

Linear Algebra II

Sheet 3 — HT21

1. Let A be each of the following matrices in turn:

$$\begin{pmatrix} 2 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 \\ -1 & 3 & 0 \\ -1 & 4 & -1 \end{pmatrix}, \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}.$$

Find all the eigenvectors of A , determine whether A is diagonalisable (over \mathbb{R}) and, if so, find an invertible real matrix P such that $P^{-1}AP$ is diagonal.

2. Define $S : M_n(\mathbb{R}) \rightarrow M_n(\mathbb{R})$ by $S(A) = A^T$. Prove that S has only two distinct eigenvalues and that its eigenvectors span $M_n(\mathbb{R})$.
3. For any polynomial $p(x) = a_0 + a_1x + \cdots + a_kx^k$ and any square matrix A , $p(A)$ is defined as $p(A) = a_0I + a_1A + \cdots + a_kA^k$. Show that if v is any eigenvector of A and $\chi_A(x)$ is the characteristic polynomial of A , then $\chi_A(A)v = 0$. Deduce that if A is diagonalisable then $\chi_A(A)$ is the zero matrix.

4. Let $M = \begin{pmatrix} -5 & 3 \\ 6 & -2 \end{pmatrix}$.

- (a) Find a diagonal matrix D and an invertible matrix P such that $M = PDP^{-1}$.
- (b) Find at least one cube root of M , by observing that if $D = E^3$ then $M = (PEP^{-1})^3$.
- (c) Express the infinite series $e^M = \sum_{n=0}^{\infty} \frac{1}{n!} M^n$ (where $M^0 = I$) as a 2×2 matrix with entries involving the constant e . (You may assume any general properties of infinite series of matrices that you need.)
5. Let V be a real n dimensional vector space, and $T : V \rightarrow V$ be a linear mapping. Show that if λ is the only eigenvalue of T and T is diagonalisable then $T = \lambda I$.

Now let V be the vector space of real polynomials in x of degree at most d where $d > 0$. Which of the following linear mappings of V into itself are diagonalisable?

- (a) $T_1 : f(x) \mapsto x \frac{df}{dx}$
- (b) $T_2 : f(x) \mapsto \frac{df}{dx}$
- (c) $T_3 : f(x) \mapsto f(x+1)$
- (d) $T_4 : f(x) \mapsto f(-x)$.