SET THEORY

Sets are collection of distinguish and unique objects.

$S_1 = \{a, e, i, o, u\}$

$S_2 := \{0, 2, 4, 6, 8\}$

"=" equality

":=" right side is defind by left side.

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Representing sets

Objects inside the set are called elements of the set.

1) Roaster Notation:

List all elements.

$S_2 := \{0, 2, 4, 6, 8\}$

$S_{100} := \{0, 2, 4, \dots, 100\}$

$\mathbb{N} := \{1, 2, 3, \dots\}$

2 is an element of S_2

2 belongs to S_2

$2 \underbrace{\in}_{} S_2$

2) set-builder Notation

$S := \{x \mid x \text{ has some property}\}$

such that — defining property of set.

Sets where objects are repeated and considered crucial are called Multi-sets.

$S_2 := \{x \mid 0 \leq x \leq 8 \text{ and } x \text{ is even}\}$

$S_{100} := \{x \mid 0 \leq x \leq 100 \text{ and } x \text{ is even}\}$

$S_1 := \{x \mid x \in \text{English alphabet}, x \text{ is a vowel}\}$

Our approach is naive ~~appr~~ set Theory.

Two sets A and B are equal if and only if $\forall a \in A$ also belongs to $\forall a \in B$.

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If P then Q → $P \Rightarrow Q$

If P is true then Q is also True.

$(a \in A \Rightarrow a \in B) \wedge (b \in B \wedge b \in A)$

If P is true then Q is true, and if Q is true then P is true.

P iff Q $P \Leftrightarrow Q$

Q iff P

$B \subseteq A$ iff $x \in B \Rightarrow x \in A$ (subset)

$B \subset A : (B \subseteq A : x \in B \Rightarrow x \in A) \wedge \exists y \in A \mid y \notin B$

Theoram:— Two sets A and B are equal iff $A \subseteq B \wedge B \subseteq A$

Set with one element are called Singletons.

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Proof:—

~~$(a \in A \Rightarrow a \in B) \wedge (b \in B \Rightarrow b \in A)$~~

$x \in A \Leftrightarrow x \in B$

$\mathbb{N} := \{0, 1, 2, 3, \dots\}$

$\mathbb{Z} := \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$

$\mathbb{Z}^+ := \{1, 2, 3, \dots\}$

$\mathbb{Q} := \{p/q \mid p, q \in \mathbb{Z} \wedge q \neq 0\}$

~~Proof:~~ Thm: For any set $A \mid |A| = n$ the power set $P(A)$ has $|P(A)| = 2^n$. 20/01/2022

Proof:— (by Induction).

$P(n)$: for a set $A \mid |A| = n$. $\exists P(A) \mid |P(A)| = 2^n$.

For $n = 0$, the set is null set $\emptyset$ and it's Power set is, $P(A) = \{\emptyset\}$ and $|P(A)| = 1$. $P(0)$ is True.

We are assuming our hypothesis is True for $n = k$. we will prove $P(k) \Rightarrow P(k+1)$. we have k-elements in the set, therefore the power set contains 2^k elements. we added a k+1 th element in the set. we can create it's power set by using the power set of kth-element set. we have to two choices either to include or exclude the new element. The size of the power set will be doubled. because half set will not contain element and half will have.

$$\underbrace{|P(A)|}_{k+1 \text{ element set}} = 2^k \times 2 = 2^{k+1}$$

$P(k) \Rightarrow P(k+1)$ $\quad \forall n \in \mathbb{N}$, $P(n)$ is True.

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$A\cup B := \{x \mid x\in A \vee x\in B\}$ $\quad A\cup B = B\cup A$

$A\cap B := \{x \mid x\in A \wedge x\in B\}$ $\quad A\cap B = B\cap A$

$A^c := \{x \mid x\notin A \wedge x\in U\}$

$A-B = A\setminus B := \{x \mid x\in A \wedge x\notin B\} = A\cap B^c$ $\quad A-B \neq B-A$

Theorem: If A and B are two sets,

~~$A-B \neq B-A$~~ $(A-B)\cap(B-A)=\emptyset$

~~$A-B = \{x \mid x\in A \wedge x\notin B\}$~~

~~$B-A = \{x \mid x\in B \wedge x\notin A\}$~~

$$(A-B)\cap(B-A) = (A\cap B^c)\cap(B\cap A^c)$$
$$= A\cap(B^c\cap B)\cap A^c$$
$$= A\cap\emptyset\cap A^c$$
$$=\emptyset$$

~~$(A-B)\cap(B-A) = \{x \mid (x\in A \wedge x\notin B) \wedge$~~

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$A\times B = \{(a,b) \mid a\in A \wedge b\in B\}$

$A\times B\times C = \{(a,b,c) \mid a\in A \wedge b\in B \wedge c\in C\}$

$(A\times B)\times C \neq A\times B\times C$

$$\bigtimes_{i=1}^{n} A_i = \{(a_1,a_2,\ldots,a_n) \mid \bigwedge_{i=1}^{n} a_i\in A_i\} = A_1\times A_2\times\cdots\times A_n$$

01/01/2022

Gödel's Incompletness Theorems.

P: All crows are black

~~F~~ & we don't know completly about P.

But we can assign a value to P, which is True.

This Sentence is False → Lairs Paradox.

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$\wedge$: And   If both are True then it's True.

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

V: Or   If any is True then it's True.

P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

$\neg$: Negation   If False if True.

P	$\neg P$
T	F
F	T

$\Rightarrow$: Implies   If P then Q

P only if Q

P	Q	$P \Rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

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Def:-

Toutology:= A statment is called toutology if it is always true.

Eg:- $P \vee \neg P$

خدا ہے یا خدا نہیں ہے، یہ ہمیں کچھ نہیں بتاتا۔

Def:

contradiction := A logical statement that is always false.

Self-contradiction

Eg:- $P \wedge \neg P$

$P \Rightarrow Q$

$P \vee \neg P$ is always true, if $Q \vee \neg P$ is true, and false otherwise

$P \Rightarrow Q$ is same as $\neg P \vee Q$

Equivalence:-

P: true/f $\longrightarrow$ Q: true/f

Q: true/f $\longrightarrow$ P: true/f

P is equivalent to Q and

Q is equivalent to P.

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Associative Law

$(P \wedge Q) \wedge R = P \wedge (Q \wedge R)$

$(P \vee Q) \vee R = P \vee (Q \vee R)$

Distributive Law,

$P \wedge (Q \vee R) = (P \wedge Q) \vee (P \wedge R)$

$P \vee (Q \wedge R) = (P \vee Q) \wedge (P \vee R)$

Commutative Law,

$P \vee Q = Q \vee P$

$P \wedge Q = Q \wedge P$

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Laws of Contrapositives.

$$P \Rightarrow Q \iff \neg Q \Rightarrow \neg P$$

Equivalence :=

$$(P \Rightarrow Q) \wedge (P \Leftarrow Q)$$

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(if P then Q) $\wedge$ (if Q then P)

P if and only if Q

(P is sufficent for Q) $\wedge$ (P is neccesary for Q)

Converse of an implication,

Converse of $P \Rightarrow Q$ is the statement is $Q \Rightarrow P$

$P \Rightarrow P$ is a toutology.

Proof:-

$P \Rightarrow P := P \vee \neg P = \text{True}$

$$(P \iff Q) \Rightarrow (\neg P \iff \neg Q)$$

$$(P \Rightarrow Q) \not\Rightarrow (\neg P \Rightarrow \neg Q)$$

Prove $$((P \Rightarrow Q) \wedge (Q \Rightarrow R)) \Rightarrow (P \Rightarrow R)$$

$$(Q \vee \neg P) \wedge (R \vee \neg Q)$$

$$(Q \wedge (R \vee \neg Q)) \vee (\neg P \wedge (R \vee \neg Q))$$

$$(Q \wedge R) \vee \cancel{(Q \wedge \neg Q)}^{F} \vee (\neg P \wedge R) \vee (\neg P \wedge \neg Q)$$

$$(Q \wedge R) \vee (\neg P \wedge R) \vee (\neg P \wedge \neg Q)$$

$$\cancel{(Q \wedge R) \vee (\neg P \wedge \neg Q) \vee (\neg P \wedge R)}$$

$$\cancel{(Q \wedge R) \vee (\neg P \wedge R)}$$

$$(Q \wedge R) \vee (\neg P \wedge \neg Q) \vee (\neg P \wedge R)$$

$$(Q \vee (\neg P \wedge \neg Q)) \wedge (R \vee (\neg P \wedge \neg Q)) \vee (\neg P \wedge R)$$

$$(\underbrace{(Q \vee \neg P)}_{\text{by premis.}} \wedge \underbrace{(Q \vee \neg Q)}_{T} \wedge (R \vee \neg P) \wedge \underbrace{(R \vee \neg Q)}_{\text{by premis.}}) \vee (\neg P \wedge R)$$

$$(R \vee \neg P) \vee (R \wedge \neg P)$$

~~R V (R ∧ ¬P)~~

$((R \vee \neg P) \vee R) \wedge ((R \vee \neg P) \vee \neg P)$

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$(R \vee \neg P) \wedge (R \vee \neg P)$

$R \vee \neg P$

$P \Rightarrow R$

$P \Rightarrow Q$ -T Modus

P -T Ponens

Q

$P \Rightarrow Q$ -T Modus

$\neg Q$ -T Tollens.

$\neg P$

Predicite logic

1. Universal statment. ($\forall$) Universal Quantifier.

All crows are black

$\forall x \in$ Crows, x is black. $\forall x, P(x)$

2. Existential statment. ($\exists$) Existential Quantifier.

Some crow are black

$\exists x \in$ Crows $\ni$ x is black $\exists x \ni P(x)$

such that

$\neg(\forall x, P(x)) = \exists x \ni \neg P(x)$

Predicate is not a proposition, until assigned a Domain.

$\neg(\exists x \ni P(x)) = \forall x, \underbrace{\neg P(x)}$

predicate / propositional function.

$\forall x, y, z \in \mathbb{N} \ni x^2 + y^2 = z$ False Proposition

$\forall x, y \in \mathbb{N} \ni x^2 + y^2 \in \mathbb{N}$ True Proposition.

$\exists x, y, z \in \mathbb{N} \ni x^2 + y^2 = z$ True proposition.

$\forall\,(x=\sin(\theta),\ y=\cos(\theta),\ \theta\in\mathbb{R},\ z=1),\ x^2+y^2=z$

By Pythagrous Theorem, if a and b are two sides of right triangle and c is the hypotneus, then

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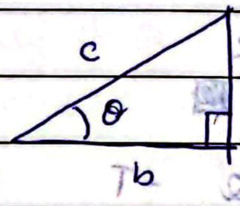
$$a^2+b^2=c^2$$



$$\Rightarrow \frac{a^2}{c^2}+\frac{b^2}{c^2}=1$$

$$\Rightarrow \left(\frac{a}{c}\right)^2+\left(\frac{b}{c}\right)^2$$

$$(\sin(\theta))^2+(\cos(\theta))^2=1$$

$$\cancel{y}^2+x^2=\cancel{z}$$

$$x^2+y^2=z$$

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Existential Proposition

- Produce an example
- Negation of this, universal proposition is false

Universal Proposition

- Enumerate over all values of Domain and check (if domain is finite and relatively small)
- If the domain is infinite or large, we verify for a general element of the domain.

Direct Proof:-

If we want to proof $P \Rightarrow Q$, we start with premise P and got to conclude Q.

$$\left.\begin{array}{l} P \Rightarrow P_1 \\ P_1 \Rightarrow P_2 \\ P_2 \Rightarrow P_3 \\ \vdots \\ P_n \Rightarrow Q \end{array}\right\}$$ By Transitive Property of Implication. $P \Rightarrow Q$.

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Proof by contraposition:—

If we want to proof $P \Rightarrow Q$, we instead proof $\neg Q \Rightarrow \neg P$.

$$\because\ (P \Rightarrow Q) \Longleftrightarrow \underbrace{(\neg Q \Rightarrow \neg P)}_{\text{Direct proof.}}$$

Prove: $(x > 1 \wedge x \in \mathbb{R}) \Rightarrow \left(\frac{1}{x^2} > 1\right)$

$\Rightarrow x > 1$

$\Rightarrow x^2 > x$ $\quad\quad \because x > 0$

$\Rightarrow x^2 > x \wedge x > 1$

$\Rightarrow x^2 > 1$

$\Rightarrow \frac{\cancel{x^2}}{\cancel{x^2}} > \frac{1}{x^2}$ $\quad\quad \because x \neq 0 \wedge x^2 > 1$

$\Rightarrow 1 > \frac{1}{x^2}$ $\quad$ ▨

Direct proof.

Prove: $\{\forall x \in \mathbb{Z},\ 3x + 7 = 1 \pmod 2\} \Rightarrow x = 0 \pmod 2$

Direct Proof:

$\Rightarrow 3x + 7 = 2k + 1$ $\quad\quad k \in \mathbb{N}$

$\Rightarrow 3x + 7 = 2k - 6$

$\Rightarrow 3x = 2(k-3)$

$\Rightarrow x = 2\underbrace{\left(\frac{k-3}{3}\right)}_{\text{stuck}}$ $\quad$ ~~we can not~~ It's hard to prove that this factor is an integer.

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Proof by Contrapositive: —

$\forall x \in \mathbb{Z}$, if x is odd then $3x+7$ is even.

$\forall x \in \mathbb{Z}, \exists k \in \mathbb{Z} \ni x = 2k+1$

$\Rightarrow x = 2k+1 \quad k \in \mathbb{Z}$

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$\Rightarrow 3x+7 = 3(2k+1)+1$

$= \cancel{3}6k+3+1$

$= 6k+4$

$3x+7 = 2(3k+2)$

$3x+7$ has a factor of $2 \Rightarrow 3x+7$ is even.

$\blacksquare$

Proof by contradiction: —

If I want to ~~imply~~ proof $P \Rightarrow Q$ and it's hard, I can instead prove. $P \wedge \neg Q$ or $\{P \not\Rightarrow Q\}$ is False.

$$(P \Rightarrow Q) \iff \neg(P \wedge \neg Q)$$

Prove: there are infinitly many primes.

If S is the set of prime, then S is finite

$S = \{p_1, p_2, \cdots, p_n\}$

$x = p_1 p_2 \cdots p_n + 1$

~~$p \in \mathbb{Z}$~~ ~~$p \in S \nmid x$~~ $p \in S \wedge p \nmid x$, therefore there must be a prime which is not is set but divides x.

$\lightning$

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~~Prov~~ $\sqrt{2}$ is irrational.

Proof: —

$\sqrt{2} \in \mathbb{R} \Rightarrow \sqrt{2} \notin \mathbb{Q}$ $\qquad Q := \{\frac{p}{q} \mid p, q \in \mathbb{Z} \wedge q \neq 0\}$

Lets suppose $\sqrt{2} \in \mathbb{R} \wedge \sqrt{2} \in \mathbb{Q}$. $\exists p, q \in \mathbb{Z} \ni \gcd(p,q) = 1 \wedge p/q = \sqrt{2} \wedge q \neq 0$

$\sqrt{2} = \frac{p}{q}$

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$\Rightarrow \quad 2 = \frac{p^2}{q^2} \Rightarrow 2q^2 = p^2 \quad \because q \neq 0$ ①

$\Rightarrow \quad p^2 = 0 \pmod 2 \Rightarrow p = 0 \pmod 2 \Rightarrow p = 2k \quad k \in \mathbb{Z}$

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① $\Rightarrow \quad 2q^2 = (2k)^2 \Rightarrow$

$\Rightarrow \cancel{2}q^2 = \cancel{4}^{2}k^2$

$q^2 = 2k^2$

q is even or $q = 2j \quad j \in \mathbb{Z}$

This contradicts with assumption that $\gcd(p,q) = 1$, but now the $\gcd(p,q) = 2$

Hence this contradicts. $\sqrt{2} \notin \mathbb{Q}$. ↯

~~Ex~~

Among any three consecutive integers, one of them is divisible by 3.

Proof by cases:-

The three consecutive integers can be of form.

cases. $\begin{cases} 3a, & 3a+1, & 3a+2 \\ 3a-1, & 3a, & 3a+1 \\ 3a-2, & 3a-1, & 3a \end{cases}$

case 1, first number $\equiv 0 \pmod 3$

case 2, Second number $\equiv 0 \pmod 3$

case 3, third number $\equiv 0 \pmod 3$

$\blacksquare$

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Good proof:-

$x, x+1, x+2$ are 3 consecutive integers $\Rightarrow 3|x \vee 3|x+1 \vee 3|x+2$.

Case 1:

$3|x$ $\Rightarrow$. We are done.

Case 2:

$3 \nmid x \Rightarrow x = 3k+1 \vee x = 3k+2$

Case 2.1:

$\cancel{3 \nmid x+2} \Rightarrow x = 3k+1$

$\Rightarrow x+2 = 3k+3$

$\Rightarrow x+2 = 3(k+2)$

$\Rightarrow 3 | x+2$

Case 2.2:

$\Rightarrow x = 3k+2$

$\Rightarrow x+1 = 3k+3$

$\Rightarrow x+1 = 3(k+1)$

$3 | x+1$

QED.

For $a, b \in \mathbb{R}$ prove that $ab = 0 \Rightarrow a = 0 \vee b = 0$

Proof by Contradiction:-

Lets suppose, $a b \in \mathbb{R} \ni ab = 0 \wedge a \neq 0 \wedge b \neq 0$

if $a \neq 0$

Multiplying of both sides by b

$ab \neq b(0)$

$ab \neq 0$

But we supposed that $ab = 0$. This contradicts with our assumption.

Henced for $\forall a, b \in \mathbb{R} \ni ab = 0 \Rightarrow a = 0 \vee b = 0$.

write these statements in english, their negation in english and Identify the difference between them. $(1 \wedge 2)$

1. $\forall x\in\mathbb{R}, \exists y\in\mathbb{R} \ni x+y=0$

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2. $\exists y\in\mathbb{R} \ni \forall x\in\mathbb{R}, x+y=0$

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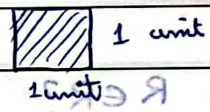
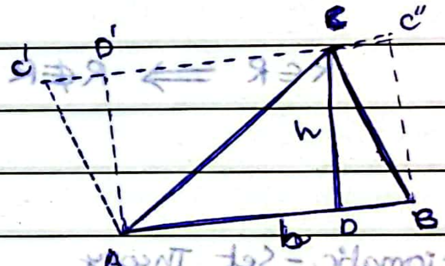
Prove that a triangle with base b and height h has area $= \frac{1}{2}bh$

Given that area of a unit square := 1 sq. unit.



Let ABCD be a rectangle with, $\overline{AB}=b$ and ~~BC = h~~ $\overline{CD}=h$

$\overline{CD}\perp\overline{AB}$.



Construction:-

~~AC' is a line~~ $\overline{AC'} \parallel \overline{BC}$ such that $m\overline{AC'} = m\overline{BC}$

$\Delta ABC = \frac{1}{2} \square AD'C''B$

$\forall \varepsilon>0, \exists \delta>0 \ni \forall x\in\mathbb{R}, |x-a|<\delta \Rightarrow |f(x)-f(a)|<\varepsilon$

For all epsilon greater than zero, there exists a delta such that, for all real number ~~|x-a|<δ~~ if $|f(x)-f(a)|<\varepsilon$ if $|x-a|<\delta$

$\exists \varepsilon>0 \ni \forall \delta>0 \ni \exists x\in\mathbb{R} \ni |x-a|<\delta \wedge |f(x)-f(a)|\geq\varepsilon$

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Naïve Set Theory:
Set: An unordered collection of unique objects.

~~There can be a sets~~

$\exists A \ni A \notin A$

Are there any.

$\exists A \ni A \in A$?

Russel's ~~Ro~~ Paradox (1903)

Consider $R := \{ A \mid A \text{ is a set } A \notin A \}$

~~Is~~ Does $R \in R$?

$R \in R \implies R \notin R$

~~Atox~~ Axiomatic-Set Theory

Zermelo-Frankel-Choice ~~Th~~ Set Theory.

Axiom of Choice: From any non-empty set we can always chose an element.

Axiom: A proposition that can not be proven, taken to be true.
Definition: An iff proposition where a concept is asigned a new name.
Theorem: A proposition that can be proven ~~by~~ from axioms.
Lemma: Theorems that are build towards a bigger theorem.
Corollary: Theorems that follows from a bigger theorem.

$\forall a, b, d \in \mathbb{Z} \ni \gcd(a,b) = d \ \exists x, y \in \mathbb{Z} \ni ax + by = d$

Gauss' Lemma.

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<u>Relation:</u>

a relation R from set A to B, such that it is a subset of $A \times B$,

$$(R : A \to B) \Longleftrightarrow (R \subseteq A \times B)$$

$$\neg \left(R : A \to B \Longleftrightarrow R : B \to A \right)$$

$$R : A \to B \land (a,b) \in R \Rightarrow \underbrace{aRb}_{\text{a is related to b.}}$$

$A := \{1, 2\}$

$R : A \to A$

$R = \{(1,1), (1,2), (2,1), (2,2)\}$

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(2, 4), (2, 6), (3, 6), (2, 0), (1, 1), (2, 2)

(1, 0), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6)

(3, 3), (4, 4), (5, 5), (6, 6), (3, 0)

Properties on <u>relations</u>

Relations on a set.

$R : A \to A$

1. R is reflexive, iff $(a,a) \in R \ \forall a \in A$ or $aRa, \forall a \in A$

Eg. The ~~…~~ relation "subset of" on the collection of all sets.

$B \subseteq B \Rightarrow (B, B) \in R$

2. R is called symmetric iff $(a,b)\in R \Rightarrow (b,a)\in R$

iff $aRb \Rightarrow bRa$

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3. R is called anti-symmetric, iff $(a,b)\in R \wedge (b,a)\in R \Rightarrow a=b$

$<$

eg. The relation "less than" on ~~relation~~ real numbers.

There is no case justifying the premise, if premise is false then conclusion is always true.

4. R is called transitive, iff $aRb \wedge bRc \Rightarrow aRc$.

The "equals to" relation on any set A is,

1. Reflexive
2. Symmetric
3. Anti Symmetric
4. Transitive

But this is a circular argument, because we have used "=" in defining the above relations.

def:-

$R: A \to A$ is called equivalence, if it is reflexive, symmetric and transitive.

	R	S	A	T
$R_1 = \{(a,b) \mid a,b\in \mathbb{Z} \wedge a=\lvert b\rvert\}$	×	×	✓	✓
$R_2 = \{(a,b) \mid a,b\in \mathbb{Z} \wedge a=b+2\}$	×	×	✓	×
$R_3 = \{(a,b) \mid a,b\in \mathbb{Z} \wedge a+b\leq 3\}$	×	✓	×	×

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3. $\forall n \in \mathbb{Z},\ n^2+5n+13 \equiv 1 \pmod 2$

Let $n = 2k$

$$(2k)^2+5(2k)+13 = 4k^2+10k+13$$
$$= 4k^2+10k+12+1$$
$$= 2(2k^2+5k+6)+1$$
$$\equiv 1 \pmod 2$$

Let $n = 2k+1$

$$(2k+1)^2+5(2k+1)+13 = 4k^2+4k+1+10k+5+13$$
$$= 4k^2+14k+19$$
$$= 4k^2+14k+18+1$$
$$= 2(2k^2+7k+9)+1$$
$$\equiv 1 \pmod 2$$

∎

5. $\forall x \in \mathbb{Z},\ 7x+9 \equiv 0 \pmod 2 \Rightarrow x \equiv 1 \pmod 2$

Proof:-

$7x+9 \equiv 0 \pmod 2$

$7x \equiv -9 \pmod 2$

$7x \equiv 1 \pmod 2$

$\cancel{7}x \equiv \cancel{7} \pmod 2$

$x \equiv 1 \pmod 2$

By contraposition.

$x \equiv 0 \pmod 2$

$7x \equiv 0 \pmod 2$

$7x+9 \equiv 9 \pmod 2$

$7x+9 \equiv 1 \pmod 2$

∎

6. Prove $(a+b)^2 = a^2+b^2$ is not an algebriac identity.

$a = 1,\ b = 2$

$(a+b)^2 \neq a^2+b^2$

$$\big((aRb \wedge bRa) \Rightarrow a=b\big) \Leftrightarrow \big((aRb \wedge a\neq b) \Rightarrow b\not R a\big)$$

Operation of Relations.

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$\forall R: A \to B \wedge \forall S: A \to B$

$R \cup S : A \to B$

$R \cap S : A \to B$

$R^c : \cancel{A} = (A \times B) - R ; A \to B$

$R \times S \subseteq (A \times B) \times (A \times B).$

$R \times S := \{ ((a_r, b_r), (a_s, b_s)) \mid (a_r, b_r) \in R \wedge (a_s, b_s) \in S \}$

Composition of Relations

if $R: A \to B$

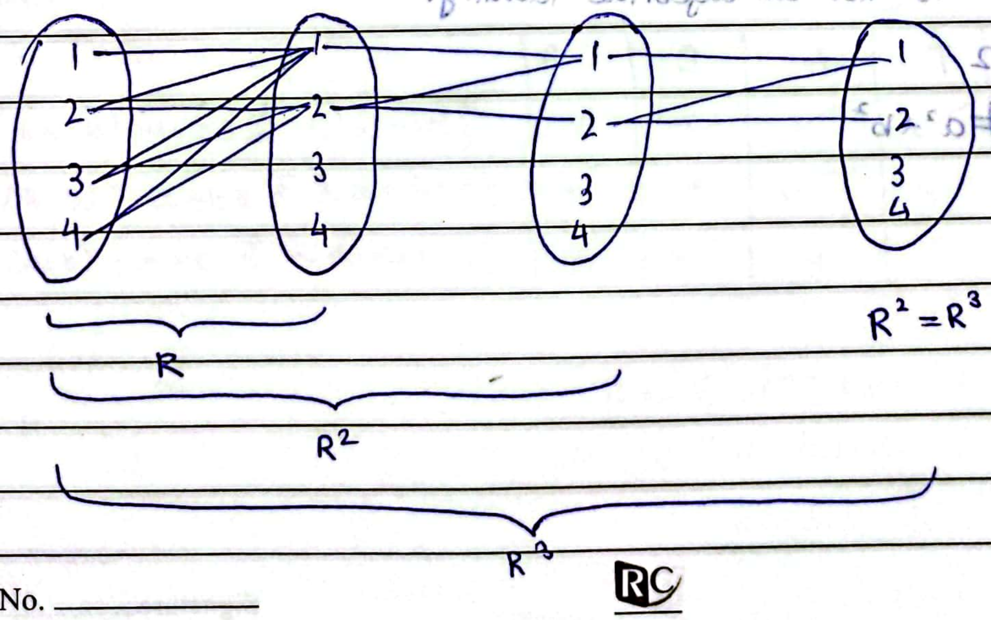
and $S: B \to C$

then $S \circ R : A \to C$

$\forall a \in A \wedge \forall c \in C$, $aRb \wedge bSc \Rightarrow aSRc$

$SR = \{ (1,\alpha), (1,\gamma), (1,\beta), (2,\beta), (4,\alpha), (4,\gamma) \}$

$$(R: A \to A) \Rightarrow \Big(R^n : A \to A \ni R^n = \underbrace{R \circ R \circ R \circ \ldots \circ R}_{n} := \{ (a,b) \mid aR^n b \}\Big)$$



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Representations of relations.

1. Digraph (directed graphs)

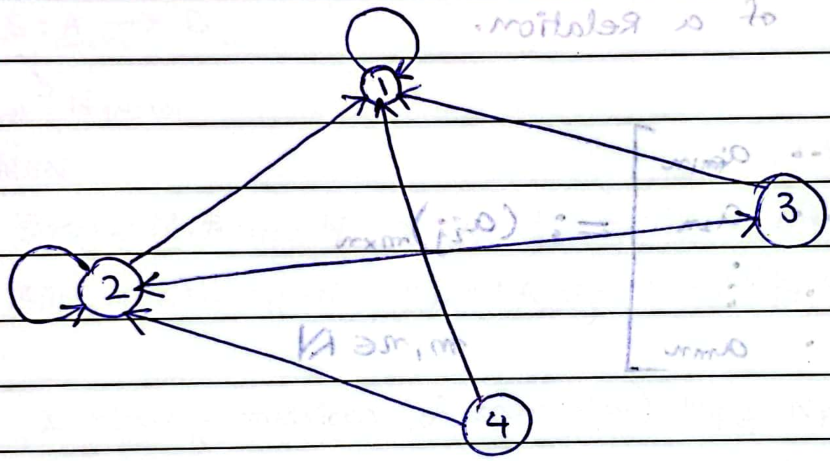
$D = (V, E)$

$V :=$ set of all vertices

~~$E := V \times$~~ $E \subseteq V \times V$

element of E are called edges.

$e_{ij} := (v_i, v_j) \in E \Rightarrow v_i$ is initial node $\wedge$ v_j is terminal node.



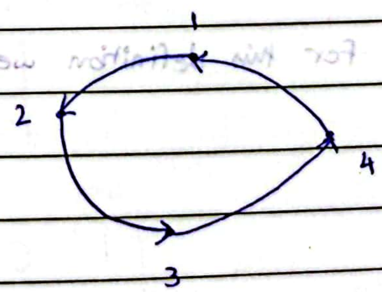
$(R : A \to B) \Rightarrow (D_R := (A \cup B, R))$

A relation is

- Reflexive if all nodes in digraph have self loop.
- Symmetric if digraph is bidirected for all edges.
- Anti-Symmetric if the digraph is unidirected.
- Transitive if $a \to b$ and $b \to c$, $a \to c$.

$A = \{1, 2, 3, 4\}$

$R : A \to A = \{(1,2), (2,3), (3,4), (4,1)\}$



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Prove that $1 = 0.999\ldots$

R: $\emptyset \to \emptyset$ is Reflexive, Symmetric, anti-symmetric and transitive.

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Let, $S = 0.999\ldots$

$$10S = 9.999\ldots$$
$$10S = 9 + 0.999\ldots$$
$$10S = 9 + S$$
$$(10-1)S = 9$$
$$9S = 9$$
$$S = 1$$

Matrix Representation of a Relation.

$$M_{m\times n} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} =: (a_{ij})_{m\times n}$$

$m, n \in \mathbb{N}$

Binary Matrices.

$$\forall m,n \in \mathbb{N}\ \forall M_{m\times n} := (a_{ij})_{m\times n},\ a_{ij} = \begin{cases} 0 \\ 1 \end{cases} \forall i,j$$

$\cancel{A}$ $\cancel{M_R}$

$$R: A \to B \Rightarrow (M_R)_{|A|\times|B|} := (a_{ij})_{|A|\times|B|} \ni a_{ij} = \begin{cases} \cancel{0}1 & \text{if } \overset{A}{\underset{\Psi}{}} a_i \, R \, \overset{B}{\underset{\Psi}{}} b_j \\ 0 & \text{else.} \end{cases}$$

For this definition we will give some order to set A and B.

R $\qquad$ M_R

Reflexive.

$aRa \; \forall a \in A$ $\qquad$ $a_{ii} = 1 \; \forall i$.

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Symmetric

$aRb \Rightarrow bRa$ $\qquad$ $\forall i,j$, $a_{ij} = a_{ji}$

Anti-Symmetric

$aRb \land a \neq b \Rightarrow bRa$ $\qquad$ $a_{ij} = 1 \Rightarrow a_{ji} = 0$

Tansitivity.

$aRb \land bRc \Rightarrow aRc$ $\qquad$ $a_{ij} = 1 \land a_{jk} = 1 \Rightarrow a_{ik} = 1$

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$R, S : A \longrightarrow B$

$M_{|A| \times |B|}$ $\quad$ $N_{|A| \times |B|}$

$\forall p_{ij} \in M \cup N$

$\exists m_{ij} \in M, \exists n_{ij} \in N, \; m_{ij} = 1 \lor n_{ij} = 1 \Longrightarrow p_{ij} = 1$

$\forall p_{ij} \in M \cap N, m_{ij} \in M, n_{ij} \in N, \; m_{ij} = 1 \land n_{ij} = 1 \Longrightarrow p_{ij} = \underline{1}$

For 2 binary matrices (of same size) $M_{p \times q}$ $N_{p \times q}$, their join is defined as

$$(M \lor N)_{p \times q} := (\underset{\in M}{m_{ij}} \lor \underset{\in N}{n_{ij}})$$

Their meet is defined as,

$$(M \land N)_{p \times q} := (\underset{\in M}{m_{ij}} \land \underset{\in N}{n_{ij}})_{p \times q}$$

Example,

$$M = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \; N = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$M \lor N = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \; M \land N = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Theorem:— Let $R, S : A \to B$ with binary representation matrix ~~rep~~ $M_{p\times q}$ and $N_{p\times q}$ respectively. The binary representation of

i) $R \cup S$ is $M \vee N$

ii) $R \cap S$ is $M \wedge N$

i).

$(a,b) \in R \cup S$

$\Leftrightarrow (a,b) \in R \vee (a,b) \in S$

$\Leftrightarrow (m_{ab} = 1 \text{ in } M) \vee (n_{ab} = 1 \text{ in } N)$

$\Leftrightarrow (m_{ab} \vee n_{ab}) = 1 \text{ in } M \vee N$

$(a,b) \in R \cap S$

$\Leftrightarrow (a,b) \in R \wedge (a,b) \in S$

$\Leftrightarrow (m_{ab} = 1 \text{ in } M) \wedge (n_{ab} = 1 \text{ in } N)$

$\Leftrightarrow (m_{ab} \wedge m_{ab}) = 1 \text{ in } M \wedge N$

Let $M_{m\times k}$ and $N_{k\times n}$ be binary matrices. Then their boolean product is defined as,

$$(M \odot N)_{m\times n} = (c_{ij})_{m\times n}$$

$$c_{ij} = \bigvee_{p=1}^{k} (a_{ip} \wedge b_{pj})$$

Let $\forall (a,c) \in R \circ S$ & $A \to C$,

$R: A \to B$

$S: B \to C$.

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$\Leftrightarrow \exists b \ni (a,b) \in R \wedge (b,c) \in S \Rightarrow (a,c) \in R \circ S$

$\Leftrightarrow \forall p, (m_{ip} = 1 \text{ in } M) \wedge (n_{pj} = 1 \text{ in } N)$

$\Leftrightarrow \bigvee_{p=1}^{k} (m_{ip} \wedge n_{pj}) = 1 \text{ in } M \odot N$

If R is an equivalent relation ~~we use~~ "~" for R i.e (~~for~~ me) $a \sim b$ means (a is equivalent to b) in R.

<u>Examples:</u>

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Relation of Partial Order

A partially ordered set is a pair $P(X, \preccurlyeq)$ where X is a set and $\preccurlyeq$ a ~~relation~~ partial order on $X \ni \preccurlyeq$ is reflexive, transitive and antisymmetric.

$\forall x \in X, x \preccurlyeq x$

$\forall x, y \in X, x \preccurlyeq y \wedge y \preccurlyeq x \Rightarrow x = y$

$\forall x, y, z \in X, x \preccurlyeq y \wedge y \preccurlyeq z \Rightarrow x \preccurlyeq y$

<u>def</u>: A partially ordered set $P(X, \preccurlyeq)$ is shortly called a poset. If for $a, b \in X$ we have $a \not\preccurlyeq b \wedge b \not\preccurlyeq a$ then we call a and b incomparable and write $a \| b$

$\forall a, b \in X, a \preccurlyeq b \wedge a \neq b \Rightarrow a \prec b$

~~$\forall a, b \nexists z \ni a \prec$~~

$\forall a, b \in X, a \prec b, \nexists z \in X \ni a \prec z \prec b \Rightarrow a \prec\!\cdot\, b$

$a \prec\!\cdot\, b$ is called cover relation.

Homework:-

$X = \{1, 2, 3\}$

$P: (P(X), \subseteq)$

t pairs. find all coverrelations, and incomparable relations.

$P(X) = \{ \{\}, \{1\}, \{2\}, \{3\}, \{1,2\}, \{2,3\}, \{1,3\}, \{1,2,3\} \}$ 31/03/2022

$\forall A \in P(X)$

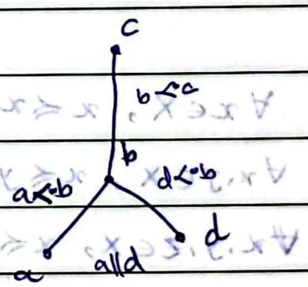
$P(P(X)\}; \subseteq) = \{ (\emptyset, A), (A, A), (\{1\}, \{1,2\}), (\{1\}, \{1,3\}), (\{2\}, \{1,2\}), (\{2\}, \{2,3\}), (\{3\}, \{1,3\}), (\{3\}, \{2,3\}), (A, X) \}$

$\{2\} \subseteq \{2,3\} \subseteq X \Rightarrow \{2\} \not\subseteq\cdot X$

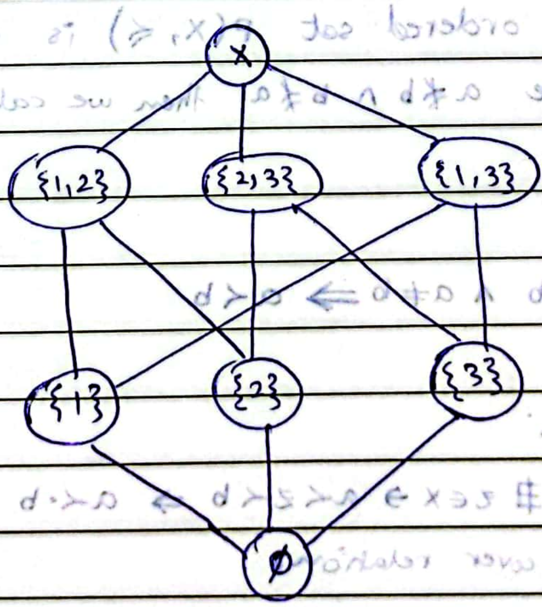
$\emptyset \subseteq\cdot \{1\}$

Hasse Diagram (Representation of Posets)

1. We draw elements of X as nodes / vertices.
2. Only cover relations are drawn with an edge.
3. If $a \preccurlyeq b$ then a is drawn below b.
4. There is an upward path from $x \in X$ to $y \in X$ if $x \prec y$
5. Transitive and reflexive edges are not drawn.



Hasse Diagram for above Homework.



$X = \{1, 2, 3, 4, 6\}$ $\forall A \in X$

$\preccurlyeq = \{(1,A), (2,4), (2,6), (3,6)\}$

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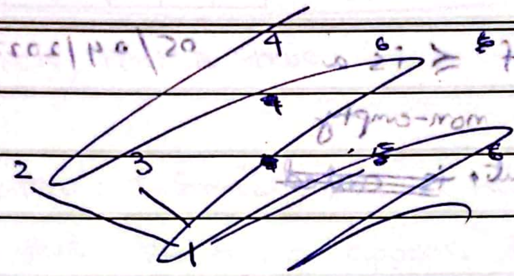

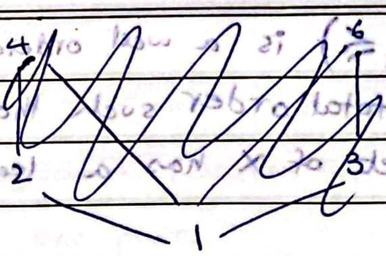

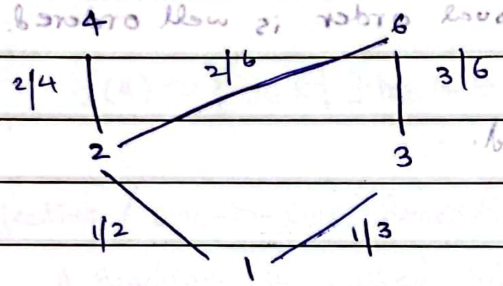

~~Def~~ ~~Po~~ For $P(X, \preccurlyeq)$ poset

1). $x \in X$ is called minimal if $\nexists\, a \in X \ni a \prec x$

2) $x \in X$ is called maximal if $\nexists\, a \in X \ni x \prec a$

3) $x \in X$ is called least element if $\forall a \in X$, $x \preccurlyeq a$

4) $x \in X$ is called greatest element if $\forall a \in X$, $a \preccurlyeq x$

Theorem:— If there exists a greatest / least element in any poset $(X, \preccurlyeq)$ then it is unique.

Proof:—

we assume that they are not unique, let say they are a and b.

$$\forall c \in X,\ a \preccurlyeq c \wedge b \preccurlyeq c$$

b is also in set therefore, $a \preccurlyeq b$, and a is also in set $b \preccurlyeq a$

This contradicts, which means. ~~4. least is~~ least is unique.

Def: A poset $(X, \preceq)$ is called total order if $\forall a, b \in X$ either $a \preceq b$ or $b \preceq a$, i.e. there are no incompariable elements.

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Def: $(X, \preceq)$ is a well ordered set if $\preceq$ is a ~~tot~~ total order such that every non-empty subset of X has a least element. ~~is called~~

Eg: Any finite set of numbers with usual order is well ordered.

Non-Eg: ~~$\mathbb{Z} \subseteq \mathbb{R}$~~ is not well ordered.
$(\mathbb{R}, \leq)$

$\mathbb{Z} \subseteq \mathbb{R}$, $(a, b) \subseteq \mathbb{R}$

↳ soppose ϵ is the least element then $\frac{\epsilon}{2} < \epsilon \in (0,1)$.

Well-ordering Principal (Axiom): ~~set of~~ $\mathbb{N}$ is well-ordered.

Principal of Mathematical Induction

Goal: we want to prove $P(n)$ $\forall n \in \mathbb{N}$

1. Prove $P(1)$ is true.
2. Prove $P(k)$ is true $\Rightarrow$ $P(k+1)$ is true, for $k \in \mathbb{N}$

$\Rightarrow$ $P(n)$ true for all $n \in \mathbb{N}$

Theorem:—

well-ordering Principal $\Leftrightarrow$ Principal of Mathematical Induction.

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Principal of strong Induction

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$P(1)$ is true

$(P(1) \wedge P(2) \wedge \cdots \wedge P(k)) \Rightarrow P(k+1)$

$\forall n \in \mathbb{N}$, $P(n)$ is true

Function : Informally, its a ~~rule of assignment~~ relation from A to B such that $\forall a \in A$, a appears in exactly one pair.

$f(A) := \{ b \in B \mid \exists\, a \in A \ni (a,b) \in f \}$

Injective / one-to-one functions.

A function is called injective if $\forall\, b \in f(A)$, $\exists!\, a \in A$

$\forall a_i, a_j \in A$, $f(a_i) = f(a_j) \Rightarrow a_i = a_j$.

or

$\forall a_i, a_j \in A$, $a_i \neq a_j \Rightarrow f(a_i) \neq f(a_j)$

Surjective / Onto functions.

A function is called surjective, if $f(A) = B$;

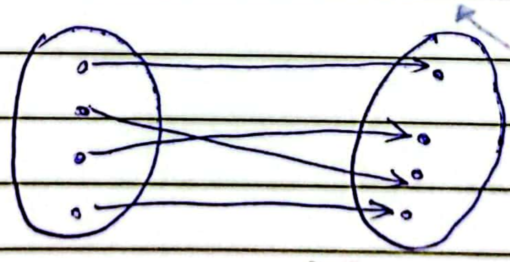
$\forall b \in B$, $\exists\, a \in A \ni f(a) = b$

Bijective Functions / one-to-one correspondence.

A function that is Injective and surjective both

Cardinality of a set 12/04/2022

If there exists a bijection between sets A and B, then we say A and B have same cardinality.



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Finite set

if there is a bijection between set A and set $\{1, 2, 3, \ldots, n\}$ for some $n \in \mathbb{N}$.

Infinite set.

A set is infinite if it is not finite.

Cardinality

It is a property shared by all sets that have a bijection possible among them, and not shared by any other set that does not have a possible bijection with them.

$|A| :=$ cardinality of set A

Finite set (Redefinition).

A set X is finite set if theres ~~exists~~ no bijection possible b/w X and any proper subset of X

Infinite set (Redifinition)

A set X is infinite, if there exists a bijection b/w X and some proper subset of X.

A natural number is a symbol given to an equivalence class of the equivalence relation "having same cardinality" containing finite sets. 14/04/2022

~~$\aleph_0 = |\mathbb{N}|$~~

$\aleph_0 := |\mathbb{N}|$

countable infinity

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$\mathbb{Q}$ is countable.

Proof:-

Let $g : \mathbb{N} \to \mathbb{Q}^+$

$$\mathbb{Q}^+ = \{ p/q \mid p, q \in \mathbb{N}/\{0\} \}$$

q \ p	1	2	3	4	5	6	...
1	1/1	2/1	3/1	4/1	5/1	6/1	...
2	1/2	2/2	3/2	4/2	5/2	6/2	...
3	1/3	2/3	3/3	4/3	5/3	6/3	...
4	1/4	2/4	3/4	4/4	5/4	6/4	...
5	1/5	2/5	3/5	4/5	5/5	6/5	...
6	1/6	2/6	3/6	4/6	5/6	6/6	...
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋱

Thm.

① Adding or removing finitely many elements from a countably infinite set does not change its cardinality.

$$n + \aleph_0 = \aleph_0$$

② The union of two countabley ~~sets~~ infinite sets is still countabley infinite.

$$\aleph_0 + \aleph_0 = \aleph_0$$

③ Countable Union of countably infinite set is also countable

$\bigcup_{i \in \mathbb{N}} A_i$, where each A_i is countable.

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Uncountable sets:—

An infinite set which is not countable is then called uncountable.

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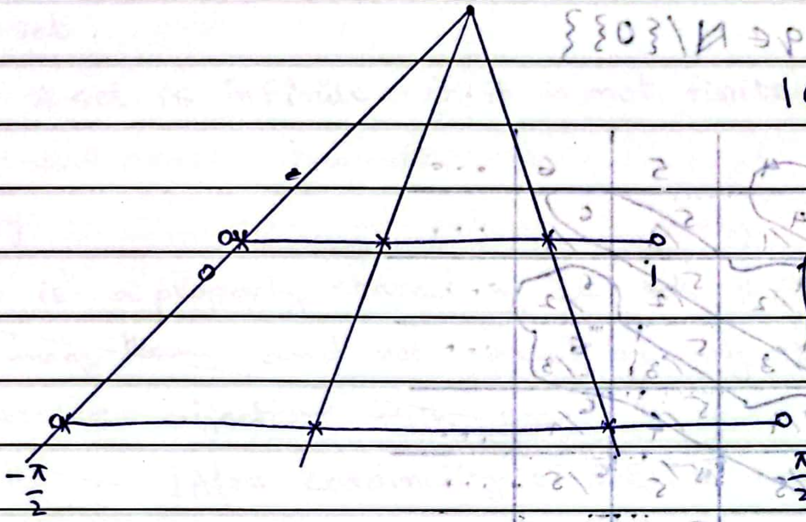

$|(0,1)| = |(-\frac{\pi}{2}, \frac{\pi}{2})|$

$y = \frac{\pi}{1} x - \frac{\pi}{2}$

$\boxed{y = \pi\left(x - \frac{1}{2}\right)}$

$y = \tan(x)$

$(0,1)$ interval is uncountable.

Proof:—

claim:—

Every element $d \in (0,1)$ can be written as,

$d = 0.d_1 d_2 d_3 d_4 \ldots$

where $d_i \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$

i.e. every number b/w $(0,1)$ has ~~decimal representation~~ an infinite decimal representation.

Contors' Diagonalization Argument:—

Suppose there is a bijection b/w $\mathbb{N}$ and $(0,1)$ the f will be,

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$$
\begin{array}{l}
1 \xrightarrow{f} 0.d_{11}d_{12}d_{13}d_{14}\cdots \\
2 \longrightarrow 0.d_{21}d_{22}d_{23}d_{24}\cdots \\
3 \longrightarrow 0.d_{31}d_{32}d_{33}d_{34}\cdots \\
4 \longrightarrow 0.d_{41}d_{42}d_{43}d_{44}\cdots \\
\vdots
\end{array}
$$

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Suppose a new number $a = 0.a_1a_2a_3a_4\cdots \ni a_i \neq d_{ii}$.

$\aleph_1 > \aleph_0$ $|\mathbb{N}| = \aleph_0$

$|\mathbb{R}| =$ 21/04/2022

B/w any two non-empty sets. we can always make a function.

$f: A \to B$.

since $B \neq \emptyset$, $\exists b \in B \ni \forall a \in A,\ f(a) = b$ ⊠

B/w any two non-empty sets there always exists a function which is either injective /one-to-one or surjective/ onto.

<u>Proof:-</u> Let $f: A \to B$ be a function neither injective nor surjective.

$\exists a_i, a_j \in A \ni f(a_i) = f(a_j) \wedge a_i \neq a_j$

$\exists b \in B \ni \forall a \in A,\ f(a) \neq b$

Create a function $g: A \to B$ such that,

$$
g(a) = \begin{cases} f(a) & a \neq a_2 \\ b & a = a_2 \end{cases}
$$

Do this iteratively until f modified is either injective or surjective.

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Def :-

Let A and B be non-empty sets

i) $|A| = |B|$ when $\exists f: A \to B$ and f is bijective.

ii) $|A| < |B|$ when $\nexists f: A \to B$ and f is surjective.

iii) $|A| > |B|$ when $\nexists f: A \to B$ and f is injective.

iv) $|\emptyset| \leq |A|$, $\forall$ sets A.

$$\aleph_0 = |\mathbb{N}| \qquad \aleph_0 < \aleph_1$$
$$\aleph_1 = |\mathbb{R}|$$

Homework:-

If $\exists f: A \to B \ni f$ is injective, then $\exists g: B \to A$ surjective.

If $\exists f: A \to B \ni f$ is surjective, then $\exists g: B \to A$ injective.

Theorem:-

$\forall$ set A, $|A| < |P(A)|$, where $P(A)$ is the power set of A.

Proof:-

we prove that no $f: A \to P(A)$ is surjective.

Let $g: A \to B$ be surjective.

~~T = { a∈A~~

$T = \{a \mid a \in A \wedge a \notin g(a)\} \subseteq A$.

~~T ∈ P(A)~~ $T \in P(A)$

Since g is surjective, $\exists\, y \in A \ni g(y) = T$.

if $y \in T = \{a \mid a \in A \wedge a \notin g(a)\}$ ~~then y~~ $\Rightarrow y \notin g(y) = T$

then, $y \notin T = \{a \mid a \in A \wedge a \notin g(a)\}$

$\Rightarrow y \in g(y) = T$

$\Rightarrow y \in T$

$\lightning$

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This proves that we can have bigger infinities,

$$\underbrace{|\mathbb{N}|}_{\text{countable infinity}} < \underbrace{|P(\mathbb{N})|}_{\text{continoum}} < \underbrace{|P(P(\mathbb{N}))| < |P(P(P(\mathbb{N})))| < \cdots}_{\text{uncountable Infinity}}$$

$$\aleph_0 = |\mathbb{N}|,\quad \aleph_1 = |P(\mathbb{N})|,\quad \aleph_2 = |P(P(\mathbb{N}))|,\quad \aleph_3 = |P(P(P(\mathbb{N})))|$$

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Theorem:-

A tree $T = (V, E)$ with $|V| = n$ and $|E| = n-1$.

Proof:- $P(n): |V| = n \Rightarrow |E| = n-1$

Base case:-

$$|V| = 1 \Rightarrow |E| = 0$$

Hypothesis.

$$\bigwedge_{i=1}^{k-1} (|V| = i \Rightarrow |E| = i-1) \Rightarrow (|V| = k \Rightarrow |E| = k-1)$$

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Thm; A connected graph $G = (V, E)$ has an Euler circuit iff $\forall v \in V$, $\deg(v)$ is even.

Proof:-

Euler circuit $\Rightarrow$ $\deg(v)$ is even is abvious.

Everytime a vertex appears in a ~~graph~~ cycle, two of it's edges are consumed.

If in $G:(V,E)$, $\forall v \in V$, $\deg(v) \equiv 0 \pmod 2$ $\Rightarrow$ G has an euler path.

Proof:— Consider the maximal path P in G.

Claim:— The maximal path is actually a cycle.

Lets say P starts at vertex u and ends at v. The number of edges of v that appear in path P is odd. $\deg(v)$ is odd.
i.e. $\exists\, w \in V$ such that vw is not on the path P (since G is connected.