$a_1x + a_2y = b$ → Linear equation with 2-unknown

$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$ → Linear equation with n-unknown.

Date: 10/01/2022

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$S = \{s_1, s_2, \ldots, s_n\}$ is a solution set iff $x_1 = s_1$, $x_2 = s_2, \ldots, x_n = s_n$.

Linear Equation is an equation with degree 1 in all variables.

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}\right\}$$

m- System of Linear Equations with n-unknown.

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} \Rightarrow A\vec{x} = \vec{b}$$

coefficent matrix

constant vector.

unknown vector

$$[A \;\; b] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{bmatrix}$$ ← Augmented matrix.

Order of of a matrix

= # of rows × # of cols.

$A = [a_{ij}]_{m\times n}$ $B = [b_{ij}]_{m\times n}$

$A + B = [a_{ij} + b_{ij}]_{m\times n}$

$A - B = [a_{ij} - b_{ij}]_{m\times n}$

For addition the order of matrixes must be same, so that corresponding elements can be added.

Date: 12/01/2022

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If $A = [a_{ij}]_{m\times n}$ and $B = [b_{ij}]_{p\times q}$ then $A = B$ iff $a_{ij} = b_{ij}$ and $m = p$ and $n = q$.

If $A = [a_{ij}]_{m\times n}$ and $B = [b_{ij}]_{p\times q}$ the $C = AB$ iff $n = p$ and $c_{ij} = \sum_{r=1}^{n} a_{ir} b_{rj}$ and $C = [c_{ij}]_{m\times q}$

$AB \neq BA$

If $A = [a_{ij}]_{m\times n}$ and $c \in \mathbb{R}$ the $cA = [ca_{ij}]_{m\times n}$

$$A = \begin{bmatrix} 40 & 30 & 81 & 0 & 21 & 47 \\ 0 & 12 & 78 & 50 & 50 & 96 \\ 10 & 0 & 0 & 27 & 43 & 78 \end{bmatrix}$$

$$B = \begin{bmatrix} 20 & 19 & 18 & 1 & 12 & 74 \\ 2 & 21 & 87 & 50 & 49 & 96 \\ 10 & 1 & 2 & 72 & 34 & 87 \end{bmatrix}$$

$$C = \begin{bmatrix} 20 & 3 & 18 & 1 & 12 & 45 \\ 1 & 12 & 78 & 5 & 0 & 6 \\ 1 & 2 & 3 & 6 & 8 & 99 \end{bmatrix}$$

$$A + B + C = \begin{bmatrix} 80 & 52 & 117 & 2 & 45 & 166 \\ 3 & 45 & 243 & 105 & 99 & 198 \\ 21 & 3 & 5 & 105 & 85 & 264 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 4 & 1 & 4 & 3 \\ 0 & -1 & 3 & 1 \\ 2 & 7 & 5 & 2 \end{bmatrix}$$

$$AB = \begin{bmatrix} 12 & 27 & 30 & 13 \\ 8 & -4 & 26 & 12 \end{bmatrix}$$

$$A = \begin{bmatrix} -1 & 0 \\ 2 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix} \qquad AB = \begin{bmatrix} -1 & -2 \\ 11 & 4 \end{bmatrix}$$

$$BA = \begin{bmatrix} 3 & 6 \\ -3 & 0 \end{bmatrix}$$

For what A and B $(A+B)^2 = A^2+B^2+2AB$.

Proof. Let A and B be a $n\times n$ square matrix.

$$(A+B)^2 = (A+B)(A+B)$$

$$= A(A+B)+B(A+B) \quad \text{(Distributive Law)}$$

$$(A+B)^2 = A^2+AB+BA+B^2$$

$$\cancel{A^2}+\cancel{B^2}+\cancel{2}AB = \cancel{A^2}+\cancel{AB}+BA+\cancel{B^2}$$

$$AB = BA$$

$$D = \operatorname{diag}[d_1, d_2, \dots, d_m] = \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_m \end{bmatrix} = [d_{ij}]_{m\times m}$$

$$DA = \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & d_m \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} = \begin{bmatrix} d_1a_{11} & d_1a_{12} & \cdots & d_1a_{1n} \\ d_2a_{21} & d_2a_{22} & \cdots & d_2a_{2n} \\ \vdots & \vdots & & \vdots \\ d_ma_{m1} & d_ma_{m2} & \cdots & d_ma_{mn} \end{bmatrix}$$

If B and C are inverses of a non-singular matrix A, then $B=C$. 17/01/2022 Mon.

The proposition says,

$$AB = I$$

$$C(AB) = CI$$

$$(CA)B = C \quad \text{(Assosiative)}$$

$$IB = C$$

$$B = C$$

Inverse of A is always unique.

If A and B are ~~non~~ invertible matrices, then $(AB)^{-1} = B^{-1}A^{-1}$.

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Proof:— Let

$$C = (AB)^{-1} \Leftrightarrow ABC = I$$

$$\Rightarrow A^{-1}ABC = A^{-1}I$$

$$IBC = A^{-1}$$

$$BC = A^{-1}$$

$$B^{-1}BC = B^{-1}A^{-1}$$

$$IC = B^{-1}A^{-1}$$

$$C = B^{-1}A^{-1}$$

$$\Rightarrow (AB)^{-1} = B^{-1}A^{-1}$$

QED

$$\left(\prod_{i=1}^{n} A_i\right)^{-1} = \prod_{i=1}^{n} A_{n+1-i}^{-1}$$

For a invertible matrix A.

$$(A^{-1})^{-1} = A$$

Proof:—

$$(A^{-1})^{-1} = A$$

$$A^{-1}(A^{-1})^{-1} = A^{-1}A$$

$$(A^{-1}A)^{-1} = I$$

$$I^{-1} = I$$

$$I = I$$

$$(A^{-1})^{-1}A^{-1} = AA^{-1}$$

$$(AA^{-1})^{-1} = I$$

$$(I)^{-1} = I$$

$$I = I$$

QED

If A is invertible matrix, then

$$(A^n)^{-1} = (A^{-1})^n$$

Proof:—

$$(A^n)^{-1} = (A^{-1})^n$$

$$(\underbrace{A \cdot A \cdots A}_{n})^{-1} = (A^{-1})^n$$

$$\underbrace{A^{-1}A^{-1}\cdots A^{-1}}_{n} = (A^{-1})^n$$

$$(A^{-1})^n = (A^{-1})^n$$

QED.

For a $m \times n$ matrices A and B.

$$(A^T)^T = A$$
$$(A+B)^T = A^T + B^T$$

If A is a symmetrix matrix,

$$A^T = A$$

If $A^T = -A$ then A is skew symmetric. It's diagonals are always zero.

If $\vec{x}_1$ and $\vec{x}_2$ are solution of $n \times n$ full Rank matrix A satisfying equation,

$$A\vec{x} = \vec{b} \Leftrightarrow \vec{x} = A^{-1}\vec{b}$$

Proof:—

$$\vec{x}_1 = A^{-1}b \wedge \vec{x}_2 = A^{-1}b$$
$$\vec{x}_1 = \vec{x}_2$$

QED.

19/01/2022

If θ is operator of Elementary row operation then,

$$\theta(A) = EA$$

where A is a $m \times n$ matrix and E is $n \times m$ matrix corresponding to that Elementary Row operation.

For $A\vec{x} = \vec{b}$ if $\vec{b}$ is a null vector then the system of equation is called Homogenous. Coversly, for $\vec{b} \neq \vec{0}$ then it is non-homogenous.

For series of row operation,

$$\theta_n \theta_{n-1} \cdots \theta_2 \theta_1 (A) = E_n E_{n-1} \cdots E_2 E_1 A$$

and if,

$$E_n E_{n-1} \cdots E_2 E_1 A = I \Rightarrow A^{-1} = E_n E_{n-1} \cdots E_2 E_1$$

$$A = \begin{bmatrix} 3 & 4 & 5 \\ 1 & -1 & 2 \\ 2 & 1 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 3 & 0 \\ -2 & 0 & 3 \end{bmatrix} A = \begin{bmatrix} 3 & 4 & 5 \\ 0 & -7 & 1 \\ 0 & -5 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & 1 \\ 0 & -5 & 7 \end{bmatrix} E_1 A = \begin{bmatrix} 3 & -21 & 0 \\ 0 & -12 & 0 \\ 0 & 0 & -12 \end{bmatrix}$$

$$\begin{bmatrix} \frac{1}{3} & -\frac{7}{12} & 0 \\ 0 & -1/12 & 0 \\ 0 & 0 & -1/12 \end{bmatrix} E_2 E_1 A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E_3 E_2 E_1 = \begin{bmatrix} \frac{1}{3} & -\frac{7}{12} & 0 \\ 0 & -1/12 & 0 \\ 0 & 0 & -1/12 \end{bmatrix} \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & 1 \\ 0 & -5 & 7 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -1 & 3 & 0 \\ -2 & 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1/3 & -7/12 & 0 \\ 0 & -1/12 & 0 \\ 0 & 0 & -1/12 \end{bmatrix} \begin{bmatrix} -9 & 0 & 15 \\ -3 & 3 & 3 \\ -9 & -15 & 21 \end{bmatrix}$$

$$= \begin{bmatrix} -5/4 & -7/4 & 13/4 \\ 1/4 & -1/4 & -1/4 \\ 3/4 & 5/4 & -7/4 \end{bmatrix}$$

$$x_1 + x_2 + 2x_3 = b_1$$
$$x_1 + x_3 = b_2$$
$$2x_1 + x_2 + 3x_3 = b_3$$

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Augmented Matrix is,

$$\begin{bmatrix} 1 & 1 & 2 & b_1 \\ 1 & 0 & 1 & b_2 \\ 2 & 1 & 3 & b_3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 2 & b_1 \\ 0 & 1 & 1 & b_1 - b_2 \\ 0 & -1 & -1 & b_3 - 2b_1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 2 & b_1 \\ 0 & 1 & 1 & b_1 - b_2 \\ 0 & 0 & 0 & b_3 - b_1 - b_2 \end{bmatrix}$$

$$b_3 = b_1 + b_2$$

$$x_1 + 2x_2 + 3x_3 = b_1$$
$$2x_1 + 5x_2 + 3x_3 = b_2$$
$$x_1 + 8x_3 = b_3$$

$$\begin{bmatrix} 1 & 2 & 3 & b_1 \\ 2 & 5 & 3 & b_2 \\ 1 & 0 & 8 & b_3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 & b_1 \\ 0 & 1 & -3 & b_2 - 2b_1 \\ 0 & 2 & -5 & b_1 - b_3 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 9 & 5b_1-2b_2 \\ 0 & 1 & -3 & b_2-2b_1 \\ 0 & 0 & 1 & 5b_1-2b_2-b_3 \end{array}\right]$$

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$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots & \ddots & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nj} & \cdots & a_{nn} \end{bmatrix}$$

Minor Factor of a_{ij} is calculate by deleting i^{th} row and j^{th} column and denoted by M_{ij}.

$$C_{ij} = (-1)^{i+j} M_{ij} \quad - C_{ij} \text{ is cofactor.}$$

$$\det(A) = \sum_{i=1}^{n} a_{ij} C_{ij}$$

Proof of Cramer's Rule.

~~9/02/2022~~

9/02/2022

$A^{-1}A = I_n$

where, $A = [a_{ij}]_{(n,n)}$

$I_n = [e_1 | e_2 | \cdots | e_n]$

$A = [c_1 | c_2 | \cdots | c_3]$

$A^{-1}A = [A^{-1}c_1 | A^{-1}c_2 | \cdots | A^{-1}c_3] = I_n$

$\Rightarrow A^{-1}c_1 = e_1, \; A^{-1}c_2 = e_2, \ldots, A^{-1}c_n = e_n$

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$$D_k = \begin{bmatrix} 1 & 0 & \cdots & x_1 & \cdots & 0 \\ 0 & 1 & \cdots & x_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & x_k & \cdots & 0 \\ \vdots & \vdots & & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & x_n & \cdots & 1 \end{bmatrix}$$

$$D_k = [\, e_1 \mid e_2 \mid \cdots \mid e_{k-1} \mid x \mid e_{k+1} \mid \cdots \mid e_n \,]$$

$$D_k = [\, A^{-1}c_1 \mid A^{-1}c_2 \mid \cdots \mid A^{-1}c_{k-1} \mid A^{-1}b \mid A^{-1}c_{k+1} \mid \cdots \mid A^{-1}c_n \,]$$

$$D_k = A^{-1}[\, c_1 \mid c_2 \mid \cdots \mid c_{k-1} \mid b \mid c_{k+1} \mid \cdots \mid c_n \,]$$

$$D_k = A^{-1} A_k$$

$$\det(D_k) = \det(A^{-1} A_k)$$

$$x_k = \det(A^{-1})\det(A_k)$$

$$\boxed{x_k = \frac{\det(A_k)}{\det(A)}}$$

$|\vec{u}\cdot\vec{v}| \leq \|\vec{u}\|\|\vec{v}\|$ (Chaudy - Schwartz Inequality) 14/02/2022

$$\frac{1}{\|\vec{v}\|^2}\left\| \|\vec{v}\|^2 \vec{u} - (\vec{u}\cdot\vec{v})\vec{v} \right\|^2 = \frac{1}{\|\vec{v}\|^2}\left(\|\vec{v}\|^4\|\vec{u}\|^2 - (\vec{u}\cdot\vec{v})^2\|\vec{v}\|^2 \cancel{+ (\vec{u}\cdot\vec{v})^2 \|\vec{v}\|^2} \right)$$

$$= \|\vec{v}\|^2\|\vec{u}\|^2 - (\vec{u}\cdot\vec{v})^2$$

Left hand side is greater than equal to zero, because it's positive,

$$\|\vec{v}\|^2\|\vec{u}\|^2 - (\vec{u}\cdot\vec{v})^2 \geq 0$$

$$\|\vec{u}\|^2\|\vec{v}\|^2 \geq (\vec{u}\cdot\vec{v})^2$$

$$\|\vec{u}\|\|\vec{v}\| \geq (\vec{u}\cdot\vec{v})$$

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$$\|\vec{u}+\vec{v}\|^2 = (\vec{u}+\vec{v})\cdot(\vec{u}+\vec{v})$$
$$= \vec{u}\cdot\vec{u} + \vec{v}\cdot\vec{u} + \vec{u}\cdot\vec{v} + \vec{v}\cdot\vec{v}$$
$$= \|\vec{u}\|^2 + 2(\vec{u}\cdot\vec{v}) + \|\vec{v}\|^2$$
$$\le \|\vec{u}\|^2 + 2|\vec{u}\cdot\vec{v}| + \|\vec{v}\|^2$$
$$\le \|\vec{u}\|^2 + 2\|u\|\|\vec{v}\| + \|\vec{v}\|^2$$
$$\|\vec{u}+\vec{v}\|^2 \le (\|\vec{u}\| + \|\vec{v}\|)^2$$
$$\|\vec{u}+\vec{v}\| \le \|\vec{u}\| + \|\vec{v}\|$$

Proof:- $d(\vec{u},\vec{v}) \le d(\vec{u},\vec{w}) + d(\vec{w},\vec{v})$

$$d(\vec{u},\vec{v}) = d(\vec{u},\vec{v})$$
$$\Leftrightarrow \quad = \|\vec{u}-\vec{v}\|$$
$$\Leftrightarrow \quad = \|(\vec{u}-\vec{w}) + (\vec{w}-\vec{v})\|$$
$$\Leftrightarrow \quad \le \|\vec{u}-\vec{w}\| + \|\vec{w}-\vec{v}\|$$
$$\Leftrightarrow \quad d(\vec{u},\vec{v}) \le d(\vec{u},\vec{w}) + d(\vec{w},\vec{v})$$

■

Proof:-

$$\vec{u}\cdot\vec{v} = \frac{1}{4}\|\vec{u}+\vec{v}\|^2 - \frac{1}{4}\|\vec{u}-\vec{v}\|^2$$

$$\frac{1}{4}\|\vec{u}+\vec{v}\|^2 - \frac{1}{4}\|\vec{u}-\vec{v}\|^2 = \frac{1}{4}(\vec{u}+\vec{v})\cdot(\vec{u}+\vec{v}) - \frac{1}{4}(\vec{u}-\vec{v})\cdot(\vec{u}-\vec{v})$$

$$= \frac{1}{4}\left(\vec{u}\cdot\vec{u} + \vec{v}\cdot\vec{u} + \vec{u}\cdot\vec{v} + \vec{v}\cdot\vec{v}\right)$$
$$- \frac{1}{4}\left(\vec{u}\cdot\vec{u} - \vec{v}\cdot\vec{u} - \vec{u}\cdot\vec{v} + \vec{v}\cdot\vec{v}\right)$$

$$= \frac{1}{4}\left(\cancel{\vec{u}\cdot\vec{u}} + \vec{v}\cdot\vec{u} + \vec{u}\cdot\vec{v} + \cancel{\vec{v}\cdot\vec{v}} - \cancel{\vec{u}\cdot\vec{u}} + \vec{v}\cdot\vec{u} + \vec{u}\cdot\vec{v} - \cancel{\vec{v}\cdot\vec{v}}\right)$$

$$= \frac{1}{\cancel{4}}(\cancel{4}\,\vec{u}\cdot\vec{v})$$

$$= \vec{u}\cdot\vec{v}$$

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Orthogonal vectors:-

$$\vec{u}\cdot\vec{v}=0$$

$$\vec{u}\cdot\vec{v} = \vec{v}^T\vec{u} = \vec{u}^T\vec{v}$$

Pythogrean Theorem:-

$$\|\vec{u}+\vec{v}\|^2 = \|\vec{u}\|^2+\|\vec{v}\|^2+2(\vec{u}\cdot\vec{v})$$

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$$\|\vec{u}+\vec{v}\|^2 = \|\vec{u}\|^2+\|\vec{v}\|^2 \quad \because \vec{u}\cdot\vec{v}=0$$

Proof:- $(A\vec{u})\cdot\vec{v} = \vec{u}\cdot(A^T\vec{v})$

$(A\vec{u})\cdot\vec{v} = (A\vec{u})^T\vec{v}$ $\quad \because \vec{u}\cdot\vec{v} = \vec{u}^T v$

$= u^T A^T \vec{v}$

$= u^T (A^T\vec{v})$

$(A\vec{u})\cdot\vec{v} = u\cdot(A^T\vec{v})$

Proof: $\vec{u}\cdot(A\vec{v}) = (A\vec{v})^T\vec{u}$

$= \vec{v}^T A^T \vec{u}$

$= (A^T\vec{u})\cdot\vec{v}$

Vector Space.

V is a vector space if,

- $\vec{u},\vec{v}\in V \wedge \vec{u}+\vec{v}\in V$
- $\vec{u}+\vec{v} = \vec{v}+\vec{u}$
- $\vec{u},\vec{v},\vec{w}\in V \wedge (\vec{u}+\vec{v})+\vec{w} = \vec{u}+(\vec{v}+\vec{w})$
- $\vec{u},\vec{0}\in V \ni \vec{u}+\vec{0} = \vec{0}+\vec{u} = \vec{u}$
- $\forall \vec{u}\in V\ \exists -\vec{u}\in V \ni \vec{u}+(-\vec{u}) = \vec{0} = (-\vec{u})+\vec{u}$

$\vec{0}+\vec{u}=\vec{u}+\vec{0}=\vec{u}$ ~~$\forall$~~ $\forall \vec{u}\in V$ insures V is non-empty set.

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Prove. For 2x2 real matrixes, prove that they are vector space under matrix addition.

Proof,

$\forall \alpha,\beta,\gamma \in V$ and $k\in F$

1. $\alpha+\beta\in V$
2. $\alpha+\beta=\beta+\alpha$
3. $(\alpha+\beta)+\gamma=\alpha+(\beta+\gamma)$
4. $\underline{0}+\alpha=\alpha+\underline{0}=\alpha$
5. $\alpha+(-\alpha)=(-\alpha)+\alpha=\underline{0}$, $-\alpha\in V$
6. $k\alpha\in V$
7. $k(\alpha+\beta)=k\alpha+k\beta$
8. $(k+l)\alpha=k\alpha+l\alpha$
9. $k(l\vec{u})=(kl)\vec{u}$
10. $1\vec{u}=\vec{u}$

$$\alpha+\beta=\begin{bmatrix} a_1+b_1 & a_2+b_2 \\ a_3+b_3 & a_4+b_4 \end{bmatrix}$$

$$k(\alpha+\beta)=\begin{bmatrix} k(a_1+b_1) & k(a_2+b_2) \\ k(a_3+b_3) & k(a_4+b_4) \end{bmatrix}\in V$$

$$=k\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}+k\begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}$$

$$k(\alpha+\beta)=k\alpha+k\beta\in V$$

$$(k+l)\alpha=\begin{bmatrix} (k+l)a_1 & (k+l)a_3 \\ (k+l)a_3 & (k+l)a_4 \end{bmatrix}$$

$$=k\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}+l\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}$$

$$=k\alpha+l\alpha$$

Prove. for $a,b\in F$ $\begin{bmatrix} a & 1 \\ 1 & b \end{bmatrix}$ is a vector space under matrix addition.

$$\alpha=\begin{bmatrix} a_1 & 1 \\ 1 & a_2 \end{bmatrix} \quad \beta=\begin{bmatrix} b_1 & 1 \\ 1 & b_2 \end{bmatrix}$$

$$\alpha+\beta=\begin{bmatrix} a_1+b_1 & 2 \\ 2 & a_2+b_2 \end{bmatrix}\notin V$$

Doesn't form a vector space.

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Let ~~V~~ $V := \mathbb{R}^+$. V is a vector space with $u, v \in V$ and $k \in F$

$\underline{u} + \underline{v} := uv$

$k\underline{u} := u^k$

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1. $\underline{u} + \underline{v} = uv \in V$
2. $\underline{u} + \underline{v} = \underline{v} + \underline{u} = uv \in V$
3. $(\underline{u} + \underline{v}) + \underline{w} = \underline{u} + (\underline{v} + \underline{w})$
4. $-\underline{u} = -\frac{1}{u}$

23/02/2022

Prove:- $0\vec{u} = \vec{0} \quad \forall \vec{u} \in V$

Proof,

Ax 8 $\Rightarrow$ $(k+l)\vec{u} = k\vec{u} + l\vec{u}$

$0\vec{u} + 0\vec{u} = (0+0)\vec{u}$

$0\vec{u} + 0\vec{u} = 0\vec{u}$

Ax 4 $\Rightarrow$ $u \in V \Leftrightarrow -u \in V$

$0\vec{u} + 0\vec{u} + (-0\vec{u}) = 0\vec{u} + (-0\vec{u})$

Ax 3 $\Rightarrow$ $0\vec{u} + (0\vec{u} + (-0\vec{u})) = 0\vec{u} + (-0\vec{u})$

Ax 5 $\Rightarrow$ $0\vec{u} + \vec{0} = \vec{0}$

Ax 4 $\Rightarrow$ $0\vec{u} = \vec{0}$

■

Prove :- $(-1)\vec{u} = -\vec{u} \quad \forall u \in V$

Proof :-

$(-1)\vec{u} = -\vec{u}$

$\Rightarrow \vec{u} + (-1)\vec{u} = \vec{0}$ we have to prove this

$\vec{u} + (-1)\vec{u} =$

$1\vec{u} + (-1)\vec{u}$ $\because$ Ax 10

$(1 + (-1))\vec{u}$ $\because$ Ax 8

$0\vec{u}$

$\vec{0}$ $\because$ Theorem $0\vec{u} = \vec{0}$

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A set of vector is V. Operations in V are defined as,

$(x,y)+(x',y') = (x+x', y+y')$ $\qquad (x,y),(x',y') \in V$

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$k(x,y) = (2kx, 2ky)$ $\qquad k \in F, (x,y) \in V$

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Proof:-

Axiom 1, 2, 3, 4 and 5 holds, where $\vec{0} = (0,0)$ and $-\vec{u} = (-x,-y)$

Axiom 6,

$k(x,y) = (2kx, 2ky)$ $\qquad 2\vec{u} \in V$

Axiom 7,

$$
\begin{aligned}
k(\vec{u}+\vec{v}) &= k((x,y)+(x',y')) \\
&= \cancel{k(x,y)+k(x',y')} \\
&= \cancel{(2kx,2ky)+(2kx',2ky')} \\
&= k(x+x', y+y') \\
&= (2k(x+x'), 2k(y+y')) \\
&= (2kx+2kx', 2ky+2ky') \\
&= (2kx, 2ky) + (2kx', 2ky') \\
&= k(x,y) + k(x',y') \\
&= k\vec{u} + k\vec{v}
\end{aligned}
$$

Axiom 8,

$$
\begin{aligned}
(k+l)\vec{u} &= (k+l)(x,y) \\
&= (2(k+l)x + 2(k+l)y) \\
&= (2kx, 2ky) + (2lx, 2ly) \\
&= k\vec{u} + l\vec{u}
\end{aligned}
$$

Axiom 9,

$$
\begin{aligned}
k(l\vec{u}) &= k(l(x,y)) \\
&= k\{(2lx, 2ly)
\end{aligned}
$$

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$$= (4kl x, 4kl y)$$
$$= ((2kl)(2x), (2kl)(2y))$$
$$= \cancel{2kl(2x, 2y)}$$
$$= 2kl(x, y)$$

$\therefore k(l\vec{u}) = (2kl)\vec{u}$

But $k(l\vec{u}) := (kl)\vec{u}$

$\therefore$ Axiom 9 fails. $\therefore$ V is not a vector space.

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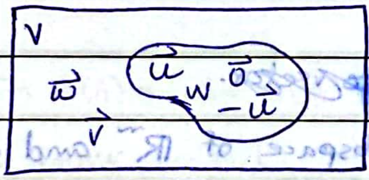
Axiom 10,

$$1\vec{u} = 1(x, y)$$
$$= (2x, 2y)$$
$$1\vec{u} \neq \vec{u}$$

Subspace:



$W \subseteq V$

$\forall u, v \in W$, $\vec{u} + \vec{v} \in W$, $k\vec{u} \in W$ $\forall k \in F$ $\Longleftrightarrow$ W is a subspace.

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If $S = \{\underline{v}_1, \underline{v}_2, \dots, \underline{v}_r\}$, then $\text{span}(S) = W$, span is the set of all ~~vecto~~ linear combination of vectors in S.

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Linear Independence

$$\sum_{i=1}^{n} k_i \underline{v}_i = 0$$

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$\forall\, k_i = 0$, then $\underline{v}_i$'s are linearly independent.

Span $S = \{ \underline{v}_1, \underline{v}_2, \dots, \underline{v}_3 \}$

if $\sum_{i=1}^{n} k_i \underline{v}_i \in W$, $\forall\, k_i \in \mathbb{R}$, & $\forall\, v_i$ then $W = \text{span}(S)$

Basis: minumum ~~n~~ vectors to span a vector space ~~are~~ called basis.

Dimension = # of vectors in Basis.

14/03/2022

~~Nullspace: Space that spans~~ ~~space spanned by zero vector.~~

Solution space of $A\underline{x} = \underline{0}$ is a subspace of $\mathbb{R}^n$ and called nullspace.
Dimension of nullspace is the number of free variables and called Nullity(A). N(A) $\Longleftrightarrow$

Column space: ~~vector space~~ Subspace of $\mathbb{R}^m$ spanned by the column of A. C(A)

Row space: Subspace of $\mathbb{R}^n$ spanned by the ~~column~~ rows of A.
$R(A) = C(A^T)$

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If $A\underline{x} = \underline{b}$ is consistent $\Leftrightarrow$ $\underline{b} \in C(A)$

Proof:-

$$A = [\underline{c}_1 | \underline{c}_2 | \cdots | \underline{c}_n]$$

$$A\underline{x} = \underline{c}_1 x_1 + \underline{c}_2 x_2 + \cdots + \underline{c}_n x_n = \underline{b}$$

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$\underline{b}$ is expresible as ~~a~~ linear combination of all columns in matrix A.

Rank = $\text{Rank}(A) = \dim C(A) = \dim C(A^T)$

$$\dim C(A) + \dim N(A) = n \ (\# \text{ of columns})$$

$$\dim C(A^T) + \dim N(A^T) = m \ (\# \text{ of rows}).$$

~~dim~~ ~~dim (R(A~~

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$\dim R(A) = \dim R(R) \qquad A \sim R$

$\dim C(A) = \dim C(R)$

$\text{rank}(A) = \text{rank}(A^T)$

Proof:-

$$\text{rank}(A) = \dim R(A) = \dim C(A^T) = \text{rank}(A^T)$$

Euclidean Inner Product.

$\langle \underline{u}, \underline{v} \rangle = \underline{u} \cdot \underline{v} \quad \forall \underline{u}, \underline{v} \in \mathbb{R}^n \ \forall n \in \mathbb{N}$

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$\|\underline{u}\| = \langle \underline{u}, \underline{u} \rangle^{1/2}$

Statement,

Let $f = f(x)$, $g = g(x)$, $\forall x \in \mathbb{R}$

$$\langle f, g \rangle = \int_a^b f(x)\, g(x)\, dx, \quad C[a,b]$$

Proof:-

i) $\langle f, g \rangle = \int_a^b f(x)\, g(x)\, dx = \int_a^b g(x)\, f(x)\, dx = \langle g, f \rangle$

ii) $\langle f+g, s \rangle = \int_a^b \left(f(x) + g(x)\right) s(x)\, dx = \int_a^b \left(f(x)\, s(x) dx + g(x)\, s(x)\right) dx$

$= \langle f, s \rangle + \langle g, s \rangle$

iii) $k\langle f,g\rangle = k\int_a^b f(x)g(x)\,dx = \int_a^b (kf(x))g(x)\,dx = \langle kf, g\rangle$

iv) $(\langle f,f\rangle = 0) \iff f=0$

$\Rightarrow \langle f,f\rangle = 0$

$\Rightarrow \int_a^b f^2(x)\,dx = 0$

$\Rightarrow f^2(x) = 0$

$\Rightarrow f(x) = 0$

$f = 0$

$\Leftarrow f = 0$

$\Leftarrow \langle f,f\rangle = \int_a^b (0)(0)\,dx$

$\Leftarrow \int_a^b (0)\,dx$

$\Leftarrow \langle f,f\rangle = 0$

$\langle f,f\rangle = \int_a^b f^2(x)\,dx$

$f\in[a,b] \Rightarrow f^2\in[a,b]$

$\min f^2(x) = 0$

$\max f^2(x) = M,\ M\in[a,b]$

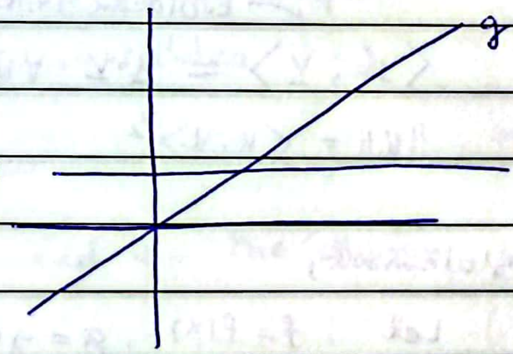
$0 \le f^2(x) \le M$

$\int_a^b 0\,dx \le \int_a^b f^2(x) \le \int_a^b M\,dx$

$0 \le \langle f,f\rangle \le M(b-a)$

~~For~~ If $\langle \underline{u},\underline{v}\rangle = 0$ then $\underline{u},\underline{v}$ are orthogonal vectors. Furthermore, if $\|\underline{u}\| = \|\underline{v}\| = 1$, then they are orthonormal.

$f(x) = \frac{1}{\sqrt{2}}$, $g(x) = \frac{\sqrt{3}}{\sqrt{2}}x$, $[-1,1]$



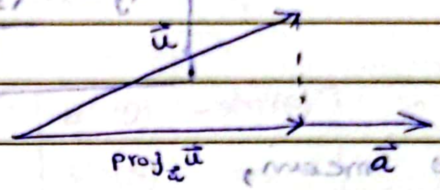
$\langle f,g\rangle = \int_{-1}^{1} \frac{1}{\sqrt{2}}\frac{\sqrt{3}}{\sqrt{2}}x\,dx$

$\langle f,g\rangle = 0$

Gram-Schmidt Process.

For set of vectors $\{\underline{u}_1, \underline{u}_2, \dots, \underline{u}_n\}$ that are bases for V, then we produce orthogonal bases $\{\underline{v}_1, \underline{v}_2, \dots, \underline{v}_n\}$ and orthogonal bases. The orthonormal bases are,

$$\left\{ \frac{\underline{v}_1}{\|\underline{v}_1\|}, \frac{\underline{v}_2}{\|\underline{v}_2\|}, \dots, \frac{\underline{v}_n}{\|\underline{v}_n\|} \right\}.$$



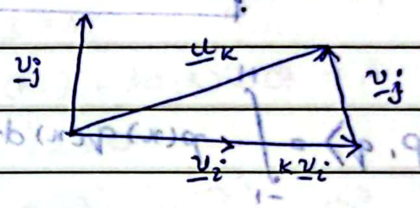
$$proj_{\underline{a}}\underline{u} = (\underline{u}\cdot\hat{\underline{a}})\hat{\underline{a}} = \frac{(\underline{u}\cdot\underline{a})\underline{a}}{\|\underline{a}\|^2}$$

<u>Proof:-</u>

$$u_k, v_i, v_j \in V, \ \forall i \neq j, \ 1 \leq i, j, k \leq n$$

$$\underline{u}_k = k\underline{v}_i + \underline{v}_j \quad \text{——— (i)}$$

Taking inner-product with $\underline{v}_i$



$$\langle \underline{u}_k, \underline{v}_i\rangle = \langle k\underline{v}_i, \underline{v}_i\rangle + \cancel{\langle \underline{v}_j, \underline{v}_i\rangle}$$

$$\langle \underline{u}_k, \underline{v}_i\rangle = k\langle \underline{v}_i, \underline{v}_i\rangle$$

$$\langle \underline{u}_k, \underline{v}_i\rangle = k\|\underline{v}_i\|^2$$

$$k = \frac{\langle \underline{u}_k, \underline{v}_i\rangle}{\|\underline{v}_i\|^2}$$

$$(i) \Rightarrow \quad \underline{u}_k = \frac{\langle \underline{u}_k, \underline{v}_i\rangle}{\|\underline{v}_i\|^2}\underline{v}_i + \underline{v}_j$$

$$\underline{v}_j = \underline{u}_k - \frac{\langle \underline{u}_k, \underline{v}_i\rangle}{\|\underline{v}_i\|^2}\underline{v}_i, \ \forall\ 1 \leq i, j, k \leq n, \ i \neq j$$

<u>Proof in n-dimension:-</u>

[redacted] We know that, $u_k \in V$ and v_i's are orthogonal basis for V. This means u_k can be written as the linear combination of $\underline{v}_i$.

$$\underline{u}_k = \sum_{i=1}^{n} a_i \underline{v}_i$$

Taking inner-product with $\underline{v}_j$, $1 \leq j \leq n$,

$$\langle \underline{u}_k, \underline{v}_j\rangle = \left\langle \sum_{i=1}^{n} a_i \underline{v}_i, \underline{v}_j \right\rangle$$

$$\langle \underline{u}_k, \underline{v}_j\rangle = \sum_{i=1}^{n} a_i \langle \underline{v}_i, \underline{v}_j\rangle$$

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$$\langle \underline{u}_k, \underline{v}_j \rangle = \sum_{i=1}^{n} a_i \delta_j^i \|\underline{v}_j\|^2$$

$$\langle \underline{u}_k, \underline{v}_j \rangle = a_j \|\underline{v}_j\|^2$$

$$\boxed{a_j = \frac{\langle \underline{u}_k, \underline{v}_j \rangle}{\|\underline{v}_j\|^2}}$$

This mean,

$$\boxed{\underline{u}_k = \sum_{i=1}^{n} \frac{\langle \underline{u}_k, \underline{v}_i \rangle}{\|\underline{v}_i\|^2} \underline{v}_i}$$

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$$\langle p, q \rangle = \int_{-1}^{+1} p(x) q(x)\, dx$$

P_2 := Polynomial space of 2^{nd} degree polynomial.

$$S = \{\underset{\underline{u}_1}{1}, \underset{\underline{u}_2}{x}, \underset{\underline{u}_3}{x^2}\}$$

$\underline{v}_1 = \underline{u}_1 = 1 \qquad \|\underline{v}_1\| = \langle 1, 1 \rangle^{1/2} = \left(\int_{-1}^{1} dx\right)^{1/2} = \sqrt{2}$

$$\|\underline{v}_3\| = \langle x^2, x^2 \rangle^{1/2} = \left(\int_{-1}^{1} x^2\, dx\right)^{1/2} = \left(\frac{x^3}{3}\Big|_{-1}^{1}\right)^{1/2} = \left(\frac{1}{3} + \frac{1}{3}\right)^{1/2} = \sqrt{\frac{2}{3}}$$

$$\underline{u} \cdot \underline{v} = \|\underline{u}\| \|\underline{v}\| \cos(\theta)$$

$$\Rightarrow \cos(\theta) = \frac{\underline{u} \cdot \underline{v}}{\|\underline{u}\| \|\underline{v}\|} = \frac{\langle \underline{u}, \underline{v} \rangle}{\|\underline{u}\| \|\underline{v}\|}$$

$$-1 \leq \cos(\theta) \leq 1$$

$$|\cos(\theta)| \leq 1$$

$$\left| \frac{\langle \underline{u}, \underline{v} \rangle}{\|\underline{u}\| \|\underline{v}\|} \right| \leq 1$$

$$\frac{|\langle \underline{u}, \underline{v} \rangle|}{\|\underline{u}\| \|\underline{v}\|} \leq 1$$

$|\langle \underline{u}, \underline{v} \rangle| \leq \|\underline{u}\| \|\underline{v}\|$

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Eigen vectors,

$Ax = \lambda x$

$Ax - \lambda x = 0$

$Ax - \lambda I x = 0$

$(A - \lambda I)x = 0$

$x \neq 0 \Rightarrow \det(A - \lambda I) = 0$

Rotation Matrix :-

$$R(\theta)_{2,} \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

Trace,

$$A = (a_{ij})_{(n \times n)}$$

$$tr(A) = \sum_{i=1}^{n} a_{ii}$$

$$tr(A) = \lambda_1 + \lambda_2 + \cdots + \lambda_n$$

$$\det(A) = \lambda_1 \lambda_2 \cdots \lambda_n$$

Proof,

Proof,

$A\underline{x} = \lambda \underline{x}$ $\quad A_{2\times 2}$

$\lambda^2 - tr(A)\lambda + \det(A) = 0$

$(A - \lambda I)x = 0$

$$\det\left(\begin{bmatrix} a-\lambda & b \\ c & d-\lambda \end{bmatrix}\right) = 0$$

$(a-\lambda)(d-\lambda) - bc = 0$

$\lambda^2 - (a+d)\lambda \overset{+ad}{-} bc = 0$

$\lambda^2 - tr(A)\lambda - bc + ad = 0$

$\lambda^2 - tr(A)\lambda + \det(A) = 0$

$$\lambda = \frac{tr(A) \pm \sqrt{(tr(A))^2 - 4\det(A)}}{2}$$

$$\lambda = \frac{(a+d) \pm \sqrt{(a+d)^2 - 4(ad-bc)}}{2}$$

$$\lambda = \frac{(a+d) \pm \sqrt{(a-d)^2 + 4bc}}{2}$$

If A has eigen-values $\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_{n-1}, \lambda_n$. then

$$Ax_i = \lambda_i x_i$$

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$$[Ax_1 \mid Ax_2 \mid Ax_3 \mid \cdots \mid Ax_{n-1} \mid Ax_n] = [\lambda_1 x_1 \mid \lambda_2 x_2 \mid \lambda_3 x_3 \mid \cdots \mid \lambda_{n-1} x_n \mid \lambda_n x_n]$$

$$A \underbrace{[x_1 \mid x_2 \mid x_3 \mid \cdots \mid x_{n-1} \mid x_n]}_{P} = \underbrace{[x_1 \mid x_2 \mid x_3 \mid \cdots \mid x_{n-1} \mid x_n]}_{P} \underbrace{\begin{bmatrix} \lambda_1 & & & & 0 \\ & \lambda_2 & & & \\ & & \lambda_3 & & \\ & & & \ddots \ \lambda_{n-1} & \\ & & & & \lambda_n \end{bmatrix}}_{D}$$

$$AP = PD$$

$$APP^{-1} = PDP^{-1}$$

$$AI = PDP^{-1}$$

$$A = PDP^{-1}$$

$$A^2 = AA = (PDP^{-1})(PDP^{-1})$$

$$= PD^2P^{-1}$$

$$A^3 = A^2A = (PD^2P^{-1})(PDP^{-1})$$

$$= PD^3P^{-1}$$

$$A^n = PD^nP^{-1}$$

18/04/2022

$$A = \begin{bmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 4-\lambda & 2 & 2 \\ 2 & 4-\lambda & 2 \\ 2 & 2 & 4-\lambda \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$(4-\lambda)^3 + 8 + 8 - 4(4-\lambda) - 4(4-\lambda) - 4(4-\lambda) = 0$$

$$\cancel{(4-\lambda)^3 - 12(4-\lambda) = -16}$$

$$\cancel{(4-\lambda)((4-\lambda)^2 - 12) = -16}$$

$x_1 = -x_2 - x_3$

$4 \quad -2 \quad -2$

$x_1 = x_2 + x_2$

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$\cancel{(4-\lambda)^3 + 16 - 12(4-\lambda) = 0}$

$\cancel{(4-\lambda)^3 + 16 - 48 + 12\lambda = 0}$

$\cancel{(4-\lambda)^3 + 12\lambda - 32 = 0}$

$\cancel{(4-\lambda)^3 + 16 - 16 + 4\lambda - 8(4-\lambda) = 0}$

$\cancel{(4-\lambda)^3 - 8(4-\lambda) + 4\lambda = 0}$

$\cancel{(4-\lambda)((4-\lambda)^2 - 8) + 4\lambda = 0}$

$(4-\lambda)(16 - 4\lambda$

$(4-\lambda)^3 + 16 - 12(4-\lambda) = 0$

$64 - 48\lambda + 12\lambda^2 - \lambda^3 + 16 - 48 + 12\lambda = 0$

$32 - 36\lambda + 12\lambda^2 - \lambda^3 = 0$

$\lambda = 8, 2, 2$

$$\begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix} \underline{v}_1 = \underline{0} \qquad \begin{bmatrix} -4 & 2 & 2 \\ 2 & -4 & 2 \\ 2 & 2 & -4 \end{bmatrix} \underline{v}_2 = \underline{0}$$

$$v_1 = c_0 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + c_1 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \qquad \cancel{v_2 = c_0 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + c_1 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}}$$

$\cancel{-2x_1 + x_2 + x_3 = 0}$

$\cancel{x_1 + x_2 - 2x_3 = 0}$

$\cancel{x_1 - 2x_2 + x_3 = 0}$

$$v_2 = c_{02} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

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$$P = \begin{bmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \Rightarrow P^{-1} = \begin{bmatrix} -1/3 & 2/6 & -1/3 \\ -1/3 & -1/3 & 2/6 \\ 1/3 & 1/3 & 1/3 \end{bmatrix}$$

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$$P^{-1}AP = D$$

$$D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 8 \end{bmatrix}$$

$T: V \longrightarrow W$ 25/04/2022

Proof :-

$T(\underline{0}) = \underline{0}$

$T(\underline{0}\underline{v}) = 0(T(\underline{v})) = \underline{0}$

$T(\underline{0}) = T(\underline{v} - \underline{v}) = T(\underline{v} + (-\underline{v})) =$

$= T(\underline{v}) + T(-\underline{v}) = T(\underline{v}) - T(\underline{v}) = 0$

$T: \mathbb{R}^n \to \mathbb{R}^m$, $\ker(T) := \{ \underline{v} \mid T(\underline{v}) = \underline{0} \}$

$J: P_1 \to \mathbb{R}$

$$J(P) = \int_{-1}^{+1} P(x)dx$$

$$= \int_{-1}^{1} (ax+b)dx$$

$$= \left(a\frac{x^2}{2} + bx\right)_{-1}^{1}$$

$0 = a\left(\frac{1}{2} - \frac{1}{2}\right) + b(1-(-1))$

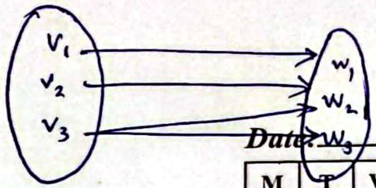
$0 = 2b$

$b = 0$

$I(\underline{v}) = \underline{v}$

$I(\underline{v} + \underline{w}) = \underline{v} + \underline{w} = I(\underline{v}) + I(\underline{w})$

$I(k\underline{v}) = k\underline{v} = kI(\underline{v})$



$\cancel{R(I) = \{\underline{v} \mid \underline{v} \in V\}}$ $R(I) = V$

$KER(I) = \{\underline{0}\}$

$T: V \longrightarrow W \Rightarrow T^{-1}: R(T) \longrightarrow V$