0.1 Modelling aircraft take-off

The set of ordinary differential equations that models the take-off run is written in the following form:

$$\begin{Bmatrix} \dot{s} \\ \dot{V} \\ \dot{\gamma} \\ \dot{h} \end{Bmatrix} = \begin{Bmatrix} f_1\big(s,\, V,\, \gamma,\, h\,;\,\alpha\big) \\ f_2\big(s,\, V,\, \gamma,\, h\,;\,\alpha\big) \\ f_3\big(s,\, V,\, \gamma,\, h\,;\,\alpha\big) \\ f_4\big(s,\, V,\, \gamma,\, h\,;\,\alpha\big) \end{Bmatrix} \qquad \text{with} \quad \begin{cases} x_1 = s \\ x_2 = V \\ x_3 = \gamma \\ x_4 = h \end{cases} \qquad \text{and} \quad u = \alpha \tag{1}$$

or, more concisely:

$$\dot{\boldsymbol{x}} = \boldsymbol{f}\big(\boldsymbol{x}\,;\,u\big) \tag{2}$$

The unknown $\boldsymbol{x} = [x_1,\, x_2,\, x_3,\, x_4]^{\mathrm{T}}$ is the vector of state variables. The input $u(t)$ is a given function of time, for $0 \le t \le t_{\text{final}}$, that corresponds to an assumed time history of the angle of attack during take-off.

The right-hand sides of system (1) are defined by the following functions:

$$f_1\big(\boldsymbol{x}\,,\,u\big) = x_2 \tag{3a}$$

$$f_2\big(\boldsymbol{x}\,,\,u\big) = \frac{g}{W}\begin{cases} T(x_2) - D(x_2,u) - \mu\big[W - L(x_2,u)\big] & \text{if } \mathcal{S}(x_2,u) < 1 \\ T(x_2)\cos u - D(x_2,u) - W\sin x_3 & \text{if } \mathcal{S}(x_2,u) \ge 1 \end{cases} \tag{3b}$$

$$f_3\big(\boldsymbol{x}\,,\,u\big) = \frac{g}{W\,x_2}\begin{cases} 0 & \text{if } \mathcal{S}(x_2,u) < 1 \\ L(x_2,u) + T(x_2)\sin u - W\cos x_3 & \text{if } \mathcal{S}(x_2,u) \ge 1 \end{cases} \tag{3c}$$

$$f_4\big(\boldsymbol{x}\,,\,u\big) = x_2\,\sin x_3 \tag{3d}$$

The thrust $T(x_2)$ is calculated by means of the interpolating function $T_{\text{tab}}\big(V_{\text{a}}\big)$ based on a table lookup algorithm — where $V_{\text{a}} = V + V_{\text{w}}$ is the airspeed and V_{w} is the wind speed (horizontal component, positive if opposite to the aircraft motion). The drag D and lift L, as functions of airspeed V_{a} and angle of attack, are given by the conventional formnulas

$$D(x_2,u) = \frac{1}{2}\,\rho\,\big(x_2 + V_{\text{w}}\cos x_3\big)^2\,S\,C_D\big(u\big)\,, \quad L(x_2,u) = \frac{1}{2}\,\rho\,\big(x_2 + V_{\text{w}}\cos x_3\big)^2\,S\,C_L\big(u\big) \tag{3e}$$

The switching function $\mathcal{S}$ of aircraft velocity and angle of attack is defined as follows:

$$\mathcal{S}(x_2,u) = \frac{L(x_2,u)}{W\cos x_3} \tag{3f}$$

The formulas (3) make the system (2) a closed set of ODEs. When the function $u(t)$ is assigned and the system is associated to a set of initial conditions a well-posed initial value problem (IVP) is formed, which can be solved numerically.

The function u can be constructed by picking the time t_{Rot} when the rotation speed V_{Rot} is reached along the ground roll. It is assumed that

$$u(t) = \begin{cases} \alpha_g & \text{if } t < t_{\text{Rot}} \\ \alpha_1(t) & \text{if } t \geq t_{\text{Rot}} \end{cases} \tag{4}$$

with a constant α_g during the ground run up to the rotation speed, and a given non-zero law $\alpha_1(t)$ for the post-rotation angle of attack time history.

In Table 1 are reported the take-off characteristic speeds and their corresponding requirements as defined by FAR 25.

Table 1 Take-off speeds and FAR 25 requirements..

Speed	Description	Requirement
V_S	fuselage total length	—
V_{MC}	minimum control speed with one engine inoperative (OEI)	—
V_1	OEI decision speed	$\geq V_{mc}$
V_{Rot}	rotation speed	$> 1.05\, V_{MC}$
V_{MU}	minimum unstick speed for safe flight	$\geq V_S$
V_{LO}	lift-off speed	$> 1.10\, V_{MU}$
		$> 1.05\, V_{MU}$ (OEI)
V_2	take-off climb speed at 35 ft	$> 1.20\, V_S$
		$> 1.10\, V_{MC}$

The drag coefficient C_D that appears in (3e) can be modelled as follows:

$$C_D = C_{D0} + \Delta C_{D0} + \left(K + \frac{G}{\pi \, \text{Æ} e} \right) C_L^2 \tag{5}$$

with a ΔC_{D0} due to flap and undercarriage

$$\Delta C_{D0} = \frac{W}{S} K_{\text{uc}} \, m_{\text{MTO}}^{-0.219} \,, \qquad K_{\text{uc}} = \begin{cases} 5.81 \cdot 10^{-5} & \text{zero flap} \\ 3.16 \cdot 10^{-5} & \text{max flap down} \end{cases} \tag{6}$$

where the maximum mass m_{MTO} is expressed in kg and the wing loading W/S is expressed in N/m^2. The term G in (5) incorporates the ground effect and is calculated as

$$G = \frac{\left(16\, h_{\text{W}}/b\right)^2}{1 + \left(16\, h_{\text{W}}/b\right)^2} \tag{7}$$

with h_{W} the height of wing above the ground. With conventional undercarriages h_{W}/b is usually between 0.1 and 0.2 when the aircraft is on the ground. When the aircraft is airborne it can be assumed $h_{\text{W}} \approx h$.