

# 1 第二次作业

## 1.1 习题1

### 1.1.1 Ex5

$$L_2(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}f(x_0) + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}f(x_1) + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}f(x_2)$$

将 $x_0 = 81, x_1 = 100, x_2 = 121, f(x_0) = 9, f(x_1) = 10, f(x_2) = 11, x = 105$ 代入上式, 得到 $L_2(105) = 10.2481203$ 。

$$\text{则 } e = \sqrt{105} - L_2(105) = 10.2469508 - 10.2481203 = -1.17 \times 10^{-3}$$

而 $R_2(x) = \frac{f^3(\xi)}{3!} \prod_{i=1}^3 (x-x_i), \xi \in [a, b]$ , 代入 $x = 105, a = x_0 = 81, x_1 = 100, b = x_2 = 121$ , 即得:

$$R_2(105) = -120\xi^{-\frac{5}{2}} \in [-2.03 \times 10^{-3}, -7.45 \times 10^{-4}]$$

$\therefore$  实际误差在误差界内。

### 1.1.2 Ex6

i	$x_i$	$f(x_i)$	一阶差商	二阶差商	三阶差商
0	-1	3			
1	2	5	$f[-1, 2] = \frac{2}{3}$		
2	3	7	$f[2, 3] = 2$	$f[-1, 2, 3] = \frac{1}{3}$	
3	4	5	$f[3, 4] = -2$	$f[2, 3, 4] = -2$	$f[-1, 2, 3, 4] = -\frac{7}{15}$

$$\therefore N_3(x) = 3 + \frac{2}{3}(x+1) + \frac{1}{3}(x+1)(x-2) - \frac{7}{15}(x+1)(x-2)(x-3), N_3(1.2) = 2.4016$$

### 1.1.3 Ex7

(1)

$$N_3(x) = 1 + 2(x-4) + (x-4)(x-1) - (x-4)(x-1)(x-3)$$

(2)

$$f(2) = N_3(2) = -7$$

$$f[1, 2, 3, 4] = \frac{f[2, 3, 4] - f[1, 3, 4]}{2-1} = -1$$

$$\therefore f[2, 3, 4] = f[1, 3, 4] - 1 = 0$$

#### 1.1.4 Ex9

$$\begin{aligned}f[2^0, 2^1] &= \frac{f(2^1) - f(2^0)}{2^1 - 2^0} = \frac{-2975 + 886}{2 - 1} = -2089 \\ \therefore f[x_0, \dots, x_n] &= \frac{f^n(\xi)}{(n)!} \\ \therefore f[2^0, \dots, 2^7] &= \frac{f^7(\xi)}{7!} = 1, \quad f[2^0, \dots, 2^8] = \frac{f^8(\xi)}{8!} = 0\end{aligned}$$

#### 1.1.5 Ex12

见例1.5,  $H_2(x) = x^2 - 3x + 5$ ,  $R_2(x) = \frac{f'''(\xi)}{3!}(x-3)(x-5)^2$ ,  $H_2(3.7) = 7.59$

T15

法 1: 设  $P_4(x) = t_0(x)f(1) + t_1(x)f'(1) + t_2(x)f(2) + t_3(x)f'(2) + t_4(x)f''(2)$ , 其中  $t_0(x), t_1(x), t_2(x), t_3(x), t_4(x)$  为次数不高于四次的多项式, 分别满足

$$t_0(1) = 1, t'_0(1) = 0, t_0(2) = 0, t'_0(2) = 0, t''_0(2) = 0$$

$$t_1(1) = 0, t'_1(1) = 1, t_1(2) = 0, t'_1(2) = 0, t''_1(2) = 0$$

$$t_2(1) = 0, t'_2(1) = 0, t_2(2) = 1, t'_2(2) = 0, t''_2(2) = 0$$

$$t_3(1) = 0, t'_3(1) = 0, t_3(2) = 0, t'_3(2) = 1, t''_3(2) = 0$$

$$t_4(1) = 0, t'_4(1) = 0, t_4(2) = 0, t'_4(2) = 0, t''_4(2) = 1$$

同 13 题, 用待定系数法, 求得

$$t_0(x) = (-3x + 2)(x - 2)^3$$

$$t_1(x) = -(x - 1)(x - 2)^3$$

$$t_2(x) = (x - 1)^2(3x^2 - 14x + 17)$$

$$t_3(x) = (x - 1)^2(x - 2)(-2x + 5)$$

$$t_4(x) = \frac{1}{2}(x - 1)^2(x - 2)^2$$

从而

$$\begin{aligned} P_4(x) &= 0.5(-3x + 2)(x - 2)^3 - 0.5(x - 1)(x - 2)^3 + (x - 1)^2(3x^2 - 14x + 17) \\ &\quad + (x - 1)^2(x - 2)(-2x + 5) + \frac{1}{2}(x - 1)^2(x - 2)^2 \\ &= 3.5x^4 - 22.5x^3 + 51.5x^2 - 49x + 17 \end{aligned}$$

法 2: 对于重结点, 实际上有  $f[\underbrace{x_0, x_0, \dots, x_0}_{n+1 \uparrow}] = \frac{f^{(n)}(x_0)}{n!}$ 。由差商的计算方法, 得差商表

1	0.5	0.5	0	-1.5	3.5
1	0.5	0.5	-1.5	2	
2	1	-1	0.5		
2	1	-1			
2	1				

$$\begin{aligned} P_4(x) &= 0.5 + 0.5(x-1) - 1.5(x-1)^2(x-2) + 3.5(x-1)^2(x-2)^2 \\ &= 3.5x^4 - 22.5x^3 + 51.5x^2 - 49x + 17 \end{aligned}$$

余项为  $\frac{f^{(5)}(\xi)}{5!}(x-1)^2(x-2)^3$ , 证明如下:

令  $R(x) = f(x) - P_4(x)$ , 则 1,2 分别为  $R(x)$  的零点, 其中 1 为二重根,2 为三重根。设

$$R(x) = k(x)(x-1)^2(x-2)^3$$

令

$$\phi(t) = f(t) - P_4(t) - k(x)(t-1)^2(t-2)^3$$

则  $\phi(t)$  至少有  $x, 1, 2$  三个零点, 其中 1 为二重,2 为三重。根据 Rolle 定理,  $\phi'(t)$  至少有  $\eta_1, \eta_2, 1, 2$  四个零点, 其中 2 为二重根。再次使用 Rolle 定理,  $\phi''(t)$  至少有  $\mu_1, \mu_2, \mu_3, 2$  四个零点。重复使用, 可得  $\phi^{(5)}(t)$  至少有一个零点  $\xi$ 。则

$$\phi^{(5)}(\xi) = f^{(5)}(\xi) - k(x)5! = 0$$

即

$$k(x) = \frac{f^{(5)}(\xi)}{5!}$$

故

$$R(x) = \frac{f^{(5)}(\xi)}{5!}(x-1)^2(x-2)^3, \quad 1 < \xi < 2$$

P46 T16

用课本 41-43 页的方法，计算得到

$$h_0 = 1, h_1 = 2, h_2 = 1$$

$$\lambda_1 = \frac{2}{3}, \mu_1 = \frac{1}{3}, d_1 = -12$$

$$\lambda_2 = \frac{1}{3}, \mu_2 = \frac{2}{3}, d_2 = 12$$

关于  $M_1, M_2$  的方程组

$$\begin{bmatrix} 2 & \frac{2}{3} \\ \frac{2}{3} & 2 \end{bmatrix} \begin{bmatrix} M_1 \\ M_2 \end{bmatrix} = \begin{bmatrix} -12 \\ 12 \end{bmatrix}$$

得到

$$M_1 = -9, M_2 = 9$$

故

$$S(x) = \begin{cases} -1.5x^3 - 9x^2 - 9.5x + 1 & x \in [-2, -1] \\ 1.5x^3 - 0.5x + 4 & x \in [-1, 1] \\ -1.5x^3 + 9x^2 - 9.5x + 7 & x \in [1, 2] \end{cases}$$
$$S(0) = 4$$

P46 T17

计算得到

$$h_0 = 1, h_1 = 1, h_2 = 2$$

$$\lambda_1 = \frac{1}{2}, \mu_1 = \frac{1}{2}, d_1 = 0$$

$$\lambda_2 = \frac{2}{3}, \mu_2 = \frac{1}{3}, d_2 = 23$$

通过  $S'(-1) = 5, S'(3) = 29$  得到的方程为

$$2M_0 + M_1 = -24$$

$$M_2 + 2M_3 = \frac{99}{2}$$

故关于  $M_0, M_1, M_2, M_3$  的方程组为

$$\begin{bmatrix} 2 & 1 & 0 & 0 \\ \frac{1}{2} & 2 & \frac{1}{2} & 0 \\ 0 & \frac{1}{3} & 2 & \frac{2}{3} \\ 0 & 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} M_0 \\ M_1 \\ M_2 \\ M_3 \end{bmatrix} = \begin{bmatrix} -24 \\ 0 \\ 23 \\ \frac{99}{2} \end{bmatrix}$$

解得

$$M_0 = -13.2273, M_1 = 2.45455, M_2 = 3.40909, M_3 = 23.0455$$

故

$$S(x) = \begin{cases} 2.61364x^3 + 1.22728x^2 - 0.386367x + 3 & x \in [-1, 0] \\ 0.15909x^3 + 1.22728x^2 - 0.386365x + 3 & x \in [0, 1] \\ 1.63637x^3 - 3.20456x^2 + 4.04545x + 1.52274 & x \in [1, 3] \end{cases}$$

$$S(2) = 9.88636$$