

## 0.1 Modelling aircraft take-off

The set of ordinary differential equations that model the take-off run are written in the following form:

$$\begin{pmatrix} \dot{s} \\ \dot{V} \\ \dot{\gamma} \\ \dot{h} \end{pmatrix} = \begin{pmatrix} f_1(s, V, \gamma, h; \alpha) \\ f_2(s, V, \gamma, h; \alpha) \\ f_3(s, V, \gamma, h; \alpha) \\ f_4(s, V, \gamma, h; \alpha) \end{pmatrix} \quad \text{with} \quad \begin{cases} x_1 = s \\ x_2 = V \\ x_3 = \gamma \\ x_4 = h \end{cases} \quad \text{and} \quad u = \alpha \quad (1)$$

or, more concisely:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, u(t)) \quad (2)$$

The unknown  $\mathbf{x} = [x_1, x_2, x_3, x_4]^T$  is the vector of state variables. The input  $u(t)$  is a given function of time, for  $0 \leq t \leq t_{\text{final}}$ , that corresponds to an assumed time history of the angle of attack during take-off.

The right-hand sides of system (1) are defined by the following functions:

$$f_1(\mathbf{x}, u) = x_2 \quad (3a)$$

$$f_2(\mathbf{x}, u) = \frac{g}{W} \begin{cases} T(x_2) - D(x_2, u) - \mu[W - L(x_2, u)] & \text{if } \mathcal{S}(x_2, u) < 1 \\ T(x_2) \cos u - D(x_2, u) - W \sin x_3 & \text{if } \mathcal{S}(x_2, u) \geq 1 \end{cases} \quad (3b)$$

$$f_3(\mathbf{x}, u) = \frac{g}{W x_2} \begin{cases} 0 & \text{if } \mathcal{S}(x_2, u) < 1 \\ L(x_2, u) + T(x_2) \sin u - W \cos x_3 & \text{if } \mathcal{S}(x_2, u) \geq 1 \end{cases} \quad (3c)$$

$$f_4(\mathbf{x}, u) = x_2 \sin x_3 \quad (3d)$$

The thrust  $T(x_2)$  is calculated by means of the interpolating function  $T_{\text{tab}}(V_a)$  based on a table lookup algorithm — where  $V_a = V + V_w$  is the airspeed and  $V_w$  is the wind speed (horizontal component, positive if opposite to the aircraft motion). The drag  $D$  and lift  $L$ , as functions of airspeed  $V_a$  and angle of attack, are given by the conventional formulas

$$D(x_2, u) = \frac{1}{2} \rho (x_2 + V_w \cos x_3)^2 S C_D(u), \quad L(x_2, u) = \frac{1}{2} \rho (x_2 + V_w \cos x_3)^2 S C_L(u) \quad (3e)$$

The switching function  $\mathcal{S}$  of aircraft velocity and angle of attack is defined as follows:

$$\mathcal{S}(x_2, u) = \frac{L(x_2, u)}{W \cos x_3} \quad (3f)$$

The formulas (3) make the system (2) a closed set of ODEs. When the function  $u(t)$  is assigned and the system is associated to a set of initial conditions a well-posed initial value problem (IVP) is formed, which can be solved numerically.

The function  $u$  can be constructed by picking the time  $t_{\text{Rot}}$  when the rotation speed  $V_{\text{rot}}$  is

reached along the ground roll. Setting

$$u(t) = \begin{cases} \alpha_{\text{ground}} & \text{if } t < t_{\text{rot}} \\ \alpha(t) & \text{if } t \geq t_{\text{rot}} \end{cases} \quad (4)$$

In Table 1 are reported the take-off characteristic speeds and their corresponding requirements as defined by FAR 25.

**Table 1** Take-off speeds and FAR 25 requirements..

Speed	Description	Requirement
$V_S$	fuselage total length	—
$V_{MC}$	minimum control speed with one engine inoperative (OEI)	—
$V_1$	OEI decision speed	$\geq V_{mc}$
$V_{Rot}$	rotation speed	$> 1.05 V_{MC}$
$V_{MU}$	minimum unstick speed for safe flight	$\geq V_S$
$V_{LO}$	lift-off speed	$> 1.10 V_{MU}$ $> 1.05 V_{MU} \text{ (OEI)}$
$V_2$	take-off climb speed at 35 ft	$> 1.20 V_S$ $> 1.10 V_{MC}$