M5 Fourier Series & PDEs

Overview

The course begins by introducing students to Fourier series, concentrating on their practical application rather than proofs of convergence. Students will then be shown how the heat equation, the wave equation and Laplace's equation arise in physical models. They will learn basic techniques for solving each of these equations in several independent variables, and will be introduced to elementary uniqueness theorems.

Problem sheet schedule

Sheet	Sections
1	1.1 – 2.1
2	2.2 – 2.5
3	2.6 – 2.9
4	3.1 – 3.5
5	3.6 – 3.9
6	4.1 – 4.4
7	4.5 – 4.11
8	5.1 – 5.8

Reading list

- [1] E. Kreyszig, Advanced Engineering Mathematics (Wiley, 10th Edition, 2011).
- [2] W. A. Strauss, Partial Differential Equations: An Introduction (Wiley, 2nd Edition, 2008).
- [3] D. W. Jordan & P. Smith, Mathematical Techniques (OUP, 4th Edition, 2008).
- [4] P. J. Olver, Introduction to Partial Differential Equations (Springer, 1st Edition, 2014).
- [5] G. F. Carrier & C. E. Pearson, Partial Differential Equations — Theory and Technique (Academic Press, 2014).

Synopsis (16 lectures)

Fourier series: Periodic, odd and even functions. Calculation of sine and cosine series. Simple applications concentrating on imparting familiarity with the calculation of Fourier coefficients and the use of Fourier series. The issue of convergence is discussed informally with examples. The link between convergence and smoothness is mentioned, together with its consequences for approximation purposes.

Partial differential equations: Introduction in descriptive mode on partial differential equations and how they arise. Derivation of (i) the wave equation of a string, (ii) the heat equation in one dimension (box argument only). Examples of solutions and their interpretation. D'Alembert's solution of the wave equation and applications. Characteristic diagrams (excluding reflection and transmission). Uniqueness of solutions of wave and heat equations.

PDEs with Boundary conditions. Solution by separation of variables. Use of Fourier series to solve the wave equation, Laplace's equation and the heat equation (all with two independent variables). Laplace's equation in Cartesian and in plane polar coordinates. Applications.

Authorship and acknowledgments

The author of these notes is **Jim Oliver**, taken largely from notes originally written by **Ruth Baker**, **Yves Capdeboscq**, **Alan Day**, **Janet Dyson** and **Peter Howell**. **Benjamin Walker** is gratefully acknowledged for help with typesetting.

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Contents

1 Introduction

1.1 Fourier series

- While developing the theory of heat conduction in the early 19th century, Jean-Baptiste Joseph Fourier kick-started a mathematical revolution by claiming that "every" real-valued function defined on a finite interval could be expanded as an infinite series of elementary trigonometric functions — cosines and sines.

- Given a function $f : \mathbb{R} \to \mathbb{R}$ that is periodic with period 2π (about which more in §2), Fourier's claim is equivalent to the assertion that there exist constants $a_0, a_1, \ldots$ and $b_1, b_2, \ldots$ in terms of which f may be expanded in the form

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \big(a_n \cos(nx) + b_n \sin(nx)\big) \quad \text{for } x \in \mathbb{R}, \tag{1.1}$$

  the infinite series on the right-hand side of the equality then being the *Fourier series* for f.

- **Remark:** In §2 it will become apparent that the choice to focus on functions of period 2π and the factor of $1/2$ in the leading term in (1.1) are both for algebraic convenience.

- Fourier's claim raises two fundamental questions:

  Question 1: If (1.1) is true, can we find the constants a_n and b_n in terms of f?

  Question 2: With these a_n and b_n, when is (1.1) true?

- The need for rigorous mathematical analysis to address these questions led to a surprisingly large proportion of the material covered in prelims, part A and beyond (*e.g.* the definition of a function, the ε-δ definition of limit, the theory of convergence of sequences and series of functions, Lebesgue integration and Cantor's set theory).

- The implications of Fourier's claim for practical applications were no less powerful or far-ranging: the decomposition led to deep and fundamental insights into numerous physical phenomena (*e.g.* mass and heat transport, vibrations of elastic media, acoustics and quantum mechanics) and continue to be exploited today in numerous fields (*e.g.* signal processing, approximation theory and control theory).

- In this course we introduce fundamental results for the pointwise convergence of Fourier series. We then follow in Fourier's footsteps by using them to construct solutions to fundamental problems involving the three most ubiquitous partial differential equations in mathematics, science and engineering: the heat equation, the wave equation and Laplace's equation.

- We begin with a motivational example illustrating the existence of a convergent Fourier series.

Example: existence of a convergent Fourier series

■ Recall from Analysis I the definition of the exponential function in terms of a power series:

$$\exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!} \quad \text{for} \quad z \in \mathbb{C}. \tag{1.2}$$

■ If we let $z = \exp(\mathrm{i}\theta) = \cos\theta + \mathrm{i}\sin\theta$, where $\theta \in \mathbb{R}$, then

$$\mathrm{Re}\big(\exp(z)\big) = \mathrm{Re}\big(\exp(\cos\theta)\exp(\mathrm{i}\sin\theta)\big) = \exp(\cos\theta)\cos(\sin\theta), \tag{1.3}$$

and

$$\mathrm{Re}\big(z^n\big) = \mathrm{Re}\big(\exp(\mathrm{i}n\theta)\big) = \cos n\theta. \tag{1.4}$$

■ Taking the real part of (1.2) and using (1.3)–(1.4), we deduce that

$$\exp(\cos\theta)\cos(\sin\theta) = \sum_{n=0}^{\infty} \frac{\cos n\theta}{n!} \quad \text{for} \quad \theta \in \mathbb{R}. \tag{1.5}$$

The infinite trigonometric series on the right-hand side of this equality is the *Fourier cosine series* for the function on the left-hand side. ■

- **Question**: How would you generate an example of a convergent *Fourier sine series*?
- **Answer:** One option is to take instead the imaginary part of (1.2) on the unit circle in the complex z-plane, which would give

$$\exp(\cos\theta)\sin(\sin\theta) = \sum_{n=1}^{\infty} \frac{\sin n\theta}{n!} \quad \text{for} \quad \theta \in \mathbb{R}. \tag{1.6}$$

- **Remark:** The method of extracting from a complex-valued function with a power series representation the Fourier series representation of a real-valued function is of limited applicability for two reasons:
  (1) it can only generate the Fourier series of an infinitely differentiable real-valued function — the proof of this fact relies on material covered in part A Complex Analysis;
  (2) even within this restricted class of functions, how should the complex-valued function be chosen to obtain the Fourier series that converges to a given real-valued function?
- At the heart of this course is a much simpler and more powerful method pioneered by Fourier that allows a much wider class of functions to be represented as convergent Fourier series.

1.2 Ordinary differential equations

- Here we revise essential background concerning ordinary differential equations from Introductory Calculus.
- **Definition:** An *ordinary differential equation* (ODE) is an equation involving a function of one variable and at least one of its derivatives, *i.e.* an ODE for the function $y(x)$ may be written in the form

$$G\left(x, y(x), \frac{\mathrm{d}y}{\mathrm{d}x}, \ldots, \frac{\mathrm{d}^n y}{\mathrm{d}x^n}\right) = 0 \tag{1.7}$$

  for some function G and some positive integer n.
- **Definition:** The function y is called the *dependent variable* and x the *independent variable*.
- **Definition:** The *order* of an ODE is the order of the highest order derivative that it contains, *e.g.* the order of (1.7) is n.
- **Definition:** An ODE is *linear* if the dependent variable and its derivatives appear in terms with degree at most one. An equation that is not linear is *nonlinear*.
- **Definition:** The most general nth-order linear ODE for $y(x)$ takes the form

$$\mathcal{L}y(x) = f(x), \tag{1.8}$$

  where $f(x)$ is a given *forcing function* and $\mathcal{L}$ is the *linear differential operator* defined by

$$\mathcal{L}y(x) = a_n(x)\frac{\mathrm{d}^n y}{\mathrm{d}x^n} + \cdots + a_1(x)\frac{\mathrm{d}y}{\mathrm{d}x} + a_0(x)y(x) \tag{1.9}$$

  for some coefficients $a_0(x)$, $a_1(x)$, …, $a_n(x)$ with $a_n(x) \neq 0$.
- **Definition:** The ODE (1.8) is called *homogeneous* if the right-hand side f is identically zero and if not then it is called *inhomogeneous*.
- Since differentiation is distributive, the differential operator $\mathcal{L}$ is *linear* in the sense that

$$\mathcal{L}\big[\alpha_1 y_1(x) + \alpha_2 y_2(x)\big] = \alpha_1 \mathcal{L}y_1(x) + \alpha_2 \mathcal{L}y_2(x)$$

  for any constants α_1, $\alpha_2 \in \mathbb{R}$ and any (suitably differentiable) functions $y_1(x)$, $y_2(x)$.
- **Definition:** A consequence of the linearity of $\mathcal{L}$ is the *Principle of Superposition* that the linear combination of two or more solutions is also a solution for a linear homogeneous ODE — but not for a linear inhomogeneous ODE nor a nonlinear ODE.

- In Introductory Calculus you studied methods to find the *general solution* of first- and second-order linear ODEs, *e.g.*
  - the integrating factor method for first-order linear inhomogeneous ODEs;
  - reduction of order for second-order linear homogeneous ODEs (given one solution);
  - methods for second-order linear inhomogeneous ODEs with constant coefficients.
- The linearity of $\mathcal{L}$ played a key role: the general solution of (1.8) takes the form of the superposition of the n linearly independent solutions of the homogeneous problem (called a *complementary function*), together with any solution of the inhomogeneous problem (called a *particular integral*).
- The exploitation of linearity will play a similar fundamental role in the methods we shall use to solve the heat equation, the wave equation and Laplace's equation.
- In Introductory Calculus you applied your toolbox of ODE methods to solve
  - *initial value problems* (IVPs) in which an nth-order ODE is supplemented by n initial conditions at some point x_0;
  - *boundary value problems* (BVPs) in which an nth-order ODE is supplemented by a total of n boundary conditions at two distinct points between which the ODE pertains.
- In general the method was to determine the general solution of the ODE and then to try to choose the n arbitrary constants that it contains to satisfy the n initial or boundary conditions. This does not work in general because a solution may not exist or if a solution exists it may not be unique.

Example: non-existence for an ODE BVP

- Consider the boundary value problem for $y(x)$ given by

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + y = 0 \quad \text{for } 0 < x < 2\pi, \tag{1.10}$$

  with $y(0) = 1$ and $y(2\pi) = 0$.
- The ODE has general solution $y(x) = A\cos x + B\sin x$, but the constants A and B cannot be chosen to satisfy the boundary conditions, so there is no solution. ■

Example: non-uniqueness for an ODE BVP

- Suppose now $y(x)$ satisfies the boundary value problem given by (1.10), but with the boundary conditions $y(0) = 0$ amd $y(2\pi) = 0$.
- The ODE again has general solution $y(x) = A\cos x + B\sin x$, but now the boundary conditions are satisfied for $A = 0$ and arbitrary $B \in \mathbb{R}$, so the solution is not unique. ■

- Questions of existence and uniqueness of ordinary differential equations will be a central theme in *e.g.* part A Differential Equations 1.
- **Question:** Why discuss here the issues of existence and uniqueness?
- **Answer:** Because we face precisely the same issues when solving a partial differential equation, so we should keep them in mind.

1.3 Partial differential equations

- We now introduce partial differential equations building on the terminology outlined in §1.2.
- **Definition:** A *partial differential equation* (PDE) is an equation for an unknown function of two or more *independent variables* that involves at least one partial derivative of that function. The unknown function is called the *dependent variable*.

- **Definition:** The *order* of a PDE is the order of the highest order partial derivative that it contains.
- **Definition:** A PDE is *linear* if the dependent variable and its partial derivatives appear in terms with degree at most one. An equation that is not linear is *nonlinear*.
- In this course we focus on the case in which there are two independent variables: (x, y) or (x, t), where in applications x and y often represent spatial variables and t often represents time.

Example: general and linear first-order PDEs

- A first-order PDE for $u(x, y)$ may be written in the form

$$G(x, y, u, u_x, u_y) = 0 \tag{1.11}$$

for some function G, where here and hereafter we use subscripts as shorthand for partial derivatives, *i.e.* $u_x = \partial u/\partial x$ etc.

- The most general first-order linear PDE for $u(x, y)$ is an equation of the form

$$a_1 u_x + a_2 u_y + a_3 u = f, \tag{1.12}$$

where a_1, a_2, a_3 and f are given functions of (x, y). The PDE (1.12) is *homogeneous* if $f = 0$ and *inhomogeneous* otherwise. ■

Example: general and linear second-order PDEs

- A second-order PDE for $u(x, y)$ may be written in the form

$$H(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}) = 0 \tag{1.13}$$

for some function H.

- The most general second-order linear PDE for $u(x, y)$ is an equation of the form

$$a_1 u_{xx} + a_2 u_{xy} + a_3 u_{yy} + a_4 u_x + a_5 u_y + a_6 u = f, \tag{1.14}$$

where a_1, a_2, $\ldots$, a_6 and f are given functions of (x, y). The PDE (1.14) is *homogeneous* if $f = 0$ and *inhomogeneous* otherwise. ■

Example: some important PDEs

- There are many important PDEs and we give some examples of them below:

transport equation for $u(x, t)$:	$u_t + xtu_x = 0;$
inviscid Burger's equation for $u(x, t)$:	$u_t + uu_x = 0;$
heat equation for $u(x, t)$:	$u_t = \kappa u_{xx};$
Fisher's equation for $u(x, t)$:	$u_t = \kappa u_{xx} + ru(1 - u);$
viscous Burger's equation for $u(x, t)$:	$u_t + uu_x = \nu u_{xx};$
porous medium equation for $u(x, t)$:	$u_t = (u^m u_x)_x;$
thin-film equation for $u(x, t)$:	$u_t + (u^m u_{xxx})_x = 0;$
wave equation for $u(x, t)$:	$u_{tt} = c^2 u_{xx};$
plate equation for $u(x, t)$:	$u_{tt} + \alpha^2 u_{xxxx} = 0;$
Eikonal equation for $u(x, y)$:	$u_x^2 + u_y^2 = 1;$
Laplace's equation for $u(x, y)$:	$u_{xx} + u_{yy} = 0;$
Poisson's equation for $u(x, y)$:	$u_{xx} + u_{yy} = f(x, y).$

- **Exercise:** What is the order of each PDE? Which are linear and which nonlinear? ■

Example: some more important PDEs

- There are many more important PDEs studied throughout the mathematics course using a range of mathematical techniques, *e.g.*

    - Euler's equations for inviscid fluid flow (A10);
    - Schrodinger's equation for the wave function in quantum mechanics (A11);
    - Euler-Lagrange equations in the calculus of variations (ASO);
    - Navier-Stokes equations for viscous fluid flow (B5.3);
    - Turing's reaction-diffusion equations for pattern formation (B5.5);
    - Maxwell's equations of electromagnetism (B7.2);
    - Black-Scholes' equation for derivative pricing (B8.3).

- Many of these PDEs may be written concisely using the *vector differential operators*, which are introduced in Multivariable Calculus. ■

- The PDEs in the last two examples encode a model of a physical or real-world process and arise as part of the *mathematical modelling process*, which we summarise as follows:

    (1) Start from a *physical* or *real-world* problem.

    (2) Use physical or non-physical principles (*e.g.* Newton's laws, energy conservation, no arbitrage) to translate it into mathematics — this involves developing appropriate mathematical technology (*e.g.* calculus).

    (3) Use empirical laws to derive a soluble mathematical model (*e.g.* Hooke's law or Newton's law of universal gravitation in Dynamics; Fourier's law in heat conduction).

    (4) Solve the mathematical model — again this involves developing mathematical techniques (*e.g.* methods to solve ODEs and PDEs).

    (5) Use mathematical results to make predictions about the real system — usually these can only be sensible if there exists a unique solution to the underling mathematical problem.

- In this course we will illustrate the mathematical modelling process by

    - deriving from physical principles the heat equation, the wave equation and Laplace's equation;
    - using Fourier series methods to construct and analyse solutions to physical problems involving them.

- The physical problems we shall consider will often take the form of

    - *initial boundary value problems* (IBVPs) for the heat and wave equations in which suitable *boundary conditions* and *initial conditions* will need to be prescribed;
    - *boundary value problems* (BVPs) for Laplace's equation in which a suitable *boundary condition* will need to be prescribed.

- In each physical problem we will

    - establish *existence* by constructing explicitly a solution;
    - prove *uniqueness* by showing that the difference between any two solutions much vanish.

  We will demonstrate thereby that we have *correctly specified* the number and form of boundary and/or initial conditions.

- The course will finish with a brief introduction to the notion of *well-posedness* of a PDE problem, which in addition to existence and uniqueness of a solution requires the solution to depend *continuously* on the boundary and/or initial data in some suitable sense.

- We wrap up with an example of an IBVP for the heat equation that illustrates the connection to Fourier series and the practical need to answer the fundamental questions in §1.1.

Example: IBVP for the heat equation

- In a suitably scaled mathematical model for heat conduction along a thin metal wire, the temperature $T(x,t)$ satisfies the heat equation

$$\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2} \quad \text{for} \quad 0 < x < \pi, \ t > 0, \tag{1.15}$$

with the boundary conditions

$$T(0,t) = 0, \quad T(\pi,t) = 0 \quad \text{for} \quad t > 0, \tag{1.16}$$

and the initial condition

$$T(x,0) = \exp(\cos x)\sin(\sin x) \quad \text{for} \quad 0 < x < \pi, \tag{1.17}$$

where x measures distance along the centreline of the wire and t measures time.

- We delay until later on a description of the modelling assumptions underlying the mathematical model that is encoded in the IBVP (1.15)–(1.17).

- We also delay until later on a description of the method that may be used to construct the series solution

$$T(x,t) = \sum_{n=1}^{N} b_n \exp(-n^2 t)\sin(nx) \tag{1.18}$$

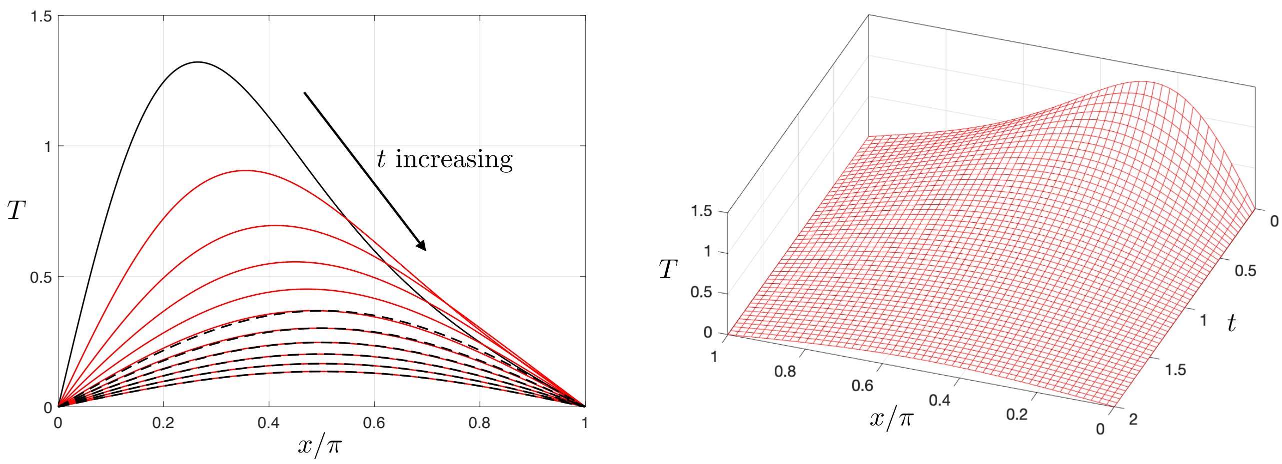
that satisfies (1.15) and (1.16) for all b_1, b_2,..., $b_n \in \mathbb{R}$ and positive integers N, which may be verified by direct substitution. Our focus here is on the following question.

- **Question:** how should we pick N and the constants b_n?

- **Answer:** If we choose $N = \infty$ and $b_n = 1/n!$, then (1.18) satisfies (1.17) by (1.6), so a solution of the IBVP (1.15)–(1.17) would appear to be given by

$$T(x,t) = \sum_{n=1}^{\infty} \frac{1}{n!} \exp(-n^2 t)\sin(nx). \tag{1.19}$$

- On the left below we plot the initial profile (1.17) (black line); snap shots of the series solution (1.19) truncated to 5 terms for $t = 0.2, 0.4, \ldots, 2$ (red lines); and the leading term of the series solution, $\sin(x)\exp(-t)$, for $t = 1, 1.2, \ldots, 2$ (dashed lines). On the right below we plot the series solution (1.19) truncated to 5 terms oriented to give a good view.



- The left-hand plot illustrates that the exponential decay of the truncated series solution is well approximated by its leading term for $t \geq 1$, the higher-order terms decaying exponentially more rapidly because of the factor of $\exp(-n^2 t)/n!$ in the nth-term in (1.19).

- **Question**: But what about other initial conditions? ■

Notes

(1) It is straightforward to check that the infinite series in (1.19) satisfies the boundary conditions (1.16) and initial condition (1.17) by substituting the values $x = 0$, $x = \pi$ and $t = 0$.

(2) Comparison methods from Analysis II may be used to show that for $t > 0$ all of the partial derivatives of the infinite series in (1.19) exist and may be computed by term-by-term differentiation. Since each term in the infinite series in (1.19) satisfies the heat equation (1.15), it follows that so too does their infinite sum. Hence, the infinite series (1.19) is indeed a solution of the IBVP (1.15)–(1.17).

(3) This means that truncating it after a sufficiently large number of terms will result in a good approximation to the solution, *e.g.* 5 terms gives a relative error of less than 0.1% in the plots above.

(4) In this course our focus will be on the formal derivation — via Fourier series methods — of infinite series solutions of the form in (1.19), rather than on addressing the delicate convergence issues that arise when it comes to showing whether or not they actually satisfy the governing partial differential equation. Such issues are of course fundamentally important and so we will comment on them from time to time.

2 Fourier Series

2.1 Periodic, even and odd functions

- The building blocks that form the partial sums of a Fourier series are cosines and sines. Not only are cosines and sines infinitely differentiable on $\mathbb{R}$, their graphs have important periodicity and symmetry properties: cos is an even periodic function, while sin is an odd periodic function. We therefore start with a refresher of what it means for a function to have these properties.

- **Definition:** The function $f : \mathbb{R} \to \mathbb{R}$ is a *periodic function* if there exists $p > 0$ such that

$$f(x+p) = f(x) \quad \text{for all } x \in \mathbb{R}. \tag{2.1}$$

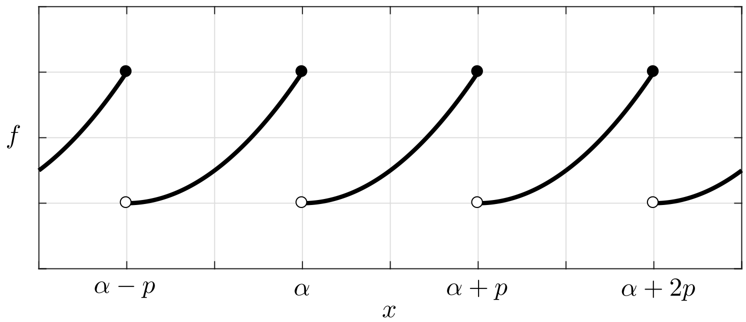
  In this case p is a *period* for f and f is called *p–periodic*. A period is not unique, but if there exists a smallest such p it is called the *prime period*.

- **Notes:**

  (1) If $f(x) = c$ for $x \in \mathbb{R}$, where c is a real constant, then f is a p-periodic function for each $p > 0$, so does not have a prime period.

  (2) Examples of periodic functions are $\cos x$, $\sin x$ with prime period 2π and $\cos(\pi x/L)$, $\sin(\pi x/L)$ with prime period $2L$ for each $L > 0$. Examples of non-periodic functions are x and x^2.

  (3) As illustrated in the figure below the graph of a p-periodic function f repeats every p along the x-axis because it is invariant to the translation $(x, y) \mapsto (x+p, y)$.



- If a function is defined on a half-open interval of length $p > 0$, *i.e.* on $(\alpha, \alpha+p]$ or $[\alpha, \alpha+p)$ for some $\alpha \in \mathbb{R}$, then we can extend it to a unique periodic function by demanding it to be periodic with period p. Formally, we define as follows the periodic extension of such a function.

- **Definition:** The *periodic extension* of the function $f : (\alpha, \alpha+p] \to \mathbb{R}$ is the function $F : \mathbb{R} \to \mathbb{R}$ defined by

$$F(x) = f\big(x - m(x)p\big) \quad \text{for } x \in \mathbb{R}, \tag{2.2}$$

  where, for each $x \in \mathbb{R}$, $m(x)$ is the unique integer such that $x - m(x)p \in (\alpha, \alpha+p]$.

- The following properties follow from the definition of a periodic function and you may find it instructive to interpret them geometrically.

- **Properties of periodic functions:** If f and g are p–periodic, then:

  (1) f, g are np–periodic for all $n \in \mathbb{N} \setminus \{0\}$;

  (2) $\alpha f + \beta g$ are p–periodic for all $\alpha, \beta \in \mathbb{R}$;

  (3) fg is p–periodic;

  (4) $f(\lambda x)$ is p/λ–periodic for all $\lambda > 0$;

  (5) $\int\limits_0^p f(x)\,\mathrm{d}x = \int\limits_\alpha^{\alpha+p} f(x)\,\mathrm{d}x$ for all $\alpha \in \mathbb{R}$.

- **Remark:** The prime period can change or cease to exist when multiplying or summing periodic functions. For example, $\cos x$ and $\sin x$ have prime period 2π, while $\cos^2 x$ and $\sin^2 x$ have prime period π and $\cos^2 x + \sin^2 x = 1$ has no prime period.

- **Definition:** The function $g : \mathbb{R} \to \mathbb{R}$ is *even* if

$$g(x) = g(-x) \quad \text{for all } x \in \mathbb{R}. \tag{2.3}$$

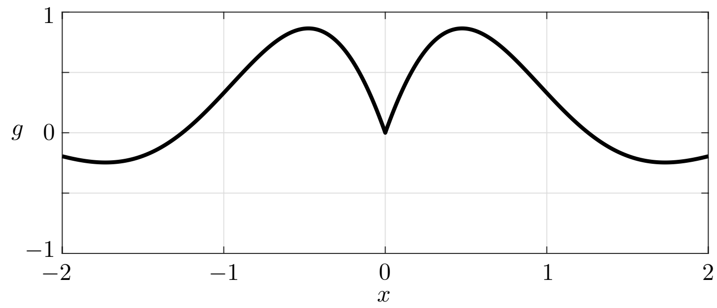
- **Definition:** The function $h : \mathbb{R} \to \mathbb{R}$ is *odd* if

$$h(x) = -h(-x) \quad \text{for all } x \in \mathbb{R}. \tag{2.4}$$

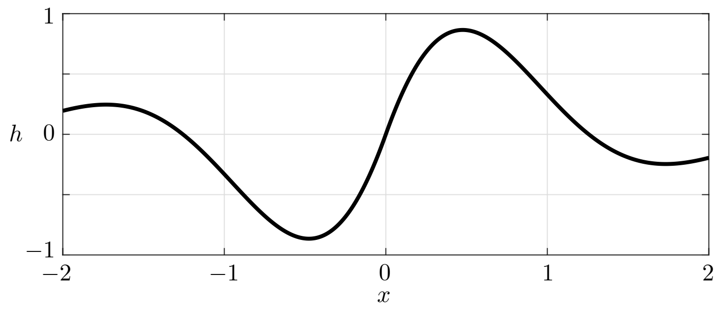
- **Notes:**

  (1) Examples of even functions are x^n for positive even integers n (hence the name even function) and $\cos(\lambda x)$ for $\lambda \in \mathbb{R}$. Examples of odd functions are x^n for positive odd integers n (hence the name odd function) and $\sin(\lambda x)$ for $\lambda \in \mathbb{R}$.

  (2) As illustrated in the figure below the graph of an even function g is symmetric about the y-axis because it is unchanged by a reflection in the y-axis, *i.e.* it is invariant under the transformation $(x, y) \mapsto (-x, y)$.



  (3) As illustrated in the figure below the graph of an odd function h is unchanged by a rotation by π radians about the origin $(x, y) = (0, 0)$, *i.e.* it is invariant under the transformation $(x, y) \mapsto (-x, -y)$; note that this transformation is equivalent to a reflection in the y-axis followed by a reflection in the x-axis, or *vice versa*.



- The following properties of even and odd functions follow from their definitions and again you may find it instructive to interpret geometrically.

- **Properties of even/odd functions:** If g, g_1 are even and h, h_1 are odd, then:

  (1) gg_1 is even, gh is odd, and hh_1 is even;

  (2) $\int\limits_{-\alpha}^{\alpha} g(x)\,\mathrm{d}x = 2\int\limits_{0}^{\alpha} g(x)\,\mathrm{d}x$ for all $\alpha \in \mathbb{R}$;

  (3) $\int\limits_{-\alpha}^{\alpha} h(x)\,\mathrm{d}x = 0$ for all $\alpha \in \mathbb{R}$;

  (4) $h(0) = 0$.

- We finish this section with a proposition.

- **Proposition:** Given a function $f : \mathbb{R} \to \mathbb{R}$ there exist unique functions $g : \mathbb{R} \to \mathbb{R}$ and $h : \mathbb{R} \to \mathbb{R}$ with g even and h odd such that $f(x) = g(x) + h(x)$ for $x \in \mathbb{R}$.

Proof:

- To prove existence note that the following functions have the required properties:

$$g(x) = \frac{1}{2}\big(f(x) + f(-x)\big), \;\; h(x) = \frac{1}{2}\big(f(x) - f(-x)\big) \quad \text{for } x \in \mathbb{R}. \tag{2.5}$$

- To prove uniqueness suppose that $f = g_1 + h_1$ and $f = g_2 + h_2$, with g_1, g_2 even and h_1, h_2 odd; then $g_1 - g_2 = h_2 - h_1$ is both even and odd, and hence must vanish on $\mathbb{R}$. ■

- **Definition:** The function g in (2.5) is the *even part* of f and h the *odd part* of f.

- **Remark:** The proof of uniqueness illustrates a common theme in the elementary uniqueness proofs in this course, namely that of showing the difference between two solutions must vanish.

2.2 Fourier series for functions of period 2π

- Let $f : \mathbb{R} \to \mathbb{R}$ be a periodic function of period 2π. We would like an expansion for f of the form

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \big(a_n \cos(nx) + b_n \sin(nx)\big) \quad \text{for } x \in \mathbb{R}, \tag{2.6}$$

  where a_0, a_1, ... and b_1, b_2, ... are constants.

- Recall the two fundamental questions raised in §1.1:

  Question 1: If (2.6) is true, can we find the constants a_n and b_n in terms of f?

  Question 2: With these a_n and b_n, when is (2.6) true?

- We address the first question in this section and the second in §2.5.

- Suppose (2.6) is true and that we can integrate it term-by-term over a period, so that

$$\int_{-\pi}^{\pi} f(x)\,\mathrm{d}x = \frac{1}{2}a_0 \int_{-\pi}^{\pi} \mathrm{d}x + \sum_{n=1}^{\infty} \left(a_n \int_{-\pi}^{\pi} \cos(nx)\,\mathrm{d}x + b_n \int_{-\pi}^{\pi} \sin(nx)\,\mathrm{d}x \right).$$

- Since, for positive integers n,

$$\int_{-\pi}^{\pi} \mathrm{d}x = 2\pi, \quad \int_{-\pi}^{\pi} \cos(nx)\,\mathrm{d}x = 0, \quad \int_{-\pi}^{\pi} \sin(nx)\,\mathrm{d}x = 0,$$

  we must have

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\,\mathrm{d}x, \tag{2.7}$$

  which determines a_0 in terms of f.

- **Notes:**

  (1) Since f is 2π-periodic we could have integrated over any interval of length 2π.

  (2) The leading term $a_0/2$ in the Fourier series (2.6) is equal to the mean of f over a period.

- In order to determine the higher-order Fourier coefficients we will need the following Lemma.

- **Lemma:** Let m and n be positive integers. Then we have the *orthogonality relations:*

$$\int_{-\pi}^{\pi} \cos(mx)\cos(nx)\,\mathrm{d}x = \pi\delta_{mn}, \tag{2.8}$$

$$\int_{-\pi}^{\pi} \cos(mx)\sin(nx)\,\mathrm{d}x = 0, \tag{2.9}$$

$$\int_{-\pi}^{\pi} \sin(mx)\sin(nx)\,\mathrm{d}x = \pi\delta_{mn}, \tag{2.10}$$

where δ_{mn} is Kronecker's delta defined by

$$\delta_{mn} = \begin{cases} 0 & \text{for } m \neq n, \\ 1 & \text{for } m = n. \end{cases}$$

Proof:

- Since the integrand in (2.8) is even, for $m \neq n$,

$$\begin{aligned}
\int_{-\pi}^{\pi} \cos(mx)\cos(nx)\,\mathrm{d}x &= \int_0^{\pi} 2\cos(mx)\cos(nx)\,\mathrm{d}x \\
&= \int_0^{\pi} \cos\big((m+n)x\big) + \cos\big((m-n)x\big)\,\mathrm{d}x \\
&= \left[\frac{\sin\big((m+n)x\big)}{m+n} + \frac{\sin\big((m-n)x\big)}{m-n}\right]_0^{\pi} \\
&= 0.
\end{aligned}$$

- If $m = n$, we have instead

$$\begin{aligned}
\int_{-\pi}^{\pi} \cos(mx)\cos(nx)\,\mathrm{d}x &= \int_0^{\pi} 2\cos^2(nx)\,\mathrm{d}x \\
&= \int_0^{\pi} 1 + \cos(2nx)\,\mathrm{d}x \\
&= \left[x + \frac{\sin(2nx)}{2n}\right]_0^{\pi} \\
&= \pi.
\end{aligned}$$

- Equation (2.9) follows immediately from the oddness of the integrand. The proof of (2.10) is similar to that of (2.8) and covered on a problem sheet. ■

- Fixing $m \in \mathbb{N} \setminus \{0\}$, multiplying (2.6) by $\cos(mx)$ and assuming that the orders of summation and integration may be interchanged, we obtain

$$\begin{aligned}
\int_{-\pi}^{\pi} f(x)\cos(mx)\,\mathrm{d}x =& \frac{1}{2}a_0 \int_{-\pi}^{\pi} \cos(mx)\,\mathrm{d}x \\
&+ \sum_{n=1}^{\infty} \left(a_n \int_{-\pi}^{\pi} \cos(mx)\cos(nx)\,\mathrm{d}x + b_n \int_{-\pi}^{\pi} \cos(mx)\sin(nx)\,\mathrm{d}x \right).
\end{aligned} \tag{2.11}$$

- Using the orthogonality relations (2.8)–(2.9), we deduce that

$$\int_{-\pi}^{\pi} f(x)\cos(mx)\,\mathrm{d}x = \frac{1}{2}a_0 \cdot 0 + \sum_{n=1}^{\infty} (a_n \pi \delta_{mn} + b_n \cdot 0) = \pi a_m,$$

so that

$$a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(mx)\,\mathrm{d}x \quad \text{for} \quad m \in \mathbb{N}\setminus\{0\}.$$

- **Question:** How would you derive a similar integral expression for b_n?
- **Answer:** By fixing $m \in \mathbb{N}\setminus\{0\}$, multiplying (2.6) by $\sin(mx)$, integrating from $x = -\pi$ to $x = \pi$ and assuming that the orders of summation and integration may be interchanged. As shown on a problem sheet, this gives

$$b_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(mx)\,\mathrm{d}x \quad \text{for} \quad m \in \mathbb{N}\setminus\{0\}.$$

- We wrap these formulae into the following definion.
- **Definition:** Let $f : \mathbb{R} \to \mathbb{R}$ be 2π-periodic and integrable on $[-\pi, \pi]$. Then, regardless of whether or not it converges, the *Fourier series* for f is defined to be the infinite series given by

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \big(a_n \cos(nx) + b_n \sin(nx)\big) \tag{2.12}$$

for $x \in \mathbb{R}$, where the *Fourier coefficients* of f are the constants a_n and b_n given by

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,\mathrm{d}x \quad \text{for } n \in \mathbb{N}, \tag{2.13}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,\mathrm{d}x \quad \text{for } n \in \mathbb{N}\setminus\{0\}. \tag{2.14}$$

- **Notes:**

(1) The integrability condition ensures the existence of the integral expressions (2.13)–(2.14) for the Fourier coefficients, which are sometimes called the *Euler formulae.*

(2) We adopt the short-hand notation

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \big(a_n \cos(nx) + b_n \sin(nx)\big) \tag{2.15}$$

to indicate that the Fourier series for f is given by the right-hand side of this expression regardless of whether or not it converges.

(3) The factor of 1/2 in the first term of the Fourier series (2.12) ensures that the formula (2.13) is the same for all non-negative integers n.

(4) It is readily shown that (2.15) may be written in the equivalent complex form

$$f(x) \sim \sum_{n=-\infty}^{\infty} c_n \mathrm{e}^{\mathrm{i}nx}, \tag{2.16}$$

where it follows from (2.13)–(2.14) that the complex Fourier coefficients c_n are given by

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\mathrm{e}^{-\mathrm{i}nx}\,\mathrm{d}x \quad \text{for } n \in \mathbb{Z}. \tag{2.17}$$

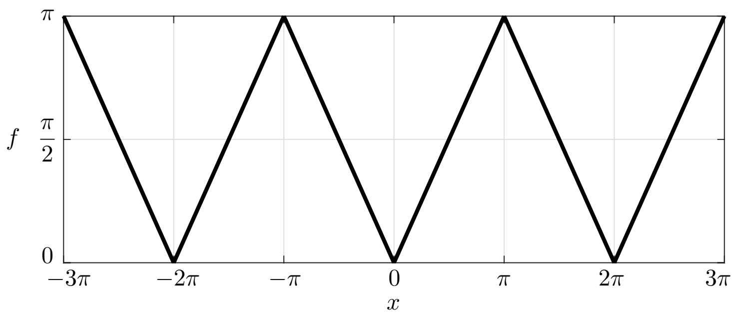
Despite the elegance of this formulation, we will work almost exclusively with (2.12)–(2.14) because it is much better suited to the PDE applications later on in the course.

Example 1

- Find the Fourier series for the 2π-periodic function f defined by

$$f(x) = |x| \quad \text{for} \quad -\pi < x \le \pi. \tag{2.18}$$

- The plot of the graph of f shows that it has a "sawtooth" profile that is piecewise linear and continuous, with corners at integer multiples of π.



- Since $f(x)$ is even, $f(x)\cos(nx)$ is even and $f(x)\sin(nx)$ is odd, so (2.13)–(2.14) give

$$a_n = \frac{2}{\pi}\int_0^\pi f(x)\cos(nx)\,\mathrm{d}x, \quad b_n = 0. \tag{2.19}$$

- For $n = 0$ direct integration gives

$$a_0 = \frac{2}{\pi}\int_0^\pi x\,\mathrm{d}x = \left[\frac{2}{\pi}\frac{x^2}{2}\right]_0^\pi = \pi. \tag{2.20}$$

- For $n \ge 1$ we use integration by parts by making an appropriate choice for the functions u and v in the identity:

$$\left[uv\right]_0^\pi = \int_0^\pi (uv)'\,\mathrm{d}x = \int_0^\pi u'v + uv'\,\mathrm{d}x. \tag{2.21}$$

- Taking $u = x$ and $v = \sin(nx)/n$, we obtain

$$\left[\frac{x}{n}\sin(nx)\right]_0^\pi = \int_0^\pi 1\cdot\frac{1}{n}\sin(nx) + x\cos(nx)\,\mathrm{d}x, \tag{2.22}$$

so that

$$\int_0^\pi x\cos(nx)\,\mathrm{d}x = -\int_0^\pi \frac{1}{n}\sin(nx)\,\mathrm{d}x = \left[\frac{\cos(nx)}{n^2}\right]_0^\pi = \frac{(-1)^n - 1}{n^2}. \tag{2.23}$$

- Since,

$$a_n = -\frac{2}{\pi}\frac{[1-(-1)^n]}{n^2} = \begin{cases} 0 & \text{for } n = 2m,\ m \in \mathbb{N}\setminus\{0\}, \\ -\dfrac{4}{\pi(2m+1)^2} & \text{for } n = 2m+1,\ m \in \mathbb{N}, \end{cases} \tag{2.24}$$

we can write (2.15) in the form

$$f(x) \sim \frac{\pi}{2} - \frac{4}{\pi}\sum_{m=0}^{\infty}\frac{\cos((2m+1)x)}{(2m+1)^2}, \tag{2.25}$$

the right-hand side being the Fourier series for f. ■

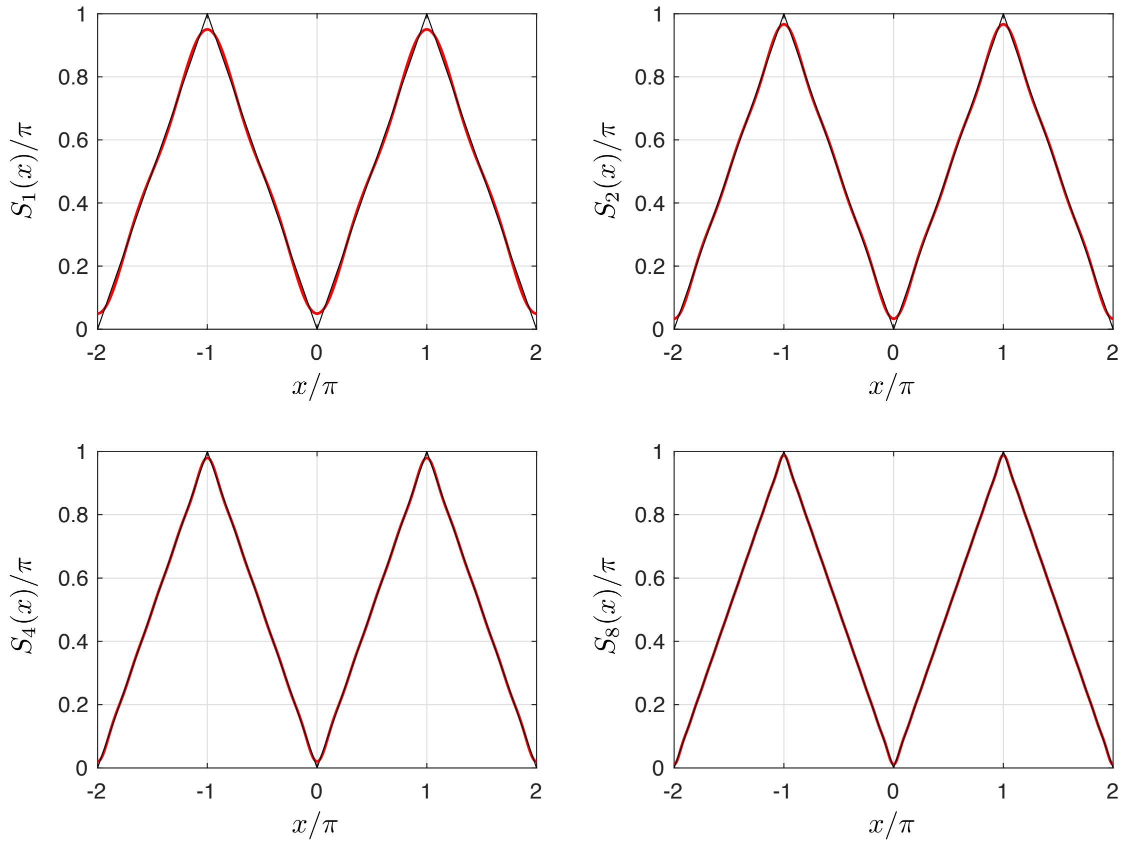
Notes

(1) The partial sums of the Fourier series for f may be defined for $N \in \mathbb{N}$ by

$$S_N(x) = \frac{\pi}{2} - \frac{4}{\pi} \sum_{m=0}^{N} \frac{\cos\left((2m+1)x\right)}{(2m+1)^2} \quad \text{for } x \in \mathbb{R}. \tag{2.26}$$

The plots below show that S_N rapidly approaches f with increasing N, suggesting that the Fourier series converges to f on $\mathbb{R}$, *i.e.*

$$\lim_{N\to\infty} S_N(x) = f(x) \quad \text{for } x \in \mathbb{R}. \tag{2.27}$$



(2) If (2.27) is true, we can pick x to evaluate the sum of a series, *e.g.* $x = 0$ gives

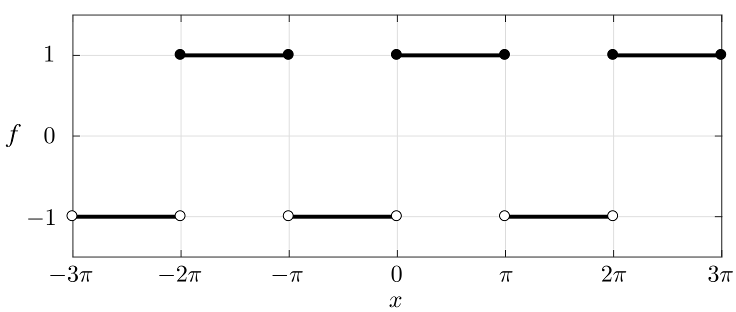
$$0 = \frac{\pi}{2} - \frac{4}{\pi} \sum_{m=0}^{\infty} \frac{1}{(2m+1)^2} \quad \Longrightarrow \quad \sum_{m=0}^{\infty} \frac{1}{(2m+1)^2} = \frac{\pi^2}{8}. \tag{2.28}$$

Example 2

- Find the Fourier Series for the 2π-periodic function f defined by

$$f(x) = \begin{cases} 1 & \text{for } 0 \le x \le \pi, \\ -1 & \text{for } -\pi < x < 0. \end{cases} \tag{2.29}$$

- The plot of the graph of f shows that it has a "square wave" profile that is piecewise linear with jump discontinuities at integer multiples of π.



- Since $f(x)$ is odd for $x/\pi \in \mathbb{R}\backslash\mathbb{Z}$, (2.13)–(2.14) imply

$$a_n = 0, \quad b_n = \frac{2}{\pi}\int_0^\pi f(x)\sin(nx)\,\mathrm{d}x. \tag{2.30}$$

- But $f(x) = 1$ for $0 < x < \pi$, so

$$b_n = \frac{2}{\pi}\int_0^\pi \sin(nx)\,\mathrm{d}x = \left[-\frac{2}{\pi}\frac{\cos(nx)}{n}\right]_0^\pi = \frac{2[1-(-1)^n]}{\pi n}. \tag{2.31}$$

- Hence, setting $n = 2m+1$ to enumerate the non-zero terms, we can write (2.15) in the form

$$f(x) \sim \frac{4}{\pi}\sum_{m=0}^{\infty}\frac{\sin((2m+1)x)}{2m+1}, \tag{2.32}$$

the right-hand side being the Fourier series for f. ■

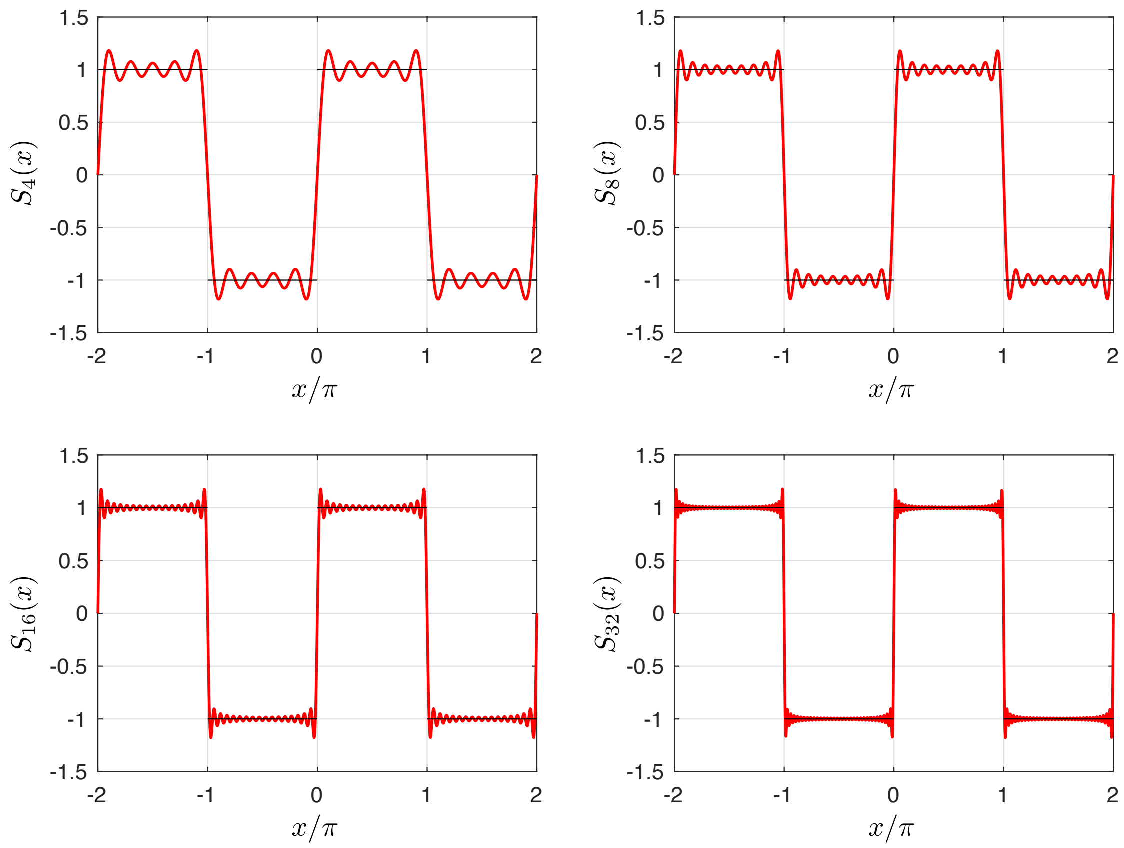
Notes

(1) The partial sums of the Fourier series for f may be defined for $N \in \mathbb{N}$ by

$$S_N(x) = \frac{4}{\pi}\sum_{m=0}^{N}\frac{\sin((2m+1)x)}{2m+1} \qquad \text{for } x \in \mathbb{R}. \tag{2.33}$$

The plots below show that S_N slowly approaches f with increasing N away from the jump discontinuities at which S_N vanishes, suggesting that

$$\lim_{N\to\infty} S_N(x) = \begin{cases} f(x) & \text{for } x/\pi \in \mathbb{R}\backslash\mathbb{Z}, \\ 0 & \text{for } x/\pi \in \mathbb{Z}. \end{cases} \tag{2.34}$$



(2) The convergence is slower than in Example 1 and there is a persistent overshoot near the discontinuities of f — this is called *Gibb's phenomenon*, about which more in §2.7.

2.3 Cosine and sine series

- Let $f : \mathbb{R} \to \mathbb{R}$ be 2π-periodic and integrable on $[-\pi, \pi]$, so that the Fourier coefficients exist.
- In numerous practical applications the relevant function f is even or odd. It is for this reason we chose to integrate from $x = -\pi$ to $x = \pi$ in (2.13) and (2.14), rather than over any other interval of length 2π, since we may then exploit immediately the symmetry of f, as we shall now describe.
- If f is even, then the integrands in (2.13) and (2.14) are even and odd, respectively, so that

$$a_n = \frac{2}{\pi}\int_0^\pi f(x)\cos(nx)\,\mathrm{d}x \quad \text{for } n \in \mathbb{N}, \tag{2.35}$$

$$b_n = 0 \quad \text{for } n \in \mathbb{N}\backslash\{0\}, \tag{2.36}$$

giving

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx), \tag{2.37}$$

i.e. f has a *Fourier cosine series.*

- If f is odd, then the integrands in (2.13) and (2.14) are odd and even, respectively, so that

$$a_n = 0 \quad \text{for } n \in \mathbb{N}, \tag{2.38}$$

$$b_n = \frac{2}{\pi}\int_0^\pi f(x)\sin(nx)\,\mathrm{d}x \quad \text{for } n \in \mathbb{N}\backslash\{0\}, \tag{2.39}$$

giving

$$f(x) \sim \sum_{n=1}^{\infty} b_n \sin(nx), \tag{2.40}$$

i.e. f has a *Fourier sine series.*

- **Remark:** Since the value of an integral is unchanged if the value of its integrand is modified at a finite number of points, we obtain exactly the same Fourier sine series for f as in (2.40) if f is odd on *e.g.* $\mathbb{R}\backslash\{k\pi : k \in \mathbb{Z}\}$, as in Example 2, rather than on the whole of $\mathbb{R}$.

2.4 Tips for evaluating the Fourier coefficients

- The worked examples and problem sheets illustrate a number of useful tips that facilitate the efficient evaluation of the integral expressions for the Fourier coefficients a_n and b_n, while minimizing the chances of making an algebraic slip. The main tips are as follows.

  (1) Exploit as early as possible any simplifications afforded by an integrand being even or odd. This will more or less half the work required.

  (2) When integrating by parts it is usually safer to write down the identity

  $$[uv]_a^b = \int_a^b (uv)'\,\mathrm{d}x = \int_a^b uv' + u'v\,\mathrm{d}x$$

  and make appropriate choices for u, v, a and b, rather than doing the calculation in your head.

  (3) Similarly, when integrating by parts twice it is usually quicker to write down the identity

  $$\left[uv' - u'v\right]_a^b = \int_a^b (uv' - u'v)'\,\mathrm{d}x = \int_a^b uv'' - u''v\,\mathrm{d}x$$

  and make appropriate choices for u, v, a and b, rather than undertaking two sequential integration by parts.

  (4) If f is a piecewise exponential or trigonometric function, it is usually quicker to evaluate the complex integral expression

  $$a_n + \mathrm{i}b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\mathrm{e}^{\mathrm{i}nx}\,\mathrm{d}x.$$

(5) Beware of special cases: do not divide by zero, Such special cases sometimes arise for the same reasons that $m = n$ is a special case in the orthogonality relations.

(6) Check that $a_n \to 0$ and $b_n \to 0$ as $n \to \infty$. This is a direct consequence of the *Riemann-Lebesgue Lemma*, which you will prove in Analysis III. Later on in this course we will be more precise about the rate of decay of the Fourier coefficients as $n \to \infty$.

2.5 Convergence of Fourier series

- **Definition:** The *right-hand limit of f at c* is

$$f(c_+) = \lim_{\substack{h \to 0 \\ h > 0}} f(c + h)$$

if it exists.

- **Definition:** The *left-hand limit of f at c* is

$$f(c_-) = \lim_{\substack{h \to 0 \\ h < 0}} f(c + h)$$

if it exists.

- **Notes:**

(1) $f(c_+)$ can only exist if f is defined on $(c, c + \epsilon)$ for some $\epsilon > 0$.

(2) $f(c_-)$ can only exist if f is defined on $(c - \epsilon, c)$ for some $\epsilon > 0$.

(3) $f(c)$ need not be defined for $f(c_+)$ or $f(c_-)$ to exist.

(4) The existence part is important, *e.g.* if $f(x) = \sin(1/x)$ for $x \neq 0$, then $f(0_\pm)$ do not exist.

(5) f is continuous at c if and only if $f(c_-) = f(c) = f(c_+)$.

(6) In Example 2, f is continuous for $x/\pi \in \mathbb{R}\backslash\mathbb{Z}$ with $f(0_\pm) = \pm 1$ and $f(\pi_\pm) = \mp 1$.

- **Definition:** f is *piecewise continuous* on $(a, b) \subseteq \mathbb{R}$ if there exists a finite number of points $x_1, \ldots, x_m \in \mathbb{R}$ with $a = x_1 < x_2 < \ldots < x_m = b$ such that

(i) f is defined and continuous on (x_k, x_{k+1}) for all $k = 1, \ldots, m - 1$;

(ii) $f(x_{k+})$ exists for $k = 1, \ldots, m - 1$;

(iii) $f(x_{k-})$ exists for $k = 2, \ldots, m$.

- **Notes:**

(1) Note that f need not be defined at its exceptional points $x_1, \ldots, x_m$.

(2) The functions in Examples 1 and 2 are piecewise continuous on any interval $(a, b) \subset \mathbb{R}$.

- **Fourier Convergence Theorem:** Let $f : \mathbb{R} \to \mathbb{R}$ be 2π-periodic, with f and f' piecewise continuous on $(-\pi, \pi)$. Then, the Fourier coefficients

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx)\, \mathrm{d}x \quad (n \in \mathbb{N}), \tag{2.41}$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx)\, \mathrm{d}x \quad (n \in \mathbb{N} \setminus \{0\}) \tag{2.42}$$

exist, and

$$\frac{1}{2}\big(f(x_+) + f(x_-)\big) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \big(a_n \cos(nx) + b_n \sin(nx)\big) \quad \text{for } x \in \mathbb{R}. \tag{2.43}$$

- **Notes on the hypotheses:**

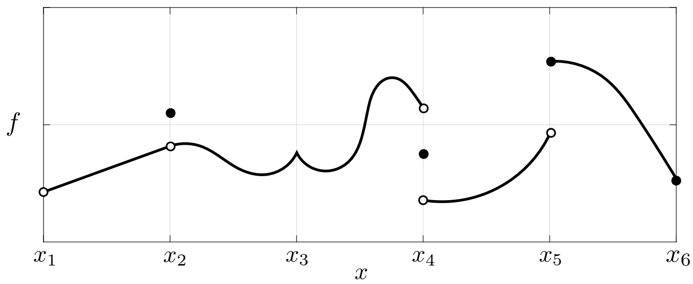
  (1) If f and f' are piecewise continuous on $(-\pi, \pi)$, then there exist $x_1, \ldots, x_m \in \mathbb{R}$ with $-\pi = x_1 < x_2 < \ldots < x_m = \pi$ such that

  (i) f and f' are continuous on (x_k, x_{k+1}) for $k = 1, \ldots, m-1$.

  (ii) $f(x_{k+})$ and $f'(x_{k+})$ exist for $k = 1, \ldots, m-1$.

  (iii) $f(x_{k-})$ and $f'(x_{k-})$ exist for $k = 2, \ldots, m$.

  (2) Thus, in any period f, f' are continuous except possibly at a finite number of points. At each such point f' need not be defined, and one or both of f and f' may have a jump discontinuity, as illustrated for some of the possibilities in the schematic below.



  (3) For example, if

$$f(x) = \begin{cases} x^{1/2} & \text{for } 0 \leq x \leq \pi, \\ 0 & \text{for } -\pi < x < 0, \end{cases} \tag{2.44}$$

  then

$$f'(x) = \begin{cases} \frac{1}{2}x^{-1/2} & \text{for } 0 < x < \pi, \\ 0 & \text{for } -\pi < x < 0, \\ \text{undefined} & \text{for } x = 0, \pi. \end{cases} \tag{2.45}$$

  Hence, while f is piecewise continuous on $(-\pi, \pi)$, f' is not because $f'(0_+)$ does not exist.

- **Notes on the convergence result:**

  (1) The partial sums of the Fourier series are defined for $N \in \mathbb{N}\backslash\{0\}$ by

$$S_N(x) = \frac{a_0}{2} + \sum_{n=1}^{N} \big(a_n \cos(nx) + b_n \sin(nx)\big) \quad \text{for } x \in \mathbb{R}. \tag{2.46}$$

  The theorem states that the partial sums converge pointwise in the sense that

$$\lim_{N\to\infty} S_N(x) = \frac{1}{2}\big(f(x_+) + f(x_-)\big) \quad \text{for } x \in \mathbb{R}. \tag{2.47}$$

  (2) If f has a jump discontinuity at x, so that $f(x_+) \neq f(x_-)$, then the Fourier series converges to $\big(f(x_+) + f(x_-)\big)/2$, *i.e.* the average of the left- and right-hand limits of f at x.

  (3) If f is continuous at x, then $f(x_-) = f(x) = f(x_+)$ and the Fourier series converges to $f(x)$.

  (4) If we redefined f to be equal to the average of its left- and right-hand limits at each of its jump discontinuities, then the Fourier series would converge instead to f on $\mathbb{R}$.

  (5) If f is defined only on *e.g.* $(-\pi, \pi]$, then the Fourier Convergence Theorem holds for its 2π-periodic extension.

  (6) The Fourier Convergence Theorem implies that

$$\frac{1}{2}\big(g(x_+) + g(x_-)\big) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) \quad \text{for} \quad x \in \mathbb{R}, \tag{2.48}$$

$$\frac{1}{2}\big(h(x_+) + h(x_-)\big) = \sum_{n=1}^{\infty} b_n \sin(nx) \quad \text{for} \quad x \in \mathbb{R}, \tag{2.49}$$

  where $g : \mathbb{R} \to \mathbb{R}$ is the even part of f and $h : \mathbb{R} \to \mathbb{R}$ is the odd part of f defined in (2.5).

- **Notes on the proof:** While the proof is not examinable, it is amenable to methods from Prelims Analysis as follows.

  (1) Use the integral expressions for the Fourier coefficients and properties of periodic, even and odd functions to manipulate the partial sums into the form

$$S_N(x) - \frac{1}{2}\big(f(x_+) + f(x_-)\big) = \int_0^\pi F(x,t)\sin\left[\left(N+\frac{1}{2}\right)t\right] \mathrm{d}t, \tag{2.50}$$

  where

$$F(x,t) = \frac{1}{\pi}\left(\frac{f(x+t)-f(x_+)}{t} + \frac{f(x-t)-f(x_-)}{t}\right)\left(\frac{t}{2\sin(t/2)}\right). \tag{2.51}$$

  (2) Use the *Mean Value Theorem* (of Analysis II) to show that $F(x,t)$ is a piecewise continuous function of t on $(0,\pi)$, and hence deduce from the *Riemann-Lebesgue Lemma* (of Analysis III) that

$$\int_0^\pi F(x,t)\sin\left[\left(N+\frac{1}{2}\right)t\right] \mathrm{d}t \to 0 \quad \text{as } N\to\infty. \tag{2.52}$$

- **Notes on differentiability and integrability:**

  (1) The Fourier series can be integrated termwise under weaker conditions, e.g. if f is 2π-periodic and piecewise continuous on $(-\pi,\pi)$, then the Fourier Convergence Theorem implies

$$\int_0^x f(s)\,\mathrm{d}s = \int_0^x \frac{1}{2}a_0\,\mathrm{d}s + \sum_{n=1}^\infty \int_0^x \big(a_n\cos(ns) + b_n\sin(ns)\big)\,\mathrm{d}s \quad \text{for } x\in\mathbb{R}, \tag{2.53}$$

  this function being 2π-periodic if and only if $a_0 = 0$.

  (2) However, we need stronger conditions to differentiate termwise, *e.g.* if f is 2π-periodic and continuous on $\mathbb{R}$ with both f' and f'' piecewise continuous on $(-\pi,\pi)$, then the Fourier Convergence Theorem implies

$$\frac{1}{2}\big(f'(x_+) + f'(x_-)\big) = \sum_{n=1}^\infty \frac{\mathrm{d}}{\mathrm{d}x}\big(a_n\cos(nx) + b_n\sin(nx)\big) \quad \text{for } x\in\mathbb{R}. \tag{2.54}$$

Examples 1 and 2 revisited

- Recall the 2π-periodic function of Example 1 which we defined by setting

$$f(x) = |x| \quad \text{for } -\pi < x \le \pi. \tag{2.55}$$

- We calculate

$$f'(x) = \begin{cases} 1 & \text{for } 0 < x < \pi, \\ -1 & \text{for } -\pi < x < 0, \\ \text{undefined} & \text{for } x = 0,\ \pi. \end{cases} \tag{2.56}$$

- Since both f and f' are piecewise continuous on $(-\pi,\pi)$, with f continuous on $\mathbb{R}$, the Fourier Convergence Theorem gives

$$\frac{\pi}{2} - \frac{4}{\pi}\sum_{m=0}^\infty \frac{\cos((2m+1)x)}{(2m+1)^2} = f(x) \quad \text{for} \quad x\in\mathbb{R}. \tag{2.57}$$

- Since f is piecewise continuous on $(-\pi,\pi)$, we can integrate termwise to obtain

$$\frac{4}{\pi}\sum_{m=0}^\infty \frac{\sin((2m+1)x)}{(2m+1)^3} = \int_0^x f(s) - \frac{\pi}{2}\,\mathrm{d}s \quad \text{for} \quad x\in\mathbb{R}. \tag{2.58}$$

- We calculate

$$f''(x) = \begin{cases} 0 & \text{for } 0 < x < \pi, \\ 0 & \text{for } -\pi < x < 0, \\ \text{undefined} & \text{for } x = 0,\ \pi. \end{cases} \tag{2.59}$$

- Since f is continuous on $\mathbb{R}$ and both f' and f'' are piecewise continuous on $(-\pi, \pi)$, we can differentiate termwise the Fourier series (2.57) for f to obtain

$$\frac{4}{\pi} \sum_{m=0}^{\infty} \frac{\sin\left((2m+1)x\right)}{2m+1} = \frac{1}{2}\big(f'(x_-) + f'(x_+)\big) = \begin{cases} 1 & \text{for } 0 < x < \pi, \\ -1 & \text{for } -\pi < x < 0, \\ 0 & \text{for } x = 0,\ \pi. \end{cases} \tag{2.60}$$

- The function to which this Fourier series converges is equal to the function considered in Example 2 for $x/\pi \in \mathbb{R}\backslash\mathbb{Z}$, which deals thereby with the convergence and termwise integration of the Fourier series of that function; it remains to note that, since that function is not continuous on $\mathbb{R}$, its Fourier series cannot be differentiated termwise.

2.6 Rate of convergence

- The smoother f, *i.e.* the more continuous derivatives it has, the faster the convergence of the Fourier series for f.
- If the first jump discontinuity is in the p^{th} derivative of f, with the convention that $p = 0$ if there is a jump discontinuity in f, then in general the slowest decaying a_n and b_n decay like $1/n^{p+1}$ as $n \to \infty$.
- More specifically, if the first jump discontinuity is in the p^{th} derivative of the even part of f, then in general a_n decays like $1/n^{p+1}$ as $n \to \infty$; similarly, if the first jump discontinuity is in the p^{th} derivative of the odd part of f, then in general b_n decays like $1/n^{p+1}$ as $n \to \infty$.
- For example, $p = 1$ in (2.57), $p = 2$ in (2.58) and $p = 0$ in (2.60).
- This is an extremely useful result in practice (*e.g.* for approximately 1% accuracy we need 100 terms for $p = 0$, but only 10 terms for $p = 1$) and for checking calculations (*e.g.* an erroneous contribution to a Fourier coefficient can be rapidly identified if it does not decay fast enough for large n).
- We can understand the rate of decay using integration by parts, as follows. Suppose that (i) the first jump discontinuity is in the p^{th} derivative of f with jumps at the exceptional points $x_1 < x_2 < \cdots < x_m$, where $x_1 \geq x_0 = -\pi$ and $x_m \leq x_{m+1} = \pi$; (ii) $f^{(p+1)}(x)$ is integrable on each of the intervals (x_k, x_{k+1}) for $k = 0, 1, \ldots, m$, which is often the case in practice.
- Then, repeated integration by parts gives

$$\begin{aligned}
\pi\big(a_n + \mathrm{i}b_n\big) &= \int_{-\pi}^{\pi} f(x)\mathrm{e}^{\mathrm{i}nx}\,\mathrm{d}x \\
&= \frac{1}{\mathrm{i}n}\left(\big[f(x)\mathrm{e}^{\mathrm{i}nx}\big]_{-\pi}^{\pi} - \int_{-\pi}^{\pi} f^{(1)}(x)\mathrm{e}^{\mathrm{i}nx}\,\mathrm{d}x\right) \\
&= \frac{-1}{\mathrm{i}n}\int_{-\pi}^{\pi} f^{(1)}(x)\mathrm{e}^{\mathrm{i}nx}\,\mathrm{d}x \\
&\ \vdots \\
&= \frac{(-1)^p}{(\mathrm{i}n)^p}\int_{-\pi}^{\pi} f^{(p)}(x)\mathrm{e}^{\mathrm{i}nx}\,\mathrm{d}x \\
&= \frac{(-1)^p}{(\mathrm{i}n)^p}\sum_{k=0}^{m}\int_{x_k}^{x_{k+1}} f^{(p)}(x)\mathrm{e}^{\mathrm{i}nx}\,\mathrm{d}x \\
&= \frac{(-1)^p}{(\mathrm{i}n)^{p+1}}\sum_{k=0}^{m}\left(\big[f^{(p)}(x)\mathrm{e}^{\mathrm{i}nx}\big]_{x_k}^{x_{k+1}} - \int_{x_k}^{x_{k+1}} f^{(p+1)}(x)\mathrm{e}^{\mathrm{i}nx}\,\mathrm{d}x\right).
\end{aligned} \tag{2.61}$$

- The Riemann-Lebesgue Lemma implies that each of the integrals in the sum in (2.61) tend to zero as $n \to \infty$. Since the pth-derivative $f^{(p)}(x)$ has jump discontinuities at the exceptional points, the sum in (2.61) is non-zero and finite as $n \to \infty$ in general. Hence, we recover the claimed rate of decay.

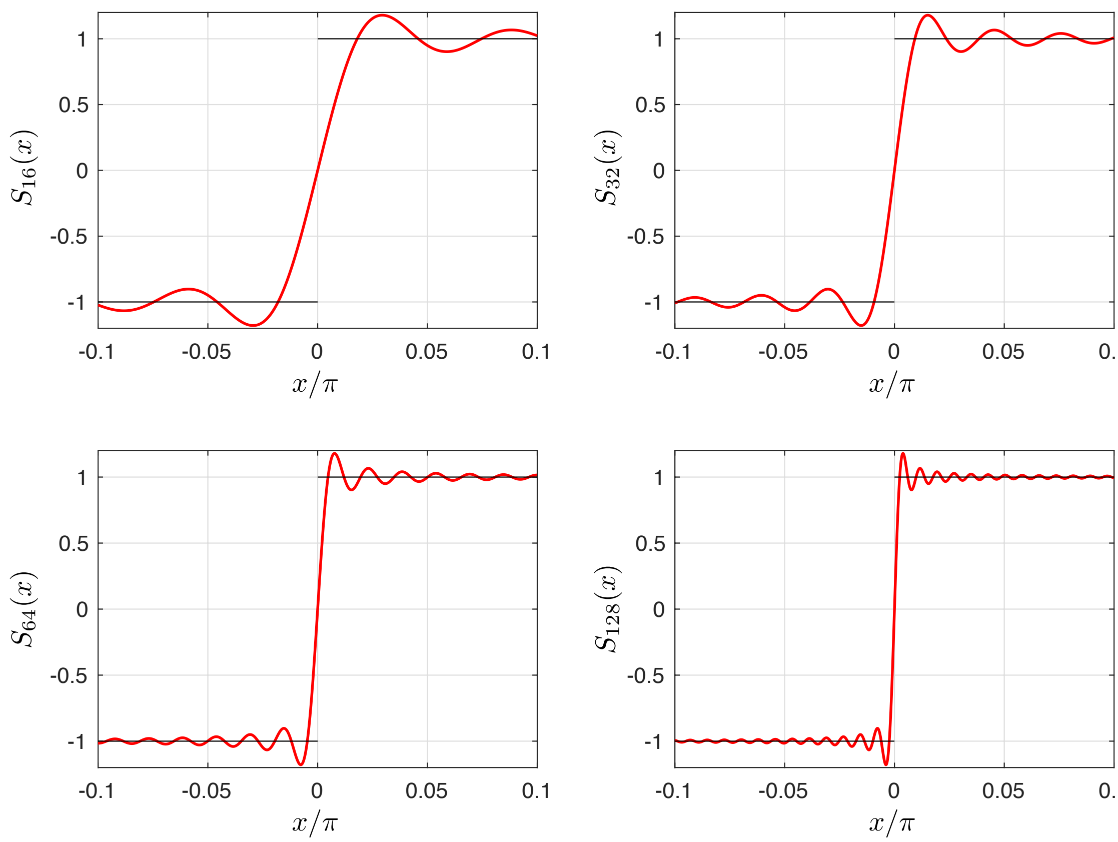
- If the Fourier coefficients decay like $1/n^{p+1}$ as $n \to \infty$ with $p \geq 1$, then the *Weierstrass M-test* of Analysis II may be used to show that the Fourier series for f *converges uniformly* to f on any interval $(a, b) \subset \mathbb{R}$.

- If the Fourier coefficients decay like $1/n$ as $n \to \infty$ (so that $p = 0$), then the partial sums of the Fourier series for f do not converge uniformly on any interval containing a jump discontinuity. Remarkably, the form of the non-uniformity is universal for such functions, being characterized by Gibb's phenomenon, as we shall now describe.

2.7 Gibb's phenomenon

- *Gibb's phenomenon* is the persistent overshoot near a jump discontinuity that we first encountered in Example 2. It happens whenever there a jump discontinuity.

- In the plots below of the partial sums (2.33) from Example 2, we have zoomed into near the jump discontinuity at the origin to illustrate the so-called "ringing" nature of the overshoot as the number of terms in the partial sum is increased.



- More generally, as the number of terms in the partial sum tends to infinity:

  - the width of the overshoot region tends to zero by the Fourier Convergence Theorem;
  - it may be shown that the total height of the overshoot region approaches $\gamma|f(x_+) - f(x_-)|$, where

$$\gamma = \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{\sin x}{x} \, \mathrm{d}x \approx 1.18, \tag{2.62}$$

  i.e. approximately a 9% overshoot top and bottom. The plots above illustrate the approach to this value, which is evidently awful for approximation purposes.

- Some geometric insight into the underlying cause of Gibb's phenomenon may be gleamed from the following manipulation of the partial sums S_N of the Fourier series for f defined in (2.46).

- We begin by substituting the integral expressions (2.41) and (2.42) for the Fourier coefficients a_n and b_n into the partial sum S_N and interchanging the orders of summation and integration, which allows it to be manipulated as follows:

$$\begin{aligned} S_N(x) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,\mathrm{d}t + \sum_{n=1}^{N}\left(\frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(nt)\,\mathrm{d}t\cos(nx) + \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(nt)\,\mathrm{d}t\sin(nx)\right) \\ &= \int_{-\pi}^{\pi} f(t)\left(\frac{1}{2\pi} + \frac{1}{\pi}\sum_{n=1}^{N}\big(\cos(nt)\cos(nx) + \sin(nt)\sin(nx)\big)\right)\mathrm{d}t \\ &= \int_{-\pi}^{\pi} f(t)\left(\frac{1}{2\pi} + \frac{1}{\pi}\sum_{n=1}^{N}\cos\big(n(t-x)\big)\right)\mathrm{d}t \end{aligned}$$

so that

$$S_N(x) = \int_{-\pi}^{\pi} f(t)D_N(t-x)\,\mathrm{d}t, \tag{2.63}$$

where the function $D_N : \mathbb{R} \to \mathbb{R}$ is defined by

$$D_N(t) = \frac{1}{2\pi} + \frac{1}{\pi}\sum_{n=1}^{N}\cos(nt) \quad \text{for } t \in \mathbb{R}. \tag{2.64}$$

- The integral in (2.63) is a *convolution integral* giving the mean of the function $f(t)$ over a period weighted by the *Dirichlet kernel* $D_N(t-x)$. Since D_N does not depend on f it encodes the operation of taking a partial sum of a Fourier series.

- For each positive integer N it follows from (2.64) that D_N is an even 2π-periodic function that is infinitely differentiable on $\mathbb{R}$ and has integral over a period equal to unity, *i.e.*

$$\int_{-\pi}^{\pi} D_N(t)\,\mathrm{d}t = 1. \tag{2.65}$$

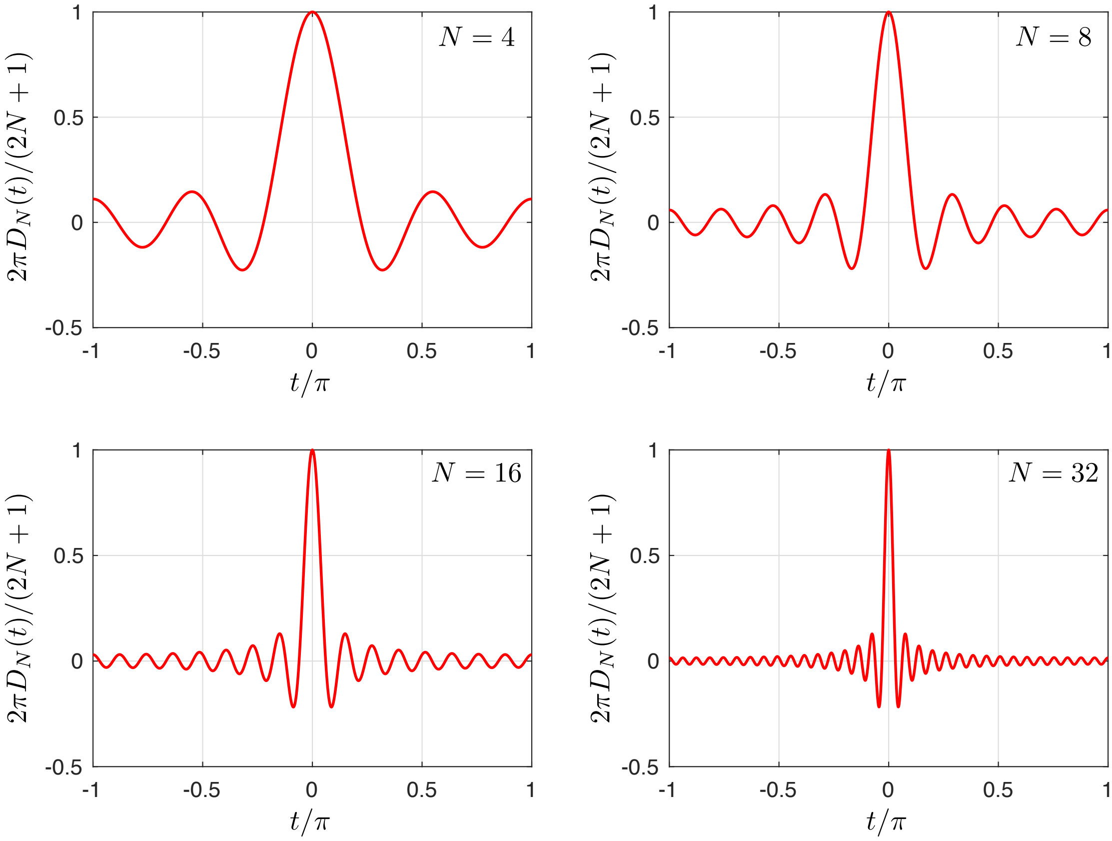
- Using a trigonometric identity we compute

$$\begin{aligned} 2\pi\sin(t/2)D_N(t) &= \sin(t/2) + \sum_{n=1}^{N} 2\cos(nt)\sin(t/2) \\ &= \sin(t/2) + \sum_{n=1}^{N}\Big(\sin\big((n+1/2)t\big) - \sin\big((n-1/2)t\big)\Big) \\ &= \sin\big((N+1/2)t\big), \end{aligned}$$

the last equality following from the fact that the preceding sum is telescoping. Hence,

$$D_N(t) = \begin{cases} \dfrac{\sin\big((N+1/2)t\big)}{2\pi\sin(t/2)} & \text{for } \dfrac{t}{2\pi} \in \mathbb{R}\backslash\mathbb{Z}. \\[2ex] \dfrac{2N+1}{2\pi} & \text{for } \dfrac{t}{2\pi} \in \mathbb{Z}. \end{cases} \tag{2.66}$$

- We plot below the graph of D_N for $N = 4$, 8, 16 and 32, illustrating that as $N \to \infty$ the main contribution of the integrand in (2.65) comes from the central lobe that lies above the interval $[-\pi,\pi]/(N+1/2)$.



- When x nears a jump discontinuity of f, it is the interaction of this jump and the rapidly oscillating Dirichlet kernel around its dominant central lobe in the convoluton integral (2.63) that results in Gibb's phenomenon or the so-called "ringing of the partial sums," with the structure of the central lobe causing the 9% overshoot as $N \to \infty$.

2.8 Functions of any period

- Suppose now $f : \mathbb{R} \to \mathbb{R}$ is a periodic function of period $2L$, where L is a positive number, not necessarily equal to π.

- We want to develop the analogous results for the Fourier series for $f(x)$. Since this will involve a series in the trigonometric functions $\cos(n\pi x/L)$ and $\sin(n\pi x/L)$, where n is a positive integer, we make the transformation

$$x = \frac{LX}{\pi}, \quad f(x) = g(X) \tag{2.67}$$

  which defines a new function $g : \mathbb{R} \to \mathbb{R}$.

- It follows that, for $X \in \mathbb{R}$,

$$g(X + 2\pi) = f\left(\frac{L}{\pi}(X + 2\pi)\right) = f\left(\frac{LX}{\pi} + 2L\right) = f\left(\frac{LX}{\pi}\right) = g(X),$$

  where we used the fact that $g(X) = f(LX/\pi)$ in the first equality and the fact that f is $2L$-periodic in the third equality.

- Hence, g is periodic with period 2π, and we can therefore use the transformation (2.67) to derive the Fourier theory for f from that for g in §2.2 — §2.7.

- In particular, if we can write

$$g(X) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \big(a_n \cos(nX) + b_n \sin(nX)\big)$$

  so that the Fourier coefficients a_n and b_n exist, then

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} g(X)\cos(nX)\,\mathrm{d}X = \frac{1}{\pi}\int_{-L}^{L} g\left(\frac{\pi x}{L}\right)\cos\left(\frac{n\pi x}{L}\right)\frac{\pi}{L}\,\mathrm{d}x = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)\,\mathrm{d}x,$$

and

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} g(X)\sin(nX)\,\mathrm{d}X = \frac{1}{\pi}\int_{-L}^{L} g\left(\frac{\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)\frac{\pi}{L}\,\mathrm{d}x = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)\,\mathrm{d}x.$$

- So if we can write

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\left(\frac{n\pi x}{L}\right) + b_n\sin\left(\frac{n\pi x}{L}\right)\right),$$

  then

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)\,\mathrm{d}x, \qquad b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)\,\mathrm{d}x.$$

- We wrap these formal calculations into the definition of the Fourier series for f.
- **Definition:** Let $f : \mathbb{R} \to \mathbb{R}$ be $2L$-periodic and integrable on $[-L, L]$. Then, regardless of whether or not it converges, the *Fourier series* for f is defined to be the infinite series given by

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\left(\frac{n\pi x}{L}\right) + b_n\sin\left(\frac{n\pi x}{L}\right)\right) \tag{2.68}$$

  for $x \in \mathbb{R}$, where the *Fourier coefficients* of f are given by

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)\,\mathrm{d}x \quad (n \in \mathbb{N}), \tag{2.69}$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)\,\mathrm{d}x \quad (n \in \mathbb{N}\setminus\{0\}). \tag{2.70}$$

- **Remark:** The formulae for the Fourier coefficients may also be derived from the Fourier series for f by assuming that the orders of summation and integration may be interchanged and using the orthogonality relations

$$\int_{-L}^{L} \cos\left(\frac{m\pi x}{L}\right)\cos\left(\frac{n\pi x}{L}\right)\,\mathrm{d}x = L\delta_{mn},$$

$$\int_{-L}^{L} \cos\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)\,\mathrm{d}x = 0,$$

$$\int_{-L}^{L} \sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)\,\mathrm{d}x = L\delta_{mn},$$

  where $n, m \in \mathbb{N}\setminus\{0\}$ and δ_{mn} is Kronecker's delta.
- We are now in a position to write down the corresponding Fourier Convergence Theorem.
- **Fourier Convergence Theorem:** Let $f : \mathbb{R} \to \mathbb{R}$ be $2L$-periodic, with f and f' piecewise continuous on $(-L, L)$. Then, the Fourier coefficients

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)\,\mathrm{d}x \quad (n \in \mathbb{N}), \tag{2.71}$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)\,\mathrm{d}x \quad (n \in \mathbb{N}\setminus\{0\}) \tag{2.72}$$

  exist, and

$$\frac{1}{2}\big(f(x_+) + f(x_-)\big) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\left(\frac{n\pi x}{L}\right) + b_n\sin\left(\frac{n\pi x}{L}\right)\right) \quad \text{for} \quad x \in \mathbb{R}. \tag{2.73}$$

- **Remark:** The notes following the Fourier Convergence Theorem for functions of period 2π in §2.5 apply with appropriate modifications to functions of period $2L$.

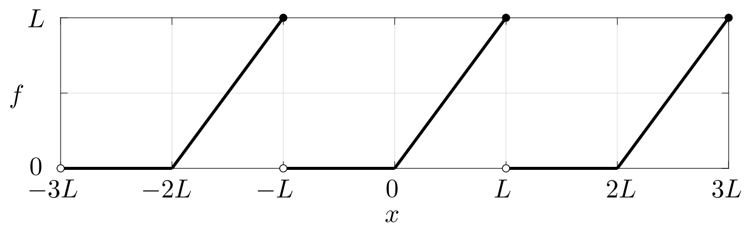
Example 3

- Consider the $2L$-periodic function f defined by

$$f(x) = \begin{cases} x & \text{for } 0 \le x \le L, \\ 0 & \text{for } -L < x < 0. \end{cases} \tag{2.74}$$

  Find the Fourier series for f and the function to which the Fourier series converges.

- The plot of the graph of f shows that it is piecewise linear with corners as $x = 2kL$ for $k \in \mathbb{Z}$ and jump discontinuities at $x = (2k+1)L$ for $k \in \mathbb{Z}$.



- By the definition of f, the Fourier coefficients are given by

$$a_n = \frac{1}{L}\int_0^L x \cos\left(\frac{n\pi x}{L}\right) \mathrm{d}x, \quad b_n = \frac{1}{L}\int_0^L x \sin\left(\frac{n\pi x}{L}\right) \mathrm{d}x. \tag{2.75}$$

- A direct integration gives $a_0 = L/2$, but for $n \ge 1$ it is a bit quicker to evaluate

$$\begin{aligned} a_n + ib_n &= \frac{1}{L}\int_0^L \underbrace{x}_{u} \underbrace{\exp\left(\frac{in\pi x}{L}\right)}_{v'} \mathrm{d}x \\ &= \left[\frac{1}{L} \underbrace{x}_{u} \underbrace{\frac{L}{in\pi}\exp\left(\frac{in\pi x}{L}\right)}_{v}\right]_0^L - \frac{1}{L}\int_0^L \underbrace{1}_{u'} \underbrace{\frac{L}{in\pi}\exp\left(\frac{in\pi x}{L}\right)}_{v} \mathrm{d}x \\ &= -\left[\frac{1}{L}\left(\frac{L}{in\pi}\right)^2 \exp\left(\frac{in\pi x}{L}\right)\right]_0^L + \frac{L}{in\pi}\exp(in\pi) \\ &= \frac{L}{n^2\pi^2}\left((-1)^n - 1\right) + \frac{iL(-1)^{n+1}}{n\pi}. \end{aligned}$$

- Hence,

$$f(x) \sim \frac{L}{4} + \sum_{m=1}^{\infty}\left(-\frac{2L}{(2m-1)^2\pi^2}\cos\left(\frac{(2m-1)\pi x}{L}\right) + \frac{L(-1)^{m+1}}{m\pi}\sin\left(\frac{m\pi x}{L}\right)\right).$$

- Since f and f' are piecewise continuous on $(-L, L)$, the Fourier Convergence Theorem implies that the Fourier series for f converges to $f(x)$ at points of continuity of f, *i.e.* for $x \neq (2k+1)L$, $k \in \mathbb{Z}$, while at the jump discontinuities the Fourier series converges to the average of the left- and right-hand limits of f, *i.e.* to $\big(f(L_+) + f(L_-)\big)/2 = (0+L)/2 = L/2$ at $x = L$ and hence at $x = (2k+1)L$, $k \in \mathbb{Z}$ by periodicity. ■

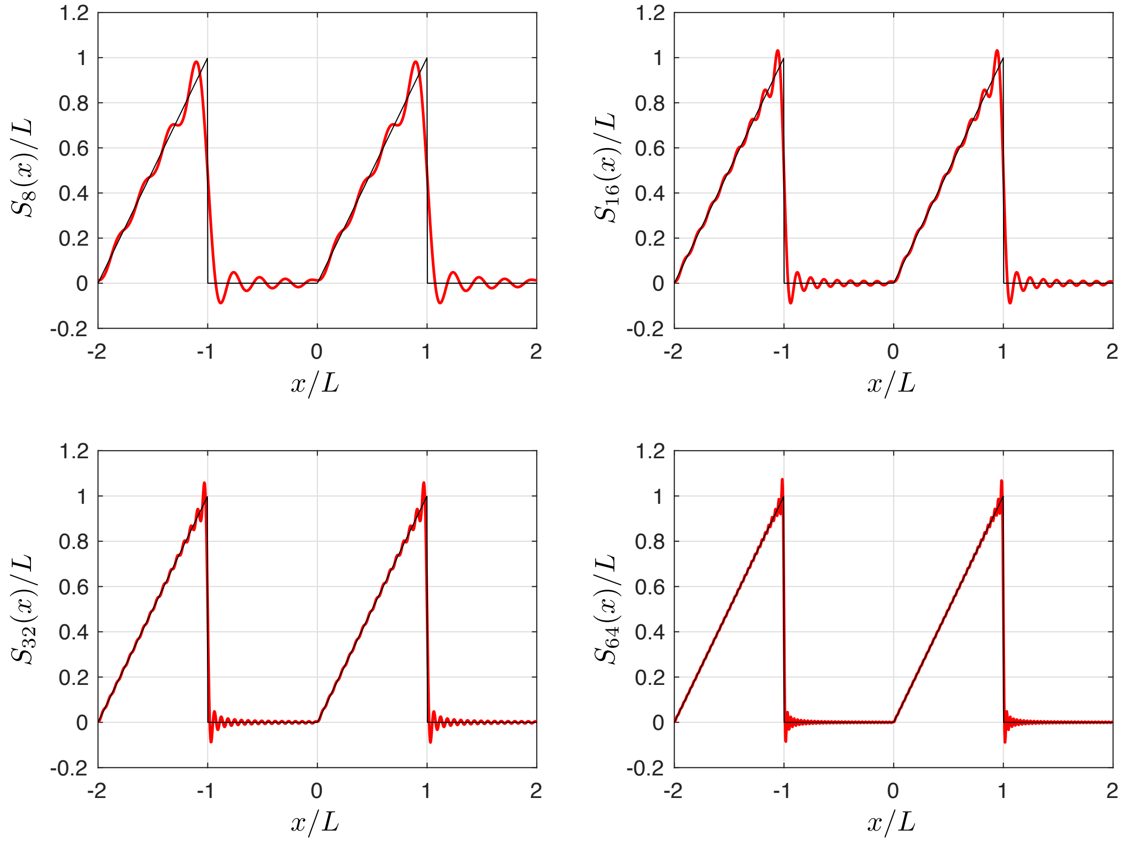
Notes

(1) The slowest decaying Fourier coefficients b_n decay as expected like $1/n$ as $n \to \infty$ because f has jump discontinuities so that $p = 0$.

(2) The partial sums of the Fourier series for f may be defined for positive integers N by

$$S_N(x) = \frac{L}{4} + \sum_{m=1}^{N} \left(-\frac{2L}{(2m-1)^2\pi^2} \cos\left(\frac{(2m-1)\pi x}{L}\right) + \frac{L(-1)^{m+1}}{m\pi} \sin\left(\frac{m\pi x}{L}\right) \right) \quad \text{for } x \in \mathbb{R}.$$

We plot below the partial sums for $N = 8$, 16, 32 and 64, which illustrates that the slow convergence away from the jump discontinuities of f is hindered by Gibb's phenomenon.



2.9 Half-range series

- In many practical applications we wish to express a given function $f : [0, L] \to \mathbb{R}$ in terms of either a Fourier cosine series or a Fourier sine series.
- This may be accomplished by extending f to be even (for only cosine terms) or odd (for only sine terms) on $(-L, 0) \cup (0, L)$ and then extending to a periodic function of period $2L$.
- We wrap these extensions and the corresponding Fourier series into the following definitions.
- **Definition:** The *even $2L$-periodic extension* $f_e : \mathbb{R} \to \mathbb{R}$ of $f : [0, L] \to \mathbb{R}$ is defined by

$$f_e(x) = \begin{cases} f(x) & \text{for } 0 \le x \le L, \\ f(-x) & \text{for } -L < x < 0, \end{cases} \tag{2.76}$$

with $f_e(x + 2L) = f_e(x)$ for $x \in \mathbb{R}$. The *Fourier cosine series* for $f : [0, L] \to \mathbb{R}$ is the Fourier series for f_e, *i.e.*

$$f_e(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right), \tag{2.77}$$

where

$$a_n = \frac{1}{L}\int_{-L}^{L} f_e(x)\cos\left(\frac{n\pi x}{L}\right)\mathrm{d}x = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)\mathrm{d}x \quad \text{for } n \in \mathbb{N}. \tag{2.78}$$

- **Definition:** The *odd $2L$-periodic extension* $f_o : \mathbb{R} \to \mathbb{R}$ of $f : [0, L] \to \mathbb{R}$ is defined by

$$f_o(x) = \begin{cases} f(x) & \text{for } 0 \leq x \leq L, \\ -f(-x) & \text{for } -L < x < 0, \end{cases} \tag{2.79}$$

  with $f_o(x + 2L) = f_o(x)$ for $x \in \mathbb{R}$. The *Fourier sine series* for $f : [0, L] \to \mathbb{R}$ is the Fourier series for f_o, *i.e.*

$$f_o(x) \sim \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right), \tag{2.80}$$

  where

$$b_n = \frac{1}{L}\int_{-L}^{L} f_e(x) \sin\left(\frac{n\pi x}{L}\right) \mathrm{d}x = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) \mathrm{d}x \quad \text{for } n \in \mathbb{N}\backslash\{0\}. \tag{2.81}$$

- **Notes:**

  (1) $f_o(x)$ is odd for $x/L \in \mathbb{R}\backslash\mathbb{Z}$ and odd (on $\mathbb{R}$) if and only if $f(0) = f(L) = 0$.

  (2) If f is continuous on $[0, L]$ and f' piecewise continuous on $(0, L)$, then the Fourier Convergence Theorem implies that

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) = f_e(x) \text{ for } x \in \mathbb{R},$$

$$\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) = \begin{cases} f_o(x) & \text{for } x/L \in \mathbb{R}\backslash\mathbb{Z}, \\ 0 & \text{for } x/L \in \mathbb{R}\backslash\mathbb{Z}. \end{cases}$$

Example 4

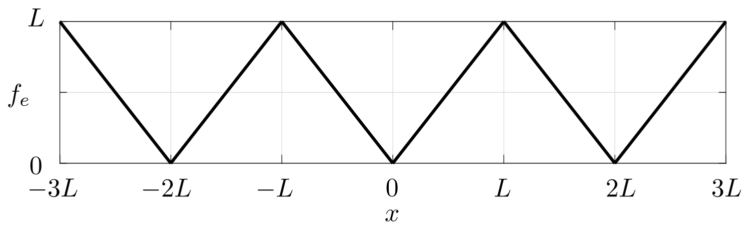
- Consider the function $f : [0, L] \to \mathbb{R}$ defined by $f(x) = x$ for $0 \leq x \leq L$. Find the Fourier cosine and sine series for f and the functions to which each of them converge on $[0, L]$. Which truncated series gives the best approximation to f on $[0, L]$?

- The even $2L$-periodic extension f_e is defined by

$$f_e(x) = \begin{cases} x & \text{for } 0 \leq x \leq L, \\ -x & \text{for } -L < x < 0, \end{cases} \tag{2.82}$$

  i.e. $f_e(x) = |x|$ for $-L < x \leq L$, with $f_e(x + 2L) = f_e(x)$ for $x \in \mathbb{R}$.

- The plot of the graph of f_e shows that it has a "sawtooth" profile that is piecewise linear and continuous, with corners at integer multiples of L.



- By (2.78), we have

$$a_n = \frac{2}{L}\int_0^L x \cos\left(\frac{n\pi x}{L}\right) \mathrm{d}x. \tag{2.83}$$

- Evaluating this integral as in Example 3 and substituting into (2.77) gives the Fourier cosine series

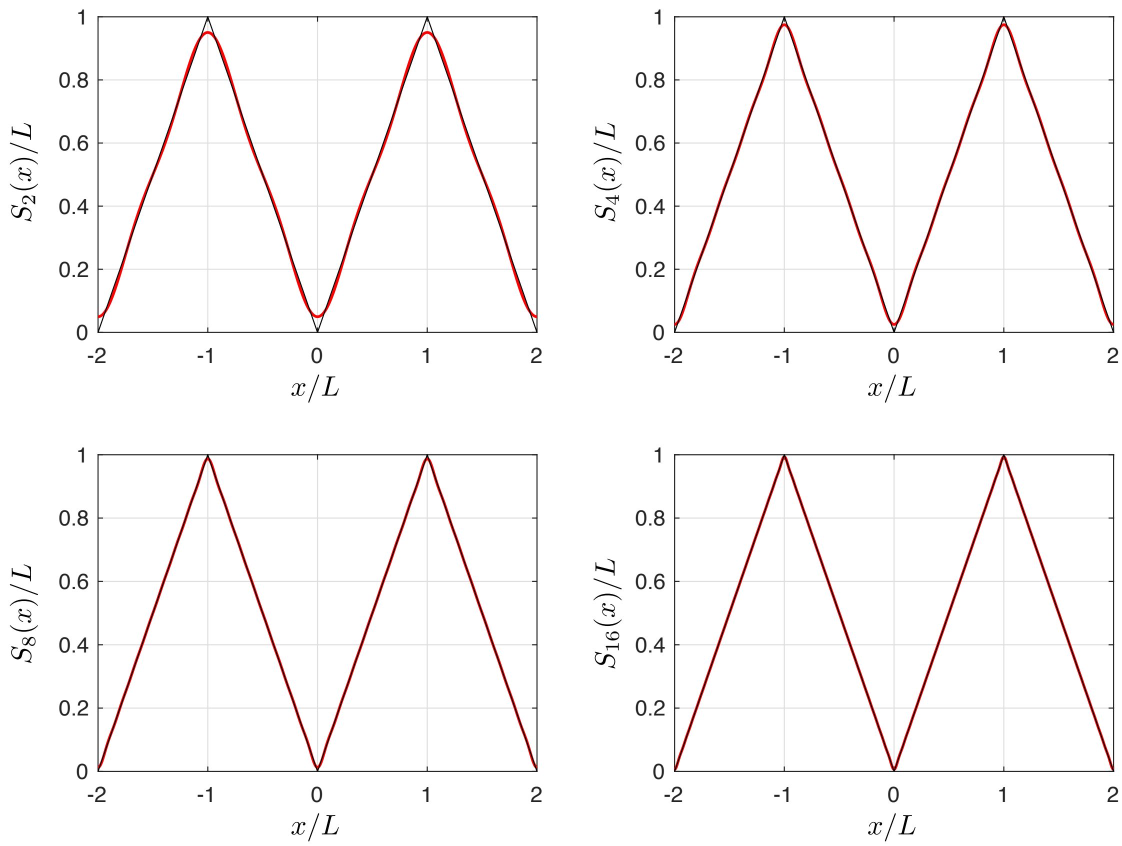
$$f_e(x) \sim \frac{L}{2} - \sum_{m=0}^{\infty} \frac{4L}{(2m+1)^2\pi^2} \cos\left(\frac{(2m+1)\pi x}{L}\right). \tag{2.84}$$

- Since f_e is continuous on $\mathbb{R}$ and f_e' is piecewise continuous on $(-L, L)$, the Fourier Convergence Theorem implies that the Fourier series for f_e converges to f_e on $\mathbb{R}$. Hence the Fourier cosine series for f converges to f on $[0, L]$

- The partial sums of the Fourier series for f_e may be defined for $N \in \mathbb{N}$ by

$$S_N(x) = \frac{L}{2} - \sum_{m=0}^{N} \frac{4L}{(2m+1)^2\pi^2} \cos\left(\frac{(2m+1)\pi x}{L}\right) \quad \text{for } x \in \mathbb{R}. \tag{2.85}$$

  We plot below the partial sums for $N = 2$, 4, 8 and 16, which illustrates their rapid convergence to f_e.

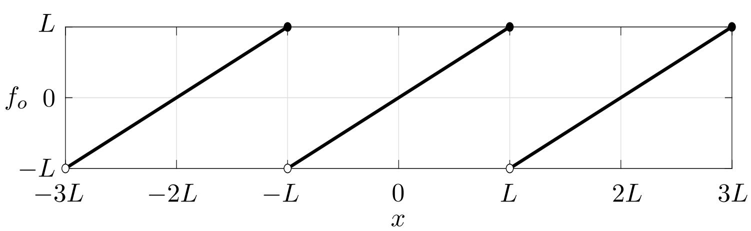


- Similarly, the odd $2L$-periodic extension f_o is defined by

$$f_o(x) = \begin{cases} x & \text{for } 0 \le x \le L, \\ -(-x) & \text{for } -L < x < 0, \end{cases} \tag{2.86}$$

  i.e. $f_o(x) = x$ for $-L < x \le L$, with $f_o(x + 2L) = f_o(x)$ for $x \in \mathbb{R}$.

- The plot of the graph of f_0 shows that it is piecewise linear with jump discontinuities at $x = (2k+1)L$ for $k \in \mathbb{Z}$.



- By (2.81), we have

$$b_n = \frac{2}{L} \int_0^L x \sin\left(\frac{n\pi x}{L}\right) \mathrm{d}x. \tag{2.87}$$

- Evaluating this integral as in Example 3 and substituting into (2.80) gives the Fourier sine series

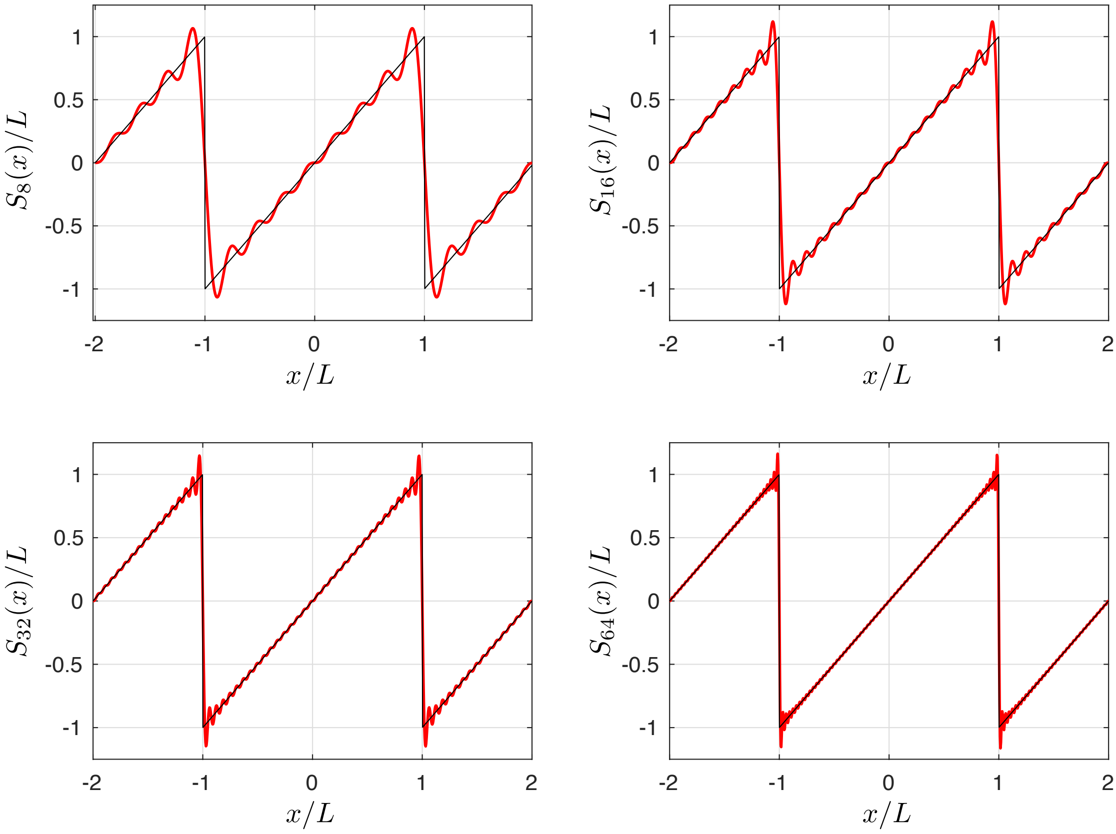
$$f_o(x) \sim \sum_{n=1}^{\infty} \frac{2L(-1)^{n+1}}{n\pi} \sin\left(\frac{n\pi x}{L}\right). \tag{2.88}$$

- Since f_o and f_o' are piecewise continuous on $(-L, L)$, the Fourier Convergence Theorem implies that the Fourier series for f_o converges to $f_o(x)$ at points of continuity of f_o, *i.e.* for $x \neq (2k+1)L$, $k \in \mathbb{Z}$, while at the jump discontinuities the Fourier converges to the average of the left- and right-hand limits of f_0, *i.e.* to $\big(f(L_+) + f(L_-)\big)/2 = (-L+L)/2 = 0$ for $x = L$ and hence for $x = (2k+1)L$, $k \in \mathbb{Z}$ by periodicity. Hence, the Fourier sine series for f converges to $f(x)$ for $0 \leq x < L$, but to 0 for $x = L$.

- The partial sums of the Fourier series for f_o may be defined for positive integers N by

$$S_N(x) = \sum_{n=1}^{N} \frac{2L(-1)^{n+1}}{n\pi} \sin\left(\frac{n\pi x}{L}\right) \quad \text{for } x \in \mathbb{R}. \tag{2.89}$$

  We plot below the partial sums for $N = 8$, 16, 32 and 64, which illustrates that the slow convergence away from the jump discontinuities of f_0 is hindered by Gibb's phenomenon.



- The truncated cosine series gives a better approximation to f on $[0, L]$ than the truncated sine series because it converges everywhere on $[0, L]$, it converges more rapidly and it does not exhibit Gibb's phenomenon. ■

- **Remark:** Let f_3 denote twice the function in Example 3, so that

$$f_3(x) \sim \frac{L}{2} - \sum_{m=1}^{\infty} \frac{4L}{(2m-1)^2\pi^2} \cos\left(\frac{(2m-1)\pi x}{L}\right) + \sum_{m=1}^{\infty} \frac{2L(-1)^{m+1}}{m\pi} \sin\left(\frac{m\pi x}{L}\right).$$

  Question: Why is the Fourier series for f_3 equal to the sum of the Fourier series for f_e and f_o?

  Answer: Because f_e is the even part of f_3 and f_o the odd part of f_3.

- This explains the rate of decay of the Fourier coefficients in Example 3, with $p = 1$ for f_e and $p = 0$ for f_0 in the notation of §2.6.

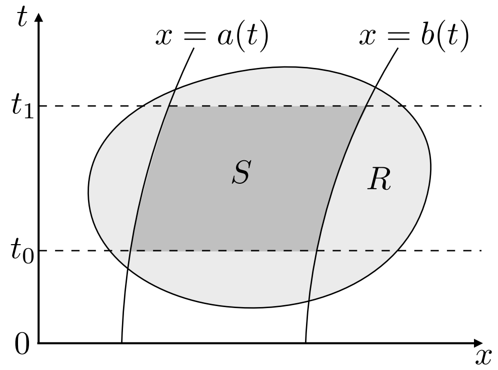
3 The heat equation

3.1 Some preliminaries

- **Fundamental Theorem of Calculus:** If $f(x)$ is continuous in a neighbourhood of a, then

$$\frac{1}{h}\int_a^{a+h} f(x)\,\mathrm{d}x \to f(a) \quad \text{as} \quad h \to 0. \tag{3.1}$$

- **Leibniz's Integral Rule:** Let $F(x,t)$ and $\partial F/\partial t$ be continuous in both x and t in some region R of the (x,t) plane containing the region $S = \{(x,t) : a(t) \le x \le b(t),\ t_0 \le t \le t_1\}$, where the functions $a(t)$ and $b(t)$ and their derivatives are continuous for $t \in [t_0, t_1]$.



Then

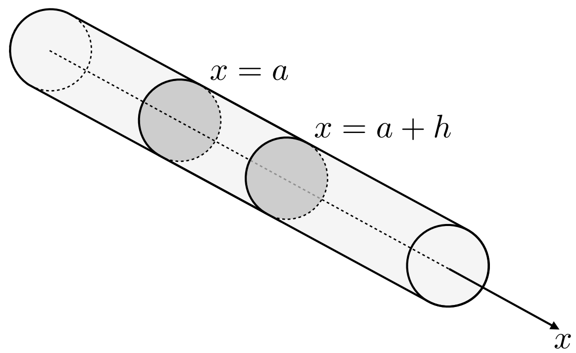
$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{a(t)}^{b(t)} F(x,t)\ \mathrm{d}x = \int_{a(t)}^{b(t)} \frac{\partial F}{\partial t}(x,t)\,\mathrm{d}x + \dot{b}(t)F(b(t),t) - \dot{a}(t)F(a(t),t). \tag{3.2}$$

As a result, if $a(t)$ and $b(t)$ are constants, then

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_a^b F(x,t)\ \mathrm{d}x = \int_a^b \frac{\partial F}{\partial t}(x,t)\,\mathrm{d}x. \tag{3.3}$$

3.2 Derivation of the one-dimensional heat equation

- Consider a rigid isotropic conducting rod (*e.g.* metal) of constant cross-sectional area A lying along the x-axis. We shall consider conservation of thermal or heat energy in the arbitrary section of the rod in $a \le x \le a+h$, where a and h are constants. The geometric setup is as illustrated in the schematic below.



- In the simplest one-dimensional model we assume that the lateral surfaces of the rod are insulated, so that no thermal energy can be transported through them and the absolute temperature T may be taken to be a function of distance x along an axis of the rod and time t. This assumption is applicable if the rod is long and thin, like a wire.

- We denote by ρ the density of the rod and by c_v the specific heat of the rod, and we assume that these material parameters are constant. The specific heat c_v of a material is the energy required to heat up a kilogram by one degree kelvin (in SI units, about which more in §3.4), so the thermal energy in the section of the rod in $a \leq x \leq a+h$ is given by

$$A \int_a^{a+h} \rho c_v T(x,t)\,\mathrm{d}x. \tag{3.4}$$

- We now introduce the heat flux $q(x,t)$ in the positive x-direction, which is the rate at which thermal energy is transported through a cross-section of the rod at station x at time t in the positive x-direction per unit cross-sectional area per unit time, *i.e.* the rate of flow of thermal energy along the rod. By definition, the rate at which thermal energy enters the section through its left-hand cross-section in the plane $x = a$ is $Aq(a,t)$. Similarly, the rate at which thermal energy leaves the section through the right-hand cross-section in the plane $x = a+h$ is $Aq(a+h,t)$. Hence, with our sign convention on the heat flux, the net rate at which thermal energy enters the section is

$$Aq(a,t) - Aq(a+h,t). \tag{3.5}$$

- Assuming insulated lateral surfaces and no external heating (*e.g.* due to microwave heating), conservation of energy states that the rate of change of the thermal energy in the section is equal to the net rate at which thermal energy enters the section, so that

$$\underbrace{\frac{\mathrm{d}}{\mathrm{d}t}\left(A\int_a^{a+h} \rho c_v T(x,t)\,\mathrm{d}x\right)}_{(1)} = \underbrace{Aq(a,t)}_{(2)} - \underbrace{Aq(a+h,t)}_{(3)}, \tag{3.6}$$

where we have labeled the three terms in order to summarize their physical significane as follows:

(1) is the time rate of change of thermal energy in the section in $a \leq x \leq a+h$;

(2) is the rate at which thermal energy enters the section through $x = a$;

(3) is the rate at which thermal energy leaves the section through $x = a+h$.

- We note this integral conservation law is also true for $h < 0$ with appropriate reinterpretation of the terms.

- Assuming T_t is continuous, Leibniz's Integral Rule with a and $a+h$ constant (*i.e.* in the form (3.3)) gives

$$\frac{\rho c_v}{h}\int_a^{a+h} T_t(x,t)\,\mathrm{d}x + \frac{q(a+h,t)-q(a,t)}{h} = 0, \tag{3.7}$$

where we have also rearranged into a form that will enable us to take the limit $h \to 0$.

- To take the limit $h \to 0$ we apply the Fundamental Theorem of Calculus (3.1) to the first term (assuming T_t is continuous in a neighbourhood of a) and use the definition of the partial derivative of q with respect to x (assuming it to exist and to be continuous at a), to obtain the partial differential equation

$$\rho c_v T_t + q_x = 0, \tag{3.8}$$

which relates the time rate of change of the temperature and the spatial rate of change of the heat flux.

- To make further progress we must decide how the heat flux $q(x,t)$ depends on the temperature $T(x,t)$. This is called a constitutive relation and cannot be deduced, relying instead on some assumptions about the physical properties of the material under consideration. An example of a simple constitutive relation is Hooke's law for the extension of a spring — we note that

  - a "thought-experiment" suggests this law is reasonable;

- it could be confirmed experimentally;
- it will almost certainly fail under "extreme" conditions.

- To close our model for heat conduction we will adopt *Fourier's Law*, which is is the constitutive law given by
$$q = -kT_x, \tag{3.9}$$
where k is the thermal conductivity of the rod, which is another material parameter that we take to be constant.

- The minus sign in Fourier's law means that thermal energy flows down the temperature gradient, *i.e.* from high to low temperatures. Physical experiments confirm that this is an excellent approximation in many practical applications. We note that a good conductor of heat (such as silver) will have a higher thermal conductivity than a poor conductor of heat (such as glass).

- Substituting (3.9) into (3.8), we arrive at the heat equation
$$\frac{\partial T}{\partial t} = \kappa \frac{\partial^2 T}{\partial x^2}, \tag{3.10}$$
where the thermal diffusivity $\kappa = k/(\rho c_v)$.

- The heat equation (3.10) is a second-order linear partial differential equation.

3.3 Initial boundary value problems

- There are numerous applications of the heat equation ranging from the diffusive transport of chemical species to the pricing of financial derivatives, the latter being governed by a *backward* heat equation called the *Black-Scholes equation.*

- In this course we focus on the modelling of the evolution of the temperature $T(x,t)$ in a metal rod of finite length L lying along the x-axis in the region $0 \leq x \leq L$.

- Suppose the metal is at room temperature $T_0 \approx 300\,\mathrm{K}$ when some large ice blocks at their melting temperature $T^* \approx 273\,\mathrm{K}$ are held instantaneously against each end of the rod at time $t = 0$. We encode this setup into a mathematical model, as follows:

  - the temperature $T(x,t)$ satisfies the heat equation (3.10) inside the rod, so that
$$\frac{\partial T}{\partial t} = \kappa \frac{\partial^2 T}{\partial x^2} \quad \text{for} \quad 0 < x < L,\ t > 0; \tag{3.11}$$
  - the effect of the ice blocks on the rod are modelled through the boundary conditions
$$T(0,t) = T^*, \quad T(L,t) = T^* \quad \text{for } t > 0; \tag{3.12}$$
  - the initial state of the temperature in the rod is fed into the initial condition
$$T(x,0) = T_0 \quad \text{for } 0 < x < L. \tag{3.13}$$

- **Notes:**

  (1) The mathematical model (3.11)–(3.13) forms an *initial boundary value problem* (IBVP) for the temperature $T(x,t)$.

  (2) The boundary conditions (3.12) are called *Dirichlet boundary conditions* because they prescribe the value of the dependent variable T. They are called *homogeneous* if $T^* = 0$ and *inhomogeneous* otherwise.

  (3) While the boundary and initial conditions were motivated on physical grounds, they can only make mathematical sense if the IBVP (3.11)–(3.13) is *well-posed* in the sense that it has a unique solution that varies continuously with the boundary and initial data (*i.e.* with T^* and T_0) in some suitable sense. We we shall return to the issue of well-posedness in §7.

(4) The total number of boundary conditions is equal to the number of spatial partial derivatives in the heat equation, which is the same count as for a typical ODE BVP. The total number of initial conditions is equal to the number of temporal derivatives in the heat equation, which is the same count as for a typical ODE IVP. These counts are typical for PDE IBVPs.

- **Definition:** The outward normal derivative of T on the boundary is equal to the directional derivative in the direction of the outward pointing unit normal, *i.e.* $-\boldsymbol{i}\cdot\boldsymbol{\nabla}T = -T_x$ on $x=0$ and $\boldsymbol{i}\cdot\boldsymbol{\nabla}T = T_x$ on $x=L$.

- Other common boundary conditions are:

  - *inhomogeneous Neumann boundary conditions* which prescribe the outward normal derivative of the dependent variable on the boundary (here proportional to the heat flux $q=-kT_x$ by Fourier's law), *e.g.*

$$-\frac{\partial T}{\partial x}(0,t)=\phi(t), \quad \frac{\partial T}{\partial x}(L,t)=\psi(t) \quad \text{for } t>0, \tag{3.14}$$

    where the functions $\phi(t)$ and $\psi(t)$ are given.

  - *inhomogeneous Robin boundary conditions* which prescribe a linear combination of the outward normal derivative and temperature at the boundary, *e.g.*

$$-\frac{\partial T}{\partial x}(0,t)+\alpha(t)T(0,t)=\phi(t), \quad \frac{\partial T}{\partial x}(L,t)+\beta(t)T(l,T)=\psi(t) \quad \text{for } t>0, \tag{3.15}$$

    where the functions $\alpha(t)$, $\phi(t)$, $\beta(t)$ and $\psi(t)$ are given.

3.4 Units and nondimensionalisation

- **Notation:** We denote by $[p]$ the dimension of the quantity p in fundamental units (M, L, T, Θ *etc*) or SI units (kg, m, s, K *etc*). We will work with the latter and recall that kelvin K is the SI unit of temperature, the newton N is the SI derived unit of force ($1\,\mathrm{N} = 1\,\mathrm{kg\,m\,s^{-2}}$), while the joule J is the SI derived unit of energy ($1\,\mathrm{J} = 1\,\mathrm{N\,m}$).

- Both sides of an equation modelling a physical process must have the same dimensions, *e.g.* in the integral conservation law (3.6),

$$[(1)] = [(2)] = [(3)] = \mathrm{J\,s^{-1}}, \tag{3.16}$$

  while in the heat equation (3.11),

$$[T_t] = [\kappa T_{xx}] = \mathrm{K\,s^{-1}}. \tag{3.17}$$

- We can exploit this fact to determine the dimensions of parameters and to check that solutions are dimensionally correct.

- For example, using Fourier's Law (3.9) we find that the dimensions of the thermal conductivity are given by

$$[k] = \frac{[q]}{[T_x]} = \frac{\mathrm{J\,m^{-2}\,s^{-1}}}{\mathrm{K\,m^{-1}}} = \mathrm{J\,K^{-1}\,m^{-1}\,s^{-1}}, \tag{3.18}$$

  and using the heat equation (3.11) we find that the dimensions of the thermal diffusivity are given by

$$[\kappa] = \frac{[T_t]}{[T_{xx}]} = \frac{\mathrm{K\,s^{-1}}}{\mathrm{K\,m^{-2}}} = \mathrm{m^2\,s^{-1}}. \tag{3.19}$$

- We summarize in the table below the SI units of all of the variables and parameters involved in the derivation of the one-dimensional heat equation.

Symbol	Quantity	SI units
x	Axial distance	m
t	Time	s
T	Absolute temperature	K
q	Heat flux in positive x-direction	$\mathrm{J\,m^{-2}\,s^{-1}}$
A	Cross-sectional area	$\mathrm{m^2}$
ρ	Rod density	$\mathrm{kg\,m^{-3}}$
c_v	Rod specific heat	$\mathrm{J\,kg^{-1}\,K^{-1}}$
k	Rod thermal conductivity	$\mathrm{J\,K^{-1}\,m^{-1}\,s^{-1}}$
κ	Rod thermal diffusivity	$\mathrm{m^2\,s^{-1}}$

- **Nondimensionalisation:** The method of scaling variables with typical values to derive dimensionless equations. These usually contain dimensionless parameters that characterise the relative importance of the physical mechanisms in the model.
- We illustrate the method with an example.

Example: nondimensionalisation of an IBVP

- Consider the IBVP for the temperature $T(x,t)$ in a metal rod of length L given by the heat equation

$$\frac{\partial T}{\partial t} = \kappa \frac{\partial^2 T}{\partial x^2} \quad \text{for} \quad 0 < x < L,\ t > 0, \tag{3.20}$$

with the inhomogeneous Dirichlet boundary conditions

$$T(0,t) = T_0, \quad T(L,t) = T_1 \quad \text{for} \quad t > 0, \tag{3.21}$$

and the initial condition

$$T(x,0) = T_2 \frac{x}{L}\left(1 - \frac{x}{L}\right) \quad \text{for} \quad 0 < x < L, \tag{3.22}$$

where T_0, T_1 snd T_2 are prescribed constant temperatures.

- **Remark:** There are five dimensional parameters, namely κ, L, T_0, T_1 and T_2.
- We can nondimensionalise by scaling

$$x = L\widehat{x}, \quad t = \tau\widehat{t}, \quad T(x,t) = T_2\widehat{T}(\widehat{x},\widehat{t}), \tag{3.23}$$

where L, τ and T_2 are a typical lengthscale, timescale and temperature, respectively, so that the quantities $\widehat{x}$, $\widehat{t}$ and $\widehat{T}$ are dimensionless.

- By the chain rule,

$$\begin{aligned} \frac{\partial T}{\partial t} &= T_2 \frac{\partial \widehat{T}}{\partial \widehat{t}} \frac{\mathrm{d}\widehat{t}}{\mathrm{d}t} = \frac{T_2}{\tau}\frac{\partial \widehat{T}}{\partial \widehat{t}}, \\ \frac{\partial T}{\partial x} &= T_2 \frac{\partial \widehat{T}}{\partial \widehat{x}} \frac{\mathrm{d}\widehat{x}}{\mathrm{d}x} = \frac{T_2}{L}\frac{\partial \widehat{T}}{\partial \widehat{x}}, \qquad \textit{etc.} \end{aligned}$$

- Hence, the dimensional problem (3.20)-(3.22) for the dimensional temperature $T(x,t)$ implies that the corresponding dimensionless problem for the dimensionless temperature $\widehat{T}(\widehat{x},\widehat{t})$ is given by

$$\frac{\partial \widehat{T}}{\partial \widehat{t}} = D \frac{\partial^2 \widehat{T}}{\partial \widehat{x}^2} \quad \text{for} \quad 0 < \widehat{x} < 1,\ \widehat{t} > 0, \tag{3.24}$$

with the boundary conditions

$$\widehat{T}(0,\widehat{t}) = \alpha_0, \quad \widehat{T}(1,t) = \alpha_1 \quad \text{for} \quad \widehat{t} > 0, \tag{3.25}$$

and the initial condition

$$\widehat{T}(\widehat{x},0) = \widehat{x}(1-\widehat{x}) \quad \text{for} \quad 0 < \widehat{x} < 1, \tag{3.26}$$

where the three dimensionless parameters D, α_0 and α_1 are defined by

$$D = \frac{\kappa\tau}{L^2}, \quad \alpha_0 = \frac{T_0}{T_2}, \quad \alpha_1 = \frac{T_1}{T_2}. \tag{3.27}$$

- We can further reduce the number of dimensionless parameters to two by choosing the timescale τ so that $D = 1$, *i.e.* by choosing $\tau = L^2/\kappa$, which is the timescale for conductive transport of heat over a distance L because it balances both terms in (3.24).

- With this choice of timescale, we note that if $\widehat{T}(\widehat{x}, \widehat{t}; \alpha_0, \alpha_1)$ is a solution of (3.24)-(3.26), then a solution of (3.20)-(3.22) is given by
$$\frac{T}{T_2} = \widehat{T}\left(\frac{x}{L}, \frac{\kappa t}{L^2}; \frac{T_0}{T_2}, \frac{T_1}{T_2}\right). \tag{3.28}$$
i.e. T/T_2 must be a function of x/L and $\kappa t/L^2$.

- This means that we can compare heat problems on different scales. For example, two IBVPs that are identical except for L and κ will exhibit the same behaviour on the same timescale if and only if L^2/κ is the same in each problem. ■

3.5 Heat conduction in a finite rod

- Consider the initial boundary value problem for the temperature $T(x,t)$ in a metal rod of length L given by the heat equation
$$\frac{\partial T}{\partial t} = \kappa \frac{\partial^2 T}{\partial x^2} \quad \text{for} \quad 0 < x < L,\ t > 0, \tag{3.29}$$
with the homogeneous Dirichlet boundary conditions
$$T(0,t) = 0, \quad T(L,t) = 0 \quad \text{for} \quad t > 0, \tag{3.30}$$
and the initial condition
$$T(x,0) = f(x) \quad \text{for} \quad 0 < x < L, \tag{3.31}$$
where the initial temperature profile $f(x)$ is given.

- We will construct a solution using *Fourier's method*, which consists of the following three steps:
  - (I) Use the method of separation of variables to find the countably infinite set of nontrivial separable solutions satisfying the partial differential equation (3.29) and boundary conditions (3.30), each containing an arbitrary constant.
  - (II) Use the principle of superposition — that the sum of any number of solutions of a linear problem is also a solution (assuming convergence) — to form the general series solution that is the infinite sum of the separable solutions.
  - (III) Use the theory of Fourier series to determine the constants in the general series solution for which it satisfies the initial condition (3.31).

- **Notes:**
  - (1) Both the partial differential equation (3.29) and and the boundary conditions (3.30) are linear since, if T_1 and T_2 satisfy them, then so does $\alpha_1 T_1 + \alpha_2 T_2$ for all α_1, $\alpha_2 \in \mathbb{R}$.
  - (2) To verify that the resulting infinite series is actually a solution of the heat equation, we need it to converge sufficiently rapidly that T_t and T_{xx} can be computed by termwise differentiation.

Step (I) Find all nontrivial separable solutions of the PDE and BCs

- We begin by seeking a nontrivial separable solution of the form $T = F(x)G(t)$ for which the partial differential equation (3.29) gives
$$F(x)G'(t) = \kappa F''(x)G(t),$$
with a prime $'$ denoting here and hereafter the derivative with respect to the argument.

- Separating the variables by assuming $F(x)G(t) \neq 0$ therefore gives
$$\frac{F''(x)}{F(x)} = \frac{G'(t)}{\kappa G(t)}. \tag{3.32}$$

- The left-hand side of this expression is independent of t, while the right-hand side is independent of x. Since the left-hand side is equal to the right-hand side, they must both be independent of x and t, and therefore equal to a constant, $-\lambda \in \mathbb{R}$ say.

- The boundary condition at $x = 0$ implies that $F(0)G(t) = 0$ for $t > 0$. Since we're seeking solutions T that are nontrivial (*i.e.* not identically equal to zero), there must exist a time $t > 0$ such that $G(t) \neq 0$, and hence we must impose on $F(x)$ the boundary condition $F(0) = 0$. Similarly, the boundary condition at $x = L$ implies that $F(L) = 0$.

- In summary, we have deduced that $F(x)$ satisfies the boundary value problem given by the ordinary differential equation
$$-F''(x) = \lambda F(x) \quad \text{for} \quad 0 < x < L, \tag{3.33}$$
with the boundary conditions
$$F(0) = 0, \quad F(L) = 0, \tag{3.34}$$
where $\lambda \in \mathbb{R}$.

- Now we need to find all $\lambda \in \mathbb{R}$ such that the boundary value problem (3.33)-(3.34) for $F(x)$ has a nontrivial solution.

- Since the general solution of (3.33) is different for (i) $\lambda < 0$, (ii) $\lambda = 0$ and (iii) $\lambda > 0$, there are three cases to consider.

- <u>Case (i): $\lambda = -\omega^2$ ($\omega > 0$ wlog)</u>
  - If $F'' - \omega^2 F = 0$, then $F(x) = A\cosh(\omega x) + B\sinh(\omega x)$, where $A, B \in \mathbb{R}$.
  - The boundary conditions (3.34) then require $A = 0,\ B\sinh(\omega L) = 0$, so that $F = 0$.

- <u>Case (ii): $\lambda = 0$</u>
  - If $F'' = 0$, then $F(x) = A + Bx$, where $A, B \in \mathbb{R}$.
  - The boundary conditions (3.34) then require $A = 0,\ BL = 0$, so that $F = 0$.

- <u>Case (iii): $\lambda = \omega^2$ ($\omega > 0$ wlog)</u>
  - If $F'' + \omega^2 F = 0$, then $F(x) = A\cos(\omega x) + B\sin(\omega x)$, where $A, B \in \mathbb{R}$.
  - The boundary conditions (3.34) then require $A = 0,\ B\sin(\omega L) = 0$.
  - Since $B \neq 0$ for nontrivial F, we must have $\sin \omega L = 0$, *i.e.* $\omega L = n\pi$ for some $n \in \mathbb{N} \setminus \{0\}$.

- Hence, the nontrivial solutions of the BVP (3.33)–(3.34) are given for positive integers n by
$$F(x) = B\sin\left(\frac{n\pi x}{L}\right), \quad \lambda = \frac{n^2\pi^2}{L^2}, \tag{3.35}$$
where B is an arbitrary constant.

- Since $G(t)$ satisfies the ordinary differential equation $G' = -\lambda\kappa G$, we deduce that
$$G(t) = C\exp\left(-\lambda\kappa t\right), \tag{3.36}$$
where $C \in \mathbb{R}$.

- Since $T(x,t) = F(x)G(t)$, we conclude that the nontrivial separable solutions of the heat equation (3.29) that satisfy the boundary conditions (3.30) are given by
$$T_n(x,t) = b_n \sin\left(\frac{n\pi x}{L}\right)\exp\left(-\frac{n^2\pi^2\kappa t}{L^2}\right), \tag{3.37}$$
where n is a positive integer, b_n is a constant (equal to BC above) and we have introduced the subscript n on T_n and b_n to enumerate the countably infinite set of such solutions.

Step (II) Apply the principle of superposition

- Since (3.29)–(3.30) are linear, a formal application of the principle of superposition implies that the general series solution is given by

$$T(x,t) = \sum_{n=1}^{\infty} T_n(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) \exp\left(-\frac{n^2\pi^2\kappa t}{L^2}\right). \tag{3.38}$$

Step (III) Use the theory of Fourier series to satisfy the IC

- The initial condition (3.31) can only be satisfied by the general series solution (3.38) if

$$f(x) = T(x,0) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) \quad \text{for} \quad 0 < x < L, \tag{3.39}$$

  so that we need to find the Fourier sine series for f on $[0, L]$

- The theory of Fourier series implies that the Fourier coefficients b_n are given by

$$b_n = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) \mathrm{d}x \quad \text{for} \quad n \in \mathbb{N} \setminus \{0\}, \tag{3.40}$$

- Hence, we have derived a solution in the form of an infinite trigonometric series.

Notes

(1) The boundary value problem (3.33)–(3.34) is an *eigenvalue problem* in which the unknown parameter λ is called an *eigenvalue* and the corresponding non-trivial solution $F(x)$ an *eigenfunction*. The expansions (3.38) and (3.39) are therefore called *eigenfunction expansions*. That there are a discrete countably infinite set of eigenvalues and corresponding eigenfunctions (given by (3.35) for positive integers n) is a property of the boundary value problem that is explained by *Sturm-Liouville theory* of *e.g.* part A Differential Equations 2.

(2) The integral expressions for the Fourier coefficients in (3.40) may be derived by assuming that the orders of summation and integration may be interchanged and using the orthogonality relations

$$\int_0^L \sin\left(\frac{m\pi x}{L}\right) \sin\left(\frac{n\pi x}{L}\right) \mathrm{d}x = \frac{L}{2}\delta_{mn} \quad \text{for } m,\, n \in \mathbb{N} \setminus \{0\}. \tag{3.41}$$

(3) Since $T_n(x,t)$ decays exponentially as $n \to \infty$ for $t > 0$, comparison methods from Analysis II may be used to show that if the Fourier coefficients b_n are merely bounded for all n, then the general series solution (3.38) has partial derivatives of all orders for $t > 0$ that may be computed by term-by-term differentiation. It follows from the Fourier convergence theorem that if f and f' are piecewise continuous on $(0, L)$, then the infinite series solution given by (3.38) and (3.40) is indeed a solution of the initial boundary value problem (3.29)–(3.31). Thus, Fourier's method can accommodate even jump discontinuities in the initial temperature profile, the heat equation acting to instantaneously "smooth" them out.

(4) If the initial temperature profile has a jump discontinuity, then the truncated series solution for $T(x,t)$ will exhibit Gibb's phenomenon at $t = 0$, and hence at sufficiently small times $t \ll L^2/\kappa$ by continuity. In principle this deficiency can be avoided at some fixed $t > 0$ by keeping enough terms. In contrast, the exponential decay of $T_n(x,t)$ with $n^2\kappa t/L^2$ means that the solution will be well approximated by the leading-term $T_1(x,t)$ at sufficiently long times $t \gg L^2/\kappa$.

- We illustrate notes (3) and (4) with an example.

Example: the smoothing effect of the heat equation

- Consider the IBVP (3.29)–(3.31) in which the initial temperature profile given by

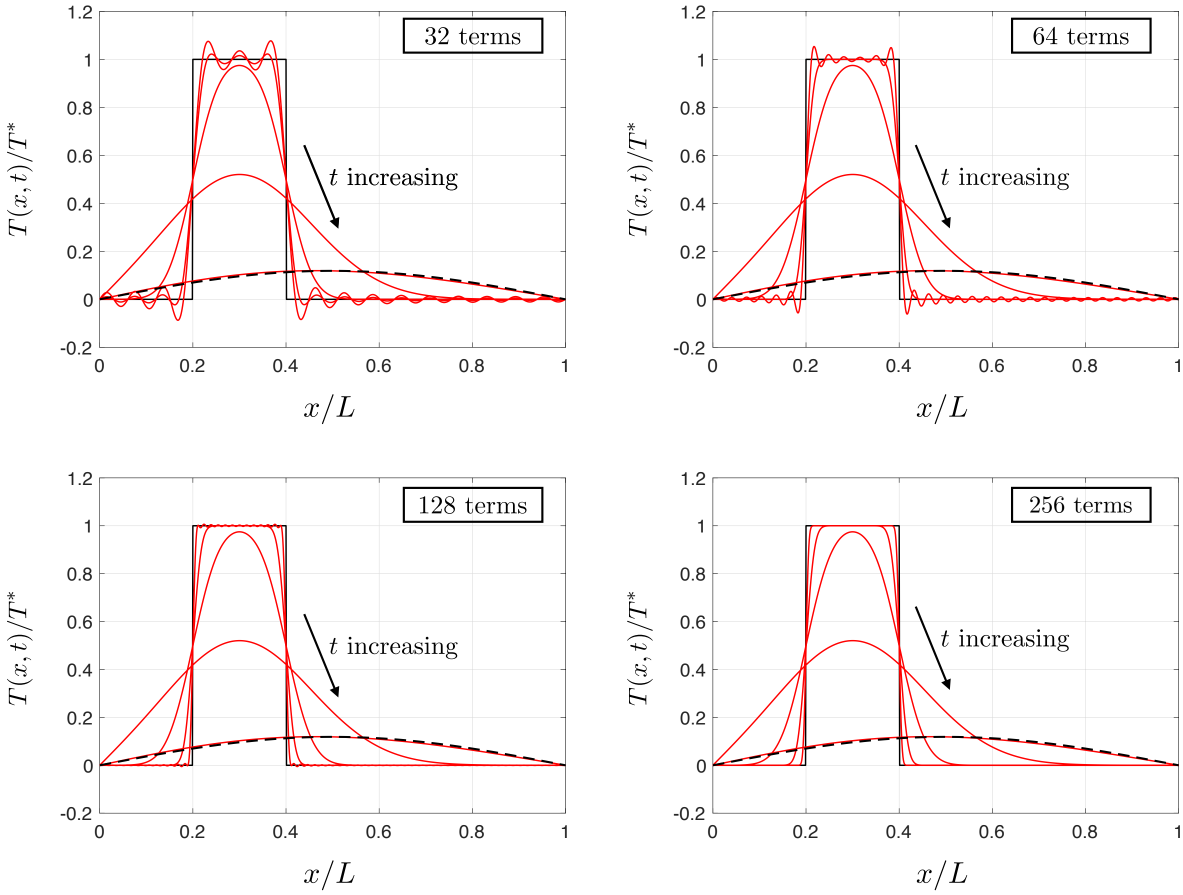
$$f(x)=\begin{cases} T^* & \text{for } L_1<x<L_2,\\ 0 & \text{otherwise,}\end{cases} \tag{3.42}$$

where T^*, L_1 and L_2 are constants, so that the Fourier coefficients are given by

$$b_n=\frac{2}{L}\int_{L_1}^{L_2} T^*\sin\left(\frac{n\pi x}{L}\right)\mathrm{d}x=\frac{2T^*}{n\pi}\left(\cos\left(\frac{n\pi L_1}{L}\right)-\cos\left(\frac{n\pi L_2}{L}\right)\right)\quad \text{for } n\in\mathbb{N}\backslash\{0\}. \tag{3.43}$$

- We plot below snapshots of the partial sums of the truncated series solution (red lines) with 32, 64, 128 and 256 terms at times t given by $\kappa t/L^2=10^{-5}$, 10^{-4}, 10^{-3}, 10^{-2} and 10^{-1} for $L_1/L=0.2$, $L_2/L=0.4$. The jump conditions in the initial temperature profile at $L_1/L=0.2$ and $L_2/L=0.4$ are indicated by vertical black lines.

- As the number of terms increases we see that Gibb's phenomenon is suppressed more rapidly. Any profile that is oscillatory or not positive for $0<x<L$, $t>0$ is a poor approximation of the solution, so we see that only the plot with 256 terms is acceptable for the times chosen. The final snap shot in each case is close to $T_1(x,t)$ (dashed line) for which $\pi^2\kappa t/L^2=\pi^2/10$.



- The early time behaviour of the temperature for the initial profile (3.42) is captured much more effectively by the *asymptotic solution*

$$T(x,t)\approx\frac{T^*}{\sqrt{4\pi\kappa t}}\int_{L_1}^{L_2}\exp\left(-\frac{(s-x)^2}{4\kappa t}\right)\mathrm{d}s, \tag{3.44}$$

which is valid as $t\to 0+$. The asymptotic solution does not exhibit Gibb's phenomenon and tends to the initial profile (3.42) as $t\to 0+$ except at the jump discontinuities where it tends to $T^*/2$. The asymptotic solution is the superposition of *fundamental solutions of the heat equation* and may be derived systematically using the *method of matched asymptotic expansions* — see part A Differential Equations 2 and Integral Transforms. ■

3.6 Uniqueness Theorem

- In the last section we considered the IBVP for the temperature $T(x,t)$ given by the heat equation
$$\frac{\partial T}{\partial t} = \kappa \frac{\partial^2 T}{\partial x^2} \quad \text{for} \quad 0 < x < L,\ t > 0, \tag{3.45}$$
with the homogeneous Dirichlet boundary conditions
$$T(0,t) = 0, \quad T(L,t) = 0 \quad \text{for} \quad t > 0, \tag{3.46}$$
and the initial condition
$$T(x,0) = f(x) \quad \text{for} \quad 0 < x < L, \tag{3.47}$$
where the initial temperature profile $f(x)$ is given.
- We used Fourier's method to construct an infinite series solution, but is it the only solution?
- **Uniqueness Theorem:** The IBVP (3.45)–(3.47) has at most one solution.

Proof:

- Our strategy is to show that the difference between any two solutions much vanish. Thus, we suppose that $T(x,t)$ and $\widetilde{T}(x,t)$ are solutions to (3.45)–(3.47) and let $W(x,t) = T(x,t) - \widetilde{T}(x,t)$.
- By linearity, (3.45)–(3.47) imply that $W(x,t)$ satisfies the heat equation
$$\frac{\partial W}{\partial t} = \frac{\partial T}{\partial t} - \frac{\partial \widetilde{T}}{\partial t} = \kappa \frac{\partial^2 T}{\partial x^2} - \kappa \frac{\partial^2 \widetilde{T}}{\partial x^2} = \kappa \frac{\partial^2 W}{\partial x^2} \quad \text{for} \quad 0 < x < L,\ t > 0, \tag{3.48}$$
with the boundary conditions
$$W(0,t) = T(0,t) - \widetilde{T}(0,t) = 0, \quad W(L,t) = T(L,t) - \widetilde{T}(L,t) = 0 \quad \text{for} \quad t > 0, \tag{3.49}$$
and the initial condition
$$W(x,0) = T(x,0) - \widetilde{T}(x,0) = f(x) - f(x) = 0 \quad \text{for} \quad 0 < x < L. \tag{3.50}$$
- The trick is to analyse the integral $I(t)$ defined by
$$I(t) := \frac{1}{2} \int_0^L W(x,t)^2 \, \mathrm{d}x. \tag{3.51}$$
- Evidently $I(t) \geq 0$ for $t \geq 0$ and $I(0) = 0$ by (3.50).
- But, for $t > 0$,
$$\begin{aligned}
\frac{\mathrm{d}I}{\mathrm{d}t} &= \int_0^L W \frac{\partial W}{\partial t} \, \mathrm{d}x && \text{(by Liebniz Integral Rule (3.3))} \\
&= \int_0^L W \kappa \frac{\partial^2 W}{\partial x^2} \, \mathrm{d}x && \text{(by (3.48))} \\
&= \left[\kappa W \frac{\partial W}{\partial x} \right]_{x=0}^{x=L} - \kappa \int_0^L \frac{\partial W}{\partial x} \frac{\partial W}{\partial x} \, \mathrm{d}x && \text{(by integration by parts)} \\
&= -\kappa \int_0^L \left(\frac{\partial W}{\partial x} \right)^2 \mathrm{d}x && \text{(by (3.49))} \\
&\leq 0
\end{aligned}$$
which means that $I(t)$ cannot increase, so that $I(t) \leq I(0) = 0$ for $t \geq 0$.
- Since $I(t) \geq 0$ and $I(t) \leq 0$ for $t \geq 0$, we deduce that $I(t) = 0$ for $t \geq 0$, and hence that $W(x,t) = 0$ for $0 \leq x \leq L$, $t \geq 0$ (assuming continuity of W there). ■

Notes

(1) Since W is the temperature in a metal rod whose initial temperature is everywhere zero and whose ends are held at zero temperature thereafter, on physical grounds we expect the rod to remain at zero temperature, *i.e.* $W = 0$ for $0 \leq x \leq L$ and $t \geq 0$, which is precisely what we showed to prove uniqueness.

(2) The proof works for any boundary conditions for which it is possible to show that

$$\left[\kappa W \frac{\partial W}{\partial x}\right]_{x=0}^{x=L} \leq 0. \tag{3.52}$$

Examples include inhomogeneous Dirichlet and Neumann boundary conditions.

3.7 Inhomogeneous Dirichlet boundary conditions

- Consider the initial boundary value problem for the temperature $T(x,t)$ given by the heat equation

$$\frac{\partial T}{\partial t} = \kappa \frac{\partial^2 T}{\partial x^2} \quad \text{for} \quad 0 < x < L,\ t > 0, \tag{3.53}$$

with the inhomogeneous Dirichlet boundary conditions

$$T(0,t) = T_L, \quad T(L,t) = T_R \quad \text{for} \quad t > 0, \tag{3.54}$$

and the initial condition

$$T(x,0) = f(x) \quad \text{for} \quad 0 < x < L, \tag{3.55}$$

where T_L and T_R are prescribed constant temperatures, not both zero, and the initial temperature profile $f(x)$ is given.

- Lets try to apply Fourier's method. In step (I) we need to find the nontrivial separable solutions $T(x,t) = F(x)G(t)$ of the heat equation (3.53) and boundary conditions (3.54). But the latter would require

$$F(0)G(t) = T_L, \quad F(L)G(t) = T_R \quad \text{for} \quad t > 0, \tag{3.56}$$

forcing G to be constant. It follows that the only nontrivial separable solution satisfying the boundary conditions is the time-independent or steady-state solution (about which more shortly). Since this cannot satisfy the initial condition (3.55), Fourier's method fails because the boundary conditions (3.54) are not homogeneous.

- However, we can transform the problem into one amenable to Fourier's method, as follows.

- On physical grounds, we conjecture that $T(x,t) \to S(x)$ as $t \to \infty$, where $S(x)$ is the aforementioned steady-state solution of (3.53)–(3.54), which satisfies

$$0 = \kappa \frac{\mathrm{d}^2 S}{\mathrm{d}x^2} \quad \text{for} \quad 0 < x < L, \tag{3.57}$$

with $S(0) = T_L$ and $S(L) = T_R$. Thus, $S(x)$ has the linear temperature profile given by

$$S(x) = T_L\left(1 - \frac{x}{L}\right) + T_R\left(\frac{x}{L}\right). \tag{3.58}$$

- **Remark:** In steady state thermal energy is conducted along the rod with constant heat flux

$$q = -k\frac{\partial T}{\partial x} = \frac{k(T_L - T_R)}{L}, \tag{3.59}$$

so that heat flows steadily in the positive x-direction for $T_L > T_R$.

- We now observe that if we let

$$T(x,t) = S(x) + U(x,t), \tag{3.60}$$

then by linearity (3.53)–(3.55) imply that $U(x,t)$ satisfies the initial boundary value problem given by the heat equation

$$\frac{\partial U}{\partial t} = \kappa \frac{\partial^2 U}{\partial x^2} \quad \text{for} \quad 0 < x < L,\ t > 0, \tag{3.61}$$

with the homogeneous Dirichlet boundary conditions

$$U(0,t) = 0, \quad U(L,t) = 0 \quad \text{for} \quad t > 0, \tag{3.62}$$

and the initial condition

$$U(x,0) = f(x) - S(x) \quad \text{for} \quad 0 < x < L. \tag{3.63}$$

- The initial boundary value problem (3.61)–(3.63) for $U(x,t)$ is amenable to Fourier's method: we solved it in §3.4 to find the solution given by

$$U(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) \exp\left(-\frac{n^2\pi^2\kappa t}{L^2}\right), \tag{3.64}$$

where

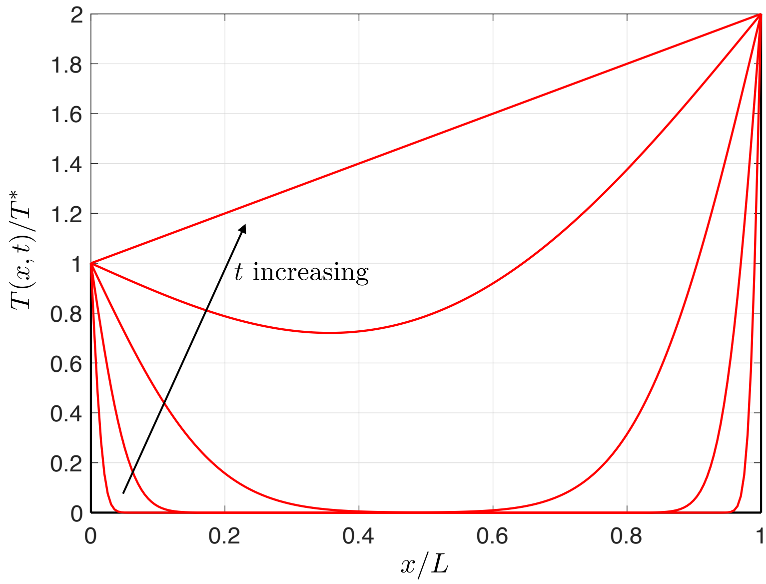
$$b_n = \frac{2}{L}\int_0^L (f(x) - S(x)) \sin\left(\frac{n\pi x}{L}\right) \mathrm{d}x = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)\mathrm{d}x - \frac{2}{n\pi}(T_L - (-1)^n T_R), \tag{3.65}$$

which verifies our conjecture that $T(x,t) \to S(x)$ as $t \to \infty$ because $U(x,t) \to 0$ as $t \to \infty$.

- **Remark:** The parameters T_L and T_R in the boundary conditions (3.54) ended up in the initial condition (3.63) — hence the method is sometimes called 'the method of shifting the data.'

Example: infinite speed of propagation

- Consider the IBVP (3.53)–(3.55) with $f(x) = 0$, $T_L = T^*$ and $T_R = 2T^*$.
- We plot below snapshots of the partial sums of the truncated series solution with 128 terms for $\kappa t/L^2 = 0$ (black line) and $\kappa t/L^2 = 10^{-4}$, 10^{-3}, 10^{-2}, 10^{-1}, 1 (red lines). The profiles illustrate the manner in which heat conduction rapidly drives the temperature toward the linear steady-state temperature profile.



- Since the temperature is zero for $0 < x < L$ at $t = 0$, but everywhere positive for $t > 0$, the effect of the boundary conditions is felt everywhere instantaneously — the heat equation propagates information with infinite speed.

3.8 Homogeneous Neumann boundary conditions

- Consider the IBVP for the temperature $T(x,t)$ given by the heat equation

$$\frac{\partial T}{\partial t} = \kappa \frac{\partial^2 T}{\partial x^2} \quad \text{for} \quad 0 < x < L,\ t > 0, \tag{3.66}$$

with the homogeneous Neumann boundary conditions

$$\frac{\partial T}{\partial x}(0,t) = 0, \quad \frac{\partial T}{\partial x}(L,t) = 0 \quad \text{for} \quad t > 0, \tag{3.67}$$

and the initial condition

$$T(x,0) = f(x) \quad \text{for} \quad 0 < x < L. \tag{3.68}$$

- **Remark:** The ends of the rod are thermally insulated because the heat flux $q = -k\partial T/\partial x$ vanishes there.

- Fourier's method is applied on problem sheet 5 to show that the solution is given by

$$T(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) \exp\left(-\frac{n^2\pi^2\kappa t}{L^2}\right), \tag{3.69}$$

where the constants a_n are the Fourier coefficients of the Fourier cosine series for f given by

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right) \mathrm{d}x. \tag{3.70}$$

- **Notes:**

(1) The constant separable and steady-state solution $T = a_0/2$ of (3.66)–(3.67) comes from the case in which the separation constant is zero.

(2) The Uniqueness Theorem in §3.6 may be adapted to show that the IBVP (3.66)–(3.68) has no more than one solution.

(3) Integrating (3.66) from $x = 0$ to $x = L$ and applying the boundary conditions (3.67), we find that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^L \rho c_v T(x,t)\,\mathrm{d}x = \left[k\frac{\partial T}{\partial x}\right]_{x=0}^{x=L} = 0, \tag{3.71}$$

which is an expression representing global conservation of energy: the thermal energy stored in the rod is constant because all of its surfaces are insulated. Integrating (3.71) and applying the initial condition (3.68) gives

$$\int_0^L \rho c_v T(x,t)\,\mathrm{d}x = \int_0^L \rho c_v f(x)\,\mathrm{d}x \quad \text{for } t > 0, \tag{3.72}$$

an expression that may be derived directly from the solution (3.69) assuming that the orders of summation and integration may be interchanged.

(4) The exponentially decaying modes in (3.69) imply that the temperature

$$T(x,t) \to \frac{a_0}{2} = \frac{1}{L}\int_0^L f(x)\,\mathrm{d}x \quad \text{as } t \to \infty, \tag{3.73}$$

i.e. the temperature tends to the mean of the initial temperature profile. This is because the rod is insulated so that heat conduction acts to drive the temperature toward the steady-state solution in which T is spatially uniform.

Example: trapped heat

- Consider the IBVP (3.66)–(3.68) in which the initial temperature profile is given by

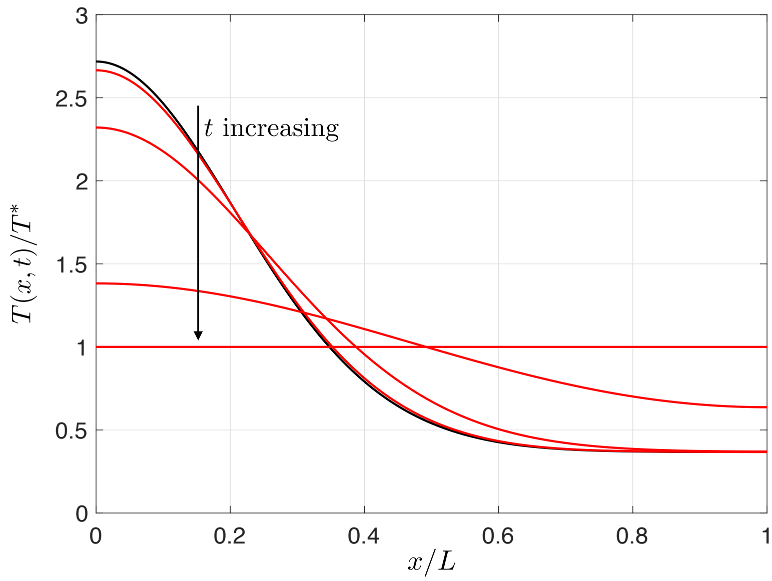
$$f(x) = T^* \exp\big(\cos(\pi x/L)\big)\cos\big(\sin(\pi x/L)\big) \quad \text{for } 0 < x < L, \tag{3.74}$$

where T^* is a positive constant.

- Since $a_0 = 2T^*$ and $a_n = T^*/n!$ for $n \geq 1$ by (1.5), the solution is given by

$$T(x,t) = T^* + \sum_{n=1}^{\infty} \frac{T^*}{n!} \cos\left(\frac{n\pi x}{L}\right) \exp\left(-\frac{n^2\pi^2\kappa t}{L^2}\right). \tag{3.75}$$

- We plot below snapshots of the partial sums of the truncated series solution with 6 terms for $\kappa t/L^2 = 0$ (black line) and $\kappa t/L^2 = 10^{-3}$, 10^{-2}, 10^{-1}, 1 (red lines), illustrating the rapid evolution toward the spatially uniform steady-state in which $T = T^*$. Since the thermal energy of the rod is conserved according to (3.72), the area under each curve is the same.



3.9 Inhomogeneous heat equation and boundary conditions

- Consider the IBVP for the temperature $T(x,t)$ in a rod of length L given by the inhomogeneous heat equation

$$\rho c_v \frac{\partial T}{\partial t} = k\frac{\partial^2 T}{\partial x^2} + Q(x,t) \quad \text{for} \quad 0 < x < L,\ t > 0, \tag{3.76}$$

with the inhomogeneous Neumann boundary conditions

$$-kT_x(0,t) = q_L(t), \quad -kT_x(L,t) = -q_R(t) \quad \text{for} \quad t > 0, \tag{3.77}$$

and the initial condition

$$T(x,0) = f(x) \quad \text{for} \quad 0 < x < L, \tag{3.78}$$

where $Q(x,t)$ is the rate of volumetric heating, $q_L(t)$ is the heat flux into the left-hand end, $q_R(t)$ is the heat flux into the right-hand end and $f(x)$ is the initial temperature profile, each of these functions being prescribed.

- **Notes:**

  (1) The Uniqueness Theorem in §3.6 may be adapted to show that the initial boundary value problem (3.76)–(3.78) has no more than one solution.

(2) Integrating (3.76) across the rod and applying the boundary conditions (3.77), we find that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^L \rho c_v T(x,t)\,\mathrm{d}x = q_L(t) + q_R(t) + \int_0^L Q(x,t)\,\mathrm{d}x, \tag{3.79}$$

which is an expression representing global conservation of energy: the thermal energy stored in the rod increases or decreases at the net rate at which thermal energy is supplied to the rod by the heat flux through its ends and by volumetric heating.

- In general Fourier's method cannot be used to solve the IBVP for $T(x,t)$ because the heat equation and boundary conditions are inhomogeneous, *i.e.* $Q(x,t)$, $q_L(t)$ and $q_R(t)$ are non-zero. We now describe a generalization of Fourier's method that works.

- We deal first with the boundary conditions: if we let

$$T(x,t) = S(x,t) + U(x,t), \tag{3.80}$$

where

$$S(x,t) = q_L(t)\frac{(x-L)^2}{2kL} + q_R(t)\frac{x^2}{2kL}, \tag{3.81}$$

say, is chosen to satisfy the boundary conditions (3.77), then by linearity the initial boundary value problem (3.76)–(3.78) for $T(x,t)$ implies that the initial boundary value problem for $U(x,t)$ is given by

$$\rho c_v \frac{\partial U}{\partial t} = k\frac{\partial^2 U}{\partial x^2} + \widetilde{Q}(x,t) \quad \text{for} \quad 0 < x < L,\ t > 0, \tag{3.82}$$

with the homogeneous Neumann boundary conditions

$$U_x(0,t) = 0, \quad U_x(L,t) = 0 \quad \text{for} \quad t > 0, \tag{3.83}$$

and the initial condition

$$U(x,0) = \widetilde{f}(x) \quad \text{for} \quad 0 < x < L, \tag{3.84}$$

where the functions

$$\widetilde{Q}(x,t) = Q(x,t) + k\frac{\partial^2 S}{\partial x^2} - \rho c_v \frac{\partial S}{\partial t}, \quad \widetilde{f}(x) = f(x) - S(x,0) \tag{3.85}$$

are known in terms of $Q(x,t)$, $q_L(t)$, $q_R(t)$ and $f(x)$.

- Thus, the boundary conditions have been rendered homogeneous by 'shifting the data' in the sense that both $q_L(t)$ and $q_R(t)$ have moved from the boundary conditions (3.77) for $T(x,t)$ into the heat equation (3.82) and initial conditions (3.84) for $U(x,t)$.

- If $\widetilde{Q} \equiv 0$, then we can solve the initial boundary value problem (3.82)–(3.84) for $U(x,t)$ using Fourier's method as in §3.8 to obtain the solution

$$U(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)\exp\left(-\frac{n^2\pi^2 kt}{\rho c_v L^2}\right), \qquad a_n = \frac{2}{L}\int_0^L \widetilde{f}(x)\cos\left(\frac{n\pi x}{L}\right)\mathrm{d}x, \tag{3.86}$$

where the Fourier coefficients a_n have been chosen to satisfy the initial condition (3.84).

- The series solution (3.86) for $U(x,t)$ suggests that if $\widetilde{Q}(x,t)$ is not identically zero, then we should seek a solution for $U(x,t)$ in the form of the Fourier cosine series

$$U(x,t) = \frac{U_0(t)}{2} + \sum_{n=1}^{\infty} U_n(t)\cos\left(\frac{n\pi x}{L}\right) \tag{3.87}$$

in which the Fourier coefficients $U_n(t)$ are to be determined.

- From the formulae for the Fourier coefficients of a Fourier cosine series, we deduce that $U_n(t)$ are given in terms of $U(x,t)$ by the integral expressions

$$U_n(t) = \frac{2}{L}\int_0^L U(x,t)\cos\left(\frac{n\pi x}{L}\right)\,\mathrm{d}x. \tag{3.88}$$

- **Question:** How do we derive an equation for $U_n(t)$?
- **Answer:** Inspired by the proof of the uniqueness theorem in §3.6, we differentiate $U_n(t)$ with respect to t to obtain

$$\rho c_v \frac{\mathrm{d}U_n}{\mathrm{d}t} = \frac{2}{L}\int_0^L \rho c_v \frac{\partial U}{\partial t}\cos\left(\frac{n\pi x}{L}\right)\mathrm{d}x = \frac{2}{L}\int_0^L \left(k\frac{\partial^2 U}{\partial x^2} + \widetilde{Q}\right)\cos\left(\frac{n\pi x}{L}\right)\mathrm{d}x, \tag{3.89}$$

where we used Leibniz's Integral Rule (3.3) in the first equality and the heat equation (3.82) in the second equality. Integrating by parts using the identity

$$\int_0^L uv'' - u''v\,\mathrm{d}x = \int_0^L (uv' - u'v)'\,\mathrm{d}x = \big[uv' - u'v\big]_0^L \tag{3.90}$$

with $u = U$ and $v = \cos(n\pi x/L)$ gives

$$\int_0^L U\left(-\frac{n^2\pi^2}{L^2}\cos\left(\frac{n\pi x}{L}\right)\right) - U_{xx}\cos\left(\frac{n\pi x}{L}\right)\mathrm{d}x = \left[U\left(-\frac{n\pi}{L}\right)\sin\left(\frac{n\pi x}{L}\right) - U_x\cos\left(\frac{n\pi x}{L}\right)\right]_0^L = 0$$

by the boundary conditions (3.83), so that

$$\frac{2}{L}\int_0^L U_{xx}\cos\left(\frac{n\pi x}{L}\right)\mathrm{d}x = -\frac{n^2\pi^2}{L^2}\frac{2}{L}\int_0^L U\cos\left(\frac{n\pi x}{L}\right)\mathrm{d}x = -\frac{n^2\pi^2}{L^2}U_n. \tag{3.91}$$

Hence, combining (3.89) and (3.91), we find that $U_n(t)$ is governed by the ODE

$$\rho c_v\frac{\mathrm{d}U_n}{\mathrm{d}t} + \frac{kn^2\pi^2}{L^2}U_n = \widetilde{Q}_n(t) \quad \text{for} \quad t > 0, \tag{3.92}$$

where the coefficients of the Fourier cosine series for $\widetilde{Q}(x,t)$ are defined by

$$\widetilde{Q}_n(t) = \frac{2}{L}\int_0^L \widetilde{Q}(x,t)\cos\left(\frac{n\pi x}{L}\right)\mathrm{d}x = \frac{2}{L}\int_0^L \left(Q(x,t) + k\frac{\partial^2 S}{\partial x^2} - \rho c_v\frac{\partial S}{\partial t}\right)\cos\left(\frac{n\pi x}{L}\right)\mathrm{d}x. \tag{3.93}$$

- The initial condition (3.84) for $U(x,t)$ implies that the initial condition for $U_n(t)$ is given by

$$U_n(0) = \frac{2}{L}\int_0^L \widetilde{f}(x)\cos\left(\frac{n\pi x}{L}\right)\,\mathrm{d}x = \frac{2}{L}\int_0^L \big(f(x) - S(x,0)\big)\cos\left(\frac{n\pi x}{L}\right)\,\mathrm{d}x. \tag{3.94}$$

- Using an integrating factor, we find that the solution of (3.92) subject to (3.94) may be written in the form

$$U_n(t) = \left(\frac{1}{\rho c_v}\int_0^t \widetilde{Q}_n(s)\mathrm{e}^{\kappa_n s}\,\mathrm{d}s + U_n(0)\right)\mathrm{e}^{-\kappa_n t}, \tag{3.95}$$

where $\kappa_n = n^2\pi^2\kappa/L^2$ in terms of the thermal diffusivity $\kappa = k/(\rho c_v)$.
- In summary, we have been able to solve analytically the initial boundary value problem (3.76)–(3.78) for $T(x,t)$: by (3.80) and (3.87), the solution is given by

$$T(x,t) = S(x,t) + \frac{U_0(t)}{2} + \sum_{n=1}^{\infty} U_n(t)\cos\left(\frac{n\pi x}{L}\right), \tag{3.96}$$

where $S(x,t)$ is given by (3.81) and $U_n(t)$ by (3.95).

- **Notes:**

  (1) If $\widetilde{Q}(x,t) = 0$ in (3.85), then $\widetilde{Q}_n(t) = 0$ in (3.93) and we recover from (3.94)–(3.95) the solution (3.86) for $U_n(t)$ obtained by Fourier's method.

  (2) By (3.81) and (3.96), the total thermal energy in the rod is given by

$$\int_0^L \rho c_v T(x,t)\,\mathrm{d}x = \rho c_v \left(\frac{L^2 q_L}{6k} + \frac{L^2 q_R}{6k} + \frac{LU_0(t)}{2} \right), \tag{3.97}$$

  from which it follows that the ordinary differential equation for $U_0(t)$ in (3.92) is equivalent to the expression (3.79) representing global conservation of energy.

  (3) The derivation of the ODE (3.92) for $U_n(t)$ may also be accomplished by multiplying the heat equation (3.82) by $\cos(n\pi x/L)$ and integrating from $x = 0$ to $x = L$ to obtain

$$\int_0^L \left(\rho c_v \frac{\partial U}{\partial t} - k\frac{\partial^2 U}{\partial x^2} - \widetilde{Q}(x,t) \right) \cos\left(\frac{n\pi x}{L}\right) \mathrm{d}x = 0; \tag{3.98}$$

  the ODE then follows upon applying Leibniz's integral rule to the U_t term and integrating by parts the U_{xx} term.

  (4) **Question:** What are the advantages of expanding U as a Fourier cosine series rather than T?
  Answer: Expanding T as a Fourier cosine series is equivalent to expanding S as a Fourier cosine series in (3.96), which cannot improve the accuracy of the approximate solution that would be obtained by truncation. In general the method of shifting the data (to render homogeneous the boundary conditions) results in a solution that converges more rapidly, especially if Gibb's phenomenon can be avoided by doing so.

Example: sinusoidal forcing

- Consider the initial boundary value problem (3.76)–(3.78) in the case in which

$$q_L(t) = q^* \sin(\omega t), \quad q_R(t) = 0, \quad Q(x,t) = 0, \quad f(x) = 0, \tag{3.99}$$

  where q^* and ω are positive constants, as if the left-hand end of the rod were radiated sinusoidally.

- By (3.93),

$$\widetilde{Q}_n(s) = \begin{cases} \dfrac{2q^*}{L}\sin(\omega s) - \dfrac{\omega L q^*}{3\kappa}\cos(\omega s) & \text{for } n = 0, \\ -\dfrac{2\omega L q^*}{\kappa n^2\pi^2}\cos(\omega s) & \text{for } n \geq 1, \end{cases} \tag{3.100}$$

  while (3.94) gives $U_n(0) = 0$ for $n \geq 0$.

- Substituting these expressions into (3.95) and integrating gives

$$U_n(t) = \begin{cases} \dfrac{2\kappa T^*}{\omega L^2}\big(1 - \cos(\omega t)\big) - \dfrac{T^*}{3}\sin(\omega t) & \text{for } n = 0, \\ \dfrac{2\omega T^*}{n^2\pi^2(\kappa_n^2 + \omega^2)}\Big(\kappa_n\cos(\omega t) + \omega\sin(\omega t) - \kappa_n\exp(-\kappa_n t)\Big) & \text{for } n \geq 1, \end{cases} \tag{3.101}$$

  where we defined the temperature $T^* = Lq^*/k$.

- Substituting into (3.96), we find that the solution may be written in the form

$$T(x,t) = T_\infty(x,t) + V(x,t), \tag{3.102}$$

  where

$$\begin{aligned} T_\infty(x,t) &= T^*\sin(\omega t)\frac{(x-L)^2}{2L^2} + \frac{\kappa T^*}{\omega L^2}\big(1 - \cos(\omega t)\big) - \frac{T^*}{6}\sin(\omega t) \\ &\quad + \sum_{n=1}^{\infty} \frac{2\omega T^*}{n^2\pi^2(\kappa_n^2 + \omega^2)}\Big(\kappa_n\cos(\omega t) + \omega\sin(\omega t)\Big)\cos\left(\frac{n\pi x}{L}\right) \end{aligned} \tag{3.103}$$

and

$$V(x,t) = -\sum_{n=1}^{\infty} \frac{2\kappa_n \omega T^*}{n^2\pi^2(\kappa_n^2+\omega^2)} \exp(-\kappa_n t) \cos\left(\frac{n\pi x}{L}\right). \tag{3.104}$$

- Since $V(x,t)$ decays exponentially with t, the solution settles down rapidly to a periodic solution $T_\infty(x,t)$ with frequency ω. Since $V(x,t)$ satisfies the homogeneous versions of the heat equation (3.76) and boundary conditions (3.77), the long-time solution $T_\infty(x,t)$ satisfies the same heat equation, but the inhomogeneous boundary conditions given by (3.77) and (3.99). We now show that these properties of the long-time solution can be used to construct it directly.

- The trick is to seek a complex-valued separable solution $\mathrm{e}^{\mathrm{i}\omega t}F(x)$ with frequency ω. Substituting this *ansatz* into the heat equation (3.76) with $Q(x,t)=0$, we find that it can only be satisfied if

$$\kappa F'' = \mathrm{i}\omega F \quad \text{for } 0 < x < L. \tag{3.105}$$

- Seeking an exponential solution $F(x) = \mathrm{e}^{\lambda x}$ to (3.105) gives the auxiliary equation $\lambda^2 = \mathrm{i}\omega/\kappa$, so that

$$\lambda = \pm\sqrt{\frac{\omega}{\kappa}}\mathrm{e}^{\mathrm{i}\pi/4} = \pm\sqrt{\frac{\omega}{2\kappa}}(1+\mathrm{i}), \tag{3.106}$$

giving the general solution

$$F(x) = A\mathrm{e}^{\nu(1+\mathrm{i})x/L} + B\mathrm{e}^{-\nu(1+\mathrm{i})x/L}, \tag{3.107}$$

where A and B are arbitrary complex constants and $\nu = L\sqrt{\omega/2\kappa}$ is a dimensionless parameter.

- We now observe that if we impose on F the boundary conditions $-kF'(0) = q^*$ and $F'(L) = 0$, then

$$T_p(x,t) = \mathrm{Im}\big(\mathrm{e}^{\mathrm{i}\omega t}F(x)\big) \tag{3.108}$$

satisfies both the heat equation and boundary conditions given by (3.76), (3.77) and (3.99) because taking the imaginary part commutes with partial differentiation.

- The resulting solution for $F(x)$ may then be written in the form

$$F(x) = \frac{T^* \cosh\big(\nu(1+\mathrm{i})(1-x/L)\big)}{\nu(1+\mathrm{i})\sinh\big(\nu(1+\mathrm{i})\big)}, \tag{3.109}$$

so that

$$T_p(x,t) = \mathrm{Im}\left(\frac{T^* \cosh\big(\nu(1+\mathrm{i})(1-x/L)\big)}{\nu(1+\mathrm{i})\sinh\big(\nu(1+\mathrm{i})\big)}\mathrm{e}^{i\omega t}\right), \tag{3.110}$$

satisfies (3.76) and (3.77) subject to (3.99).

- **Question:** How are the solutions $T_\infty(x,t)$ and $T_p(x,t)$ related?

- **Answer:** Having found a particular solution $T_p(x,t)$ satisfying the heat equation and boundary conditions, we see that we could also solve the initial boundary value problem (3.76)–(3.78) for $T(x,t)$ by setting

$$T(x,t) = T_p(x,t) + W(x,t), \tag{3.111}$$

since then $W(x,t)$ satisfies

$$\rho c_v \frac{\partial W}{\partial t} = k\frac{\partial^2 W}{\partial x^2} \quad \text{for} \quad 0 < x < L,\ t > 0, \tag{3.112}$$

with

$$W_x(0,t) = 0, \quad W_x(L,t) = 0 \quad \text{for} \quad t > 0, \tag{3.113}$$

and

$$W(x,0) = -T_p(x,0) \quad \text{for} \quad 0 < x < L, \tag{3.114}$$

i.e. the boundary conditions are rendered homogeneous by the *ansatz* (3.111) while retaining the homogeneity of the heat equation, in contrast to the *ansatz* (3.80) which results in homogeneous boundary conditions but at the expense of an inhomogeneous heat equation.

- The initial boundary value problem (3.112)–(3.114) for $W(x,t)$ may be solved using Fourier's method as in §3.8, giving the solution

$$W(x,t) = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n \cos\left(\frac{n\pi x}{L}\right) \exp\left(-\frac{n^2\pi^2 kt}{\rho c_v L^2}\right), \qquad c_n = -\frac{2}{L}\int_0^L T_p(x,0)\cos\left(\frac{n\pi x}{L}\right)\,\mathrm{d}x, \quad (3.115)$$

so that

$$W(x,t) \to \frac{c_0}{2} = -\frac{1}{L}\int_0^L T_p(x,0)\,\mathrm{d}x = \frac{\kappa T^*}{\omega L^2} \quad \text{as } t \to \infty. \quad (3.116)$$

- We can now invoke uniqueness of the initial boundary value problem (3.76)–(3.78) for $T(x,t)$ to deduce that

$$T_\infty(x,t) + V(x,t) = T_p(x,t) + W(x,t) \quad \text{for } 0 \le x \le L,\ t \ge 0. \quad (3.117)$$

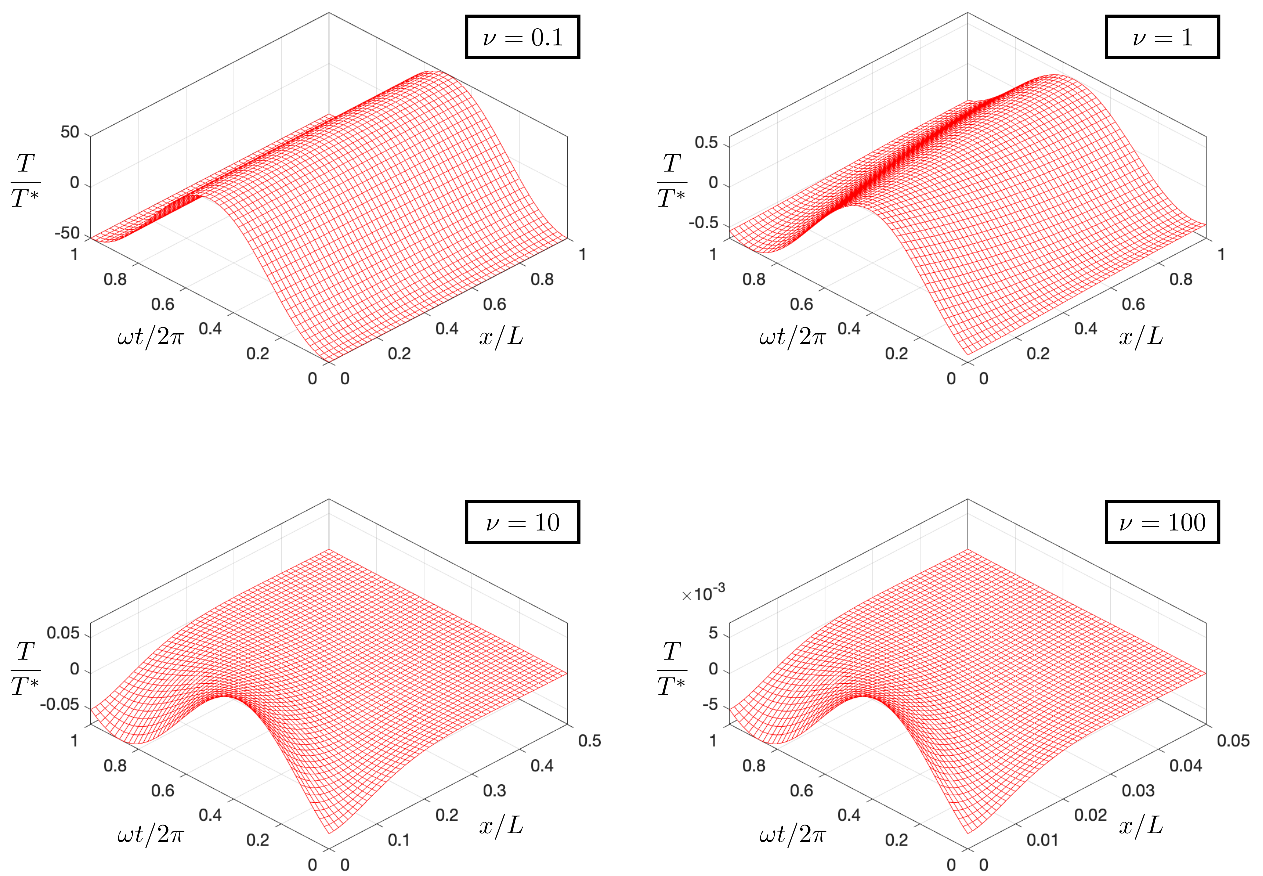
But $V(x,t) \to 0$ and $W(x,t) \to \kappa T^*/\omega L^2$ as $t \to \infty$, which can only be the case if

$$T_\infty(x,t) = T_p(x,t) + \frac{\kappa T^*}{\omega L^2} \quad \text{for } 0 \le x \le L,\ t \ge 0 \quad (3.118)$$

because both $T_\infty(x,t)$ and $T_p(x,t)$ are periodic in t with frequency ω.

- **Remark:** This argument saves us from the unwieldy algebraic manipulations that would otherwise be required to establish the relationship (3.118), *e.g.* by showing that the Fourier cosine coefficients of each side are identical at say $t = 0$.

- The plots below show a period of oscillation of $T_p(x,t)$ for $\nu = 0.1$, 1, 10 and 100. The plots illustrate that the heat flux imposed at $x = 0$ generates a temperature profile that is almost spatially uniform for small ν, but penetrates only partially and inside a thin boundary layer of thickness of order L/ν for large ν. This is in accordance with the physical interpretation of $\nu = L/\sqrt{2\kappa/\omega}$ as the ratio of the length of the rod L to the typical distance thermal energy conducts in a period of oscillation (since there is a balance in the heat equation when x and t are scaled with $\sqrt{\kappa/\omega}$ and $1/\omega$, respectively). That the shape of the profiles for $\nu = 10$ and $\nu = 100$ are almost identical is because the response in the thin boundary layer is as if the rod were semi-infinite.

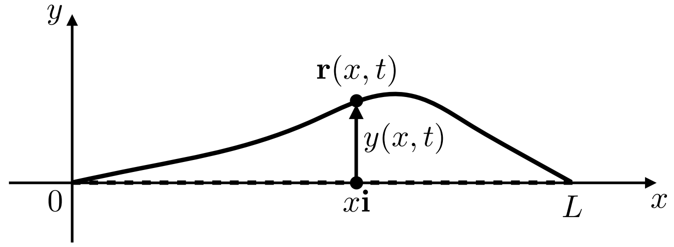


■

4 The wave equation

4.1 Derivation of the one-dimensional wave equation

- Consider the small transverse vibrations of a homogeneous extensible elastic string stretched initially along the x-axis at time $t = 0$ to a length L.
- A point at $x\boldsymbol{i}$ at time $t = 0$ is displaced to $\boldsymbol{r}(x,t) = x\boldsymbol{i} + y(x,t)\boldsymbol{j}$ at time $t > 0$, where the transverse displacement $y(x,t)$ is to be determined. We illustrate the geometrical setup in the schematic below.



- Consider the section of string in the fixed region $a \le x \le a+h$, where a and h are arbitrary constants.
- The linear momentum of the section of the string in $a \le x \le a+h$ is

$$\int_a^{a+h} \rho \frac{\partial \boldsymbol{r}}{\partial t}\, \mathrm{d}x, \tag{4.1}$$

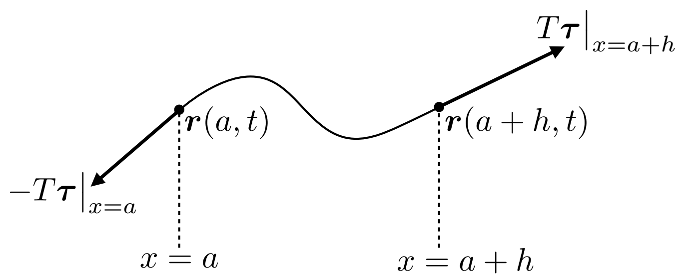
where ρ is the constant line density of the string (with $[\rho] = \mathrm{kg\,m^{-1}}$).

- The string offers no resistance to bending (*cf.* a ruler) in the sense that the string to the right of the point $\boldsymbol{r}(x,t)$ exerts at that point a tangential force $T(x,t)\boldsymbol{\tau}(x,t)$ on the string to the left, where $T(x,t)$ is the tension ($[T] = \mathrm{N} = \mathrm{kg\,m\,s^{-2}}$) and $\boldsymbol{\tau} = \boldsymbol{r}_x/|\boldsymbol{r}_x|$ is the unit tangent vector pointing in the positive x-direction. Note that Newton's third law implies that the string to the left of the point $\boldsymbol{r}(x,t)$ exerts at that point a tangential force $-T(x,t)\boldsymbol{\tau}(x,t)$ on the string to the right.
- Assuming the tension is so large that the effects of gravity and air resistance may be neglected, the forces acting on the ends of the section of string in $a \le x \le a+h$ are

  (i) the force $T(a+h,t)\boldsymbol{\tau}(a+h,t)$ exerted at the right-hand end at $\boldsymbol{r}(a+h,t)$ by the string to the right of the section;

  (ii) the force $-T(a,t)\boldsymbol{\tau}(a,t)$ exerted at the left-hand end at $\boldsymbol{r}(a,t)$ by the string to the left of the section.

  We illustrate the forces and where they act on the section in the schematic below.



- We are now in a position to apply Newton's Second Law, which states that the rate of change of the linear momentum of the section of string in $a \le x \le a+h$ is equal to the net force acting on it, so that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_a^{a+h} \rho \frac{\partial \boldsymbol{r}}{\partial t}\, \mathrm{d}x\right) = T(a+h,t)\boldsymbol{\tau}(a+h,t) - T(a,t)\boldsymbol{\tau}(a,t). \tag{4.2}$$

- Assuming $\boldsymbol{r}_{tt}$ is continuous, Leibniz's Integral Rule (3.3) with a and $a+h$ constant gives

$$\frac{1}{h}\int_a^{a+h} \rho\frac{\partial^2 \boldsymbol{r}}{\partial t^2}\,\mathrm{d}x = \frac{T(a+h,t)\boldsymbol{\tau}(a+h,t) - T(a,t)\boldsymbol{\tau}(a,t)}{h}, \tag{4.3}$$

where we divided by h in anticipation of taking the limit $h \to 0$.

- To take the limit $h \to 0$, we apply the Fundamental Theorem of Calculus (3.1) to the first term (assuming $\boldsymbol{r}_{tt}$ is continuous in a neighbourhood of a) and use the definition of the partial derivative of $T\boldsymbol{\tau}$ with respect to x (assuming it to exist and to be continuous at a) to obtain

$$\rho\frac{\partial^2 \boldsymbol{r}}{\partial t^2} = \frac{\partial}{\partial x}(T\boldsymbol{\tau}). \tag{4.4}$$

- Recalling the definitions of $\boldsymbol{r}$ and $\boldsymbol{\tau}$, it follows that

$$\rho\frac{\partial^2 y}{\partial t^2}\boldsymbol{j} = \frac{\partial}{\partial x}\left(\frac{T\boldsymbol{i} + Ty_x\boldsymbol{j}}{(1+y_x^2)^{1/2}}\right). \tag{4.5}$$

- But we are also assuming that the transverse displacement is small in the sense that the slope of the string is small, *i.e.* $|y_x| \ll 1$.

- Since a Taylor expansion gives

$$\left(1+y_x^2\right)^{1/2} = 1 + \frac{1}{2}(y_x)^2 + \cdots \quad \text{for} \quad |y_x| \ll 1, \tag{4.6}$$

to a first approximation, *i.e.* neglecting quadratic and higher order terms,

$$\rho\frac{\partial^2 y}{\partial t^2}\boldsymbol{j} = \frac{\partial}{\partial x}(T\boldsymbol{i} + Ty_x\boldsymbol{j}). \tag{4.7}$$

- **Remark:** We call (4.7) the *linearized* version of (4.5).

- The x-component of (4.7) implies that the tension T is spatially uniform, but could vary with time t, *e.g.* as when tuning a guitar string. We shall take the tension T to be constant, which is the case in many practical applications.

- The y-component of (4.7) then implies that

$$\rho\frac{\partial^2 y}{\partial t^2} = T\frac{\partial^2 y}{\partial x^2}, \tag{4.8}$$

giving the *wave equation*

$$\frac{\partial^2 y}{\partial t^2} = c^2\frac{\partial^2 y}{\partial x^2}, \tag{4.9}$$

where $c = \sqrt{T/\rho}$ is the *wave speed* (for reasons that will become apparent).

- The wave equation (4.8) is a second-order linear partial differential equation.

4.2 Units and nondimensionalisation

- In §4.1 we showed that the small transverse displacement $y(x,t)$ of an elastic string is governed by wave equation

$$\frac{\partial^2 y}{\partial t^2} = c^2\frac{\partial^2 y}{\partial x^2}, \tag{4.10}$$

where $c > 0$ is the constant wave speed.

- Consider the units of the variables (x, t and y) and parameter (c) in the wave equation. Since

$$[y_{tt}] = \mathrm{m\,s^{-2}}, \quad [y_{xx}] = \mathrm{m\,m^{-2}}, \tag{4.11}$$

  it follows that

$$\left[c^2\right] = \frac{[y_{tt}]}{[y_{xx}]} = \mathrm{m^2\,s^{-2}}, \tag{4.12}$$

  so that $[c] = \mathrm{m\,s^{-1}}$, *i.e.* c has the units of speed.

- **Question:** On what timescale does a displacement travel a distance L?

- **Answer:** If we nondimensionalize by scaling $x = L\hat{x}$, $t = t_0\hat{t}$, $y = H\hat{y}(\hat{x}, \hat{t}\,)$, then the wave equation becomes

$$\frac{H}{t_0^2}\frac{\partial^2 \hat{y}}{\partial \hat{t}^2} = \frac{Hc^2}{L^2}\frac{\partial^2 \hat{y}}{\partial \hat{x}^2}; \tag{4.13}$$

  the terms balance giving

$$\frac{\partial^2 \hat{y}}{\partial \hat{t}^2} = \frac{\partial^2 \hat{y}}{\partial \hat{x}^2} \tag{4.14}$$

  provided $t_0 = L/c$, which is therefore the timescale for a displacement to travel a distance L.

4.3 Normal modes of vibration for a finite string

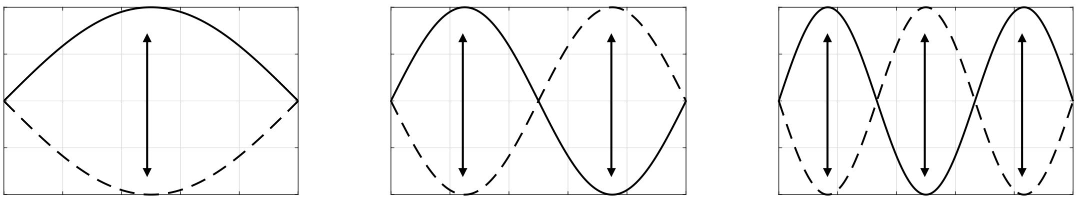
- Suppose an elastic string is stretched between $x = 0$ and $x = L$ and the ends held fixed, so that the small transverse displacement $y(x,t)$ of the string is governed by the wave equation

$$\frac{\partial^2 y}{\partial t^2} = c^2\frac{\partial^2 y}{\partial x^2} \quad \text{for} \quad 0 < x < L, \tag{4.15}$$

  with the boundary conditions

$$y(0,t) = 0 \quad y(L,t) = 0. \tag{4.16}$$

- An experiment with a slinky suggests there exist discrete modes of vibration, as illustrated in the schematic below.



- To analyse mathematically the possible modes of vibration, we seek nontrivial separable solutions of the form $y = F(x)G(t)$.

- Substituting this expression into the wave equation (4.15) gives $F(x)G''(t) = c^2F''(x)G(t)$, so we may separate the variables for $FG \neq 0$ to obtain

$$\frac{F''(x)}{F(x)} = \frac{G''(t)}{c^2G(t)}. \tag{4.17}$$

- The left-hand side of this expression is independent of t, while the right-hand side is independent of x. Since the left-hand side is equal to the right-hand side, they must both be independent of x and t, and therefore equal to a constant, $-\lambda \in \mathbb{R}$ say.

- The boundary condition at $x = 0$ in (4.16) implies that $F(0)G(t) = 0$ for $t > 0$. Since we're seeking solutions y that are nontrivial, there must exist a time $t > 0$ such that $G(t) \neq 0$, and hence we must impose on $F(x)$ the boundary condition $F(0) = 0$. Similarly, the boundary condition at $x = L$ in (4.16) implies that $F(L) = 0$.

- In summary, we have deduced that $F(x)$ and λ satisfies the boundary value problem given by the ordinary differential equation
$$-F''(x) = \lambda F(x) \quad \text{for} \quad 0 < x < L, \tag{4.18}$$
with the boundary conditions
$$F(0) = 0, \quad F(L) = 0. \tag{4.19}$$

- We solved this problem in §3.4: the nontrivial solutions are given for positive integers n by
$$F(x) = B\sin\left(\frac{n\pi x}{L}\right), \quad \lambda = \left(\frac{n\pi}{L}\right)^2, \tag{4.20}$$
where B is an arbitrary constant; since $G'' + \lambda c^2 G = 0$, the corresponding solution for $G(t)$ is given by
$$G(t) = C\cos\left(\frac{n\pi ct}{L}\right) + D\sin\left(\frac{n\pi ct}{L}\right), \tag{4.21}$$
where C and D are arbitrary constants.

- Since $T(x,t) = F(x)G(t)$, we conclude that the nontrivial separable solutions or the *normal modes* of (4.15)–(4.16) are given for positive integers n by
$$y_n(x,t) = \sin\left(\frac{n\pi x}{L}\right)\left(a_n\cos\left(\frac{n\pi ct}{L}\right) + b_n\sin\left(\frac{n\pi ct}{L}\right)\right) \tag{4.22}$$
where a_n and b_n are arbitrary constants (with $a_n = BC$ and $b_n = BD$) and we have introduced the subscript n to enumerate the countably infinite set of such solutions.

- **Notes:**

  (1) The normal mode $y_n(x,t)$ is periodic in t with prime period
  $$p = \frac{2\pi}{n\pi c/L} = \frac{2L}{nc} \tag{4.23}$$
  and *frequency* or *pitch*
  $$\frac{1}{p} = \frac{nc}{2L}. \tag{4.24}$$

  (2) The first normal mode y_1 is called the *fundamental mode*, with associated *fundamental frequency* $c/(2L)$. All of the other modes have a frequency that is an integer multiple of the fundamental frequency.

  (3) The predictions are consistent with the slinky experiment.

  (4) The normal modes are an example of a standing wave because y_n is equal to a function of x multiplied by an oscillatory function of time.

4.4 Initial boundary value problem for a finite string

- Consider the initial boundary value problem for the small transverse displacement $y(x,t)$ of an elastic string given by the wave equation
$$\frac{\partial^2 y}{\partial t^2} = c^2\frac{\partial^2 y}{\partial x^2} \quad \text{for} \quad 0 < x < L,\ t > 0, \tag{4.25}$$
with the Dirichlet boundary conditions
$$y(0,t) = 0, \quad y(L,t) = 0 \quad \text{for} \quad t > 0, \tag{4.26}$$
and the two initial conditions
$$y(x,0) = f(x), \quad \frac{\partial y}{\partial t}(x,0) = g(x) \quad \text{for} \quad 0 < x < L, \tag{4.27}$$
where the initial transverse displacement $f(x)$ and the initial transverse velocity $g(x)$ are given.

- **Remark:** The total number of boundary (initial) conditions is equal to the number of spatial (temporal) partial derivatives in the wave equation.

- We will use Fourier's method to find a series solution.

Step (I): Find all nontrivial separable solutions of the PDE and BCs

- We found above that these are the normal modes given for positive integers n by

$$y_n(x,t) = \sin\left(\frac{n\pi x}{L}\right)\left(a_n\cos\left(\frac{n\pi ct}{L}\right) + b_n\sin\left(\frac{n\pi ct}{L}\right)\right), \tag{4.28}$$

  where a_n and b_n are arbitrary real constants.

Step II: Apply the principle of superposition

- Since (4.25)–(4.26) are linear, we can superimpose the normal modes (assuming convergence) to obtain the general series solution

$$y(x,t) = \sum_{n=1}^{\infty} y_n(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right)\left(a_n\cos\left(\frac{n\pi ct}{L}\right) + b_n\sin\left(\frac{n\pi ct}{L}\right)\right). \tag{4.29}$$

Step III: Use the theory of Fourier series to satisfy the ICs

- The initial conditions (4.27) can only be satisfied by (4.29) if

$$f(x) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right) \text{ for } 0 < x < L, \tag{4.30}$$

$$g(x) = \sum_{n=1}^{\infty} \frac{n\pi c}{L} b_n \sin\left(\frac{n\pi x}{L}\right) \text{ for } 0 < x < L. \tag{4.31}$$

- Hence, a_n is the nth Fourier coefficient of the Fourier sine series for f, while $n\pi c b_n/L$ is the nth Fourier coefficient of the Fourier sine series for g, *i.e.*, for positive integers n,

$$a_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)\mathrm{d}x, \quad \frac{n\pi c}{L}b_n = \frac{2}{L}\int_0^L g(x)\sin\left(\frac{n\pi x}{L}\right)\mathrm{d}x. \tag{4.32}$$

Example: plucking a guitar string

- Consider the initial boundary value problem (4.25)–(4.27) with

$$f(x) = \begin{cases} 2hx/L & \text{for } 0 \le x \le L/2, \\ 2h(L-x)/L & \text{for } L/2 \le x \le L, \end{cases} \qquad g(x) = 0, \tag{4.33}$$

  where h is a constant.

- By (4.32), we find

$$a_n = \frac{2}{L}\int_0^{L/2} \frac{2hx}{L}\sin\left(\frac{n\pi x}{L}\right)\mathrm{d}x + \frac{2}{L}\int_{L/2}^{L} \frac{2h(L-x)}{L}\sin\left(\frac{n\pi x}{L}\right)\mathrm{d}x = \frac{8h}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right) \tag{4.34}$$

  after an integration by parts (see problem sheet 6), while $b_n = 0$.

- Since

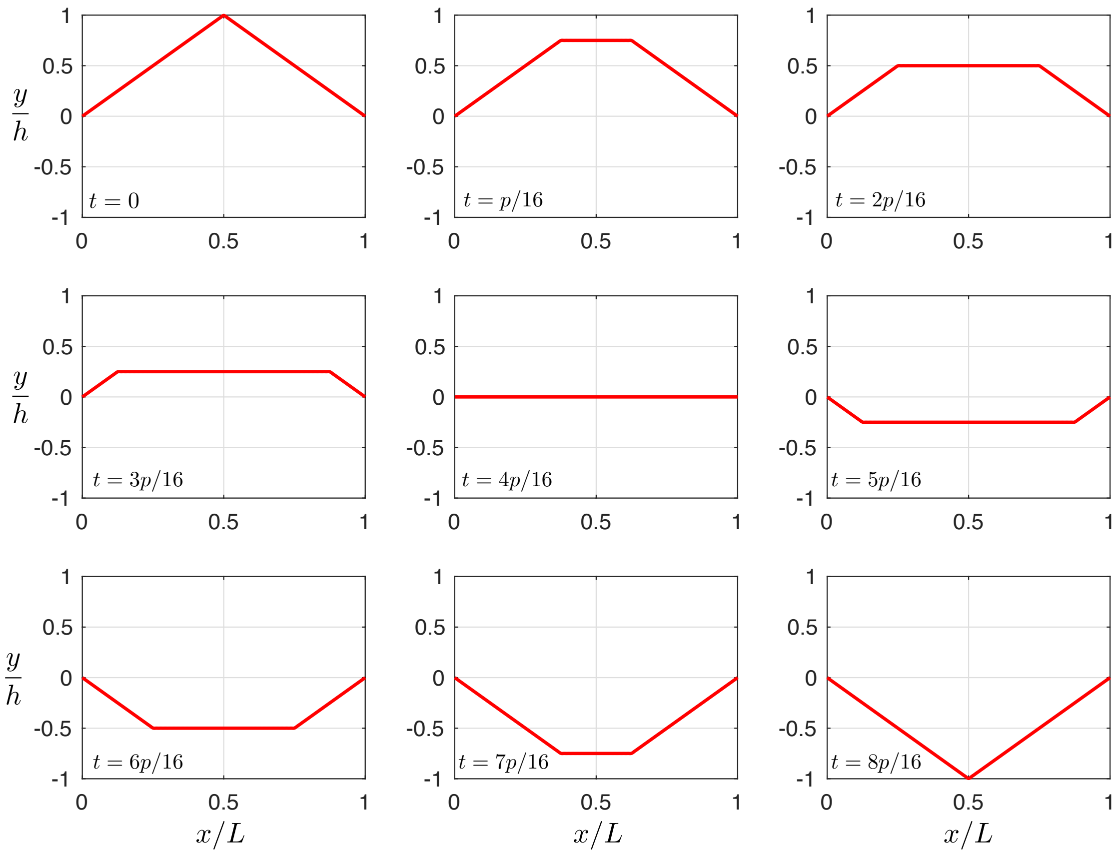
$$\sin\left(\frac{n\pi}{2}\right) = \begin{cases} 0 & \text{for } n = 2m,\ m \in \mathbb{N}\setminus\{0\}, \\ (-1)^m & \text{for } n = 2m+1,\ m \in \mathbb{N}, \end{cases}$$

  it follows from (4.29) that a series solution is given by

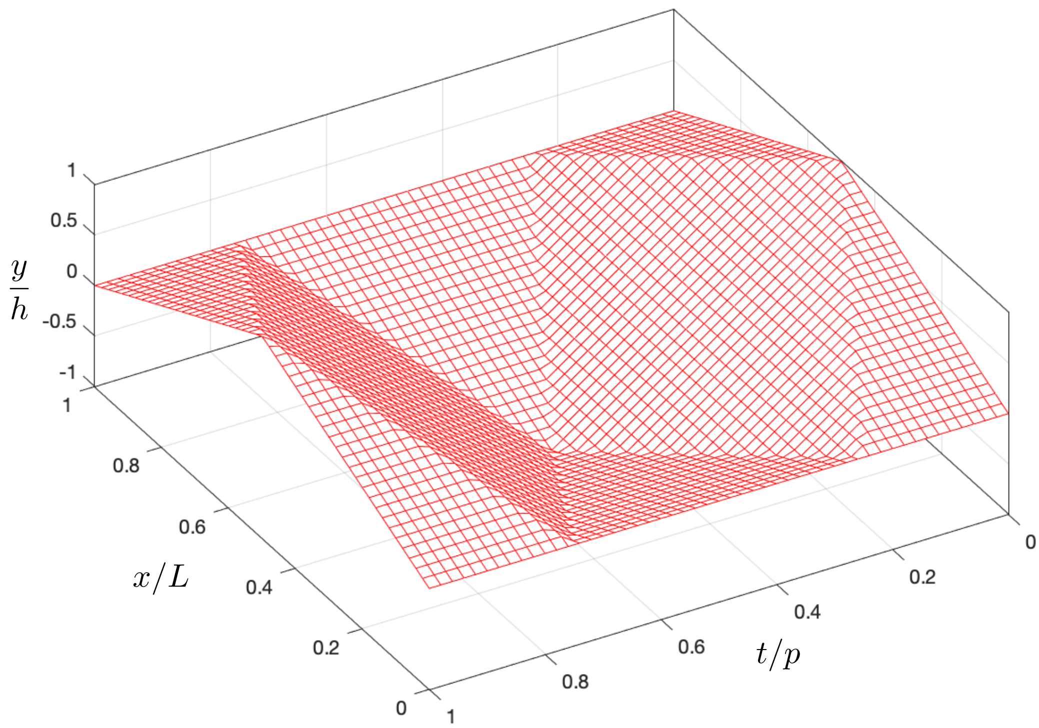
$$y(x,t) = \frac{8h}{\pi^2}\sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)^2}\sin\left(\frac{(2m+1)\pi x}{L}\right)\cos\left(\frac{(2m+1)\pi ct}{L}\right), \tag{4.35}$$

  so that $p = 2L/c$ is the prime period of the oscillation.

- We plot below snapshots of the series solution truncated to 128 terms over the first half-period, which illustrates the persistence of corners moving with speed c.



- The mesh plot below shows the series solution again truncated to 128 terms, but this time over the first period, with the orientation chosen for a good view.



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Example: hammering a piano string

- Consider the initial boundary value problem (4.25)–(4.27) with

$$f(x) = 0, \quad g(x) = \begin{cases} v & \text{for } L_1 \le x \le L_2, \\ 0 & \text{otherwise,} \end{cases} \tag{4.36}$$

where v, L_1 and L_2 are constants.

- By (4.32), we have $a_n = 0$ and

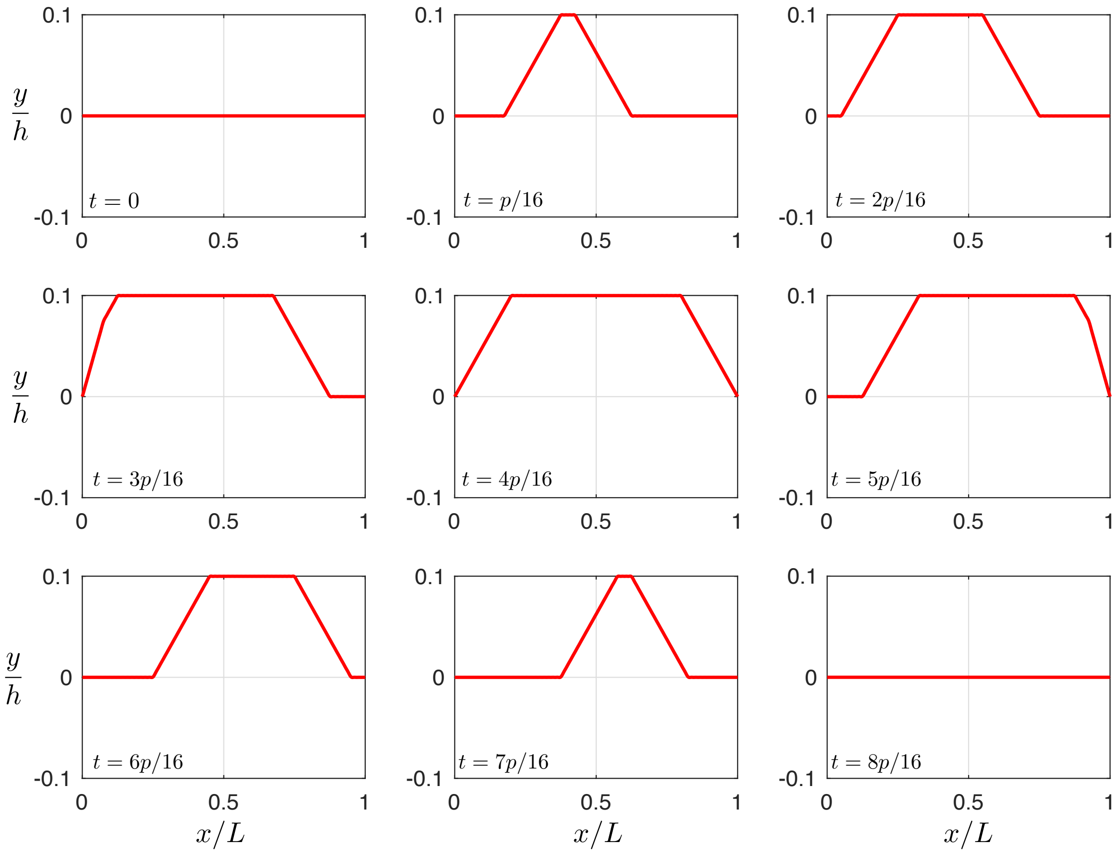
$$\frac{n\pi c}{L} b_n = \frac{2}{L} \int_{L_1}^{L_2} v \sin\left(\frac{n\pi x}{L}\right) \mathrm{d}x = \frac{2v}{n\pi} \left[\cos\left(\frac{n\pi L_1}{L}\right) - \cos\left(\frac{n\pi L_2}{L}\right)\right]. \tag{4.37}$$

- It follows from (4.29) that a series solution is given by

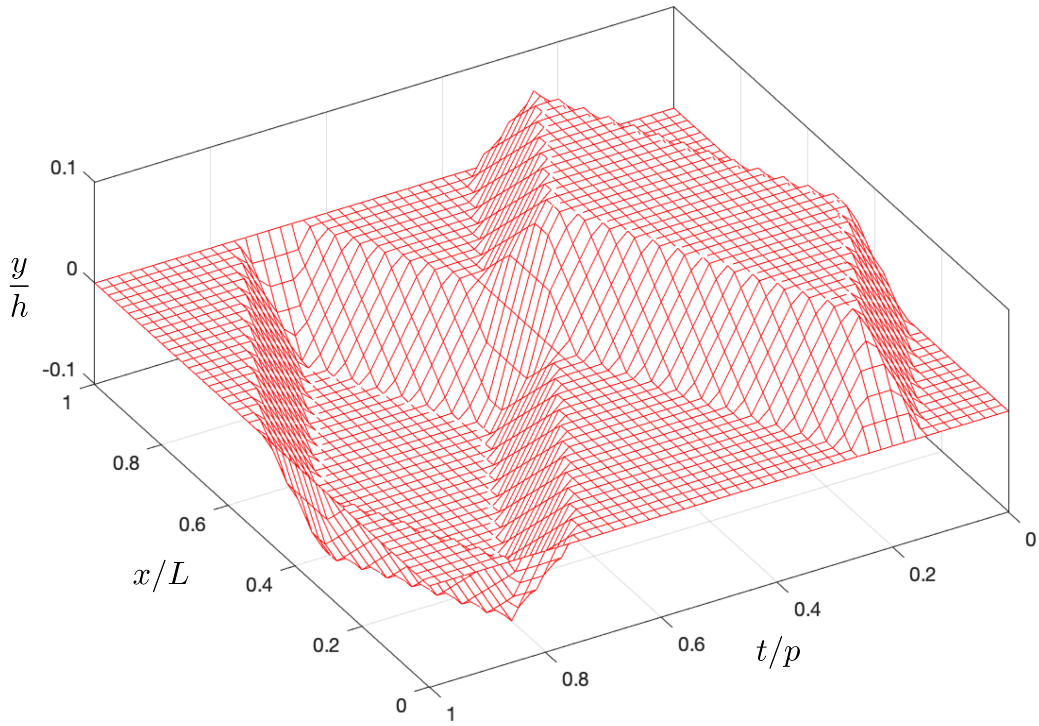
$$y(x,t) = \frac{2h}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \left[\cos\left(\frac{n\pi L_1}{L}\right) - \cos\left(\frac{n\pi L_2}{L}\right)\right] \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{n\pi ct}{L}\right), \tag{4.38}$$

where $h = vL/c$ and we see that the prime period of the oscillation is again $p = 2L/c$.

- We plot below show snapshots of the evolution of the series solution truncated to 128 terms for $L_1/L = 0.3$, $L_2/L = 0.5$ over the first half-period, which again illustrates the persistence of corners moving with speed c.



- The mesh plot below shows the series solution again truncated to 128 terms, but this time over the first period, with the orientation chosen for a good view.



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Notes:

(1) Both the guitar solution (4.35) and piano solution (4.38) contain persistent corners travelling with speed c. This means that neither solution can be twice continuously differentiable with respect to x or t, and hence a so-called *classical solution* of the wave equation. However, if we were to modify the initial data by smoothing off the corners and jump discontinuities in small neighbourhoods of these irregularities in such a way that the new initial data is infinitely differentiable (*e.g.* by truncating the Fourier or Fejér series for the original initial data after a large number of terms), then the new solutions would also be infinitely differentiable, and hence classical solutions, and they would be "close" in some sense to the original solutions. Hence, we do not want to discount the series solutions we have found, but to view them them instead as motivation to *weaken* the sense in which a function can be a solution of a PDE — the resulting notion of a *weak solution* forms the basis for the modern theory of PDEs that can be studied further on in the course in *e.g.* B4.3 and B5.2.

(2) The differences in the makeup of the normal modes for the guitar and piano solutions contribute to the different timbres of the musical instruments

4.5 Conservation of energy

- Suppose an elastic string is stretched between $x = 0$ and $x = L$ along the x-axis to a line density ρ and a tension T, so that its small transverse displacement $y(x,t)$ is governed by the wave equation

$$\rho \frac{\partial^2 y}{\partial t^2} = T \frac{\partial^2 y}{\partial x^2} \quad \text{for} \quad 0 < x < L, \ t > 0, \tag{4.39}$$

with the Dirichlet boundary conditions

$$y(0,t) = 0, \quad y(L,t) = 0 \quad \text{for} \quad t > 0, \tag{4.40}$$

and the initial conditions

$$y(x,0) = f(x), \quad \frac{\partial y}{\partial t}(x,0) = g(x) \quad \text{for} \quad 0 < x < L, \tag{4.41}$$

where the initial transverse displacement is $f(x)$ and the initial transverse velocity is $g(x)$.

- Recall that the point of the string that lies at $x\mathbf{i}$ in its so-called *reference configuration* is displaced transversely to the point with position vector $\mathbf{r}(x,t) = x\mathbf{i} + y(x,t)\mathbf{j}$. We note that when we impose the initial conditions (4.41), we must deform the string from its reference configuration along the x-axis to have transverse displacement $y(x,0) = f(x)$ and we must impart on the string the transverse velocity given by $y_t(x,0) = g(x)$.

- The kinetic energy of the string is given by

$$\int_0^L \frac{1}{2}\rho\,|\boldsymbol{r}_t|^2\,\mathrm{d}x = \int_0^L \frac{1}{2}\rho y_t^2\,\mathrm{d}x. \tag{4.42}$$

- The elastic potential energy of the string is the product of tension and extension, and therefore given by

$$T\left(\int_0^L |\boldsymbol{r}_x|\,\mathrm{d}x - L\right) = T\int_0^L \left(1+y_x^2\right)^{\frac{1}{2}} - 1\,\mathrm{d}x. \tag{4.43}$$

Since the transverse displacement is small in the sense that $|y_x| \ll 1$, a Taylor expansion gives

$$\left(1+y_x^2\right)^{\frac{1}{2}} - 1 = \frac{1}{2}y_x^2 + \cdots . \tag{4.44}$$

Hence, to a first approximation (*i.e.* neglecting cubic and higher order terms), the elastic potential energy is given by

$$\int_0^L \frac{1}{2}Ty_x^2\,\mathrm{d}x. \tag{4.45}$$

- **Definition:** The energy of the string is defined to be the sum of its kinetic and elastic potential energies, and given by

$$E(t) = \int_0^L \frac{1}{2}\rho y_t^2 + \frac{1}{2}Ty_x^2\,\mathrm{d}x. \tag{4.46}$$

- **Proposition:** If $y(x,t)$ satisfies the wave equation (4.39) and the boundary conditions (4.40), then the energy $E(t)$ is constant for $t > 0$.

Proof:

- The idea is to show that the derivative of $E(t)$ is equal to zero.
- By Leibniz's Integral Rule (3.3),

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \int_0^L \frac{\partial}{\partial t}\left(\frac{1}{2}\rho y_t^2 + \frac{1}{2}Ty_x^2\right)\mathrm{d}x = \int_0^L \rho y_t y_{tt} + Ty_x y_{xt}\,\mathrm{d}x.$$

- Substituting for ρy_{tt} from the wave equation (4.39), we deduce that

$$\begin{aligned}\frac{\mathrm{d}E}{\mathrm{d}t} &= \int_0^L Ty_t y_{xx} + Ty_x y_{xt}\,\mathrm{d}x \\ &= \int_0^L (Ty_t y_x)_x\,\mathrm{d}x \\ &= [Ty_t y_x]_{x=0}^{x=L}.\end{aligned}$$

- Since each of the boundary conditions in (4.40) may be differentiated with respect to t to give $y_t(0,t) = 0$ and $y_t(L,t) = 0$ for $t > 0$, we deduce that $\mathrm{d}E/\mathrm{d}t = 0$. ■

Notes

(1) We have shown that the energy of the elastic string is conserved during its motion, with the kinetic and elastic potential energy being transferred back and forth as the string oscillates.

(2) The energy of the string is set by the initial conditions in (4.41) to be given by

$$E(t) = E(0) = \int_0^L \frac{1}{2}\rho(g(x))^2 + \frac{1}{2}T(f'(x))^2 \, \mathrm{d}x. \tag{4.47}$$

(3) The energy of the nth normal mode $y_n(x,t)$ is given by

$$E_n(t) = \int_0^L \frac{1}{2}\rho\left(\frac{\partial y_n}{\partial t}\right)^2 + \frac{1}{2}T\left(\frac{\partial y_n}{\partial x}\right)^2 \mathrm{d}x. \tag{4.48}$$

Since $y_n(x,t)$ satisfies the wave equation (4.39) and the boundary conditions (4.40) by construction, it follows that its energy is conserved during its motion and given by

$$E_n(t) = E_n(0) = \frac{n^2\pi^2\rho c^2 b_n^2}{4L} + \frac{n^2\pi^2 T a_n^2}{4L}, \tag{4.49}$$

where in the last equality we substituted for $y_n(x,0)$ from (4.22) and integrated.

(4) Recalling (4.30)–(4.31) and assuming convergence, Parseval's Identity for g and f' imply that

$$\int_0^L \frac{1}{2}\rho g(x)^2 + \frac{1}{2}Tf'(x)^2 \, \mathrm{d}x = \sum_{n=1}^{\infty}\left(\frac{n^2\pi^2\rho c^2 b_n^2}{4L} + \frac{n^2\pi^2 T a_n^2}{4L}\right), \tag{4.50}$$

and hence that

$$E(t) = E(0) = \sum_{n=1}^{\infty} E_n(0) = \sum_{n=1}^{\infty} E_n(t), \tag{4.51}$$

i.e. the energy of the elastic string is made up of that in its normal modes.

4.6 Uniqueness Theorem

- **Uniqueness Theorem:** The IBVP (4.39)–(4.41) has at most one solution.

Proof:

- Our strategy is to show that the difference between any two solutions much vanish.
- We suppose that $y(x,t)$ and $\widetilde{y}(x,t)$ are solutions to (4.39)–(4.41) and let

$$w(x,t) = y(x,t) - \widetilde{y}(x,t) \tag{4.52}$$

be their difference.

- By linearity, (4.39)–(4.41) imply that $w(x,t)$ satisfies the wave equation

$$\frac{\partial^2 w}{\partial t^2} = c^2\frac{\partial^2 w}{\partial x^2} \quad \text{for} \quad 0 < x < L, \; t > 0, \tag{4.53}$$

with the boundary conditions

$$w(0,t) = 0, \quad w(L,t) = 0 \quad \text{for} \quad t > 0, \tag{4.54}$$

and the initial conditions

$$w(x,0) = 0, \quad \frac{\partial w}{\partial t}(x,0) = 0 \quad \text{for} \quad 0 < x < L. \tag{4.55}$$

- The trick now is to analyse the energy $E(t)$ associated with $w(x,t)$, which is given by

$$E(t) = \int_0^L \frac{1}{2}\rho w_t^2 + \frac{1}{2}T w_x^2 \, \mathrm{d}x. \tag{4.56}$$

- Since w satisfies (4.53) and (4.54), the energy $E(t)$ is conserved. But $E(0) = 0$ by (4.55), so

$$\int_0^L \frac{1}{2}\rho w_t^2 + \frac{1}{2}T w_x^2 \, \mathrm{d}x = 0 \quad \text{for} \quad t \geq 0. \tag{4.57}$$

- We deduce that $w_t = w_x = 0$ for $0 < x < L$, $t > 0$ assuming w_t and w_x are continuous on the region $R = \{(x,y) : 0 \leq x \leq L,\ t \geq 0\}$. Since (4.54) and (4.55) imply that $w = 0$ on the boundary of R, we deduce that $w = 0$ or $y = \widetilde{y}$ on R. ■

- **Remark:** Since w is the small transverse displacement of an elastic string whose initial transverse displacement and velocity are everywhere zero and whose ends are fixed thereafter, on physical grounds we expect the string to remain stationary along the x-axis, *i.e.* $w = 0$ for $0 \leq x \leq L$ and $t \geq 0$, which is what we showed to prove uniqueness.

4.7 Inhomogeneous wave equation and boundary conditions

- Consider the initial boundary value problem for the small transverse displacement $y(x,t)$ of an elastic string given by the forced wave equation

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} + F(x,t) \quad \text{for} \quad 0 < x < L,\ t > 0, \tag{4.58}$$

with the inhomogeneous Dirichlet boundary conditions

$$y(0,t) = \phi(t), \quad y(L,t) = \psi(t) \quad \text{for} \quad t > 0, \tag{4.59}$$

and the initial conditions

$$y(x,0) = f(x), \quad \frac{\partial y}{\partial t}(x,0) = g(x) \quad \text{for} \quad 0 < x < L, \tag{4.60}$$

where the forcing functions $F(x,t)$, $\phi(t)$ and $\psi(t)$, as well as $f(x)$ and $g(x)$, are given.

- In general Fourier's method cannot be used to solve the IBVP for y because the wave equation and boundary conditions are inhomogeneous (*i.e.* F, ϕ and ψ are non-zero). However, we can construct a solution by adapting Fourier's method in the same way as we did for the heat equation in §3.9.

- We deal first with the boundary conditions: if we let

$$y(x,t) = \phi(t)\left(1 - \frac{x}{L}\right) + \psi(t)\frac{x}{L} + Y(x,t), \tag{4.61}$$

then by linearity the initial boundary value problem (4.58)–(4.60) for $T(x,t)$ implies that the initial boundary value problem for $Y(x,t)$ is given by

$$\frac{\partial^2 Y}{\partial t^2} = c^2 \frac{\partial^2 Y}{\partial x^2} + \widetilde{F}(x,t) \quad \text{for} \quad 0 < x < L,\ t > 0, \tag{4.62}$$

with the homogeneous Dirichlet boundary conditions

$$Y(0,t) = 0, \quad Y(L,t) = 0 \quad \text{for} \quad t > 0, \tag{4.63}$$

and the initial conditions

$$Y(x,0) = \widetilde{f}(x), \quad \frac{\partial Y}{\partial t}(x,0) = \widetilde{g}(x) \quad \text{for} \quad 0 < x < L, \tag{4.64}$$

where the functions

$$\widetilde{F}(x,t) = F(x,t) - \ddot{\phi}(t)\left(1 - \frac{x}{L}\right) - \ddot{\psi}(t)\frac{x}{L}, \tag{4.65}$$

$$\widetilde{f}(x) = f(x) - \phi(0)\left(1 - \frac{x}{L}\right) - \psi(0)\frac{x}{L}, \tag{4.66}$$

$$\widetilde{g}(x) = g(x) - \dot{\phi}(0)\left(1 - \frac{x}{L}\right) - \dot{\psi}(0)\frac{x}{L}, \tag{4.67}$$

are known in terms of $F(x,t)$, $\phi(t)$, $\psi(t)$, $f(x)$ and $g(x)$.

- Thus, the boundary conditions have been rendered homogeneous by shifting the data in the sense that both $\phi(t)$ and $\psi(t)$ have moved from the boundary conditions (4.59) for $y(x,t)$ into the wave equation (4.62) and initial conditions (4.64) for $Y(x,t)$.
- If $\widetilde{F}(x,t) = 0$, then we can solve the initial boundary value problem (4.62)–(4.64) for $Y(x,t)$ using Fourier's method as in §4.4 to obtain the infinite series solution (4.29) in which the Fourier coeffocients a_n and b_n are given by (4.32), but with $f(x)$ and $g(x)$ replaced by $\widetilde{f}(x)$ and $\widetilde{g}(x)$.
- This solution suggests that if $\widetilde{F}(x,t)$ is not identically zero, then we should seek a solution for $Y(x,t)$ in the form of the Fourier sine series

$$Y(x,t) = \sum_{n=1}^{\infty} Y_n(t)\sin\left(\frac{n\pi x}{L}\right), \tag{4.68}$$

where the Fourier sine coefficients $Y_n(t)$ are to be determined and given in terms of $Y(x,t)$ by

$$Y_n(t) = \frac{2}{L}\int_0^L Y(x,t)\sin\left(\frac{n\pi x}{L}\right)\,\mathrm{d}x. \tag{4.69}$$

- Multiplying the wave equation (4.62) by $\sin(n\pi x/L)$ and integrating from $x = 0$ to $x = L$ gives

$$\int_0^L \left(\frac{\partial^2 Y}{\partial t^2} - c^2\frac{\partial^2 Y}{\partial x^2} - \widetilde{F}(x,t)\right)\sin\left(\frac{n\pi x}{L}\right)\,\mathrm{d}x = 0. \tag{4.70}$$

- For the Y_{tt} term we use Leibniz's integral rule (3.3) and for the Y_{xx} term we integrate by parts twice using the boundary conditions (4.63). The result is the constant coefficient, second-order, linear ODE for $Y_n(t)$ given by

$$\frac{\mathrm{d}^2 Y_n}{\mathrm{d}t^2} + \omega_n^2\, Y_n = \widetilde{F}_n(t) \quad \text{for } t > 0, \tag{4.71}$$

where the natural frequencies ω_n and the Fourier coefficients $\widetilde{F}_n(t)$ of the Fourier sine series for $\widetilde{F}(x,t)$ are given by

$$\omega_n = \frac{n\pi c}{L}, \tag{4.72}$$

$$\widetilde{F}_n(t) = \frac{2}{L}\int_0^L \widetilde{F}(x,t)\sin\left(\frac{n\pi x}{L}\right)\mathrm{d}x. \tag{4.73}$$

- The initial conditions (4.64) for $Y(x,t)$ imply that the initial conditions for $Y_n(t)$ are given by

$$Y_n(0) = \frac{2}{L}\int_0^L \widetilde{f}(x)\sin\left(\frac{n\pi x}{L}\right)\mathrm{d}x, \tag{4.74}$$

$$\frac{\mathrm{d}Y_n}{\mathrm{d}t}(0) = \frac{2}{L}\int_0^L \widetilde{g}(x)\sin\left(\frac{n\pi x}{L}\right)\mathrm{d}x. \tag{4.75}$$

- In summary, by (4.61) and (4.68), the solution of the initial boundary value problem (4.58)–(4.60) for $y(x,t)$ is given by

$$y(x,t) = \phi(t)\left(1 - \frac{x}{L}\right) + \psi(t)\frac{x}{L} + \sum_{n=1}^{\infty} Y_n(t)\sin\left(\frac{n\pi x}{L}\right), \tag{4.76}$$

where $Y_n(t)$ is governed by the ODE (4.71) with initial conditions (4.74)–(4.75).

- **Remark:** The initial value problem for $Y_n(t)$ may be solved using the method of variation of parameters of part A Differential Equations II, giving

$$Y_n(t) = Y_n(0)\cos(\omega_n t) + \frac{1}{\omega_n}\frac{\mathrm{d}Y_n}{\mathrm{d}t}(0)\sin(\omega_n t) + \frac{1}{\omega_n}\int_0^t \widetilde{F}_n(s)\sin\omega_n(t-s)\,\mathrm{d}s. \tag{4.77}$$

Example: sinusoidal forcing

- Consider the case in which the string is at rest along the x-axis when we start to oscillate the left-hand end sinusoidally with frequency ω at time $t = 0$, so that $y(x,t)$ is governed by the initial boundary value problem (4.58)–(4.60) with

$$F(x,t) = 0, \qquad \phi(t) = h\sin(\omega t), \qquad \psi(t) = 0, \qquad f(x) = 0, \qquad g(x) = 0,$$

where h and ω are positive constants.

- In this case, we find that

$$\widetilde{F}(x,t) = h\omega^2\sin(\omega t)\left(1 - \frac{x}{L}\right), \quad \widetilde{f}(0) = 0, \quad \widetilde{g}(x) = -h\omega\left(1 - \frac{x}{L}\right), \tag{4.78}$$

giving

$$\widetilde{F}_n(t) = \frac{2}{L}\int_0^L h\omega^2\sin(\omega t)\left(1 - \frac{x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)\mathrm{d}x = \frac{2h\omega^2}{\pi n}\sin\omega t. \tag{4.79}$$

- Hence, the initial value problem for $Y_n(t)$ given by

$$\frac{\mathrm{d}^2 Y_n}{\mathrm{d}t^2} + \omega_n^2 Y_n = \frac{2h\omega^2}{\pi n}\sin\omega t \quad \text{for } t > 0, \tag{4.80}$$

with

$$Y_n(0) = 0, \quad \frac{\mathrm{d}Y_n}{\mathrm{d}t}(0) = -\frac{2}{L}\int_0^L h\omega\sin(\omega t)\left(1 - \frac{x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)\mathrm{d}x = -\frac{2h\omega}{\pi n}. \tag{4.81}$$

- The solution for $Y_n(t)$ is given by

$$Y_n(t) = \begin{cases} \dfrac{2h\omega}{n\pi(\omega_n^2 - \omega^2)}\Big(\omega\sin(\omega t) - \omega_n\sin(\omega_n t)\Big) & \text{for } \omega \neq \omega_n, \\[2ex] -\dfrac{h}{n\pi}\Big(\omega_n t\cos(\omega_n t) + \sin(\omega_n t)\Big) & \text{for } \omega = \omega_n. \end{cases} \tag{4.82}$$

- Hence, putting everything together, there are two cases to consider.

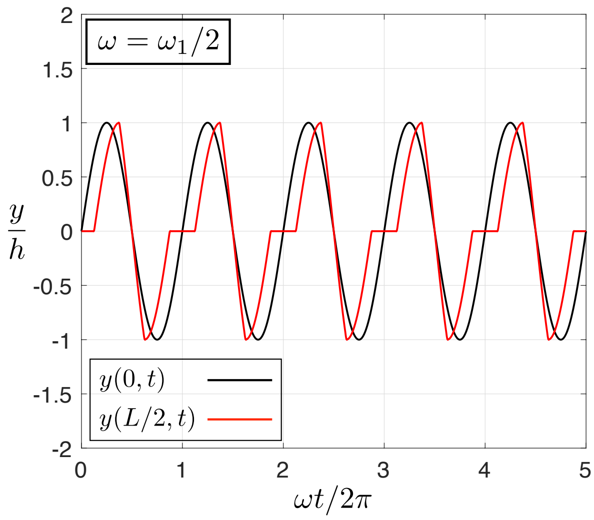
- Case (i) $\omega \neq \omega_n$ for all positive integers n:

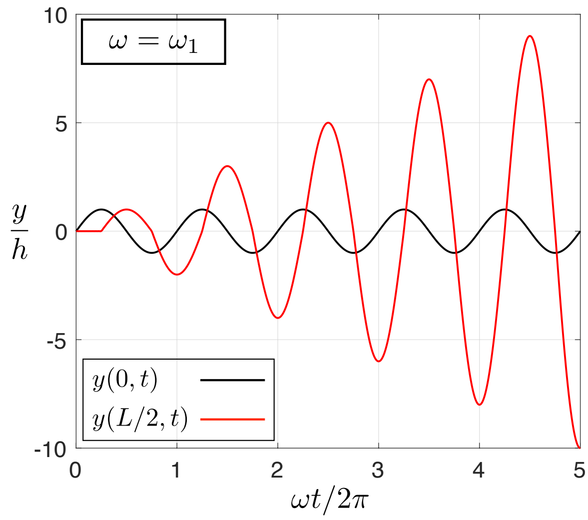
$$y(x,t) = h\sin(\omega t)\left(1 - \frac{x}{L}\right) + \sum_{n=1}^{\infty}\frac{2h\omega}{\pi n(\omega_n^2 - \omega^2)}\Big(\omega\sin(\omega t) - \omega_n\sin(\omega_n t)\Big)\sin\left(\frac{n\pi x}{L}\right). \tag{4.83}$$

- Case (ii) $\omega = \omega_p$ for some positive integer p:

$$\begin{aligned} y(x,t) \;=\; & h\sin(\omega_p t)\left(1 - \frac{x}{L}\right) + \sum_{\substack{n=1 \\ n\neq p}}^{\infty}\frac{2h\omega_p}{\pi n(\omega_n^2 - \omega_p^2)}\Big(\omega_p\sin(\omega_p t) - \omega_n\sin(\omega_n t)\Big)\sin\left(\frac{n\pi x}{L}\right) \\ & -\frac{h}{\pi p}\Big(\omega_p t\cos(\omega_p t) + \sin(\omega_p t)\Big)\sin\left(\frac{p\pi x}{L}\right). \end{aligned} \tag{4.84}$$

- **Question:** What is the essential difference between the two cases?
- **Answer:** In case (i) $y(x,t)$ is bounded for all $t \geq 0$, while in case (ii) $y(x,t)$ grows without bound (because $Y_p(t)$ does) as $t \to \infty$. That the amplitude grows without bound when the system is forced at a natural frequency is called *resonance*.
- We plot below the forcing $y(0,t)$ and response $y(L/2,t)$ over five periods for (i) $\omega = \omega_1/2$ (left-hand plot) and (ii) $\omega = \omega_1$ (right-hand plot), which were obtained by truncating the series solution to 32 terms. We see a periodic solution in case (i) and linear growth of the amplitude in case (ii). Note that the horizontal red segments are an artefact of the speed of propagation of information being c for the wave equation, *e.g.* the string has zero displacement initially and it takes until time $t = L/2c$ for the effect of the forcing at $x = 0$ to reach $x = L/2$.





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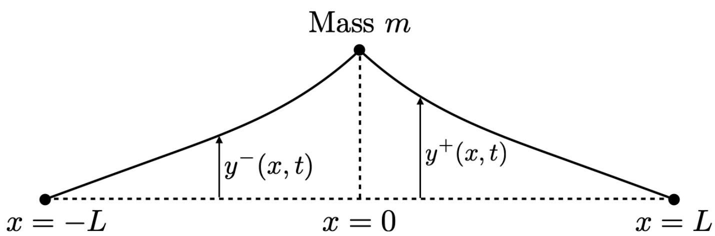
4.8 Normal modes for a weighted string

- An elastic string of length $2L$ has its ends fixed at $(x,y) = (\pm L, 0)$ and a point particle of mass m is attached to the mid-point, as illustrated in the schematic below.



- We seek here the normal modes of vibration.
- Since the transverse displacements are assumed to be small (in the sense that $|y_x| \ll 1$) and the tension T in the elastic string is assumed to be constant, the horizontal components of the forces exerted by the string on the point particle will balance to a first approximation, so we need only consider the transverse displacement of the point particle, $Y(t)$ say.
- We let $y^-(x,t)$ and $y^+(x,t)$ denote the small transverse displacements for $-L \leq x < 0$ and $0 < x \leq L$, respectively.
- Then y^- and y^+ must satisfy the wave equations

$$\frac{\partial^2 y^-}{\partial t^2} = c^2 \frac{\partial^2 y^-}{\partial x^2} \quad \text{for} \quad -L < x < 0, \tag{4.85}$$

$$\frac{\partial^2 y^+}{\partial t^2} = c^2 \frac{\partial^2 y^+}{\partial x^2} \quad \text{for} \quad 0 < x < L, \tag{4.86}$$

and the boundary conditions

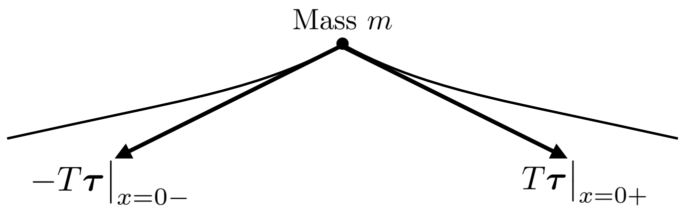
$$y^{-}(-L,t) = 0, \tag{4.87}$$

$$y^{+}(L,t) = 0. \tag{4.88}$$

- **Question:** What conditions hold at $x = 0$?
- **Answer:** There are two. Firstly, since the point particle is attached to the string, we require

$$y^{-}(0_{-},t) = Y(t) = y^{+}(0_{+},t). \tag{4.89}$$

Secondly, the string exerts on the point particle the forces illustrated below (neglecting gravity and air resistance as in the wave equations above).



Here $\boldsymbol{\tau}$ is the right-pointing unit tangent vector to the string given by

$$\boldsymbol{\tau} = \frac{\boldsymbol{i} + y_x\boldsymbol{j}}{(1+y_x^2)^{1/2}},$$

where $y = y^{-}$ for $-L < x < 0$ and $y = y^{+}$ for $0 < x < L$.
Hence, applying Newton's Second Law to the point particle in the y-direction gives

$$m\frac{\mathrm{d}^2Y}{\mathrm{d}t^2} = \big(T\boldsymbol{\tau}(0_{+},t) - T\boldsymbol{\tau}(0_{-},t)\big)\cdot\mathbf{j}.$$

We now apply the assumption that the slope y_x is everywhere small to linearize this boundary condition: since

$$\left(1+y_x^2\right)^{1/2} = 1 + \frac{1}{2}(y_x)^2 + \cdots \quad \text{for} \quad |y_x| \ll 1, \tag{4.90}$$

we deduce that to a first approximation

$$m\frac{\mathrm{d}^2Y}{\mathrm{d}t^2} = Ty_x^{+}(0_{+},t) - Ty_x^{-}(0_{-},t). \tag{4.91}$$

- To find the normal modes we seek nontrivial separable solutions of (4.85)–(4.91) of the form

$$y^{\pm} = F_{\pm}(x)G(t), \tag{4.92}$$

since we must choose the same time dependence for both y^{-} and y^{+} in order to satisfy (4.89).

- In the usual manner we may deduce from (4.85)–(4.86) that there is a real constant λ such that

$$\frac{F_{\pm}''(x)}{F_{\pm}(x)} = \frac{G''(t)}{c^2G(t)} = -\lambda, \tag{4.93}$$

i.e.

$$G_{\pm}'' + \lambda c^2 G_{\pm} = 0, \tag{4.94}$$

and

$$F_{\pm}'' + \lambda F_{\pm} = 0. \tag{4.95}$$

- Since we're seeking nontrivial solutions, it follows from (4.87)–(4.88) that

$$F_{-}(-L) = 0, \; F_{+}(L) = 0. \tag{4.96}$$

- Similarly (4.89) gives
$$F_-(0_-) = F_+(0_+), \tag{4.97}$$
while (4.91) implies
$$mF_\pm(0)G''(t) = T(F_+'(0_+) - F_-'(0_-))G(t);$$
eliminating the time dependence in this expression using (4.95) gives
$$-\lambda m F_\pm(0) = \rho\big(F_+'(0_+) - F_-'(0_-)\big), \tag{4.98}$$
where we also used $c^2 = T/\rho$ to eliminte T.

- Since we are seeking non-trivial oscillatory solutions, we now focus on the case in which λ is positive by setting $\lambda = \omega^2$, where $\omega > 0$ without loss of generality.

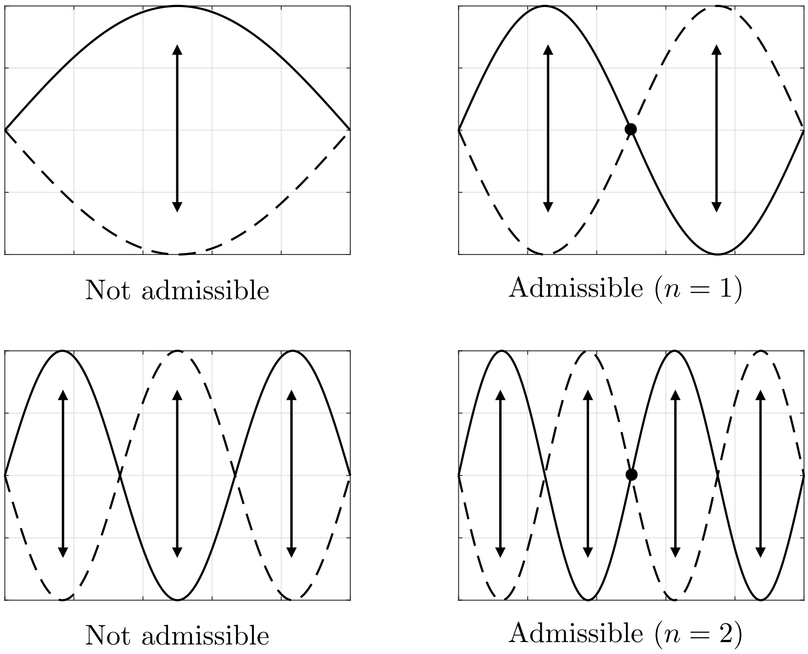
- By (4.94), we then have $G(t) = C\cos(\omega ct + \epsilon)$, where ϵ is an arbitrary constant and we may take $C = 1$ without loss of generality.

- Moreover, (4.95) gives
$$\begin{aligned} F_-'' + \omega^2 F_- &= 0 \text{ for } -L < x < 0, && (4.99)\\ F_+'' + \omega^2 F_+ &= 0 \text{ for } 0 < x < L, && (4.100)\end{aligned}$$
so that (4.96) imply
$$\begin{aligned} F_-(x) &= A\sin\big(\omega(L+x)\big), && (4.101)\\ F_+(x) &= B\sin\big(\omega(L-x)\big), && (4.102)\end{aligned}$$
where A and B are arbitrary real constants.

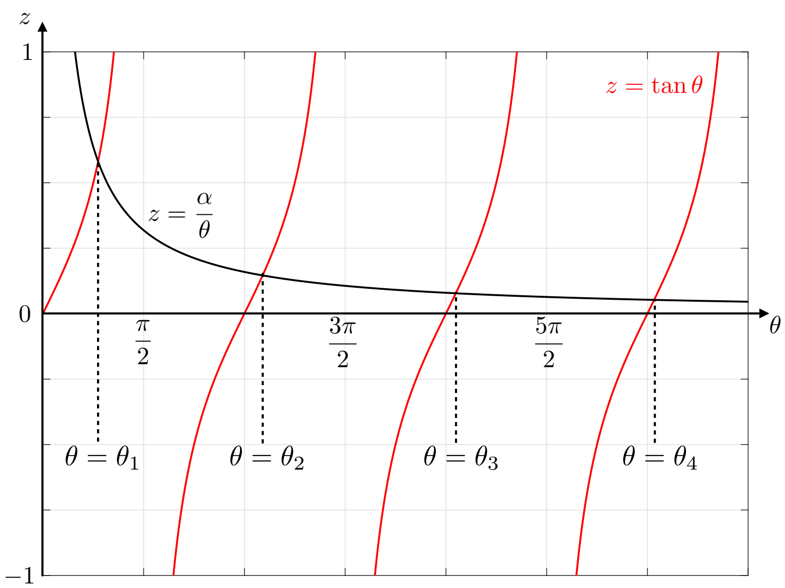
- Substituting these expressions for $F_\pm(x)$ into the boundary conditions (4.97) and (4.98) at $x = 0$, we obtain two linear algebraic equations for A and B that may be written in the form
$$\underbrace{\begin{bmatrix} \sin\omega L & -\sin\omega L \\ \rho\cos\omega L - m\omega\sin\omega L & \rho\cos\omega L \end{bmatrix}}_{M} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{4.103}$$

- For nontrivial solutions for $F_\pm(x)$, we need
$$\begin{bmatrix} A \\ B \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{4.104}$$
and hence for the matrix M to be singular: setting $\det(M) = 0$, we deduce that ω must satisfy
$$\sin\omega L\,(2\rho\cos\omega L - m\omega\sin\omega L) = 0. \tag{4.105}$$
Hence, there are two cases depending on which of the factors vanishes in this solvability condition: either (i) $\sin\omega L = 0$ or (ii) $2\rho\cos\omega L - m\omega\sin\omega L = 0$.

- <u>Case (i) $\sin\omega L = 0$</u>
  - We deduce immediately that $\omega = n\pi/L$, where n is a positive integer.
  - Then (4.103) gives $B = -A$, so that the normal modes are given by
  $$\begin{aligned} y_-(x,t) &= A\sin\big(\omega(L+x)\big)\cos(\omega ct + \epsilon), && (4.106)\\ y_+(x,t) &= -A\sin\big(\omega(L-x)\big)\cos(\omega ct + \epsilon). && (4.107)\end{aligned}$$
  - This means that the normal modes are the same as for a string of length $2L$ with a node at $x = 0$, *i.e.* the point particle is stationary and remains at the origin, as illustrated for the first few such modes in the schematic below.



- <u>Case (ii) $2\rho\cos\omega L - m\omega\sin\omega L = 0$</u>
  - If we scale $\omega = \theta/L$, then θ satisfies the transcendental equation
    $$\tan\theta = \frac{\alpha}{\theta}, \tag{4.108}$$
    where the dimensionless parameter $\alpha = 2L\rho/m$ is the ratio of the mass of the string to that of the point particle.
  - By plotting the graph of the left- and right-hand sides of (4.108), as illustrated below for $\alpha = 1$, we can convince ourselves that there are countably many roots
    $$\theta_1 < \theta_2 < \theta_3 < \cdots,$$
    with $(n-1)\pi < \theta_n < (n-1/2)\pi$ and $\theta_n/(n-1) \to \pi+$ as $n \to \infty$.



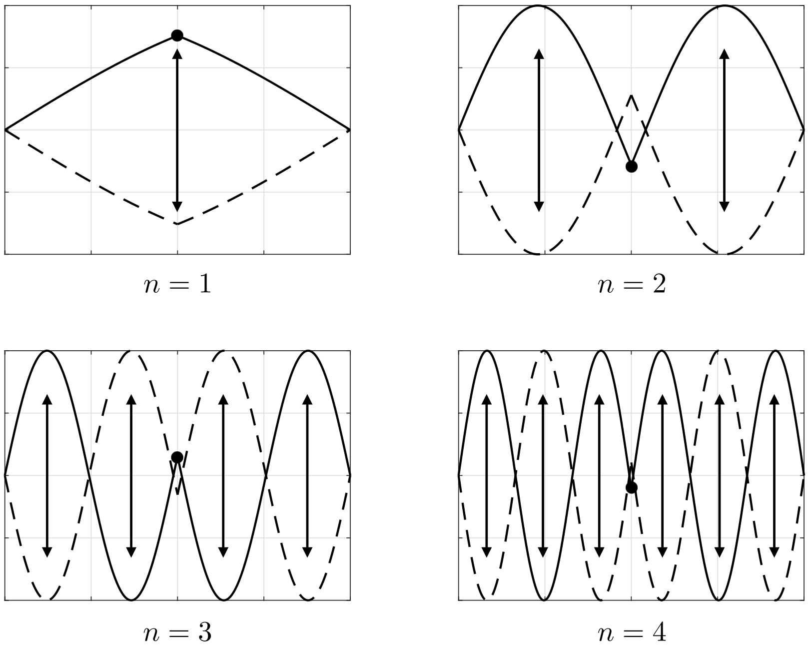
  - Hence, there are countably many natural frequencies $\omega c = \theta_n c/L$, where n is a positive integer.

- Now (4.103) gives $B = A$, so that the normal modes are given by

$$y_-(x,t) = A\sin\big(\omega(L+x)\big)\cos(\omega ct+\epsilon), \tag{4.109}$$

$$y_+(x,t) = A\sin\big(\omega(L-x)\big)\cos(\omega ct+\epsilon). \tag{4.110}$$

- This means that the string is symmetric about $x = 0$, as illustrated for the first few such modes in the schematic below.



4.9 General solution to the wave equation

- It is a remarkable fact that it is possible to write down all solutions of the wave equation

$$\frac{\partial^2 y}{\partial t^2} = c^2\frac{\partial^2 y}{\partial x^2}, \tag{4.111}$$

where we recall that the parameter $c > 0$ is the wave speed.

- To verify this fact we introduce new independent variables

$$\xi = x - ct, \quad \eta = x + ct, \tag{4.112}$$

and seek a solution in which

$$y(x,t) = Y(\xi,\eta). \tag{4.113}$$

- The chain rule implies

$$y_x = Y_\xi\xi_x + Y_\eta\eta_x = Y_\xi + Y_\eta,$$

$$y_t = Y_\xi\xi_t + Y_\eta\eta_t = -cY_\xi + cY_\eta,$$

giving

$$y_{xx} = (Y_\xi + Y_\eta)_\xi\xi_x + (Y_\xi + Y_\eta)_\eta\eta_x = Y_{\xi\xi} + 2Y_{\xi\eta} + Y_{\eta\eta},$$

$$y_{tt} = (-cY_\xi + cY_\eta)_\xi\xi_t + (-cY_\xi + cY_\eta)_\eta\eta_t = c^2(Y_{\xi\xi} - 2Y_{\xi\eta} + Y_{\eta\eta}),$$

where we assumed $Y_{\xi\eta} = Y_{\eta\xi}$, so that

$$\frac{\partial^2 y}{\partial t^2} - c^2\frac{\partial^2 y}{\partial x^2} = -4c^2\frac{\partial^2 Y}{\partial\xi\partial\eta}.$$

- Hence, in the new variables (ξ, η) the wave equation (4.111) transforms to the partial differential equation
$$\frac{\partial^2 Y}{\partial \xi \partial \eta} = 0,$$
i.e.
$$\frac{\partial}{\partial \xi}\left(\frac{\partial Y}{\partial \eta}\right) = 0.$$

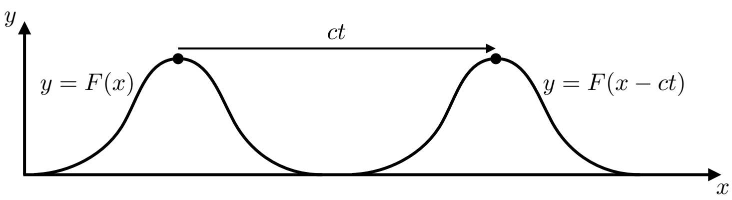
- Thus, $\partial Y/\partial \eta$ is independent of ξ and is a function of η only, say $G'(\eta)$, *i.e.*
$$\frac{\partial Y}{\partial \eta} = G'(\eta),$$
and so
$$\frac{\partial}{\partial \eta}\left[Y - G(\eta)\right] = 0.$$

- Thus, $Y - G(\eta)$ is a function of ξ only, say $F(\xi)$, and therefore
$$Y - G(\eta) = F(\xi),$$
giving
$$y(x,t) = F(x - ct) + G(x + ct), \tag{4.114}$$
where F and G are arbitrary twice continuously differentiable functions.

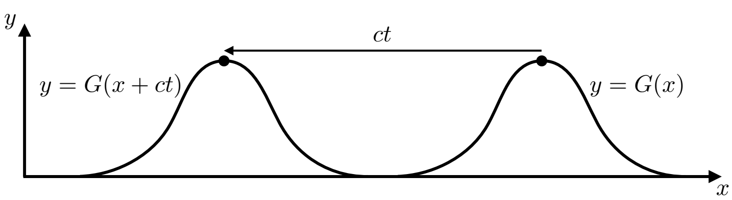
Notes

(1) It is straightforward to use the chain rule to verify that (4.114) is a solution. We have shown that all solutions must be of this form.

(2) We note that $F(x - ct)$ is a travelling wave of constant shape moving in the positive x-direction with speed c, as illustrated in the sketch below in which the initial profile $y = F(x)$ at $t = 0$ is translated a distance ct to the right at time t.



(3) We note that $G(x + ct)$ is a travelling wave of constant shape moving in the negative x-direction with speed c, as illustrated in the sketch below in which the initial profile $y = G(x)$ at $t = 0$ is translated a distance ct to the left at time t.



(4) The general solution is therefore the superposition of left- and right-travelling waves each moving with speed c, which is the reason the parameter c is called the wave speed. It follows that the wave equation propagates information at constant speed c in contrast to solutions of the heat equation in which information propagates at infinite speed.

Example: wave reflection

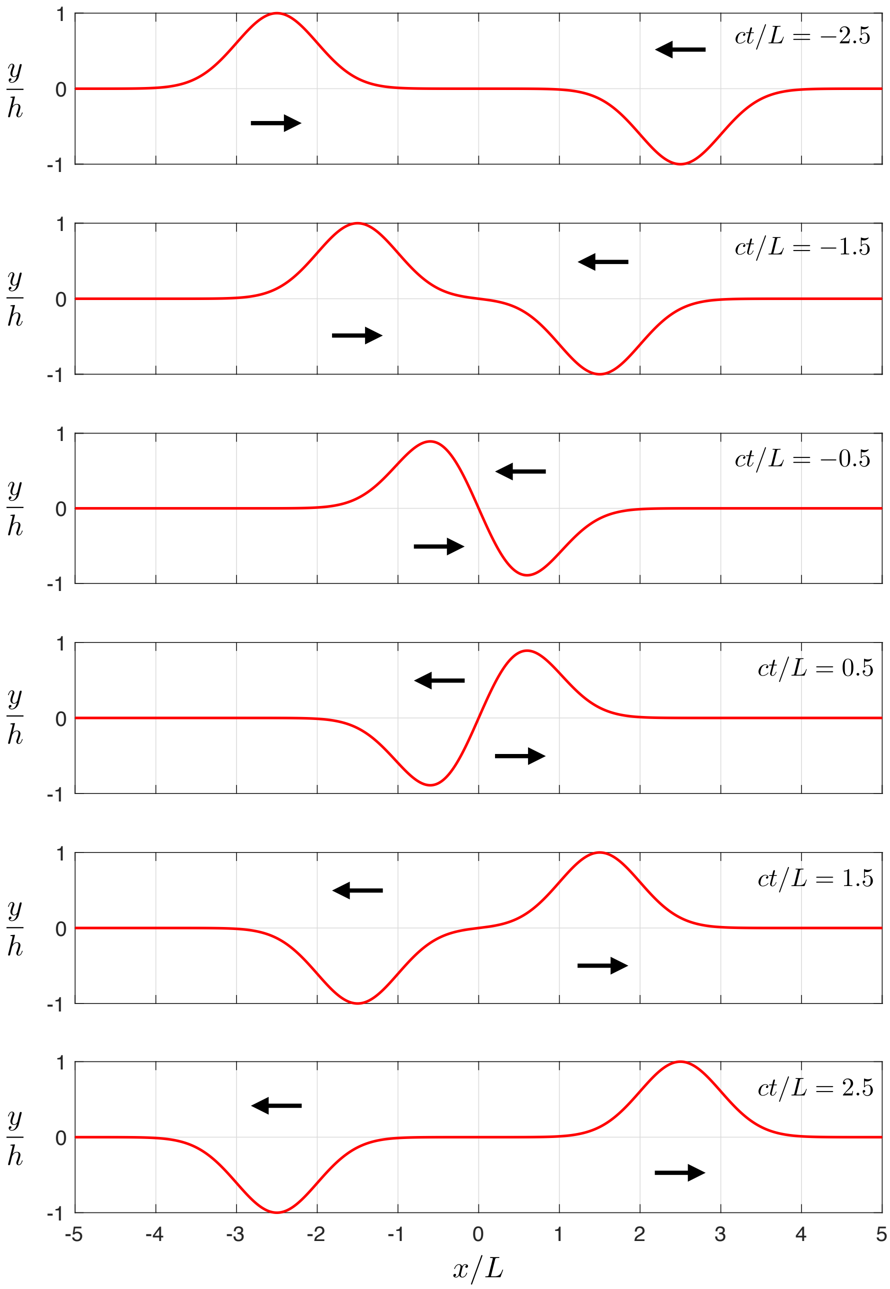
- A string occupies $-\infty < x \leq 0$ and is fixed at $x = 0$. A wave $y(x,t) = f(x - ct)$ is incident from $x < 0$. Find the reflected wave.

- In the general solution of the wave equation (4.114), we take $F = f$ and G to be found.

- The boundary condition $y(0,t) = 0$ is to be satisfied for all t. Hence, $f(-ct) + G(ct) = 0$ for all t, and so $G(\theta) = -f(-\theta)$ for all θ. Thus,

$$y(x,t) = \underbrace{f(x-ct)}_{\text{incident wave}} - \underbrace{f(-x-ct)}_{\text{reflected wave}}. \tag{4.115}$$

- The snapshots below illustrates the reflection of an incident wave by plotting (4.115) for $f(x) = h\exp(-x^2/L^2)$, where h and L are positive constants. The arrows indicated the direction of travel with speed c of the incident and reflected waves. Focussing on $x \leq 0$, we see that the reflected wave has the same shape and speed as the incident wave, but the opposite sign and direction of travel.



■

4.10 Waves on an infinite string: D'Alembert's formula

- Consider the initial value problem for the small transverse displacement $y(x,t)$ of an elastic string given by the wave equation

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \quad \text{for} \quad -\infty < x < \infty,\ t > 0, \tag{4.116}$$

with the initial conditions

$$y(x,0) = f(x), \quad \frac{\partial y}{\partial t}(x,0) = g(x) \quad \text{for} \quad -\infty < x < \infty, \tag{4.117}$$

where the initial transverse displacement $f(x)$ and the initial transverse velocity $g(x)$ are given.

- The general solution of the wave equation (4.116) is given by (4.114), so it remains to determine the functions F and G for which it satisfies the initial conditions (4.117).

- Substituting (4.114) into (4.117) gives

$$F(x) + G(x) = f(x), \tag{4.118}$$

$$-cF'(x) + cG'(x) = g(x). \tag{4.119}$$

- The expression (4.119) integrates to give

$$-F(x) + G(x) = \frac{1}{c}\int_0^x g(s)\,\mathrm{d}s + a, \tag{4.120}$$

where a is an arbitrary constant.

- Subtracting and adding (4.118) and (4.120), we deduce that F and G are given by

$$F(x) = \frac{1}{2}\left(f(x) - \frac{1}{c}\int_0^x g(s)\,\mathrm{d}s - a\right),$$

$$G(x) = \frac{1}{2}\left(f(x) + \frac{1}{c}\int_0^x g(s)\,\mathrm{d}s + a\right).$$

- Thus,

$$\begin{aligned} y(x,t) &= \frac{1}{2}\left(f(x-ct) - \frac{1}{c}\int_0^{x-ct} g(s)\,\mathrm{d}s - a\right) + \frac{1}{2}\left(f(x+ct) + \frac{1}{c}\int_0^{x+ct} g(s)\,\mathrm{d}s + a\right) \\ &= \frac{1}{2}\big(f(x-ct) + f(x+ct)\big) + \frac{1}{2c}\left(\int_{x-ct}^0 g(s)\,\mathrm{d}s + \int_0^{x+ct} g(s)\,\mathrm{d}s\right), \end{aligned}$$

and we arrive at *D'Alembert's Formula*

$$y(x,t) = \frac{1}{2}\big(f(x-ct) + f(x+ct)\big) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,\mathrm{d}s. \tag{4.121}$$

- **Notes:**

  (1) The argument shows that, for given f and g, the initial value problem has one and only one solution, *i.e.* existence and uniqueness.

  (2) We note that uniqueness may also be proved by energy conservation under the additional assumption that y_t, $y_x \to 0$ sufficiently rapidly as $x \to \pm\infty$ that we can ensure the existence of the energy

$$E(t) = \int_{-\infty}^{\infty} \frac{\rho}{2}y_t^2 + \frac{T}{2}y_x^2\,\mathrm{d}x.$$

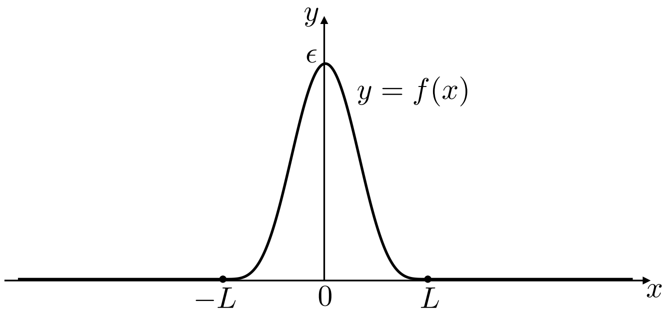
Example 1

- Suppose that f and g are given by

$$f(x) = \begin{cases} \epsilon \cos^4\left(\dfrac{\pi x}{2L}\right) & \text{for } |x| \leq L, \\ 0 & \text{otherwise}, \end{cases} \qquad g(x) = 0, \tag{4.122}$$

where ϵ and L are positive constants.

- **Remark:** As illustrated in the sketch below, f, f', f'' and f''' are continuous on $\mathbb{R}$ and f is *compactly supported* because it vanishes outside of a closed bounded interval.



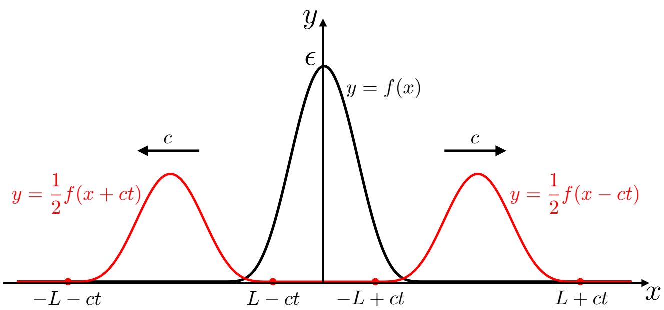
- By D'Alembert's formula (4.121) the solution of the initial boundary value problem (4.116)–(4.117) is given by

$$y(x,t) = \frac{1}{2}\big(f(x-ct) + f(x+ct)\big), \tag{4.123}$$

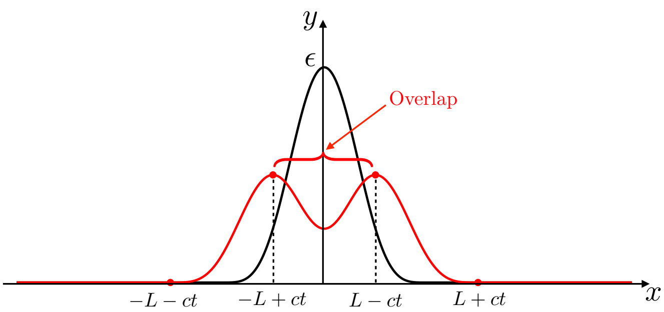
- **Remark:** The solution is a classical solution because it is twice continuously differentiable with respect to x and t and satisfies (4.116)–(4.117).

- We can sketch the solution $y(x,t)$ at a fixed time $t > 0$ using the geometrical properties of its travelling wave components.

  - For $\underline{ct > L}$, the supports of $f(x-ct)$ and $f(x+ct)$ do not overlap, as illustrated below.



  - For $\underline{0 < ct < L}$, the supports of $f(x-ct)$ and $f(x+ct)$ overlap, as illustrated below.



- The derivation of explicit formulae for the solution therefore requires some careful bookkeeping for which it is easier to think geometrically rather than algebraically. ■

4.11 Characteristic diagrams

- Let us ask how the solution at a point $P:(x_0, t_0)$ in the upper half of the (x,t)-plane depends upon the data f and g.

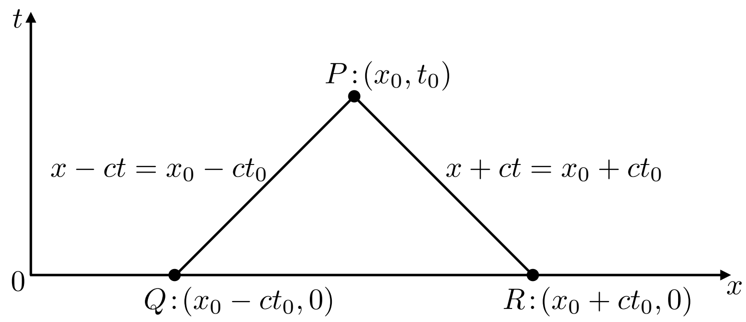
- By D'Alembert's Formula (4.121), we have

$$y(x_0, t_0) = \frac{1}{2}[f(x_0 - ct_0) + f(x_0 + ct_0)] + \frac{1}{2c}\int_{x_0 - ct_0}^{x_0 + ct_0} g(x)\,\mathrm{d}x, \tag{4.124}$$

  which may be written in the form

$$y(P) = \frac{1}{2}\big(f(Q) + f(R)\big) + \frac{1}{2c}\int_Q^R g(s)\,\mathrm{d}s, \tag{4.125}$$

  where Q and R are the points $(x_0 - ct_0, 0)$ and $(x_0 + ct_0, 0)$, respectively, on the x-axis, as illustrated in the sketch below.



- We note the deliberate abuse of notation in (4.125) to aid the geometric interpretation of D'Alembert's formula (4.121).

- **Definition:** The lines $x \pm ct = x_0 \pm ct_0$ are the *characteristic lines* through $P:\ (x_0, t_0)$.

- It follows from (4.125) that $y(P)$ depends only on

  (i) f though the values f takes at Q and R;

  (ii) g though the values g takes on the x-axis between Q and R.

  This motivates the following definition.

- **Definition:** The interval $[x_0 - ct_0, x_0 + ct_0]$ of the x-axis between Q and R is called the *domain of dependence* of $P:(x_0, t_0)$

- If f or g are modified outside the domain of dependence of P, then $y(P)$ is unchanged.

- We can exploit the geometric interpretation of (4.125) to construct explicit formulae for the solution: the contribution to $y(P)$ from f and g changes at points on the x-axis where f and g change their analytic behaviour.

- Hence, given a particular f and g, the first task is to identify such points on the x-axis and sketch the characteristic lines $x \pm ct =$ constant through each of them — this is the *characterisrtic diagram.*

- The characteristic diagram divides the (x,t)-plane into regions in which the contributions from f and g may be different: the second task is to evaluate $y(P)$ for P in each of these regions.

Example 1 revisited

- Since g vanishes in this case, (4.125) becomes

$$y(P) = \frac{1}{2}\big(f(Q) + f(R)\big), \tag{4.126}$$

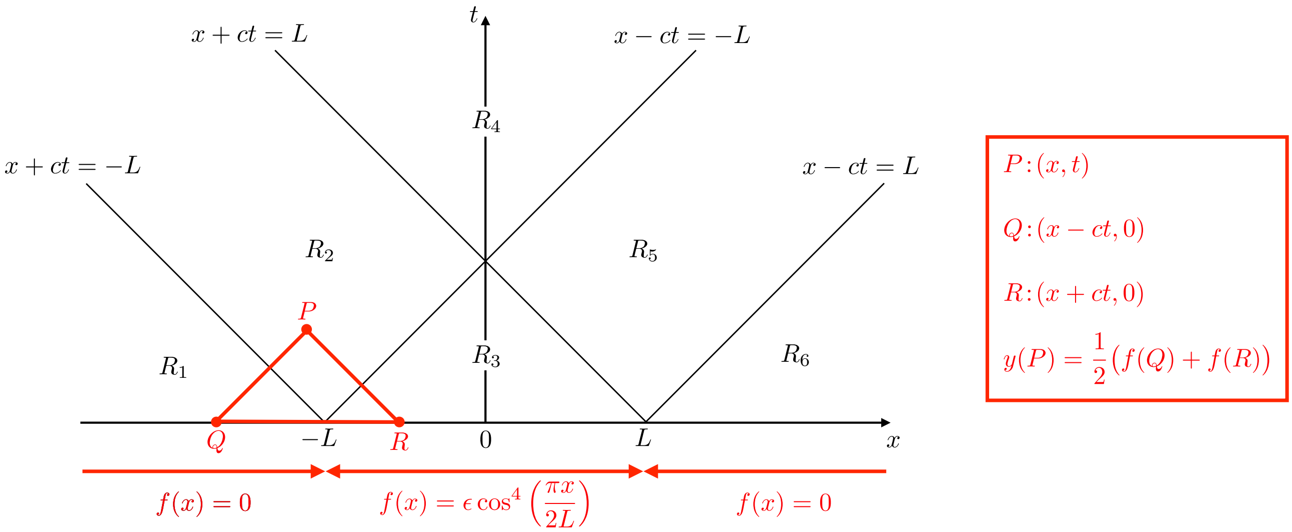
  where Q and R are the left- and right-hand intersections with the x-axis of the characteristic lines though P, as illustrated in the diagram above.

- Recall that f is given by

$$f(x) = \begin{cases} \epsilon\cos^4\left(\dfrac{\pi x}{2L}\right) & \text{for } |x| \le L, \\ 0 & \text{otherwise,} \end{cases} \tag{4.127}$$

  so that it is compactly supported with support $(-L, L)$, and therefore changes its analytic behaviour at the points $(-L, 0)$ and $(L, 0)$ on the x-axis in the (x,t)-plane.

- The characteristics through these points are $x \pm ct = -L$ and $x \pm ct = L$ and they divide the upper-half of the (x,t)-plane into six regions $R_1, \ldots, R_6$, forming the characteristic diagram illustrated below.



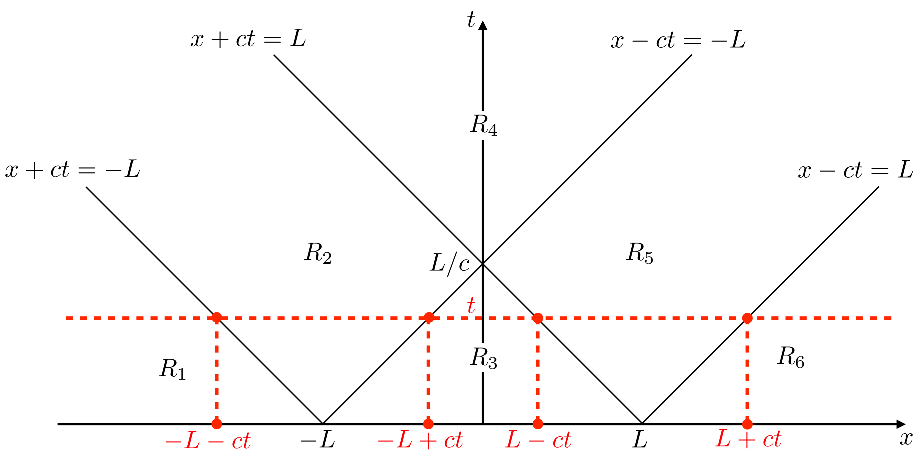
- In particular, we let

$$\begin{aligned}
R_1 &= \{(x,t) : t > 0,\ x + ct < -L\}, \\
R_2 &= \{(x,t) : t > 0,\ -L \le x + ct \le L,\ x - ct \le -L\}, \\
R_3 &= \{(x,t) : t > 0,\ x - ct > -L,\ x + ct < L\}, \\
R_4 &= \{(x,t) : t > 0,\ x - ct < -L,\ x + ct > L\}, \\
R_5 &= \{(x,t) : t > 0,\ -L \le x - ct \le L,\ x + ct \ge L\}, \\
R_6 &= \{(x,t) : t > 0,\ x - ct > L\}.
\end{aligned}$$

- **Remark:** By including the dividing characteristics in regions R_2 and R_5 we have ensured that each point (x,t) in the upper half plane belongs to one and only one region. The choice to have regions R_2 and R_5 contain their bounding characteristics is arbitrary if the solution is everywhere continuous, as it is in this example. The choice is not arbitrary if the solution is anywhere discontinuous, though in this case the solution would not be a classical solution — nevertheless, if we were to entertain it, then any discontinuity would move with speed c and hence along a dividing characteristic.

- As illustrated in the figure above, since PQ is parallel to the characteristics $x - ct = \pm L$, while PR is parallel to the characteristics $x + ct = \pm L$, we may construct the solution with the aid of the characteristic diagram by drawing on it the triangle PQR for P in each of the different regions.

- Thus, the locations of Q and R on the x-axis dictate their contributions to (4.126), as follows:
  - if $P \in R_1$, then Q and R lie to the left of $(-L, 0)$, so that $f(Q) = f(R) = 0$, giving
$$y(x,t) = 0 \quad \text{for } (x,t) \in R_1;$$
  - if $P \in R_2$, then Q lies at or to the left of $(-L, 0)$, while R lies at or between $(-L, 0)$ and $(L, 0)$, so that $f(Q) = 0$ and $f(R) = f(x + ct) = \epsilon \cos^4(\pi(x + ct)/2L)$, giving
$$y(x,t) = \frac{\epsilon}{2}\cos^4\left(\frac{\pi}{2L}(x+ct)\right) \quad \text{for } (x,t) \in R_2;$$
  - if $P \in R_3$, then Q and R lie between $(-L, 0)$ and $(L, 0)$, so that $f(Q) = f(x - ct) = \epsilon \cos^4(\pi(x - ct)/2L)$ and $f(R) = f(x + ct) = \epsilon \cos^4(\pi(x + ct)/2L)$, giving
$$y(x,t) = \frac{\epsilon}{2}\cos^4\left(\frac{\pi}{2L}(x-ct)\right) + \frac{\epsilon}{2}\cos^4\left(\frac{\pi}{2L}(x+ct)\right) \quad \text{for } (x,t) \in R_3;$$
  - if $P \in R_4$, then Q lies to the left of $(-L, 0)$ and R lies to the right of $(L, 0)$, so that $f(Q) = f(R) = 0$, giving
$$y(x,t) = 0 \quad \text{for } (x,t) \in R_4;$$
  - if $P \in R_5$, then Q lies at or between $(-L, 0)$ and $(L, 0)$, while R lies at or to the right of $(L, 0)$, so that $f(Q) = f(x - ct) = \epsilon \cos^4(\pi(x - ct)/2L)$ and $f(R) = 0$, giving
$$y(x,t) = \frac{\epsilon}{2}\cos^4\left(\frac{\pi}{2L}(x-ct)\right) \quad \text{for } (x,t) \in R_5;$$
  - if $P \in R_6$, then Q and R lie to the right of $(L, 0)$, so that $f(Q) = f(R) = 0$, giving
$$y(x,t) = 0 \quad \text{for } (x,t) \in R_6.$$
- In order to plot snapshots of the solution at some fixed time $t > 0$, we draw the corresponding horizontal line on the characteristic diagram and then write down the solution in the various different regions it crosses, *e.g.* for $0 \le t \le L/c$, the horizontal line crosses all but region R_4, as shown below.



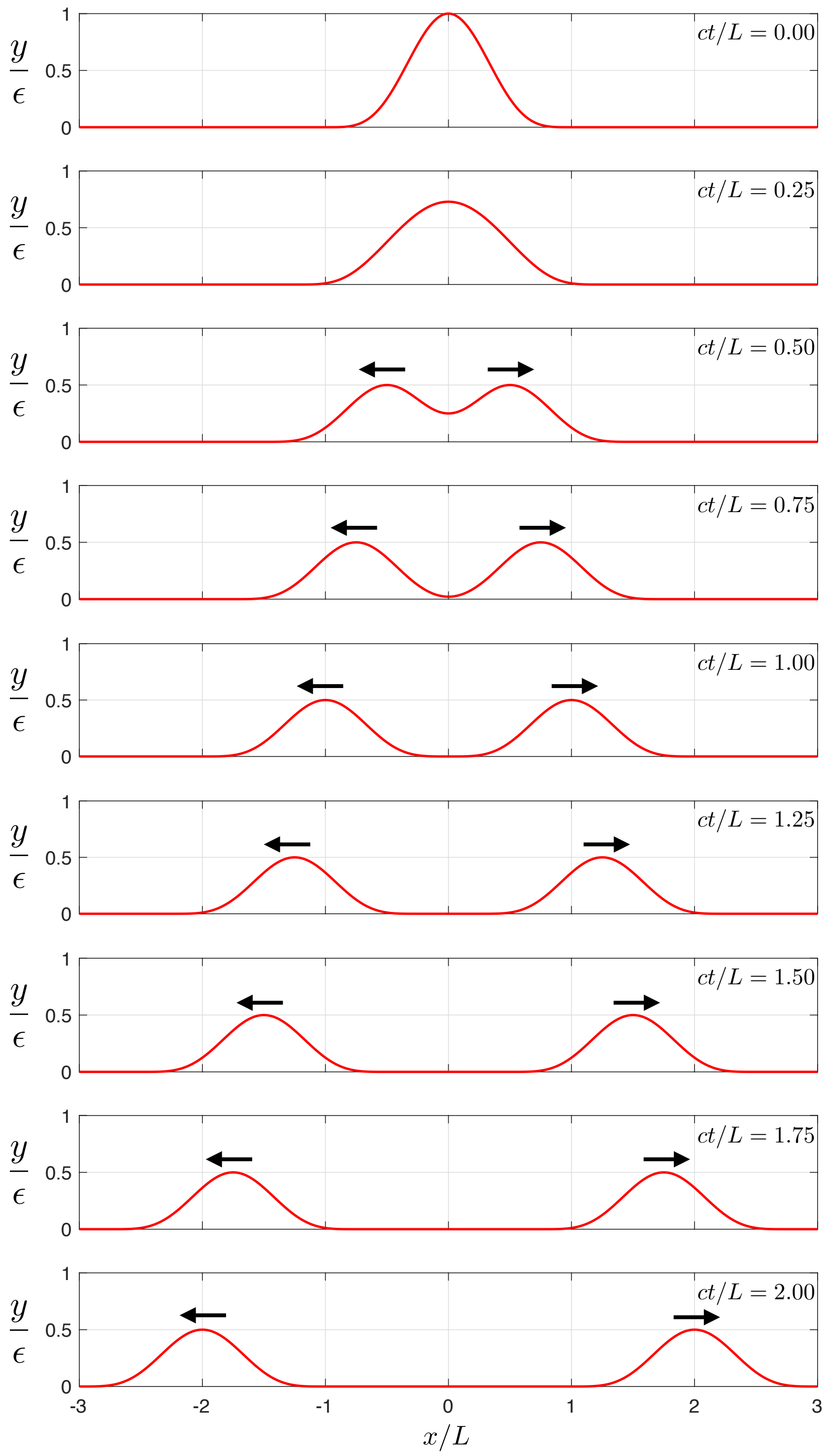
- We find that for $0 < t \le L/c$,

$$y(x,t) = \begin{cases} 0 & \text{for } x < -L - ct, & (R_1) \\ \frac{\epsilon}{2}\cos^4\left(\frac{\pi}{2L}(x+ct)\right) & \text{for } -L - ct \le x \le -L + ct, & (R_2) \\ \frac{\epsilon}{2}\cos^4\left(\frac{\pi}{2L}(x-ct)\right) + \frac{\epsilon}{2}\cos^4\left(\frac{\pi}{2L}(x+ct)\right) & \text{for } -L + ct < x < L - ct, & (R_3) \\ \frac{\epsilon}{2}\cos^4\left(\frac{\pi}{2L}(x-ct)\right) & \text{for } L - ct \le x \le L + ct, & (R_5) \\ 0 & \text{for } x > L + ct, & (R_6) \end{cases}$$

- Similarly, we find that for $t > L/c$,

$$y(x,t) = \begin{cases} 0 & \text{for } x < -L - ct, & (R_1) \\ \dfrac{\epsilon}{2}\cos^4\left(\dfrac{\pi}{2L}(x+ct)\right) & \text{for } -L - ct \le x \le L - ct, & (R_2) \\ 0 & \text{for } L - ct < x < -L + ct, & (R_4) \\ \dfrac{\epsilon}{2}\cos^4\left(\dfrac{\pi}{2L}(x-ct)\right) & \text{for } -L + ct \le x \le L + ct, & (R_5) \\ 0 & \text{for } x > L + ct. & (R_6) \end{cases}$$

- We plot below snapshots of the solution to illustrate the formation of two distinct compactly supported waves, one moving to the right with speed c and one to the left with speed c, each of them being the same shape as the initial profile, but half the amplitude. The arrows indicate the direction of travel of the waves.



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Example 2

- Suppose that f and g are given by

$$f(x)=0, \qquad g(x)=\begin{cases} vx/L & \text{for } |x|\le L, \\ 0 & \text{otherwise,}\end{cases} \tag{4.128}$$

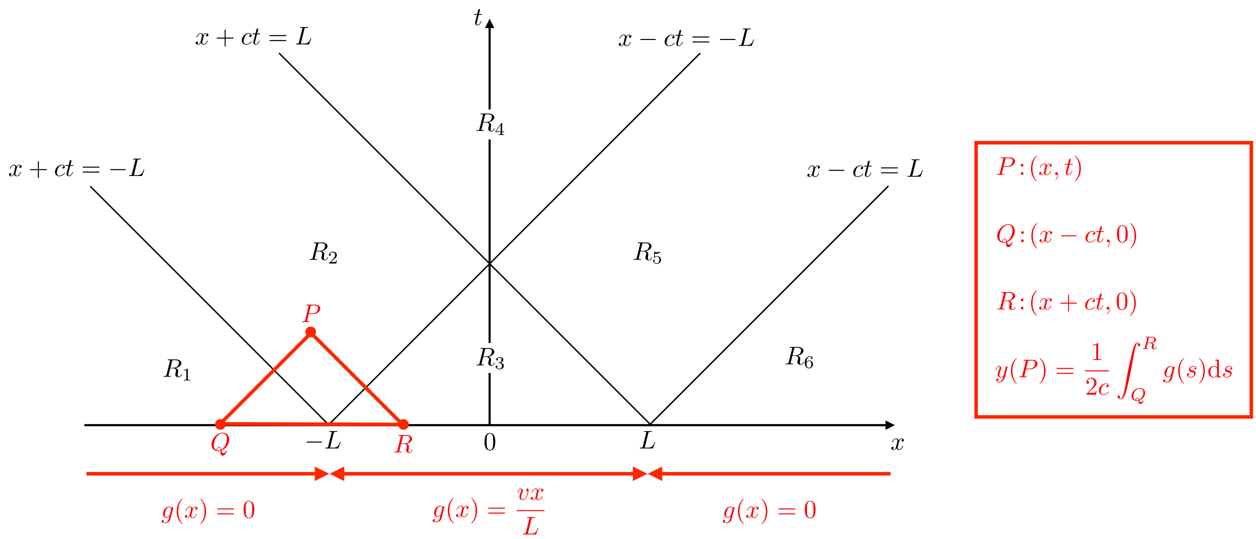
  where L and v are positive constants.

- By (4.125), we now have

$$y(P)=\frac{1}{2c}\int_Q^R g(s)\,\mathrm{d}s, \tag{4.129}$$

  where again Q and R are the left- and right-hand intersections with the x-axis of the characteristic lines though P, as illustrated in the diagram above.

- Since g is compactly supported with support $(-L,L)$, it changes its analytic behaviour at the points $(-L,0)$ and $(L,0)$ on the x-axis in the (x,t)-plane. The characteristic diagram is therefore identical to that in Example 1, with characteristics $x\pm ct=$ constant through the points $(\pm L,0)$, which divides the upper-half of the (x,t)-plane into six regions R_1, R_2, …, R_6 that we take to be the same as in Example 1, as illustrated below.



- Since PQ is parallel to the characteristics $x-ct=\pm L$, while PR is parallel to the characteristics $x+ct=\pm L$, the locations of Q and R on the x-axis dictate their contributions to (4.129), as follows:

  - if $P\in R_1$, then Q and R lie to the left of $(-L,0)$, giving

$$y(x,t)=\frac{1}{2c}\int_{x-ct}^{x+ct} 0\,\mathrm{d}s=0 \quad \text{for } (x,t)\in R_1;$$

  - if $P\in R_2$, then Q lies at or to the left of $(-L,0)$, while R lies at or between $(-L,0)$ and $(L,0)$, giving

$$y(x,t)=\frac{1}{2c}\int_{x-ct}^{-L} 0\,\mathrm{d}s+\frac{1}{2c}\int_{-L}^{x+ct}\frac{vs}{L}\,\mathrm{d}s=\frac{v}{4Lc}\left((x+ct)^2-L^2\right) \quad \text{for } (x,t)\in R_2;$$

  - if $P\in R_3$, then Q and R lie between $(-L,0)$ and $(L,0)$, giving

$$y(x,t)=\frac{1}{2c}\int_{x-ct}^{x+ct}\frac{vs}{L}\,\mathrm{d}s=\frac{v}{4Lc}\left((x+ct)^2-(x-ct)^2\right)=\frac{vxt}{L} \quad \text{for } (x,t)\in R_3;$$

- if $P \in R_4$, then Q lies to the left of $(-L, 0)$ and R lies to the right of $(L, 0)$, giving

$$y(x,t) = \frac{1}{2c} \int_{x-ct}^{-L} 0 \, \mathrm{d}s + \frac{1}{2c} \int_{-L}^{L} \frac{vs}{L} \, \mathrm{d}s + \frac{1}{2c} \int_{L}^{x+ct} 0 \, \mathrm{d}s = 0 \quad \text{for } (x,t) \in R_4;$$

- if $P \in R_5$, then Q lies at or between $(-L, 0)$ and $(L, 0)$, while R lies at or to the right of $(L, 0)$, giving

$$y(x,t) = \frac{1}{2c} \int_{x-ct}^{L} \frac{vs}{L} \, \mathrm{d}s + \frac{1}{2c} \int_{L}^{x+ct} 0 \, \mathrm{d}s = \frac{v}{4Lc} \left(L^2 - (x-ct)^2\right) \quad \text{for } (x,t) \in R_5;$$

- if $P \in R_6$, then Q and R lie to the right of $(L, 0)$, giving

$$y(x,t) = \frac{1}{2c} \int_{x-ct}^{x+ct} 0 \, \mathrm{d}s = 0 \quad \text{for } (x,t) \in R_6.$$

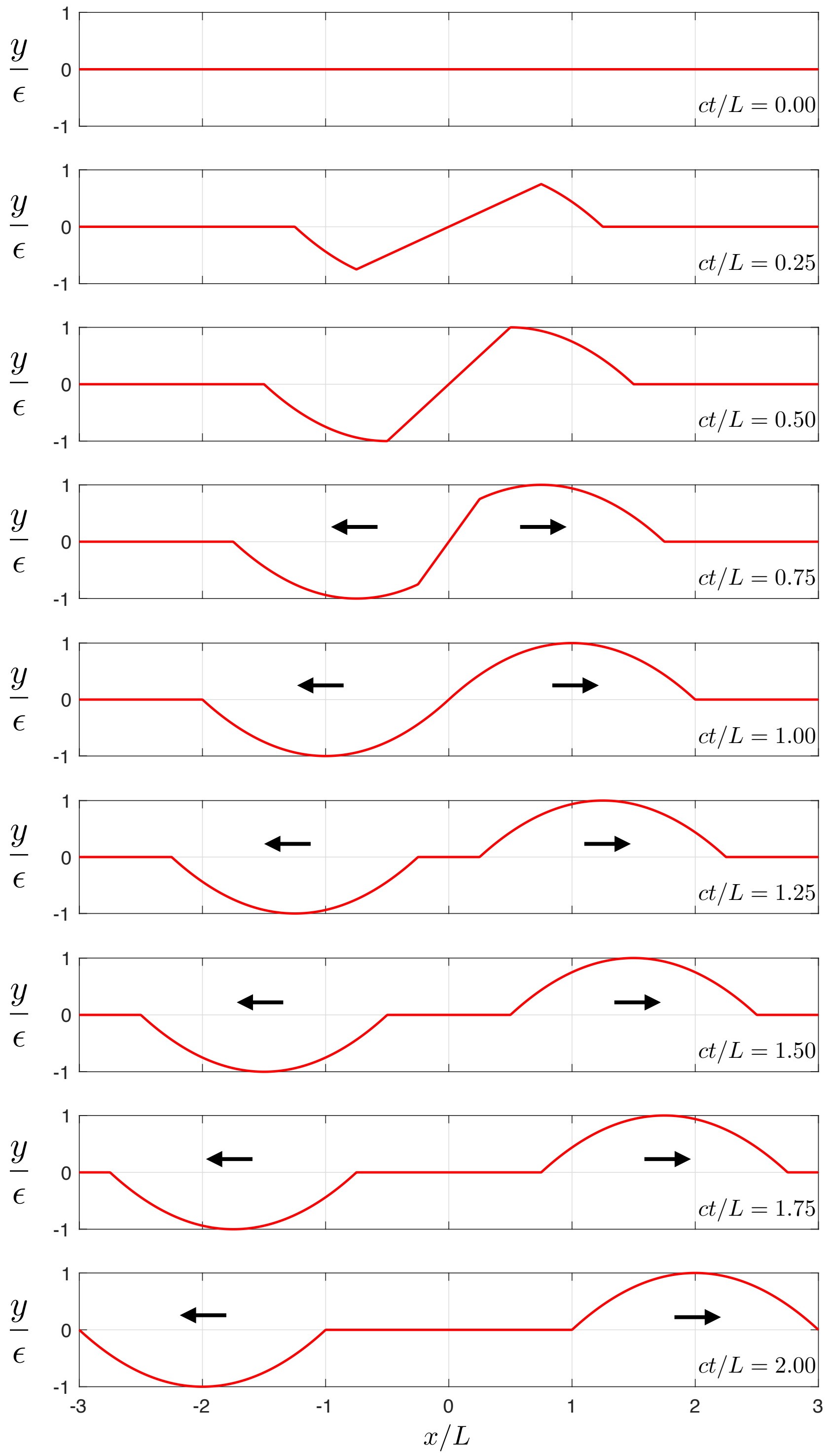
- We deduce that for $0 < t \leq L/c$,

$$y(x,t) = \begin{cases} 0 & \text{for } x < -L - ct, & (R_1) \\ \dfrac{v}{4Lc} \left((x+ct)^2 - L^2\right) & \text{for } -L - ct \leq x \leq -L + ct, & (R_2) \\ \dfrac{vxt}{L} & \text{for } -L + ct < x < L - ct, & (R_3) \\ \dfrac{v}{4Lc} \left(L^2 - (x-ct)^2\right) & \text{for } L - ct \leq x \leq L + ct, & (R_5) \\ 0 & \text{for } x > L + ct, & (R_6) \end{cases}$$

- Similarly, we find that for $t > L/c$,

$$y(x,t) = \begin{cases} 0 & \text{for } x < -L - ct, & (R_1) \\ \dfrac{v}{4Lc} \left((x+ct)^2 - L^2\right) & \text{for } -L - ct \leq x \leq L - ct, & (R_2) \\ 0 & \text{for } L - ct < x < -L + ct, & (R_4) \\ \dfrac{v}{4Lc} \left(L^2 - (x-ct)^2\right) & \text{for } -L + ct \leq x \leq L + ct, & (R_5) \\ 0 & \text{for } x > L + ct. & (R_6) \end{cases}$$

- We plot below snapshots of the solution with $\epsilon = vL/16c$ to illustrate the formation of two distinct compactly supported waves, one moving to the right with speed c and one with the opposite sign to the left with speed c. The arrows indicate the direction of travel of the waves.



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- **Notes:**

  (1) Since f is even in Example 1 and g is odd in Example 2, $y(x,t)$ is an even function of x in Example 1 and an odd function of x in Example 2. This provides a useful check of the solutions.

  (2) While the solution that we constructed in Example 1 is twice continuously differentiable with respect to x and t and hence a classical solution, the solution in example 2 contains corners (moving with speed c) and hence is not a classical solution. As mentioned at the end of §4.4, while we do not discount such solutions, we must wait for a more sophisticated theory of PDEs in order to make sense of them.

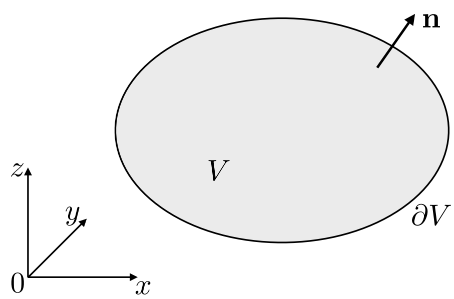
5 Laplace's equation

5.1 Preliminaries

- **Divergence Theorem:** Let V be a region of $\mathbb{R}^3$ with a piecewise smooth boundary ∂V. Let $\mathbf{F}(x, y, z)$ be a vector field with continuous first-order partial derivatives on $V \cup \partial V$. Then

$$\iiint_V \boldsymbol{\nabla} \cdot \mathbf{F} \, \mathrm{d}V = \iint_{\partial V} \mathbf{F} \cdot \mathbf{n} \, \mathrm{d}S, \tag{5.1}$$

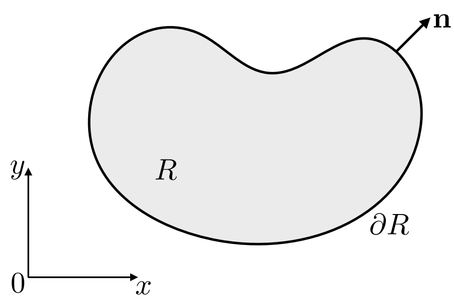
where $\mathbf{n}$ is the outward pointing unit normal to ∂V.



- **Green's Theorem in the plane:** Let R be a region in the (x, y)-plane, whose boundary ∂R is a piecewise smooth simple closed curve. Let $\mathbf{G}(x, y)$ be a vector field with continuous first-order partial derivatives on $R \cup \partial R$. Then

$$\iint_R \boldsymbol{\nabla} \cdot \mathbf{G} \, \mathrm{d}x \, \mathrm{d}y = \int_{\partial R} \mathbf{G} \cdot \mathbf{n} \, \mathrm{d}s, \tag{5.2}$$

where $\mathbf{n}$ is the outward pointing unit normal to ∂R in the (x, y)-plane.

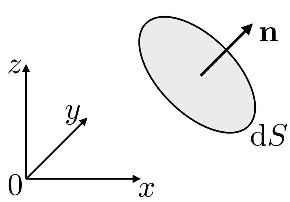


5.2 Derivation of the three-dimensional heat equation

- We begin by recalling from Multivariable Calculus the derivation of the three-dimensional heat equation because it introduces all of the quantities that we shall need.

- Let $T(\mathbf{x}, t)$ be the absolute temperature in a rigid isotropic conducting material (*e.g.* metal), with constant density ρ and specific heat c_v.

- Let $\mathbf{q}(\mathbf{x}, t)$ be the heat flux vector, so that $\mathbf{q} \cdot \mathbf{n} \, \mathrm{d}S$ is the rate at which thermal energy is transported through a surface element $\mathrm{d}S$ in the direction of the unit normal $\mathbf{n}$ that orients it.



- Let R be a fixed region in the conducting material whose boundary ∂R has outward pointing unit normal $\mathbf{n}$, as in the statement of the Divergence Theorem.

- We suppose that the material is heated volumetrically at a prescribed rate $Q(\mathbf{x}, t)$ per unit volume, so that conservation of thermal energy in V is given by

$$\underbrace{\frac{\mathrm{d}}{\mathrm{d}t}\iiint_V \rho c_v T \,\mathrm{d}V}_{(1)} = \underbrace{\iint_{\partial V} \mathbf{q}\cdot(-\mathbf{n})\,\mathrm{d}S}_{(2)} + \underbrace{\iiint_V Q\,\mathrm{d}V}_{(3)}, \tag{5.3}$$

  where (1) is the time rate of change of the thermal energy in V, (2) is the net rate at which thermal energy enters V through ∂V and (3) is the net rate of volumetric heating of V.

- Assuming T_t to be continuous on $V \cup \partial V$, so that we can differentiate under the integral sign in term (1), and applying the Divergence Theorem (5.1) with $\mathbf{F} = \mathbf{q}$ to term (2), we obtain

$$\iiint_V \rho c_v \frac{\partial T}{\partial t} + \boldsymbol{\nabla}\cdot\mathbf{q} - Q\,\mathrm{d}V = 0. \tag{5.4}$$

- Since V is arbitrary, the integrand in (5.4) must vanish if it is continuous, so we obtain the PDE

$$\rho c_v \frac{\partial T}{\partial t} + \boldsymbol{\nabla}\cdot\mathbf{q} = Q \tag{5.5}$$

  relating the time rate of change of the temperature, the divergence of the heat flux vector and the rate of volumetric heating Q.

- A closed model for heat conduction is obtained by prescribing a constitutive law relating the heat flux vector $\mathbf{q}$ and the temperature T. *Fourier's Law* states that thermal energy is transported down the temperature gradient, with

$$\mathbf{q} = -k\boldsymbol{\nabla}T, \tag{5.6}$$

  where k is the constant thermal conductivity. Recall from Introductory Calculus that $-\boldsymbol{\nabla}T$ points in the direction in which T decreases most rapidly.

- Substituting (5.6) into (5.5), we deduce that T satisfies the three-dimensional inhomogeneous or forced heat equation given by

$$\rho c_v \frac{\partial T}{\partial t} = k\boldsymbol{\nabla}^2 T + Q. \tag{5.7}$$

5.3 Steady three-dimensional heat conduction

- In steady state the temperature T and volumetric heating Q are independent of time t, so that the heat equation (5.7) reduces to *Poisson's equation*

$$-k\boldsymbol{\nabla}^2 T = Q, \tag{5.8}$$

  while conservation of energy (5.3) becomes

$$\iint_{\partial V} \mathbf{q}\cdot\mathbf{n}\,\mathrm{d}S = \iiint_V Q\,\mathrm{d}V, \tag{5.9}$$

  i.e. the net rate at which thermal energy is supplied to a region by volumetric heating is equal to the net rate at which thermal energy is conducted out though its boundary. This result holds locally for any region V, as well as globally for the whole material.

- If in addition there is no volumetric heating, so that $Q = 0$, Poisson's equation becomes *Laplace's equation*

$$\boldsymbol{\nabla}^2 T = 0, \tag{5.10}$$

  while conservation of energy (5.9) becomes

$$\iint_{\partial V} \mathbf{q}\cdot\mathbf{n}\,\mathrm{d}S = 0, \tag{5.11}$$

  i.e. the net rate at which thermal energy is conducted though the boundary of any region must vanish.

5.4 Steady two-dimensional heat conduction

- In this course we consider two-dimensional steady-state solutions of the heat equation.
- Setting $T = T(x, y)$ and $Q = Q(x, y)$, where (x, y) are Cartesian coordinates, we obtain from (5.8) Poisson's equation in the plane,

$$-k\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) = Q. \tag{5.12}$$

- In the absence of volumetric heating, we recover from (5.12) Laplace's equation in the plane,

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0. \tag{5.13}$$

- In terms of plane polar coordinates (r, θ) defined by $(x, y) = (r\cos\theta, r\sin\theta)$, (5.13) becomes

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2} = 0 \quad \text{for} \quad r > 0. \tag{5.14}$$

- We will use Fourier's method to construct solutions to boundary value problems for Laplace's equation in the plane in terms of both Cartesian and plane polar coordinates.
- If Laplace's equation holds in some region R, as in the statement of Green's Theorem in the plane, then we will need to prescribe a boundary condition on the boundary ∂R of R.
- **Definition:** The *outward normal derivative* of T on the boundary ∂R is the directional derivative of T in the direction of the outward pointing unit normal $\boldsymbol{n}$ to ∂R, and hence given by

$$\frac{\partial T}{\partial n} = \boldsymbol{n} \cdot \boldsymbol{\nabla} T \quad \text{on } \partial R. \tag{5.15}$$

- Common boundary conditions for Laplace's equation and Poisson's equation are:
  - a *Dirichlet boundary condition* in which the temperature is prescribed on the boundary,

  $$T = f \quad \text{on } \partial R; \tag{5.16}$$

  - a *Neumann boundary condition* in which the outward normal derivative of the temperature (or equivalently the heat flux $\mathbf{q}\cdot\mathbf{n} = -k\partial T/\partial n$) is prescribed on the boundary,

  $$\frac{\partial T}{\partial n} = -\frac{q}{k} \quad \text{on } \partial R; \tag{5.17}$$

  - a *Robin boundary condition* in which a linear relationship between the temperature and its outward normal derivative is prescribed on the boundary,

  $$\frac{\partial T}{\partial n} + \alpha T = \beta \quad \text{on } \partial R; \tag{5.18}$$

  here the functions f, q, α and β in (5.16)–(5.18) are prescribed on the boundary ∂R.

- **Remark:** Since (5.12) is equivalent to $\boldsymbol{\nabla}\cdot\mathbf{q} = Q$ by Fourier's law (5.6), Green's Theorem in the plane (5.2) with $\mathbf{F} = \mathbf{q}$ implies that

$$\int_{\partial R} \mathbf{q}\cdot\mathbf{n}\,\mathrm{d}s = \iint_R \boldsymbol{\nabla}\cdot\mathbf{q}\,\mathrm{d}x\,\mathrm{d}y = \iint_R Q\,\mathrm{d}x\,\mathrm{d}y, \tag{5.19}$$

which is the two-dimensional version of (5.9) and has two important consequences:

(1) if $Q = 0$, then the net heat flux through the boundary (per unit distance in the z-direction) must vanish, *i.e.*

$$\int_{\partial R} \mathbf{q}\cdot\mathbf{n}\,\mathrm{d}s = 0; \tag{5.20}$$

(2) if we impose the Neumann boundary conditions (5.17), then there can only be a steady-state solution if the net heat flux though the boundary equals the net rate of volumetric heating (per unit distance in the z-direction), *i.e.*

$$\int_{\partial R} q\,\mathrm{d}s = \iint_R Q\,\mathrm{d}x\,\mathrm{d}y, \tag{5.21}$$

since otherwise the temperature must change.

5.5 Boundary value problems in Cartesian coordinates

- An infinite straight metal rod has a rectangular cross-section whose sides are of length a and b. The temperature $T(x, y)$ in each cross-section satisfies the boundary value problem given by Laplace's equation

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0 \quad \text{for} \quad 0 < x < a,\ 0 < y < b, \tag{5.22}$$

with the Dirichlet boundary conditions

$$\begin{aligned} T(0, y) &= 0 \quad && \text{for} \quad 0 < y < b, && (5.23) \\ T(a, y) &= 0 \quad && \text{for} \quad 0 < y < b, && (5.24) \\ T(x, 0) &= 0 \quad && \text{for} \quad 0 < x < a, && (5.25) \\ T(x, b) &= f(x) \quad && \text{for} \quad 0 < x < a, && (5.26) \end{aligned}$$

where $f(x)$ is the prescribed temperature at which the top face of the rod is held.

- We construct a solution to the boundary value problem using Fourier's method, as follows.

Step (I) Find all nontrivial separable solutions of the PDE and homogeneous BCs

- We begin by finding all nontrivial separable solutions of Laplace's equation (5.22) and the homogeneous boundary conditions (5.23)-(5.25).

- Substituting $T(x, y) = F(x)G(y)$ into (5.22) and dividing through by $F(x)G(y) \neq 0$ gives

$$\frac{F''(x)}{F(x)} = -\frac{G''(y)}{G(y)}. \tag{5.27}$$

- The left-hand side of this expression is independent of y, while the right-hand side is independent of x. Since the left-hand side is equal to the right-hand side, they must both be independent of x and y, and therefore equal to a constant, $-\lambda \in \mathbb{R}$ say.

- Hence, $-F'' = \lambda F$ for $0 < x < a$, with (5.23) and (5.24) giving the boundary conditions $F(0) = 0$ and $F(a) = 0$ for nontrivial G.

- We solved this problem for F in §3.4: the nontrivial solutions are given for positive integers n by

$$F(x) = B \sin\left(\frac{n\pi x}{a}\right) \qquad \lambda = \left(\frac{n\pi}{a}\right)^2, \tag{5.28}$$

where B is an arbitrary constant.

- Since $G'' - \lambda G = 0$, the corresponding solution for $G(y)$ that satisfies (5.25) is given by

$$G = C \sinh\left(\frac{n\pi y}{a}\right), \tag{5.29}$$

where C is an arbitrary constant.

- Hence, the nontrivial separable solutions of (5.22) subject to (5.23)–(5.25) are given for positive integers n by

$$T_n(x, y) = b_n \sin\left(\frac{n\pi x}{a}\right) \sinh\left(\frac{n\pi y}{a}\right), \tag{5.30}$$

where $b_n = BC$ are real constants and we have introduced the subscript n on T_n and b_n to enumerate the countably infinite set of such solutions

- **Remark:** In contrast to the wave equation for which the nontrivial separable solutions are the product of trigonometric functions in x and trigonometric functions in t, the nontrivial separable solutions of Laplace's equation are products of trigonometric functions in x with hyperbolic functions in y or *vice versa*.

Step (II) Apply the principle of superposition

- Since (5.22)–(5.25) are linear, we can superimpose the separable solutions (assuming convergence) to obtain the general series solution

$$T(x,y) = \sum_{n=1}^{\infty} T_n(x,y) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{a}\right) \sinh\left(\frac{n\pi y}{a}\right). \tag{5.31}$$

Step (III) Use the theory of Fourier series to satisfy the inhomogeneous BC

- The boundary condition (5.26) on the top side can only be satisfied if

$$f(x) = \sum_{n=1}^{\infty} b_n \sinh\left(\frac{n\pi b}{a}\right) \sin\left(\frac{n\pi x}{a}\right) \quad \text{for} \quad 0 < x < a, \tag{5.32}$$

  so that the theory of Fourier series gives

$$b_n \sinh\left(\frac{n\pi b}{a}\right) = \frac{2}{a} \int_0^a f(x) \sin\left(\frac{n\pi x}{a}\right) \mathrm{d}x \tag{5.33}$$

  for positive integers n.

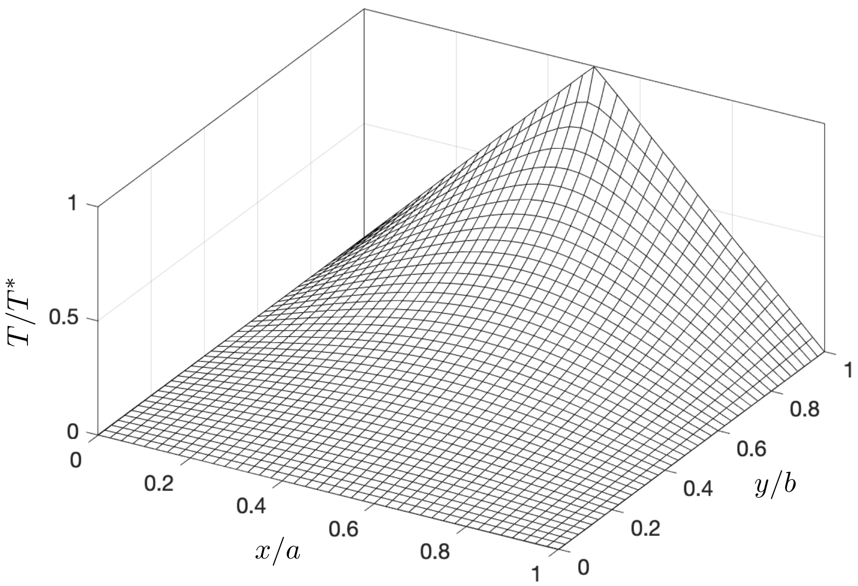
- **Remark:** If f satisfies the conditions of the Fourier Convergence Theorem, then it may be shown that the infinite series solution given by (5.31) and (5.33) is termwise infinitely differentiable with respect to x and y inside the rectangular domain $0 < x < a$, $0 < y < b$, so that it satisfies Laplace's equation there.

Example

- If the boundary data is given by $f(x) = T^*(1 - |2x/a - 1|)$, where T^* is a constant temperature, then (5.31) and (5.33) give the Fourier coefficients

$$T(x,y) = \frac{8T^*}{\pi^2} \sum_{m=0}^{\infty} \frac{(-1)^m \sin\big((2m+1)\pi x/a\big) \sinh\big((2m+1)\pi y/a\big)}{(2m+1)^2 \sinh\big((2m+1)\pi b/a\big)}. \tag{5.34}$$

- We plot below the series solution truncated to 100 terms, which illustrates the "smoothing out" of the corner in the boundary data.



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5.6 Boundary value problems in plane polar coordinates

- Recall that in plane polar coordinates (r, θ), Laplace's equation for $T(r, \theta)$ becomes

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2} = 0 \text{ for } r > 0. \tag{5.35}$$

- We start by finding all nontrivial separable solutions of the form $T(r, \theta) = F(r)G(\theta)$.
- Since T is a single-valued function of position on $r > 0$, we require $G(\theta)$ to be 2π-periodic.
- Substituting $T(r, \theta) = F(r)G(\theta)$ into (5.35) we obtain

$$F''G + \frac{1}{r}F'G + \frac{1}{r^2}FG'' = 0. \tag{5.36}$$

- Separating the variables by dividing through by $F(r)G(\theta)/r^2 \neq 0$ gives

$$\frac{r^2 F''(r) + rF'(r)}{F(r)} = -\frac{G''(\theta)}{G(\theta)}. \tag{5.37}$$

- The left-hand side of this expression is independent of θ, while the right-hand side is independent of r. Since the left-hand side is equal to the right-hand side, they must both be independent of r and θ, and therefore equal to a constant, $\lambda \in \mathbb{R}$ say.
- Hence, we need to find all $\lambda \in \mathbb{R}$ for which $G''(\theta) + \lambda G(\theta) = 0$ has a nontrivial, 2π-periodic, solution $G(\theta)$. There are three cases to consider.
- Case (i) $\lambda = -\omega^2$ ($\omega > 0$ wlog)
  - If $G'' - \omega^2 G = 0$, then $G(\theta) = A\cosh\omega\theta + B\sinh\omega\theta$, where A, $B \in \mathbb{R}$.
  - If G is 2π-periodic, then $G(0) = G(\pm 2\pi)$, which implies $A = A\cosh 2\pi\omega \pm B\sinh 2\pi\omega$, so that $A(\cosh 2\pi\omega - 1) = 0$ and $B\sinh 2\pi\omega = 0$, giving $A = B = 0$ and $G = 0$.
- Case (ii) $\lambda = 0$
  - If $G'' = 0$, then $G(\theta) = A + B\theta$, where A, $B \in \mathbb{R}$.
  - If G is 2π-periodic, then $B = 0$, but arbitrary A is admissible.
  - For $\lambda = 0$, $r^2 F'' + rF' = 0$, so $(rF')' = 0$, giving $r = c + d\log r$ for $r > 0$, where c, $d \in \mathbb{R}$.
  - We conclude that for $\lambda = 0$ there is a nontrivial, 2π-periodic, separable solution in $r > 0$ of the form

$$T_0 = A_0 + B_0 \log r, \tag{5.38}$$

    where $A_0 = cA$ and $B_0 = dA$ are real constants. Since the solution (5.38) is independent of θ it is called the *cylindrically-symmetric solution* of (5.35).
- Case (iii) $\lambda = \omega^2$ ($\omega > 0$ wlog)
  - If $G'' + \omega^2 G = 0$, then $G(\theta) = R\cos(\omega\theta + \Phi)$, where R, $\Phi \in \mathbb{R}$.
  - If G is nontrivial, then $R \neq 0$ and G has prime period $p = 2\pi/\omega$. Hence, G can only be nontrivial and 2π-periodic if there exists a positive integer n such that $np = 2\pi$, *i.e.* $\omega = n$ for some positive integer n, which the graph of G would reveal to be a geometrically obvious result.
  - In anticipation of the need to write the solution in the form of a Fourier series, we write the resulting solution for $\omega = n$ in the form $G(\theta) = A\cos n\theta + B\sin n\theta$, where $A = R\cos\Phi$, $B = -R\sin\Phi$ are arbitrary real constants.
  - If $\lambda = \omega^2 = n^2$, then we obtain for $F(r)$ Euler's ODE in the form

$$r^2 F'' + rF' - n^2 F = 0 \quad \text{for} \quad r > 0. \tag{5.39}$$

- As in Introductory Calculus, we derive the general solution of this ODE by making the change of variable $r = e^t$, $F(r) = W(t)$. By the chain rule,

$$\frac{\mathrm{d}W}{\mathrm{d}t} = \frac{\mathrm{d}F}{\mathrm{d}r}\frac{\mathrm{d}r}{\mathrm{d}t} = r\frac{\mathrm{d}F}{\mathrm{d}r}, \tag{5.40}$$

  so that

$$\frac{\mathrm{d}^2W}{\mathrm{d}t^2} = \frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}F}{\mathrm{d}r}\right)\frac{\mathrm{d}r}{\mathrm{d}t} = r\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}F}{\mathrm{d}r}\right) = r^2F'' + rF' = n^2F = n^2W. \tag{5.41}$$

  Hence, $W = Ce^{nt} + De^{-nt}$, where C, $D \in \mathbb{R}$, and we conclude that the general solution for $F(r)$ is given by

$$F(r) = Cr^n + Dr^{-n}. \tag{5.42}$$

- **Remark:** An alternative method is to seek a solution of the form $F(r) = r^\mu$ for which $\mu(\mu - 1) + \mu - \mu^2 = 0$, so that $\mu = \pm n$, from which follows the general solution (5.42).
- We conclude that for $\lambda = \omega^2$ there are a countably infinite set of nontrivial, 2π-periodic, separable solution in $r > 0$ given for positive integers n by

$$T_n = (A_nr^b + B_nr^{-n})\cos n\theta + (C_nr^n + D_nr^{-n})\sin n\theta, \tag{5.43}$$

  where $A_n = AC$, $B_n = AD$, $C_n = BC$, $D_n = BD$ are real constants and we have introduced the subscript n on T_n and these constants to enumerate the countably infinite set of such solutions.

- Superimposing the nontrivial, separable solutions in $r > 0$, namely (5.38) and (5.43), we obtain the general series solution

$$T(r,\theta) = A_0 + B_0\log r + \sum_{n=1}^{\infty}\Big((A_nr^n + B_nr^{-n})\cos n\theta + (C_nr^n + D_nr^{-n})\sin n\theta\Big). \tag{5.44}$$

- **Notes:**

  (1) The solutions $\log r$, $r^{-n}\cos n\theta$ and $r^{-n}\sin n\theta$ are unbounded as $r \to 0+$, and hence not defined at $r = 0$. This means that these solutions are not admissible if the origin belongs to the domain in which T is defined.

  (2) Similarly, if the domain in which T is defined extends to infinity and T is bounded there, then the solutions $\log r$, $r^n\cos n\theta$ and $r^n\sin n\theta$ are not admissible. We illustrate these results below with some concrete examples.

Example 1

- Consider the boundary value problem for T given by

$$\boldsymbol{\nabla}^2T = 0 \quad \text{in} \quad a < r < b, \tag{5.45}$$

  with

$$T = T_0^\star \quad \text{on} \quad r = a, \qquad T = T_1^\star \quad \text{on} \quad r = b, \tag{5.46}$$

  where a and b are constant radii, while $T_0^\star$ and $T_1^\star$ are constant temperatures.

- It follows from (5.45) that the general series solution (5.44) pertains, so that the boundary conditions (5.46) can only be satisfied if

$$T_0^\star = A_0 + B_0\log a + \sum_{n=1}^{\infty}\left((A_na^n + B_na^{-n})\cos n\theta + (C_na^n + D_na^{-n})\sin n\theta\right), \tag{5.47}$$

$$T_1^\star = A_0 + B_0\log b + \sum_{n=1}^{\infty}\left((A_nb^n + B_nb^{-n})\cos n\theta + (C_nb^n + D_nb^{-n})\sin n\theta\right), \tag{5.48}$$

  for $-\pi < \theta \leq \pi$, say.

- Since the Fourier coefficients of a Fourier series are unique, we can equate them on the left- and right-hand sides of these equalities to obtain, for positive integers n,

$$\begin{bmatrix} 1 & \log a \\ 1 & \log b \end{bmatrix} \begin{bmatrix} A_0 \\ B_0 \end{bmatrix} = \begin{bmatrix} T_0^\star \\ T_1^\star \end{bmatrix}, \qquad \begin{bmatrix} a^n & a^{-n} \\ b^n & b^{-n} \end{bmatrix} \begin{bmatrix} A_n \\ B_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \qquad \begin{bmatrix} a^n & a^{-n} \\ b^n & b^{-n} \end{bmatrix} \begin{bmatrix} C_n \\ D_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \tag{5.49}$$

giving, since $a < b$,

$$\begin{bmatrix} A_0 \\ B_0 \end{bmatrix} = \frac{1}{\log\left(\frac{b}{a}\right)} \begin{bmatrix} \log b & -\log a \\ -1 & 1 \end{bmatrix} \begin{bmatrix} T_0^\star \\ T_1^\star \end{bmatrix}, \qquad \begin{bmatrix} A_n \\ B_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \qquad \begin{bmatrix} C_n \\ D_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{5.50}$$

so that

$$T = \frac{T_0^\star \log b - T_1^\star \log a}{\log(b/a)} + \frac{T_1^\star - T_0^\star}{\log\left(\frac{b}{a}\right)} \log r. \tag{5.51}$$

i.e. a cylindrically symmetric solution. ■

Notes

(1) The solution (5.51) may be written in the form

$$\frac{T}{T_0^\star} = \frac{\log(r/b)}{\log(a/b)} + \frac{T_1^\star}{T_0^\star}\frac{\log(r/a)}{\log(b/a)}. \tag{5.52}$$

Since all of the fractions in this expression are dimensionless, it is dimensionally correct.

(2) We could have sought a circularly-symmetric solution $T = T(r)$ from the outset because the boundary data is independent of θ. However, the method above generalises to $T_0^\star$ and $T_1^\star$ being functions of θ.

Example 2

- Consider the boundary value problem for T given by

$$\boldsymbol{\nabla}^2 T = 0 \quad \text{in} \quad r < a, \tag{5.53}$$

with

$$T(a, \theta) = T^\star \sin^3\theta \quad \text{for} \quad -\pi < \theta \le \pi, \tag{5.54}$$

where a is a constant radius and $T^\star$ is a constant temperature.

- Since T satisfies Laplace's equation in $r < a$, it must be twice differentiable with respect to x and y in a neighbourhood of the origin, and therefore continuous and bounded at the origin.

- Hence, the general series solution (5.44) pertains, but with $B_0 = 0$ and $B_n = D_n = 0$ for positive integers n.

- The boundary condition (5.54) can only be satisfied if

$$T^\star \sin^3\theta = A_n + \sum_{n=1}^{\infty} \left(A_n a^n \cos n\theta + C_n a^n \sin n\theta\right) \quad \text{for} \quad -\pi < \theta \le \pi. \tag{5.55}$$

- Since the Fourier series for the left-hand side of this expression is given by the identity

$$T^\star \sin^3\theta = \frac{3T^\star}{4}\sin\theta - \frac{T^\star}{4}\sin 3\theta, \tag{5.56}$$

we can equate Fourier coefficients to deduce that

$$C_1 a = \frac{3T^\star}{4}, \; C_3 a^3 = -\frac{T^\star}{4} \tag{5.57}$$

while the remainder must vanish.

- Hence, a solution is given by

$$T = \frac{3T^\star}{4}\left(\frac{r}{a}\right)\sin\theta - \frac{T^\star}{4}\left(\frac{r}{a}\right)^3 \sin 3\theta. \tag{5.58}$$

- **Question:** What is the heat flux out of the disc through $r = a$?

- **Answer:** The heat flux vector $\boldsymbol{q} = -k\boldsymbol{\nabla}T$ according to Fourier's Law (5.6) and we need the component in the direction of the outward pointing unit normal $\boldsymbol{n} = \boldsymbol{e}_r$ to the boundary $r = a$, namely

$$\boldsymbol{q}\cdot\boldsymbol{n}|_{r=a} = (-k\boldsymbol{\nabla}T)\cdot\boldsymbol{e}_r|_{r=a} = -k\frac{\partial T}{\partial r}(a,\theta) = -k\left(\frac{3T^\star}{4a}\sin\theta - \frac{3T^\star}{4a}\sin 3\theta\right), \tag{5.59}$$

where in the last equality we substituted the solution (5.58). Since there is no volumetric heating, the net heat flux though $r = a$ must vanish according to (5.20), *i.e.*

$$\int_{r=a} \mathbf{q}\cdot\mathbf{n}\,\mathrm{d}s = 0, \tag{5.60}$$

which may be verified by substituting for (5.59) and integrating. ■

5.7 Poisson's Integral Formula

- Consider the boundary value problem for T given by

$$\boldsymbol{\nabla}^2 T = 0 \quad \text{in} \quad r < a, \tag{5.61}$$

with

$$T(a,\theta) = f(\theta) \quad \text{for} \quad -\pi < \theta \leq \pi, \tag{5.62}$$

where a is a constant radius and the temperature profile f is given.

- As in Example 2, the general series solution of (5.61) is given by (5.44), but with $B_0 = 0$ and $B_n = D_n = 0$ for positive integers n. Replacing A_0 with $A_0/2$ for algebraic convenience, we have

$$T(r,\theta) = \frac{A_0}{2} + \sum_{n=1}^{\infty}\Big(A_n r^n \cos n\theta + C_n r^n \sin n\theta\Big). \tag{5.63}$$

- The boundary condition (5.62) can only be satisfied if

$$f(\phi) = \frac{A_0}{2} + \sum_{n=1}^{\infty}\left(A_n a^n \cos(n\phi) + C_n a^n \sin(n\phi)\right) \text{ for } -\pi < \phi \leq \pi, \tag{5.64}$$

where we replaced the dummy variable θ with ϕ in anticipation of the following analysis.

- The theory of Fourier series then gives the Fourier coefficients

$$a^n A_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\phi)\cos(n\phi)\,\mathrm{d}\phi \quad \text{for } n \in \mathbb{N}, \tag{5.65}$$

$$a^n C_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\phi)\sin(n\phi)\,\mathrm{d}\phi \quad \text{for } n \in \mathbb{N}\backslash\{0\}. \tag{5.66}$$

- While these integral expressions can evaluated in simple cases (such as in Example 2), it is a remarkable fact that the series solution may be summed for a wide class of functions f (namely those that are sufficiently regular that the following analysis is valid).

- We being by substituting the Fourier coefficients (5.65)–(5.66) into the series solution (5.63) and assuming that the orders of summation and integration may be interchanged, *viz.*

$$T(r,\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\phi)\,\mathrm{d}\phi + \sum_{n=1}^{\infty}\left(\frac{1}{\pi}\int_{-\pi}^{\pi}\left(\frac{r}{a}\right)^n [\cos(n\theta)\cos(n\phi) + \sin(n\theta)\sin(n\phi)] f(\phi)\,\mathrm{d}\phi\right)$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi}\left(\frac{1}{2} + \sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n \cos n(\theta-\phi)\right) f(\phi)\,\mathrm{d}\phi. \tag{5.67}$$

- Now, if we let $\alpha = \theta - \phi$ and $z = \frac{r}{a}\mathrm{e}^{\mathrm{i}\alpha}$, then

$$\begin{aligned}
\frac{1}{2} + \sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n \cos n\alpha &= \mathrm{Re}\left(\frac{1}{2} + \sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n \mathrm{e}^{\mathrm{i}n\alpha}\right) \\
&= \mathrm{Re}\left(\frac{1}{2} + \sum_{n=1}^{\infty} z^n\right) \\
&= \mathrm{Re}\left(\frac{1}{2}\frac{1+z}{1-z}\right) \\
&= \mathrm{Re}\left(\frac{1}{2}\frac{a + r\mathrm{e}^{\mathrm{i}\alpha}}{a - r\mathrm{e}^{\mathrm{i}\alpha}}\right) \\
&= \mathrm{Re}\left(\frac{1}{2}\frac{(a + r\cos\alpha + \mathrm{i}r\sin\alpha)(a - r\cos\alpha + \mathrm{i}r\sin\alpha)}{(a - r\cos\alpha - \mathrm{i}r\sin\alpha)(a - r\cos\alpha + \mathrm{i}r\sin\alpha)}\right) \\
&= \frac{1}{2}\frac{(a + r\cos\alpha)(a - r\cos\alpha) + (\mathrm{i}r\sin\alpha)^2}{(a - r\cos\alpha)^2 + (r\sin\alpha)^2} \\
&= \frac{a^2 - r^2}{2(a^2 - 2ar\cos\alpha + r^2)},
\end{aligned} \tag{5.68}$$

where the summation of the geometric series in the third equality is valid for $|z| < 1$, *i.e.* $0 \le r < a$.

- Substituting (5.68) into (5.67), we obtain *Poisson's Integral Formula* in the form

$$T(r,\theta) = \frac{(a^2 - r^2)}{2\pi}\int_{-\pi}^{\pi}\frac{f(\phi)\,\mathrm{d}\phi}{a^2 - 2ar\cos(\theta-\phi) + r^2}, \tag{5.69}$$

which is valid for $0 \le r < a$.

Notes

(1) The value of T at the centre of the disc is given by

$$T(0,\theta) = \frac{1}{2\pi}\int_0^{2\pi} f(\phi)\,\mathrm{d}\phi, \tag{5.70}$$

which is the mean value of T over the boundary.

(2) More generally, we can now see that if T satisfies Laplace's equation in some region R and if $D(x,y,a)$ is a disk inside R with centre (x,y) and radius a, then

$$T(x,y) = \frac{1}{2\pi a}\int_{\partial D(x,y,a)} T\,\mathrm{d}s, \tag{5.71}$$

where $\partial D(x,y,a)$ is the boundary of $D(x,y,a)$ and $\mathrm{d}s$ an element of arclength. That the mean value over a circle is equal to its value at the centre is called the *mean-value property of Laplace's equation* and has important consequences. For example, it explains why solutions of Laplace's equation are infinitely differentiable, since local averages over a circle vary smoothly as the circle moves.

5.8 Uniqueness Theorems

- We will state and prove uniqueness theorems for the two-dimensional inhomogeneous Dirichlet and Neumann problems and illustrate with examples their implications for the application of Fourier's method.

- **Uniqueness Theorem (Dirichlet problem):** Consider the Dirichlet problem for $T(x, y)$ given by

$$-k\boldsymbol{\nabla}^2 T = Q \quad \text{in} \quad R, \tag{5.72}$$

with

$$T = f \quad \text{on} \quad \partial R, \tag{5.73}$$

where R is a path-connected region as in the statement of Green's theorem in the plane, Q is a given function on R and f is a given function on ∂R. Then the boundary value problem has at most one solution.

Proof:

- Let W be the difference between two solutions, then (5.72)–(5.73) imply

$$\boldsymbol{\nabla}^2 W = 0 \quad \text{in } R, \tag{5.74}$$

with

$$W = 0 \quad \text{on } \partial R. \tag{5.75}$$

- The trick is to apply Green's theorem in the plane (5.2) with $\boldsymbol{F} = W\boldsymbol{\nabla}W$ to obtain the integral identify

$$\iint\limits_R \boldsymbol{\nabla}\cdot(W\boldsymbol{\nabla}W)\,\mathrm{d}x\,\mathrm{d}y = \int\limits_{\partial R}(W\boldsymbol{\nabla}W)\cdot\boldsymbol{n}\,\mathrm{d}s. \tag{5.76}$$

- Since $\boldsymbol{\nabla}^2 W = 0$ in R,

$$\boldsymbol{\nabla}\cdot(W\boldsymbol{\nabla}W) = W\boldsymbol{\nabla}^2 W + \boldsymbol{\nabla}W\cdot\boldsymbol{\nabla}W = |\boldsymbol{\nabla}W|^2 \quad \text{in } R; \tag{5.77}$$

since $W = 0$ on ∂R,

$$W\boldsymbol{\nabla}W\cdot\boldsymbol{n} = 0 \quad \text{on } \partial R. \tag{5.78}$$

- Substituting (5.77)–(5.78), the integral identity (5.76) becomes

$$\iint\limits_R |\boldsymbol{\nabla}W|^2\,\mathrm{d}x\,\mathrm{d}y = 0. \tag{5.79}$$

- Assuming $\boldsymbol{\nabla}W$ is continuous on $R\cup\partial R$, (5.79) implies that $\boldsymbol{\nabla}W = \mathbf{0}$ on R, so that W is constant on R because R is path connected.

- But $W = 0$ on ∂R, so assuming W is continuous on $R\cup\partial R$, the constant must vanish, and we deduce that $W = 0$ on $R\cup\partial R$. ■

Example 1

- Find T such that $\boldsymbol{\nabla}^2 T = 0$ in $r < a$ with $T = T^\star x/a$ on $r = a$, where a and $T^\star$ are constants.

- If we can find any solution, then the uniqueness theorem guarantees it is the only solution.

- We could proceed via Fourier's method or Poisson's Integral Formula, but it is quicker to spot that the solution is simply $T = T^\star x/a$. ■

- **Uniqueness Theorem (Neumann problem):** Consider the Neumann problem for $T(x, y)$ given by

$$k\boldsymbol{\nabla}^2 T + Q = 0 \quad \text{in} \quad R, \tag{5.80}$$

with

$$-k\frac{\partial T}{\partial n} = q \quad \text{on} \quad \partial R, \tag{5.81}$$

where R is a bounded and path-connected region as in the statement of Green's theorem in the plane, Q is a given function on R and q is a given function on ∂R. Then the boundary value problem has no solution unless Q and q satisfy the solvability condition

$$\iint_R Q\,\mathrm{d}x\,\mathrm{d}y = \int_{\partial R} q\,\mathrm{d}s. \tag{5.82}$$

When a solution exists, it is not unique: any two solutions differ by a constant.

Proof:

- Suppose there is a solution T, then

$$\iint_R Q\,\mathrm{d}x\,\mathrm{d}y = -\frac{1}{k}\iint_R \boldsymbol{\nabla}\cdot\boldsymbol{\nabla}T\,\mathrm{d}x\,\mathrm{d}y = -\frac{1}{k}\iint_{\partial R} \boldsymbol{n}\cdot\boldsymbol{\nabla}T\,\mathrm{d}s = -\frac{1}{k}\int_{\partial R}\frac{\partial T}{\partial n}\,\mathrm{d}s = \int_{\partial R} q\,\mathrm{d}s, \tag{5.83}$$

where we used (5.80) in the first equality, Green's theorem in the plane (5.2) with $\boldsymbol{F} = \boldsymbol{\nabla}T$ in the second equality and (5.81) in the final equality.

- Now let W be the difference between two solutions, so that linearity gives

$$\boldsymbol{\nabla}^2 W = 0 \quad \text{in} \quad R, \tag{5.84}$$

with

$$\frac{\partial W}{\partial n} = 0 \quad \text{on} \quad \partial R. \tag{5.85}$$

- Then, as in the uniqueness proof for the Dirichlet problem,

$$\begin{aligned}
\iint_R |\boldsymbol{\nabla}W|^2\,\mathrm{d}x\,\mathrm{d}y &= \iint_R W\boldsymbol{\nabla}^2 W + \boldsymbol{\nabla}W\cdot\boldsymbol{\nabla}W\,\mathrm{d}x\,\mathrm{d}y \\
&= \iint_R \boldsymbol{\nabla}\cdot(W\boldsymbol{\nabla}W)\,\mathrm{d}x\,\mathrm{d}y \\
&= \int_{\partial R} W\boldsymbol{\nabla}W\cdot\boldsymbol{n}\,\mathrm{d}s \\
&= \int_{\partial R} W\frac{\partial W}{\partial n}\,\mathrm{d}s \\
&= 0,
\end{aligned} \tag{5.86}$$

where we used (5.84) in the first equality, Green's theorem in the plane (5.2) with $\boldsymbol{F} = W\boldsymbol{\nabla}W$ in the second equality and (5.85) in the final equality.

- Assuming $\boldsymbol{\nabla}W$ is continuous on $R\cup\partial R$, (5.86) implies that $\boldsymbol{\nabla}W = \boldsymbol{0}$ on R, so that W is constant on R because R is path connected. Hence, W is constant on $R\cup\partial R$, assuming W is continuous there, *i.e.* any two solutions differ by a constant. ■

- **Remark:** The solvability condition (5.83) is precisely the global energy balance expressed in (5.21)!

Example 2

- Find T such that

$$\boldsymbol{\nabla}^2 T = 0 \quad \text{in } r < a, \tag{5.87}$$

with

$$-\,k\frac{\partial T}{\partial n}(a,\theta) = q(\theta) \quad \text{for} -\pi < \theta \leq \pi, \tag{5.88}$$

where the heat flux $q(\theta)$ is given.

- As in §5.5 the general series solution of Laplace's equation in $r < a$ is given by

$$T = A_0 + \sum_{n=1}^{\infty} \left(A_n r^n \cos n\theta + C_n r^n \sin n\theta\right). \tag{5.89}$$

so the boundary condition on $r = a$ can be satisfied only if

$$q(\theta) = \sum_{n=1}^{\infty} \big(-knA_n a^{n-1}\cos n\theta - knC_n a^{n-1}\sin n\theta\big) \quad \text{for} \quad -\pi < \theta \leq \pi. \tag{5.90}$$

- The theory of Fourier series then requires

$$0 = \frac{1}{\pi}\int_{-\pi}^{\pi} q(\theta)\,\mathrm{d}\theta, \tag{5.91}$$

while for positive integers n

$$-knA_n a^{n-1} = \frac{1}{\pi}\int_{-\pi}^{\pi} q(\theta)\cos n\theta\,\mathrm{d}\theta, \tag{5.92}$$

$$-knC_n a^{n-1} = \frac{1}{\pi}\int_{-\pi}^{\pi} q(\theta)\sin n\theta\,\mathrm{d}\theta. \tag{5.93}$$

- Hence, there are two cases:

  (i) if q is such that (5.91) is not satisfied, then there is no solution;

  (ii) if q is such that (5.91) is satisfied, then there is a solution but it is not unique because A_0 is arbitrary (while the other Fourier coefficients are uniquely determined). ■

Notes

(1) This conclusion is in agreement with the Uniqueness Theorem, which also guarantees that in case (ii) we've found all possible solutions.

(2) In case (i) there is no solution because the temperature cannot be in steady state if the net heat flux through $r = a$ is a non-zero.

(3) In case (ii) there can be a steady state solution because the net heat flux through $r = a$ vanishes, but we cannot pin down the temperature without additional information — in practice this would usually be provided by the evolution toward the steady state.

5.9 A boundary value problem for Poisson's equation

- An infinite straight metal rod of constant thermal conductivity k has a square cross-section whose sides are of length L. The temperature $T(x,y)$ in each cross-section $R = \{(x,y) : 0 < x, y < L\}$ satisfies the boundary value problem given by Poisson's equation

$$-k\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) = Q \quad \text{in } R, \tag{5.94}$$

with the Dirichlet boundary condition

$$T = 0 \quad \text{on } \partial R, \tag{5.95}$$

where the rate of volumetric heating $Q(x,y)$ is given on R.

- We may construct a solution using Fourier series, as follows.

- Motivated by the success of the expansion (4.68) in §4.7 and the form of the boundary conditions (5.95), we suppose that $T(x,y)$ may be expanded as the Fourier sine series

$$T(x,y) = \sum_{m=1}^{\infty} T_m(y) \sin\left(\frac{m\pi x}{L}\right), \tag{5.96}$$

where for positive integers m the Fourier coefficients are given by

$$T_m(y) = \frac{2}{L}\int_0^L T(x,y)\sin\left(\frac{m\pi x}{L}\right)\, dx. \tag{5.97}$$

- Suppose further that for each positive integer m, $T_m(y)$ may be expanded as the Fourier sine series

$$T_m(y) = \sum_{n=1}^{\infty} B_{mn} \sin\left(\frac{n\pi y}{L}\right), \tag{5.98}$$

where for positive integers n the Fourier coefficients are given by

$$B_{mn} = \frac{2}{L}\int_0^L T_m(y)\sin\left(\frac{n\pi y}{L}\right)\, dy. \tag{5.99}$$

- Substituting (5.98) into (5.96) and (5.97) into (5.99), we see that we are seeking a solution for $T(x,y)$ in the form of the doubly-infinite Fourier sine series

$$T(x,y) = \sum_{m,n=1}^{\infty} B_{mn} \sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi y}{L}\right), \tag{5.100}$$

where the Fourier coefficients are given for positive integers m and n by

$$B_{mn} = \left(\frac{2}{L}\right)^2 \int_0^L\int_0^L T(x,y)\sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi y}{L}\right)\, dx\, dy. \tag{5.101}$$

Evidently (5.100) satisfies the boundary conditions (5.95).

- To determine the dependence of the Fourier coefficients B_{mn} on $Q(x,y)$, we multiply Poisson's equation (5.94) by $\sin(m\pi x/L)\sin(n\pi y/L)$ and integrate over R to obtain

$$\int_0^L\int_0^L \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{Q}{k}\right)\sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi y}{L}\right)\, dx\, dy = 0. \tag{5.102}$$

- Integrating by parts using the boundary conditions (5.95) gives

$$\int_0^L \frac{\partial^2 T}{\partial x^2}\sin\left(\frac{m\pi x}{L}\right) dx = -\frac{m^2\pi^2}{L^2}\int_0^L T\sin\left(\frac{m\pi x}{L}\right) dx, \tag{5.103}$$

$$\int_0^L \frac{\partial^2 T}{\partial y^2}\sin\left(\frac{n\pi y}{L}\right) dx = -\frac{n^2\pi^2}{L^2}\int_0^L T\sin\left(\frac{n\pi y}{L}\right) dx. \tag{5.104}$$

- Substituting (5.103)–(5.104) into (5.102) gives

$$\int_0^L\int_0^L\left(-\frac{m^2\pi^2}{L^2}T-\frac{n^2\pi^2}{L^2}T+\frac{Q}{k}\right)\sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi y}{L}\right)\,\mathrm{d}x\,\mathrm{d}y=0, \tag{5.105}$$

so that

$$B_{mn}=\frac{L^2Q_{mn}}{k\pi^2(m^2+n^2)}, \tag{5.106}$$

where the Fourier coefficients of the doubly-infinite Fourier sine series for $Q(x,y)$ are defined by

$$Q_{mn}=\left(\frac{2}{L}\right)^2\int_0^L\int_0^L Q(x,y)\sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi y}{L}\right)\,\mathrm{d}x\,\mathrm{d}y. \tag{5.107}$$

- Hence, the solution of the boundary value problem (5.94)–(5.95) is given by (5.100) with B_{mn} given in terms of $Q(x,y)$ by (5.106)–(5.107).

Example

- Consider the boundary value problem for $T(x,y)$ given by (5.94)–(5.95) with $Q=Q^*$, a constant.
- Evaluating (5.107) we obtain

$$Q_{mn}=Q^*\left(\frac{2}{L}\right)^2\left(\frac{L}{m\pi}(1-(-1)^m)\right)\left(\frac{L}{n\pi}(1-(-1)^n)\right), \tag{5.108}$$

so that (5.106) gives

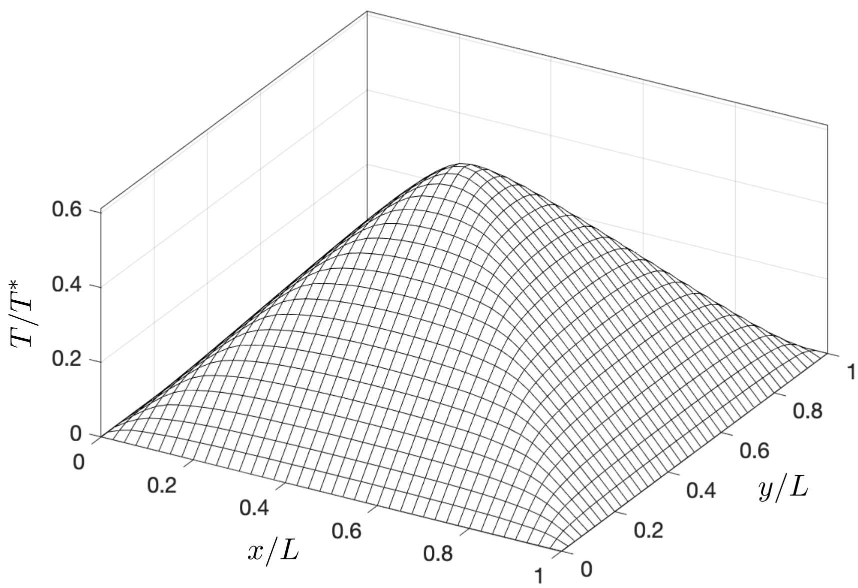
$$B_{mn}=\frac{4L^2Q^*}{\pi^4k}\frac{\big(1-(-1)^m\big)\big(1-(-1)^n\big)}{mn(m^2+n^2)}. \tag{5.109}$$

- Setting $m=2i+1$ and $n=2j+1$ to enumerate the non-zero terms and substituting into (5.100), we deduce that

$$T(x,y)=T^*\sum_{i,j=0}^{\infty}\frac{\sin\big((2i+1)\pi x/L\big)\sin\big((2j+1)\pi y/L\big)}{(2i+1)(2j+1)\big((2i+1)^2+(2j+1)^2\big)}. \tag{5.110}$$

where $T^*=16L^2Q^*/\pi^4k$.

- We plot below (5.110) truncated symmetrically to 100 terms, which illustrates that there is a maximum of the temperature at the centre of the square.



Notes

(1) The functions

$$T_{mn}(x,y) = B_{mn} \sin\left(\frac{m\pi x}{L}\right) \sin\left(\frac{n\pi y}{L}\right) \qquad (m,\, n \in \mathbb{N}\backslash\{0\}) \tag{5.111}$$

form the doubly-infinite set of eigenfunctions of the eigenvalue problem for T and $\lambda \in \mathbb{R}$ given by

$$-\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) = \lambda T \text{ in } R \quad \text{with} \quad T = 0 \text{ on } \partial R, \tag{5.112}$$

the eigenvalue $\lambda = \lambda_{mn}$ corresponding to the eigenfunction $T_{mn}(x,y)$ being given by

$$\lambda_{mn} = \frac{m^2\pi^2}{L^2} + \frac{n^2\pi^2}{L^2}. \tag{5.113}$$

The problem (5.112) is the two-dimensional generalization of the eigenvalue problem (3.33)–(3.34) and the solution (5.100) is another example of an *eigenfunction expansion*.

(2) Such is the importance of Laplace's equation that the skylight crystals in the Mathematical Institute are based on an eigenfunction for the opening R in the floor.

Photo: South Crystal Skylight © David Hawgood
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6 Well-posedness

- **Definition:** A problem is said to be *well-posed* if the following three conditions are satisfied:
  (1) EXISTENCE — there is a solution;
  (2) UNIQUENESS — there is no more than one solution;
  (3) CONTINUOUS DEPENDENCE — the solution depends continuously on the data.
- The first is obvious: there is no point in trying to find a solution that does not exist.
- As for the second, if a problem is physically motivated, and the solution represents a physical quantity, then we would expect it to have a unique well-defined value at each point. If it does not, it suggests that a boundary condition or other constraint is missing from the problem.
- To illustrate the final condition, suppose we vary the initial temperature profile $f(x)$ in the IBVP (3.29)–(3.31) by a small amount and ask whether the corresponding variation in the solution is similarly small. If it is not, then the numerical solution of the problem is practically impossible, since any numerical errors in $f(x)$, however small, would lead to large errors in the solution.

6.1 The heat equation

- Consider the dimensionless initial value problem for $T(x,t)$ given by
$$\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2} \quad \text{for } -\infty < x < \infty,\ t > 0, \tag{6.1}$$
with the initial condition
$$T(x,0) = f(x) \quad \text{for } -\infty < x < \infty, \tag{6.2}$$
where $f(x)$ is given.
- If $f(x) = 0$, then the trivial solution $T(x,t) = 0$ satisfies the initial value problem.
- Tychonoff (1935) showed that a nontrivial solution for $f(x) = 0$ is given by
$$T(x,t) = \sum_{k=0}^{\infty} \frac{g^{(k)}(t)x^{2k}}{(2k)!}, \tag{6.3}$$
where
$$g(t) = \begin{cases} \exp(-1/t^2) & \text{for } t > 0, \\ 0 & \text{for } t = 0. \end{cases} \tag{6.4}$$
- Since there is more than one solution to the initial value problem (6.1)–(6.2), it is not well-posed — it is *ill-posed*. However, well-posedness can be established by imposing additional regularity conditions on the growth of $T(x,t)$ as $x \to \pm\infty$.
- For example, if $T(x,t)$ is assumed to be bounded for $-\infty < x < \infty$, $t > 0$ and $f(x)$ to be piecewise continuous on any interval $(a,b) \subset \mathbb{R}$, then it may be shown that the unique solution is given by
$$T(x,t) = \int_{-\infty}^{\infty} \frac{f(s)}{\sqrt{4\pi t}} \exp\left(-\frac{(s-x)^2}{4t}\right) \mathrm{d}s, \tag{6.5}$$
i.e. the superposition of fundamental solutions of the heat equation weighted by the initial temperature profile. Continuous dependence on the initial data may then be established as follows.
- Let $\epsilon > 0$ and suppose that the initial data $f = f_1$ and $f = f_2$ are close together in the sense that
$$|f_1(x) - f_2(x)| < \epsilon \quad \text{for} -\infty < x < \infty.$$
Then the corresponding solutions $T_1(x,t)$ and $T_2(x,t)$ are close together because (6.5) implies that
$$|T_1(x,t) - T_2(x,t)| \le \int_{-\infty}^{\infty} \frac{|f_1(s) - f_2(s)|}{\sqrt{4\pi t}} \exp\left(-\frac{(s-x)^2}{4t}\right) \mathrm{d}s \le \int_{-\infty}^{\infty} \frac{\epsilon}{\sqrt{4\pi t}} \exp\left(-\frac{(s-x)^2}{4t}\right) \mathrm{d}s = \epsilon$$
for $-\infty < x < \infty$, $t > 0$. In this sense (3) holds.

6.2 The wave equation

- Consider the dimensionless initial value problem for $y(x,t)$ given by

$$\frac{\partial^2 y}{\partial t^2} = \frac{\partial^2 y}{\partial x^2} \quad \text{for} \quad -\infty < x < \infty,\ t > 0, \tag{6.6}$$

with the initial conditions

$$y(x,0) = f(x), \quad \frac{\partial y}{\partial t}(x,0) = g(x) \quad \text{for} \quad -\infty < x < \infty, \tag{6.7}$$

where $f(x)$ and $g(x)$ are given.

- By D'Alembert's formula (with $c = 1$) there exists a unique solution given by

$$y(x,t) = \frac{1}{2}\big(f(x-t) + f(x+t)\big) + \frac{1}{2}\int_{x-t}^{x+t} g(s)\,\mathrm{d}s. \tag{6.8}$$

- Hence, (1) and (2) hold.

- Suppose

$$|f(x)| < \delta \quad \text{and} \quad |g(x)| < \delta \quad \text{for} \ -\infty < x < \infty, \tag{6.9}$$

where $\delta > 0$. Then (6.8) implies that

$$\begin{aligned}
|y(x,t)| &= \left|\frac{1}{2}f(x-t) + \frac{1}{2}f(x+t) + \frac{1}{2}\int_{x-t}^{x+t} g(s)\,\mathrm{d}s\right| \\
&\leq \frac{1}{2}\big|f(x-t)\big| + \frac{1}{2}\big|f(x+t)\big| + \frac{1}{2}\int_{x-t}^{x+t} \big|g(s)\big|\,\mathrm{d}s \\
&\leq \frac{1}{2}\delta + \frac{1}{2}\delta + \frac{1}{2}\int_{x-t}^{x+t} \delta\,\mathrm{d}s \\
&= \frac{1}{2}\delta + \frac{1}{2}\delta + \frac{1}{2}\cdot 2t\delta \\
&= (1+t)\delta \quad \text{for} \ -\infty < x < \infty,\ t \geq 0,
\end{aligned} \tag{6.10}$$

where we used the triangle inequality.

- Consider the initial value problems with initial data $(f,g) = (f_1, g_1)$ and $(f,g) = (f_2, g_2)$ and corresponding solutions $y_1(x,t)$ and $y_2(x,t)$, respectively.

- Suppose that we are interested in making predictions in the time interval $t \in (0, t_0)$ and let $\epsilon > 0$.

- It follows from (6.8)–(6.10) with $f = f_1 - f_2$, $g = g_1 - g_2$, $y = y_1 - y_2$ and $\delta = \epsilon/(1+t_0)$ that if the data (f_1, g_1) is close to the data (f_2, g_2) in the sense that

$$|f_1(x) - f_2(x)| < \frac{\epsilon}{1+t_0} \quad \text{and} \quad |g_1(x) - g_2(x)| < \frac{\epsilon}{1+t_0} \quad \text{for} \ -\infty < x < \infty, \tag{6.11}$$

then the corresponding solutions $y_1(x,t)$ and $y_2(x,t)$ are close together in the sense that

$$|y_1(x,t) - y_2(x,t)| \leq (1+t)\frac{\epsilon}{1+t_0} < \epsilon \quad \text{for} \ -\infty < x < \infty,\ 0 < t < t_0. \tag{6.12}$$

- Hence, (3) holds in this sense and we conclude that the initial value problem for the wave equation is well-posed.

6.3 Laplace's equation

- By contrast the corresponding initial value problem for Laplace's equation is *ill-posed*.
- Consider the initial value problem for $y(x,t)$ given by

$$\frac{\partial^2 y}{\partial x^2} + \frac{\partial^2 y}{\partial t^2} = 0, \quad -\infty < x < \infty,\ t > 0, \tag{6.13}$$

$$y(x,0) = f(x), \quad \frac{\partial y}{\partial t}(x,0) = g(x), \quad -\infty < x < \infty, \tag{6.14}$$

  where $f(x)$ and $g(x)$ are given.

- If we take the initial data $(f,g) = (f_1,g_1)$ and $(f,g) = (f_2,g_2)$ given by

$$f_1(x) = 0, \quad g_1(x) = 0, \quad f_2(x) = 0, \quad g_2(x) = \delta \cos(x/\delta),$$

  where $\delta > 0$, then corresponding solutions are given by

$$y_1(x,t) = 0, \quad y_2(x,t) = \delta^2 \cos\left(\frac{x}{\delta}\right) \sinh\left(\frac{t}{\delta}\right).$$

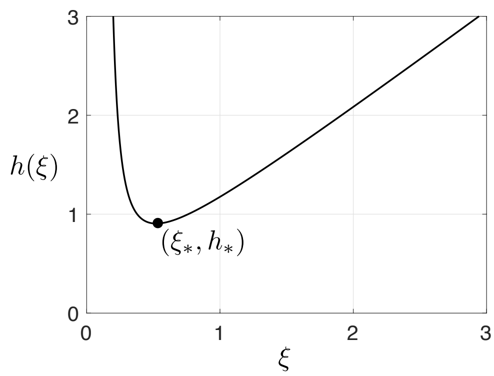
- Again suppose we want to make predictions in $0 < t < t_0$.
- Observe that

$$|f_1(x) - f_2(x)| = 0 < \delta, \quad |g_1(x) - g_2(x)| = \delta \left|\cos\left(\frac{x}{\delta}\right)\right| \le \delta \quad \text{for } -\infty < x < \infty.$$

- If we define $h(\xi) = \xi^2 \sinh(1/\xi)$, then

$$\max_{t\in[0,t_0]} \max_{x\in\mathbb{R}} \left|y_1(x,t) - y_2(x,t)\right| = \max_{t\in[0,t_0]} \delta^2 \sinh\left(\frac{t}{\delta}\right) = t_0^2 \left(\frac{\delta}{t_0}\right)^2 \sinh\left(\frac{t_0}{\delta}\right) = t_0^2 h(\delta/t_0).$$

- As illustrated by the plot below, the function $h(\xi)$ is bounded below by $h_* = h(\xi_*) \approx 0.905$ for $\xi > 0$, elementary calculus giving the location of the minimum to be ξ_*, where $\xi_* \approx 0.522$ is the unique positive root of the transcendental equation $2\tanh(1/\xi_*) = 1/\xi_*$.



- We deduce that

$$\min_{\delta>0} t_0^2\, h(\delta/t_0) = t_0^2 \min_{\xi>0} h(\xi) = h_* t_0^2. \tag{6.15}$$

- Since the maximum of $\left|y_1(x,t) - y_2(x,t)\right|$ for $\infty < x < \infty$, $0 \le t \le t_0$ is bounded below by a positive constant (namely $h_* t_0^2$) for $\delta > 0$, we cannot make

$$|y_1(x,t) - y_2(x,t)| < \epsilon \quad \text{for } -\infty < x < \infty,\ 0 < t < t_0 \tag{6.16}$$

  by making δ suitably small for all $\epsilon > 0$.

- Hence, the initial value problem for Laplace's equation is ill-posed because it fails condition (3).

- Instead of imposing two initial conditions, we should have imposed $y(x,t)$ on $t=0$ as well as in the far-field as $x^2+t^2 \to \infty$.

- For example, switching back to spatial coordinates (x,y), suppose that $T(x,y)$ satisfies

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0 \quad \text{for } -\infty < x < \infty,\ y > 0, \tag{6.17}$$

  with boundary condition

$$T(x,0) = f(x) \quad \text{for } -\infty < x < \infty, \tag{6.18}$$

  and the far-field boundary condition

$$T(x,y) \to 0 \quad \text{as } x^2+y^2 \to \infty,\ y > 0, \tag{6.19}$$

  where $f(x)$ is given.

- It may be shown that the boundary value problem (6.17)–(6.19) is well-posed given sufficient regularity of f, *e.g.* if f is piecewise continuous on any interval $(a,b) \subset \mathbb{R}$. In this case it may be shown that the solution is given by

$$T(x,y) = \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{f(s)}{(s-x)^2+y^2}\, \mathrm{d}s. \tag{6.20}$$

- **Remark:** The solutions (6.5) and (6.20) are derived in part A Integral Transforms using a *Fourier transform* – a powerful generalization of Fourier series.

7 Summary

(1) Introduction

- History: Fourier's revolutionary claim.
- Revised ODEs: nomenclature and pre-requisite material.
- Introduced PDEs: nomenclature and how they arise.
- Motivated need to study Fourier series to solve PDE problems.

(2) Fourier Series

- Periodic, even and odd functions and periodic extensions.
- Euler's formulae for Fourier coefficients via orthogonality relations.
- Statement of a powerful pointwise convergence theorem.
- Related rate of convergence to smoothness.
- Discussed Gibb's phenomenon - try to avoid!
- Problem sheets imparted "familiarity with the calculation of Fourier coefficients."

(3) Heat equation

- Derivation in 1D.
- Simple solutions.
- Units and nondimensionalisation.
- Fourier's method for IBVPs.
- Generalised to inhomogeneous heat equation and BCs.
- Uniqueness.

(4) Wave equation

- Derivation in 1D with gravity and air resistance.
- Normal modes and natural frequencies.
- Fourier's method for IBVPs - plucked and hammered strings.
- Forced wave equation with inhomogeneous BCs.
- Normal modes for weighted strings.
- D'Alembert's solution and characteristic diagrams.
- Uniqueness.

(5) Laplace's equation

- Fourier's method for BVPs in (x, y) and (r, θ).
- Poisson's Integral Formula for Dirichlet problem on a disk.
- Uniqueness of Dirichlet problem.
- Nonexistence and nonuniqueness of Neumann problem.

(6) Well-posedness

- Introduced concepts developed later on in course.