# 1 第四次作业

## 1.1 习题6

### 1.1.1 Ex9.1

$$\begin{gathered}\int_{-1}^{1}\int_{-1}^{1} xydxdy \\ m=n=4, h=k=\frac{1}{2} \\ I=hk\sum_{i=0}^{m}\sum_{j=0}^{n} c_{i,j} f\left(x_i, y_j\right)\end{gathered} \tag{1}$$

代入计算得到

| $f(x,y)$ | -1 | $-\frac{1}{2}$ | 0 | $\frac{1}{2}$ | 1 |
|---|---|---|---|---|---|
| -1 | 1 | $\frac{1}{2}$ | 0 | $-\frac{1}{2}$ | -1 |
| $-\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{4}$ | 0 | $-\frac{1}{4}$ | $-\frac{1}{2}$ |
| 0 | 0 | 0 | 0 | 0 | 0 |
| $\frac{1}{2}$ | $-\frac{1}{2}$ | $-\frac{1}{4}$ | 0 | $\frac{1}{4}$ | $\frac{1}{2}$ |
| 1 | -1 | $-\frac{1}{2}$ | 0 | $\frac{1}{2}$ | 1 |

| $c_{i,j}$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| 0 | $\frac{1}{4}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{4}$ |
| 1 | $\frac{1}{2}$ | 1 | 1 | 1 | $\frac{1}{2}$ |
| 2 | $\frac{1}{2}$ | 1 | 1 | 1 | $\frac{1}{2}$ |
| 3 | $\frac{1}{2}$ | 1 | 1 | 1 | $\frac{1}{2}$ |
| 4 | $\frac{1}{4}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{4}$ |

$$I=0$$

### 1.1.2 Ex11

$f'(0.02)$向前差商

$$f'(0.02)\approxeq\frac{7-9}{0.04-0.02}=-100$$

向后差商

$$f'(0.02)\approxeq\frac{9-11}{0.02-0.00}=-100$$

$f'(0.06)$向后差商

$$f'(0.06)\approxeq\frac{10-7}{0.06-0.04}=150$$

## 1.2 习题7

### 1.2.1 Ex1

由

$$\frac{y_{n+1}-y_n}{h}\approxeq y'(x_n)=x_n+y_n^2$$

有

$$y_{n+1} = h\ y_n^2 + y_n + h\ x_n$$

即

$$y(0) = 1$$
$$y(0.1) = 1.1$$
$$y(0.2) = 1.231$$
$$y(0.3) = 1.402536$$
$$y(0.4) = 1.629247$$
$$y(0.5) = 1.934692$$

#### 1.2.2 Ex7

取 $h = 0.1$，剖分结点为 $0, 0.1, 0.2, 0.3, 0.4, 0.5$，用二阶 Runge-Kutta 格式

$$\begin{cases} y_{n+1} = y_n + hk_2 \\ k_1 = f\left(x_n, y_n\right) \\ k_2 = f\left(x_n + \frac{h}{2}, y_n + \frac{h}{2}k_1\right) \end{cases}$$

起步计算 $y(0.1)$

$$k_1 = -0 \times 1 = 0$$

$$k_2 = -0.05 \times 1 = -0.05$$

$$y(0.1) = 1 + 0.1 \times (-0.05) = 0.995$$

代入 Adams 显式公式，得到

$$y(0.2) = 0.995 + \frac{0.1}{2} \times (3 \times (-0.1 \times 0.995) - (-0 \times 1)) = 0.980075$$

类似可得

$$y(0.3) = 0.980075 + \frac{0.1}{2} \times (3 \times (-0.2 \times 0.980075) - (-0.1 \times 0.995)) = 0.955655$$
$$y(0.4) = 0.955655 + \frac{0.1}{2} \times (3 \times (-0.3 \times 0.955655) - (-0.2 \times 0.980075)) = 0.922455$$
$$y(0.5) = 0.922455 + \frac{0.1}{2} \times (3 \times (-0.4 \times 0.922455) - (-0.3 \times 0.955655)) = 0.881445$$

P138 T10(2)

由单位区间上的两点 Gauss-Legendre 公式

$$\int_{-1}^{1} f(x)dx = f(-\frac{\sqrt{3}}{3}) + f(\frac{\sqrt{3}}{3})$$

作变换得到一般区间上的两点 Gauss-Legendre 公式

$$\int_{a}^{b} f(x)dx = \frac{b-a}{2}\left(f(\frac{a+b}{2} - \frac{\sqrt{3}}{3} \times \frac{b-a}{2}) + f(\frac{a+b}{2} + \frac{\sqrt{3}}{3} \times \frac{b-a}{2})\right)$$

所以

$$\begin{aligned}\int_{-3}^{1} f(x)dx &= 2 \times \left(f(-1 - \frac{\sqrt{3}}{3} \times 2) + f(-1 + \frac{\sqrt{3}}{3} \times 2)\right) \\ &= 2\left((-1 - \frac{2\sqrt{3}}{3})^5 + (-1 - \frac{2\sqrt{3}}{3}) + (-1 + \frac{2\sqrt{3}}{3})^5 + (-1 + \frac{2\sqrt{3}}{3})\right) \\ &= -96.8889\end{aligned}$$

P138 T15

将 $f(-h), f(2h)$ 在 $x=0$ 处展开

$$f(-h)=f(0)-hf'(0)+\frac{h^2}{2}f''(0)-\frac{h^3}{6}f'''(0)+\dots \quad (1)$$

$$f(2h)=f(0)+2hf'(0)+2h^2f''(0)+\frac{4h^3}{3}f'''(0)+\dots \quad (2)$$

代入

$$f'(0)=c_1f(-h)+c_2f(0)+c_3f(2h)$$

得

$$f'(0)=(c_1+c_2+c_3)f(0)+(-hc_1+2hc_3)f'(0)+(\frac{h^2}{2}c_1+2h^2c_3)f''(0)+...$$

比较两端系数

$$\begin{cases} c_1+c_2+c_3 &= 0 \\ -hc_1+2hc_3 &= 1 \\ \frac{h^2}{2}c_1+2h^2c_3 &= 0 \end{cases}$$

解得

$$\begin{cases} c_1 &= -\frac{2}{3h} \\ c_2 &= \frac{1}{2h} \\ c_3 &= \frac{1}{6h} \end{cases}$$

同理，将 (1),(2) 代入

$$f''(0)=d_1f(-h)+d_2f(0)+d_3f(2h)$$

得

$$f''(0)=(d_1+d_2+d_3)f(0)+(-hd_1+2hd_3)f'(0)+(\frac{h^2}{2}d_1+2h^2d_3)f''(0)+...$$

比较两端系数

$$\begin{cases} d_1 + d_2 + d_3 & = 0 \\ -hd_1 + 2hd_3 & = 0 \\ \frac{h^2}{2} d_1 + 2h^2 d_3 & = 1 \end{cases}$$

解得

$$\begin{cases} d_1 & = \frac{2}{3h^2} \\ d_2 & = -\frac{1}{h^2} \\ d_3 & = \frac{1}{3h^2} \end{cases}$$

(或用插值型微分公式，或代入多项式基)