$\{a, b, c, d\}$ $\{x \mid x \in \mathbb{R} \wedge \cos(x) > \frac{1}{2}\}$

Date: 12/01/2022

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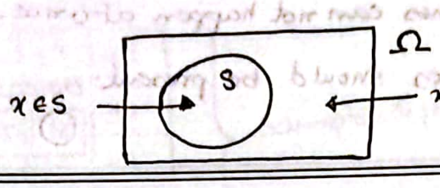


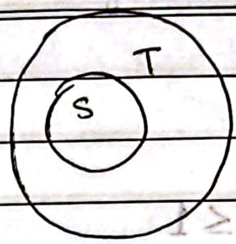


$S^c$ or $S' = \{x \mid x \notin S \wedge x \in \Omega\}$

$\Omega^c = \emptyset = \{\} =$ Null set

$S \subset T : x \in S \rightarrow x \in T$

If ~~A∩A~~ $\bigcap_{i=1}^{n} A_i = \emptyset$ then all

$A_i$'s are disjoint.

If $S \cup T = \Omega$ and $S \cap T = \emptyset$ then $S$ and $T$ are partition of $\Omega$.


DeMorgan's Law

$S \cup T = \{x \mid x \in S \vee x \in T\}$

$S \cap T = \{x \mid x \in S \wedge x \in T\}$

$$\left(\bigcap_n S_n\right)^c = \bigcup_n S_n^c$$

$$\left(\bigcup_n S_n\right)^c = \bigcap_n S_n^c$$

$$\bigcup_n S_n = \{x \mid \bigvee_n x \in S_n\}$$

$$\bigcap_n S_n = \{x \mid \bigwedge_n x \in S_n\}$$

$x \in (A \cap B)^c$

$x \notin A \cap B$

$x \notin A \wedge x \notin B$

$x \in A^c \vee x \in B^c$

$x \in A^c \cup B^c$

Sample space: set of all possible outcomes.

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Sample space must be

- Mutually exclusive $\leftrightarrow$ two outcomes can not happen at once.
- collectively exhaustive $\leftrightarrow$ all outcomes should be present.
- At "right" granularity.



<u>Axioms :-</u>

$$0 \leq P(A) \leq 1$$

$$P(\Omega) = 1 \quad \text{(Normalization)}$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) \leq P(A) + P(B)$$

Event : $\forall A \mid A \subseteq \Omega$

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$$P\left(\bigcup_n A_n\right) = \sum_n P(A_n) \quad \text{iff} \quad \cancel{P}\bigcap_n A_n = \emptyset$$

we know that,

$$A \cup A^c = \Omega \wedge A \cap A^c = \emptyset$$

$$P(A \cup A^c) = P(\Omega)$$

$$P(A) + P(A^c) = 1$$

$$P(A^c) = 1 - P(A) \leq 1$$

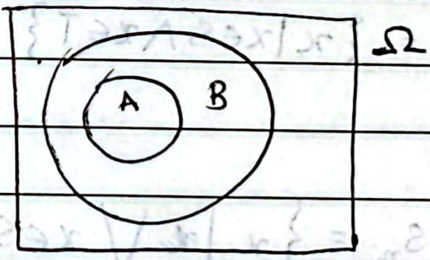
$$P(A \cup A^c) = P(A) + P(A^c) - P(A \cap A^c)$$

$$P(\Omega) = \cancel{P(A)} + 1 - \cancel{P(A)} - P(\emptyset)$$

$$\cancel{1} = \cancel{1} - P(\emptyset)$$

$$P(\emptyset) = 0$$

$$A \subseteq B \Longleftrightarrow P(A) \leq P(B)$$




$$B = A \cup (A^c \cap B)$$

~~P(B)~~

$$P(B) = P(A \cup (A^c \cap B))$$

$$P(B) = P(A) + P(A^c \cap B) \geq P(A)$$

Two Tetrahedral dies are rolled.

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Y = second Roll

| Y \ X | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| 4 | 14 | 24 | 34 | 44 |
| 3 | 13 | 23 | 33 | 43 |
| 2 | 12 | 22 | 32 | 42 |
| 1 | 11 | 21 | 31 | 41 |

X = First Roll

Sample Space

Each event is equi-likely

$$P(X=1) = P(\{11, 12, 13, 14\})$$
$$= P(\{11\} \cup \{12\} \cup \{13\} \cup \{14\})$$
$$= P(\{11\}) + P(\{12\}) + P(\{13\}) + P(\{14\})$$
$$= \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16}$$

$$P(X=1) = \frac{1}{4}$$

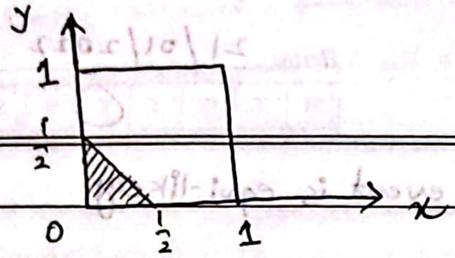
$$P(\min(X,Y)=4) = P(\{44\})$$
$$= \frac{1}{16}$$

$$P(Z=2) = P(\{22, 32, 42, 23, 24\})$$
$$= P(\{22\} \cup \{32\} \cup \{42\} \cup \{23\} \cup \{24\})$$
$$= P(\{22\}) + P(\{32\}) + P(\{42\}) + P(\{23\}) + P(\{24\})$$
$$= 1/16 + 1/16 + 1/16 + 1/16 + 1/16$$
$$P(\min(X,Y)=2) = \frac{5}{16}$$

$$P(X>Y) = P(\{21, 31, 41, 32, 42, 43\})$$
$$= \frac{3}{8}$$

$$P(X+Y=0 \pmod 2) = P(\{11, 31, 22, 42, 13, 33, 24, 44\})$$
$$= \frac{1}{2}$$

~~Sample Space~~ Event A $= \{(x,y) \mid x, y \in [0,1] \wedge x+y \le \frac{1}{2}\}$




sample space: $x, y \in [0,1]$



$$P(A) = \frac{1}{2}\int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}} dx\,dy = \frac{1}{8}.$$

Discrete but infinite sample space.

Sample space $= \mathbb{N} = \Omega$

Probability Law : $P(n) = \dfrac{1}{2^n}$

$$P(\Omega) = \sum_{n=1}^{\infty} \frac{1}{2^n} = 1$$

Countable Additivity Axiom

$$P\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} P(A_n)$$

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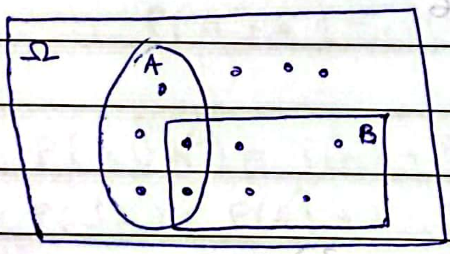
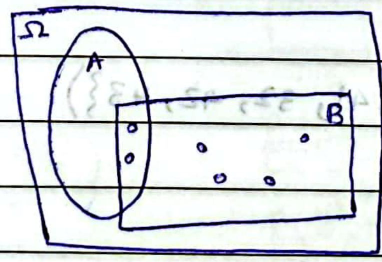
Conditional Probability.




$P(A) = \dfrac{5}{12}$, $P(B) = \dfrac{6}{12}$

$P(A|B)$ = Probability of A given B has occurred.

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \qquad P(B) > 0$$

$$P(A|B) = \frac{P(A\cap B)}{P(B)}$$



$P(A|B)\,P(B) = P(A\cap B)$ ⟶ Multiplication Rule

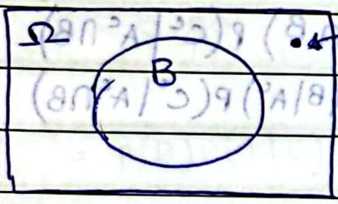
$P(A|B)\,P(B) = P(B\cap A)$

$P(A|B)\,P(B) = \frac{P(B\cap A)}{P(A)} \times P(A) \quad \because P(A) > 0$

$$\boxed{P(A|B)\,P(B) = P(B|A)\,P(A)}$$

Baye's Theorem

If $\Omega$ is a sample space | $|\Omega| \in \mathbb{N}$ and $B$ is an event $B :\neq \emptyset$.
The discrete uniform Law for $\Omega$ does not hold after $B$ occurs.



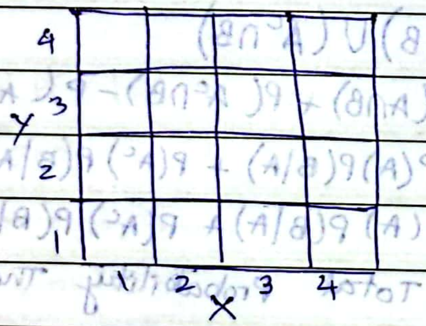

$= x \in \Omega - B$

For $B$ has not occured yet, the $P(\{x\}) = \frac{1}{|\Omega|}$

After B occurs, $P(\{x\}) = 0$.

y = Second Roll

X = First Roll




$M = \max(x, y)$

$B : \min(x, y) = 2$

$P(M = 1 | B) = 0$

$P(A|B) \geq 1$

$P(\Omega|B) = 1$

$P(A|A) = 1$

If $A\cap C = \emptyset \Rightarrow P(A\cup C|B) = P(A|B) + P(C|B)$

$$P(A\cup C|B) = \frac{P((A\cup C)\cap B)}{P(B)}$$

$$= \frac{P((A\cap B)\cup(C\cap B))}{P(B)}$$

$$= \frac{P(A\cap B)+P(C\cap B)-P(A\cap B\cap C)}{P(B)}$$

$0<P(B)$

$$= \frac{P(A\cap B)}{P(B)}+\frac{P(C\cap B)}{P(B)}-\frac{P((A\cap C)\cap B)}{P(B)}$$

$$P(A\cup C|B) = P(A|B)+P(C|B)-P(A\cap C|B)$$

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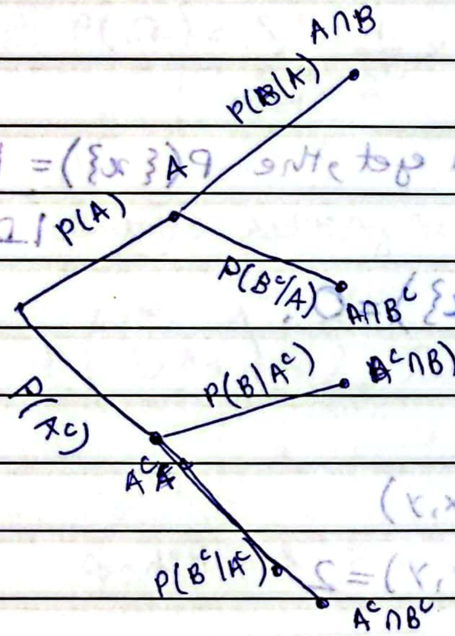



$$P(A^c\cap B\cap C^c) = P(A^c\cap B)\,P(C^c|A^c\cap B)$$

$$= P(A^c)\,P(B|A^c)\,P(C^c|A^c\cap B)$$

$=$

$$P\left(\bigcap_{i=1}^{n} A_i\right) = P(A_1)\prod_{i=2}^{n} P\left(A_i \,\Big|\, \bigcap_{j=1}^{i-1} A_j\right)$$

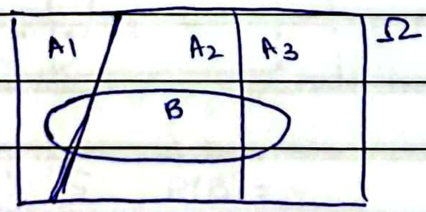
$$B=(A\cap B)\cup(A^c\cap B)$$

$$P(B)=P(A\cap B)+P(A^c\cap B)-P(A\cap B\cap A^c\cap B)$$

$$P(B)=P(A)P(B|A)+P(A^c)P(B|A^c)-0$$

$$P(B)=P(A)P(B|A)+P(A^c)P(B|A^c)$$

Total Probability Theorem.




$A_1\cup A_2\cup A_3=\Omega \wedge A_1\cap A_2\cap A_3=\emptyset$

$A_1, A_2$ and $A_3$ are partitions of $\Omega$

$B=\Omega\cap B$

$B=(A_1\cup A_2\cup A_3)\cap B$

$B=(A_1\cap B)\cup(A_2\cap B)\cup(A_3\cap B)$

$P(B) = P(A_1 \cap B) + P(A_2 \cap B) + P(A_3 \cap B)$

$P(B) = P(A_1)P(B|A_1) + P(A_2)P(B|A_2) + P(A_3)P(B|A_3)$

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If, $\bigcup_{i=1}^{n} A_i = \Omega \wedge \bigcap_{i=1}^{n} A_i = \emptyset$

if $B \subset \Omega$

$\Rightarrow P(B) = \sum_{i=1}^{n} P(A_i) P(B|A_i) \rightarrow$ Total Probability Theorem

$T :=$ test is positive

$D :=$ has desease.

$P(T|D) = 0.95$

$P(\neg T|\neg D) = 0.95$

$P(D) = 0.001$

$P(T) = P(D)P(T|D) + P(\neg D)P(T|\neg D)$

$P(T) = 0.001 \times 0.95 + (1 - 0.001)(0.95)$

$P(T) = 0.95(\cancel{0.001} + 1 - \cancel{0.001})$

$P(T) = 0.95$

$P(D|T) = \dfrac{P(D)\,P(T|D)}{P(T)}$

$= \dfrac{\cancel{0.45}\ 0.001 \times \cancel{0.95}}{\cancel{0.001}\ 0.95}$

$P(D|T) = 0.001$

$P(H) = p \wedge P(T) = 1-p$

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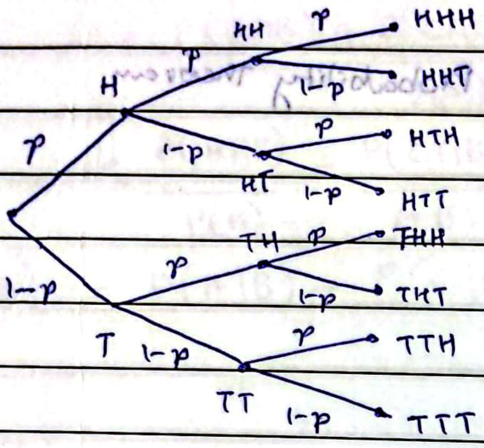



$P(T|H) = P(T)$

$P(T|T\cap H) = P(T)$

$P(1\text{ head}) = P(\{HTT, THT, TTH\}) = P(\{HTT\}) + P(\{THT\}) + P(\{TTH\})$

$= p(1-p)^2 + p(1-p)^2 + p(1-p)^2$

$P(1\text{ head}) = 3p(1-p)^2$

$$P(\text{First is head} \mid 1\text{Head}) = \frac{P(\text{First is head} \cap 1\text{Head})}{P(1\text{Head})}$$

$$= \frac{\cancel{p(1-p)^2}}{3\cancel{(1-p)^2 p}}$$

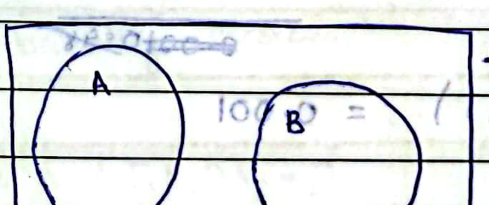
$$P(\text{First is head} \mid 1\text{ Head}) = \frac{1}{3}$$

Independence of 2 ~~set~~ events.

$P(A\cap B) = P(A)\,P(B)$

iff

$P(A|B) = P(A) \wedge P(B|A) = P(B)$




$P(A) > 0 \quad P(B) > 0$

$P(A\cap B) = 0 \Rightarrow P(A)P(B) = 0$ but $P(A) > 0$ and $P(B) > 0$ this contradict

There fore A and B are dependent.

If A be an event, a subset of the sample space $\Omega$. Are A and $\Omega$ independent?

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$$A \subseteq \Omega$$

$$P(A|\Omega) = P(A)$$

$$P(\Omega|A) = P(\Omega) = 1$$

$$P(\Omega \cap A) = P(A)P(\Omega) = P(A)$$

They are independent.

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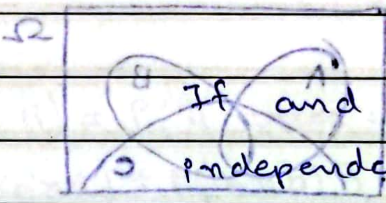
$$A = (A \cap B^c) \cup (A \cap B)$$



$$P(A) = P((A \cap B^c) \cup (A \cap B))$$

$$= P(A \cap B^c) + P(A \cap B)$$

$$P(A) = P(A \cap B^c) + P(A)P(B)$$

$$P(A \cap B^c) = P(A)(1 - P(B))$$

$$P(A \cap B^c) = P(A)P(B^c) \text{ iff } P(A \cap B) = P(A)P(B)$$

If A and B are independent then A and $B^c$ are also independent.

$$(1-P(A))(1-P(B)) = 1 - (P(A)+P(B)) + P(A)P(B)$$

$$= 1 - (P(A \cup B) + P(A \cap B)) + P(A)P(B)$$

$$= 1 - P(A \cup B) - P(A \cap B) + P(A)P(B)$$

$$= 1 - P(A \cup B) \cancel{- P(A)P(B)} \cancel{+ P(A)P(B)}$$

$$= 1 - P(A \cup B)$$

$$(1-P(A))(1-P(B)) = P(A^c \cap B^c)$$

If A and B are independent then $A^c$ and $B^c$ are also independent.

Given three events A, B, and C.

$$P(A\cap B|C) = P(A|C)\,P(B|C)$$

if this equation holds, then A and B are independent given C.

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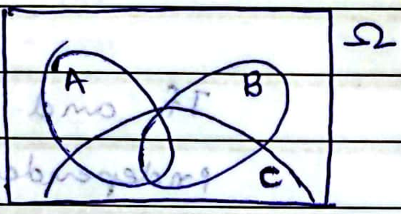
$$P(A\cap B|C) = P(A|C)P(B|C) \not\Rightarrow P(A\cap B|C^c) = P(A|C^c)P(B|C^c)$$

$$\therefore\ P(A\cap B|C^c) = 0$$

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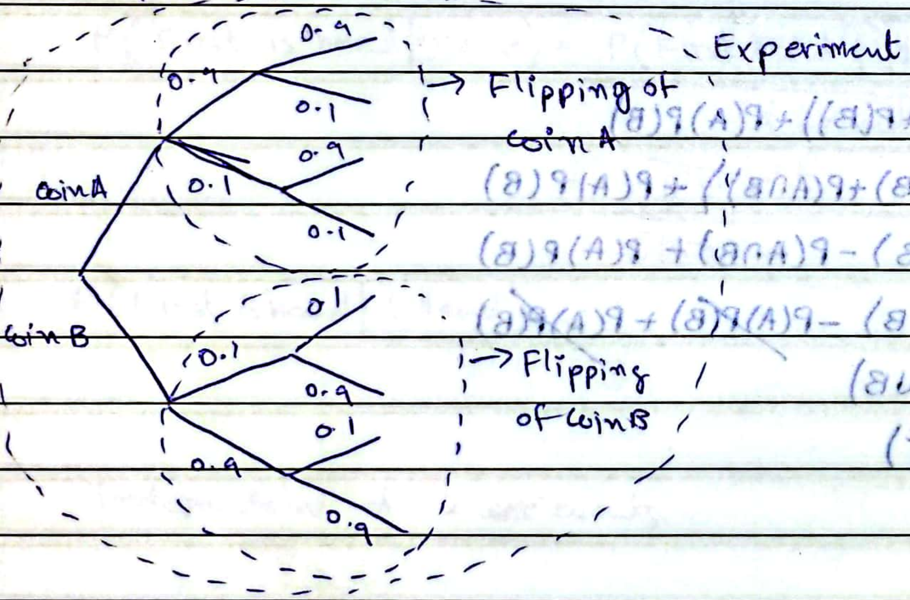
$P(H \mid \text{coin } A) = 0.9$

$P(\text{coin } A) = P(\text{coin } B) = \frac{1}{2}$

$P(H \mid \text{coin } B) = 0.1$




$$P(\text{toss } 11 = H) = P(\text{coin } A)\,P(\text{toss } 11 = H \mid \text{coin } A) + P(\text{coin } B)\,P(\text{toss } 11 = H \mid \text{coin } B)$$

$$= \left(\frac{1}{2}\right)(0.9) + \left(\frac{1}{2}\right)(0.1)$$

$$= \frac{1}{2}$$

$$P(\text{toss } 11 = H \mid \text{first 10 tosses are } H) = \frac{P(\text{toss } 11 = H \cap \text{first 10 tosses are } H)}{P(\text{first 10 tosses are } H)}$$

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$$P(\text{first 10 tosses are } H) = P(\text{coin } A)\,P(\text{first 10 tosses are } H \mid \text{coin } A) + P(\text{coin } (B))\,P(\text{first 10 tosses are } H \mid \text{coin } B)$$

$$= \frac{1}{2}(0.9)^{10} + \frac{1}{2}(0.1)^{10}$$

$$P(\text{first 10 tosses are } H) = 0.1743$$

$$P(\text{toss } 11 = H \cap \text{first 10 tosses are } H) = P(11 \text{ tosses are Heads})$$

$$= P(\text{coin } A)\,P(11 \text{ tosses are heads} \mid \text{coin } A) + P(\text{coin } B)\,P(11 \text{ tosses are heads} \mid \text{coin } B)$$

$$= \frac{1}{2}(0.9)^{11} + \frac{1}{2}(0.1)^{11}$$

$$P(\text{toss } 11 = H \cap \text{first 10 tosses are } H) = 0.1569$$

$$P(\text{toss } 11 = H \mid \text{first 10 tosses are } H) = 0.899$$

Independence of Collection of sets,

$A_1, A_2, A_3, \dots, A_n$ $\qquad P\left(\bigcap_i A_i\right) \stackrel{?}{=} \prod_i P(A_i)$

$\cancel{P\left(\bigcap_{k=1}^{n} A_k\right) = \prod_{k=1}^{n} P(A_k)}$

For $|\Omega| = n$, if we have to arrange $k$ elements such that $k \le n$
~~the possib~~ in such a way that order matters, then

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if repitition is allowed $:= n^k$

if repitition not allowed $:= \dfrac{n!}{(n-k)!} = {}^{n}P_{k}$

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for any set $A \ni |A| = n$ if ~~B containing~~ $C \in B \ni |C| = k$ and $C \subset A$, then $|B| = \binom{n}{k}$.

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

Let $P(H) = p$ and $P(T) = 1-p$     Independent tosses.

$P(HTTHHH) = p^4(1-p)^2$

$P(\text{particular seq. of H and T}) = p^{\#\text{ of H}}(1-p)^{\#\text{ of T}}$

$P(\text{a particular k-head seq. with n flip}) = p^k(1-p)^{n-k}$

$P(\text{k heads in n flip}) = \binom{n}{k} p^k (1-p)^{n-k}$

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Number of partition of a set of size $n$ into $n_1, n_2, \ldots, n_r$ sized sets such that, $n_1 + n_2 + \cdots + n_r = n$, will be, $\dfrac{n!}{n_1!\, n_2! \cdots n_r!}$

$$\binom{n}{n_1, n_2, \ldots, n_r} = \frac{n!}{n_1!\, n_2! \cdots n_r!}$$

| | with replacement | without replacement |
|---|---|---|
| Order matters | $n^r$ | ${}^nP_r$ |
| Order doesn't matter | $\binom{n+k-1}{k}$ | ${}^nC_r$ |

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# Discrete Random Variables

$$X : \underset{\text{Sample Space}}{\Omega} \longrightarrow \underset{\text{Range}}{\mathbb{R}}$$

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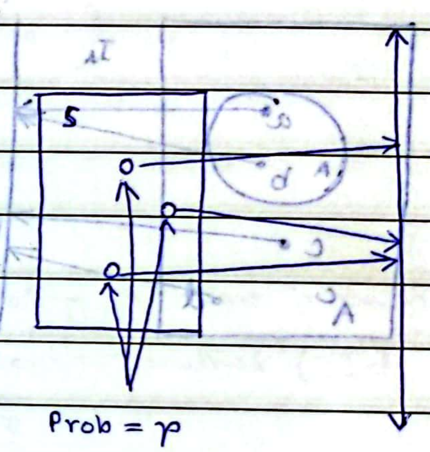
$$\omega \in \Omega, \quad X(\omega) = x$$

There can be more than one random variables on a sample space. Any function of those random variables is also a random variable.




## Probability Mass Function (PMF)

$$p_X(x) := P(X = x)$$
$$= P(\{\omega \mid \omega \in \Omega \wedge X(\omega) = x\})$$

$$p_X(x) \geq 0$$

$$\sum_x p_X(x) = 1$$

| Y = Second Roll | | | | |
|---|---|---|---|---|
| 4 | 14 | 24 | 34 | 44 |
| 3 | 13 | 23 | 33 | 43 |
| 2 | 12 | 22 | 32 | 42 |
| 1 | 11 | 21 | 31 | 41 |
| | 1 | 2 | 3 | 4 |

X = First Roll

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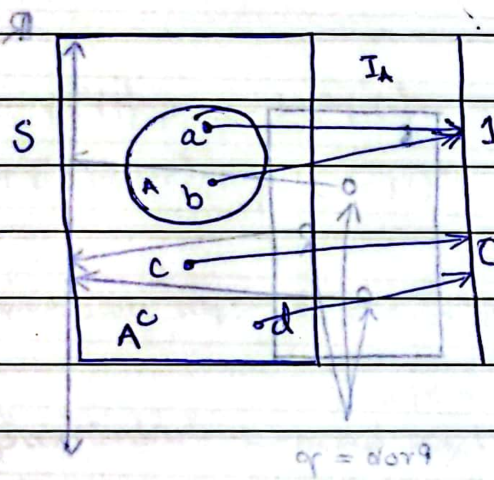
## Bernoulli Random variable.

parameter:= $p$

$\cancel{X : \{0,1\} \to \{B}$

$$X := \begin{cases} 1 & \text{with probability } p \\ 0 & \text{with probability } 1-p. \end{cases}$$

Used when there is one trial with two possibilities.




$I_A$: Indicator Random variable.

$I_A(a) = 1$

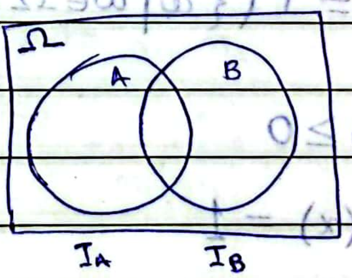
$I_A(b) = 1$

$I_A(c) = 0$

$I_A(d) = 0$

$$p_{I_A}(1) = P(\underbrace{I_A = 1}_{\text{The outcome is from event A.}}) = P(A)$$

$I_A + I_B$ is not an IRV, because in $A \cap B$ ~~IRV~~ $I_A + I_B = 2$, which is absurd.




$I_A I_B$ is an IRV.

$$p_{I_A I_B}(1) = P(I_A I_B = 1) = P(A \cap B)$$

Discrete Uniform random variable.

parameter: $a, b \rightarrow$ ending number.
$\uparrow$ starting number

Model of complete ignorance



$\Omega := \{a, a+1, a+2, \dots, b-1, b\}$

Random Variable: $X(\omega) = \omega$

$$p_X(\omega) = P(X=\omega) = \frac{1}{b-(a+1)-1}$$

Special case:-

$a=b \qquad p_X(\omega) = P(X=\omega) = 1$

X is called constant / Deterministic Random variable.

Binomial Random Variable

Parameters: $n :=$ number of independent trials.

$p \in [0,1]$

conditions: n trials are independent, and the probability of success in each trial remain same. we call it Independent - Identical (IId)

Random Varaible X: # of success obsserved.

$$p_X(k) = P(X=k) = \binom{n}{k} p^k (1-p)^{n-k}$$

23/02/2022

Geometric Random Variable.

Tells the probability of number of trials took until first success.

number of failures until success.

P(success) = $p$

$$p_X(k) = P(X=k) = (1-p)^{k-1} p$$

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X is a G.R.V.

$$P(X \geq 10) = \sum_{i=10}^{\infty} P(X=i)$$

$$= \sum_{i=10}^{\infty} (1-p)^{i-1} p$$

$$= p\left((1-p)^9 + (1-p)^{10} + \cdots\right)$$

$$= \frac{p\,(1-p)^9}{1-(1-p)}$$

$$= \frac{p(1-p)^9}{\cancel{1}-\cancel{1}+p}$$

$$P(X \geq 10) = (1-p)^9$$

$$P(X \geq 10) = 1 - P(X < 10)$$

$$= 1 - \sum_{i=1}^{9} P(X=i)$$

$$= 1 - \sum_{i=1}^{9} (1-p)^{i} p$$

$$= 1 - p\,\frac{1-(1-p)^9}{1-(1-p)}$$

$$= 1 - \frac{\cancel{p}\left(1-(1-p)^9\right)}{\cancel{p}}$$

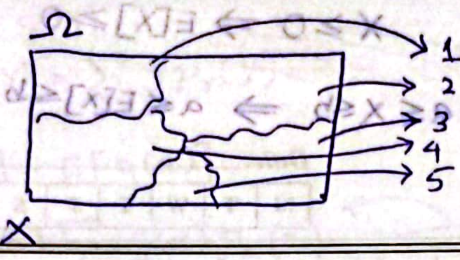
$$= \cancel{1}-\cancel{1}+(1-p)^9$$

$$P(X \geq 10) = (1-p)^9$$

Expectation.

also called ppopulation averege,

$$E[X] = \sum_{x} x\, p_X(x)$$




Probabilistic Model

(i) — Sample space.

Probability law

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Random variable maps some subset of ~~sap~~ sample space to a unique number. In order to identify events!

Variance

Tells how much data is spreaded.

$$Var(X) = E[X^2] - (E[X])^2 = E[(X-\mu)^2] = \sigma^2$$

Poisson Random Variable.

$$p_X(x) = P(X=x) = \frac{e^{-\lambda}\lambda^x}{x!}$$

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Expectation:

$$E[X] = \sum_x x\, p_X(x)$$

Bernoulli Random Variable

$$E[X] = (0)p_X(0) + (1)p_X(1)$$

$$E[X] = p$$

Uniform Random variable

$$E[X] = (0)p_X(0) + (1)p_X(1) + \cdots + (n)p_X(n)$$

$$= \frac{0}{n+1} + \frac{1}{n+1} + \cdots + \frac{n}{n+1}$$

$$= \frac{1}{n+1} \times (1+2+3+\cdots+n)$$

$$= \frac{1}{n+1} \times \frac{n(n+1)}{2}$$

$$E[X] = n/2$$

Binomial Random Variable

$X \geq 0 \Rightarrow E[X] \geq 0$

$$E[X] = \sum_{k=0}^{n} k \binom{n}{k} p^k (1-p)^{n-k} \quad -(i)$$

$a \leq X \leq b \Rightarrow a \leq E[X] \leq b$

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$$= \sum_{k=0}^{n} \frac{k\, n!}{k!\,(n-k)!} p^k (1-p)^{n-k}$$

$-1 \leq X \leq 2$

$-2 \leq Y \leq 3$

$$= \sum_{k=0}^{n} \frac{n!}{(k-1)!\,(n-k)!} p^k (1-p)^{n-k}$$

$+2 \geq -Y \geq -3$

$$= \sum_{k=0}^{n} \frac{n!\,(n-k+1)}{(k-1)!\,(n-k+1)!} p^k (1-p)^{n-k}$$

$-1 \leq X \leq 2$

$-3 \leq -Y \leq +2$

$$= \sum_{k=0}^{n} \binom{n}{k-1} (n-k+1)\, p^k (1-p)^{n-k}$$

$-4 \leq X-Y \leq 4$

$$E[X] = \sum_{k=0}^{n} (n-k) \binom{n}{n-k} p^{n-k} (1-p)^k \quad -(ii)$$

$$2E[X] = \sum_{k=0}^{n} (n-k) \binom{n}{n-k} p^{n-k} (1-p)^k + \sum_{k=0}^{n} k \binom{n}{k} p^k (1-p)^{n-k}$$

$$= \sum_{k=0}^{n} \left\{ (n-k) \binom{n}{k} p^n \left(\frac{1-p}{p}\right)^k \right\} + \sum_{k=0}^{n} \left\{ k \binom{n}{k} (1-p)^n \left(\frac{p}{1-p}\right)^k \right\}$$

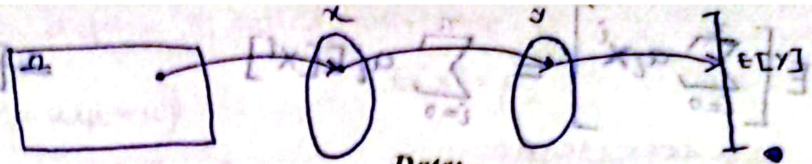
$$= \sum_{k=0}^{n} \binom{n}{k} \left\{ (n-k)\, p^n \left(\frac{1-p}{p}\right)^k + k\, (1-p)^n \left(\frac{p}{1-p}\right)^k \right\}$$

$$\cancel{E[X] = (1-p)^n \sum_{k=0}^{n} \binom{n}{k} k \left(\frac{p}{1-p}\right)^k}$$

$$\cancel{E[X] = \frac{p(1-p)^{n+1}}{p} \sum_{k=0}^{n} \binom{n}{k} \frac{d}{dp}}$$

$Y = g(X)$

$E[Y] = \sum_y y\, p_Y(y)$

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$E[Y] = E[g(X)]$

$= \sum_x g(x)\, p_X(x)$ Expected value Rule

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$\sum_y \sum_{x: g(x)=y} g(x)\, p_X(x)$

$E[X^2] \neq (E[X])^2$

$= \sum_y \sum_{x: g(x)=y} y\, p_X(x)$

$= \sum_y y \underbrace{\sum_{x: g(x)=y} p_X(x)}$ → Summing over particular value of y.

$= \sum_y y\, p_Y(y)$

$= E[Y]$

Uniform Random variable.

$\exists\ \Omega = \{-1, 0, 1, 2\} \ni \forall \exists\ X(\Omega) \rightarrow R,\ E[X^4] = ?$

$E[X^4] = \sum_{x \in \{-1,0,1,2\}} x^4 p_X(x) = \sum_{x \in \{-1,0,1,2\}} x^4 \left(\frac{1}{4}\right)$

$= \frac{1}{4}\left((-1)^4 + (0)^4 + (1)^4 + (2)^4\right)$

$E[X^4] = 4.5$

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$$E\left[\sum_{i=0}^{n} a_i x^i\right] = \sum_{i=0}^{n} a_i E[x^i]$$ ~~$E[g(x)] = g(E[x])$~~

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Proof:

Einstien Notation.

$$E[a_i x^i] = \sum_x a_i x^i p_X(x)$$ Expected value rule

$$= a_i \sum_x x^i p_X(x)$$

$$= a_i E[x^i]$$

If $E[X]=2$ and $E[X^2]=7$

$$E[8-X] = 8 - E[X]$$
$$= 8-2$$
$$= 6$$

$$E[(X-3)(X+3)] = E[X^2-9]$$
$$= E[X^2] - 9E[1]$$
$$= 7-9$$
$$= -2$$

Poisson Random Variable

$$p_X(x) = \frac{e^{-\lambda}\lambda^x}{x!}$$

$$E[X] = \sum_{x=0}^{\infty} \frac{x e^{-\lambda}\lambda^x}{x!} = \sum_{x=1}^{\infty} \frac{e^{-\lambda} x \lambda^x}{x!}$$

$$= \sum_{x=0}^{\infty} \frac{e^{-\lambda}\lambda^{x+1}}{x!}$$

$$= e^{-\lambda}\lambda \sum_{x=0}^{\infty} \frac{\lambda^x}{x!}$$

$$= e^{-\lambda}\lambda e^{\lambda}$$

$$E[X] = \lambda$$

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Varience:- The amount of data disperse is called variance.

$E[X]=\mu$

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$$var(X) = E[(X-\mu)^2] = E[X^2 - 2X\mu + \mu^2]$$
$$= E[X^2] - 2\mu E[X] + \mu^2$$
$$= E[X^2] - 2\mu\mu + \mu^2$$
$$= E[X^2] - \mu^2$$
$$var(X) = E[X^2] - (E[X])^2$$

$$SD(X) = \sqrt{Var(X)} = \sigma(X)$$

Variance is conserved under linear translation of PMFs at x-axis.

$$Var(X+b) = Var(X)$$

Proof:-

$$E[X+b] = \mu + b$$
$$Var(X+b) = E[((X+b)-(\mu+b))^2]$$
$$= E[(X-\mu)^2]$$
$$= Var(X)$$

$$Var(aX) = a^2 Var(X)$$

Proof:-

$$E[aX] = a\mu$$
$$Var(aX) = E[(aX - a\mu)^2]$$
$$= a^2 E[(X-\mu)^2]$$
$$Var(aX) = a^2 Var(X)$$

Bernoulli Random Variable:-

$$Var(X) = E[(X-\mu)^2]$$
$$= E[(X-p)^2]$$
$$= (0-p)^2(1-p) + (1-p)^2(p)$$
$$= p(1-p)(\cancel{p} + 1 - \cancel{p})$$
$$Var(X) = p(1-p)$$

For $p=1$ or $0$ in $X \sim Ber(p)$ $Var(X) = 0$, means the experiment is bias.

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Uniform Random Variable.

$X: \{0, 1, 2, \dots, n\} \longrightarrow R$

~~$\mu = \frac{n}{2}$~~

$$\cancel{Var(X) = E[X^2] - (E[X])^2}$$

$$\cancel{= \sum_{i=0}^{n} i^2 \frac{1}{n} - \left(\frac{n}{2}\right)^2}$$

$$\cancel{= \frac{1}{n+1} \sum_{i=0}^{n} i^2 - \frac{n^2}{4}}$$

$$\cancel{= \frac{1}{n+1} \frac{n(n+1)(2n+1)}{6} - \frac{n^2}{4}}$$

$$E[X] = \sum_{i=0}^{n} i \frac{1}{n+1} = \frac{n}{2} = \mu$$

$$Var(X) = E[X^2] - (E[X])^2$$

$$= \sum_{i=0}^{n} i^2 \frac{1}{n+1} - \frac{n^2}{4}$$

$$= \frac{1}{\cancel{n+1}} \times \frac{n\cancel{(n+1)}(2n+1)}{6} - \frac{n^2}{4}$$

$$= \frac{n}{2}\left(\frac{2n+1}{3} - \frac{n}{2}\right)$$

$$= \frac{n}{2}\left(\frac{4n+2-3n}{6}\right)$$

$$Var(X) = \frac{n}{12}(n+2)$$

if $X: \{a, a+1, a+2, \dots, b\} \rightarrow R$

$$Var(X) = \frac{(b-a)}{12}(b-a+2)$$

Conditional PMFs and Expectations.

~~$p_X(x) = P(X=x)$ Unconditional PMF~~

~~$p_{X|A}(x) = P$~~

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$$p_X(x) = P(X=x) = P(\{\omega : X(\omega) = x\})$$

$$p_{X|A}(x) = P(X=x|A) = P(\{\omega : X(\omega)=x\} \cap \{\omega' : \omega' \in A\})$$

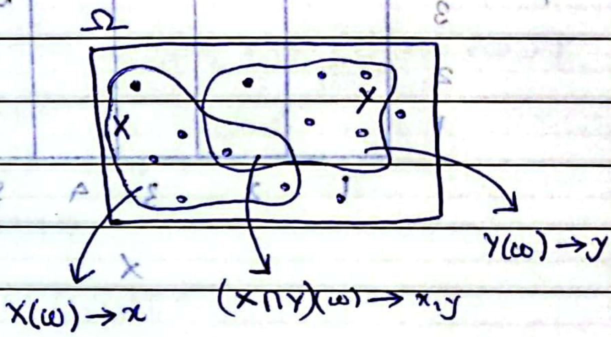
$$\sum_x p_{X|A}(x) = 1 \qquad E[g(X)|A] = \sum_x g(x)\, p_{X|A}(x)$$

$$E[X|A] = \sum_x x\, p_{X|A}(x)$$

09/03/2022.

Joint PMFs.

$$p_{X,Y}(x,y) = P((X=x) \wedge (Y=y))$$




$$p_{X,Y}(x,y) = P(\{\omega : X(\omega)=x \wedge Y(\omega)=y\})$$

$$\sum_x \sum_y p_{X,Y}(x,y) = 1$$

| Y \ X | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| 4 | 1/20 | 2/20 | 2/20 | |
| 3 | 2/20 | 4/20 | 1/20 | 2/20 |
| 2 | | 1/20 | 3/20 | 1/20 |
| 1 | | 1/20 | | |

$$p_{X,Y}(1,3) = \frac{2}{20}$$

$$p_X(4) = \sum_{y} p_{X,Y}(4,2) + p_{X,Y}(4,3) = \frac{3}{20}$$

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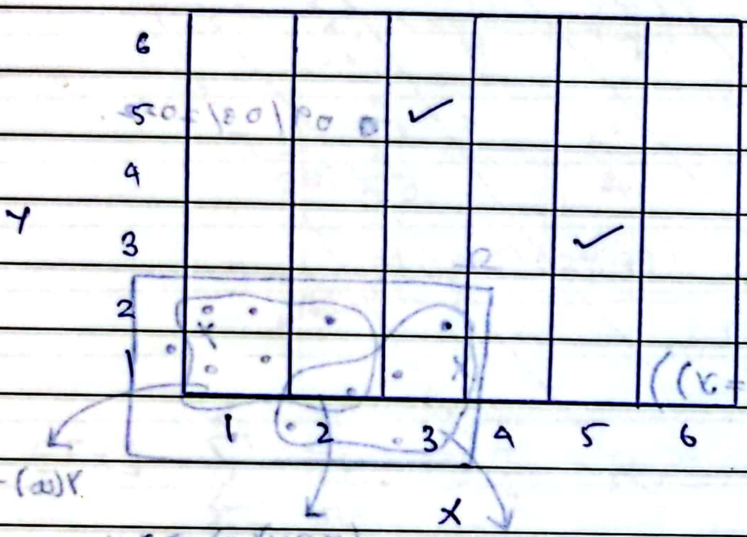
Marginal PMF

$$p_Y(y) = \sum_x p_{X,Y}(x,y)$$

For 2 rolls of $D_6$,

$$p_{X,Y}(3,5) = P(X=3 \text{ and } Y=5) = P\{\omega : \omega \in \Omega \wedge \min(\omega) = 3 \wedge \max(\omega) = 5\}$$

$$= \frac{2}{36}$$




$$p_{X_1, X_2, \dots, X_n}(x_1, x_2, \dots, x_n) = P\left(\bigwedge_{i=1}^{n} (X_i = x_i)\right)$$

$$= P\left(\left\{\omega : \bigwedge_{i=1}^{n} (X_i(\omega) = x_i)\right\}\right)$$

$$\sum_{x_1}\sum_{x_2}\cdots\sum_{x_n} p_{X_1,X_2,\dots,X_n}(x_1, x_2, \dots, x_n) = 1$$

$$\sum_{x_1}\sum_{x_2}\cdots\sum_{x_{i-1}}\sum_{x_{j+1}}\cdots\sum_{x_n} p_{X_1,X_2,\dots,X_n}(x_1, x_2, \dots, x_n) = p_{X_i, X_{i+1}, \dots, X_{j-1}, X_j}(x_i, x_{i+1}, \dots, x_{j-1}, x_j)$$

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Functions of multi random variable.

$$Y = g(X_1, X_2, \ldots, X_n)$$

$$p_Y(y) = P(Y=y) = P(g(x_1, x_2, \ldots, x_n) = y)$$

$$= \sum_{(x_1, x_2, \ldots, x_n) : g(x_1, x_2, \ldots, x_n) = y} g(x_1, x_2, \ldots, x_n)\, p_{X_1, X_2, \ldots, X_n}(x_1, x_2, \ldots, x_n)$$

$$E[Y] = \sum_{x_1} \sum_{x_2} \cdots \sum_{x_n} g(x_1, x_2, \ldots, x_n)\, p_{X_1, X_2, \ldots, X_n}(x_1, x_2, \ldots, x_n)$$

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$$\sum_{v \in \{0,1\}} \sum_{w \in \{0,1,2\}} c(v+w) = 1$$

$$E\left[\sum_i X_i\right] = \sum_i E[X_i]$$

$$c \sum_{v \in \{0,1\}} (3v + 3) = 1$$

$$c(3+6) = 1$$

$$c = \frac{1}{9}$$

$$p_v(1) = \sum_{w \in \{0,1,2\}} \frac{1}{9}(1+w)$$

$$= \frac{1}{9}(6)$$

$$= \frac{2}{3}$$

$$\cancel{\sum_i E[X_i]}$$

$$p_{X|Y}(x|y) = P(X=x \mid Y=y) = \frac{P(X=x, Y=y)}{P(Y=y)} = \frac{p_{X,Y}(x,y)}{p_Y(y)}$$

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$p_{X|Y}(1,1) = 0$

$p_{X|Y}(2,2) = \frac{1}{5}$

$$p_{X,Y}(x,y) = p_Y(y)\, p_{X|Y}(x|y)$$

$$p_{X,Y}(x,y) = p_X(x)\, p_{Y|X}(y|x)$$

$$p_X(x)\, p_{Y|X}(y|x) = p_Y(y)\, p_{X|Y}(x|y)$$

$$p_{X,Y|Z}(x,y|z) = \frac{P(X=x, Y=y, Z=z)}{P(Z=z)}$$

$$= \frac{P(X=x, Z=z)}{P(Z=z)} \; \frac{P(X=x, Y=y, Z=z)}{P(X=x, Z=z)}$$

$$= p_{X|Z}(x|z)\, p_{Y|X,Z}(y|x,z)$$

Conditional Expectation.

$$\cancel{E[X \mid Y=y] = \sum_x x} \qquad E[g(X) \mid Y=y] = \sum_x g(x)\, p_{X|Y}(x|y)$$

$$\cancel{p_X(x) = \sum_i p_X(x)} \qquad p_X(x) = \sum_{y_i} p_Y(y_i)\, p_{X|Y}(x|y_i)$$

$$\sum_x x\, p_X(x) = \sum_x x \sum_y p_Y(y)\, p_{X|Y}(x|y)$$

$$E[X] = \sum_y p_Y(y) \sum_x x\, p_{X|Y}(x|y)$$

$$E[X] = \sum_y p_Y(y)\, E[X \mid Y=y]$$

D.R.Vs Independence.

$$p_{X,Y}(x,y) = p_X(x)\, p_Y(y)$$

$$p_{X|Y}(x|y) = p_X(x) \qquad p_{Y|X}(y|x) = p_Y(y)$$

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$$p_{X_1,X_2,\dots,X_n}(x_1,x_2,\dots,x_n) = \prod_{i=1}^{n} p_{X_i}(x_i)$$

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If $X$ and $Y$ are independent

$$E[XY] = E[X]E[Y] \qquad p_{X,Y}(x,y) = p_X(x)\, p_Y(y)$$

$$var(X+Y) = E((X+Y)^2) = E(X^2+Y^2+2XY) \qquad \because E[X]=0 \wedge E[Y]=0$$

$$= E(X^2)+E(Y^2)+2E(X)E(Y)$$

$$= E(X^2)+E(Y^2)$$

$$= E(X^2)-(E(X))^2 + E[Y^2] - (E[Y])^2$$

$$= var(X) + var(Y)$$

Continous Random Variable.

- Outcomes will be taken out from some intervals,

$P(a \le X \le b)$

- Instead of PMF (Probability Mass Function) : $p_X(x)$

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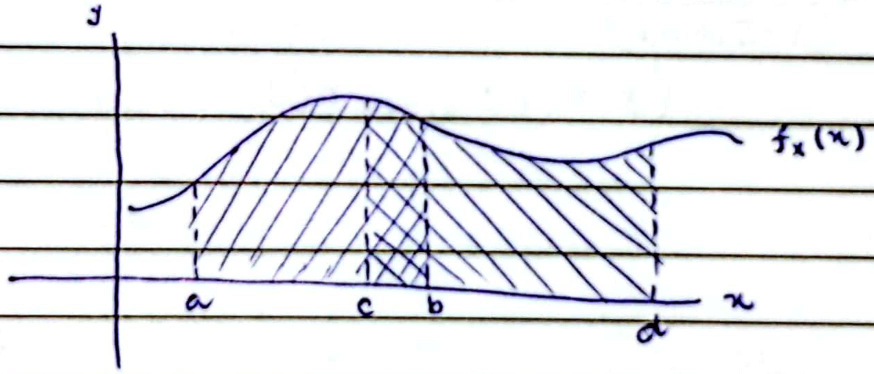
we have PDF (Probability Density Function) : $f_X(x)$

$$P(a \le X \le b) := \begin{cases} \displaystyle\sum_{x \in [a,b]} p_X(x) & \text{if } X \text{ is discrete.} \\ \displaystyle\int_a^b f_X(x)\,dx & \text{if } X \text{ is Continous.} \end{cases}$$

Valid PDF:—

- $f_X(x) \geq 0$      $f_X(x)$ could be continous or not.

- $\displaystyle\int_{-\infty}^{+\infty} f_X(x)\,dx = 1$




$P\big((a \le X \le b) \cup (c \le X \le d)\big) = P(a \le X \le b) + P(c \le X \le d) - P(c \le X \le b)$

$P(a \le X \le b) = P\big((X=a) \cup (a < X < b) \cup (X=b)\big)$

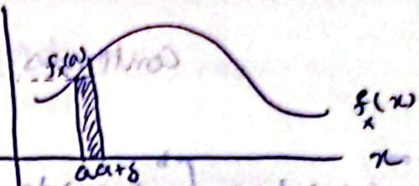
$= P(X=a) + P(a < X < b) + P(X=b)$

$= 0 + P(a < X < b) + 0$

$P(a \le X \le b) = P(a < X < b)$

$\cancel{f_X(a)}$ $\quad \delta \to 0$

$P(a \le X \le a+\delta) \approx f_X(a)\delta$




$$f_X(a) \approx \frac{P(a \le X \le a+\delta)}{\delta}$$

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$f_X(a)$ is density of the probability, that's why it is called Probability Density Function.

example:-

$$f_X(x) := \begin{cases} c(1-x) & \text{if } x \in [0,1] \\ 0 & \text{else.} \end{cases}$$

$$\int_{-\infty}^{+\infty} f_X(x)\,dx = \int_{-\infty}^{0} f_X(x)\,dx + \int_0^1 f_X(x)\,dx + \int_1^{+\infty} f_X(x)\,dx$$

$$= \int_0^1 c(1-x)\,dx$$

$$= c\left(x - \frac{x^2}{2}\right)\Big|_{x=0}^{x=1}$$

$$= c\left(1 - \frac{1}{2}\right) - c(0)$$

$$1 = \frac{c}{2}$$

$$2 = c$$

$$P\left(X \le \frac{1}{2}\right) = \int_{-\infty}^{1/2} f_X(x)\,dx = \left(\int_{-\infty}^{0} + \int_0^{1/2}\right) f_X(x)\,dx$$

$$= \int_0^{1/2} 2(1-x)\,dx$$

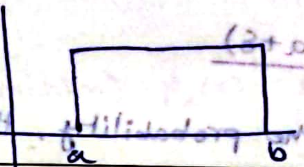
$$= 2\left(x - \frac{x^2}{2}\right)\Big|_{x=0}^{x=1/2}$$

$$= \cancel{2}\left(\frac{1}{\cancel{2}}\right)\left(1 - \frac{1}{4}\right)$$

$$P\left(X \le \frac{1}{2}\right) = \frac{3}{4}$$

Continuous | ~~PDF~~ uniform PDF.

$$\int_{-\infty}^{\infty} f_x(x)\,dx = \int_a^b f_x(x)\,dx = \underbrace{f_x(x)}^{\text{constant}} \int_a^b dx$$



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$$f_x(x)(b-a) = 1$$

$$f_x(x) = \frac{1}{b-a}$$

$$f_x(x) = \begin{cases} 2c & \text{if } 0 \le x \le 1 \\ c & \text{if } 1 < x \le 3 \\ 0 & \text{else} \end{cases}$$

$$\int_{-\infty}^{\infty} f_x(x)\,dx = \left( \cancelto{0}{\int_{-\infty}^{0}} + \int_0^1 + \int_1^3 + \cancelto{0}{\int_3^{+\infty}} \right) f_x(x)\,dx$$

$$1 = \int_0^1 2c\,dx + \int_1^3 c\,dx$$

$$1 = 2c(1) + c(3-1)$$

$$1 = 2c + 2c$$

$$c = \frac{1}{4}$$

$$P\left(\tfrac{1}{2} \le x \le \tfrac{3}{2}\right) = \int_{1/2}^{3/2} f_x(x)\,dx$$

$$= \left( \int_{1/2}^{1} + \int_1^{3/2} \right) f_x(x)\,dx$$

$$= \int_{1/2}^{1} \frac{1}{2}\,dx + \int_1^{3/2} \frac{1}{4}\,dx$$

$$= \frac{1}{2}\left(1 - \frac{1}{2}\right) + \frac{1}{4}\left(\frac{3}{2} - 1\right)$$

$$= \frac{1}{2}\left(\frac{1}{2}\right) + \frac{1}{4}\left(\frac{1}{2}\right)$$

$$= \frac{1}{2}\left(\frac{1}{2} + \frac{1}{4}\right)$$

$$\boxed{P\left(\tfrac{1}{2} \le x \le \tfrac{3}{2}\right) = \frac{3}{8}}$$

$$E[X] = \int_{-\infty}^{+\infty} x f_X(x)\,dx \qquad E[g(X)] = \int_{-\infty}^{+\infty} g(x) f_X(x)\,dx$$

$\uparrow$ continuous Random variable.

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$$f_X(x) = \begin{cases} \frac{1}{26}(4x+1) & \text{if } 2 \le x \le 4 \\ 0 & \text{else} \end{cases}$$

$$E[X] = \int_{-\infty}^{+\infty} x\, f_X(x)\,dx = \int_2^4 x \frac{1}{26}(4x+1)\,dx$$

$$= \frac{1}{26}\int_2^4 (4x^2 + x)\,dx$$

$$= \frac{1}{26}\left(\frac{4x^3}{3} + \frac{x^2}{2}\right)\Bigg|_{x=2}^{x=4}$$

$$= \frac{1}{26}\left(\frac{4}{3}(4)^3 + \frac{4^2}{2}\right) - \frac{1}{26}\left(\frac{4}{3}(2)^3 + \frac{2^2}{2}\right)$$

$$= \frac{140}{39} - \frac{19}{39}$$

$$= \frac{121}{39}$$

$E[X] = 3.10256\ldots$

$$\text{var}(X) = E[(X-\mu)^2] = \int_{-\infty}^{+\infty} (x-\mu)^2 f_X(x)\,dx$$

$X \ge 0 \Rightarrow E[X] \ge 0$

$\sigma_X = \sqrt{\text{var}(X)}$

$a \le X \le b \Rightarrow a \le E[X] \le b$

$\text{var}(aX+b) = a^2\,\text{var}(X)$

$$\text{var}(X) = \int_{-\infty}^{\infty} \left(x - \frac{121}{39}\right)^2 f_X(x)\,dx$$

$$= \int_2^4 \left(x - \frac{121}{39}\right)^2 \frac{1}{26}(4x+1)\,dx$$

$$= \frac{1}{26}\int_2^4 \left(x^2 - \frac{242}{39}x + \frac{14641}{1521}\right)(4x+1)\,dx$$

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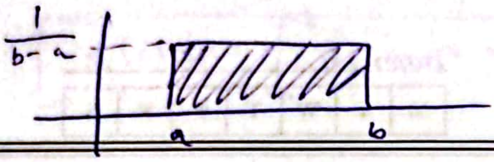
$$\mathrm{var}(X) = \frac{491}{1521} = 0.3228\ldots$$



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$$E[X] = \int_{-\infty}^{\infty} x f_X(x)\,dx = \int_a^b x \frac{1}{b-a}\,dx = \frac{b+a}{2}$$

$$\mathrm{var}(X) = \frac{(b-a)^2}{12}$$

Cummulative Distribution Function.

$$F_X(x) = P(X \leq x) = \int_{-\infty}^{x} f_X(t)\,dt \iff \frac{dF_X}{dx}(x) = \frac{dP}{dx}(X \leq x) = f_X(x)$$

$$b \geq a \Rightarrow P(X \leq b) = P(X \leq a) + P(a < X \leq b)$$

$$y \geq x \Rightarrow F_X(y) \geq F_X(x) \qquad p_X(k) = F_X(k) - F_X(k-1)$$

$$x \to \infty \Rightarrow F_X(x) \to 1$$

$$x \to -\infty \Rightarrow F_X(x) \to 0$$

$$f_X(x) = \begin{cases} \frac{1}{26}(4x+1) & 2 \leq x \leq 4 \\ 0 & \text{else.} \end{cases}$$

$$F_X(x) = \begin{cases} \frac{1}{26}(2x^2+x) - \frac{5}{13} & 2 \leq x \leq 4 \\ 0 & \text{else.} \end{cases}$$

$$P(1 \leq X \leq 3) = F_X(3) - F_X(1)$$

$$= \frac{1}{26}(2(3)^2+3) - \frac{5}{13}$$

$$= \frac{11}{26}$$

Exponential Random Functions:-

$$f_X(x) = \begin{cases} \boxed{\lambda e^{-\lambda x}} & x \geq 0 \\ 0 & x < 0 \end{cases} \qquad \lambda > 0$$

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$$E[X] = \cancel{\sum_{k=0}^{\infty} k e^{-\lambda k}\lambda} \int_{-\infty}^{+\infty} x \lambda e^{-x\lambda}\, dx = \lambda \int_0^{\infty} x e^{-x\lambda}\, dx$$

$$= \cancel{\lambda \sum_{k=0}^{\infty} k e^{-\lambda k}} \qquad = \frac{1}{\lambda} \int_0^{\infty} (x\lambda) e^{-x\lambda}\, d(x\lambda)$$

$$= \cancel{\lambda \frac{d}{d\lambda}\left(-\sum_{k=0}^{\infty} e^{-\lambda k}\right)} \qquad = \frac{\Gamma(2)}{\lambda}$$

$$= \cancel{-\lambda \frac{d}{d\lambda}} \qquad = \frac{1!}{\lambda}$$

$$= \frac{1}{\lambda}$$

$$\cancel{\mathrm{Var}(X) = \int_{-\infty}^{\infty} \left(x - \frac{1}{\lambda}\right)^2 \lambda e^{-\lambda x}\, dx}$$

$$\mathrm{Var}(X) = E[X^2] - (E[X])^2$$

$$= \int_0^{\infty} x^2 \lambda e^{-\lambda x}\, dx - \frac{1}{\lambda^2}$$

$$= \frac{1}{\lambda^2} \int_0^{\infty} (\lambda x)^2 e^{-\lambda x}\, d(\lambda x) - \frac{1}{\lambda^2}$$

$$= \frac{\Gamma(3)}{\lambda^2} - \frac{1}{\lambda^2}$$

$$\mathrm{Var}(X) = \frac{1}{\lambda^2}$$

$$\boxed{\text{Seeing Theory}}$$

$$f_X(x) = \begin{cases} 2e^{-2x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

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$$F_X(x) = \int_{-\infty}^{x} f_X(t)\,dt = \int_0^x 2e^{-2t}\,dt$$

$$= \frac{2}{-2} e^{-2t} \Big|_0^x$$

$$F_X(x) = 1 - e^{-2x}$$

$$F_X(x) = \begin{cases} 1 - e^{-2x} & x > 0 \\ 0 & x \leq 0 \end{cases}$$

$$E[(3X+1)^2] = E[9X^2 + 6X + 1]$$

$$= 9E[X^2] + 6E[X] + 1$$

$$= 9\left(\frac{2}{2^2}\right) + 6\left(\frac{1}{2}\right) + 1$$

$$E[(3X+1)^2] = 8.5$$

Normal Random variable

$$f_X(x \mid \mu, \sigma) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right)$$

$$f_X(x \mid 0, 1) = \frac{1}{\sqrt{2\pi}} \exp\left(-x^2\right)$$ ; Standard Normal Random variable

$\mathcal{N}(\mu, \sigma^2) \rightarrow$ NoRV.

$$X \sim N(\mu,\sigma^2) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\left(\frac{x-\mu}{\sigma}\right)^2\right)$$

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$$= \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{x^2-2x\mu+\mu^2}{\sigma^2}\right)$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{x^2}{\sigma^2}\right)\exp\left(\frac{2x\mu}{\sigma^2}\right)\exp\left(\frac{-\mu^2}{\sigma^2}\right)$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \left(\exp(-x^2)\right)^{1/\sigma^2} \exp\left(\frac{2x\mu-\mu^2}{\sigma^2}\right)$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \left(\frac{\sqrt{2\pi}}{\sqrt{2\pi}}\exp(-x^2)\right)^{1/\sigma^2} \exp\left(\frac{2x\mu-\mu^2}{\sigma^2}\right)$$

$$= \frac{1}{\sigma\sqrt{2\pi}} (2\pi)^{2/\sigma^2} \left(\frac{1}{\sqrt{2\pi}}e^{-x^2}\right)^{1/\sigma^2} \exp\left(\frac{2x\mu-\mu^2}{\sigma^2}\right)$$

$$N(\mu,\sigma^2) = \frac{1}{\sigma\cancel{\sqrt{2\pi}}} (2\pi)^{2/\sigma^2-1} \left(N(0,1)\right)^{1/\sigma^2} \exp\left(\frac{2x\mu-\mu^2}{\sigma^2}\right)$$

$X \sim N(\mu,\sigma^2)$

Let $Y = \dfrac{X-\mu}{\sigma}$

$$E[Y] = \frac{1}{\sigma}\left(E[X]-\mu\right) = 0$$

$$var(Y) = var\left(\frac{X-\mu}{\sigma}\right) = \frac{1}{\sigma^2} var(X) = 1$$

13/04/2022

Conditional PDFs:

$$\cancel{f_{X|A}(x)} \quad f_{X|Y}(x,y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$$

$$\cancel{\int_B f_{X|Y}(x,y)\,dy = \int_B \frac{f_{X,Y}(x,y)}{f_Y(y)}\,dy = f_{X|B}(x)}$$

$\cancel{f_X(x|A) = \int_X f_{X|A}(x)\,dx \le \int_X f_{X|A}}$

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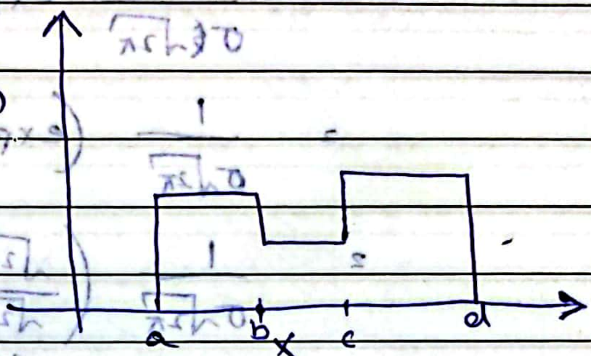
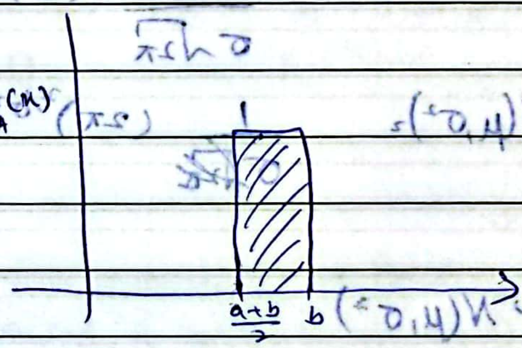
$$E[g(X)|A] = \int x f_{X|A}(x)\,dx$$

$f_X(x)$ is not defined, we only know the intervals.




A: $\frac{a+b}{2} \le X \le b$.

$$f_{X|A}(x) = \frac{f_X(x)}{P(A)}$$




The area of shaded rectangle must be 1. therefore,

$$f_{X|A}(x) = \frac{2}{b-a}$$

$$E[X|A] = \int_{\frac{a+b}{2}}^{b} x \frac{2}{b-a}\,dx = \frac{a+3b}{4}$$

$$E[X^2|A] = \int_{\frac{a+b}{2}}^{b} \frac{2}{b-a} x^2\,dx = \frac{7b^2+4ab+a^2}{12}$$

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Memorylessness of exponential PDF

Light-bulb follows exponential random variable.

T is the lifetime of Bulb.

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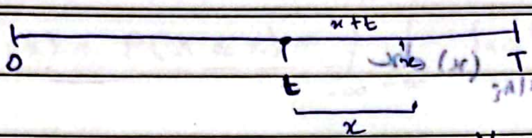


$$P(T>t) = e^{-\lambda t}$$

The remaining life time of bulb is $T-t$. Now select an arbitary point from interval ~~[t, T-t]~~ $x \in [T-t, T]$. What is the probability of lifetime / burning out of the bulb at $x$, given it was open for time $t$.

$$P(T-t > x \mid T>t) = P(T>x+t \mid T>t)$$

$$= \frac{P((T>x+t) \cap (T>t))}{P(T>t)}$$

$$= \frac{P(T>x+t)}{P(T>t)}$$

$$= \frac{e^{-\lambda(x+t)}}{e^{-\lambda t}}$$

$$= e^{-\lambda x}$$

$$\boxed{P(T>x+t \mid T>t) = P(T>x)}$$

It's like bulb ~~des~~ doesn't know it was turned on for some time t.

If $\{A_1, A_2, A_3, \dots, A_n\}$ are partition sets of $A$, and $B \subseteq \Omega$, then,

$$P(B) = \sum_{i=1}^{n} P(A_i) P(B \mid A_i)$$

$$p_X(x) = \sum_{i=1}^{n} P(A_i)\, p_{X|A_i}(x)$$

$$F_X(x) = \cancel{P(X \geq x)}\ P(x \leq X) = \sum_{i=1}^{n} P(A_i) P(X \leq x \mid A_i)$$

$$F_X(x) = \sum_{i=1}^{n} P(A_i) F_{X|A_i}(x)$$

$$\frac{d}{dx}\left(F_X(x)\right) = \sum_{i=1}^{n} P(A_i)\,\frac{d}{dx}\left(F_{X|A_i}(x)\right)$$

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$$\boxed{f_X(x) = \sum_{i=1}^{n} P(A_i)\, f_{X|A_i}(x)}$$

$$\int x\, f_X(x)\,dx = \sum_{i=1}^{n} P(A_i) \int x\, f_{X|A_i}(x)\,dx$$

$$\boxed{E[X] = \sum_{i=1}^{n} P(A_i)\, E[X|A_i]}$$

18/04/2022

Joint PDFs,

$B \subset \mathbb{R}^2$

$$P\big((X,Y)\in B\big) = \int_B f_{X,Y}(x,y)\,dx\,dy$$

~~$B \subset \mathbb{R}^2$~~ $B \subseteq \mathbb{R}^n,\ \forall n\in\mathbb{N}$

~~$P(x\in$~~ $P(B) = \int_B f_{X_1,X_2,\dots,X_n}(x_1,x_2,\dots,x_n)\,d^n x$

$X_1, X_2, \dots, X_n$ are jointly continous, if they fulfill below requirments,

1. $f_{X_1,X_2,\dots,X_n}(x_1,x_2,\dots,x_n) \geq 0,\ \forall x_i \in \mathbb{R}$

2. $\int_{\mathbb{R}^n} f_{X_1,X_2,\dots,X_n}(x_1,x_2,\dots,x_n)\,d^n x = 1$

From ~~pdf~~ joint to the marginal

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$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy$$

$$F_X(x) = P(X \leq x) = \int_{-\infty}^{x}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy\,dx$$

$$f_X(x) = P(A_1) f_{X|A_1}(x) + P(A_2) f_{X|A_2}(x) + \cdots + P(A_n) f_{X|A_n}(x)$$

$$\int_{-\infty}^{x} f_X(t)\,dt = \sum_{i=1}^{n} P(A_i) \int_{-\infty}^{x} f_{X|A_i}(t)\,dt$$

$$F_X(x) = \sum_{i=1}^{n} P(A_i) F_{X|A_i}(x)$$

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$$E[g(x_1, x_2, \cdots, x_n)] = \int_{\mathbb{R}^n} g(x_1, x_2, \cdots, x_n) f_{x_1, x_2, \ldots, x_n}(x_1, x_2, \cdots, x_n)\, d^n x$$

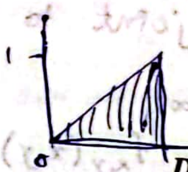
Joint CDF

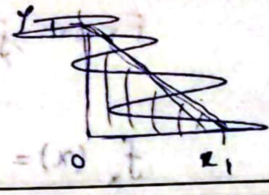
$$F_{X,Y}(x,y) = P(X \leq x,\ Y \leq y) = \int_{-\infty}^{y}\int_{-\infty}^{x} f_{X,Y}(x_1, x_2)\,dx_1\,dx_2$$

$$\frac{\partial F_{X,Y}(x,y)}{\partial y} = \int_{-\infty}^{x} f_{X,Y}(x_1, y)\,dx_1 \qquad \frac{\partial F_{X,Y}(x,y)}{\partial x} = \int_{-\infty}^{y} f_{X,Y}(x, x_2)\,dx_2$$

$$\frac{\partial^2 F_{X,Y}(x,y)}{\partial x\,\partial y} = \frac{\partial^2 F_{X,Y}(x,y)}{\partial y\,\partial x} = f_{X,Y}(x,y)$$

Conditional PDF.





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$$\boxed{f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}}$$

$$f_Y(y)\,f_{X|Y}(x|y) = f_{X,Y}(x,y)$$

$$= \frac{f_{X,Y}(x,y)}{f_X(x)}\,f_X(x)$$

$$\boxed{f_Y(y)\,f_{X|Y}(x|y) = f_X(x)\,f_{Y|X}(y|x)}$$ Continous Baye's Rule

$$P(X\in A \mid Y=y) = \int_A f_{X|Y}(x|y)\,dx$$

$$f_{X,Y}(x,y) = \begin{cases} cxy & \text{if } 0\le x\le y\le 1 \\ 0 & \text{else.} \end{cases}$$

$$\cancel{\int_0^1\int_0^x cxy\,dy\,dx = 1}$$

$f_{X|Y}(2|0.5) = 0$

$f_{X|Y}(0.5|2) = 0$

$$\cancel{c\int_0^1 \frac{xy^2}{2}\Big|_0^x dx = 1}$$

$$\int_0^1\int_0^x cxy\,dx\,dx = 1 \Rightarrow c = \frac{1}{8}$$

$$\cancel{\frac{c}{2}\int_0^1 x^3\,dx = 1}$$

$$\cancel{\frac{c}{2}\frac{x^4}{4}\Big|_0^1 = 1}$$

$$\cancel{\frac{c}{2}\frac{1}{4} = 1}$$

$$\cancel{c = 8}$$

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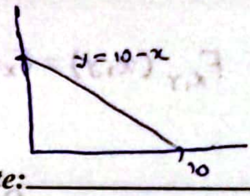
$$f_{X,Y}(x,y) = \begin{cases} 1/50 & \text{if } x,y \geq 0,\ x+y \geq 10 \\ 0 & \text{else.} \end{cases}$$






$$f_Y(y) = \int_0^{10-y} \frac{1}{50}\,dx = \frac{10-y}{50}$$

$$f_{X|Y}(x,y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} \quad \text{if } x,y \geq 0,\ x+y \geq 10$$

$$= \frac{1/50}{\frac{10-y}{50}} \qquad f_{X|Y}(x|2) = \frac{1}{8}$$

$$\boxed{f_{X|Y}(x|y) = \frac{1}{10-y}}$$

General Random Variable. 25/04/2022

Independence.

$$f_{X,Y}(x,y) = f_X(x)\, f_Y(y)$$

$$E[XY] = E[X]E[Y]$$

$$var(X+Y) = var(X) + var(Y)$$

$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{8}(1-x^2)(3-y) & \text{if } -1 \leq x,y \leq 1 \\ 0 & \text{else.} \end{cases}$$

$$F_{X,Y}(x,y) = F_X(x)\,F_Y(y) \iff f_{X,Y}(x,y) = f_X(x)\,f_Y(y)$$

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Baye's Rule.

$$f_{X|Y}(x|y) = \frac{f_X(x)\,f_{Y|X}(y|x)}{f_Y(y)}$$

$$f_Y(y) = \int f_X(x')\,f(y|x')\,dx'$$

If K is discrete and X is continous random variable.

$$f_X(x)\,p_{K|X}(k|x) = p_K(k)\,f_{X|K}(x|k)$$

$$p_{K|X}(k|x) = \frac{p_K(k)\,f_{X|K}(x|k)}{f_X(x)} \qquad f_{X|K}(x|k) = \frac{f_X(x)\,p_{K|X}(k|x)}{p_K(k)}$$

$$f_X(x) = \sum_{k'} p_K(k')\,f_{X|K}(x|k') \qquad p_K(k) = \int f_X(x')\,p_{K|X}(k|x')\,dx'$$

29/04/2022

$X$ ↑ discrete $\qquad$ $Y$ ↑ continous.

$$P(X=x,\ y \le Y \le y+\delta) = P(X=x)\,P(y \le Y \le y+\delta \mid X=x)$$

$$\approx p_X(x)\,f_{Y|X}(y|x)\,\delta$$

$$= P(y \le Y \le y+\delta)\,P(X=x \mid y \le Y \le y+\delta)$$

$$\approx f_Y(y)\,\delta\ p_{X|Y}(x|y)$$

$$p_X(x)\,f_{Y|X}(y|x) = f_Y(y)\,p_{X|Y}(x|y)$$