Assignment (5)

(1) observable must be linear and Hermitian:

a) $\hat{A}\Psi(x) = \Psi^2(x) + \Psi^*(x)$

$\hat{A}[c\Psi(x)] = \underbrace{c^2\Psi^2(x) + c^*\Psi^*(x)}_{\text{not linear}}$ NO//

b) $\hat{A}\Psi(x) = \cos(x)\Psi(x) + x\Psi(x) = [\cos(x) + x]\Psi(x)$

$\hat{A}[c\Psi(x)] = c[\cos(x)\Psi(x) + x\Psi(x)]$

$\hat{A}[\Psi_1(x) + \Psi_2(x)] = \cos(x)\Psi_1(x) + x\Psi_1(x) + \cos(x)\Psi_2(x) + x\Psi_2(x) =$

$\int \Psi_1^*(x)\hat{A}\Psi_2(x)\,dx = \int \Psi_1^*(x)[\cos(x)+x]\Psi_2(x)\,dx = \int [\cos(x)+x]^*\Psi_1^*(x)\Psi_2(x)\,dx$ YES//

$= \int \{[\cos(x)+x]\Psi_1(x)\}^*\Psi_2(x)\,dx$

$= \int \{\hat{A}\Psi_1(x)\}^*\Psi_2(x)\,dx$

c) $\hat{A}\Psi(x) = \Psi'(x) = \frac{d\Psi(x)}{dx}$

$\hat{A}[c\Psi(x)] = c\Psi'(x)$

$\hat{A}[\Psi_1(x) + \Psi_2(x)] = \Psi_1'(x) + \Psi_2'(x)$

$\int \Psi_1^*(x)\hat{A}\Psi_2(x)\,dx = \int \Psi_1^*(x)\Psi_2'(x)\,dx \neq \int \left[\frac{d\Psi_1(x)}{dx}\right]^*\Psi_2(x)\,dx$ NO//

(2) $\Psi(x) = \begin{cases} A(1-x^2) & -1 \le x \le 1 \\ 0 & \text{other} \end{cases}$

a) $\int_{-1}^{1}\Psi^*(x)\Psi(x)\,dx = \int_{-1}^{1}A^2(1-x^2)^2\,dx = A^2\int_{-1}^{1}(1-2x^2+x^4)\,dx = A^2\left[x - \frac{2x^3}{3} + \frac{x^5}{5}\right]_{-1}^{1} dx = A^2\left[1+1-\frac{2}{3}-\frac{2}{3}+\frac{1}{5}+\frac{1}{5}\right]$

$A^2\left[\frac{30-20+6}{15}\right] = A^2\left[\frac{16}{15}\right] = 1$

$A = \pm\sqrt{\frac{15}{16}} = \pm\frac{\sqrt{15}}{4}$ //

b) $\hat{H}\Psi(x) = E\Psi(x) \rightarrow [\hat{H} - E]\Psi(x) = 0$

$\left[-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x) - E\right]\Psi(x) = 0$

$\frac{d\Psi}{dx} = \frac{d}{dx}(A - Ax^2) = -2Ax$

$\frac{d^2\Psi}{dx} = -2A$

$[V(x) - E]\Psi(x) = \frac{\hbar^2}{2m}\frac{d^2}{dx^2}\Psi(x)$

$V(x) - E = \left[\frac{\hbar^2}{2m}\frac{d^2\Psi(x)}{dx^2}\right]\left[\frac{1}{\Psi(x)}\right]$

$V(x) = E + \frac{\hbar^2}{2m\Psi(x)}\frac{d^2\Psi(x)}{dx^2} = E + \frac{\hbar^2}{2mA(1-x^2)}(-2A) = E - \frac{\hbar^2}{m(1-x^2)}$

$V(x=0) = E - \frac{\hbar^2}{m(1)} = 0$

$E = \frac{\hbar^2}{m}$

$V(x) = \frac{\hbar^2}{m} - \frac{\hbar^2}{m(1-x^2)} = \frac{\hbar^2}{m}\left[\frac{1-x^2-1}{1-x^2}\right] = \frac{\hbar^2}{m}\left[\frac{x^2}{x^2-1}\right]$

$\hat{H} = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{\hbar^2}{m}\left[\frac{x^2}{x^2-1}\right]$ //

c) $E = \int_{-1}^{1}\Psi^*(x)\hat{H}\Psi(x)\,dx = \int_{-1}^{1}A(1-x^2)\left[-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{\hbar^2}{m}\left(\frac{x^2}{x^2-1}\right)\right]A(1-x^2)\,dx$

$= A^2\int_{-1}^{1}(1-x^2)\left[-\frac{\hbar^2}{2m}(-2) + \frac{\hbar^2}{m}(-x^2)\right]dx = A^2\int_{-1}^{1}\frac{\hbar^2}{m}(1-x^2)^2\,dx$

$= \frac{\hbar^2}{m}[1] = \frac{\hbar^2}{m}$ //

d) $KE = \int_{-1}^{1}\Psi^*(x)\left[-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\right]\Psi(x)\,dx = -\frac{\hbar^2}{2m}\int_{-1}^{1}A(1-x^2)(-2A)\,dx = \frac{2A^2\hbar^2}{2m}\int_{-1}^{1}1-x^2\,dx = \frac{\cancel{2}A^2\hbar^2}{\cancel{2}m}\left[x - \frac{x^3}{3}\right]_{-1}^{1} = \frac{\frac{15}{16}\hbar^2}{m}\left[1+1-\frac{1}{3}-\frac{1}{3}\right]$

$= \frac{15}{16}\frac{\hbar^2}{m}\left(\frac{4}{3}\right) = \frac{5\hbar^2}{4m}$ //

e) $PE = \int_{-1}^{1}\Psi^*(x)\left[E - \frac{\hbar^2}{m(1-x^2)}\right]\Psi(x)\,dx = \int_{-1}^{1}A(1-x^2)\left[E - \frac{\hbar^2}{m(1-x^2)}\right]A(1-x^2)\,dx$

$= \int_{-1}^{1}A^2\left[E(1-2x^2+x^4) - \frac{\hbar^2}{m}(1-x^2)\right]dx = \frac{15}{16}\left[\frac{\hbar^2}{m}\left(\frac{16}{15}\right) - \frac{\hbar^2}{m}\left(\frac{4}{3}\right)\right] = \frac{\hbar^2}{m} - \frac{\hbar^2}{m}\left(\frac{5}{4}\right) = E - KE$

$= -\frac{1}{4}\frac{\hbar^2}{m}$ //

f) $E'' = \sum_{m=1}^{\infty}\frac{\left[\int \Psi_0^*(x)V(x)\Psi_m(x)\,dx\right]\left[\int \Psi_m^*(x)V(x)\Psi_0(m)\,dx\right]}{E_0 - E_m}$; $\Psi'' = \sum_{m=1}^{\infty}\frac{\int \Psi_m^*(x)V(x)\Psi_0'(x)\,dx - E_0'\int \Psi_m^*\Psi_0'(x)\,dx}{E_0 - E_m}\Psi_m^*(x)$ //

(3) $V(x) = \begin{cases} \frac{V_0 x}{a} & 0 \le x \le a \\ 0 & \text{other} \end{cases}$

$E' = \int_0^a \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi x}{a}\right)\left[\frac{V_0 x}{a}\right]\sqrt{\frac{2}{a}}\sin\left(\frac{n\pi x}{a}\right)dx = \frac{2}{a}\left(\frac{V_0}{a}\right)\underbrace{\int_0^a x\sin^2\left(\frac{n\pi x}{a}\right)dx}_{\frac{a^2}{4}} = \frac{2V_0}{a^2}\left(\frac{a^2}{4}\right) = \frac{V_0}{2}$ //

$E_{TOTAL} = \frac{h^2}{8ma^2} + \frac{V_0}{2}$ //

(4) $V(x) = \frac{1}{2}kx^2 + \underbrace{\frac{1}{6}\gamma_3 x^3 + \frac{1}{24}\gamma_4 x^4}_{\text{perturbation}}$.

$E' = \int_{-\infty}^{\infty}\left(\frac{\sqrt{km}}{\pi\hbar}\right)^{1/4}\exp\left(-\frac{\sqrt{km}}{2\hbar}x^2\right)\left(\frac{1}{6}\gamma_3 x^3 + \frac{1}{24}\gamma_4 x^4\right)\left(\frac{\sqrt{km}}{\pi\hbar}\right)^{1/4}\exp\left(-\frac{\sqrt{km}}{2\hbar}x^2\right)dx$

$= \left(\frac{\sqrt{km}}{\pi\hbar}\right)^{1/2}\left[\underbrace{\int_{-\infty}^{\infty}\frac{1}{6}\gamma_3 \underbrace{x^3}_{\text{odd}}\underbrace{\exp\left(-\frac{\sqrt{km}}{\hbar}x^2\right)}_{\text{even}}dx}_{\text{odd} = 0} + \int_{-\infty}^{\infty}\frac{1}{24}\gamma_4 x^4\exp\left(-\frac{\sqrt{km}}{\hbar}x^2\right)dx\right]$

$= \left(\frac{\sqrt{km}}{\pi\hbar}\right)^{1/2}\left(\frac{\gamma_4}{24}\right)\left[\underbrace{\int_{-\infty}^{\infty}x^4\exp\left(-\frac{\sqrt{km}}{\hbar}x^2\right)dx}_{\frac{1\times 3}{\left(2\frac{\sqrt{km}}{\hbar}\right)^2}\sqrt{\frac{\pi}{\frac{\sqrt{km}}{\hbar}}}}\right] = \left(\frac{\sqrt{km}}{\pi\hbar}\right)^{1/2}\left(\frac{\gamma_4}{24}\right)\left(\frac{3\hbar^2}{4km}\right)\left(\frac{\pi\hbar}{\sqrt{km}}\right)^{1/2} = \frac{\gamma_4\hbar^2}{32km}$ //