Assignment 2

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CS/PHY-314/300 Quantum Computing

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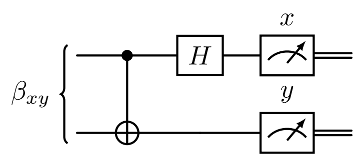
1. (15 points) Given one of the following four bell states, i.e.

$$|\Phi^+\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}$$
$$|\Phi^-\rangle = \frac{|00\rangle - |11\rangle}{\sqrt{2}}$$
$$|\Psi^+\rangle = \frac{|01\rangle + |10\rangle}{\sqrt{2}}$$
$$|\Psi^+\rangle = \frac{|01\rangle - |10\rangle}{\sqrt{2}}$$

(a) (5 points) Design a quantum circuit that measures the bell states, (i.e. perfectly distinguishes between each of the four bell states) with perfect probability.

Solution:



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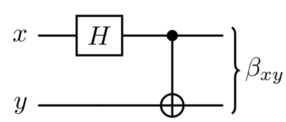
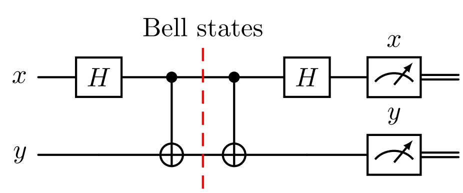
(b) (5 points) Show with analysis (for each of the four bell states) the output of the quantum circuit in (a).

Solution: For analysis, I will use the notation β_{xy} for bell states. The circuit to generate bell states is,



We can see that two operations are being applied in some order to the two given inputs, first is the Hadamard gate followed by a controlled not gate. Therefore to invert these operations we have to apply their inverses in reverse order, which means we will first apply a controlled not gate then a Hadamard gate, after which we will apply the inverse of the Hadamard gate i.e. Hadamard gate itself, and after which we will measure the qubits.



This means that for a unique pair of x and y we would have a unique bell state. At the end of the circuit, we are getting back those unique pairs which indicate that this circuit uniquely identifies the bell states.

Bell State	x	y
$\vert\Phi^+\rangle$	0	0
$\vert\Phi^-\rangle$	1	0
$\vert\Psi^+\rangle$	0	1
$\vert\Psi^+\rangle$	1	1

(c) (5 points) Write a jupyter notebook (.ipynb) file that implements the circuit in (a). Simulate the circuit 1000 times for each of the four bell states and show the output as a histogram. For reference, the code for generating a bell state is given at the URL `https://quantumai.google/cirq/start/basics`.

Solution: Code is available at this link.

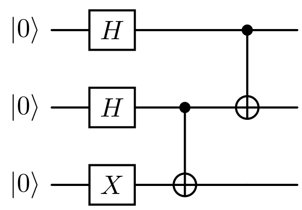
2. (25 points) This exercise develops (step-by-step) a teleportation protocol using GHZ-like qubits.

(a) (5 points) Design a quantum circuit that outputs the following state $|\phi_{GHZlike}\rangle$,

$$|\phi_{GHZlike}\rangle = \frac{1}{2}(|001\rangle + |010\rangle + |100\rangle + |111\rangle)$$

given the input qubits as $|000\rangle$.

Solution:



(b) (5 points) Consider the following scenario where Alice wants to send an arbitrary qubit $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ to Bob with the help of a facilitator Carlos.
Show that the tensor product of the state $|\psi\rangle$ with the qubit $|\phi_{GHZlike}\rangle$ can be simplified to:

$$\begin{aligned}|\psi\rangle \otimes |\phi_{GHZlike}\rangle = &\frac{1}{2\sqrt{2}} \quad (|\phi^+\rangle \otimes X|\psi\rangle + |\phi^-\rangle \otimes XZ|\psi\rangle + |\psi^+\rangle \otimes |\psi\rangle + |\psi^-\rangle \otimes Z|\psi\rangle)|0\rangle \\ &+\frac{1}{2\sqrt{2}} \quad (|\phi^+\rangle \otimes |\psi\rangle + |\phi^-\rangle \otimes Z|\psi\rangle + |\psi^+\rangle \otimes X|\psi\rangle + |\psi^-\rangle \otimes XZ|\psi\rangle)|1\rangle \quad (1)\end{aligned}$$

Solution: From the left-hand side, we have,

$$\begin{aligned}|\psi\rangle &= \alpha|0\rangle + \beta|1\rangle \\ |\phi_{\text{GHZlike}}\rangle &= \frac{1}{2}(|001\rangle + |010\rangle + |100\rangle + |111\rangle) \\ \Longrightarrow \quad |\psi\rangle \otimes |\phi_{\text{GHZlike}}\rangle &= (\alpha|0\rangle + \beta|1\rangle) \otimes \frac{1}{2}(|001\rangle + |010\rangle + |100\rangle + |111\rangle) \\ |\psi\rangle \otimes |\phi_{\text{GHZlike}}\rangle &= \frac{\alpha}{2}(|0001\rangle + |0010\rangle + |0100\rangle + |0111\rangle) + \frac{\beta}{2}(|1001\rangle + |1010\rangle + |1100\rangle + |1111\rangle)\end{aligned}$$

From the right-hand side, we have,

$$\begin{aligned}
|\phi^+\rangle \otimes X|\psi\rangle &= \frac{1}{\sqrt{2}}(\beta|000\rangle + \alpha|001\rangle + \beta|110\rangle + \alpha|111\rangle) \\
|\phi^-\rangle \otimes XZ|\psi\rangle &= \frac{1}{\sqrt{2}}(-\beta|000\rangle + \alpha|001\rangle + \beta|110\rangle - \alpha|111\rangle) \\
|\psi^+\rangle \otimes |\psi\rangle &= \frac{1}{\sqrt{2}}(\alpha|010\rangle + \beta|011\rangle + \alpha|100\rangle + \beta|101\rangle) \\
|\psi^-\rangle \otimes Z|\psi\rangle &= \frac{1}{\sqrt{2}}(\alpha|010\rangle - \beta|011\rangle - \alpha|100\rangle + \beta|101\rangle)
\end{aligned}$$

$$\begin{aligned}
&|\phi^+\rangle \otimes X|\psi\rangle + |\phi^-\rangle \otimes XZ|\psi\rangle + \\
&\quad |\psi^+\rangle \otimes |\psi\rangle + |\psi^-\rangle \otimes Z|\psi\rangle = \sqrt{2}(\alpha|001\rangle + \alpha|010\rangle + \beta|101\rangle + \beta|110\rangle)
\end{aligned}$$

$$\begin{aligned}
\implies &(|\phi^+\rangle \otimes X|\psi\rangle + |\phi^-\rangle \otimes XZ|\psi\rangle + \\
&|\psi^+\rangle \otimes |\psi\rangle + |\psi^-\rangle \otimes Z|\psi\rangle)|0\rangle = \sqrt{2}(\alpha|0010\rangle + \alpha|0100\rangle + \beta|1010\rangle + \beta|1100\rangle) \quad (2)
\end{aligned}$$

$$\begin{aligned}
|\phi^+\rangle \otimes |\psi\rangle &= \frac{1}{\sqrt{2}}(\alpha|000\rangle + \beta|001\rangle + \alpha|110\rangle + \beta|111\rangle) \\
|\phi^-\rangle \otimes Z|\psi\rangle &= \frac{1}{\sqrt{2}}(\alpha|000\rangle - \beta|001\rangle - \alpha|110\rangle + \beta|111\rangle) \\
|\psi^+\rangle \otimes X|\psi\rangle &= \frac{1}{\sqrt{2}}(\beta|010\rangle + \alpha|011\rangle + \beta|100\rangle + \alpha|101\rangle) \\
|\psi^-\rangle \otimes XZ|\psi\rangle &= \frac{1}{\sqrt{2}}(-\beta|010\rangle + \alpha|011\rangle + \beta|100\rangle - \alpha|101\rangle)
\end{aligned}$$

$$\begin{aligned}
&|\phi^+\rangle \otimes |\psi\rangle + |\phi^-\rangle \otimes Z|\psi\rangle + \\
&\quad |\psi^+\rangle \otimes X|\psi\rangle + |\psi^-\rangle \otimes XZ|\psi\rangle = \sqrt{2}(\alpha|000\rangle + \alpha|011\rangle + \beta|100\rangle + \beta|111\rangle)
\end{aligned}$$

$$\begin{aligned}
\implies &(|\phi^+\rangle \otimes |\psi\rangle + |\phi^-\rangle \otimes Z|\psi\rangle + \\
&|\psi^+\rangle \otimes X|\psi\rangle + |\psi^-\rangle \otimes XZ|\psi\rangle)|1\rangle = \sqrt{2}(\alpha|0001\rangle + \alpha|0111\rangle + \beta|1001\rangle + \beta|1111\rangle) \quad (3)
\end{aligned}$$

Adding equation 2 and 3 and scaling them by $\frac{1}{2\sqrt{2}}$ gives us,

$$\begin{aligned}
&\frac{1}{2\sqrt{2}}(|\phi^+\rangle \otimes X|\psi\rangle + |\phi^-\rangle \otimes XZ|\psi\rangle + |\psi^+\rangle \otimes |\psi\rangle + |\psi^-\rangle \otimes Z|\psi\rangle)|0\rangle \\
&+\frac{1}{2\sqrt{2}}(|\phi^+\rangle \otimes |\psi\rangle + |\phi^-\rangle \otimes Z|\psi\rangle + |\psi^+\rangle \otimes X|\psi\rangle + |\psi^-\rangle \otimes XZ|\psi\rangle)|1\rangle = \\
&\frac{\alpha}{2}(|0001\rangle + |0010\rangle + |0100\rangle + |0111\rangle) + \frac{\beta}{2}(|1001\rangle + |1010\rangle + |1100\rangle + |1111\rangle)
\end{aligned}$$

Hence proofed.

(c) (5 points) Using the above simplification, our objective is to design a quantum teleportation protocol. Consider the following partial protocol. Note that these are steps with some missing components to be filled in by you in parts (ii) and (iv):

(i) Alice, Bob and Carlos each have a single qubit. Furthermore, Alice also possesses the arbitrary qubit $|\psi\rangle$.

(ii) Using these 3 input qubits and applying some operations in the form of a circuit, the state $\phi_{GHZlike}$ is produced. **Your task is to state these operations as a circuit.**

(iii) The tensor product of Alice's arbitrary qubit $|\psi\rangle$ with the entire system is given in (a) of this question.

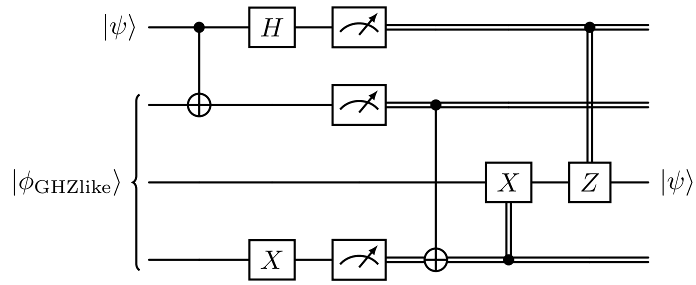
(iv) What are some of the steps that now need to be taken by Alice and Carlos for her arbitrary qubit to be teleported to Bob? State the teleportation protocol.

Solution: The circuit to produce $\phi_{GHZlike}$ will be the same as part a's circuit.
Alice possess 2 qubits $|\psi\rangle$ and the first qubit from $\phi_{GHZlike}$. Bob possesses the second qubit from $\phi_{GHZlike}$. And Carlos possesses the 3rd qubit from $\phi_{GHZlike}$. Now we build a protocol for Bob to construct $|\psi\rangle$. The following protocol teleports $|\psi\rangle$ to Bob: The protocol would be as follows:

1. Alice applies the Controlled NOT where $|\psi\rangle$ is the control bit and the first qubit from $\phi_{GHZlike}$ is the target.
2. Alice applies the Hadamard gate to her first qubit then measures both of her qubits.
3. Carlos applies X gate on the 3rd qubit he posses from $\phi_{GHZlike}$ and then measure the result.
4.
   - If Alice measures $|00\rangle$ and Carlos measures $|0\rangle$ then Bob has obtained $|\psi\rangle$.
   - If Alice measures $|00\rangle$ and Carlos measures $|1\rangle$ then Bob applies X gate on his qubit and obtain $|\psi\rangle$.
   - If Alice measures $|01\rangle$ and Carlos measures $|0\rangle$ then Bob applies X gate on his qubit and obtain $|\psi\rangle$.
   - If Alice measures $|01\rangle$ and Carlos measures $|1\rangle$ then Bob has obtained $|\psi\rangle$.
   - If Alice measures $|10\rangle$ and Carlos measures $|0\rangle$ then Bob applies Z gate on his qubit and obtain $|\psi\rangle$.
   - If Alice measures $|10\rangle$ and Carlos measures $|1\rangle$ then Bob first applies X gate on his qubit then apply Z on the result and obtains $|\psi\rangle$.
   - If Alice measures $|11\rangle$ and Carlos measures $|0\rangle$ then Bob first applies X gate on his qubit then apply Z on the result and obtains $|\psi\rangle$.
   - If Alice measures $|11\rangle$ and Carlos measures $|1\rangle$ then Bob applies Z gate on his qubit and obtain $|\psi\rangle$.

(d) (5 points) Construct the circuit corresponding to the above teleportation protocol.

Solution:



(e) (5 points) **Programming Component**

Write a jupyter notebook (`.ipynb`) file that implements the teleportation circuit stated above in Cirq. Simulate the circuit for 1000 times for various inputs. For reference, the code for quantum teleportation is given at the URL `https://quantumai.google/cirq/experiments/textbook_algorithms#quantum_teleportation`.

Solution: Code is available at this link.

3. (5 points) [**Extending the Deutsch-Jozsa Algorithm**] For the Deutsch Jozsa Algorithm, we are given an input function $f : \Sigma^n \to \Sigma$ where $\Sigma = \{0, 1\}$ such that we have a promise, i.e., the function f is either one of these two cases:

- the function f for the first half of the input bits is zero and for the second half of the input bits is 1. An Example: Suppose f is a function from $\Sigma^3 \to \Sigma$. Then $f(000) = f(001) = ... = f(011) = 0$ and $f(100) = f(101) = ... = f(111) = 1$. Note that the function is balanced.
- the **sum of the number of times** for which $f = 1$ in the first half of the input bits and the number of times for which $f = 0$ in the second half of the bits is $N/2$ where $N = 2^n$. An Example: Suppose f is a function from $\Sigma^3 \to \Sigma$. Then $N = 2^3 = 8$, so $N/2 = 4$. An example of such a function is, let $f(000) = 1, f(001) = 0, f(010) = 1, f(011) = 1, f(100) = 1, f(101) = 0, f(110) = 1, f(111) = 1$. Now since, in the first half, there are three values (indices) for which $f = 1$, and in the second half there is only one value (index) for which $f = 0$, therefore $3 + 1 = 4 = 8/2 = N/2$. An example of a function *not* satisfying the above requirement is a constant function, since for a constant function, this sum will be either 0 or N.

Modify the Deutsch-Jozsa Algorithm to distinguish between these two cases. State the modifications and construct the modified circuit.

Solution: We first formalize our definition of f.

Let $f : \{0, 1\}^n \to \{0, 1\}$ be a function such that f is one of the two kinds:

Either type 1:

$$f = \begin{cases} 0 & x \in \{0\} \circ \{0,1\}^{n-1} \\ 1 & x \in \{1\} \circ \{0,1\}^{n-1} \end{cases}$$

or type 2:

$$f = \sum_{x \in \{0\} \circ \{0,1\}^{n-1}} f(x) + \sum_{x \in \{1\} \circ \{0,1\}^{n-1}} 1 - f(x) = 2^{n-1}$$

We now use this definition of f to construct a new function $g : \{0,1\}^n \to \{0,1\}$ such that:

$$g = \begin{cases} f(x) & x \in \{0\} \circ \{0,1\}^{n-1} \\ 1 - f(x) & x \in \{1\} \circ \{0,1\}^{n-1} \end{cases}$$

Now note that when f is of type 1 g becomes a constant function such that $\forall x \in \{0,1\}^n,\ g(x) = 0$.

Now when f is of the type 2 from definition of g, g will be the follow kind:

$$g = \sum_{x \in \{0,1\}^n} g(x) = 2^{n-1}$$

So we have that when f is of the type 2 g becomes a balanced function.

We have an oracle U_f which computes f, using this we construct an oracle U_g that computes g. So now the problem becomes "is g constant or balances" which is the same problem the Deutsch-Jozsa Algorithm solves. Let $\mathbf{C}$ be the following quantum circuit: So now the following quantum algorithm solves this problem

Input: A function $f : \{0,1\}^n \to \{0,1\}$ with a promise that f is type 1 or type 2

Output: "1" if f is type 1 and "0" if f is type 2

1. Let $g : \{0,1\}^n \to \{0,1\}$ such that

$$g = \begin{cases} f(x) & x \in \{0\} \circ \{0,1\}^{n-1} \\ 1 - f(x) & x \in \{1\} \circ \{0,1\}^{n-1} \end{cases}$$

2. Construct oracle U_g using oracle U_f.
3. Construct quantum circuit $\mathbf{C}$ using U_g.
4. Run $\mathbf{C}$, if all 0s are measured then output "1" else output "0".