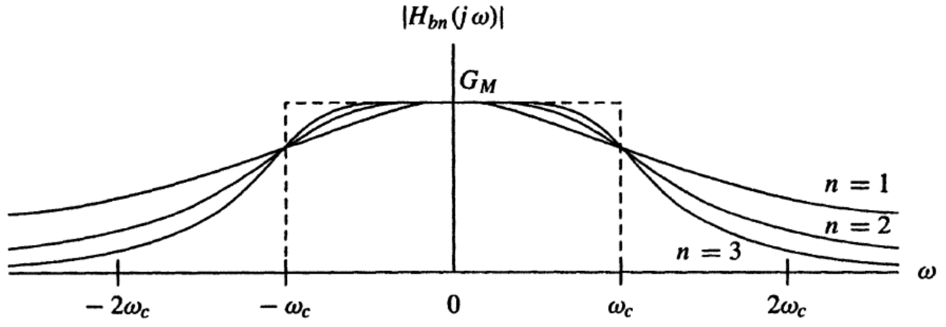
(a) First Order



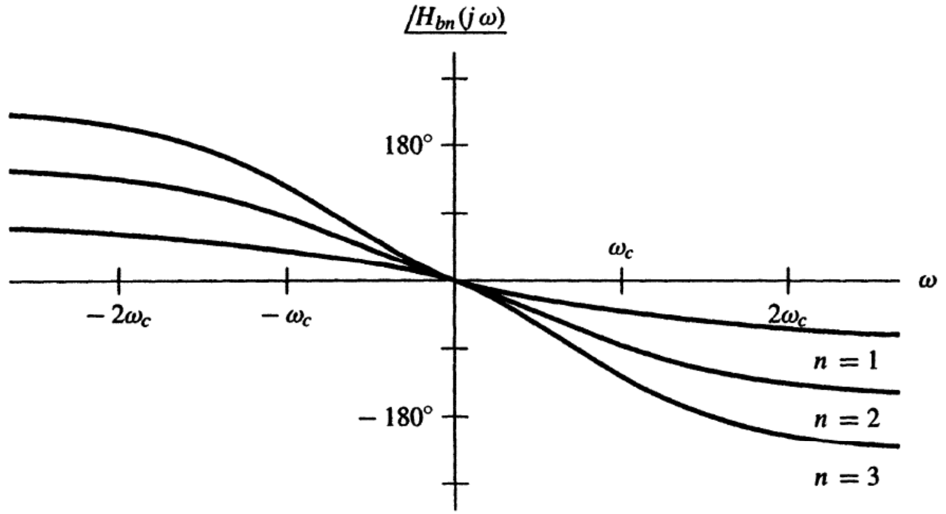
(b) Second Order

Analog Filters – Butterworth filter

Figure 8.14
Amplitude and Phase Responses for First-, Second-, and Third-Order Low-Pass Butterworth Filters



(a) Amplitude Response



(b) Phase Response

Order의 증가: ideal에 근접, 하지만… (1) 필요한 소자가 많아짐, (2) delay가 커짐

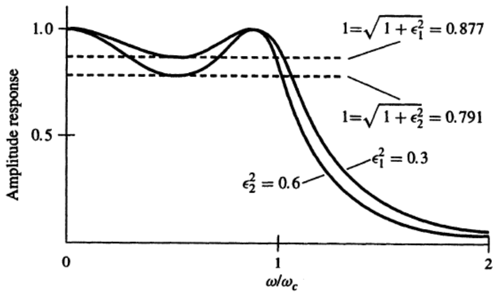
Analog Filters – Chebyshev filter

Definition

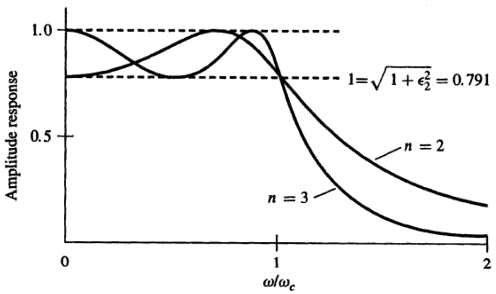
A low-pass Chebyshev filter has the amplitude response

$$|H_{cn}(j\omega)| = G_M/\sqrt{1+\varepsilon^2 T_n^2(\omega/\omega_c)} \quad (8.26)$$

where $T_n(x)$ is the nth-order Chebyshev polynomial, ε is a constant, $n \geq 1$ is the order of the filter, and the subscript c denotes a Chebyshev filter.



(a) Fixed Filter Order ($n = 3$)



(b) Fixed Value of ϵ^2 ($\epsilon^2 = 0.6$)

Butterworth보다 cutoff frequency에서 좀 더 sharp 하지만 ripple 발생

Variation in amplitude between the maximum and minimum values: amplitude-response passband ripple

$$r = 10\log(1+\varepsilon^2)$$

Analog Filters – Chebyshev filter

Transfer function for the first two orders of low-pass Chebyshev filters with unity DC gain and a passband ripple of 2 dB are

$$H_{c1}(s) = \frac{1.3076\omega_c}{1.3076\omega_c + s}$$

$$H_{c2}(s) = \frac{0.6368\omega_c^2}{0.6368\omega_c^2 + 0.8038\omega_c s + s^2}$$

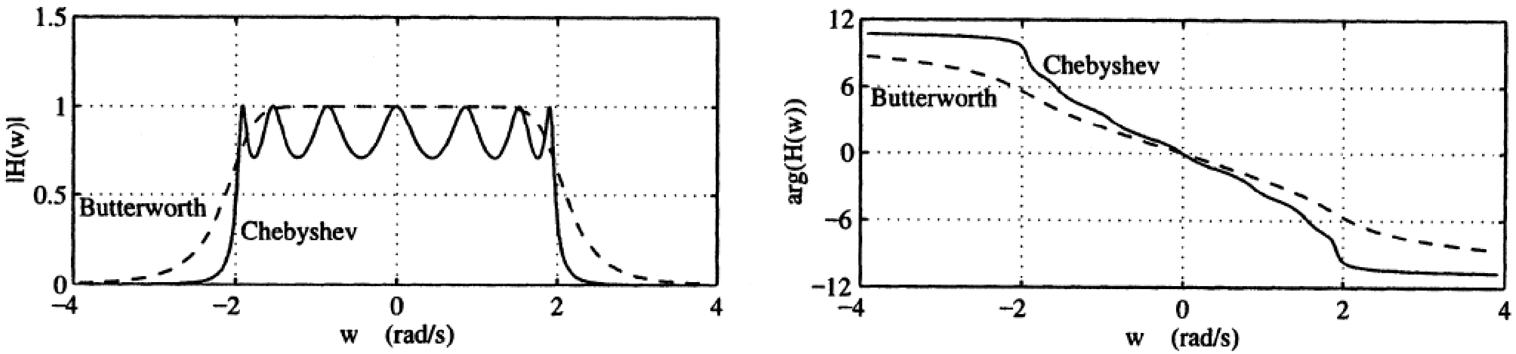


Figure 8.18 Amplitude and Phase Response for Seventh-Order Butterworth and Chebyshev Filters

High-Pass, Bandpass, and Band-Reject Approximations

Low pass filter → nonlinear frequency transformation
→ highpass, bandpass, band-reject approximation

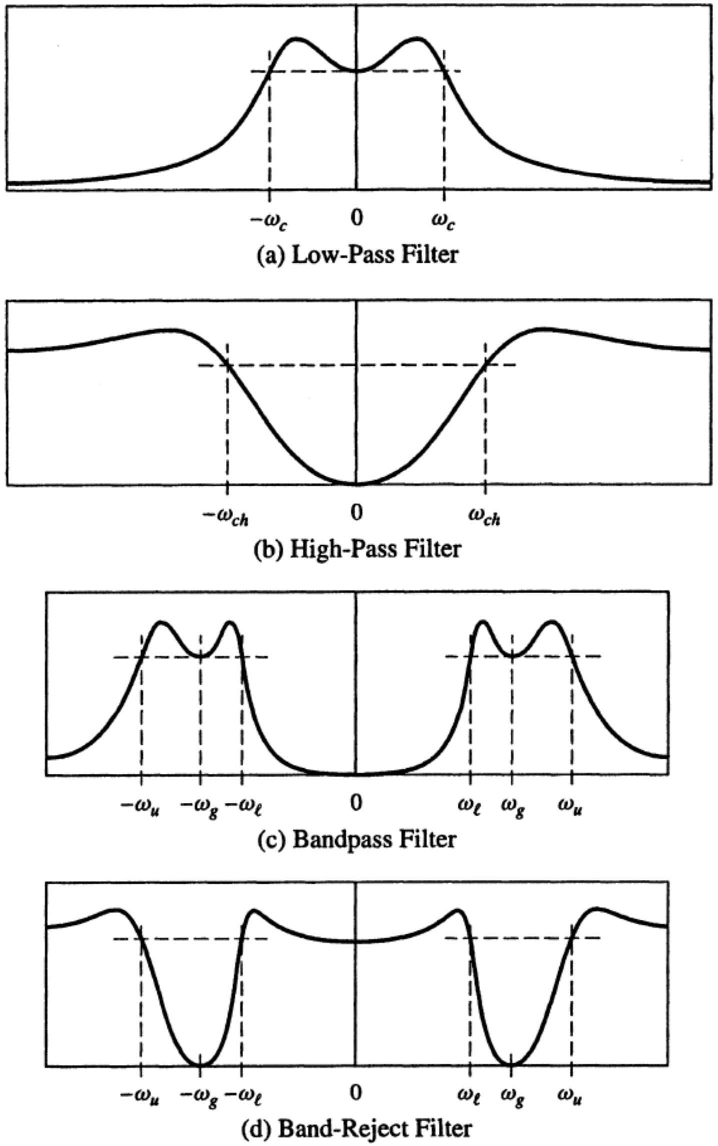
$$H_H(j\omega) = H_L(j\omega_L)\Big|_{\omega_L=\omega_c\omega_{ch}/\omega}$$

for a high-pass filter,

$$H_B(j\omega) = H_L(j\omega_L)\Big|_{\omega_L=\omega_c(\omega^2-\omega_u\omega_\ell)/\omega(\omega_u-\omega_\ell)}$$

for a bandpass filter, and

$$H_R(j\omega) = H_L(j\omega_L)\Big|_{\omega_L=\omega_c\omega(\omega_u-\omega_\ell)/(\omega^2-\omega_u\omega_\ell)}$$



Filter Design Procedure

Step 1: Choice of filter type: Butterworth or Chebyshev (depend on the relative importance of passband gain variation and cutoff sharpness)

Step 2: Selection of the filter cutoff frequency, ω_c.

Step 3: Determine the filter order (use Curves of the gain in the normalized low-pass filter stopband)

Step 4: Choose denominator coefficients for a normalized filter of the chosen order. E.g., Table 8.1

Step 5: Consideration of Gain

$$|a_o| = b_o G_{DC}$$ (Buttorworth, odd-order Chebyshev)

$$|a_o| = b_o G_M / \sqrt{1+\varepsilon^2}$$ (even-order Chebyshev)

Step 6: Frequency Scaling

Table 8.1 Denominator Polynomial Coefficients for Normalized Low-Pass Filters of Order $n = 1$ Through $n = 4$

n	b_o	b_1	b_2	b_3
Butterworth				
1	1.0000	—	—	—
2	1.0000	1.4142	—	—
3	1.0000	2.0000	2.0000	—
4	1.0000	2.6131	3.4142	2.6131
0.5-dB Passband Ripple Chebyshev ($\varepsilon^2 = 0.1220$)				
1	2.8628	—	—	—
2	1.5162	1.4256	—	—
3	0.7157	1.5439	1.2529	—
4	0.3791	1.0255	1.7169	1.1974
1.0-dB Passband Ripple Chebyshev ($\varepsilon^2 = 0.2589$)				
1	1.9652	—	—	—
2	1.1025	1.0977	—	—
3	0.4913	1.2384	0.9883	—
4	0.2756	0.7426	1.4539	0.9528
3.0-dB Passband Ripple Chebyshev ($\varepsilon^2 = 1.0000$)				
1	1.0000	—	—	—
2	0.7071	0.6436	—	—
3	0.2500	0.9277	0.5961	—
4	0.1768	0.4039	1.1685	0.5805

Low-pass filter design

$$H_N(s_N) = \frac{a_o}{b_o + b_1 s_N + \ldots + b_{n-1} s_N^{n-1} + s_N^n}$$

Normalized low-pass filter (cutoff freq. $\omega_c = 1$ rad/s)

$$|a_o| = b_o G_{DC}$$ (positive unless signal inversion is desired)

$$|a_o| = b_o G_M / \sqrt{1+\varepsilon^2}$$

$$\boxed{\begin{aligned} H_L(s) &= H_N(s_N)\big|_{s_N = s/\omega_c} \\ &= \frac{a_o}{b_o + b_1(s/\omega_c) + \ldots + (s/\omega_c)^n} \end{aligned}}$$

일반적으로는 Matlab, 각종 cad 프로그램 등등에서 기본적으로 제공. 요즘에는 손으로 계산하지는 않음 → 원하는 ripple 크기, order 수, transition band의 size 등을 입력하면 필터의 계수를 계산해서 output으로 출력해 줌: 함수 찾아볼 것

참고: elliptic filter: stopband에서의 ripple을 허용하여 좀 더 sharp하게

Table 8.1 Denominator Polynomial Coefficients for Normalized Low-Pass Filters of Order $n = 1$ Through $n = 4$

n	b_o	b_1	b_2	b_3
Butterworth				
1	1.0000	—	—	—
2	1.0000	1.4142	—	—
3	1.0000	2.0000	2.0000	—
4	1.0000	2.6131	3.4142	2.6131
0.5-dB Passband Ripple Chebyshev ($\varepsilon^2 = 0.1220$)				
1	2.8628	—	—	—
2	1.5162	1.4256	—	—
3	0.7157	1.5439	1.2529	—
4	0.3791	1.0255	1.7169	1.1974
1.0-dB Passband Ripple Chebyshev ($\varepsilon^2 = 0.2589$)				
1	1.9652	—	—	—
2	1.1025	1.0977	—	—
3	0.4913	1.2384	0.9883	—
4	0.2756	0.7426	1.4539	0.9528
3.0-dB Passband Ripple Chebyshev ($\varepsilon^2 = 1.0000$)				
1	1.0000	—	—	—
2	0.7071	0.6436	—	—
3	0.2500	0.9277	0.5961	—
4	0.1768	0.4039	1.1685	0.5805

Filter Design

Design a first-order Chebyshev low-pass filter with 3-dB passband ripple, maximum gain of unity, and cutoff frequency of 100 Hz

Analog Filter의 설계

$$H_{aN}(s_N) = \frac{a_0}{1+s_N}$$

$$a_0 = G_M b_0 = (1)(1) = 1 \quad \longleftarrow \quad \text{maximum gain of unity}$$

$$\boxed{\begin{aligned} H_L(s) &= H_N(s_N)\big|_{s_N = s/\omega_c} \\ &= \frac{a_o}{b_o + b_1(s/\omega_c) + \ldots + (s/\omega_c)^n} \end{aligned}}$$

$$H_a(s) = H_{aN}(s_N)\big|_{s_N = s/2\pi(100)} = \frac{1}{1+(s/200\pi)} = \frac{628.32}{628.32+s}$$

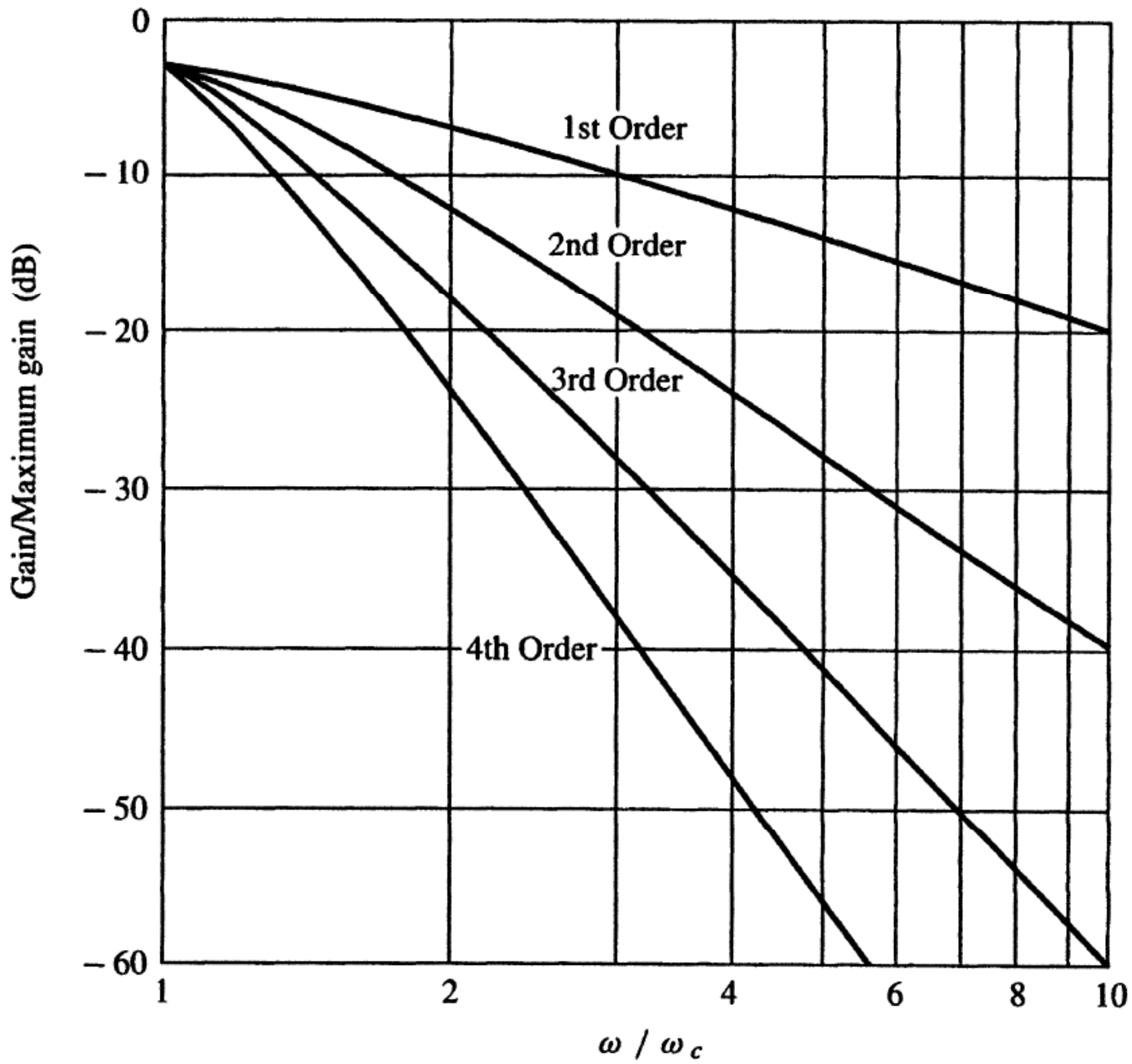
(cutoff = 100 Hz)

Table 8.1 Denominator Polynomial Coefficients for Normalized Low-Pass Filters of Order $n = 1$ Through $n = 4$

n	b_o	b_1	b_2	b_3
Butterworth				
1	1.0000	—	—	—
2	1.0000	1.4142	—	—
3	1.0000	2.0000	2.0000	—
4	1.0000	2.6131	3.4142	2.6131
0.5-dB Passband Ripple Chebyshev ($\varepsilon^2 = 0.1220$)				
1	2.8628	—	—	—
2	1.5162	1.4256	—	—
3	0.7157	1.5439	1.2529	—
4	0.3791	1.0255	1.7169	1.1974
1.0-dB Passband Ripple Chebyshev ($\varepsilon^2 = 0.2589$)				
1	1.9652	—	—	—
2	1.1025	1.0977	—	—
3	0.4913	1.2384	0.9883	—
4	0.2756	0.7426	1.4539	0.9528
3.0-dB Passband Ripple Chebyshev ($\varepsilon^2 = 1.0000$)				
1	1.0000	—	—	—
2	0.7071	0.6436	—	—
3	0.2500	0.9277	0.5961	—
4	0.1768	0.4039	1.1685	0.5805

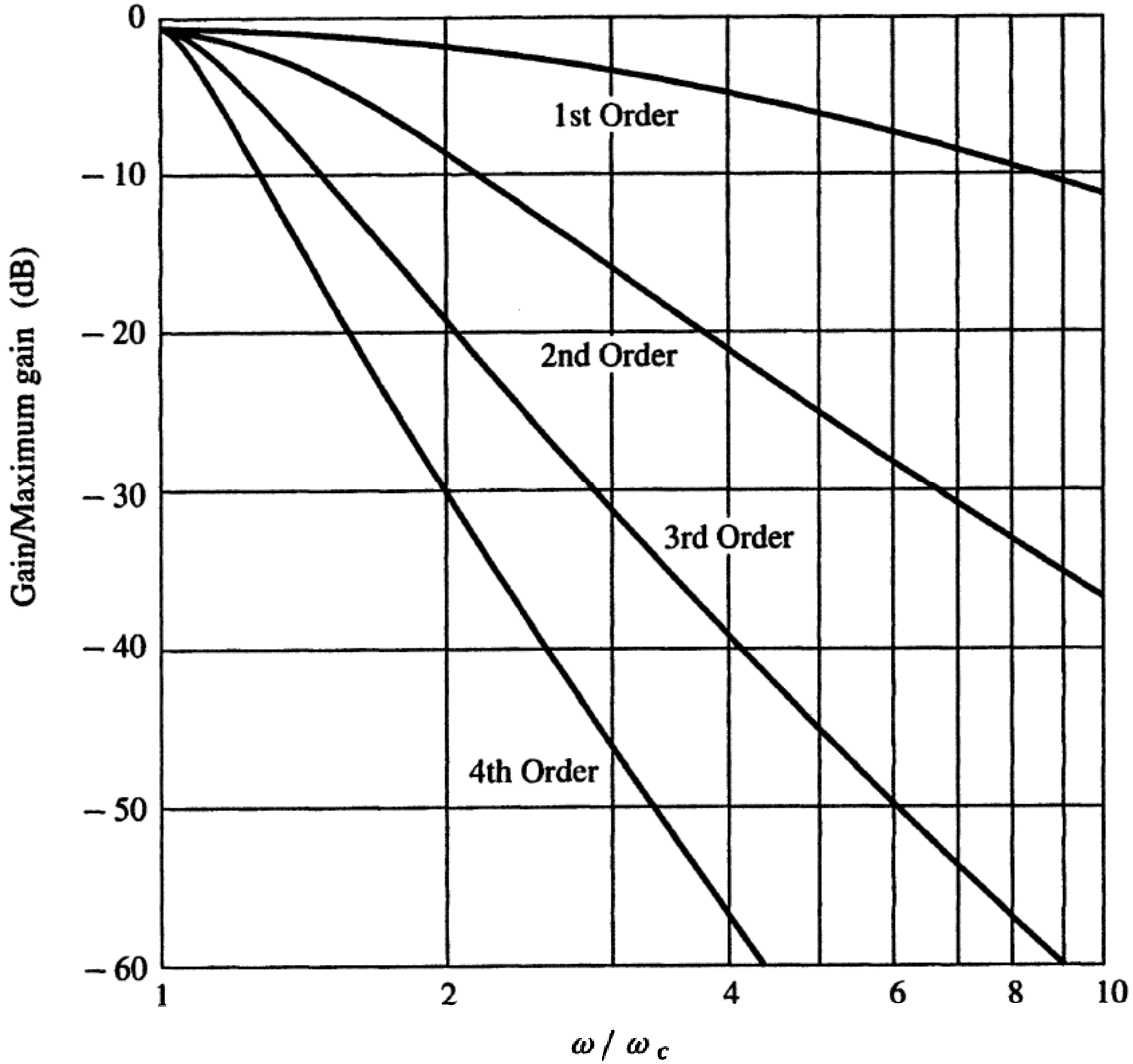
Stopband Gain Graphs to Determine Filter Order

Figure 8.20
Stopband Gain as a Function of Normalized Frequency $\omega_N = \omega/\omega_c$ for a Normalized Low-Pass Butterworth Filter



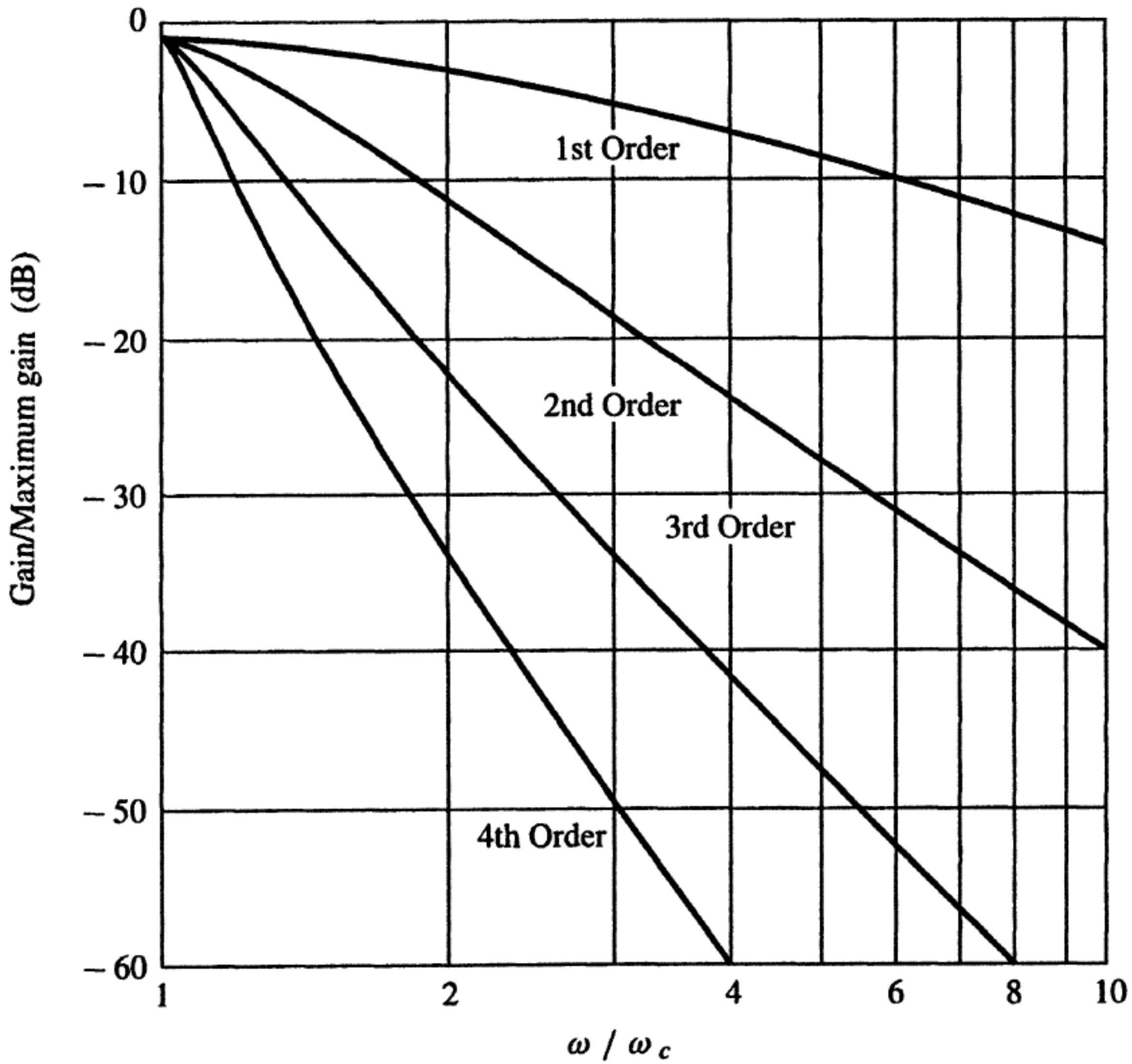
Stopband Gain Graphs to Determine Filter Order

Figure 8.21
Stopband Gain as a Function of Normalized Frequency $\omega_N = \omega/\omega_c$ for a Normalized Low-Pass Chebyshev Filter with a 0.5-dB Passband Ripple



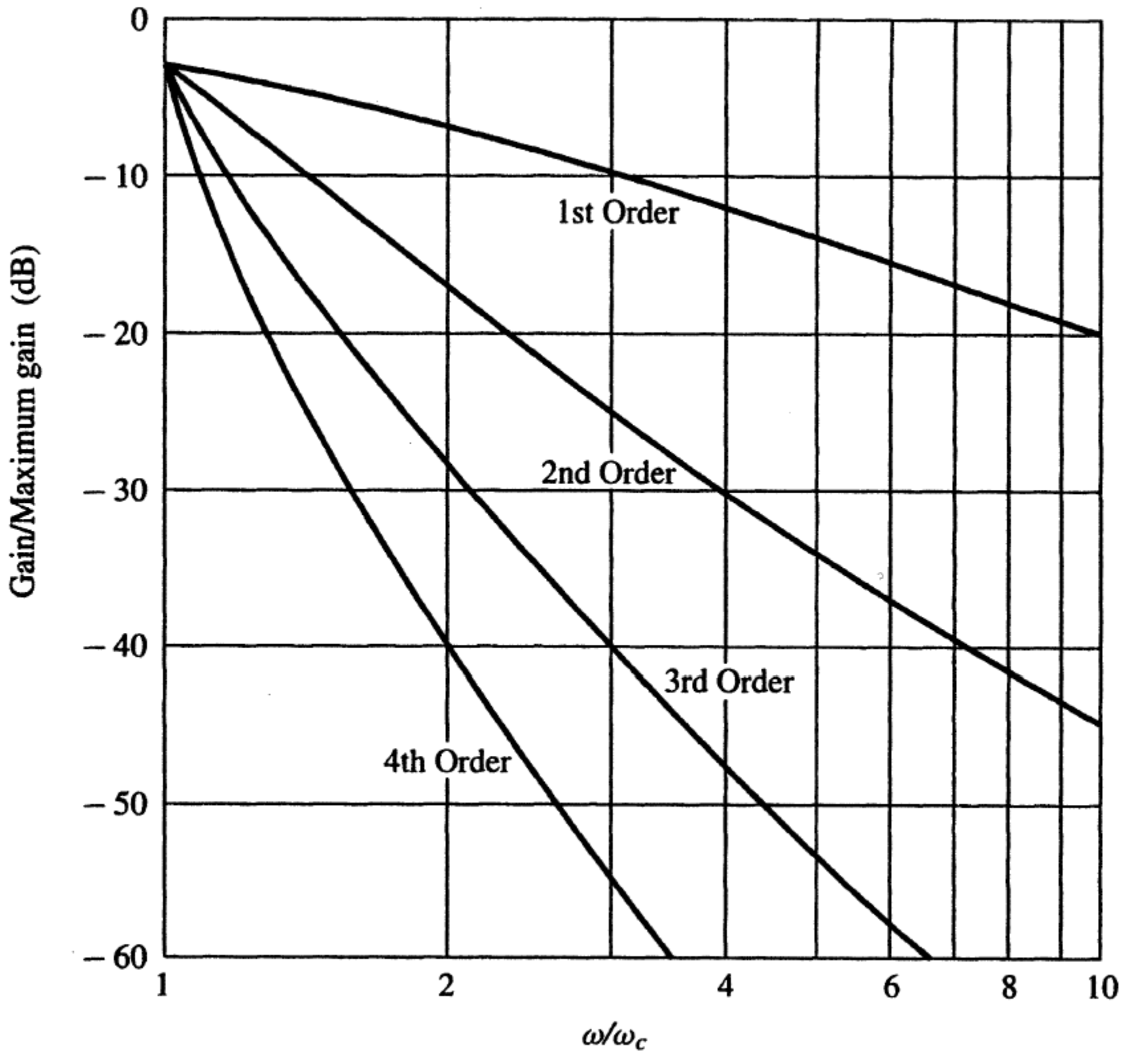
Stopband Gain Graphs to Determine Filter Order

Figure 8.22
Stopband Gain as a Function of Normalized Frequency $\omega_N = \omega/\omega_c$ for a Normalized Low-Pass Chebyshev Filter with a 1.0-dB Passband Ripple



Stopband Gain Graphs to Determine Filter Order

Figure 8.23
Stopband Gain as a Function of Normalized Frequency $\omega_N = \omega/\omega_c$ for a Normalized Low-Pass Chebyshev Filter with a 3.0-dB Passband Ripple



Example

We need a low-pass filter to reduce noise that is present on a signal received from a strain gauge that monitors a bridge beam's deflection. The filter is to satisfy the specifications.

1. Maximum gain equals 2.
2. Gain variation less than or equal to 3 dB from DC to 50 Hz.
3. Gain less than or equal to -50 dB with respect to maximum gain for $f \geq 250$ Hz.

Design the filter with Butterworth filter

Filter cutoff frequency was set as

$$\omega_c = 2\pi(50) = 100\pi \text{ rad/s}$$

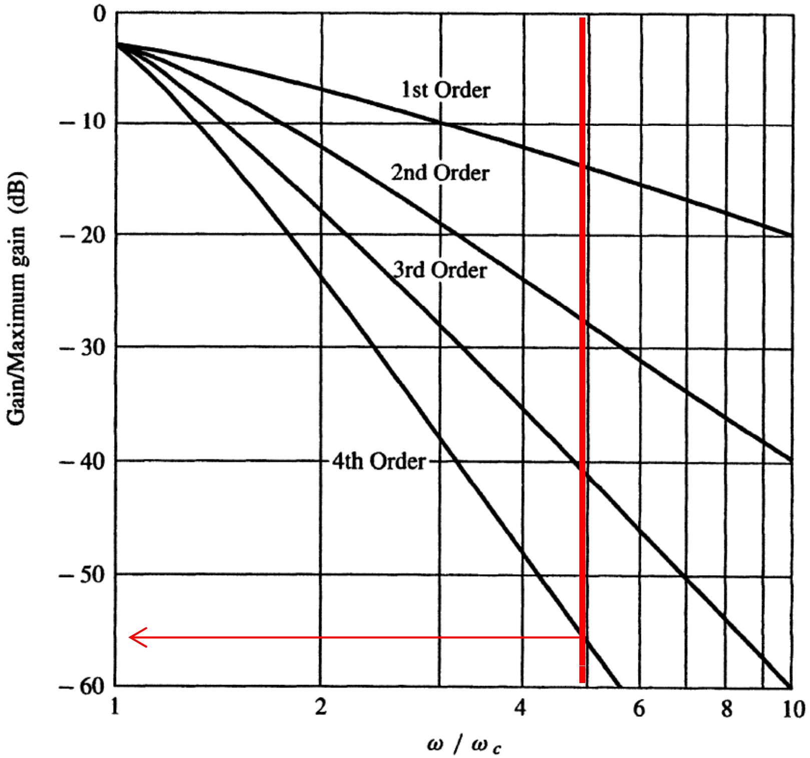
Filter stopband frequency was set as

$$\omega_1 = 2\pi(250) = 500\pi \text{ rad/s}$$

Next is "**Determination of filter order**"

Stopband Gain Graphs to Determine Filter Order

Figure 8.20
Stopband Gain as a Function of Normalized Frequency $\omega_N = \omega/\omega_c$ for a Normalized Low-Pass Butterworth Filter



4th order is the minimum order

$$\frac{\omega_1}{\omega_c} = \frac{500\pi}{100\pi} = 5$$

3. Gain less than or equal to -50 dB with respect to maximum gain for $f \geq 250$ Hz.

Example – Cont'd

$$H_N(s_N) = \frac{a_o}{b_o + b_1 s_N + \ldots + b_{n-1} s_N^{n-1} + s_N^n}$$

n	b_o	b_1	b_2	b_3
Butterworth				
1	1.0000	—	—	—
2	1.0000	1.4142	—	—
3	1.0000	2.0000	2.0000	—
4	1.0000	2.6131	3.4142	2.6131

$$H_N(s_N) = \frac{a_o}{1 + 2.6131 s_N + 3.4142 s_N^2 + 2.6131 s_N^3 + s_N^4}$$

Since maximum gain equals 2,

$$a_o = b_o G_M = (1)(2) = 2$$

$$\boxed{\begin{aligned} H_L(s) &= \frac{2}{1 + 2.6131\left(\frac{s}{100\pi}\right) + 3.4142\left(\frac{s}{100\pi}\right)^2 + 2.6131\left(\frac{s}{100\pi}\right)^3 + \left(\frac{s}{100\pi}\right)^4} \\ &= \frac{1.948 \times 10^{10}}{9.741 \times 10^9 + (8.102 \times 10^7)s + (3.370 \times 10^5)s^2 + 820.9s^3 + s^4} \end{aligned}}$$

Example 2 – Chebyshev filter

We need a low-pass filter to reduce noise that is present on a signal received from a strain gauge that monitors a bridge beam's deflection. The filter is to satisfy the specifications.

1. Maximum gain equals 2.
2. Gain variation less than or equal to 3 dB from DC to 50 Hz.
3. Gain less than or equal to −50 dB with respect to maximum gain for $f \geq 250$ Hz.

Design the filter with Chebyshev filter

Filter cutoff frequency was set as

$$\omega_c = 2\pi(50) = 100\pi \ \text{rad/s}$$

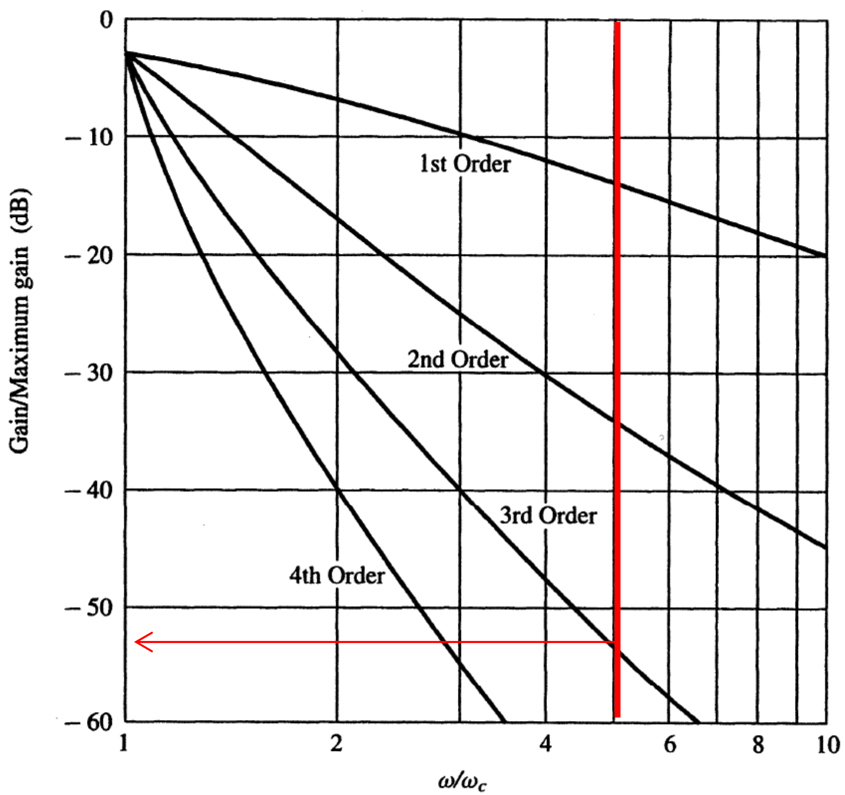
Filter stopband frequency was set as

$$\omega_1 = 2\pi(250) = 500\pi \ \text{rad/s}$$

Next is "**Determination of filter order**"

Stopband Gain Graphs to Determine Filter Order

Figure 8.23
Stopband Gain as a Function of Normalized Frequency $\omega_N = \omega/\omega_c$ for a Normalized Low-Pass Chebyshev Filter with a 3.0-dB Passband Ripple



3rd order is the minimum order

$$\frac{\omega_1}{\omega_c} = \frac{500\pi}{100\pi} = 5$$

3. Gain less than or equal to –50 dB with respect to maximum gain for $f \geq 250$ Hz.

Example – Cont'd

$$H_N(s_N) = \frac{a_o}{b_o + b_1 s_N + \ldots + b_{n-1} s_N^{n-1} + s_N^n}$$

3.0-dB Passband Ripple Chebyshev ($\varepsilon^2 = 1.0000$)				
1	1.0000	—	—	—
2	0.7071	0.6436	—	—
3	0.2500	0.9277	0.5961	—
4	0.1768	0.4039	1.1685	0.5805

$$H_N(s_N) = \frac{a_o}{0.2506 + 0.9283 s_N + 0.5972 s_N^2 + s_N^3}$$

Since maximum gain equals 2,

$$a_o = b_o G_M = (0.2506)(2) = 0.5012$$

$$H_L(s) = \frac{0.5012}{0.2506 + 0.9283\left(\frac{s}{100\pi}\right) + 0.5972\left(\frac{s}{100\pi}\right)^2 + \left(\frac{s}{100\pi}\right)^3}$$

$$= \frac{1.554 \times 10^7}{7.770 \times 10^7 + (9.162 \times 10^4)s + 187.6 s^2 + s^3}$$

Results

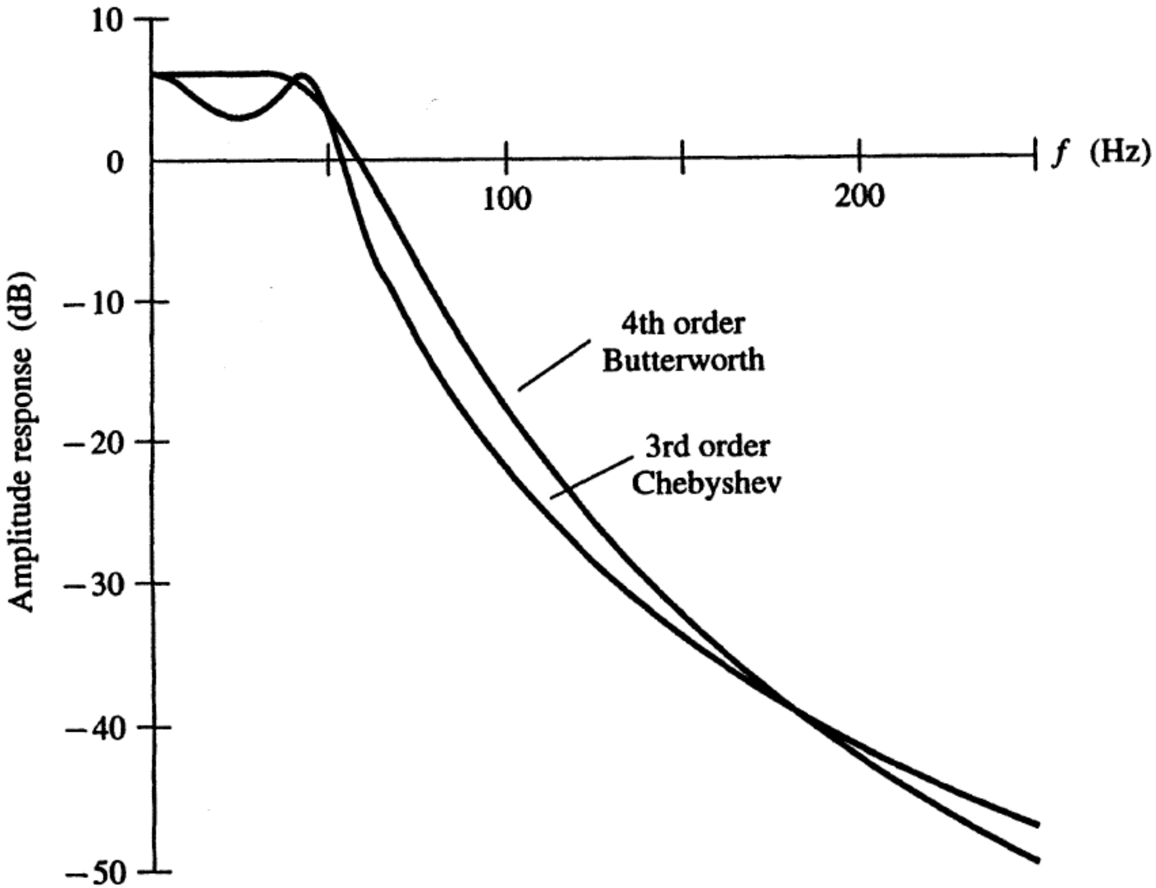


Figure 8.24 Butterworth and Chebyshev Low-Pass Filter Amplitude Responses for Example 8.6