0.1 Modelling aircraft take-off

The set of ordinary differential equations that model the take-off run is the following:

$$\begin{Bmatrix} \dot{s} \\ \dot{V} \\ \dot{\gamma} \\ \dot{h} \end{Bmatrix} = \begin{Bmatrix} f_1\big(s,\ V,\ \gamma,\ h\ ;\ \alpha\big) \\ f_2\big(s,\ V,\ \gamma,\ h\ ;\ \alpha\big) \\ f_3\big(s,\ V,\ \gamma,\ h\ ;\ \alpha\big) \\ f_4\big(s,\ V,\ \gamma,\ h\ ;\ \alpha\big) \end{Bmatrix} \quad \text{with} \quad \begin{cases} x_1 = s \\ x_2 = V \\ x_3 = \gamma \\ x_4 = h \end{cases} \quad \text{and} \quad u = \alpha \tag{1}$$

that can be written in the concise form:

$$\dot{\boldsymbol{x}} = \boldsymbol{f}\big(\boldsymbol{x}\ ,\ u(t)\big) \tag{2}$$

where $\boldsymbol{x} = [x_1,\ x_2,\ x_3,\ x_4]^{\mathrm{T}}$ is the vector of state variables and $u(t)$ is a given function of time, for $0 \le t \le t_{\text{final}}$.

The functions defining the right-hand sides of system (1) are defined as follows:

$$f_1\big(\boldsymbol{x}\ ,\ u(t)\big) = x_2 \tag{3a}$$

$$f_2\big(\boldsymbol{x}\ ,\ u(t)\big) = \frac{g}{W}\begin{cases} T(x_2) - D(x_2,u) - \mu\big[W - L(x_2,u)\big] & \text{if } \mathcal{S}(x_2,u) < 1 \\ T(x_2)\cos u - D(x_2,u) - W\sin x_3 & \text{if } \mathcal{S}(x_2,u) \ge 1 \end{cases} \tag{3b}$$

$$f_3\big(\boldsymbol{x}\ ,\ u(t)\big) = \frac{g}{W\,x_2}\begin{cases} 0 & \text{if } \mathcal{S}(x_2,u) < 1 \\ L(x_2,u) + T(x_2)\sin u - W\cos x_3 & \text{if } \mathcal{S}(x_2,u) \ge 1 \end{cases} \tag{3c}$$

$$f_4\big(\boldsymbol{x}\ ,\ u(t)\big) = x_2\ \sin x_3 \tag{3d}$$

The thrust $T(x_2)$ is calculated by means of the interpolating function $T_{\text{tab}}\big(V_{\text{a}}\big)$ based on a table lookup algorithm — where $V_{\text{a}} = V + V_{\text{w}}$ is the airspeed. The drag D and lift L, as functions of airspeed V_{a} and angle of attack, are given by the conventional formnulas

$$D(x_2,u) = \frac{1}{2}\,\rho\,\big(x_2 + V_{\text{w}}\big)^2\,S\,C_D\big(u\big)\ , \qquad L(x_2,u) = \frac{1}{2}\,\rho\,\big(x_2 + V_{\text{w}}\big)^2\,S\,C_L\big(u\big) \tag{3e}$$

The switching function $\mathcal{S}$ of aircraft velocity and angle of attack is defined as follows:

$$\mathcal{S}(x_2,u) = \frac{L(x_2,u)}{W\cos x_3} \tag{3f}$$

The formulas (3) make the system (2) a closed set of ODEs. When the function $u(t)$ is assigned and the system is associated to a set of initial conditions a well-posed initial value problem (IVP) is formed, which can be solved numerically.

The function u can be constructed by picking the time t_{rot} when the rotation speed V_{rot} is reached along the ground roll. Setting

$$u(t) = \begin{cases} \alpha_{\mathrm{ground}} & \text{if } t < t_{\mathrm{rot}} \\ \alpha(t) & \text{if } t \geq t_{\mathrm{rot}} \end{cases} \tag{4}$$