Real Analysis

Date: 24/08/2022

M	T	W	T	F	S	S

A	$\neg A$
T	F
F	T

A	B	$A \vee B$	$A \wedge B$	$A \Rightarrow B$
T	T	T	T	T
T	F	T	F	F
F	T	T	F	T
F	F	F	F	T

$A \Rightarrow B$ is read as,

1. B can not be false, if A is true.
2. If A then B.
3. A only if B.
4. A is sufficent for B.
5. B is mecessary for A.
6. B whenever A.

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- Tautology is a statement, that is always True. $P \vee \neg P$
- Contradictions is a statement that is always False. $P \wedge \neg P$
- Equivalent statements, P and Q are equivalent when they have same truth value.

$$(P \Rightarrow Q) \wedge (\neg P \Rightarrow \neg Q)$$

$$(P \Rightarrow Q) \wedge (Q \Rightarrow P)$$

$$P \Leftrightarrow Q$$

↑ Equivalent symbol.

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Contrapositive :-

$$(P \Rightarrow Q) \Leftrightarrow (\neg Q \Rightarrow \neg P)$$

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$P \Rightarrow Q$ Modus Ponens

$\underline{P}$

Q

$P \Rightarrow Q$ Transitivity of Implication. / Hypothetical syllogism.

$\underline{Q \Rightarrow R}$

$P \Rightarrow R$

$P \Rightarrow Q$ Modus Tollens.

$\underline{\neg Q}$

$\neg P$

$$\big((P \Rightarrow Q) \wedge (Q \Rightarrow R)\big) \Rightarrow (P \Rightarrow R)$$

Proof:-

$(P \Rightarrow Q) \wedge (Q \Rightarrow R)$

$\Leftrightarrow (\neg P \vee Q) \wedge (\neg Q \vee R)$

$\Leftrightarrow [\neg P \wedge (\neg Q \vee R)] \vee [Q \wedge (\neg Q \vee R)]$

$\Leftrightarrow (\neg P \wedge \neg Q) \vee (\neg P \wedge R) \vee (Q \wedge \neg Q) \vee (Q \wedge R)$

$\Leftrightarrow (\neg P \wedge \neg Q) \vee (\neg P \wedge R) \vee (Q \wedge R)$

$\Leftrightarrow (\neg P \wedge \neg Q) \vee (Q \wedge R) \vee (\neg P \wedge R)$

$\Leftrightarrow \big([\neg P \vee (Q \wedge R)] \wedge [\neg Q \vee (Q \wedge R)]\big) \vee (\neg P \wedge R)$

$\Leftrightarrow [\underbrace{(\neg P \vee Q)}_{T} \wedge (\neg P \vee R) \wedge \underbrace{(\neg Q \vee Q)}_{T} \wedge \underbrace{(\neg Q \vee R)}_{T}] \vee (\neg P \wedge R)$

$\Leftrightarrow [T \wedge (\neg P \vee R) \wedge T \wedge T] \vee (\neg P \wedge R)$

$\Leftrightarrow (\neg P \vee R) \vee (\neg P \wedge R)$

$\Leftrightarrow R \vee \big((\neg P \vee \neg P) \wedge (\neg P \vee R)\big)$

$\Leftrightarrow R \vee (\neg P \wedge (\neg P \vee R))$

$\Leftrightarrow (\neg P \vee R) \wedge (R \vee (\neg P \vee R)) \Leftrightarrow (\neg P \vee R) \vee (\neg P \vee R)$

$\Leftrightarrow \cancel{(\neg P \vee R) (\vee (T \wedge R)} \Leftrightarrow \neg P \vee R$

$\Leftrightarrow \cancel{(\neg P \vee R)} \Leftrightarrow P \Rightarrow R$

Proof by contradiction

To prove if P then Q

ie. $(P \Rightarrow Q) \Longleftrightarrow (\neg P \vee Q) \Longleftrightarrow \neg(P \wedge \neg Q)$

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Proof by Disjoint conclusion.

$$\left(P \Rightarrow Q \vee R\right) \Leftrightarrow \left((P \wedge \neg Q) \Rightarrow R\right)$$

- Logic with $\neg, \wedge, \vee, \Rightarrow$ is called zeroth order logic or propositional logic.
- Logic with quantifiers is called first order logic or, Predicate logic.

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$$\neg(\forall x \in U, P(x)) = \exists x \in U \ni \neg P(x)$$

$$\neg(\exists x \in U \ni P(x)) = \forall x \in U, \neg P(x)$$

Statement:

$P \Rightarrow Q$

Contrapositive: $\neg Q \Rightarrow \neg P$

Converse: $Q \Rightarrow P$

Inverse: $\neg P \Rightarrow \neg Q$

statement,

$\forall x > 1, \; x^2 > 1$

Proof:-

Let $x > 1 \Rightarrow x > 0$

$x^2 > x > 1$

$x^2 > 1$

~~One to fn~~ One-to-One function:-/Injective $f : X \longrightarrow Y$

$\forall a, b \in X, \; f(a) = f(b) \Leftrightarrow a = b$

Onto function:- / surjective.

$\forall b \in Y, \exists a \in X \ni f(a) = b$

Bijective / one to one - onto.

Injective $\wedge$ surjective.

Two set have same cardinality iff there exits a bijective function between them.

Cardinality:–

It's a property shared b/w sets.

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Thm | B/w any two non-empty sets their exists a function.

Proof:–

Let A, B some sets, we can make a function $f: A \to B$ such that $\forall a \in A$; $f(a) = b$ for some $b \in B$. $\square$

Thm | B/w any two ~~sets~~ non-empty sets their exists ~~either~~ an injective or surjective function.

Proof (by contradiction):–

Let A, B two sets and Let $f: A \to B$ be a such function that f is neither injective,

i.e. $\exists a_i, a_j \in A \ni (f(a_1) = f(a_2)) \wedge (a_i \neq a_2)$

nor surjective,

$\exists b \in B \ni \forall a \in A, f(a) \neq b.$

Now make a new function g

$$g = \begin{cases} g(a) = f(a) & \forall a \neq a_2 \\ g(a_2) = b \end{cases}$$

Keep modifying this untill you can't. i.e. you got either injective or surjective function. $\square$

~~|A|=|B|~~ For two sets A and B.

~~$\nexists f: A \to B$~~ $\exists f : A \to B \ni f$ is ~~in~~ bijective $\Leftrightarrow |A| = |B|$

$\nexists f : A \to B \ni f$ is ~~not~~ surjective $\Leftrightarrow |A| < |B|$

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$\nexists f : A \to B \ni f$ is injective $\Rightarrow |A| > |B|$

Equivalence Relation ($\sim$) on set A.

$\sim \subseteq A \times A \ni \forall a, b, c \in A$.

$a \sim a$ Reflexive

$a \sim b \Leftrightarrow b \sim a$ symmetric

$(a \sim b) \wedge (b \sim c) \Rightarrow a \sim c$ Transitive.

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Let $\mathbb{F}$ be a set with two operations $(+, \times)$ defined on it, such that, ~~$a, b, c \in \mathbb{F}$,~~

1. $a + b \in \mathbb{F} \quad \forall a, b \in \mathbb{F}$
2. $a \times b \in \mathbb{F} \quad \forall a, b \in \mathbb{F}$

} Closure Property.

3. $a + b = b + a \quad \forall a, b \in \mathbb{F}$
4. $(a+b)+c = a+(b+c) = a+b+c \quad \forall a, b, c \in \mathbb{F}$
5. $\exists 0 \in \mathbb{F} \ni a + 0 = a \quad \forall a \in \mathbb{F}$
6. $\forall a \in \mathbb{F}, \exists -a \in \mathbb{F} \ni a + (-a) = 0$

7. $a \times b = b \times a \quad \forall a, b \in \mathbb{F}$
8. $(a \times b) \times c = a \times (b \times c) = a \times b \times c \quad \forall a, b, c \in \mathbb{F}$
9. $\exists 1 \in \mathbb{F} \ni a \times 1 = a \quad \forall a \in \mathbb{F}$
10. $\forall a \neq 0 \in \mathbb{F}, \exists a^{-1} \in \mathbb{F} \ni a \times a^{-1} = 1$

11. $\forall a, b, c \in \mathbb{F}, \quad a \times (b + c) = (a \times b) + (a \times c)$

~~F is called the~~ Then $(\mathbb{F}, +, \times)$ is called field.

Prove:

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Theorem:– If $x^2 = x$ in any field then $x = 0$ and $x = 1$

Proof:–

~~Let $x \in F$, $\exists\, 0 \in F$~~

~~$x + 0 = x$~~

~~we can multiply x on both side.~~

~~$x \times (x + 0) = x \times x$~~

~~$x \times x + x \times 0 = x \times x$~~

~~$x^2 + 0 = x^2$~~

~~$x^2 + 0 + (-x^2) = x^2 + (-x^2)$~~

~~$x + 0 = x$~~

~~$x \times (x + 0) = x \times x$~~

~~$x \times x + x \times 0 = x^2$~~

~~$x^2 + 0 = x^2$~~

~~$a \times (b + c) = a \times b + a \times c$~~

~~$= a \times b + (a \times -b)$~~

~~$b + c = 0$~~

~~$c = -b$~~

$(-1) \times x = -x$

$x + (-1) \times x = x - x$

$x \times 1 + (-1) \times x = 0$

$x \times (1 + 1) = 0$

$x \times 0 = 0$

$0 = 0$

$x + (-x) + (-1)x = -x$

$x \times 1 + (-1) \times x = -x$

$$(-x)^2 = (-x)\times(-x)$$
$$= (-1\times x)\times(-1\times x)$$
$$= \underline{(-1\times -1)}\times x\times x$$

$$-1 = -1$$

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$$1+(-1)=0$$
$$(-1)\times(1+(-1)) = 0\times(-1)$$
$$(-1\times 1)+(-1)^2 = 0$$
$$-1+(-1)^2 = 0$$
$$1-1+(-1)^2 = 0+1$$
$$0+(-1)^2 = 1$$
$$(-1)^2 = 1$$

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Ordering on sets:

A partially ordered set $P=(S,\preccurlyeq)$ is a sub set of $S\times S$ such that,

A relation R on set S is $R\subseteq S\times S$.

$\forall s\in S, (s,s)\in P$ — $\forall s\in S, s\preccurlyeq s$

$(s,t)\in P \wedge (t,s)\in P \Rightarrow s=t$ — $s\preccurlyeq t \wedge t\preccurlyeq s \Rightarrow t=s$

$(s,t)\in P \wedge (t,u)\in P \Rightarrow (s,u)\in P$ — $s\preccurlyeq t \wedge t\preccurlyeq u \Rightarrow s\preccurlyeq u$

It is also called Linear Order

A total order on a set S is a partial order such that $\forall s,t\in S$ we have either,

1. $s \prec t$
2. $s = t$
3. $s \succ t$

} Trichotomy.

A set with a total order defined over it is called totally ordered set.

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If S is a totally ordered set, and $T \subseteq S$, we call $t \in T$ the least element of T if $\forall s \in T$, $t \leq s$.

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Total order on $\mathbb{N}$.

If $m, n \in \mathbb{N}$, we say $m < n$ iff

① If n is an element of form $m+1, m+1+1, m+1+1+1, \dots$

or

② No function from a set of m elements to a set of n elements is surjective.

or

③ No function from a set n elements to a set of m elements is injective.

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Well Ordering Principle

A totally ordered set S, is called Well Ordered if, for any non-empty subset $T \subseteq S$, T has a least elements.

Non-example: $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{Q}^+$

Example: Every finite set with a total order defined over it.

~~Is $\mathbb{N}$~~

Is $\mathbb{N}$ is well order? No we can not prove this. We axiomatically say that $\mathbb{N}$ is well ordered.

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Principle of Induction

Let $P(n)$ ~~is true~~ means that some property is true for n^{th} integer.

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If 1. $P(1)$ is true.

2. $P(k) \Rightarrow P(k+1)$
is true is true.

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Principle of Strong Induction.

1. $P(1)$
2. $P(1) \wedge P(2) \wedge \dots \wedge P(k) \Rightarrow P(k+1)$
3. $P(n) \quad \forall n \in \mathbb{N}$

Order on a field $\mathbb{F}$

compare elements to $0 \in \mathbb{F}$ $a<b$ iff $b-a>0$

A non-empty set $\mathbb{P} \subseteq \mathbb{F}$ is called the set of positives if

1. $\forall a, b \in \mathbb{P}$, $a+b \in \mathbb{P} \wedge ab \in \mathbb{P}$
2. $\forall a \in \mathbb{F}$, $(a \in \mathbb{P}) \Leftrightarrow (a=0) \Leftrightarrow (-a \in \mathbb{P})$

Then the pair $(\mathbb{F}, \mathbb{P})$ is an ordered field, and $\forall a, b \in \mathbb{F}$ $a<b$ iff $b-a \in \mathbb{P}$

If $a \in \mathbb{P}$, we call a positive element of $\mathbb{F}$, and if $-a \in \mathbb{P}$ we call a negative ~~number~~ element of $\mathbb{F}$.

Thm: In any ordered field $(\mathbb{F}, \mathbb{P})$ $\forall a \in \mathbb{F}$, $a^2 \in \mathbb{P}$.

Corollary: In $(\mathbb{F}, \mathbb{P})$, $1 \in \mathbb{P}$

Thm: In $(\mathbb{F}, \mathbb{P})$, the product of a positive and a negative element is negative.

Proof: Let $a \in \mathbb{P}$ & $b \notin \mathbb{P} \Rightarrow -b \in \mathbb{P}$

$a(-b) \in \mathbb{P} \Rightarrow -(ab) \in \mathbb{P} \Rightarrow ab \notin \mathbb{P}$

Corollary: $\forall a \in \mathbb{P}$, $a^{-1} \in \mathbb{P}$:

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Thm: In $(\mathbb{F}, \mathbb{P})$

1. If $a<b$, then $\forall c \in \mathbb{F}$, $a+c < b+c$
2. $a<b \wedge \forall c \in \mathbb{P} \Rightarrow ac < bc$
3. $a<b \wedge \forall c \notin \mathbb{P} \Rightarrow bc < ac$
4. $a<b \Rightarrow a < \frac{a+b}{2} < b$
5. $a<b \wedge b<c \Rightarrow a<c$

1\. Proof: $a<b \Rightarrow \forall c \in F, a+c<b+c$

$b-a>0$

$\Rightarrow b-a+0>0$

$b-a+c-c>0$

$(b+c)-a-c>0$

$(b+c)-(a+c)>0$

$a+c<b+c$

2\. Proof. $a<b \wedge \forall c \in \mathbb{P} \Rightarrow ac<bc$

$(b-a) \in \mathbb{P} \wedge c \in \mathbb{P}$

$(b-a)c \in \mathbb{P} \Rightarrow (bc-ac) \in \mathbb{P} \Rightarrow ac<bc.$

3\. Proof: $a<b \wedge \forall c \notin \mathbb{P} \Rightarrow bc<ac.$

$(b-a) \in \mathbb{P}$, $c \notin \mathbb{P} \Rightarrow -c \in \mathbb{P}.$

$(b-a)(-c) \in \mathbb{P}$

$(ac-bc) \in \mathbb{P}$

$\Rightarrow bc<ac$

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Thm: In $(\mathbb{F}, \mathbb{P})$ the elements, $1, 1+1+1, 1+1+1, \dots$, all belongs to $\mathbb{P}$ are distinct.

This tells us any ordered field is always infinite, and there is a copy of $\mathbb{N}$ lying in the field, as well as of $\mathbb{Z}$ that means, there is a copy of $\mathbb{Q}$. This means $\mathbb{Q}$ is the smallest ordered field.

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Proof:

Let,

$$\underbrace{1+1+1+\dots+1}_{k \in \mathbb{N}} = \sum_{i=1}^{k} 1$$

By induction, $1 \in \mathbb{P}$ proved earlier. (Base case.)

By hypothesis, assume $\sum_{i=1}^{k} 1 \in \mathbb{P}$ for some $k \in \mathbb{N}$.

$$\sum_{i=1}^{k+1} 1 = \underbrace{\sum_{i=1}^{k} 1}_{\substack{\in \mathbb{P} \\ \text{(By Hypothesis)}}} + \underbrace{1}_{\substack{\in \mathbb{P} \\ \text{(By Base case)}}} \in \mathbb{P} \quad \text{(By closure).}$$

Thm: All elements in $(\mathbb{F}, \mathbb{P})$ are distinct.

$$\sum_{i=1}^{k} 1 \neq \sum_{i=1}^{l} 1 \iff k \neq l$$

we will prove,

$$\sum_{i=1}^{k} 1 = \sum_{i=1}^{l} 1 \iff k = l.$$

Proof:

$$\sum_{i=1}^{k} 1 = \sum_{i=1}^{l} 1 \iff \sum_{i=1}^{k} 1 - \sum_{i=1}^{l} 1 = 0$$

$$\iff k = l$$

and,

$$\sum_{i=1}^{k} 1 \neq \sum_{i=1}^{l} 1 \iff k \neq l.$$

In an ordered field ~~we have~~ (IF, IP) we have a function;

$$|x| = \begin{cases} x & \text{if } x = 0 \lor x \in IP \\ -x & \text{if } -x \in IP \end{cases}$$

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we call this absolute value of x. By definition of absolute value: $x \leq |x|$

Thm: The triangle inequality.

In (IF, IP), $\forall x, y \in F$

$$\big||x| - |y|\big| \leq |x+y| \leq |x| + |y|$$

Proof:—

$$(|x+y|)^2 = (x+y)^2 \cancel{= x^2 + y^2 + 2xy}$$

$$= (x+y)(x+y)$$

$$= xx + xy + yx + yy$$

$$= x^2 + xy + xy + y^2$$

$$= x^2 + 2xy + y^2$$

$$\leq x^2 + 2|x||y| + y^2$$

$$\leq = |x|^2 + 2|x||y| + |y|^2$$

$$\cancel{(|x+y|)^2} \leq (|x| + |y|)^2$$

$$(|x+y|)^2 \leq (|x| + |y|)^2$$

$$|x+y| \leq |x| + |y|$$

$$\cancel{(|x+y|)^2 = x^2+2xy+y^2}$$

Measure Theory parhni hai

$$\cancel{(|x+y|)^2 = (|x|-|y|)^2 + 4xy}$$

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$$\cancel{||x|-|y|| \leq |x+y|}$$

In $(\mathbb{F}, \mathbb{P})$ $\forall x, y \in \mathbb{F}$ we define distance b/w x and y as,

$$|x-y| = |y-x|$$

$\forall x, y, z \in (\mathbb{F}, \mathbb{P})$, $|x-y| \leq |x-z| + |z-y|$, shortest path b/w two points is the direct path b/w them.

In $(\mathbb{F}, \mathbb{P})$, a metric is a two variable function $d(x, y)$ $\ni$

1. $\forall x, y \in \mathbb{F}$, $d(x,y) \geq 0$
2. $d(x,y) = 0 \iff x = y$
3. $\forall x, y, z \in \mathbb{F}$, $d(x,y) \leq d(x,z) + d(z,x)$

In ~~$(\mathbb{F}, \mathbb{P})$~~ any totally ordered set S we define intervals as,

$\forall a, b \in \mathbb{F}$

1. $(a, b) := \{x \mid x \in S \land a < x < b\}$
2. $(a, b] := \{x \mid x \in S \land a < x \leq b\}$
3. $[a, b) := \{x \mid x \in S \land a \leq x < b\}$
4. $[a, b] := \{x \mid x \in S \land a \leq x \leq b\}$
5. $(a, \infty) := \{x \mid x \in S \land a < x\}$
6. $(-\infty, b) := \{x \mid x \in S \land x < b\}$ — Open Rays.
7. $[a, \infty) := \{x \mid x \in S \land a \leq x\}$
8. $(-\infty, b] := \{x \mid x \in S \land x \leq b\}$
9. $(-\infty, \infty) :=$ ~~$(\mathbb{F}, \mathbb{P})$~~ S — Closed Rays.

Rays.

Thm: If $x, y \in (a,b) \subseteq (\mathbb{F}, \mathbb{P})$, then $|x-y| < b-a$.

Proof:

$$a < x < b$$
$$a < y < b$$
$$-b < -y < -a$$
$$a-b < x-y < b-a$$
$$-(b-a) < x-y < b-a$$
$$|x-y| < b-a$$

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Neighbourhood:

In $(\mathbb{F}, \mathbb{P})$, Let $(a,b) := \{x \mid x \in \mathbb{F} \wedge a < x < b\}$, then let $c = \frac{a+b}{2}$ (mid point of a,b). and $\epsilon = \frac{b-a}{2}$

$$(a,b) = \{x \mid x \in \mathbb{F} \wedge |x-c| < \epsilon\}$$

Proof:, $\forall \sigma \in \{x \mid x \in \mathbb{F} \wedge |x-c| < \epsilon\} \Leftrightarrow$

$$|\sigma - c| < \epsilon$$
$$\Leftrightarrow \left|\sigma - \frac{a+b}{2}\right| < \frac{b-a}{2}$$
$$\Leftrightarrow -\frac{b-a}{2} < \sigma - \frac{a+b}{2} < \frac{b-a}{2}$$
$$\Leftrightarrow \frac{-\cancel{b}+a}{2} + \frac{a+\cancel{b}}{2} < \sigma < \frac{b-\cancel{a}}{2} + \frac{\cancel{a}+b}{2}$$
$$\Leftrightarrow a < \sigma < b$$
$$\Leftrightarrow \sigma \in \{x \mid x \in \mathbb{F} \wedge a < x < b\}$$

Definition:-

Let $c \in (\mathbb{F}, \mathbb{P})$, $\varepsilon > 0, \varepsilon \in \mathbb{F}$, then the ε-neighbourhood of c is defined to be the following interval,

$$\mathcal{N}_\varepsilon(c) := \{x \mid x \in \mathbb{F} \wedge |x-c| < \varepsilon\}$$

Definition:—

A neighbourhood of $c \in (\mathbb{F}, \mathbb{P})$ is any set $S \subseteq \mathbb{F} \ni S$ contains some ϵ-neighbourhood of c. S is a neighbourhood of c iff $\exists\, \varepsilon > 0 \ni \mathcal{N}_\varepsilon(c) \subseteq S$.

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Example: For $x \in (a,b)$ give an $\varepsilon > 0 \ni \mathcal{N}_\varepsilon(x) \subseteq S$.

Proof:—

Let $\varepsilon = \arg\min\{x-a,\ b-x\}$

$$\begin{aligned}\mathcal{N}_\varepsilon(x) &= \{y \mid y \in \mathbb{F} \wedge |y-x| < \varepsilon\} \\ &= \{y \mid y \in \mathbb{F} \wedge x-\varepsilon < y < x+\varepsilon\} \\ &\subseteq \{y \mid y \in \mathbb{F} \wedge a < y < b\} \\ &= (a,b)\end{aligned}$$

S a neighbourhood of $x \Rightarrow x \in S$

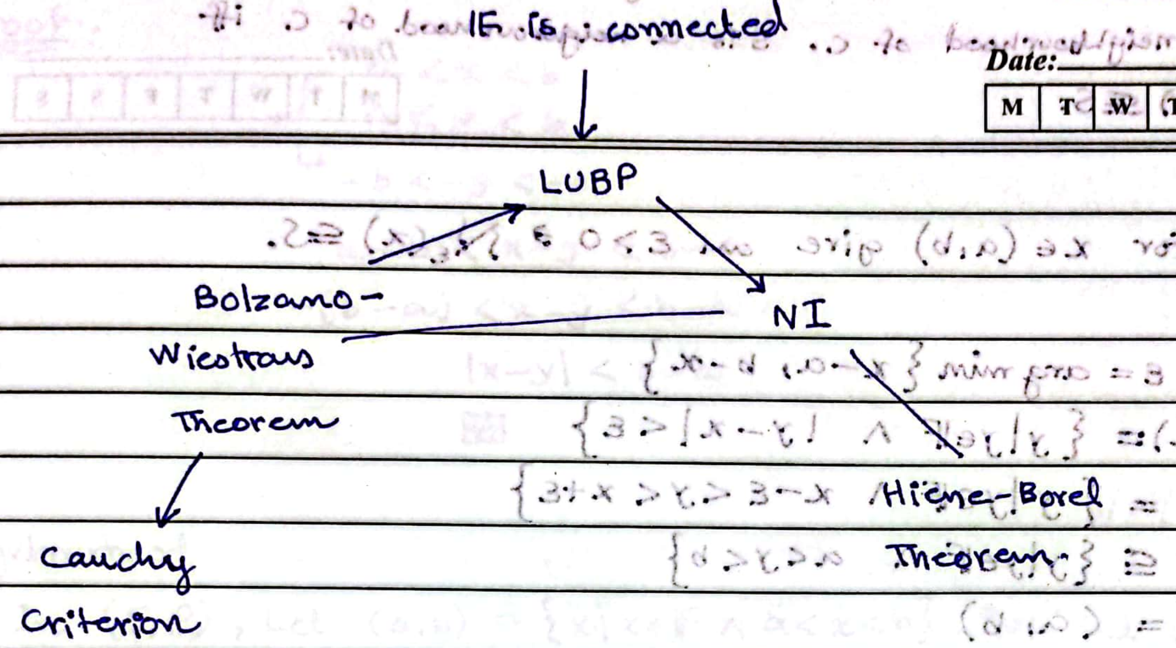
HW: Theorem: An open interval is a neighbourhood of each element of the interval.

—: THE BIG THEOREM :—

In any ordered field $(\mathbb{F}, \mathbb{P})$ the following are equivalent.

1. $\mathbb{F}$ has the least upper bound property.
2. $\mathbb{F}$ has Archimedean Property and Nested Interval Property.
3. The Bolzano-Weiestrass Theorem holds in $\mathbb{F}$.
4. The Heine-Boral Theorem holds in $\mathbb{F}$.
5. The Cauchy Criterion holds in $\mathbb{F}$.
6. $\mathbb{F}$ is connected.

If $\mathbb{F}$ follows at least one of them, it follows all of them. We can say that $\mathbb{F}$ is complete or in other words $\mathbb{F}$ has no holes. Means infinestimally small exists and we can do calculus on Real Numbers.



Upper Bound and Lower Bound.

In ordered set S let $T \subset S$, the $u \in S$ if it exist is called an upperbound of T if $\forall t \in T$, $t \leq u$. Then $l \in S$ is called lower bound of T if $\forall t \in T$, $l \leq t$.

A least upper bound of T is $u \in S$, if

i. u is an upper-bound i.e. $\forall t \in T$, $t \leq u$

ii. for every upper bound u' of T, $u \leq u'$

Then, we write $\sup(T) = u$ sup := supermum

A ~~least~~ greatest lower bound of T is $l \in S$, if

i. l is an ~~uppe~~ lower-bound i.e. $\forall t \in T$, $l \leq t$.

ii for every lower bound l' of T, $l' \leq l$.

Then, we write $\inf(T) = l$ inf := infimium.

$\sup([0,1]) = 1$ $\inf([0,1]) = 0$

$\sup(\emptyset) = -\infty$ $\inf(\emptyset) = +\infty$

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Def min and max.

$\sup(T) \in T \iff \max(T) = \sup(T)$

$\inf(T) \in T \iff \min(T) = \inf(T)$

Example,

$H := \{ \frac{1}{n} \mid n \in \mathbb{N} \}$

$\inf(H) = 0$ but $\inf(H) \notin H$ $\nexists \min(H)$

$\sup(H) = 1$ and $\sup(H) \in H$ $\therefore \max(H) = 1$

Superimum. $T \subseteq S$

$u = \sup(T) \iff (\forall t \in T, t \leq u) \wedge (\underset{S}{v \in} v < u \Rightarrow \exists t \in T \ni t > v)$

Infimum. $T \subseteq S$

$\ell = \inf(T) \iff (\forall t \in T, \ell \leq t) \wedge (\underset{S}{v \in} v > \ell \Rightarrow \exists t \in T \ni t < \ell)$.

Theorem: $T \subseteq S$

$\cancel{u = \sup(T) \iff \forall \varepsilon > 0, \forall t \in T, (t \leq u + \varepsilon) \wedge (t > u - \varepsilon)}$

$\cancel{u = \sup(T) \iff \forall \varepsilon > 0, \forall t \in T, u - \varepsilon < t \leq u + \varepsilon}$

$u = \sup(T) \iff \forall \varepsilon > 0, (\forall t \in T, t \leq u + \varepsilon) \wedge (\exists t \in T \ni u - \varepsilon < t)$

Proof:-

$u = \sup(T)$

Assume $\exists x \in T \ni u < x$

$u < \frac{u+x}{2} < x$

Let $\varepsilon = \frac{x-u}{2} > 0$

$u + \varepsilon = \frac{u+x}{2} < x$

$\therefore x \notin T$

$\therefore u$ is an upper bound of T.

By $\forall \varepsilon > 0, \exists t \in T \ni u - \varepsilon < t$.

$\forall v < u$, for $\varepsilon = u - v > 0$

and therefore v is not an upper bound.

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Def: An ordered field $(\mathbb{F}, \mathbb{P})$ has the least upper bound property if ~~$\forall T$~~ for all non-empty subsets T of $\mathbb{F}$, that has an upperbound in $\mathbb{F}$, T has a least-upper bound i.e. $\sup(T)$ ~~exists~~ $\in \mathbb{F}$ exists.

in $\mathbb{F}$

Thm: $\mathbb{Q}$ does not have the least-upper bound property. 10/10/2022

Proof:-

Let $S := \{x \mid x \in \mathbb{Q} \wedge 0 < x^2 < 2\}$

$1 \in S \Rightarrow S \neq \emptyset$

2 is an upperbound of S.

claim 1: $\sup(S) = \sqrt{2}$

claim 2: $\sqrt{2} \notin \mathbb{Q}$ (we have proved this in Discrete Mathematics).

Proving claim 2: we would show ~~$u+\varepsilon$ is an~~ $u+\varepsilon$ is an upper-bound of set S, and $u-\varepsilon$ is not an upperbound of S.

Let $y \in \mathbb{Q} \ni y^2 < 2$

we prove that $\exists z \in \mathbb{Q}, z > 0 \ni (y+z)^2 < 2$

$$y^2 + 2yz + z^2 < 2$$

$$z(2y+z) < 2 - y^2$$

we know that $y \in S$ and $y^2 < 2 \Rightarrow y < 2$

I will prove it later.

Archimedian property.

~~if~~ ~~$\forall x \in \mathbb{F}$~~ In any ordered field is ~~a~~ said to have Archimedian property, if $\forall x \in \mathbb{F}, \exists y \in \mathbb{N} \ni x < y$

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~~Thm~~ Thm: In $(\mathbb{F}, \mathbb{P})$, LUBP $\Rightarrow$ AP — 12/10/2022

i.e. $\forall x \in \mathbb{F}, \exists y \in \mathbb{N} \ni x < y$

Proof: We have to prove that $\mathbb{N}$ is unbounded in $\mathbb{F}$. For contradiction let say $\mathbb{N}$ is bounded.

$$\Rightarrow \exists x \in \mathbb{F} \ni \forall n \in \mathbb{N}, n \leq x$$

$\mathbb{N}$ is bounded, therefore there exists an least upper-bound of $\mathbb{N}$.

$$\Rightarrow \exists y \in \mathbb{F} \ni y = \sup(\mathbb{N})$$

$$\Rightarrow \underbrace{(\forall n \in \mathbb{N}, n \leq y)}_{\text{upper-bound}} \wedge \underbrace{(\forall a \in \mathbb{F}, a < y, \exists n_a \in \mathbb{N} \ni a < n_a)}_{\text{all upper-bounds are greater than that.}}$$

$$\Rightarrow \exists n \in \mathbb{N} \ni y - 1 < n \Leftrightarrow y < n+1$$

We have a contradiction. Therefore, $\mathbb{N}$ is not bounded in $\mathbb{F}$.

$\blacksquare$

Corallary I: If $(\mathbb{F}, \mathbb{P})$ has a LUBP, then $\forall x \in \mathbb{P}, \exists n_x \in \mathbb{N} \ni n_x - 1 < x \leq n_x$

~~By~~ Proof:

$\mathbb{F}$ has LUBP, therefore it has AP. $\forall x \in \mathbb{P}, \exists m \in \mathbb{N} \ni x < m$. Let $X := \{m \mid m \in \mathbb{N} \wedge x \leq m\} \subseteq \mathbb{N}$. By Well-ordering Principle X has a least element, say m_x. Then $x \leq m_x$ by definition, but $m_x - 1 \notin X$ because, m_x is the least element therefore it must be less that x. Hence,

$$m_x - 1 < x \leq m_x$$

$\blacksquare$

Corallary II: $\forall x \in \mathbb{P} \Rightarrow \exists n_x \in \mathbb{N} \ni \frac{1}{n_x} < x$.

Proof: By AP, $\forall x \in \mathbb{P}$, $\exists n_x \in \mathbb{N} \ni x < n_x$

$x \in \mathbb{P} \Leftrightarrow x^{-1} \in \mathbb{P}$.

Therefore, according to AP, $\exists n_x \in \mathbb{N} \ni x^{-1} < n_x$.

$$\Leftrightarrow x x^{-1} < x n_x$$

$$\Leftrightarrow 1 < x n_x$$

$$\Leftrightarrow n_x^{-1} 1 < x n_x n_x^{-1}$$

$$\Leftrightarrow \frac{1}{n_x} < x$$

Corallary III: $\forall x, y \in \mathbb{P}$, $\exists n \in \mathbb{N} \ni xn > y$

Proof:—

$x, y \in \mathbb{P} \Longleftrightarrow x^{-1} \in \mathbb{P}$.

$\Rightarrow x^{-1} y \in \mathbb{P}$.

By AP, $\forall x, y \in \mathbb{P}$, $\exists n \in \mathbb{N} \ni x^{-1} y < n$

$$\Leftrightarrow x x^{-1} y < xn$$

$$\Leftrightarrow 1y < xn$$

$$\Leftrightarrow y < xn$$

Corallary IV: Every non-empty interval in $\mathbb{F}$ has a rational element and an irrational element.

Proof: Let the interval be $(a, b) := \{x \mid (x \in \mathbb{F}) \wedge (a < x < b) \wedge (a \neq b)\}$

We know that, $0 < b - a \in \mathbb{P}$ and $1 \in \mathbb{P}$. By Corallary III $\exists n \in \mathbb{N}$ $\ni n(b-a) > 1$. we do not know that a, b are rational or irrational but we know that, $nb - na > 1$, but nb and na can be negative, therefore $\exists m \in \mathbb{Z} \ni na < m < nb$.

$$\Rightarrow a < \frac{m}{n} < b.$$

A set is called Dense in $\mathbb{F}$ if it's intersection with every non-empty interval of $\mathbb{F}$ is non-empty.

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Density Theorem

Corallary: $\mathbb{Q}$ and $\mathbb{I}$ are both Dense in $\mathbb{R}$.

Let D a non-empty subset of $\mathbb{R}$. Show that removing an element from D, still make it dense.

Proof:-

$D \cap (c,d) \neq \emptyset$

Remove a from D,

$$D' = D \setminus \{a\}$$

suppose,

$$D' \cap (c,d) = \emptyset \Rightarrow D \cap (c,d) = \emptyset \Rightarrow D \text{ is not dense.}$$

we have proved that,

$$D \setminus \{a\} \text{ not dense} \Rightarrow D \text{ not dense.}$$

Corallary: Removing finitely many elements from a dense set still makes it dense.

Corallary: A finite set is never dense.

Corallary: All infinite sets are not dense.

Nest

A sequence of sets $\{S_i\}$ $S_1, S_2, S_3, \dots$ is called nest iff

$$S_1 \supseteq S_2 \supseteq S_3 \supseteq \dots$$

Big Theorem:

Least upper bound property $\Rightarrow$ Nested Interval Property.

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Nested Interval property.

In $(\mathbb{F}, \mathbb{P})$ with Least upper bound property, every nest of bounded, closed, non-empty intervals $\{I_n\}$ has

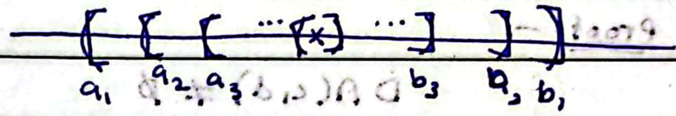
1. $\bigcap_{n\in\mathbb{N}} I_n \neq \emptyset$

$\{I_n\} = I_1, I_2, I_3, \ldots$

where $I_i = [a_i, b_i] \ni a_i \leq b_i$.

we know that,

$a_i \leq a_j \leq b_j \leq b_i \quad \forall i \leq j$



Show $a_i \leq b_j \quad \forall i, j \in \mathbb{N}$.

Proof: —

$i = j \Rightarrow a_i \leq b_i$

$i < j \Rightarrow a_i \leq a_j \leq b_j$

$i > j \Rightarrow a_i \leq b_i \leq b_j$

Then consider, $A = \{a_1, a_2, a_2 \ldots\}$. A is bounded above since, $\forall i \in \mathbb{N}, a_i \leq b_1$. Therefore, sup(A) exists due to least upper bound property. 24/10/2022

Claim $\sup(A) \in I_j$, $\forall j \in \mathbb{N}$. Since for each $I_j = [a_j, b_j]$, $\underbrace{a_j \leq \sup(A)}_{1}$, b_j is an upper bound of A i.e. $\forall a_i \in A$, $a_i \leq b_j \Rightarrow \underbrace{\sup(A) \leq b_j}_{2}$.

1 and 2 implies: $a_j \leq \sup(A) \leq b_j$, $\forall j \in \mathbb{N}$

$\Rightarrow \sup(A) \in \bigcap_{j\in\mathbb{N}} I_j$

$\Rightarrow \bigcap_{j\in\mathbb{N}} I_j \neq \emptyset$

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2. If infimum of lengths of I_j is zero. Then $\bigcap_{j\in\mathbb{N}} I_j$ is a unique element.

$S :=$ set of lengths of $I_j = \{b_j - a_j \mid j \in \mathbb{N}\}$

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Let $x > y \ni x, y \in \bigcap_{j\in\mathbb{N}} I_j \Rightarrow \forall j \in \mathbb{N},\ x, y \in I_j \Rightarrow \forall j \in \mathbb{N},\ x - y \leq b_j - a_j$

$\Rightarrow x - y \leq 0$ because $\inf(S) = 0$. $x-y$ is the lower bound of $S \Rightarrow x - y \leq \inf(S)$.

we arrive to a contradiction, therefore $x = y$.

Cluster Points

26/10/2022

Let $S \subseteq \mathbb{F} \ni p \in S$

~~i. $\exists \varepsilon > 0 \ni \forall x \in (p-\varepsilon, p+\varepsilon), x \in S$~~

~~ii. $\exists \varepsilon > 0 \ni$~~

i. $\exists \varepsilon > 0 \ni S$ contains all points of $(p-\varepsilon, p+\varepsilon)$.

ii. $\exists \varepsilon > 0 \ni S$ contains all ~~points of~~ but finitely many points of $(p-\varepsilon, p+\varepsilon)$.

iii. $\exists \varepsilon > 0 \ni S$ contains finitely many points of $(p-\varepsilon, p+\varepsilon)$.

iv. $\exists \varepsilon > 0 \ni S$ contains some ~~other~~ point other than p in $(p-\varepsilon, p+\varepsilon)$.

v. $\forall \varepsilon > 0$, S contains infinitely many points of $(p-\varepsilon, p+\varepsilon)$.

vi. $\forall \varepsilon > 0$, S contain ~~infinitely many points of~~ a point other than p of $(p-\varepsilon, p+\varepsilon)$. i.e. $S \cap ((p-\varepsilon, p+\varepsilon) \setminus \{p\}) \neq \emptyset$

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Def:-

Point s_0 is a cluster point of S if.

(i) $\forall \varepsilon>0$, S contains infinitely many points of $(s_0-\varepsilon, s_0+\varepsilon)$ OR

(ii) $\forall \varepsilon>0$, S contains some point other than s_0 of $(s_0-\varepsilon, s_0+\varepsilon)$

Proof:-

(i) $\Rightarrow$ (ii) is obvious

$\forall \varepsilon>0$, $S \cap (s_0-\varepsilon, s_0+\varepsilon)$ is infinite

$\Rightarrow S \cap (s_0-\varepsilon, s_0+\varepsilon)\setminus\{s_0\}$ is infinite

$\Rightarrow S \cap (s_0-\varepsilon, s_0+\varepsilon)\setminus\{s_0\} \neq \emptyset$ which is exactly (ii)

Conversly, (ii) $\Rightarrow$ (i)

(ii) $\Rightarrow S \cap (s_0-\varepsilon, s_0+\varepsilon)\setminus\{s_0\} \neq \emptyset$

~~Let $x_0 \in S \cap \mathcal{N}_\varepsilon(s_0)\setminus\{s_0\}$~~

~~Let, $\varepsilon_1 = |s_0 - x_0| > 0$.~~

~~by (ii) again $S \cap \mathcal{N}_{\varepsilon_1}(s_0)\setminus\{s_0\} \neq \emptyset$~~

$\forall n \in \mathbb{N}$, Let $x_n \in S \cap \mathcal{N}_{\varepsilon_n}(s_0)\setminus\{s_0\}$

Let $\varepsilon_{n+1} := |s_0 - x_n| > 0$

Then, $x_{n+1} \in S \cap \mathcal{N}_{\varepsilon_{n+1}}(s_0)\setminus\{s_0\}$

we know that,

$\mathcal{N}_{\varepsilon_{n+1}}(s_0) \subseteq \mathcal{N}_{\varepsilon_n}(s_0)$

$\Rightarrow \{x_n \mid n \in \mathbb{N}\} \subseteq S \cap \mathcal{N}_{\varepsilon}(s_0)\setminus\{s_0\}$

$\Rightarrow$

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Def:-
The set of cluster points of $S \subseteq \mathbb{R}$ is called the derived set of S and is written as S'.

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Examples:

$\mathbb{N}' = \phi$

S is finite $\Rightarrow S' = \phi$

$(a,b)' = [a,b]$

$[a,b]' = [a,b]$

$\mathbb{Q}' = \mathbb{R}$

$\mathbb{I}' = (\mathbb{R}/\mathbb{Q})' = \mathbb{R}$

$H := \{ \frac{1}{n} \mid n \in \mathbb{N} \} \Rightarrow H' = \{0\}$

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~~Bo~~

BOLZANO - WEIERSTRASS THEOREM

Every infinite bounded set in $\mathbb{R}$ has a cluster point.

Thm:- If $(\mathbb{F}, \mathbb{P})$ is an ordered field with Nested-Interval Property then every infinite, bounded set has a cluster point.

Proof:-

If $(\mathbb{F}, \mathbb{P})$, given $S \subseteq \mathbb{F} \ni S$ is infinite and bounded.
Let u and l be the upper and lower bound of S respectively.
Let $I_0 = [l_0, u_0]$, let $m = \frac{l_0 + u_0}{2}$

If $[l, m]$ is infinite, then

$I_1 = [l_0, m_0]$ and $S_1 = [l_0, m_0] \cap S$

Otherwise, let,

$I_1 = [m_0, u_0]$ and $S_1 = [m_0, u_0] \cap S$.

and repeat the process for S_1 and I_1 and this would make a nest of intervals that non-empty, close and bounded.

Iteratively,

Let $m_i = \frac{u_i + l_i}{2}$

If ~~$[l_i, m_i]$ is inf~~ $[l_i, m_i] \cap S_i$ is infinite, then $I_{i+1} = [l_i, m_i]$

Otherwise, ~~$[m_i$~~ $I_{i+1} = [m_i, u_i]$

and, $S_{i+1} = S_i \cap I_{i+1}$

$[I_i] = I_0 \supseteq I_1 \supseteq I_2 \supseteq \dots$ is a nest of closed, bounded, infinite (non-empty) interval. Each time the length of the intervals is halfed of the orignal one, therefore the infimum of set of lengths of intervals is zero.

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$$\Rightarrow \bigcap_{i \in \mathbb{N} \cup \{0\}} I_i = \{x\}$$

Claim, x is a ~~finite~~ cluster point of S,

$\forall \varepsilon > 0$, $(x-\varepsilon, x+\varepsilon) \cap$ is infinite.

$\exists n \in \mathbb{N} \ni I_n \subseteq (x-\varepsilon, x+\varepsilon)$

This is because the length of $(x-\varepsilon, x+\varepsilon)$ is 2ε, and the infimum of set of lengths of Intervals I_i ~~are~~ is zero. This means there would be some ~~n for~~ $n \in \mathbb{N}$ for which the length of interval I_n would be less than 2ε. Now we know that $I_n \cap S$ is infinite and ~~$I_n \cap S$ is~~ $I_n \subseteq (x-\varepsilon, x+\varepsilon)$ that implies $S \cap (x-\varepsilon, x+\varepsilon)$ is infinite.

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Sequences:—

A sequence $x : \mathbb{N} \to \mathbb{R}$, we denote it by (x_n) and $x(i) =: x_i$

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Limit of sequences:—

A number L is called limit of (x_n) if $\forall \varepsilon > 0, \exists N_\varepsilon \in \mathbb{N} \ni$ $n > N_\varepsilon \Rightarrow |x_n - L| < \epsilon$

Alternate definition:—

L is limit of (x_n) iff $\forall$ V neighbourhoods of L, $\exists N_V \in \mathbb{N}$ $\ni n > N_V \Rightarrow x_n \in V$.

Def: A sequence is eventually in a set, if $\exists N \in \mathbb{N} \ni$ $\forall n \geq N, x_n \in$

Def:— A sequence is frequently in a set S, if $\forall N \in \mathbb{N}, \exists n > N \ni x_n \in S$.

Def:— (Alternate definition):—

L is limit of (x_n) if (x_n) is eventually in every neighbourh V of L.

Theorem 9.6:

Limit of a convergent sequence is unique.

Proof:

Let $\lim x_n = L$ and $\lim x_n = M$, where ~~such that~~ $L \neq M$.

Let, $\epsilon = \dfrac{|M-L|}{2}$, then,

$$N_\varepsilon(L) \cap N_\varepsilon(M) = \emptyset$$

But (x_n) eventually belongs to both, therefore a contradiction arises.

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Thm 9.7:— A convergent sequence is bounded.

Proof:— Let $\lim x_n = L$, then $\exists N \ni \forall n > N, |x_n - L| < 1$

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$\Rightarrow L - 1 < x_n < L + 1$

then, $\max\{x_1, x_2, \ldots, x_N, L+1\}$ is an upper bound of (x_n),

and, $\min\{x_1, x_2, \ldots, x_N, L-1\}$ is a lower bound of (x_n).

$\Rightarrow$ (x_n) is bounded. ■

Thm 9.9:—

a) If (x_n) converges, and $x_n \geq 0$ $\forall n \in \mathbb{N}$, ~~then~~ $\lim x_n \geq 0$.

Let $\lim x_n = M < 0$. $(-\infty, 0)$ is a neighbourhood of M and $x_n \notin (-\infty, 0)$ for any n. ↯

b) If (x_n) and (y_n) are convergent $\ni x_n < y_n$, $\forall n$, $\lim x_n \leq \lim y_n$.

Let $\lim x_n = L$, $\lim y_n = M$, if $M < L$, then let,

$\varepsilon = \frac{L-M}{2}$,

$U = (M - \varepsilon, M + \varepsilon)$

$V = (L - \varepsilon, L + \varepsilon)$

and $U \cap V = \phi$

$\forall z \in U$ and $\forall w \in V$, $z < w$, since (y_n) is eventually in U and (x_n) is eventually in V. $\exists n \ni y_n < x_n$. ↯

c) Squeeze theorem / sandwich theorem.

If $(x_n), (y_n)$ are convergent sequences $\ni \lim x_n = \lim y_n = L$ and let $(z_n) \ni x_n \leq z_n \leq y_n$, $\forall n \in \mathbb{N}$, then $\lim z_n = L$.

Since (x_n) and (y_n) are eventually in every ε-neighbourhood of L $\Rightarrow L - \varepsilon < x_n \leq y_n < L + \varepsilon \Rightarrow (z_n)$ is eventually in every neighbourhood of L.

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Sequences and cluster points.

c is called cluster point of (x_n) if c is the cluster point of range $\{x_n\}$ of (x_n).

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Lemma 9.12. A convergent sequence can have at most one cluster point, its limit.

Proof:— Let $\lim x_n = L$, let $a \neq L$, we show that a cannot be a cluster point of (x_n).

Let $\varepsilon = \frac{|L-a|}{2}$, then $(a-\varepsilon, a+\varepsilon) \cap (L-\varepsilon, L+\varepsilon) = \phi$

$\exists N \in \mathbb{N} \ni \forall n > N, x_n \in (L-\varepsilon, L+\varepsilon)$

$\Rightarrow$ $(a-\varepsilon, a+\varepsilon) \cap \{x_n\}$ can have at most $x_1, x_2, \dots, x_N$, i.e. finitely many elements.

Thm 9.13: $(A \Rightarrow B \vee C) \Leftrightarrow (A \wedge \neg B \Rightarrow C)$ ~~$A \Rightarrow A$~~

If $\lim x_n = L$ then one of the following is true.

i) (x_n) is eventually equal to L.

xor

ii) L is the cluster point of ~~$\{x_n\}$~~ (x_n).

Proof:—

Let $\lim x_n = L$ and (x_n) is not eventually equal to L.

$\Rightarrow$ $\exists$ infinitely many $n \in \mathbb{N} \ni x_n \neq L$.

Also, $\forall \varepsilon > 0, \exists N_\varepsilon \in \mathbb{N} \ni n > N_\varepsilon, x_n \in (L-\varepsilon, L+\varepsilon)$.

$\Rightarrow \forall \varepsilon > 0, (L-\varepsilon, L+\varepsilon) \setminus \{L\} \cap \{x_n\} \neq \emptyset$

$\Rightarrow$ L is the cluster point of (x_n). ∎

Subsequence :-

A sequence (a substructure) that is also a sequence

Let $n: \mathbb{N} \to \mathbb{N}$ such that n is strictly increasing ($n(k+1) > n(k)$). Let $x: \mathbb{N} \to \mathbb{R}$, then a subsequence of x is the map $x \circ n : \mathbb{N} \to \mathbb{R}$.

$$\cancel{x \circ I}\ x \circ n(k) = x_{n_k}$$

For sequence $x_1, x_2, \ldots, x_n, \ldots$, a subsequence $x_{n_1}, x_{n_2}, \ldots, x_{n_k}, \ldots$

Thm 9.18 :-

If a sequence converges then so does every subsequence of it, and they all converges to the same limit.

Proof :- Let $\lim x_n = L$, i.e. $\forall$ V neighbourhood of L $\exists N \in \mathbb{N} \ni \forall k > N$, $x_k \in V$. Since, $n_k \geq k > N$, $x_{n_k} \in V$.

Corollary :. If there are two subsequences of a sequence that converges to different limits then the sequence is divergent.

CHAPTER 10

Def. A sequence is increasing if $x_{n+1} \geq x_n$ $\forall n \in \mathbb{N}$ and is strictly increasing if $x_{n+1} > x_n$ $\forall n \in \mathbb{N}$. Similarly, decreasing and strictly decreasing.

Def :- A sequence is called monotone if it is either increasing or decreasing.

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Lemma 10.2 :-

a) Any cluster point of an increasing sequence is an upperbound for the sequence.

b) Any cluster point of decreasing sequences is a lower bound for sequences.

Proof:-

Let x cluster point of (x_n) and let x not be an upper bound of (x_n).

$\Rightarrow \exists\, k \in \mathbb{N} \ni x < x_k \leq x_{k+1} \leq x_{k+2} \leq \dots$

$(-\infty, x_k)$ is a neighbourhood of x.

$(-\infty, x_k) \cap \{x_k\}$ is finite $= \{x_1, x_2, \dots, x_{k-1}\}$ $\Rightarrow\!\Leftarrow$

Thm: Every bounded eventually mono-tone sequence is convergent.

Proof: Let (x_n) seq, if $\{x_n\}$ is finite, then 21/011/2022

the seq (x_n) is eventually constant.

$\Rightarrow$ Every constant seq has a limit

$\Rightarrow$ the sequence is convergent.

if $\{x_n\}$ is infinite, then $\{x_n\}$ is a bounded infinite set and by Bolzano-Wiestrais Theorem, (x_n) has a cluster point. By Lemma 10.2 the cluster point of (x_n) is also the upperbound of (x_n).

claim c is the limit point of (x_n). By definition of cluster points, $\forall \varepsilon > 0$, $\{x_n\} \cap (c-\varepsilon, c+\varepsilon) \setminus \{c\} \neq \emptyset \Rightarrow \exists\, N \in \mathbb{N} \ni x_N \in (c-\varepsilon, c+\varepsilon) \setminus \{c\}$

$\forall n > N,\ x_N \leq x_n < c \Rightarrow \forall \varepsilon > 0, \exists\, N \in \mathbb{N} \ni \forall n > N,$ $x_n \in (c-\varepsilon, c+\varepsilon) \Rightarrow \lim x_n = c$

Thm 10·6 : If c is a cluster point of (x_n) then there exists a subsequence (x_{n_k}) converging to the cluster point of c.

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Proof :— since c cluster point of (x_n).

In particular $\forall k \in \mathbb{N}$. $\{x_n\} \cap (c-\frac{1}{k}, c+\frac{1}{k})$ is infinite.

Let,

$S_k := \{x_n\} \cap (c-\frac{1}{k}, c+\frac{1}{k})$.

Let x_{n_1} be any ~~an~~ element of S_1. Iteratively, pick $x_{n_k} \in S_k \ni n_k > n_{k-1}$ then the sequence $(x_{n_k}) = x_{n_1}, x_{n_2}, \ldots$ converges to c.

Since, $\forall \varepsilon > 0, \exists k \in \mathbb{N} \ni \frac{1}{k} < \varepsilon$, By Archimedian Property.

Then, whenever $n_i > n_k$,

$$x_{n_i} \in S_i \subseteq S_k \subseteq (c-\frac{1}{k}, c+\frac{1}{k}) \subseteq N_\varepsilon(c)$$

BOLZANO-~~WIES~~ WEIERSTRASS THEOREM FOR SEQUENCES

Every bounded seq in $\mathbb{R}$ has a converging subseq in $\mathbb{R}$.

Proof:— If $\{x_n\}$ is infinite, then it is infinite and bounded $\xrightarrow{\text{original BW}}$ (x_n) has a cluster point. Then 10.6 $\Rightarrow$ a subseq exists that converges to this cluster point.

Otherwise, if $\{x_n\}$ is finite, this mean there are infinitly many $x_{n_k} = L$ (same number).

Then $(x_n) = L, L, L, \ldots$ is constant subseq and therefore convergent.

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Def Cauchy Sequence.

A sequence is called cauchy sequence iff $\forall \varepsilon>0, \exists N\in\mathbb{N} \ni \forall m\geq n>N, |x_m-x_n|<\varepsilon$

Eg:— $1, \frac{1}{2}, \frac{1}{3}, \ldots$ is cauchy sequence.

$\forall \varepsilon>0, \exists N\in\mathbb{N} \ni \forall n>N, \frac{1}{n}<\varepsilon$

Therefore, $\forall m\geq n>N$,

$$\left|\frac{1}{m}-\frac{1}{n}\right| = \frac{1}{n}-\frac{1}{m} < \frac{1}{n} < \varepsilon$$

Eg:— $x_n = 1+\frac{1}{2}+\frac{1}{2^2}+\cdots+\frac{1}{2^n}$ is cauchy sequence.

$\forall \varepsilon>0, \exists N\in\mathbb{N} \ni m\geq n>N, |x_m-x_n|<\varepsilon$

$$2x_n = 2+1+\frac{1}{2}+\frac{1}{2^2}+\cdots+\frac{1}{2^{n-1}}+\frac{1}{2^n}-\frac{1}{2^n}$$

$$2x_n = 2+x_n-\frac{1}{2^n}$$

$$x_n = 2-\frac{1}{2^n}$$

$$|x_m-x_n| = \left|\cancel{2}-\frac{1}{2^m}-\cancel{2}+\frac{1}{2^n}\right|$$

$$= \left|\frac{1}{2^n}-\frac{1}{2^m}\right|$$

$$= \frac{1}{2^n}-\frac{1}{2^m}$$

$$< \frac{1}{2^n}$$

$$< \frac{1}{2^N}$$

$$< \varepsilon$$

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Eg:— $x_n = \ln(n)$ is not ~~Cauchy~~ Cauchy sequence.

$$\frac{\ln(m)-\ln(n)}{m-n} = (\ln(c))' = \frac{1}{c} \quad \text{for } n<c<m$$

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Mean Value Theorem: If $f(x)$ is continous on $[n,m]$ and differentiable on (n,m) then $\exists\, c \in (n,m) \ni f'(c) = \frac{f(m)-f(n)}{m-n}$

$$c<m \Leftrightarrow \frac{1}{m} < \frac{1}{c}$$

$$\ln(m)-\ln(n) = \frac{1}{c}(m-n) > \frac{1}{m}(m-n) = 1-\frac{n}{m}$$

whenever we fix n and start increasing m, $\frac{n}{m} \to 0$. Therefore $x_n = \ln(n)$ is not a Cauchy sequence. $\boxtimes$

Thm: Every convergent sequence is a Cauchy sequence.

Proof.— (x_n) is convergent $\Rightarrow \exists L \ni \lim x_n = L$

$$\Rightarrow \forall \tfrac{\varepsilon}{2}>0,\ \exists N\in\mathbb{N} \ni \forall n>N,\ |x_n - L| < \tfrac{\varepsilon}{2}$$

$$\Rightarrow \forall m,n>N,\ |x_m - x_n| = |x_m - L - x_n + L|$$

$$\leq |x_m - L| + |x_n - L|$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$$

$$= \varepsilon$$

$\boxtimes$

Lemma 10.14: If a subsequence of a Cauchy sequence converges then the original seq also converges (to the same limit).

Proof.—

$$\forall \varepsilon>0,\ \exists N_1 \in \mathbb{N} \ni m,n \geq N_1,\ |x_m - x_n| < \frac{\varepsilon}{2}$$

forall (x_{n_k}),

$$\exists L\in\mathbb{R}.\ \forall \tfrac{\varepsilon}{2}>0,\ \exists N_2 \ni n_k > N_2,\ |x_{n_k} - L| < \tfrac{\varepsilon}{2}$$

Let $N = \max\{N_1, N_2\}$, then when $m, n_k > N$

$$|x_m - L| = |x_m - x_{n_k} + x_{n_k} - L|$$

$$\le |x_m - x_{n_k}| + |x_{n_k} - L|$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$$

$$= \varepsilon$$

Lemma 10.15: Every cauchy sequence is bounded

Proof:

Since, sequence is cauchy, for $\varepsilon = 1$, $\exists N \in \mathbb{N} \ni m, n > N$, $|x_m - x_n| < 1$. In particular $|x_m - x_{N+1}| < 1$

$\Rightarrow \forall m, n > N$, $x_{N+1} - 1 < x_m < x_{N+1} + 1$

$\Rightarrow \forall n \in \mathbb{N}$, $x_n \ge \min\{x_1, x_2, \dots, x_N, x_{N+1} - 1\}$, is the lower bound of seq.

$\therefore x_n \le \max\{x_1, x_2, \dots, x_N, x_{N+1} + 1\}$ is an upper bound of sequence.

~~Thm. Cauchy~~ Cri.

CAUCHY CRITERION

A sequence in $\mathbb{R}$ is convergent iff it is cauchy.

It is trivial.

TOPOLOGY

A Topology on a set X is a collection τ of subsets of X.

i.e. $\tau = \{X_i \mid \forall i \in \Lambda, X_i \subseteq X\}$

That satifies,

i) $\phi \in \tau \wedge X \in \tau$

ii) Arbitary union of elements of τ belong to τ.

$$\bigcup_i X_i$$

$$X_i \in \tau \Rightarrow \bigcup X_i \in \tau$$

iii) Finite intersection of elements of τ belongs to τ.

$$x_i \in \tau \Rightarrow \bigcap_i x_i \in \tau$$

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we usually write (X, τ) for topology.

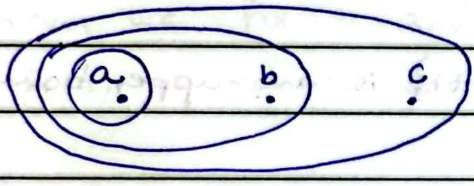
Examples:—

- For any set X, the set $\tau = \{\phi, X\}$ is a topology, called trivial / indiscrete topology.

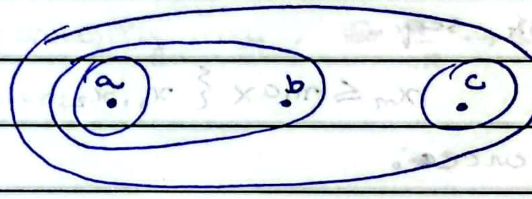
~~For~~

- ~~For~~ For any set X, $P(X)$ is a topology, called discrete topology.

Eg. $X = \{a, b, c\}$, $\tau_1 = \{\phi, X, \{a\}, \{a, b\}\}$

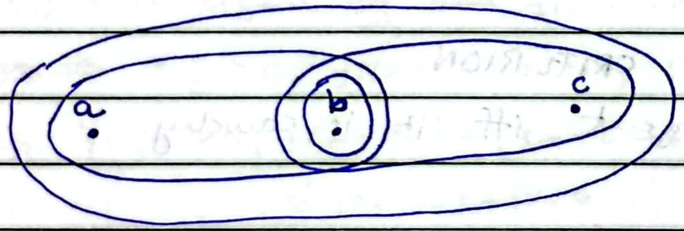


A topology



not a topology.

$\{a\} \cup \{c\} \notin \tau_1$

$\tau_2 =$ 

Exercise:— How many topologies can we make on a ~~fint~~ finite set?

Def If (X, τ) is a topology, the elements of τ are called open sets of (X, τ).

Def:— Given a topology (X, τ), a collection β of non-empty, open sets of (X, τ) is called the basis of τ iff

i) $\forall x \in X, \exists B \in \beta \ni x \in B$

ii) $\forall B_1, B_2 \in \beta, \forall x \in B_1 \cap B_2, \exists B_3 \in \beta \ni x \in B_3 \subseteq B_1 \cap B_2$

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iii) $\forall O_i \in \tau, \exists B_i \in \beta \ni B_i \subseteq O_i$

<u>Thm</u>: Given a basis β of (X, τ), the topology τ is generated by simply taking arbitrary unions of elements of β.

<u>Proof</u>:—

Given $B_{ij} \in \beta$, $O_i = \bigcup_j B_{ij} \in \tau$.

Real Analysis – MATH304

FALL 2022

Final Exam

Instructor: Rameez Ragheb **Total Points: 40**

Tuesday, 13th December, 2022

PROBLEM 1. Let $t_1 = \sqrt{2}$ and $t_n = \sqrt{2 + t_{n-1}}$ for $n = 2, 3, 4, \ldots$

a. Prove that (t_n) is a monotone increasing sequence. (4 points)
b. Prove that $t_n \leq 2$ for all $n \in \mathbb{N}$ (3 points)
c. Show that $\lim\,(t_n)$ exists. (2 points)
d. Find $\lim\,(t_n)$. (4 points)

PROBLEM 2.

a. Show that if S is dense in the real line and a finite number of points are removed from S, the resulting set is also dense in the real line. (3 points)
b. Show that "if S is dense in the real line then showing that removing one point from S still makes the new set dense" is enough to prove part a. (3 points)
c. For a set S, let S' denote the derived set of S. Show that if $S \subseteq T$, then $S' \subseteq T'$. (4 points)

PROBLEM 3.

a. What does it mean for a sequence (x_n) to not be Cauchy (Hint: All you have to do is negate the definition of a Cauchy sequence)? (2 points)
b. Let (x_n) be a Cauchy sequence such that (x_n) is an integer for all $n \in \mathbb{N}$. Show that (x_n) is eventually constant. (4 points)

PROBLEM 4.

a. Let τ be the collection containing $\mathbb{R}, \emptyset$ and subsets X of $\mathbb{R}$ such that $\mathbb{R}\backslash X$ is finite i.e. τ has elements, the set $\mathbb{R}$, the empty set, and all such subsets of $\mathbb{R}$ whose complements are finite. Show that τ is a topology on $\mathbb{R}$. (4 points)
b. Show that the open interval $(0, 3)$ in $\mathbb{R}$ can be written as a countable union of closed intervals of $\mathbb{R}$. (4 points)
c. Let T consist of $\mathbb{R}, \emptyset$ and all sets of the form $[a, \infty)$. Show that T is *not* a topology on $\mathbb{R}$. (3 points)

In the elder days of Art,
Builders wrought with greatest care
Each minute and unseen part;
For the Gods see everywhere.

~Henry Wadsworth Longsfellow