Assignment no. 01

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CS/PHY-314/300 Quantum Computing

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1. (10 points) Given a qubit $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$, and an operator denoted by a matrix U, prove that

   (a) (5 points) if U is unitary, then U preserves the norm of the qubit after application, i.e., $|||\psi\rangle|| = ||U|\psi\rangle||$.

Solution: By the definition of an $n \times n$ Unitary matrix,

$$U \in M_{n\times n} \iff U^{\dagger}U = I_n \tag{1}$$

Proof. We are given a unitary matrix U and a qubit $|\psi\rangle$, where unitary means that U satisfies $U^{\dagger}U = I$, where $U^{\dagger}$ is the conjugate transpose of U, and I is the identity matrix.
We want to prove that the norm of the qubit $|\psi\rangle$ remains the same after applying the unitary operator U, i.e., $||U|\psi\rangle|| = |||\psi\rangle||$.
To prove this, we'll use the properties of inner products. Consider the following steps:

$$\begin{aligned}
||U|\psi\rangle||^2 &= \langle U|\psi\rangle \, |U|\psi\rangle\rangle \quad \text{(Definition of norm)}\\
&= (U|\psi\rangle)^{\dagger} U|\psi\rangle \quad \text{(Definition of inner product)}\\
&= |\psi\rangle^{\dagger} U^{\dagger}U|\psi\rangle \quad \text{(Properties of conjugate transpose)}\\
&= |\psi\rangle^{\dagger} I|\psi\rangle \quad \text{(Unitary property of } U\text{)}\\
&= |\psi\rangle^{\dagger} |\psi\rangle \quad \text{(Identity matrix property)}\\
&= \langle\psi|\psi\rangle \quad \text{(Definition of inner product)}\\
&= |||\psi\rangle||^2 \quad \text{(Definition of norm)}
\end{aligned}$$

Taking the square root of both sides of $||U|\psi\rangle||^2 = |||\psi\rangle||^2$, we get:

$$||U|\psi\rangle|| = |||\psi\rangle||$$

This concludes the proof that a unitary operator preserves the norm of a qubit.

$\square$

   (b) (5 points) Alternately, if an operator U is applied on a qubit that preserves the norm of the qubit, then it must be unitary.

Solution:

Proof. Assume U is an operator that preserves the norm of a qubit, i.e., $||U|\psi\rangle|| = |||\psi\rangle||$. We aim to show that U is unitary.
Starting with the preservation of norm:

$$\begin{aligned}
||U|\psi\rangle|| &= |||\psi\rangle||\\
||U|\psi\rangle||^2 &= |||\psi\rangle||^2\\
(U|\psi\rangle)^{\dagger}U|\psi\rangle &= \langle\psi|\psi\rangle
\end{aligned}$$

Using properties of conjugate transpose:

$$|\psi\rangle^{\dagger} U^{\dagger}U|\psi\rangle = \langle\psi|\psi\rangle$$

Simplifying further:

$$\begin{aligned}
\langle\psi| U^{\dagger}U|\psi\rangle &= \langle\psi|\psi\rangle\\
|\psi\rangle\langle\psi| U^{\dagger}U|\psi\rangle &= |\psi\rangle\langle\psi|\psi\rangle
\end{aligned}$$

Finally, we arrive at the conclusion:

$$U^\dagger U |\psi\rangle = |\psi\rangle$$
$$U^\dagger U = I_n$$

Thus, we have shown that if an operator preserves the norm of a qubit, it must be unitary. $\square$

2. (10 points) Consider a Bloch Sphere representation of a qubit $|\psi\rangle = \cos(\theta/2)|0\rangle + e^{(i\phi)}\sin(\theta/2)|1\rangle$ as shown here.

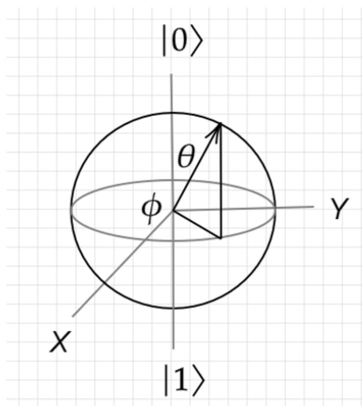
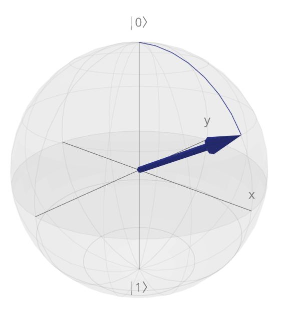


Figure 1: Bloch sphere representation of an arbitrary qubit state $|\psi\rangle = \cos(\theta/2)|0\rangle + e^{(i\phi)}\sin(\theta/2)|1\rangle$

On the Bloch Sphere, plot the following qubits (show a plot similar to the diagram above):

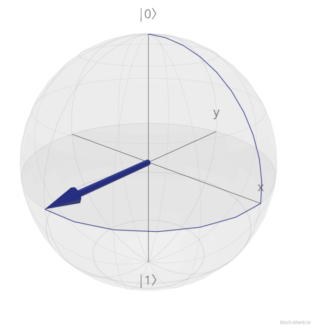
(a) (2 points) $\frac{\sqrt{3}}{2}|0\rangle + \frac{1}{2}|1\rangle$

Solution:



(b) (3 points) $\frac{1+i}{2}|0\rangle + \frac{1-i}{2}|1\rangle$

Solution:



Next, consider the following operations:

(i) $R(\theta_1) = \begin{bmatrix} \cos(\theta_1) & -\sin(\theta_1) \\ \sin(\theta_1) & \cos(\theta_1) \end{bmatrix}$

(ii) $S(\phi_2) = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\phi_2} \end{bmatrix}$

Given an arbitrary qubit on the X-Z plane (i.e.) its amplitudes have no imaginary component,

(a) (2 points) what is the effect of applying $R(\theta_1)$ to it? In particular state the effect when $\theta_1 = \pi/2$?

Solution: An arbitrary qubit can be expressed without an imaginary component as follows:

$$|\psi\rangle = \cos\left(\frac{\theta}{2}\right)|0\rangle + \sin\left(\frac{\theta}{2}\right)|1\rangle$$

Now, let's apply the $R_1(\theta)$ operation to it:

$$\begin{aligned} R(\theta_1)|\psi\rangle &= \begin{bmatrix} \cos(\theta_1) & -\sin(\theta_1) \\ \sin(\theta_1) & \cos(\theta_1) \end{bmatrix} \left(\cos\left(\frac{\theta}{2}\right)|0\rangle + \sin\left(\frac{\theta}{2}\right)|1\rangle\right) \\ &= \left(\cos(\theta_1)\cos\left(\frac{\theta}{2}\right) - \sin(\theta_1)\sin\left(\frac{\theta}{2}\right)\right)|0\rangle \\ &\quad + \left(\sin(\theta_1)\cos\left(\frac{\theta}{2}\right) + \cos(\theta_1)\sin\left(\frac{\theta}{2}\right)\right)|1\rangle \\ &= \cos\left(\theta_1 + \frac{\theta}{2}\right)|0\rangle + \sin\left(\theta_1 + \frac{\theta}{2}\right)|1\rangle \end{aligned}$$

This result demonstrates that the qubit has undergone a rotation by θ_1 around the y-axis. Notably, when $\theta_1 = \frac{\pi}{2}$, the qubit will swap probabilities.

(b) (3 points) what is the effect of applying $S(\phi_2)$ to it? In particular state the effect when $\phi_2 = \pi/2$?

Solution: Now, let's apply $S(\phi_2)$:

$$\begin{aligned} S(\phi_2)|\psi\rangle &= \begin{bmatrix} 1 & 0 \\ 0 & e^{i\phi_2} \end{bmatrix} \left(\cos\left(\frac{\theta}{2}\right)|0\rangle + \sin\left(\frac{\theta}{2}\right)|1\rangle\right) \\ &= \cos\left(\frac{\theta}{2}\right)|0\rangle + e^{i\phi_2}\sin\left(\frac{\theta}{2}\right)|1\rangle \end{aligned}$$

This result demonstrates that the qubit undergoes a rotation by ϕ_2 around the z-axis. Notably, when $\phi_2 = \frac{\pi}{2}$, the entire qubit undergoes a rotation of 90 around the z-axis.

3. (10 points) So far we have studied qubits defined using two basis states: $|0\rangle$ and $|1\rangle$. We have also seen two related states: the $|+\rangle$ state and the $|-\rangle$ state. Answer the following questions with reasons:

(a) (2 points) Do the $|+\rangle$ state and the $|-\rangle$ state form a pair of orthonormal states?

Solution: To prove the aforementioned statement, we will start by proving both $|+\rangle$ and $|-\rangle$ are normal and will then prove that they are orthogonal by showing that their inner product equals zero.

$$\| \, |+\rangle \, \| = \left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 = 1$$
$$\| \, |-\rangle \, \| = \left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{-1}{\sqrt{2}}\right)^2 = 1$$

Hence proved that both of these are normal.

$$\begin{aligned}
\langle +|-\rangle &= \left(\frac{1}{\sqrt{2}}\langle 0| + \frac{1}{\sqrt{2}}\langle 1|\right)\left(\frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle\right) \\
&= \frac{1}{2}\langle 0|0\rangle - \frac{1}{2}\langle 0|1\rangle + \frac{1}{2}\langle 1|0\rangle - \frac{1}{2}\langle 1|1\rangle \\
&= \frac{1}{2} - 0 + 0 + \frac{1}{2} \\
&= 0
\end{aligned}$$

Hence proved that $|+\rangle$ and $|-\rangle$ are orthonormal by being normal and orthogonal.

(b) (2 points) Can the $|+\rangle$ state and the $|-\rangle$ state be used as the basis states?

Solution: In order for some vectors to qualify for basis vectors, they must be linear independent so yes they can be used as basis states since neither of them can be obtained by using a linear combination of the other.

(c) If the answer to both (a) and (b) is Yes, express the following qubits as a linear combination of the $|+\rangle$ state and the $|-\rangle$ state:

i. (2 points) $|0\rangle$,

Solution:
$$|0\rangle = \frac{1}{\sqrt{2}}|+\rangle + \frac{1}{\sqrt{2}}|-\rangle$$

ii. (2 points) $|1\rangle$,

Solution:
$$|1\rangle = \frac{1}{\sqrt{2}}|+\rangle - \frac{1}{\sqrt{2}}|-\rangle$$

iii. (2 points) an arbitrary qubit: $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$.

Solution:

$$|\psi\rangle = (\alpha + \beta)\frac{1}{\sqrt{2}}\,|+\rangle + (\alpha - \beta)\frac{1}{\sqrt{2}}\,|-\rangle$$

4. (10 points) The Pauli Matrices are given by:

$$X = \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$
$$Y = \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$
$$Z = \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Show that the following relations hold; where H is the Hadamard operation.

(a) (3 points) $HX = ZH$,

Solution: Let us proof that $HX = ZH$,

Proof.

$$HX = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$
$$HX = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

Lets refactor it to get what we want:

$$HX = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

Now, since

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

and

$$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

We proved that

$$HX = ZH$$

□

(b) (3 points) $HY = -YH$,

Solution: We would use the same technique to proof this as we did earlier:

Proof.

$$HY = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$
$$HX = \frac{1}{\sqrt{2}} \begin{pmatrix} i & -i \\ -i & -i \end{pmatrix}$$

Lets refactor it to get what we want:

$$HX = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

Now, since

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

and

$$Y = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$$

so

$$-Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

We proved that

$$HY = -YH$$

□

(c) (3 points) $HZ = XH$.

Solution:

Proof. Since we know that

$$HX = ZH$$

$$\begin{aligned} HX &= ZH \\ H^{-1}HX &= H^{-1}ZH \\ XH &= H^{-1}ZHH \end{aligned}$$

Now Suppose for the time being if $H^{-1} = H$, then we can prove that:

$$\begin{aligned} XH &= HZHH^{-1} \\ XH &= HZ \end{aligned}$$

Therefore, we now need to show that $H = H^{-1}$, we can show that $H = H^{\dagger}$ which in turn proves that $H = H^{-1}$;

$$H^{\dagger} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}^{\dagger}$$

After taking the conjugate and applying transpose function, it remains same i-e;

$$\begin{aligned} H^{\dagger} &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \\ H^{\dagger} &= H \end{aligned}$$

Hence we proved that

$$HZ = XH$$

□

(d) (1 point) Using these properties, show that $HXHYHZ = ZYXH$.

Solution:

Proof.

$$HXHYHZ = (HX)HY(HZ)$$
$$HXHYHZ = (ZH)HY(XH)$$
$$HXHYHZ = ZH^{-1}HYXH$$
$$HXHYHZ = ZYXH$$

$\square$

5. (10 points) We have seen earlier that if we are given one of the two quantum states $|+\rangle$ and $|-\rangle$ such that it is not known whether it is a $|+\rangle$ and $|-\rangle$, we can distinguish between them *perfectly* by applying a Hadamard operation followed by measurement.

   Suppose we are randomly sent one of the following two states:

   (a) $\frac{1}{2}|0\rangle + \frac{\sqrt{3}}{2}|1\rangle$

   (b) $\frac{\sqrt{3}}{2}|0\rangle - \frac{1}{2}|1\rangle$

   Devise a procedure to find out which one of the two states has been sent that performs better than a random choice. More specifically, given one of the two qubits, by performing your operation(s) followed by a measurement, the probability of finding out which one of the states is greater than random guessing i.e. greater than 1/2.

Solution: Consider two given qubits:

$$|\psi_1\rangle = \frac{1}{2}|0\rangle + \frac{\sqrt{3}}{2}|1\rangle$$

$$|\psi_2\rangle = \frac{\sqrt{3}}{2}|0\rangle - \frac{1}{2}|1\rangle$$

We aim to construct a matrix, denoted as M, that can distinguish between $|\psi\rangle$ and $|\phi\rangle$. This matrix M is designed such that $M|\psi\rangle = |0\rangle$ and $M|\phi\rangle = |1\rangle$.

Let

$$M = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}$$

Now, we want to satisfy the following equations:

$$M|\psi_1\rangle = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \begin{pmatrix} 1/2 \\ \sqrt{3}/2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = |0\rangle$$

$$M|\psi_2\rangle = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \begin{pmatrix} \sqrt{3}/2 \\ -1/2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = |1\rangle$$

This leads to the following system of equations:

$$x_{11}\frac{1}{2} + x_{12}\frac{\sqrt{3}}{2} = 1, \quad x_{21}\frac{1}{2} + x_{22}\frac{\sqrt{3}}{2} = 0, \quad x_{11}\frac{\sqrt{3}}{2} - x_{12}\frac{1}{2} = 0, \quad x_{21}\frac{\sqrt{3}}{2} - x_{22}\frac{1}{2} = 1$$

Solving this system of linear equations simultaneously, we obtain the unique solution:

$$x_{11} = \frac{1}{2}, \quad x_{12} = \frac{\sqrt{3}}{2}, \quad x_{21} = \frac{\sqrt{3}}{2}, \quad x_{22} = -\frac{1}{2}$$

Therefore, our matrix M is:

$$M = \begin{pmatrix} 1/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{pmatrix}$$

We can verify that M satisfies the desired conditions:

$$M\left|\psi\right\rangle = \begin{pmatrix} 1/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{pmatrix} \begin{pmatrix} 1/2 \\ \sqrt{3}/2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \left|0\right\rangle$$

$$M\left|\phi\right\rangle = \begin{pmatrix} 1/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{pmatrix} \begin{pmatrix} \sqrt{3}/2 \\ -1/2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \left|1\right\rangle$$