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A simplified algorithm to model aircraft acceleration during take-off

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Aircraft take-off performance, which quantifies the aircraft's capability of accelerating to become airborne within the runway constraints, is crucial to the safety of the manoeuvre. As a result, scheduled calculations are performed prior to dispatch to ensure that runway lengths are indeed adequate. The same equations, however, cannot readily be used for real-time performance monitoring and a simplified algorithm is thus required. The use of a second order approximation of the velocity profile is proposed. An assessment of the goodness of fit using numerical analysis confirms that, for the operational envelope of typical aircraft, this approximation is sufficiently accurate to warrant use in performance estimation.

Nomenclature

a	=	longitudinal acceleration
g	=	acceleration due to gravity
k_1, k_2	=	constants
m	=	aircraft mass
t	=	time
t_R	=	time to rotation in the take-off run
t_{ROT}	=	rotation time during take-off
v_w	=	headwind component of windspeed
A, B, C	=	coefficients
C_D	=	coefficient of drag
C_L	=	coefficient of lift
M, N	=	roots of the 2nd order polynomial defining acceleration during take-off
S	=	wing reference area
S_G	=	distance covered to rotation during take-off
S_{ROT}	=	distance covered during rotation at take-off
T	=	thrust
V_g	=	aircraft ground speed
V_{LOF}	=	lift-off speed
V_R	=	rotation speed
V_2	=	climb safety speed
W	=	aircraft weight
Y_i	=	i-th observed independent variable
$\overline{Y}$	=	mean value of the observed independent variable ($=\sum Y_i/n$)
$\hat{Y}_i$	=	i-th independent variable of the fitted curve obtained through regression
μ	=	runway rolling friction
ρ	=	air density
θ	=	runway slope

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I. Introduction

In order to take-off, an aircraft must accelerate down the runway and achieve an airspeed allowing it to become airborne and climb away over any obstacles. This fundamental requirement is, in practice, satisfied by ensuring that the aircraft is dispatched at a weight that will allow the installed thrust to adequately accelerate the aircraft to the target airspeeds within the constraints of the runway from which the take-off will be attempted. Whilst commercial constraints require operators to dispatch aircraft with a maximum amount of fuel and payload, sufficient leeways need to be allowed for to reduce the risk of hitting obstacles to a level that is acceptable. It is the responsibility of the operator to ensure that the runway from which the take-off will be attempted is indeed adequate. To this effect, scheduled performance calculations are carried out prior to dispatch, from which the maximum weight at which the aircraft can be safely dispatched to allow it to become airborne within the runway constraints and achieve an adequate climb gradient that will ensure it will avoid any obstacles is calculated.

14CFR125 operations require operators to allow for the contingency of engine failure during take-off. Consequently, Part 25 certified aircraft must be capable of also successfully[‡] completing the take-off attempt if an engine failure is experienced at any stage during the run. This effectively requires the attempt to be rejected if failure is experienced at low speeds early on in the run, since the reduced thrust will not be sufficient to allow the aircraft to become airborne within the remaining length of the runway. If, instead, the failure is experienced towards the end of the run, the take-off is continued on the grounds that insufficient runway will be available to bring the aircraft to a halt in time. This implies that the aircraft must have the necessary excess thrust installed in order to ensure that it can still accelerate to the scheduled airspeed and achieve a minimum positive climb once the engine has failed.

The take-off run can therefore be seen to consist of two stages, namely an initial stage during which the run must be aborted if an anomaly is detected, and a final stage in which the run must be always continued (unless the aircraft is clearly not airworthy). The critical point dividing the two stages of the run is identified as the decision speed V_1 (Figure 1).

In scheduled performance, aircraft take-off performance is measured in terms of key distances required to reach salient points during the take-off attempt. Three distances are defined: the take-off run required (TORR), the take-off distance required

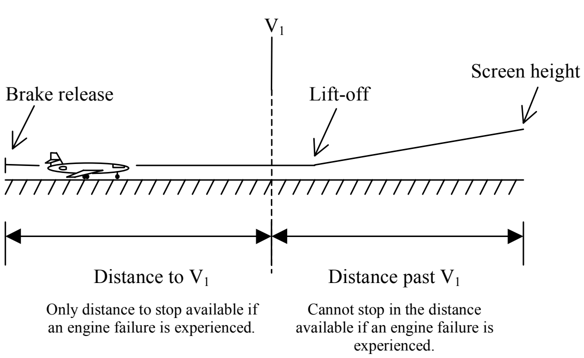

Figure 1: The decision speed V_1 as a point of no return during take-off.

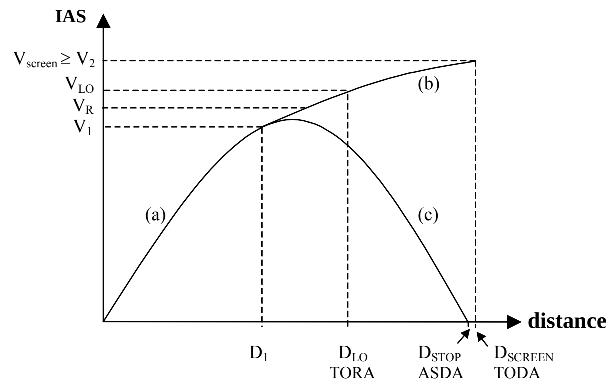

Figure 2: Minimum takeoff performance characteristic and the V-speed technique.
(a) Acceleration to V_1, all engines operative (AEO). Sufficient runway is available to bring the aircraft safely to a halt if the run is aborted. In the event of an engine failure, the run must be aborted as insufficient runway is available to allow the aircraft to lift off or reach screen height within the remaining runway distances (TORA/TODA).
(b) Acceleration past V_1 to V_R, V_{LO} and V_2. The aircraft becomes airborne before TORA and achieves screen height before the end of the runway (TODA). The aircraft is too fast and too far down the runway to be brought to rest in the remaining distance if an engine fails. Aircraft scheduled performance, however, allows the aircraft to achieve screen height even if an engine failure is experienced at any time.
(c) Run aborted at V_1. This is the latest point along the run at which the take-off may be aborted. The aircraft is brought to rest in ASDA using the retarding mechanisms allowed for by regulation. V_1 consequently defines a 'point of no return'.

‡ A takeoff attempt is, in this text, defined as successful if the manoeuvre is completed without an accident and unsuccessful otherwise.

(TODR) and the accelerate-stop distance required (ASDR). TORR is the distance from brake release required (or allowed for) for the aircraft to lift off the runway, TODR is the distance required to achieve screen height[§], whilst ASDR is the distance required from brake release to bring the aircraft to a halt if the run is aborted at V_1, the latest possible moment (Figure 2). Each runway is described by the relevant distances available, referred to as TORA, TODA and ASDA respectively. Safety is then assured in scheduled performance calculations by ensuring that the required distances fit within those available.

The TODR and TORR are defined in Part 25 regulations as the larger of a) the expected distances in the event of an engine failure a V_1 and b) the expected distances under normal (all engines operating) conditions, factored by 15%. The expected performance is statistically the average that is expected and therefore there is a 50% probability of the aircraft exceeding these distances. This estimate is referred to as a *gross* value. A 15% leeway, resulting in a quantity referred to as a *net* value, is introduced to reduce this probability to 1 in 10^7. This is because early studies have indicated that the expected variations in the actual conditions are expected to result in a random (Gaussian) probability distribution of performance with a standard deviation of 3%. 15% represents 5 standard deviations, which covers all but 1 in 10^7 of cases. In the calculation of ASDR, the gross performance distances are considered.

The rationale behind the above requirement is that the 'normal', all-engines-operating condition is expected to be the situation in operation. Consequently, net performance is allowed for to ensure an acceptably low probability of exceeding these distances. Exceedance may result in the aircraft hitting obstacles, which, in turn, may lead to personal injury or death and it is for this reason that this probability is required to be so low. The probability of engine failure at V_1, however, is considered to be sufficiently low that even if gross performance estimates for TODR, TORR and ASDR is allowed for, which, theoretically, will result in exceedence on half of the occasions, the combined probability of such an occurrence is also of the order of 1 in 10^7. Whilst larger leeways would increase safety by increasing the probability of a successful take-off, they increase the cost of operations by either requiring longer runways than would otherwise be necessary[**] or else by restricting aircraft in the fuel and pay loads[††] they take on. The amount of total leeway allowed for is, in effect, intended to strike a balance between the two opposing interests, resulting in what to date has been judged as an acceptable safety record and satisfactory operational costs. It is for this reason that gross performance is allowed for in the engine failure case.

II. Scheduled Performance Calculations

For the purposes of scheduled runway distance estimation, the all-engines-operating, continued take-off manoeuvre is segmented in a number of phases [1]:

- The *acceleration phase*, during which the aircraft is accelerating along the runway with the nose-wheel on the ground. The attitude of the aircraft is essentially constant in this phase.
- The *rotation phase*, during which the aircraft is rotated to the appropriate attitude to support lift-off.
- The *transition phase*, during which the aircraft is airborne and pitching up, prescribing a curved path in the vertical plane.
- The *climb phase*, during which the aircraft is climbing at a steady pitch angle and climb gradient. As the take-off is considered complete at 35ft above the runway datum, the climb phase may or may not be included in the take-off manoeuvre.
- In the case of a rejected take-off (RTO), the *deceleration phase*, during which the aircraft is decelerating to a halt following the rejection of the run.

The equation describing the motion of the aircraft in the acceleration phase defines acceleration as a function of velocity and environmental and operational conditions (Equation 1).

$$a = \frac{T - \left[\frac{1}{2}\rho S\left(C_D - \mu C_L\right)\right]\left(V_g - v_w\right)^2 - W\left[\sin\theta - \mu\cos\theta\right]}{m} \qquad \text{Equation 1}$$

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§ The screen height is the clearance height above any obstacles, currently 35ft for take-offs from dry runways and 15ft for wet runways.

** Longer runways involve higher construction and maintenance costs.

†† The take-off distances required can be shortened by reducing the take-off weight.

The thrust T is also a function of airspeed (V_g-v_w) and may be expressed as a quadratic function of this parameter [2]. The distance covered to rotation is then obtained by multiplying the instantaneous acceleration with velocity and integrating with respect to velocity to rotation speed (Equation 2) [1, 3, 4]:

$$S_G = \int_0^{V_R - v_w} \frac{V_g}{a} dV_g \qquad \text{Equation 2}$$

The distance covered during rotation can be calculated by using the average velocity during the manoeuvre [3]:

$$S_{ROT} = \frac{(V_R + V_{LOF})}{2} t_{ROT} \qquad \text{Equation 3}$$

The time of rotation t_{ROT} is very dependent on the aircraft geometry and piloting technique. Consequently, it is estimated for a particular aircraft through flight test and the fairing of test data. V_{LOF} is linked to V_R.

The distance covered between lift-off and 35ft is obtained through a procedure similar to that for rotation, as it is also very dependent on aircraft geometry and piloting technique. This distance covers the transition and climb phases. The velocity used in this calculation is the average between the lift-off speed and the climb safety speed V_2, since the achievement of V_2 before 35ft is a requirement in Part 25 regulations.

Other distances associated with a rejected run and engine failure are also estimated in scheduled performance, but these are not relevant to this discussion. It is relevant to emphasise in this discussion that these and all post rotation distances are highly dependent of piloting reaction, technique, accuracy and repeatability. As a result, associated methods, whilst providing a good estimate of what distances need to be allowed for (and are therefore appropriate for scheduled performance calculations), are not very reliable at estimating the actual distances that the aircraft will cover, as they do not take specific pilot performance into account.

III. The Scope of a Simplified Model

The scheduled performance calculations provide confidence and assurance prior to dispatch. During the actual manoeuvre, however, the crew has no objective means with which to ensure that the expected levels of performance are indeed being met. Indeed, currently crews only have their perception to depend on, with the pilot monitoring (PM) monitoring the airspeed indicator, speed trend vector and engine instruments whilst the pilot handling (PH) visually assessing the progress down the runway in conjunction with the perceived acceleration. Such a method of assessing aircraft performance is not sufficiently objective. Indeed, this limitation has been demonstrated in several accidents and incidents, where pilots failed to identify the seriousness of the poor level of acceleration during take-off.

The scope of monitoring the performance of the aircraft during take-off is to ensure that the actual acceleration of the aircraft is adequate to ensure that the actual runway distances requirements are indeed within the limits allowed for by scheduled performance. In this way the crew can be confident that the take-off manoeuvre is indeed adequately safe.

When attempting to predict runway distance requirements in real-time, the uncertainties associated with post-rotation distance predictions render the value of such predictions questionable. The distances covered during the rotation, transition and climb-out phases are highly dependent on piloting technique and therefore cannot be estimated with an accuracy that is adequate for monitoring purposes. Also, the distance covered during the deceleration phase is highly dependent on tyre status, brake performance and runway condition, whilst the time between the acceleration and deceleration phases, during which the crew will be reacting to the failure, depends on crew response time and coordination. As a result any estimate of these distances would also be adequate.

The authors are of the opinion that the highest value in real-time performance monitoring is in predicting the runway that will be covered during the acceleration phase. This phase of take-off is the only phase in which aircraft conditions are steady and thus can support the accurate prediction of runway distances.

The equation describing the motion of the aircraft in the acceleration phase is presented in Equation 1. This equation defines acceleration as a function of environmental and operational conditions. The integration of acceleration with respect to velocity to the target velocity (Equation 2) will then provide an estimate of the distance that the aircraft will cover to the said velocity. The accuracy of the parameter values used in Equation 1 is of paramount importance in this context, since resulting errors in the acceleration estimate will be integrated and thus result in the rapid divergence of the distance estimate from the true value. Presently, a number of parameters in

Equation 1 cannot be determined or estimated with sufficient accuracy in real-time. One of the most important parameters in this category is the aircraft's actual weight, which is not measured in operation. Such limitations compromise the effectiveness of Equation 1 when used in real-time performance prediction. Indeed, use of scheduled values for the relevant parameters would relegate the calculation to a re-calculation of scheduled performance. The standard deviation of the prediction error would therefore be expected to also be of the order of 3%. This is inadequate, as it would result in too many false warnings and missed warnings. False warnings are generated when the actual performance is adequate (within net performance) but the prediction suggests otherwise and missed warnings occur when actual performance is inadequate whilst the prediction suggests it is adequate (Figure 3).

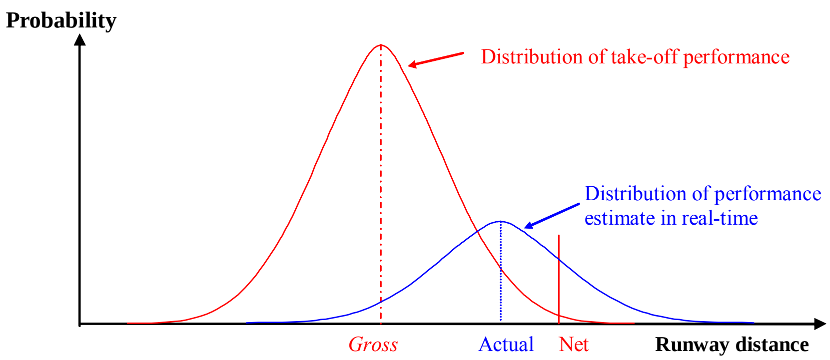

Figure 3: Scheduled performance uncertainty compared with real-time prediction uncertainty.
In the situation presented, actual performance is within net allowances and therefore adequate. The real-time estimate has a probability distribution as shown in blue, indicating that, during that particular run, the area under the graph beyond the net limit represents the probability of the generation of a (false) warning that actual performance is inadequate. This illustrates the importance of the standard deviation of the distribution of the performance estimate in real-time being much smaller than that of scheduled performance if the probabilities of false warnings and missed warnings are to be acceptably low. The situation illustrated has comparable standard deviations and is inadequate.

The importance of the accuracy of the distance estimates has been recognized by many sectors of the aviation community. The British Air Registration Board suggested, in the late 1950s, that predictions of the take-off distance required should have a standard deviation of 2% [5]. Aerospace standard AS-8044, defining minimum performance standards for take-off performance monitors, requires 'the probability that TOPM system tolerances will, of themselves, cause an error greater than ±5% in the apparent all-engine operating takeoff distance to rotation speed shall be 0.01 percent or less' [6]. This 'basic' accuracy is of the same order as the ARB recommendation. Assuming a normal distribution of instrumentation error, a 99% probability is covered by 2.3 standard deviations of the distribution and a ±5% error over this range is equivalent to a standard deviation of 2.2%. In a study of the impact such uncertainties have on the generation of false and missed warnings, the authors have identified the need for performance monitors to be more accurate, particularly as the run progresses and the risk of overrun increases. Accordingly, the authors have established a more stringent in-house performance standard [7].

In an attempt to mitigate the limitations that compromise the confidence of the conventional method of estimating aircraft performance, a novel method of modeling aircraft performance in the acceleration phase was invented at Cranfield University. The approach is based upon continuously monitoring the run and using the actual profile to predict performance further down the run. This is a very robust method of monitoring, since during the acceleration phase, the future performance can reasonably be assumed to be an extrapolation of past performance during the same run. The authors consider the concept of curve fitting using the method of least squares (curvi-linear regression) applied to the performance history profile of the actual run as best suited to the accurate determination of achieved performance, on which future performance can be estimated.

IV. Selection of a suitable model

Accordingly, a study of the candidate characteristics that would be best appropriate for the task was carried out. Ideally, the expression in question should be readily and accurately expressed as a second or third order polynomial. The expression of acceleration presented in Equation 1 is an obvious candidate in this respect, as it can be presented as a second order function of velocity. Since the thrust function T is normally described as a second order polynomial in terms of airspeed (V_g+v_w), Equation 1 can also be re-written as a second order polynomial in terms of airspeed. However, it can also be re-written in terms of the ground speed V_g and assuming the wind speed v_w is a constant. Variations in actual wind speed due to gusts cannot be predicted and therefore the assumption is valid for the purposes discussed herein. Therefore,

$$a = AV_g^{\ 2} + BV_g + C \qquad \text{Equation 4}$$

where coefficients A, B and C are functions of the terms in Equation 1.

A major difficulty associated with curve-fitting this function is that acceleration is very sensitive to disturbances that are normally experienced during take-off. Figure 4 presents a typical acceleration profile recorded from the output of a Litton Inertial Reference System during an actual take-off. It is clearly evident that when used in the 'classical' manner, where constant values for parameters such as the coefficients of lift, drag and rolling friction, thrust function and wind speed are used, Equation 1 is inadequate, as most of the significant disturbances are not accounted or compensated for by the equation.

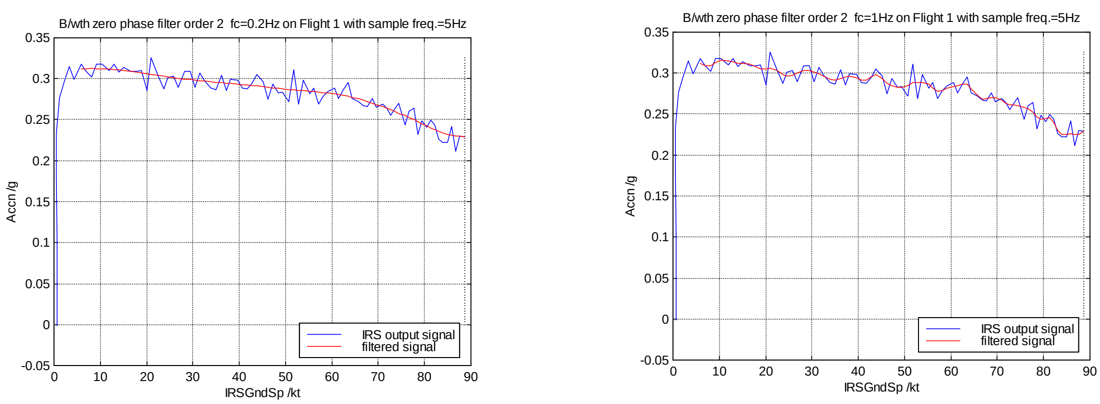

Figure 4: The acceleration profile recorded during an actual run (Jetstream-100), superimposed with the responses of a 0.2Hz (left) and a 1Hz (right) low-pass filter.

The acceleration profile shown in Figure 4 also expresses a significant noise component with a fundamental frequency of the order of 2Hz superimposed on a fluctuation of a much lower frequency. This parasitic component is difficult to filter out for prediction purposes because it lies in the frequency band relevant to the aircraft response. Figure 4 illustrates the effect of filtering with two cut-off frequencies, namely 1Hz and 0.2Hz. As a result, any attempt to curve-fit this characteristic will not readily yield useful information for the purpose of real-time parameter estimation and prediction of performance further down the run.

The acceleration profile is generally steady with a gradual drop-off along the run, mainly due to the increase in drag as speed increases. If the acceleration were constant,

- the Velocity-Time profile would be linear
- the Velocity-Distance relationship would be a square law
- the Distance-Time relationship would be a square law

As the acceleration is not constant, the above characteristics are analytically modeled by complex functions. In simpler graphical terms, however, the drop in acceleration would increase the curvature of the characteristic accordingly, suggesting that 3rd or higher order polynomials would be appropriate to model the relationships. It should be recalled that complex functions can be adequately modeled by polynomials over specific ranges through the use of MacLaurin and Taylor series. High order polynomials, however, are not preferred for prediction purposes as they can easily diverge from the actual characteristic in forward-prediction. Consequently, the velocity-time profile would be the candidate of choice in this work.

Indeed, the velocity-Time profile is fairly linear (Figure 5) and suggests that it can be readily modeled by a second order polynomial over the complete range from standstill up to rotation velocity, the limit to which the acceleration phase extends.

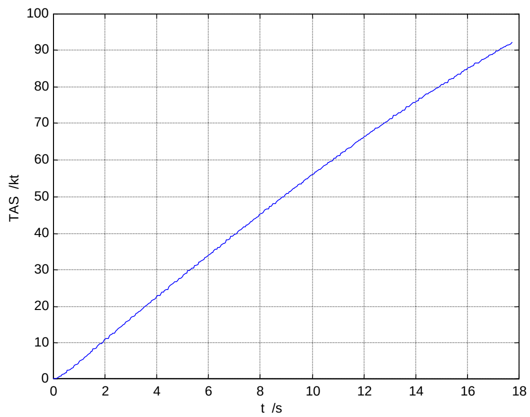

Figure 5: The Velocity-Time profile of an actual take-off (Jetstream-100)

The relationship between the airspeed (or ground speed) and time can be obtained analytically. As the acceleration profile curves downwards due to the dominating drag effects, the second order coefficient A in Equation 4 is always negative. The first order coefficient B is mainly dependent on the thrust function (momentum drag effect) and is therefore normally also negative [2], whilst coefficient C represents the static thrust-to-weight ratio and is thus positive. The graph of Equation 4 is consequently as presented in Figure 6, clearly indicating that the equation has two real roots, one positive and one negative.

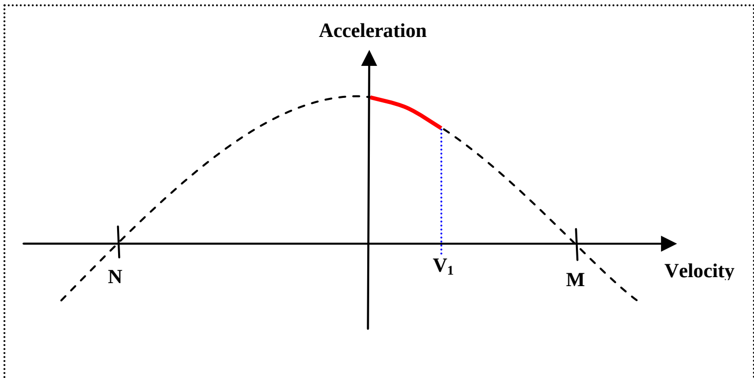

Figure 6: The acceleration as a second order function of velocity.

Defining the two roots M and N, with the former being the positive root, M would be the maximum attainable velocity if the aircraft were to continue indefinitely in the ground-roll. M is also numerically smaller than N since the first order coefficient B is negative. This results in a slope at the point where the curve cuts the y-axis that is negative and the curve is therefore asymmetric about the y-axis. The two roots are given by:

$$M = \frac{-B - \sqrt{B^2 - 4AC}}{2A} \quad , \quad N = \frac{-B + \sqrt{B^2 - 4AC}}{2A} \qquad \text{Equations 5}$$

The acceleration a can thus be expressed in the form:

$$a = \frac{dV_g}{dt} = A\left(V_g - M\right)\left(V_g - N\right) \qquad \text{Equation 6}$$

Integrating from standstill to any ground speed V_g, yields the time t to that speed:

$$\int_0 \frac{dV_g}{A\left(V_g - M\right)\left(V_g - N\right)} = t \qquad \text{Equation 7}$$

It can be shown that, through expansion into partial fractions,

$$\frac{1}{\left(V_g - M\right)\left(V_g - N\right)} = \left(\frac{1}{M-N}\right)\left(\frac{1}{V_g - M}\right) - \left(\frac{1}{M-N}\right)\left(\frac{1}{V_g - N}\right) \qquad \text{Equation 8}$$

$$\Rightarrow \quad \int_0 \frac{dV_g}{\left(V_g - M\right)} - \int_0 \frac{dV_g}{\left(V_g - N\right)} = A\left(M - N\right)t \qquad \text{Equation 9}$$

Solving, $$\left[\ln\left|V_g - M\right| - \ln\left|V_g - N\right|\right]_0 = A\left(M - N\right)t \qquad \text{Equation 10}$$

$$\Rightarrow \quad \ln\left(\frac{M - V_R}{V_g - N}\right) = A\left(M - N\right)t + \ln\left|\frac{M}{N}\right| \qquad \text{Equation 11}$$

Taking exponents, $$\frac{M - V_g}{V_g - N} = \left|\frac{M}{N}\right| e^{A[M-N]t} \qquad \text{Equation 12}$$

Now defining $k_1 = A\left[M - N\right]$ and $k_2 = \frac{M}{|N|}$ and re-arranging equation 12 to get the velocity V_g as the subject of the formula in terms of time t,

$$V_g = \frac{M\left\{1 - e^{k_1 t}\right\}}{1 + k_2 e^{k_1 t}} \qquad \text{Equation 13}$$

For values of M, k_1 and k_2 within the normal operational envelope of jet aircraft, the expression in Equation 13 can be closely approximated to a second order polynomial function of time t.

V. Numerical validation

A numerical study using Matlab® was carried out to confirm the validity of this approximation. To this effect, velocity profiles (ground speed vs. time) were generated from Equation 13 using reasonable values of coefficients A, B and C. A second order polynomial was then fitted onto each profile and the goodness of fit analysed.

Under normal conditions, the thrust-to-weight ratio of a transport-category aircraft is of the order of 0.25g to 0.3g. Consequently, four values of C where chosen, namely 0.20g, 0.25g, 0.30g and 0.35g, to cover all expected operational conditions. Similarly, a rotation speed of 170kt was selected, as most aircraft are normally rotated within this speed.

The negative coefficients A and B contribute to a drop in acceleration along the run. This drop in acceleration cannot exceed half the static thrust-to-mass ratio (which is equal to the coefficient C). This is because an aircraft is required to maintain the climb-safety speed on encountering an engine failure. In the worst case scenario, a two-engined aircraft would lose half of its thrust. Consequently, at V_2 the drag cannot be more than 50% of the applied thrust, otherwise the aircraft would decelerate on the failure of one engine. Referring to Equation 4, therefore,

$$AV_R^2 + BV_R \leq C/2 \qquad \text{Equation 14}$$

Coefficient A is determined by three components. The aerodynamic drag, which increases with speed, contributes to a negative value of A. The reduction in rolling friction due to load alleviation from the undercarriage, however, has a positive effect, as does the ram air effect on engine thrust, which results in an increase in thrust as the aircraft accelerates. The most significant component, however, is the aerodynamic drag, which ensures an overall negative value for A[‡‡]. During the ground run, the angle of attack would be very small, of the order of 2° to 3°. The angle of attack at the end of rotation is highly dependent on aircraft geometry, but is generally of the order of 10°. If all the decelerating forces were due to lift-induced drag, then these would be proportional to aircraft angle of attack. Consequently, the drag at the start of rotation can be expected to be not more than one third of that at the end of rotation. As a result, the maximum decelerating component due to aerodynamic drag at any point of the ground run, quantified by the term AV_R^2 in Equation 14, should not exceed $C/6$. A limit of $C/4$ was therefore selected for this work.

Coefficient B is mainly due to momentum drag and is normally small. Indeed, for the Rolls-Royce RB211-535E4, the momentum drag at 90m/s is approximately 23% of the static thrust[§§]. Consequently, the limit of B was, in this work, taken such that BV_R does not exceed 40% of C.

The summary of ranges of coefficients A, B and C is presented in Table 1.

Coefficient	Minimum	Maximum
A	$-0.25C/V_R^2$	0
B	$-0.4C/V_R$	0
C	0.20g	0.35g

Table 1: Summary of the ranges of the coefficient *A, B* and *C* used in the validation.

A standard statistical method for measuring the goodness of fit involves the calculation of the R^2 statistic. This parameter is defined as:

$$R^2 = \frac{\sum\left(Y_i - \bar{Y}_i\right)^2 . \sum\left(Y_i - \hat{Y}_i\right)^2}{\sum\left(Y_i - \bar{Y}_i\right)^2} \qquad \text{Equation 15}$$

where variable Y is, in this application, the ground speed V_g. A value of unity for R^2 is obtained when the fit is perfect (that is, no residual errors are present). This is because $\sum\left(Y_i - \hat{Y}_i\right)^2 = 0$ when no residuals are present. The smallest value of R^2 obtained for all fits within the limits described is 0.999973. Although this suggests that very good fits are obtained throughout the envelope, the values are so high that any quantitative analysis of the R^2 statistic would not be meaningful.

Consequently, the goodness of fit was instead measured by:

- Determining the maximum difference in airspeed at any point between the actual profile and the fitted curve. This involved the identification of the largest residual error in the regression.
- Determining the root-mean-square (RMS) of the residuals.

The results obtained are presented in Figures 7 – 16. The RMS plots (Figures 11-14) are plotted for normalized values of A and B. The normalised parameters are AV_R^2/C and BV_R/C.

[‡‡] This discussion assumes a constant thrust setting, which may not be the case with FADEC-controlled engines, in which case, A could become positive.

[§§] This value is estimated from data in [2].

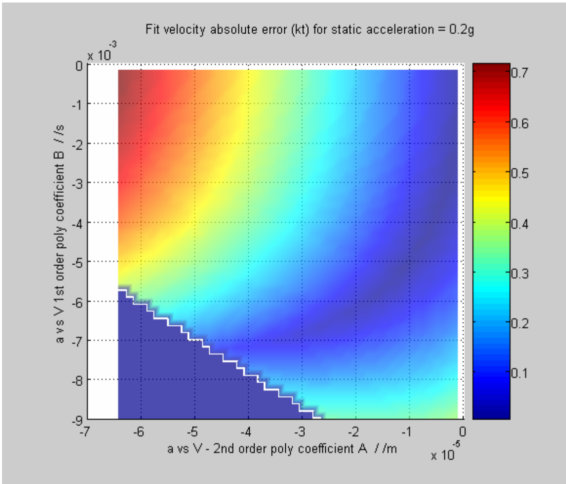

Figure 7: Maximum absolute residual error of least-square fit – *C* = 0.20g

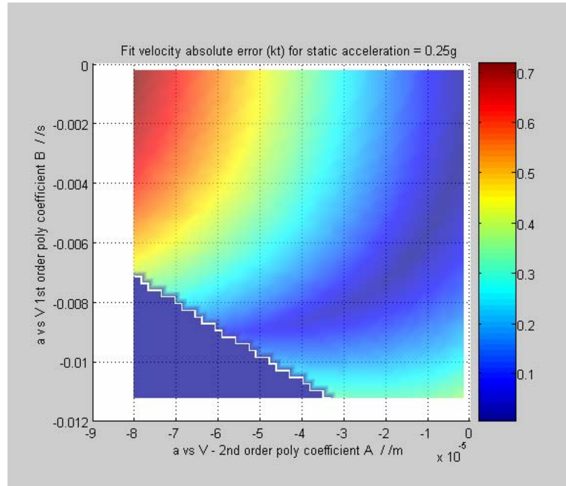

Figure 8: Maximum absolute residual error of least-square fit – *C* = 0.25g

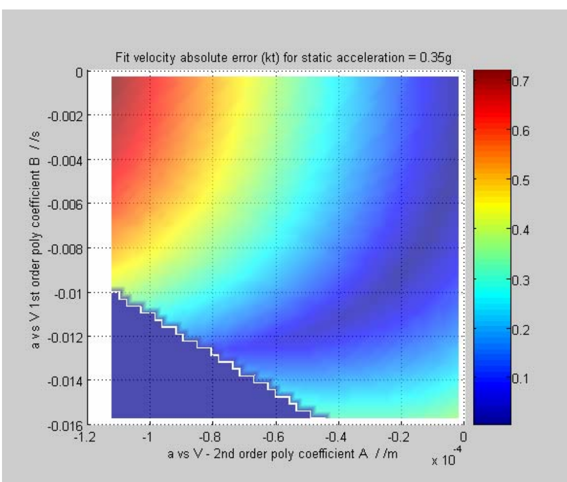

Figure 9: Maximum absolute residual error of least-square fit – *C* = 0.30g

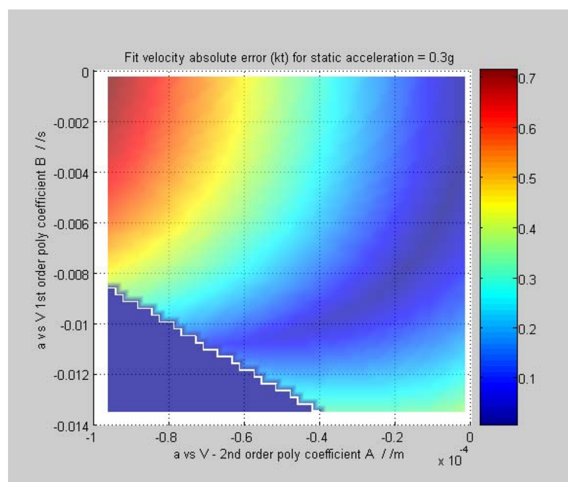

Figure 10: Maximum absolute residual error of least-square fit – *C* = 0.35g

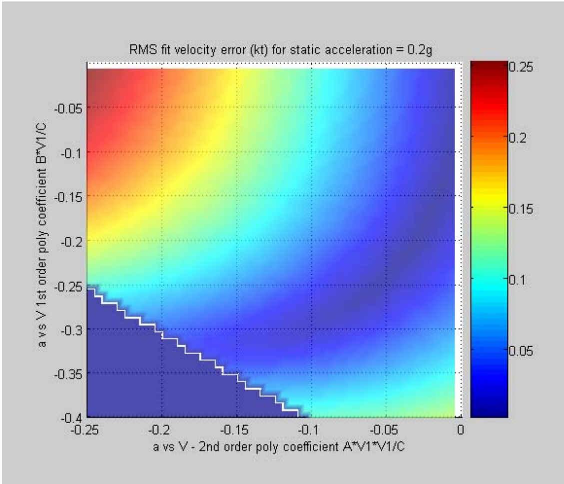

Figure 11: RMS of residual error of least-square fit – *C* = 0.20g

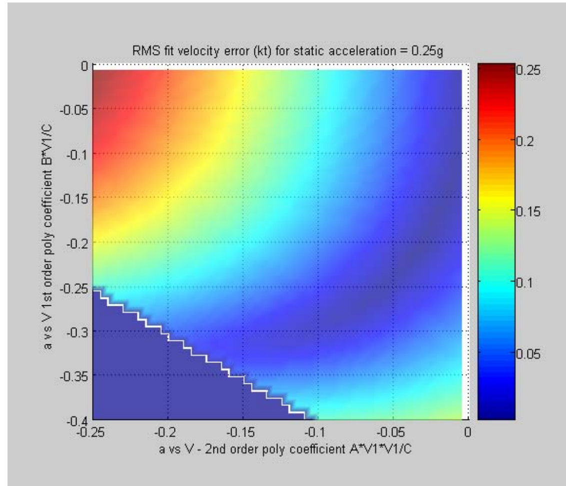

Figure 12: RMS of residual error of least-square fit – *C* = 0.25g

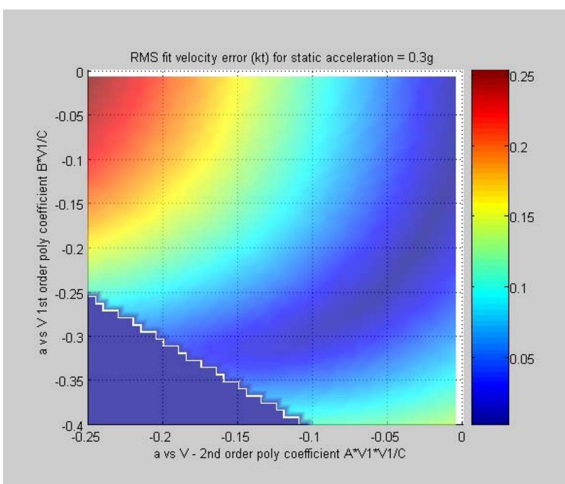

Figure 13: RMS of residual error of least-square fit – *C* = 0.30g

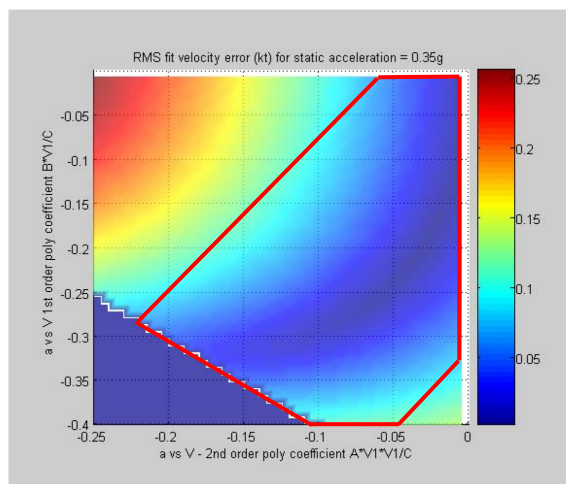

Figure 14: RMS of residual error of least-square fit – *C* = 0.35g
The red polygon defines the envelope within which the RMS fit velocity error would be expected to be less than 0.15kt.

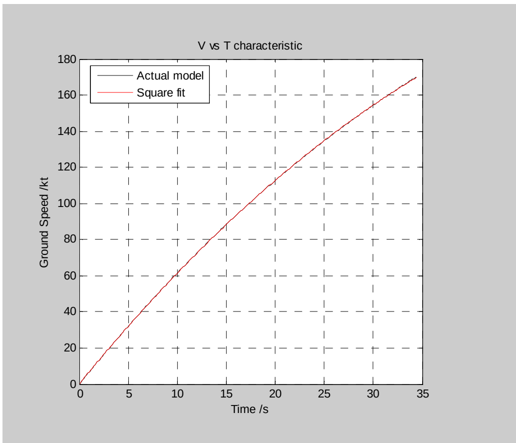

Figure 15: A typical 2nd order fit on the velocity-time characteristic (Equation 13).

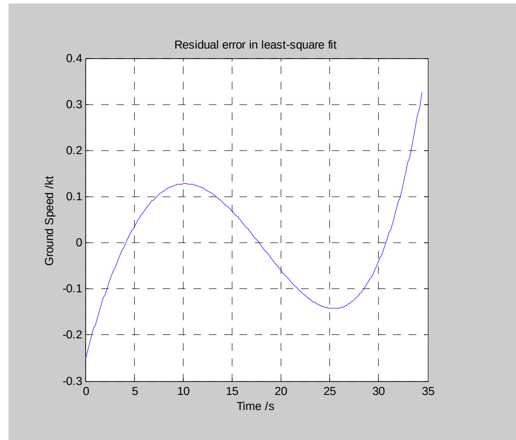

Figure 16: The residual error in a typical 2nd order least-squares fit.

VI. Discussion and Conclusion

The results presented in Figures 7 to 10 indicate that the absolute values of the residual errors do not exceed 0.7kt. Moreover, the worst fits in terms of maximum residual error occur in profiles with acceleration characteristics that have a high curvature and small linear coefficient B. When modeling thrust profiles as 2nd order polynomials, high bypass ratio engines exhibit a positive 2nd order coefficient, and this increases with bypass ratio [2]. Consequently, it can be fairly expected that the overall second order coefficient of acceleration A would be small, resulting in an effect that is smaller than that of the momentum drag of the engine at V_R.

Figures 11 to 14 present the RMS of the residuals. The results indicate that this measure is very small and in most cases below 0.15kt. This effect is noted in Figures 15 and 16, which present typical fits. Figure 15 clearly indicates the closeness of fit, whilst Figure 16 illustrates how the average of the absolute residual error is significantly smaller than the maximum and therefore confirms the small RMS value obtained. Figure 16 also shows that the average residual error is zero and this is due to the nature of curve-fitting using the method of least-squares curve.

One fundamental assumption in curvi-linear regression is that the errors, and thus the residuals, are random (Gaussian distributed) with zero mean. This is clearly not the case, as a pattern in the residual errors is exhibited in Figure 16. A similar pattern is noted in all residual plots of the combinations considered. This was expected, and is due to the fact that the coefficients A, B and C are not linearly independent.

The plots against normalised coefficients (Figures 14-17) are nearly identical in both trend and absolute values. This suggests that a single normalised plot can adequately characterise the whole operational envelope. The diagonal red boundaries presented on Figure 14 define an operational envelope within which the RMS error would be within 0.15kt. Mathematically, the boundaries are specified by:

$$-0.325\frac{C}{V_R} < \left(B - 1.75AV_R\right) < +0.105\frac{C}{V_R} \qquad \text{Equation 16}$$

where V_R = 87.46m/s (which corresponds to 170kts).

In conclusion, therefore, the modeling of the velocity-time profile during the acceleration phase of take-off as a second order polynomial function of time is valid. The fit is demonstrated to be good throughout the expected operational envelope of commercial aircraft and thus warrants its use for the purpose of real-time performance estimation.

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