2017 QuantEcon Workshops

Economic Modeling with Julia

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- Modern
- Open source
- Focused on scientific computing
- High productivity and high speed

- ...sometimes

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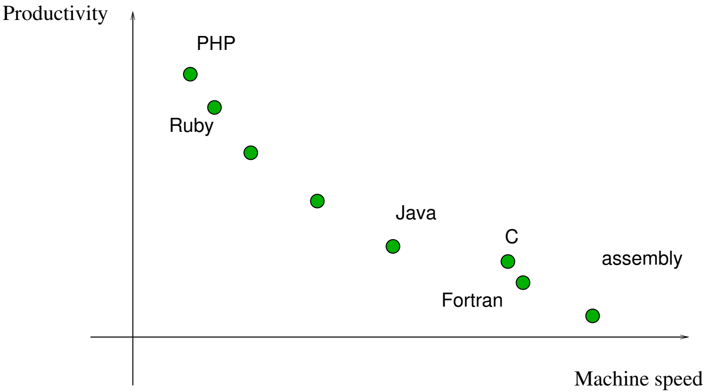
Productivity
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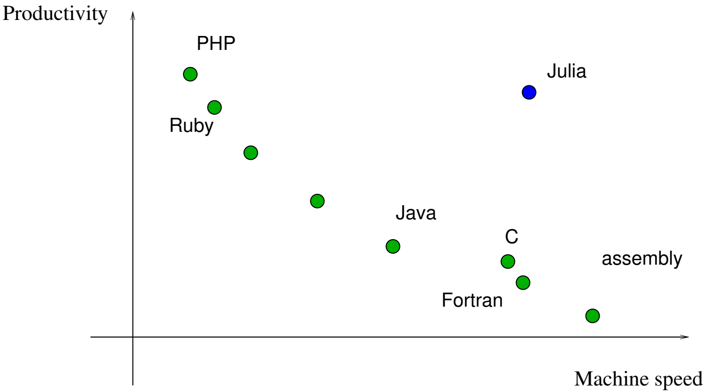
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Part 1: Julia's JIT compilation system

- See `John/Julia/julia_jit_compilation.ipynb`

Part 2: An application to recursive preference models

- Based on joint work with Jaroslav Borovička

1. Overview of problem
2. Description of model
3. Review of code

Overview of Problem

Recursive utilities models used to value consumption streams

Existence, uniqueness of solutions are parameter dependent

We lack reliable, globally convergent methods to compute solutions

Our aims:

- Provide **practical** conditions for existence / uniqueness
- Provide globally convergent computational methods

Confessions:

- We assume compact state space

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Valuation

How to model the value of a lifetime consumption path $\{C_t\}$?

We allow the path to be nonstationary

Following Hansen and Scheinkman's **multiplicative functional** formulation,

$$\ln C_{t+1} - \ln C_t = \kappa(X_{t+1}, Y_{t+1}, X_t), \tag{1}$$

where

- κ is a continuous real-valued function
- $\{X_t\}$ is a Markov process taking values in $\mathbb{X}$
- $\{Y_t\}$ is an IID innovation

EZ Preferences

The **value** V_t of consumption path $\{C_i\}_{i \geq t}$ is defined by

$$V_t = \left[\zeta C_t^{1-1/\psi} + \beta \left\{\mathcal{R}_t\left(V_{t+1}\right)\right\}^{1-1/\psi}\right]^{1/(1-1/\psi)}$$

Here $\mathcal{R}_t$ is the **KP certainty equivalent operator**

$$\mathcal{R}_t(V_{t+1}) := (\mathbb{E}_t V_{t+1}^{1-\gamma})^{1/(1-\gamma)}$$

- $\beta \in (0, 1) =$ time discount factor
- γ governs the level of relative risk aversion
- $\psi =$ elasticity of intertemporal substitution

Manipulations give

$$\left(\frac{V_t}{C_t}\right)^{1-1/\psi} = \zeta + \beta \left\{ \mathcal{R}_t \left(\frac{V_{t+1}}{C_{t+1}} \frac{C_{t+1}}{C_t} \right) \right\}^{1-1/\psi}$$

Equivalently,

$$W_t = \zeta + \beta \left\{ \mathbb{E}_t W_{t+1}^{\theta} \exp[(1-\gamma)\kappa(X_{t+1}, Y_{t+1}, X_t)] \right\}^{1/\theta}$$

where

- $\theta := (1-\gamma)/(1-1/\psi)$ and
- $W_t := (V_t/C_t)^{1-1/\psi}$

We seek a Markov solution $W_t = w(X_t)$ for some $w \colon \mathbb{X} \to \mathbb{R}$ satisfying

$$w(x) = \zeta + \beta \left\{ \int w(x')^\theta \int \exp[(1-\gamma)\kappa(x', y', x)] \nu(\mathrm{d}y') Q(x, \mathrm{d}x') \right\}^{1/\theta}$$

Equivalently, we seek fixed points in $\mathscr{C}_+$ of

$$Tw(x) = \zeta + [K w^\theta(x)]^{1/\theta}$$

where

$$Kg(x) := \beta^\theta \int g(x') \int \exp[(1-\gamma)\kappa(x', y', x)] \nu(\mathrm{d}y') Q(x, \mathrm{d}x')$$

Theorem 1

If $\theta < 0$, then the following statements are equivalent:

1. $r(K) > 1$
2. T has a fixed point in $\mathscr{C}_+$
3. T has a unique fixed point in $\mathscr{C}_+$
4. T is globally asymptotically stable on $\mathscr{C}_+$

Theorem 2

If $\theta > 0$, then the following statements are equivalent:

1. $r(K) < 1$
2. T has a fixed point in $\mathscr{C}_+$
3. T has a unique fixed point in $\mathscr{C}_+$
4. T is globally asymptotically stable on $\mathscr{C}_+$

Application: Long-Run Risk

In Bansal and Yaron (2004), consumption growth obeys

$$\ln(C_{t+1}/C_t) = \mu + z_t + \sigma_t \, \eta_{t+1}$$

where

$$z_{t+1} = \rho z_t + s_z \, \sigma_t \, \epsilon_{t+1}$$

and

$$\sigma^2_{t+1} = v \, \sigma^2_t + \bar{\sigma}^2 (1 - v) + s_\sigma \, \omega_{t+1}$$

Innovations are all $\overset{\text{IID}}{\sim} N(0, 1)$

The state can be represented as $X_t := (z_t, \sigma_t)$

Discretization

To compute $r(K)$,

- discretize $\mathbb{X}$ to $\hat{\mathbb{X}} = \{x_1, \ldots, x_M\}$
- let $\mathbf{Q}_{ij} := \mathbb{P}\{X_{t+1} = x_j \mid X_t = x_i\}$

The operator K reduces to the matrix

$$\begin{aligned} \mathbf{K}_{ij} &= \beta^\theta \int \exp[(1-\gamma)\kappa(x_j, y, x_i)] \nu(\mathrm{d}y) \mathbf{Q}_{ij} \\ &= \beta^\theta m(x_i) \mathbf{Q}_{ij} \end{aligned}$$

where

$$m(x) = m(z, \sigma) := \exp\left\{ (1-\gamma)(\mu + z) + \frac{(1-\gamma)^2 \sigma^2}{2} \right\}$$

Parameters for the Bansal and Yaron (2004) model

- $\mu = 0.0015$
- $\psi = 1.5$
- $\gamma = 10.0$
- $\beta = 0.998$
- $\rho = 0.979$
- $s_z = 0.044$
- $v = 0.987$
- $\bar{\sigma} = 0.0078$
- $s_\sigma = 0.0000026$

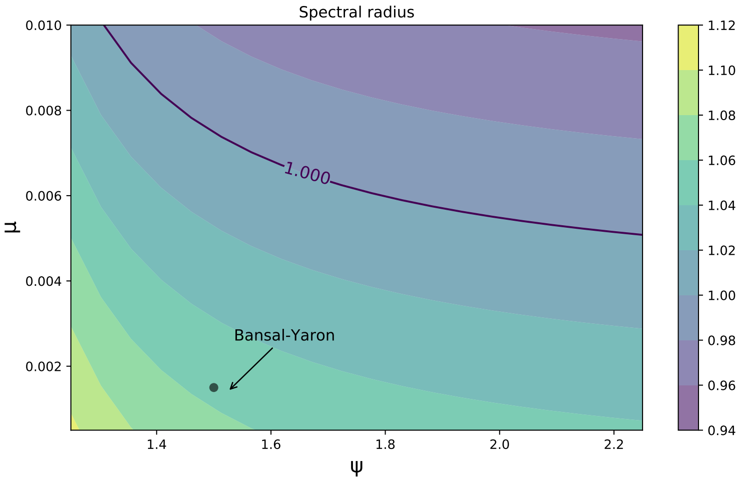

Figure: Spectral radius for the Bansal–Yaron model

Parameters for the Bansal, Kiku and Yaron (2012) model

- $\mu = 0.0015$
- $\psi = 1.5$
- $\gamma = 10.0$
- $\beta = 0.9989$
- $\rho = 0.975$
- $s_z = 0.038$
- $v = 0.999$
- $\bar{\sigma} = 0.0072$
- $s_\sigma = 0.0000028$

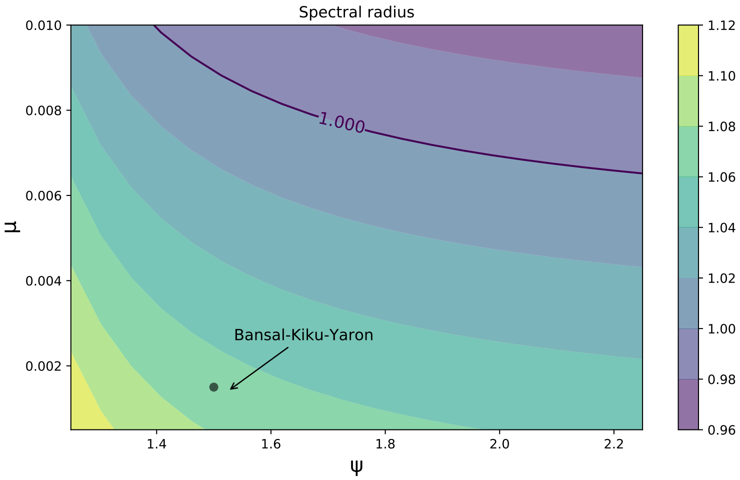

Figure: Spectral radius for the Bansal–Kiku–Yaron model

Let's look at the code...

- See `John/Julia/wams_demo.ipynb`