HW6.

习题4.

3.(1)
$$\begin{vmatrix} -1 & 3 & 2 \\ 2 & 1 & -2 \\ 3 & 6 & 2 \end{vmatrix} = \begin{vmatrix} -1 & 3 & 2 \\ 0 & 7 & 2 \\ 0 & 15 & 8 \end{vmatrix}$$

$$= \begin{vmatrix} -1 & 3 & 2 \\ 0 & 7 & 2 \\ 0 & 0 & \frac{26}{7} \end{vmatrix} = (-1)\cdot 7 \cdot \frac{26}{7} = -26$$

4.(1)
$$\begin{pmatrix} 5 & 2 & 2 & 1 \\ -1 & 3 & 0 & 7 \\ 1 & 1 & 2 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 2 & 3 \\ -1 & 3 & 0 & 7 \\ 5 & 2 & 2 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 & 2 & 3 \\ 0 & 4 & 2 & 10 \\ 0 & -3 & -8 & -14 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 2 & 3 \\ 0 & 1 & \frac{1}{2} & \frac{5}{2} \\ 0 & 0 & -\frac{13}{2} & -\frac{13}{2} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 & 2 & 3 \\ 0 & 1 & \frac{1}{2} & \frac{5}{2} \\ 0 & 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{pmatrix} \qquad \vec{x} = (-1, 2, 1)^T$$

T8

(1) 有分解

$$\begin{bmatrix} 1 & 3 & 0 & 0 \\ 2 & 7 & 3 & 0 \\ 0 & 2 & 7 & 3 \\ 0 & 0 & 2 & 7 \end{bmatrix} = \begin{bmatrix} 1 & & & \\ 2 & 1 & & \\ & 2 & 1 & \\ & & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & & \\ & 1 & 3 & \\ & & 1 & 3 \\ & & & 1 \end{bmatrix} = LU$$

求解 $Ly = b$，得

$$y = \begin{bmatrix} -2 \\ -4 \\ 2 \\ 1 \end{bmatrix}$$

求解 $Ux = y$，得

$$x = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$$

T7 法1.

(1) 由 LDL^T 分解

$$\begin{bmatrix} -6 & 3 & 2 \\ 3 & 5 & 1 \\ 2 & 1 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ -\frac{1}{3} & \frac{4}{13} & 1 \end{bmatrix} \begin{bmatrix} -6 & 0 & 0 \\ 0 & \frac{13}{2} & 0 \\ 0 & 0 & \frac{236}{39} \end{bmatrix} \begin{bmatrix} 1 & -\frac{1}{2} & -\frac{1}{3} \\ 0 & 1 & \frac{4}{13} \\ 0 & 0 & 1 \end{bmatrix} = LDL^T$$

求解 $Lz = b$，得

$$z = \begin{bmatrix} -4 \\ 9 \\ -\frac{472}{39} \end{bmatrix}$$

求解 $Dy = z$，得

$$y = \begin{bmatrix} \frac{2}{3} \\ \frac{18}{13} \\ -2 \end{bmatrix}$$

求解 $L^T x = y$，得

$$x = \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$$

法2：Doolittle分解

$$\begin{pmatrix} -6 & 3 & 2 \\ 3 & 5 & 1 \\ 2 & 1 & 6 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ -\frac{1}{3} & \frac{4}{13} & 1 \end{pmatrix} \begin{pmatrix} -6 & 3 & 2 \\ & \frac{13}{2} & 2 \\ & & \frac{236}{39} \end{pmatrix}$$

$$= LU$$

求解 $Ly=b$，得

$$y = \begin{pmatrix} -4 \\ 9 \\ -\frac{472}{39} \end{pmatrix}$$

求解 $Ux=y$，得

$$x = \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}$$

法3：Crout分解.

$$\begin{pmatrix} -6 & 3 & 2 \\ 3 & 5 & 1 \\ 2 & 1 & 6 \end{pmatrix} = \begin{pmatrix} -6 & 0 & 0 \\ 3 & \frac{13}{2} & 0 \\ 2 & 2 & \frac{236}{39} \end{pmatrix} \begin{pmatrix} 1 & -\frac{1}{2} & -\frac{1}{3} \\ 0 & 1 & \frac{4}{13} \\ 0 & 0 & 1 \end{pmatrix}$$

$$= LU$$

求解 $Ly=b$，得

$$y=\begin{pmatrix}\frac{2}{3}\\ \frac{18}{13}\\ -2\end{pmatrix}$$

求解 $Ux=y$，得

$$x=\begin{pmatrix}1\\ 2\\ -2\end{pmatrix}$$

习题5

2.(2) $B=\begin{pmatrix}5&2&2\\2&6&0\\2&0&4\end{pmatrix}=(b_{ij})$

谱半径 $\rho(B)=\max\limits_{i}|\lambda_i(B)|$

$$|\lambda I-B|=\begin{vmatrix}\lambda-5&-2&-2\\-2&\lambda-6&0\\-2&0&\lambda-4\end{vmatrix}=(\lambda-5)(\lambda-8)(\lambda-2)$$

$\rho(B)=8$

$$\|B\|_1 = \max_i \sum_{j=1}^{3} |b_{ij}| = \max\{9, 8, 6\} = 9.$$

$$\|B\|_\infty = \max_j \sum_{i=1}^{3} |b_{ij}| = \max\{9, 8, 6\} = 9.$$

$$\|B\|_2 = \sqrt{\rho(B^T B)} = \sqrt{\max_i \lambda_i(B^T B)}$$

因 B 为对称矩阵，故

$$\lambda(B^T B) = \lambda^2(B)$$

事实上，若 λ_0，X_0 为 B 的一个特征值和对应的特征向量，则

$$B X_0 = \lambda_0 X_0.$$

因 $B^T = B$

故

$$\begin{aligned} B^T B X_0 &= B(B X_0) = B(\lambda_0 X_0) \\ &= \lambda_0 B X_0 \\ &= \lambda_0 \lambda_0 X_0 \\ &= \lambda_0^2 X_0 \end{aligned}$$

故 $B^T B$ 的特征值为 $2^2, 5^2, 8^2$

$$\|B\|_2 = \sqrt{8^2} = 8.$$

T3

(1) Jacobi 迭代格式为

$$
\begin{aligned}
x_1^{k+1} &= 0.1x_2^k + 0.1 \\
x_2^{k+1} &= 0.1x_1^k + 0.1x_3^k \\
x_3^{k+1} &= 0.1x_2^k + 0.1x_4^k + 0.1 \\
x_4^{k+1} &= 0.1x_3^k + 0.2
\end{aligned}
$$

计算得

$$
X_1 = \begin{bmatrix} 0.1 \\ 0 \\ 0.1 \\ 0.2 \end{bmatrix} \quad X_2 = \begin{bmatrix} 0.1 \\ 0.02 \\ 0.12 \\ 0.21 \end{bmatrix} \quad X_3 = \begin{bmatrix} 0.102 \\ 0.022 \\ 0.123 \\ 0.212 \end{bmatrix}
$$

(2) Gauss-Seidel 迭代格式为

$$
\begin{aligned}
x_1^{k+1} &= 0.1x_2^k + 0.1 \\
x_2^{k+1} &= 0.1x_1^{k+1} + 0.1x_3^k \\
x_3^{k+1} &= 0.1x_2^{k+1} + 0.1x_4^k + 0.1 \\
x_4^{k+1} &= 0.1x_3^{k+1} + 0.2
\end{aligned}
$$

计算得

$$
X_1 = \begin{bmatrix} 0.1 \\ 0.01 \\ 0.101 \\ 0.2101 \end{bmatrix} \quad X_2 = \begin{bmatrix} 0.101 \\ 0.0202 \\ 0.12303 \\ 0.212303 \end{bmatrix} \quad X_3 = \begin{bmatrix} 0.10202 \\ 0.022505 \\ 0.123481 \\ 0.212348 \end{bmatrix}
$$

(3) Jacobi 迭代矩阵:

$$
R = \begin{bmatrix} 0 & 0.1 & 0 & 0 \\ 0.1 & 0 & 0.1 & 0 \\ 0 & 0.1 & 0 & 0.1 \\ 0 & 0 & 0.1 & 0 \end{bmatrix}
$$

或 A 严格对角占优

Gauss-Seidel 迭代矩阵:

$$
S = \begin{bmatrix} 0 & 0.1 & 0 & 0 \\ 0 & 0.01 & 0.1 & 0 \\ 0 & 0.001 & 0.01 & 0.1 \\ 0 & 0.0001 & 0.001 & 0.01 \end{bmatrix}
$$

计算得 $\rho(R)=0.161803$, $\rho(S)=0.0261803 < 1$

$$\cancel{\rho(R) = 0.402248 \quad \rho(S) = 0.161803}$$

故 Jacobi 和 Gasuu-Seidel 迭代都收敛。

或用 R,S 的 1 范数, ∞ 范数