CS 314/PHYS 300: Quantum Computing: Homework #3

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1. **Grover's Algorithm - States** In the version of Grover's Algorithm that we studied in the class (where our initial state was $|\psi\rangle$ as a uniform superposition of all states) we used:

   - an oracle 'O' to invert the sign of the required state.
   - a Grover Diffusion Operator 'G' denoted by $2|\psi\rangle\langle\psi| - I$ to amplify the amplitude of the inverted state.

   (a) Show that the Grover's Diffusion Operator 'G'

   $$2|\psi\rangle\langle\psi| - I$$

   is equivalent to

   $$H^{\otimes n}(2|0\rangle\langle 0| - I)H^{\otimes n}.$$

   Solution: We know that $|\psi\rangle = \frac{1}{\sqrt{N}}\sum_{x=0}^{N-1}|x\rangle = |+\rangle^{\otimes n} = H^{\otimes n}|0\rangle$ from this we know that $\langle\psi| = (H^{\otimes n}|0\rangle)^T = |0\rangle^T H^{\otimes nT}$, and since $H^{\otimes n}$ is unitary and $H^{\otimes n} \in \mathbb{R}^{n\times n}$, we have that $H^{\otimes nT} = H^{\otimes n}$ and $H^{\otimes n}H^{\otimes n} = I$, so then $\langle\psi| = \langle 0| H^{\otimes n}$.
   So we have that $H^{\otimes n}(2|0\rangle\langle 0| - I)H^{\otimes n} = 2H^{\otimes n}|0\rangle\langle 0| H^{\otimes n} - H^{\otimes n}IH^{\otimes n} = 2|\psi\rangle\langle\psi| - I$.

   (b) Construct (draw) a partial circuit for the Grover's Algorithm that uses this construction (i.e. $H^{\otimes n}(2|0\rangle\langle 0| - I)H^{\otimes n}$) used for the Grover Diffusion Operator.

   Solution: $|\psi\rangle$ — $\boxed{H^{\otimes n}}$ — $\boxed{2|0\rangle\langle 0| - I}$ — $\boxed{H^{\otimes n}}$ —

2. **Grover's Algorithm - Sample Run - Adapted from Cambridge University Course on Quantum Computation** ***https://www.cl.cam.ac.uk/teaching/exams/pastpapers/y2023p8q11.pdf***
   Let there be a database containing 32 elements, indexed by the binary numbers 00000 to 11111. A single element 00110 is marked.

   (a) If Grover's search algorithm is applied to find the marked element, what should the initial state be set to, and what is the state after a single Grover iterate has been applied?

   Solution: Let $|s\rangle$ be the initial state then, $|s\rangle = \frac{1}{4\sqrt{2}}(|00000\rangle + |00001\rangle + |00010\rangle + |00011\rangle + |00100\rangle + |00101\rangle + |00110\rangle + |00111\rangle + |01000\rangle + |01001\rangle + |01010\rangle + |01011\rangle + |01100\rangle + |01101\rangle + |01110\rangle + |01111\rangle + |10000\rangle + |10001\rangle + |10010\rangle + |10011\rangle + |10100\rangle + |10101\rangle + |10110\rangle + |10111\rangle + |11000\rangle + |11001\rangle + |11010\rangle + |11011\rangle + |11100\rangle + |11101\rangle + |11110\rangle + |11111\rangle)$
   After applying the oracle we get $|\psi\rangle = \frac{1}{4\sqrt{2}}(|00000\rangle + |00001\rangle + |00010\rangle + |00011\rangle + |00100\rangle + |00101\rangle - |00110\rangle + |00111\rangle + |01000\rangle + |01001\rangle + |01010\rangle + |01011\rangle + |01100\rangle + |01101\rangle + |01110\rangle + |01111\rangle + |10000\rangle + |10001\rangle + |10010\rangle + |10011\rangle + |10100\rangle + |10101\rangle + |10110\rangle + |10111\rangle + |11000\rangle + |11001\rangle + |11010\rangle + |11011\rangle + |11100\rangle + |11101\rangle + |11110\rangle + |11111\rangle)$
   The grover diffusion operator is a 32×32 matrix U_s such that:

   $$U_s = \begin{bmatrix} -\frac{15}{16} & \frac{1}{16} & \cdots & \frac{1}{16} \\ \frac{1}{16} & -\frac{15}{16} & \cdots & \frac{1}{16} \\ \vdots & \vdots & \cdots & \vdots \\ \frac{1}{16} & \frac{1}{16} & \cdots & -\frac{15}{16} \end{bmatrix}$$

   After the application of the diffusion operator we get $U_s|\psi\rangle = \frac{7}{32\sqrt{2}}|00000\rangle + \frac{7}{32\sqrt{2}}|00001\rangle +$

$\frac{7}{32\sqrt{2}}|00010\rangle + \frac{7}{32\sqrt{2}}|00011\rangle + \frac{7}{32\sqrt{2}}|00100\rangle + \frac{7}{32\sqrt{2}}|00101\rangle + \frac{23}{32\sqrt{2}}|00110\rangle + \frac{7}{32\sqrt{2}}|00111\rangle + \frac{7}{32\sqrt{2}}|01000\rangle + \frac{7}{32\sqrt{2}}|01001\rangle + \frac{7}{32\sqrt{2}}|01010\rangle + \frac{7}{32\sqrt{2}}|01011\rangle + \frac{7}{32\sqrt{2}}|01100\rangle + \frac{7}{32\sqrt{2}}|01101\rangle + \frac{7}{32\sqrt{2}}|01110\rangle + \frac{7}{32\sqrt{2}}|01111\rangle + \frac{7}{32\sqrt{2}}|10000\rangle + \frac{7}{32\sqrt{2}}|10001\rangle + \frac{7}{32\sqrt{2}}|10010\rangle + \frac{7}{32\sqrt{2}}|10011\rangle + \frac{7}{32\sqrt{2}}|10100\rangle + \frac{7}{32\sqrt{2}}|10101\rangle + \frac{7}{32\sqrt{2}}|10110\rangle + \frac{7}{32\sqrt{2}}|10111\rangle + \frac{7}{32\sqrt{2}}|11000\rangle + \frac{7}{32\sqrt{2}}|11001\rangle + \frac{7}{32\sqrt{2}}|11010\rangle + \frac{7}{32\sqrt{2}}|11011\rangle + \frac{7}{32\sqrt{2}}|11100\rangle + \frac{7}{32\sqrt{2}}|11101\rangle + \frac{7}{32\sqrt{2}}|11110\rangle + \frac{7}{32\sqrt{2}}|11111\rangle$

(b) To find the marked element with maximum probability requires N iterates in total. What is the value of N, and what is the probability of correctly finding the marked element? Show the complete working at how did you arrive at the value of probability.

Solution: The value of $N = \lceil \frac{\pi\sqrt{32}}{4} \rceil = 5$

After 5 iterations our qubit $|\psi\rangle$ is (computed with numpy): $|\psi\rangle = -0.0672893\,|00000\rangle - 0.0672893\,|00001\rangle - 0.0672893\,|00010\rangle - 0.0672893\,|00011\rangle - 0.0672893\,|00100\rangle - 0.0672893\,|00101\rangle + 0.92716593\,|00110\rangle - 0.0672893\,|00111\rangle - 0.0672893\,|01000\rangle - 0.0672893\,|01001\rangle - 0.0672893\,|01010\rangle - 0.0672893\,|01011\rangle - 0.0672893\,|01100\rangle - 0.0672893\,|01101\rangle - 0.0672893\,|01110\rangle - 0.0672893\,|01111\rangle - 0.0672893\,|10000\rangle + -0.0672893\,|10001\rangle - 0.0672893\,|10010\rangle - 0.0672893\,|10011\rangle - 0.0672893\,|10100\rangle - 0.0672893\,|10101\rangle - 0.0672893\,|10110\rangle - 0.0672893\,|10111\rangle - 0.0672893\,|11000\rangle - 0.0672893\,|11001\rangle - 0.0672893\,|11010\rangle - 0.0672893\,|11011\rangle - 0.0672893\,|11100\rangle - 0.0672893\,|11101\rangle - 0.0672893\,|11110\rangle - 0.0672893\,|11111\rangle$

Now the probability of measuring $|00110\rangle$ is $0.92716593^2 = 0.859636661752765$

(c) If the algorithm is instead run with $3N$ iterates in total, what is the probability of correctly finding the marked element? Comment on your answer.

Solution: $3N = 3 \times \lceil \frac{\pi\sqrt{32}}{4} \rceil = 15$

After 15 iterations our qubit $|\psi\rangle$ is (computed with numpy): $|\psi\rangle = 0.12841974\,|00000\rangle + 0.12841974\,|00001\rangle + 0.12841974\,|00010\rangle + 0.12841974\,|00011\rangle + 0.12841974\,|00100\rangle + 0.12841974\,|00101\rangle - 0.69911334\,|00110\rangle + 0.12841974\,|00111\rangle + 0.12841974\,|01000\rangle + 0.12841974\,|01001\rangle + 0.12841974\,|01010\rangle + 0.12841974\,|01011\rangle + 0.12841974\,|01100\rangle + 0.12841974\,|01101\rangle + 0.12841974\,|01110\rangle + 0.12841974\,|01111\rangle + 0.12841974\,|10000\rangle + 0.12841974\,|10001\rangle + 0.12841974\,|10010\rangle + 0.12841974\,|10011\rangle + 0.12841974\,|10100\rangle + 0.12841974\,|10101\rangle + 0.12841974\,|10110\rangle + 0.12841974\,|10111\rangle + 0.12841974\,|11000\rangle + 0.12841974\,|11001\rangle + 0.12841974\,|11010\rangle + 0.12841974\,|11011\rangle + 0.12841974\,|11100\rangle + 0.12841974\,|11101\rangle + 0.12841974\,|11110\rangle + 0.12841974\,|11111\rangle$

Now the probability of measuring $|00110\rangle$ is $(-0.69911334)^2 = 0.4887594621659556$

We see that running more than N iterations actually reduced the probability of measuring $|00110\rangle$

3. **Eigenvalues and Eigenvectors** Find the Eigenvalues and Eigenvectors of:

(a) The 'X' (or the not) quantum operator given by $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

Solution: The eigen values of X are,

$$\mid X - \lambda I \mid = 0 \tag{1}$$

$$\begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = 0 \tag{2}$$

$$\lambda^2 - 1 = 0 \tag{3}$$

$$\lambda_1, \lambda_2 = -1, 1 \tag{4}$$

Eigen vector for λ_1 is,

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \mathbf{v}_1 = 0 \tag{5}$$

Fixing the second variables to k we get,

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ k \end{bmatrix} = 0 \implies x_1 = -k \tag{6}$$

$$\implies \mathbf{v}_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix} \tag{7}$$

Eigen vector for λ_2 is,

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \mathbf{v}_2 = 0 \tag{8}$$

Similarly fixing the second variable to k,

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ k \end{bmatrix} = 0 \implies x_1 = k \tag{9}$$

$$\implies \mathbf{v}_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \tag{10}$$

(b) The 'H' (or the Hadamard) Operator given by: $\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$.

Solution: The eigenvalues are,

$$\mid H - \lambda I \mid = 0 \tag{11}$$

$$\begin{vmatrix} \frac{1}{\sqrt{2}} - \lambda & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} - \lambda \end{vmatrix} = 0 \tag{12}$$

$$\left(\frac{1}{\sqrt{2}} - \lambda\right)\left(-\frac{1}{\sqrt{2}} - \lambda\right) - \left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right) = 0 \tag{13}$$

$$-\frac{1}{2} + \lambda^2 - \frac{1}{2} = 0 \tag{14}$$

$$\lambda^2 = 1 \tag{15}$$

$$\lambda_1, \lambda_2 = -1, 1 \tag{16}$$

Eigen vector for λ_1 is,

$$\begin{bmatrix} 1+\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 1-\frac{1}{\sqrt{2}} \end{bmatrix} \mathbf{v}_1 = 0 \tag{17}$$

Fixing the second variable we get,

$$\begin{bmatrix} 1+\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 1-\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} x_1 \\ k \end{bmatrix} = 0 \implies x_1 = k(1-\sqrt{2}) \tag{18}$$

$$\implies \mathbf{v}_1 = \begin{bmatrix} 1-\sqrt{2} \\ 1 \end{bmatrix} \tag{19}$$

After normalizing,

$$\mathbf{v}_1 = \begin{bmatrix} \frac{1-\sqrt{2}}{\sqrt{4-2\sqrt{2}}} \\ \frac{1}{\sqrt{4-2\sqrt{2}}} \end{bmatrix} \tag{20}$$

Eigen vector for λ_2 is,

$$\begin{bmatrix} -1+\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -1-\frac{1}{\sqrt{2}} \end{bmatrix} \mathbf{v}_2 = 0 \tag{21}$$

Fixing the second variable we get,

$$\begin{bmatrix} -1+\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -1-\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} x_1 \\ k \end{bmatrix} = 0 \implies x_1 = k(1+\sqrt{2}) \tag{22}$$

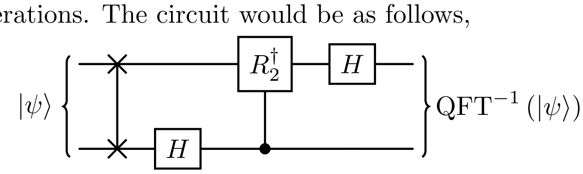
$$\implies \mathbf{v}_2 = \begin{bmatrix} \frac{1+\sqrt{2}}{\sqrt{4+2\sqrt{2}}} \\ \frac{1}{\sqrt{4+2\sqrt{2}}} \end{bmatrix} \tag{23}$$

The Eigenvalues of a square matrix A can be calculated by solving for the eigenvalues 'λ' in the equation '$\det(A - \lambda I) = 0$', and then solving for the eigenvector v using the equation '$(\lambda I - A)v = 0$'. Since we are working with qubits, ensure that the norm of your eigenvectors equals to 1. Show all your work.

4. **Inverse Quantum Fourier Transform Using Pauli Matrices and H**

   (a) Using the circuit for *inverse* quantum Fourier transform, construct (draw) the circuit for the *inverse* Fourier transform for 2 bits using the Hadamard, controlled-T and controlled-NOT gates only. Show that the original circuit for the inverse QFT and your circuit are equivalent.

Solution: The circuit for the Inverse Quantum Fourier Transform will contain a Controlled $R^\dagger$, H, and Swap operations. The circuit would be as follows,



Since we are constrained to use only the CT and CNOT gates. Therefore, from theory, we know that three CNOT gates in the alternating pattern make a swap gate. The next step is to replace the controlled $R_2^\dagger$ gate with the controlled T gate.

$$R_2 = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix} \iff R_2^\dagger = \begin{bmatrix} 1 & 0 \\ 0 & -i \end{bmatrix} \tag{24}$$

The T gate operation is defined as,

$$T = \begin{bmatrix} 1 & 0 \\ 0 & e^{\frac{i\pi}{4}} \end{bmatrix}$$

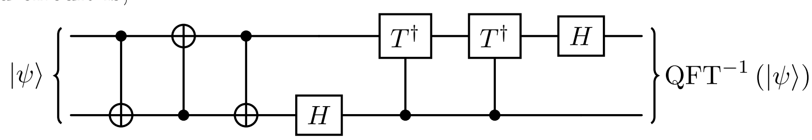
Applying two consecutive T gates will give us:

$$TT = \begin{bmatrix} 1 & 0 \\ 0 & e^{\frac{i\pi}{4}} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & e^{\frac{i\pi}{4}} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & e^{\frac{2\pi i}{4}} \end{bmatrix} = R_2 \tag{25}$$

Which implies that

$$R_2^\dagger = (TT)^\dagger = T^\dagger T^\dagger \tag{26}$$

Hence, in our original QFT^{-1} circuit, we will replace the $R_2^\dagger$ gate with two consecutive T gates. The updated circuit is,



(b) Write a Cirq function to implement this inverse quantum Fourier transform. Run your circuit for all combinations of two qubits and show that the output is correct.

Solution: Link to the code.