

High-Dimensional Economics

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Preface

To be completed. Note that “a preface or foreword deals with the genesis, purpose, limitations, and scope of the book and may include acknowledgments of indebtedness.”

This textbook is an extension of online lectures found at <https://quantecon.org/>. While those lectures provide code, this text emphasizes theory. Algorithms are described mathematically or in pseudo-code, so the text is language-agnostic. The focus is on models that are high-dimensional, due to many time periods, many states of the world, many goods or many agents.

While such diversity can be assumed away in purely theoretical models, we cannot do the same in quantitative models that seek to generate realistic predictions. Diversity matters because agents differ in many ways. Individuals differ in their incomes, households differ in their compositions and wealth levels, and firms differ in their productivities. In addition, time generates change and uncertainty. Economic agents do not make decisions without consideration of future consequences. As such, most of the models we treat are explicitly dynamic and stochastic—which leads to additional heterogeneity over products, services and technology. Predictions typically take the form of cross-sectional distributions, which are high-dimensional objects.

High-dimensional models can provide exciting and valuable insights into a broad range of social, economic and financial phenomena. At the same time, high dimensionality causes extreme challenges for quantitative analysis.

Fortunately, there has been rapid progress in our understanding of these models and the set of tools we have to solve them. From the beginnings of modern Markov process theory in the 1930s to the advances in duality and linear programming in the 1940s and 1950s, the recursive methods for control and dynamic programming developed in the 1950s and 1960s, the widespread adoption of recursive methods in economics in the 1970s and 1980s, the recent burst of research on optimal transport, Markov stability, abstract dynamic programming, stochastic control, mean field games and, finally, the latest machine learning methods combined with implementations based around specialized electronic circuits, the set of possibilities for economists tackling

high dimensional problems has never been larger. This book provides an overview and an integrated treatment.

In terms of prerequisites, we hope readers are comfortable with elementary results on sequences, series, functions, limits, linear algebra and probability, although we review them in the text. Deeper into the book, you will see an increasing amount of functional analysis. Some of this concerns fixed point theory, since we need to solve equations in which the unknown object is a function. These are classic examples of high-dimensional problems, necessitating development of some specialized machinery.

The other piece of sophisticated mathematical machinery we sometimes call on is measure theory. We have formulated the text so that the core dynamic programming theory along with many foundational models are accessible without measure. Nonetheless, as we start to drill down into topics requiring stochastic process theory and some more sophisticated parts of functional analysis, measure theory becomes essential. A quick introduction to the key ideas is given in Section 11.3.

Add a roadmap and some study options.

Full acknowledgements to be added. We are greatly indebted to Jim Savage and Schmidt Futures for their generous support. We also extend thanks to Fernando Cirelli, Rebekah Dix, Shu Hu, Fazeleh Kazemian, Flint O’Neil and Natasha Watkins for many important fixes and additions.

Common Symbols

$P \implies Q$	P implies Q
$P \iff Q$	$P \implies Q$ and $Q \implies P$
$\alpha := 1$	α is defined as equal to 1
$f \equiv 1$	function f is everywhere equal to 1
$\wp(A)$	the power set of A ; that is, the collection of all subsets of given set A
\mathbb{N} , \mathbb{Z} and \mathbb{R}	the natural numbers, integers and real numbers respectively
\mathbb{Z}_+	the nonnegative integers
\mathbb{R}_+	the nonnegative real numbers
\mathbb{R}^n	all n -tuples of real numbers
$\langle a, b \rangle$	the inner product of a and b
$\mathcal{M}_{n \times k}$	all $n \times k$ matrices
$b\mathbb{X}$	the set of bounded, real-valued functions on \mathbb{X}
$b\mathcal{E}$	the set of real-valued \mathcal{E} measurable functions on $(\mathbb{E}, \mathcal{E})$
$bc\mathbb{X}$	the set of continuous functions in $b\mathbb{X}$
$ib\mathbb{X}$	the set of increasing functions in $b\mathbb{X}$
$\text{Lip}_a(\mathbb{X}, \rho)$	the set of a -Lipschitz functions on (\mathbb{X}, ρ)
\mathcal{B}	the Borel measurable subsets of \mathbb{X}
$\mathbb{1}\{P\}$	indicator, equal to 1 if statement P is true and 0 otherwise
iid	independent and identically distributed
$X \stackrel{d}{=} Y$	X and Y have the same distribution
$X \sim F$	X has distribution F
$\mathcal{C}(\varphi, \psi)$	the set of all couplings of (φ, ψ)
e_n	the n -th canonical basis vector
$F \preceq_F G$	F first order stochastically dominates G
$F \preceq_S G$	F second order stochastically dominates G

Part I

Foundations

Chapter 1

Overview

Complete. Add a roadmap of the chapter.

1.1 The State of the Art

In the 1950s, building on recent advances in fixed point theory, linear programming, convex analysis and duality, economists such as Kenneth J. Arrow (1921-2017) and Gérard Debreu (1921-2004) constructed a strikingly broad theory of general equilibrium in free markets. In their framework, uncertainty was mapped into deterministic specifications through the use of contingent commodities. Dynamic models were reduced to static ones. States of the world and dates became just labels across goods. Such abstractions lead to fundamental insights and far reaching theorems.

Problems arose in the 1960s and 70s, however, when economists attempted to *compute* prices and quantities generated by these systems. In doing so, they ran headlong into what is now called the “curse of dimensionality.” This phrase, coined by Richard Bellman (1920-1984) during his fundamental research into dynamic optimization, refers to the exponential increase in processor cycles needed to solve computational problems to a given level of accuracy as the number of dimensions increases.

In subsequent decades, mathematicians and economists made great advances in mitigating the curse of dimensionality and solving hard computational problems. Many of their names appear in this book. These researchers understood that time provides structure (the “arrow of time”) and, when solving hard problems, structure is desperately needed. So dynamic models became dynamic again, and recursive methods moved to center stage. Economists began to think, talk and even dream in terms of recursive operations.

Now, however, the pendulum has swung back to some degree, and what was old is new again. For example, heavily parallelized algorithms, sophisticated hardware and software techniques such as automatic differentiation have set the stage for direct attacks on some high-dimensional models that have no obvious recursive structure. These are exciting developments but the challenges are by no means trivial.

For those models that do have recursive structure, including most of those treated in this text, parallel methods can be combined with recursive methods but they integrate only partially. This is because the implementations of recursive methods are iterative and sequential. Researchers are now racing to connect modern machine learning methods with existing recursive algorithms.

This book presents a summary of the current state of quantitative economic modeling, with high-dimensional problems as the common thread. The next section presents major themes. The remainder of the chapter presents a taste of the problems we will tackle, before we begin constructing our foundations.

1.2 Major Themes

One of the most important recurring topics in this text is stability. Of course, economic data exhibit all kinds of *instability*, such as recessions, financial crises, and exchange rate fluctuations. But to understand and predict, the only way that we can succeed is to detect patterns that lie underneath this apparent instability. Once we have discovered the patterns, we can model and estimate them. Then we can subject our model to random shocks or policy experiments.

The stability theory that we focus on concerns both long-run and cross-sectional regularities in dynamic stochastic systems. The best-case scenario is where some form of global stability can be established, which implies that these regularities exist and, in addition, they are a unique equilibrium prediction of the system. Obtaining these predictions with pencil and paper might not be feasible—particularly as the systems become more complex—but there are powerful ways to compute and simulate them. Although the text contains no executable computer code, it does provide algorithms. All algorithms rest on careful mathematical analysis, which is essential for accuracy and efficiency.

Complete discussion. Dynamic programming and stochastic control. Getting “Bellmanized.” More general recursive methods. Some nonrecursive techniques. Parallelization. Connection with machine learning methods.

Chapter 2

First Steps

Roadmap to be added.

2.1 Low-Dimensional Models

Roadmap to be added.

2.1.1 Notation

Let \mathbb{R} denote the **real numbers** (i.e., the set of rational and irrational numbers). For points x, y in \mathbb{R} , we sometimes use the notation

$$x \vee y := \max\{x, y\} \quad \text{and} \quad x \wedge y := \min\{x, y\}. \quad (2.1)$$

Throughout the text, an n -vector x is a tuple of n real numbers $x = (x_1, \dots, x_n)$ where $x_i \in \mathbb{R}$ for each i . In general, x is neither a row vector nor a column vector. (This extra structure is only when using matrix algebra.) We let \mathbb{R}^n be the set of all n -vectors and $\mathcal{M}_{n \times k}$ be all $n \times k$ matrices. If A is a matrix then A' is its transpose.

In general, if f and g are real-valued functions (i.e., functions taking values in \mathbb{R}) defined on some common set X and α is a scalar, then $f + g$, αf , fg , etc., have the usual interpretations: they are functions on X defined by

$$(f + g)(x) := f(x) + g(x), \quad (\alpha f)(x) = \alpha f(x), \quad \text{etc.} \quad (2.2)$$

for each $x \in X$. Similarly, $f \vee g$ and $f \wedge g$ are functions on X defined by

$$(f \vee g)(x) := f(x) \vee g(x) \quad \text{and} \quad (f \wedge g)(x) := f(x) \wedge g(x).$$

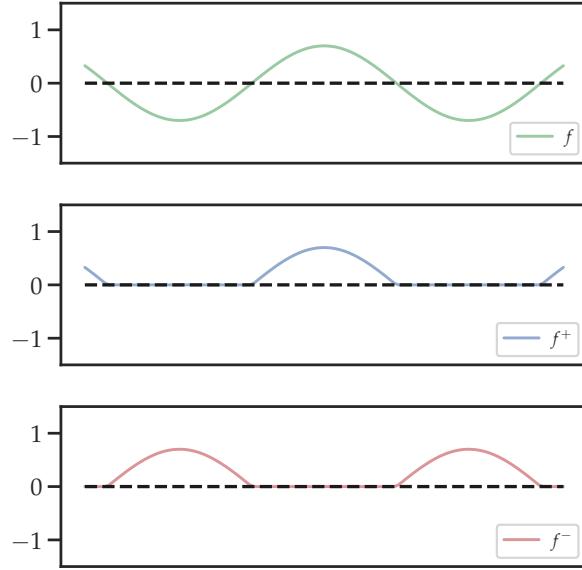


Figure 2.1: Decomposition of functions

We sometimes use the notation

$$f^+ = f \vee 0 \quad \text{and} \quad f^- = -(f \wedge 0).$$

See figure 2.1. These objects are useful because $f = f^+ - f^-$ always holds, so the pair f^+, f^- provides a decomposition of f into the difference between two nonnegative functions. The function f^+ is called the **positive part** of f , while f^- is called the **negative part**.

Throughout, a nonempty set X is called **countable** if it is finite or can be placed in one-to-one correspondence with the natural numbers \mathbb{N} . In the second case we can enumerate X by writing it as $\{x_1, x_2, \dots\}$. Any nonempty set X that fails to be countable is called **uncountable**.

2.1.2 Analysis on the Reals

Next is a short review of real analysis. The objective is to recall some definitions and results that are needed for the text. (If you require a more in depth treatment, try [Bartle and Sherbert \(2011\)](#), which is slow, careful, and beautifully written. Or, if you prefer something with a faster pace, see [Cinlar and Vanderbei \(2013\)](#).)

Regarding the order structure of \mathbb{R} , the following relationships are helpful: Given $x, y \in \mathbb{R}$ and $a \in \mathbb{R}_+$,

- (i) $x + y = x \vee y + x \wedge y$
- (ii) $|x - y| = x \vee y - x \wedge y$
- (iii) $|x - y| = x + y - 2(x \wedge y)$
- (iv) $|x - y| = 2(x \vee y) - x - y$
- (v) $a(x \vee y) = (ax) \vee (ay)$
- (vi) $a(x \wedge y) = (ax) \wedge (ay)$

For example, to show (ii), observe that, for any real values x, y , we have $x - y \leqslant x \vee y - x \wedge y$ and $y - x \leqslant x \vee y - x \wedge y$. Hence (ii) holds.

A **sequence** $\{x_n\}$ in \mathbb{R} is a mapping $n \mapsto x_n$ from \mathbb{N} to \mathbb{R} . We say that $\{x_n\}$ **converges** to a point x in \mathbb{R} and write $x_n \rightarrow x$ if

for any $\varepsilon > 0$, there is an $N \in \mathbb{N}$ such that $|x_n - x| < \varepsilon$ whenever $n \geqslant N$.

For example, if $x_n = 1 - 1/n$ and $x = 1$, then $x_n \rightarrow x$. Indeed, for any $\varepsilon > 0$, the statement $|x_n - x| < \varepsilon$ is equivalent to $n > 1/\varepsilon$. This clearly holds whenever n is sufficiently large.

If $\{x_n\}$ and $\{y_n\}$ are sequences in \mathbb{R} with $x_n \rightarrow x$ and $y_n \rightarrow y$, then

- $x_n + y_n \rightarrow x + y$ and $x_n y_n \rightarrow xy$
- $x_n \leqslant y_n$ for all n implies $x \leqslant y$
- $\alpha x_n \rightarrow \alpha x$ for any $\alpha \in \mathbb{R}$
- $x_n \vee y_n \rightarrow x \vee y$ and $x_n \wedge y_n \rightarrow x \wedge y$.

If you are not familiar with these limit laws then please review the proofs (see, e.g., [Bartle and Sherbert \(2011\)](#)). We shall use them extensively.

A sequence $\{x_n\}$ is called **bounded** if there is an $M \in \mathbb{R}$ such that $|x_n| \leqslant M$ for all $n \in \mathbb{N}$. It is called **monotone increasing** if $x_n \leqslant x_{n+1}$ for all $n \in \mathbb{N}$, and **monotone decreasing** if $x_n \geqslant x_{n+1}$ for all n . It is called **monotone** if it is either monotone increasing or decreasing. The next theorem is a deep result about the structure of \mathbb{R} , a proof for which can be found in chapter 3 of [Bartle and Sherbert \(2011\)](#).

Theorem 2.1.1. *A monotone sequence $\{x_n\}$ in \mathbb{R} converges to a point in \mathbb{R} if and only if it is bounded.*

Next let's consider **series**. Given a sequence $\{x_n\}$ in \mathbb{R} , we define $\sum_{n \geq 1} x_n$ to be the number $\lim_{N \rightarrow \infty} \sum_{n=1}^N x_n$ whenever the limit exists in \mathbb{R} . More generally, given a function g from an arbitrary countable set X to \mathbb{R} , we write $\sum_{x \in X} g(x) = M$ if there exists an enumeration $\{x_n\}_{n \in \mathbb{N}}$ of X such that absolute sum $\sum_{n \geq 1} |g(x_n)|$ is finite and, in addition, $\sum_{n \geq 1} g(x_n) = M$.¹

EXERCISE 1. Show that, if X is countable, $g \in \mathbb{R}^X$, and there exist points $x', x'' \in X$ such that $g(x') > 0$ and $g(x'') < 0$, then

$$\left| \sum_{x \in X} g(x) \right| < \sum_{x \in X} |g(x)|$$

Hint: Prove this claim for the case where X has two elements. Argue that the case with n elements follows from this result and the ordinary (weak) triangle inequality $|\sum_{x \in X} g(x)| \leq \sum_{x \in X} |g(x)|$.

2.1.3 Growth in Discrete Time

Let's put some of these ideas to work.

Most of the applications in this text start at the microeconomic level, beginning with the decisions of households, firms, and other economic entities. We then aggregate to obtain macroeconomic implications of micro decisions. Rather than restricting attention to a world with a single representative agent, we focus on settings in which agents exhibit nontrivial heterogeneities. In many cases our objective will be to understand the dynamics of a distribution flow or sequence, describing a cross-section of agents.

Nonetheless, as our first example, we begin with a traditional model, called the **Solow–Swan growth model**, in which agents are identical, aggregation is trivial, and distributions are point masses. While this set-up is unrealistic, it allows us to begin our discussion of dynamics in one dimension.

2.1.3.1 The Model

The Solow–Swan economy contains a collection of identical agents, each of whom saves the same fixed fraction of their current incomes. Savings are devoted to sustaining or increasing the physical capital stock. Capital is combined with labor to produce

¹This definition is not ambiguous because every possible enumeration leads to the same value when the absolute sum is finite (see, e.g., the rearrangement theorem in [Bartle and Sherbert \(2011\)](#), §9.1).

output, which in turn is paid out to workers and owners of capital. Some income is consumed, some saved, and we go round again. To keep things simple, we ignore population and productivity growth.

For each integer $t \geq 0$, output Y_t in period t is given by

$$Y_t = F(K_t, L_t),$$

where K_t is capital, L_t is labor and F is an aggregate production function. The function F assumed to be nonnegative and **homogeneous of degree one**, meaning that

$$F(\lambda K, \lambda L) = \lambda F(K, L) \quad \text{for all } \lambda \geq 0.$$

Production functions with this property include

- the **Cobb-Douglas** function $F(K, L) = AK^\alpha L^{1-\alpha}$ with $0 \leq \alpha \leq 1$ and
- the **CES** function $F(K, L) = \{aK^\rho + bL^\rho\}^{1/\rho}$ with $a, b, \rho > 0$.

We assume a closed economy, so domestic investment equals aggregate domestic saving. The saving rate is a positive constant s , so that aggregate investment and saving both equal sY_t . Capital depreciates. Without replenishing through investment, one unit of capital today becomes $1 - \delta$ units tomorrow. Thus, capital stock evolves according to

$$K_{t+1} = sF(K_t, L_t) + (1 - \delta)K_t.$$

Without population growth, L_t equals some constant L . Setting $k_t := K_t/L$ and using homogeneity of degree one now yields

$$k_{t+1} = s \frac{F(K_t, L)}{L} + (1 - \delta)k_t = sF(k_t, 1) + (1 - \delta)k_t.$$

With $f(k) := F(k, 1)$, the final expression for capital dynamics is

$$k_{t+1} = g(k_t) := sf(k_t) + (1 - \delta)k_t. \quad (2.3)$$

Our aim is to learn about the evolution of k_t over time, given an exogenous initial capital stock k_0 .

2.1.3.2 A Graphical Perspective

To understand the dynamics of the sequence $\{k_t\}$, we begin with a **45 degree diagram**, a useful tool for studying one-dimensional dynamics. To draw the diagram, we first need to specify the functional form for f and give values to the parameters. We

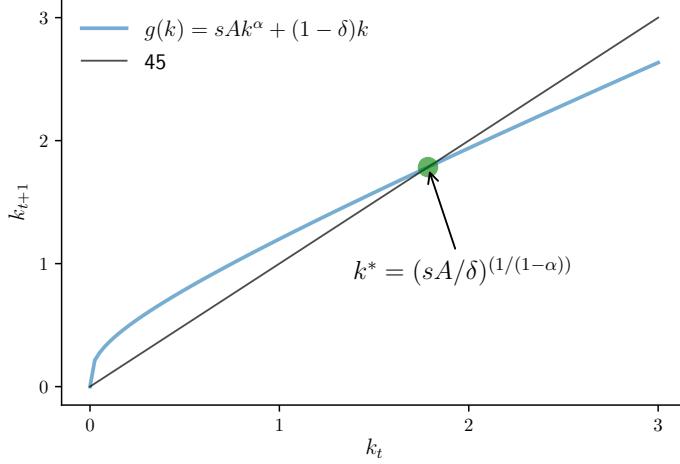


Figure 2.2: 45 degree diagram for the Solow–Swan model

choose the Cobb–Douglas specification $f(k) = Ak^\alpha$. We set $A = 2.0$, $\alpha = 0.3$, $s = 0.3$ and $\delta = 0.4$. The function g from (2.3) is then plotted, along with the 45 degree line. The result is shown in Figure 2.2.

The diagram is easily understood: If, at some k_t , the value $g(k_t)$ lies strictly above the 45 degree line, then we have $k_{t+1} = g(k_t) > k_t$. Hence capital per worker is rising. If $g(k_t)$ lies below, then capital per worker is falling. If $g(k_t) = k_t$, then we are at a **steady state** or **stationary point**, and k_t does not change with time. From a mathematical perspective, a steady state is just a **fixed point** of the map g , that is, a k such that $g(k) = k$.

From the shape of the function g in Figure 2.2, we see that there is a unique steady state. It solves $k = sAk^\alpha + (1 - \delta)k$ and hence is given by

$$k^* := \left(\frac{sA}{\delta} \right)^{1/(1-\alpha)}. \quad (2.4)$$

If initial capital is below this level then capital increases over time. If initial capital is above this level then the reverse is true. Thus, we can say that $\{k_t\}$ converges to k^* , regardless of initial capital k_0 .

Figure 2.3 shows two time paths for capital, from two distinct initial conditions, under the parameterization listed above. At this parameterization, k^* is approximately 1.78. As expected, the time paths in the figure both converge to this value. The fact that capital converges to this stationary point from all positive initial conditions indicates a form of **global stability**. We will define this term precisely in §5.3.1.1.

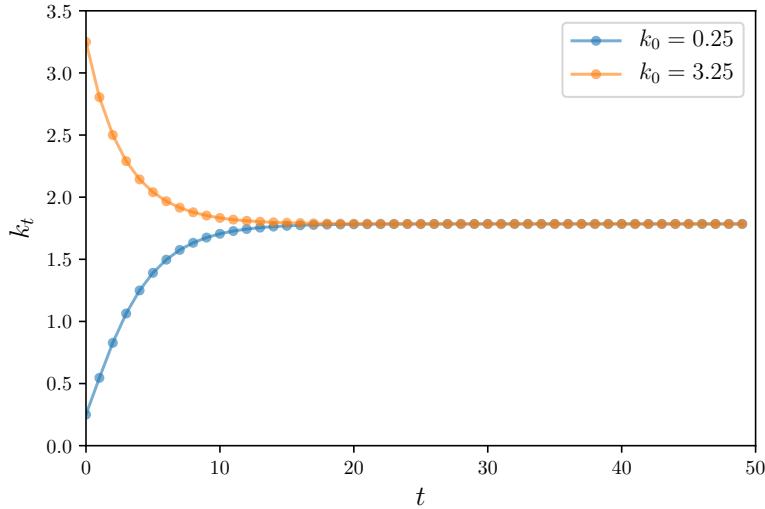


Figure 2.3: Time series for the Solow–Swan model

2.1.3.3 More General Analysis

Let's attempt a more formal analysis of the dynamics that does not rely on specifying particular values for parameters (as we did with the 45 degree diagram). In particular, we aim to prove that, if $f(k) = Ak^\alpha$ and the parameters satisfy $0 < \alpha, \delta < 1$ and $A > 0$, then $\{k_t\}$ converges to the steady state k^* whenever $k_0 > 0$. We now check this convergence claim for k_0 satisfying $k_0 \leq k^*$, with the other case left as an exercise.

We can try calculating k_t explicitly, starting with $k_1 = sAk_0^\alpha + (1 - \delta)k_0$, inserting k_1 into the right hand side of $k_2 = sAk_1^\alpha + (1 - \delta)k_1$ to obtain

$$k_2 = sA[sAk_0^\alpha + (1 - \delta)k_0]^\alpha + (1 - \delta)[sAk_0^\alpha + (1 - \delta)k_0]$$

and so on. However, the algebra becomes progressively more difficult while providing little insight. So let's try a more analytical approach.

EXERCISE 2. Prove the following for positive constant k_0 .

- If $k_0 \leq k^*$, then $\{k_t\}$ is increasing and bounded.
- If $k^* \leq k_0$, then $\{k_t\}$ is decreasing and bounded.

Since bounded monotone sequences in \mathbb{R} always converge (see Theorem 2.1.1!), it must be that $k_t \rightarrow \hat{k}$ for some $\hat{k} \in (0, \infty)$. Because g is continuous, this implies

$g(k_t) \rightarrow g(\hat{k})$. But the sequence $\{g(k_t)\}$ is just the original sequence $\{k_t\}$ with the first element removed, and hence must have the same limit point as the original. This means that $g(k_t) \rightarrow \hat{k}$ holds. Now we have both $g(k_t) \rightarrow g(\hat{k})$ and $g(k_t) \rightarrow \hat{k}$. Since limits of sequences are unique, this yields $\hat{k} = g(\hat{k})$. But k^* is the only positive number satisfying the equation $k = g(k)$, so $\hat{k} = k^*$. We already showed that $k_t \rightarrow \hat{k}$, so $k_t \rightarrow k^*$ is established.

2.1.3.4 Extensions

While the stability argument presented above is already nontrivial, its scope is limited. For one, the analysis is specialized to the Cobb–Douglas production function, which is just one of many possible specifications. It would be better if we could formulate a general argument that fits all “reasonable” production functions. For example, marginal product of capital is typically assumed to be positive and diminishing. Can we produce a stability analysis with only these assumptions? If not, what more would we require? Such an analysis would not only be more general, but would also shed more light on what stability requires.

These considerations motivate us to study stability analysis more deeply, as well as an underlying branch of mathematical analysis called fixed point theory. Once our knowledge of fixed point theory is sufficiently developed, we will see that clear answers to the questions posed above can be provided in a general setting.

One generalization we need to handle before taking the model to data is stochastic fluctuations. This will allow us to eliminate the unrealistic prediction that per-capita output $y_t = Ak_t^\alpha$ converges to a constant $y^* := A(k^*)^\alpha$. For example, we might replace the constant productivity term A with some stochastic sequence $\{A_t\}$. Figure 2.4 shows an example in which $\{A_t\}$ is IID and lognormal. (Otherwise the setting is unchanged.) Now the long run convergence obtained in the deterministic case (i.e., in Figure 2.3) breaks down. The system is hit with new shocks at each point in time and the time series no longer converge to a constant.

Although the new time series is more realistic, the apparent lack of stability is worrisome because, as emphasized in our introduction, stability analysis is important for estimation and prediction. However, if we look at Figure 2.4 carefully, we observe that the influence of initial conditions seems to die out, and the two time series eventually fluctuate in similar ranges. This hints at the fact that, for this model, stability is not lost after all. There is a higher-level notion of stability at work here. It can be shown—and we will show—that this system converges to a stochastic steady state once we refocus our attention from individual time paths, such as those shown in Figure 2.4, to the evolution of probability distributions.

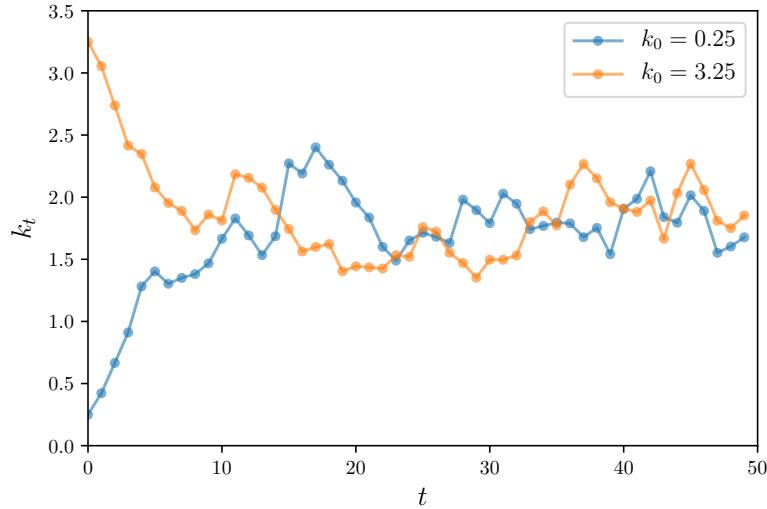


Figure 2.4: Solow–Swan dynamics with stochastic productivity

Showing this does require some investment in the underlying mathematics, since we need to study dynamics in a space of distributions that is infinite-dimensional. But once this investment is made, we will see that stochastic steady states are well defined objects that can be computed to arbitrary degrees of precision and that allow us to make rigorous probabilistic predictions about the dynamics of cross-sections and time series.

2.1.4 Growth in Continuous Time

In this section we investigate a continuous time version of the Solow–Swan growth model. We will conduct a graphical analysis that is parallel to the 45 diagram technique and also see how the smoothing provided by continuous time can sometimes aid analysis.

Recall from (2.3) that the discrete time dynamics for capital are given by $k_{t+1} = sf(k_t) + (1 - \delta)k_t$. A simple rearrangement gives the rate of change per unit of time:

$$\Delta k_t = sf(k_t) + \delta k_t \quad \text{where} \quad \Delta k_t := k_{t+1} - k_t. \quad (2.5)$$

Taking the time step to zero gives the continuous time limit

$$k'(t) = sf(k(t)) - \delta k(t). \quad (2.6)$$

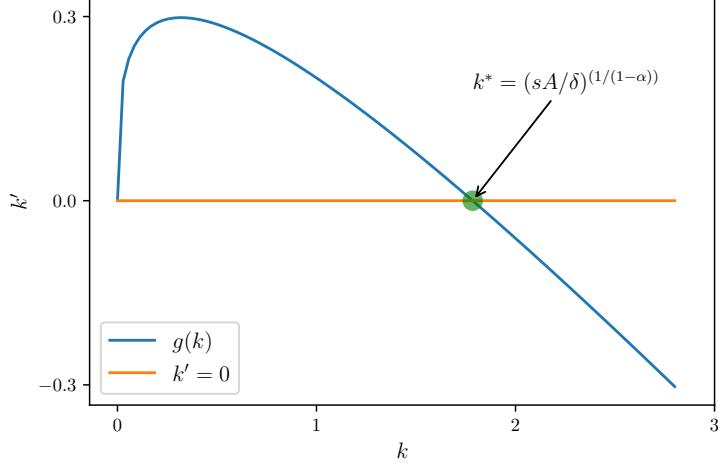


Figure 2.5: Phase diagram for $g(k) = sAk^\alpha - \delta k$

Our aim is to learn about the evolution of k_t over time, given an exogenous initial capital stock k_0 .

A **steady state** for (2.6) is value k^* at which capital is unchanging, meaning $k'(t) = 0$. Hence k^* is the k that solves $sf(k) = \delta k$. As with the discrete time case, when $f(0) = 0$ there is a trivial steady state at $k^* = 0$. Hence we restrict the state space for capital to $(0, \infty)$. Let's also restrict attention for now to the Cobb–Douglas case $f(k) = Ak^\alpha$, so k^* solves $sAk^\alpha = \delta k$. The solution is the same as the discrete time case—see (2.4).

The dynamics of the Cobb–Douglas case are represented in Figure 2.5, maintaining the parameterization we used in §2.1.3.2. Writing $k'(t) = g(k(t))$ with $g(k) = sAk^\alpha - \delta k$, values of k with $g(k) > 0$ imply that $k'(t) > 0$, so capital is increasing. When $g(k) < 0$, as happens for large values of capital stock, the opposite occurs. Once again, high marginal returns to savings at low levels of capital combined with low rates of return at high levels of capital combine to yield global stability.

This shows global stability heuristically for a fixed parameterization, but how would we show the same thing formally for a continuum of plausible parameters? In §2.1.3.3 we considered this issue for the discrete time case, finding that a neat expression for k_t was hard to obtain, and subsequently turning to a more analytical approach based around convergence of monotone trajectories.

In continuous time the process is easier: we can obtain a relatively simple expression for $k(t)$ that specifies the entire path. The first step is to let $x(t) = k(t)^{1-\alpha}$, so that $x'(t) = (1-\alpha)k(t)^{-\alpha}k'(t)$. Substituting into $k'(t) = sAk(t)^\alpha - \delta k(t)$ leads to the linear

differential equation

$$x'(t) = (1 - \alpha)(sA - \delta x(t)). \quad (2.7)$$

This equation has the exact solution

$$x(t) = \left(k_0^{1-\alpha} - \frac{sA}{\delta} \right) e^{-\delta(1-\alpha)t} + \frac{sA}{\delta}.$$

(You can confirm that this function $x(t)$ satisfies (2.7) by differentiating it.) Converting back to $k(t)$ yields

$$k(t) = \left[\left(k_0^{1-\alpha} - \frac{sA}{\delta} \right) e^{-\delta(1-\alpha)t} + \frac{sA}{\delta} \right]^{1/(1-\alpha)}.$$

Since $\delta > 0$ and $\alpha \in (0, 1)$, we see immediately that $k(t) \rightarrow k^*$ as $t \rightarrow \infty$ independent of k_0 .

Mention stochastic case, referring back to §2.1.3.4. Consider providing a brief discussion of Brownian motion and/or SDEs. Finish with a discussion of advantages and disadvantages of continuous time: Continuous time models are narrower and require discretization before numerical simulation. The stochastic case, can in some cases require layers of technical machinery that obscure key ideas. At the same time, they offer more hope of analytical solutions. They bring the power of calculus to bear and can be elegant and powerful. Ignore the evangelists who insist that one approach is better. Learn both.

2.1.5 Modeling a Pandemic

Add QE lecture and give additional references. Discretization and numerical integration. Consider simplifying parameterization. Based on [Atkeson \(2020\)](#).

2.2 Distribution Dynamics

Roadmap to be added. Emphasize the fact that we will be stepping up into high-dimensional space. Whereas our discussion of savings and capital accumulation in §2.1.3 and 2.1.4 took place in a one-dimensional state space, introducing heterogeneity will force us to confront systems that evolving in infinite-dimensional (distribution) space.

2.2.1 Wealth Dynamics

Until now we have studied settings where all agents are identical at every moment. This homogeneity assumption is convenient but limiting. For example, the dispersion of income and wealth across households cannot be studied without inserting heterogeneity. Such dispersion is of interest, partly because inequality appears to directly affect social welfare, and partly because income and wealth dispersion affects a variety of political and economic outcomes. For example, inequality can have subsidiary effects for phenomena such as productivity growth, economic fluctuations, government revenue and fiscal policy.²

Motivated by such considerations, we now introduce heterogeneity by permitting incomes and rates of return on savings to differ across households. Households' wealth levels then differ, allowing us to study the evolution of the distribution of income and wealth.

2.2.1.1 Household Wealth Dynamics

To start, we assume that a single household's wealth evolves according to

$$w_{t+1} = (1 + r_{t+1})(w_t - c_t) + y_{t+1}. \quad (2.8)$$

Here

- w_t is wealth (net asset holdings) at t ,
- c_t is current consumption,
- y_{t+1} is non-financial (labor) income received at the end of period t and
- $r_{t+1} > 0$ is the interest rate.

We are interested in how $\{w_t\}$ changes over time for this individual household, and also how the cross-section distribution of wealth evolves when wealths of a large population of households obey the dynamics in (2.8). These matters can be addressed by tracing out the implications of (2.8) via simulation or analysis. In order to do this, however, we first need to determine dynamics of the interest rate $\{r_t\}$, labor income $\{y_t\}$, and consumption $\{c_t\}$.

²For example, Glaeser et al. (2003) show how inequality alters economic and social outcomes through subversion of institutions. Further analysis of interactions between inequality and political decision making can be found in Acemoglu and Robinson (2002). For discussion of the impact of wealth dispersion on fiscal policy and economic aggregates, see, for example, Brinca et al. (2016) or Ahn et al. (2018).

Later in the book, these processes will be determined by deep primitives and equilibrium restrictions. For now, however, let us take the interest rate process $\{r_t\}$ and labor income $\{y_t\}$ as exogenously specified, supposing in particular that both are IID, independent of each other,

$$\ln(1 + r_t) \sim N(\mu_r, \sigma_r^2) \quad \text{and} \quad \ln y_t \sim N(\mu_y, \sigma_y^2).$$

We set household consumption c_t at time t to $(1 - s)w_t$, where s is a positive constant (the saving rate), thus maintaining the fixed saving rate assumption from §2.1.3. The household wealth dynamics then become

$$w_{t+1} = (1 + r_{t+1})sw_t + y_{t+1}. \quad (2.9)$$

For a baseline model, we use parameter values

$$\mu_r = 1.0, \sigma_r = 0.15, \mu_y = 1.0, \sigma_y = 0.5 \text{ and } s = 0.35.$$

These are not unreasonable values for annual frequency, at least in terms of volatility, under the assumption that household assets are mainly held in shares.³

The line labeled “baseline” in Figure 2.6 shows a time series over 200 periods generated by the model under the baseline parameters, starting from initial wealth $w_0 = 10$. The series exhibits large spikes and declines.

2.2.1.2 Sources of Volatility

It turns out that most of the volatility in the baseline series is due to volatility in $\{r_t\}$, the rate of return on assets. To see this, consider the other two lines shown in Figure 2.6. The one label “deterministic returns” shows household wealth when the sequence $\{r_t\}$ is replaced by a fixed value \bar{r} equal to the mean of r_t . The line labeled “deterministic income” is constructed in an analogous fashion, replacing r_t with y_t .

In terms of volatility, the baseline series and the series with deterministic income are almost identical. But shutting down volatility in returns rather than income has a much larger effect: the volatility of wealth falls drastically. Hence, at least for this simple model, we learn that volatility in returns on savings plays a the driving role

³In the last decade, year-on-year volatility in log labor income in the US has a standard deviation of around 0.5. See, for example, Dynan et al. (2012) and Moffitt and Zhang (2018). Regarding returns on assets, since $\ln(1+r_t) \approx r_t$ for small r_t , we can calibrate σ_r to approximate the volatility of observed returns. Volatility of returns varies greatly across asset classes, but a standard deviation of 0.15 is a reasonable estimate for volatility of yearly returns on major indices such as the Dow Jones, S&P 500 or the Nikkei.

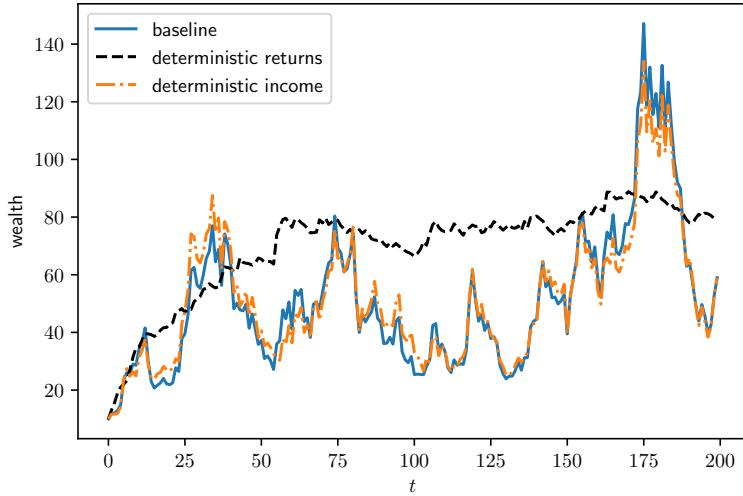


Figure 2.6: Wealth dynamics for a typical household

in wealth volatility. This turns out to be important, particularly when we turn to discussions of wealth dispersion and inequality.

Pausing momentarily, let us now ask how a more careful treatment of consumption behavior would affect these results. Might an exogenous shock that shifts up wealth be partially offset by a higher rate of consumption, thereby mitigating the upward spikes observed in Figure 2.6? To tackle such issues, we need to know how households respond to shocks. This in turn leads us to dynamic programming, which is one of the major themes of this text. Section 2.2.4 starts our discussion of dynamic programming in the context of savings and wealth.

2.2.1.3 Stability

We took some care in choosing our baseline parameters to avoid creating a model in which the wealth of a typical household diverges to $+\infty$. To see how that might occur, consider the deterministic returns version of the model, with constant returns \bar{r} , and suppose that s and \bar{r} are such that $\delta := (1 + \bar{r})s$ is greater than unity. With such a specification, we would have

$$w_t = \delta w_{t-1} + y_t \geq \delta w_{t-1} \geq \delta^2 w_{t-2} \geq \dots \geq \delta^t w_0,$$

so that $w_t \rightarrow \infty$ whenever $w_0 > 0$. Similar logic prevails in the stochastic case, as the next exercise shows.

EXERCISE 3. Let wealth obey (2.9) and let $\alpha := (1 + \mathbb{E}r_t)s$. Assume that $\mathbb{E}y_t > 0$. Show that $\alpha \geq 1$ implies $\mathbb{E}w_t \rightarrow \infty$ as $t \rightarrow \infty$. Show, conversely, that $\alpha < 1$ implies $\mathbb{E}w_t$ is bounded.

For the baseline parameterization, wealth remains bounded, since, from the formula for the mathematical expectation of a lognormal,

$$\alpha = \exp\left(\mu_r + \frac{\sigma_r^2}{2}\right)s \approx 0.96.$$

It turns out that the restriction $\alpha < 1$ that we have imposed here to avoid exponential divergence of wealth to $+\infty$ is a key stability condition that implies many useful properties for simulation and estimation. Understanding these ideas in a range of settings will be another core topic of the text.

2.2.2 Evolution of the Cross-Sectional Distribution

When economic models include heterogeneity, we can study cross-sectional distributions across agents. For example, suppose that, in model (2.9), returns on financial assets and labor income are **idiosyncratic** across households. This means that each household receives its own independent copy of the stochastic processes $\{r_t\}$ and $\{y_t\}$. Then, even if households all begin with identical assets, their wealths will soon differ. We can then study the distribution of wealth across households evolves over time. Comparing these distributions to observed cross-sectional distributions of wealth is a way to assess and improve our model.⁴

Sometimes we emphasize when shocks are idiosyncratic by writing (2.9) as

$$w_{t+1}^i = (1 + r_{t+1}^i)sw_t^i + y_{t+1}^i. \quad (2.10)$$

Here the index i refers to the i -th household. If there are n households, then the cross-sectional wealth distribution at time T is the empirical distribution of the draws $\{w_T^i\}_{i=1}^n$. Algorithm 1 shows how to generate it.

Figure 2.7 shows output from one iteration of Algorithm 1 when using the version of the model with deterministic returns on assets discussed in §2.2.1.2. Otherwise we used the baseline parameters. We set $n = 1000$, $T = 100$, and $w_0^i = 10$ for all i . In Figure 2.7, the sample $\{w_T^i\}_{i=1}^n$ is presented visually in four different ways. The first is a histogram and the other three are a kind of “smoothed histogram”.

⁴This is in the spirit of influential studies by Huggett (1993) and Aiyagari (1994), who in turn built on work by Bewley (1986).

```

Data: the initial distribution  $\{w_0^i\}_{i=1}^n$ 
1 for  $i$  in  $1, \dots, n$  do
2   for  $t$  in  $1, \dots, T$  do
3     generate independent draws of  $r_t^i$  and  $y_t^i$  ;
4     set  $w_t^i = (1 + r_t^i)sw_{t-1}^i + y_t^i$  ;
5   end
6   record the value  $w_T^i$  ;
7 end
8 return  $\{w_T^i\}_{i=1}^n$ 
```

Algorithm 1: Generating the cross-sectional distribution at time T

Before we discuss the differences between these estimates, let us consider briefly what $\{w_T^i\}_{i=1}^n$ represents. To keep things simple, suppose that all households have the same initial wealth value $w_0^i = \bar{w}$. The time- T wealth w_T^i of household i depends only on \bar{w} , the parameters in the model and the shocks $\{r_t^i, y_t^i\}$ up until time T . Since the shock processes are IID and independent across households, the collection of draws $\{w_T^i\}_{i=1}^n$ is also IID. In other words, each w_T^i is an independent draw from some fixed probability distribution, which we henceforth label ψ_T .

While ψ_T is certainly a well-defined distribution, it is also quite intractable: there is no easy way to write it down with pencil and paper. However, in the age of fast computing, it is easy to construct a large IID sample from ψ_T using Algorithm 1. Such a sample provides ample information about the underlying distribution ψ_T .

Note that the output of Algorithm 1 can be thought of as both (a) the simulated distribution of wealth in a finite cross-section of households, and (b) a collection of independent draws $\{w_T^i\}_{i=1}^n$ that we can use to estimate the underlying distribution ψ_T . It was through the lens of this second interpretation that we referred to the plots in Figure 2.7 as *estimates* of the cross-sectional wealth distribution.

2.2.2.1 Simulation and Density Estimation

The discussion above involves a foundational problem in statistics: given a collection of IID draws from a given distribution, how should we go about estimating the underlying distribution—or the density, if we think that the distribution has one? In this case, the IID draws are $\{w_T^i\}_{i=1}^n$ and the underlying distribution is ψ_T .

Figure 2.7 presents four different estimates of ψ_T from sampled data $\{w_T^i\}_{i=1}^n$. The histogram will be familiar to you but the “smoothed histograms” in the remaining subplots might not be. These are produced by feeding the sample into what is known as a

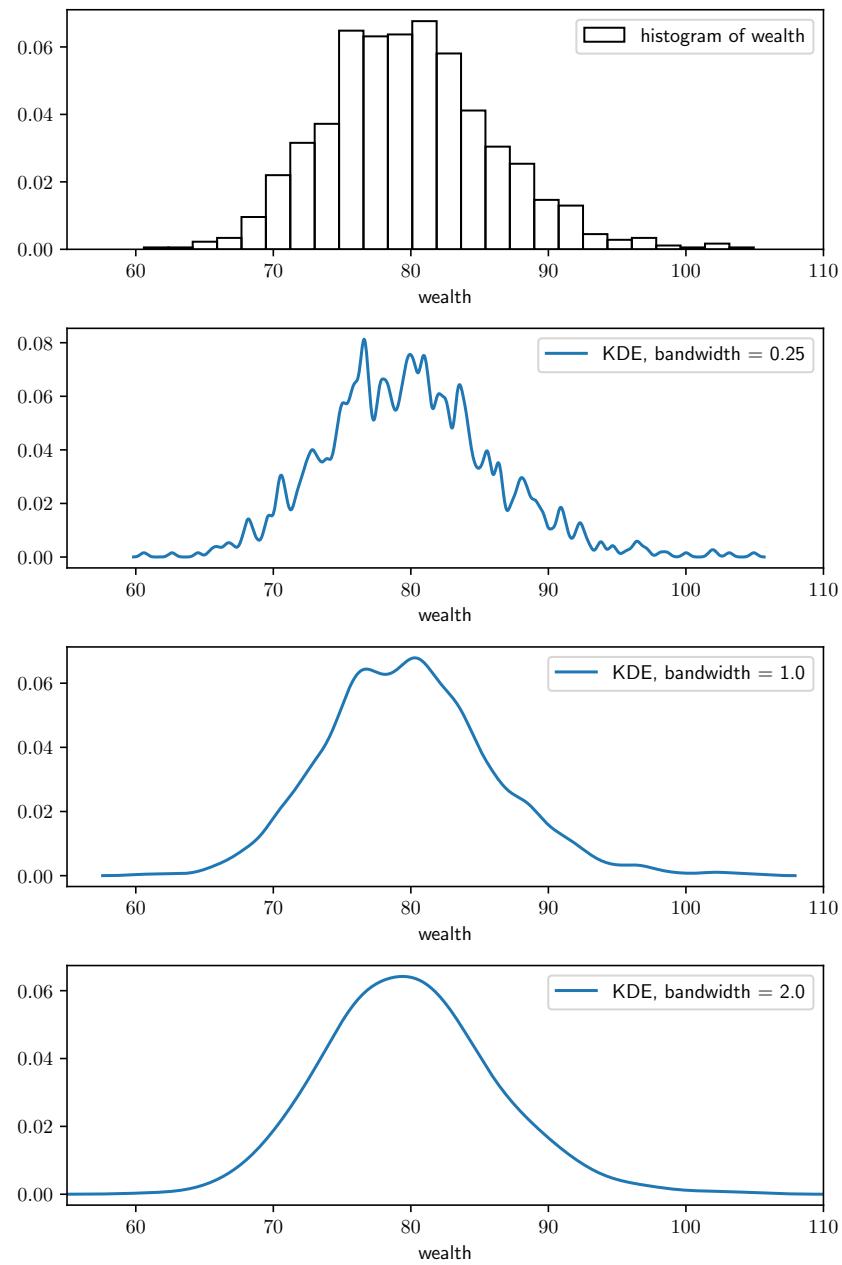


Figure 2.7: Estimates of the cross-sectional wealth distribution at $T = 100$

kernel density estimator. This estimator has a bandwidth parameter that controls the amount of smoothing imposed on the data. The figure shows how higher bandwidth increases the degree of smoothing.

None of these estimates is “better” than the other in a universal sense. Rather, they are different estimates based on different assumptions of how smooth the underlying density is. The fact that the last sub-plot is produced by imposing more smoothing means that we are implicitly assuming that the underlying density is relatively smooth (which turns out to be the case, as can be verified by increasing the sample size).

The problem of estimating distributions efficiently from simulated data is one that we will face often in this text, since most of the models we deal with are not tractable analytically. An extended discussion is given in §4.1.5.

2.2.2.2 Convergence of Cross-Sectional Distributions

In §2.2.2, we examined the cross-sectional distribution of wealth at $T = 100$. In general, we will have a different distribution of wealth at each point in time, depending on both the underlying model and the initial wealth distribution $\{w_0^i\}$. In some senses this is to be expected but in others it is problematic. For example, we prefer to have models with simple, strong predictions that can be tested against data. But here we have not just one prediction but an infinity of them—a different distribution at every point in time.

However, it turns out that, for this model, under the parameterization we have chosen, the sequence of cross-sectional wealth distributions $\{\psi_t\}$ settles down to a “stationary” wealth distribution, denoted henceforth by ψ^* . This stationary distribution is the unique long-run prediction of the model. In particular, for our parameterization, the sequence $\{\psi_t\}$ always converges to ψ^* , regardless of the initial distribution. In a sense, the stationary distribution ψ^* is the heterogeneous agent extension of the notion of steady state capital k^* presenting in §2.1.3. It is a steady state in distribution space.

The fact that $\{\psi_t\}$ always converges to ψ^* , regardless of ψ_0 , is important for computation. In particular, with this knowledge in hand, we know that we can compute an approximation to ψ^* by starting at an arbitrary initial condition and then using ψ_t for sufficiently large t to approximate ψ^* .

We formalize all of these ideas later. For now we content ourselves with a simulation that illustrates the main idea. Figure 2.8 shows estimates of ψ_t at different times t . Each one is produced using a simulated sample and a kernel density estimate. The sample size is larger, however, at 100,000 households. The distribution was produced by the version of wealth dynamics with non-stochastic returns, as in Figure 2.7. (The distributions are smoother here because the sample size we are using—the number of

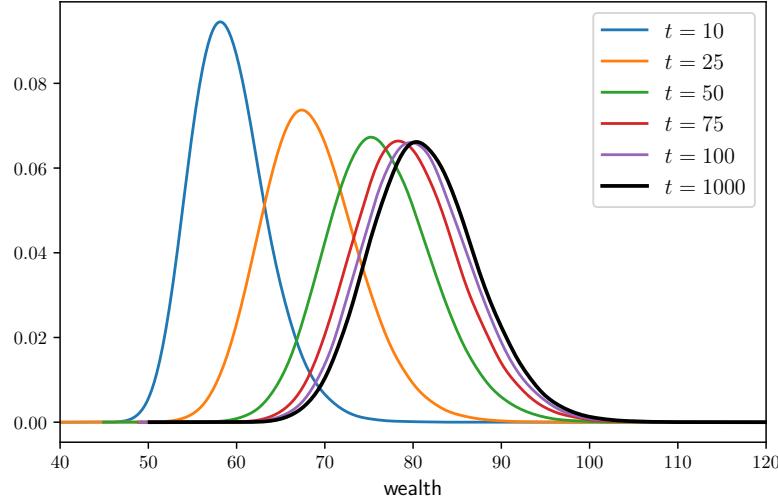


Figure 2.8: Convergence to the stationary wealth distribution

households—is much larger, providing a better estimate of the true density ψ_t .) Initial wealth in each household was set to 50.

Convergence of the sequence $\{\psi_t\}$ is evident: there is almost no difference between $t = 100$ and $t = 1000$. Moreover, even though we do not show it here, convergence toward the same limit occurs regardless of the initial wealth assigned to each household.

2.2.3 Inequality

In the previous section we used a heterogeneous agent model to generate a cross-sectional distribution of wealth from assumptions about labor income and financial returns. Now let's investigate this distribution and how it varies with model parameters. The main topic of this section will be dispersion of wealth and measures of inequality.

2.2.3.1 Dispersion and Financial Returns

Before diving into definitions, we begin with a quick comparison of outcomes for the three versions of the wealth dynamics model considered in §2.2.1.2 (baseline model, deterministic returns and deterministic income). We will examine the stationary distribution of each and compare them.

Figure 2.9 presents the stationary distributions of these three models as violin plots. Each violin plot is produced by sampling from an approximation of the stationary

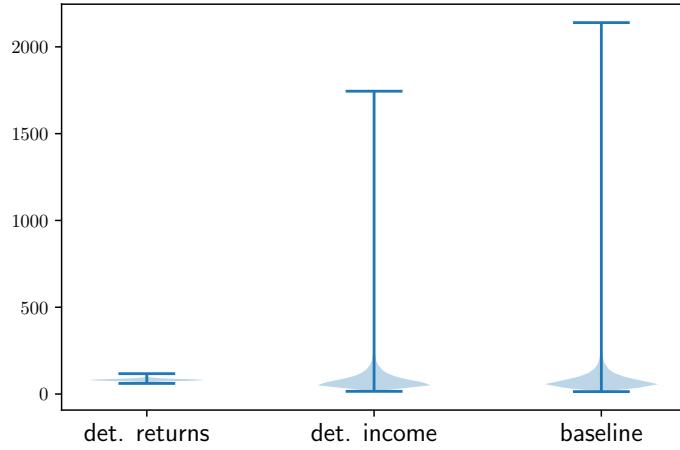


Figure 2.9: Stationary wealth distributions across model types

distribution of the model in question. We approximate stationary distributions by using the model under study to produce a sample of observations $\{w_T^i\}$ generated from Algorithm 1. In doing so, we set $T = 1000$ and simulated 10,000 households, all starting at $w_0^i = 10$.

The support of each density in the figure is represented by a vertical line with horizontal strokes at each end. These strokes show minimal and maximal observations in the sample. Each plot also shows an estimate of the density constructed using a kernel density, mirrored around the vertical line. (The density estimate is difficult to see in the deterministic case because its support is so small relative to the stochastic case.)

The main message from the data is that, while the three distributions have similar means, different sources of volatility in the dynamics produce very different levels of dispersion in the cross-section. For example, the cross-sectional distribution associated with the baseline model is far more dispersed than the model with deterministic returns. The baseline model exhibits a long upper tail.

The key reason behind these differences is that, in the baseline model (as well as the model with deterministic income), some households receive lucky draws of r_t over a number of *consecutive* periods. Those lucky draws *compound* one another, since each multiplies the wealth obtained from previous draws. We return to this important point later.

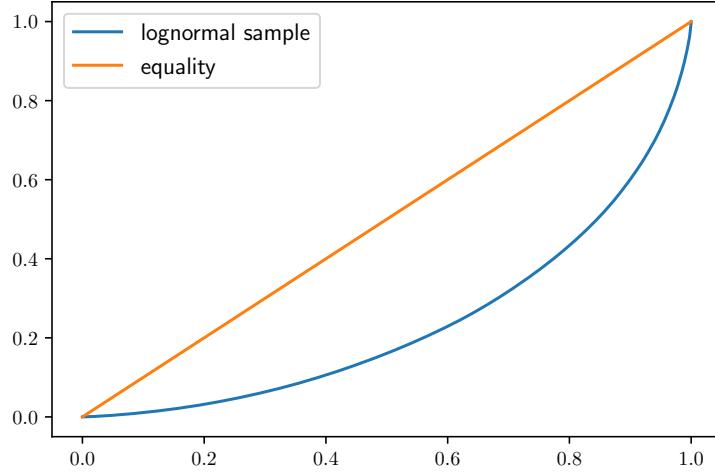


Figure 2.10: Lorenz curve for the lognormal distribution

2.2.3.2 Measures of Inequality

The different levels of dispersion in the wealth distribution seen above lead us to consider inequality. Is there a sense in which some of these distributions are more “unequal” than others?

To debate this issue on a purely scientific level, we need to be careful with our definitions. A popular function-valued measure of inequality is the **Lorenz curve**, which is, for a given sample from a population, a mapping L from $[0, 1]$ to itself such that, when the population is ranked from lowest to highest in terms of wealth, $y = L(x)$ indicates that the lowest $(100 \times x)\%$ of people have $(100 \times y)\%$ of all wealth. For an observed sample w_1, \dots, w_n , sorted from smallest to largest, the data points (x_i, y_i) for the Lorenz curve are

$$x_i = \frac{i}{n}, \quad y_i = \frac{\sum_{j \leq i} w_j}{\sum_{j \leq n} w_j}, \quad i = 1, \dots, n$$

Typically, the curve is formed from these data points using some form of interpolation.

Figure 2.10 is constructed in this way using $n = 200$ draws from the standard lognormal distribution. The straight line corresponds to perfect equality. The lognormal draws produce a less equal distribution. For example, the bottom 80% has around 40% of total wealth.

Another popular measure of income and wealth inequality—this time scalar-valued—

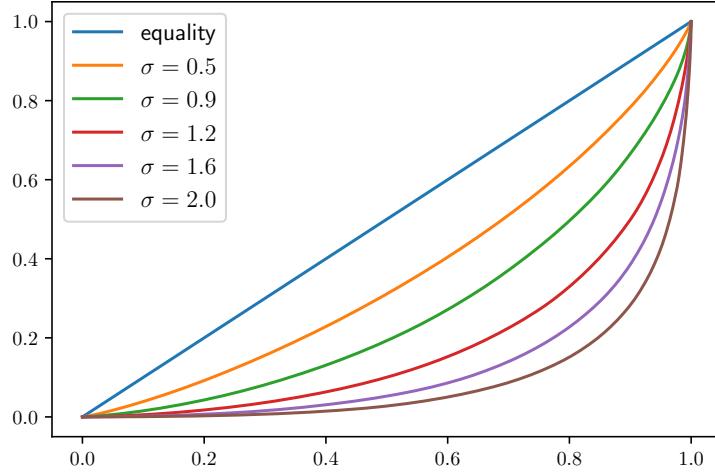


Figure 2.11: Lorenz curves for lognormal distributions, as variance increases

is the Gini coefficient, which, given the sample above, can be expressed as

$$G := \frac{\sum_{i=1}^m \sum_{j=1}^m |w_j - w_i|}{\sum_{i=1}^m w_i}. \quad (2.11)$$

The Gini coefficient is closely related to the Lorenz curve. In fact, it can be shown that its value is twice the area between the line of equality and the Lorenz curve. Thus, $G = 0$ indicates complete equality, while $G = 1$ indicates complete inequality.

To illustrate this further, we compute the Lorenz curves and Gini coefficients for the collection of lognormal distributions associated with the random variables

$$w_\sigma = \exp(\mu + \sigma Z) \quad \text{where } Z \sim N(0, 1).$$

In our calculations, σ varies between 0.2 and 4. As σ increases, so does the variance of w_σ . To focus on volatility, we adjust μ at each step to maintain the equality $\mu = -\sigma^2/2$. This implies that the mean of w_σ does not change with σ . For each σ , we generate 2,000 independent draws of w_σ and calculate the Lorenz curve and Gini coefficient. The results are shown in figure 2.11 and figure 2.12. As expected, higher variance generates more dispersion in the sample, and hence greater inequality.

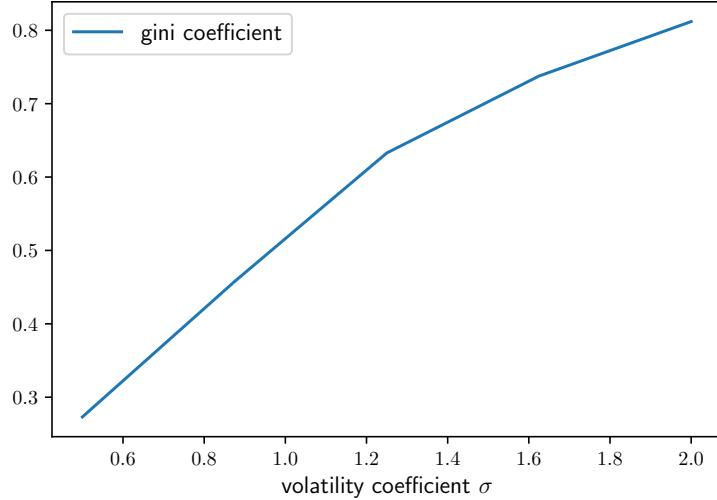


Figure 2.12: Increasing Gini coefficients as variance increases

2.2.3.3 Inequality Measures from the Data

Add observed Lorenz/Gini, for a mix of dates, in the US, Australia, Italy, Japan and other countries. Why so different over space and time?

2.2.3.4 Inequality Generated by the Model

Let's see how this compares to the stationary distributions generated by our model of wealth dynamics. Figure 2.13 shows two Lorenz curves for the wealth distribution associated with the model of wealth dynamics discussed in §2.2.1. One is for the model with deterministic returns on assets, introduced in §2.2.1.2, while the other is the baseline model. In both cases, the Lorenz curves are constructed by simulating 100,000 households forward 500 periods, starting all households from initial wealth of 1.0. 500 periods provide sufficient burn-in to generate a close approximation to the stationary distribution, so we can interpret these Lorenz curves as the prediction for steady state inequality implied by each model.

The model with risk-free returns produces a Lorenz curve with very high equality, which is a poor fit to the data. The baseline version generates greater inequality, coinciding with our intuition obtained by viewing the relative dispersion in the violin plots of Figure 2.9. However, inequality is still relatively low relative to the data. While some of this is no doubt due to factors outside the model, it is interesting to see how close we can get to the data without adding extra features.

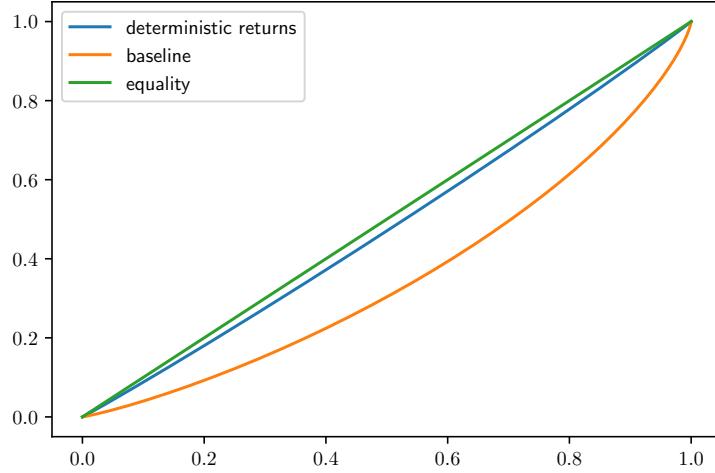


Figure 2.13: Wealth Lorenz curves from the model, risky and risk-free returns

In general, inequality rises with higher volatilities in labor income and returns. But the impact of these two parameters is not symmetric. For example, Figure 2.14 shows the impact of doubling σ_y , relative to the baseline. Despite this large increase in σ_y , the impact on the Lorenz curve is almost invisible. By comparison, the impact of doubling σ_r is far larger, as shown in Figure 2.15.

The intuition behind this result is similar to that mentioned after Figure 2.9: consecutive large draws of r_t multiply and hence compound one-another, leading to a big increase in wealth for the lucky household. This effect is reinforced when r_t has greater variance. In contrast, the affect of large realizations of y_t is additive rather than multiplicative. Thus, higher volatility in σ_r is a powerful force for generating inequality.

All of the above should be read with the following caveat in mind: We are holding the saving function fixed as we vary parameters. A better approach would be to allow the households to re-optimize as parameters change. To implement this in a model with utility maximization and to calculate households' responses to parameter changes requires dynamic programming. The theory and practice of dynamic programming is a central topic of this text.

2.2.4 Modeling Household Behavior

Add roadmap.

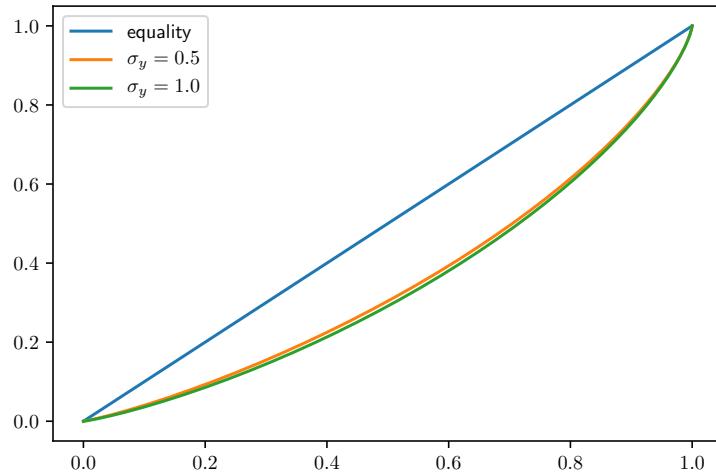


Figure 2.14: Higher inequality generated by increasing labor income volatility

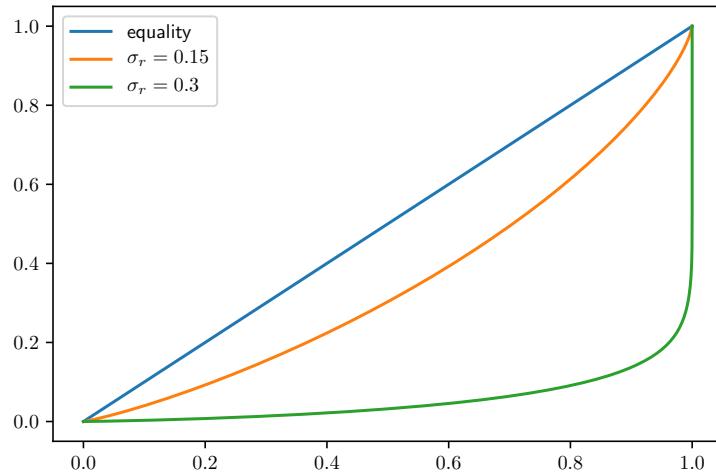


Figure 2.15: Higher inequality generated by higher returns volatility

2.2.4.1 Forward-Looking Agents

There are many ways that the simple wealth dynamics model could be improved. One would be to produce a more thoughtful model savings behavior. This is important not just for more matching of historical data, but also for policy considerations. We cannot judge how aggregate outcomes like the cross-sectional wealth distribution will respond to changes in policy unless we have some idea of how agent behavior will respond.

One way of inserting consumption behavior into (2.8) would be a statistical approach that deploys econometric or machine learning methods. While such an exercise can produce valuable insights, purely statistical approaches are inherently backward looking (unless you can time travel prior to collecting your data). This means that making statements about, say, the impact of an unprecedented and untested policy on the dynamics of the wealth distribution, will be heroic rather than sound.

One solution to this conundrum is to use agents' intertemporal choice problems to understand how they would react in settings that have not yet been observed. Let's start with an admittedly primitive model in which an agent seeks to maximize

$$\mathbb{E} \sum_{t=0}^{\infty} \beta^t u(c_t) \quad (2.12)$$

subject to (2.8) plus the nonnegativity constraints $c_t \geq 0$ and $w_t \geq 0$ for all t . (Negative wealth is not allowed in this formulation, implying that households are strongly borrowing constrained. We can accommodate negative wealth easily enough but let's put it aside for now.) Here $u(c_t)$ is the utility derived from current consumption c_t and $\beta \in (0, 1)$ is a discount factor indicating degree of impatience.

The timing for the household problem is shown in Figure 2.16, with $R_t := (1 + r_t)$. After observing w_t , the agent chooses c_t in $[0, w_t]$. Next, shocks are realized and household wealth updates. Then process now repeats.

We assume that both labor income and the interest rate are functions

$$y_t = y(z_t, \xi_t) \quad \text{and} \quad r_t = r(z_t, \zeta_t) \quad (2.13)$$

the innovations $\{\xi_t\}$ and $\{\zeta_t\}$ plus some exogenous state process $\{z_t\}$ that obeys a transition rule such as

$$z_{t+1} = az_t + b + c\eta_{t+1} \quad \text{with} \quad \{\eta_t\} \stackrel{\text{IID}}{\sim} N(0, 1) \quad (2.14)$$

The innovations are assumed to be IID over time and independent of each other and the state process $\{z_t\}$.

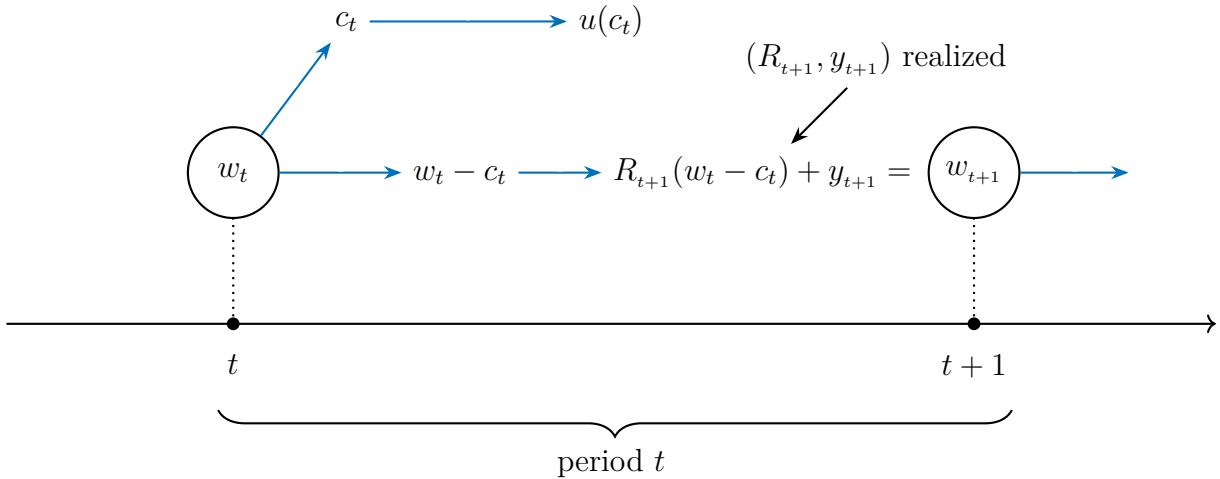


Figure 2.16: Timing for the optimal saving problem

The **value function** v^* for this problem is defined as the largest value that the objective (2.12) can take, evaluated across all of the feasible consumption paths $\{c_t\}$ that can be constructed, with initial conditions given. That is,

$$v^*(w, z) := \sup \mathbb{E} \sum_{t=0}^{\infty} \beta^t u(c_t)$$

where the supremum is over all feasible consumption paths from $(w_0, z_0) = (w, z)$. The aim of the household is to choose a feasible path for consumption that attains this supremum.

Feasible consumption paths are those that obey (2.8) on p. 15, given initial conditions (w_0, z_0) , and are **adapted** to the shocks. The last statement is an informational constraint: Consumption at c_t is not allowed to depend on information unavailable at t , such as the values $\eta_{t+1}, \eta_{t+2}, \dots$. Rather, consumption can depend only on past and current information.

In practice we take current consumption c_t to be a *function* of shocks, states, and actions observed up to and including time t . This stands to reason, since current consumption must react to past and present shocks that constrain or enhance consumption possibilities. In engineering, a mapping from the history of the state and shocks into current action is called a *closed loop control*. In economics it's called a **policy function**.

We'll show later that, for this problem, the optimal consumption policy depends only on the current state when that state is set to (w_t, z_t) . In other words, under the

optimal policy, current consumption c_t is a function of current assets and the current realization of the shock. It has no additional dependence on earlier values. Moreover, the fact that the problem has an infinite horizon and the structure is unchanging can be used to show that this optimal policy is stationary, in the sense that the mapping from current state to current consumption does not change over time. Such a policy is sometimes called a **stationary Markov policy**.

Once we have learned more about dynamic programming, we will prove that the value function satisfies a version of the **Bellman equation**, named after the mathematician Richard Bellman. In the present context, this means that, at all possible values of (w, z) ,

$$v^*(w, z) = \max_{0 \leq c \leq w} \{u(c) + \beta \mathbb{E}_z v^*(w', z')\} \quad (2.15)$$

where

$$w' := (1 + r(z', \xi'))(w - c) + y(z', \xi')$$

The symbol \mathbb{E}_z in (2.15) indicates a mathematical expectation over the random elements $r(z', \xi')$ and $y(z', \xi')$ conditional on observing $z_t = z$. Primes can be understood as representing next period values.

The Bellman equation tells us that to make the best current choice of consumption given current state $(w_t, z_t) = (w, z)$, one should optimally trade off current utility of consumption $u(c)$ against the expected *value* of resulting next period assets, appropriately discounted. When we perform this trade-off optimally we attain maximal value from the current state, which is why the left hand side of (2.15) is equal to $v^*(w, z)$.

How does (2.15) actually help us? The answer is that, like any equation that describes an unknown object (in this case v^*), it provides a restriction that can help us pin it down. In mathematics, (2.15) is called a **functional equation** because the unknown object is a function. The branch of mathematics typically deployed to solve functional equations is called fixed point theory. We begin studying fixed point results suitable for dynamic programming in [add ref](#).

This fixed point theory will provide algorithms for calculating v^* via the Bellman equation (2.15). Once we have v^* in hand we can compute the optimal policy—the best choice of consumption in any given state (w, z) —by solving the maximization problem in (2.15). Crucially, this optimization problem is only *one dimensional* at each state pair (w, z) , whereas the original problem of choosing an infinite horizon consumption path was infinite dimensional.

After we have the optimal consumption policy c^* , we can plug it into the constraint (2.8) to obtain the dynamics

$$w_{t+1} = (1 + r(z_{t+1}))(w_t - c^*(w_t, z_t)) + y(z_{t+1}) \quad (2.16)$$

Equation (2.16) determines a law of motion for wealth that we can proceed to analyze, similar to our study of the fixed savings rate case in §2.2.1.

The foregoing gives a taste of dynamic programming. We will address this topic in stages, starting with examples to build familiarity, and then working up to sophisticated theory that clarifies all the statements above in an abstract setting.

2.2.4.2 General Equilibrium Analysis

Later in the text we will shift to a general equilibrium setting, where processes like the interest rate are determined from equilibrium constraints rather than imposed exogenously. This leads to complex joint relationships that need to be untangled. For example, if households optimize as in §2.2.4.1, then the dynamics of $\{r_t\}$ will typically affect their saving and consumption. This will change the time path of wealth via (2.16). But the interest rate depends on aggregate investment, which could be equated to aggregate wealth in a closed economy. As aggregate wealth depends on household wealth, we see that the interest rate affects consumption and consumption affects the interest rate.

A way to solve this kind of problem is to work sequentially with parts of a problem, rather than trying to do everything at once. In the context of this particular problem, a common iterative process is described in algorithm 2. When convergence occurs, the prices returned clear markets and are consistent with household behavior.

```

1 fix an initial price vector  $p$  (containing interest rate, wages, etc.) ;
2 set  $\varepsilon = 1$  (a variable to track error) ;
3 while  $\varepsilon > \text{some small level of tolerance}$  do
4   deduce household behavior (e.g., consumption and saving) given  $p$  ;
5   calculate aggregate quantities given household behavior ;
6   calculate market clearing price vector  $p'$  associated with these quantities ;
7   set  $\varepsilon = \|p - p'\|$  ;
8   set  $p = p'$  ;
9 end
10 return  $p$  and associated quantities

```

Algorithm 2: Obtaining equilibrium prices and quantities

Among the objectives of this text are to work through the steps in algorithm 2 in detail and to understand their implementation so that we can calculate equilibria efficiently and accurately.

2.3 Linear Algebra

Roadmap to be added.

2.3.1 Euclidean Space

Roadmap to be added. Unify \mathbb{R}^d vs \mathbb{R}^n .

2.3.1.1 Vectors and Norms

Let's recall a few elementary facts about \mathbb{R}^d , where d is some natural number. As usual, sums and scalar products are defined pointwise, so that, for example,

$$x + y = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_d \end{pmatrix} + \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_d \end{pmatrix} := \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \\ \vdots \\ x_d + y_d \end{pmatrix} \quad \text{and} \quad \alpha x := \begin{pmatrix} \alpha x_1 \\ \alpha x_2 \\ \vdots \\ \alpha x_d \end{pmatrix}.$$

when $\alpha \in \mathbb{R}$. The **inner product** of the two vectors x and y above is defined as

$$\langle x, y \rangle := \sum_{i=1}^d x_i y_i.$$

The vectors $e_1, \dots, e_d \subset \mathbb{R}^d$ defined by

$$e_1 := \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad e_2 := \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \dots, e_d := \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$$

are called the **canonical basis vectors** of \mathbb{R}^d . This is because any vector x in \mathbb{R}^d can be written as a **linear combination** of the canonical basis vectors. Specifically, $x = \sum_{i=1}^d \langle x, e_i \rangle e_i$.

The **Euclidean norm** of $x \in \mathbb{R}^d$ is defined by $\|x\| := \sqrt{\langle x, x \rangle}$ and interpreted as the length of the vector x . For any $\alpha \in \mathbb{R}$ and any $x, y \in \mathbb{R}^d$, we have

$$\|x\| \geq 0$$

$$\|\alpha x\| = |\alpha| \|x\| \text{ and}$$

$$\|x\| = 0 \iff x = 0$$

$$\|x + y\| \leq \|x\| + \|y\|$$

(nonnegativity)	(positive homogeneity)
(positive definiteness)	(triangle inequality)

Another useful property is the **Cauchy–Schwarz inequality**

$$|\langle x, y \rangle| \leq \|x\| \cdot \|y\| \quad \text{for all } x, y \in \mathbb{R}^d.$$

The concept of convergence of sequences extends to \mathbb{R}^d via the norm. In particular, a sequence $\{x_n\}$ in \mathbb{R}^d is said to **converge** to x in \mathbb{R}^d if $\|x_n - x\|$ converges to zero in \mathbb{R} . As before, we write $x_n \rightarrow x$. One can show that this property holds if and only if each component x_n^i of the vector x_n converges in a scalar sense to the corresponding component x^i in x .

Convergence in norm is preserved under addition and scalar multiplication: if $x_n \rightarrow x$ and $y_n \rightarrow y$ in \mathbb{R}^d , while $\alpha_n \rightarrow \alpha$ in \mathbb{R} , then $x_n + y_n \rightarrow x + y$ and $\alpha_n x_n \rightarrow \alpha x$.

A subset E of \mathbb{R}^d is called **bounded** if there exists a $K < \infty$ such that $\|x\| \leq K$ for every $x \in E$. A function (and hence a sequence) taking values in \mathbb{R}^d is called bounded if its range is a bounded set.

EXERCISE 4. Show that every convergent sequence in \mathbb{R}^d is bounded.

2.3.1.2 Linear Functions and Matrices

A function $u \mapsto Au$ from \mathbb{R}^k to \mathbb{R}^n is called **linear** if

$$A(\alpha u + \beta v) = \alpha Au + \beta Av \quad \text{for all } \alpha, \beta \in \mathbb{R} \text{ and all } u, v \in \mathbb{R}^k.$$

For example, a matrix $A \in \mathcal{M}_{n \times k}$ is a linear function from \mathbb{R}^k to \mathbb{R}^n when identified with the mapping that sends a (column) vector u in \mathbb{R}^k into the vector Au in \mathbb{R}^n . In fact it is well known (see, e.g., Kreyszig (1978), §2.9) that, for *every* linear map $A: \mathbb{R}^k \rightarrow \mathbb{R}^n$, there exists a unique $M_A \in \mathcal{M}_{n \times k}$ such that

$$Au = M_A u \quad \text{for all } u \in \mathbb{R}^k. \tag{2.17}$$

Thus, the set of linear maps and the set of matrices are in one-to-one correspondence in the finite dimensional setting.

2.3.1.3 Solving Linear Systems

Let A and B be in $\mathcal{M}_{n \times n}$ and suppose that $AB = BA = I$. Then B is called the **inverse** of A , and written A^{-1} , while A is called **invertible** or **nonsingular**.

Consider the linear system $Ax = b$, where $A \in \mathcal{M}_{n \times n}$ and $b \in \mathbb{R}^n$. We seek a solution $x \in \mathbb{R}^n$. All of the following are known to be equivalent:

- (i) For each $b \in \mathbb{R}^n$, the equation $Ax = b$ has a unique solution.
- (ii) The columns of A are linearly independent.
- (iii) The linear span of the columns of A is all of \mathbb{R}^n .
- (iv) The rank of A , denoted by $\text{rank}(A)$, equals n .
- (v) The determinant of A , denoted by $\det(A)$, is nonzero.
- (vi) A is invertible.

(The definition of rank and determinant can be found in any text on linear algebra.)

2.3.1.4 Convexity

A subset C of \mathbb{R}^d is called **convex** if, for any u, v in C and any λ in $[0, 1]$, we have $\lambda u + (1 - \lambda)v \in C$. A function g from a convex subset C of \mathbb{R}^d to \mathbb{R}^k is called **convex** if

$$g(\lambda u + (1 - \lambda)v) \leq \lambda g(u) + (1 - \lambda)g(v) \quad \text{whenever } u, v \in C \text{ and } 0 \leq \lambda \leq 1,$$

and **concave** if

$$g(\lambda u + (1 - \lambda)v) \geq \lambda g(u) + (1 - \lambda)g(v) \quad \text{whenever } u, v \in C \text{ and } 0 \leq \lambda \leq 1.$$

The function g is called, respectively, **strictly convex** or **strictly concave** if, in addition, the inequalities become strict whenever $0 < \lambda < 1$.

EXERCISE 5. Let $\{g_\alpha\}_{\alpha \in \Lambda}$ is a collection of real-valued functions defined on a convex subset C of \mathbb{R}^d . Show that,

- (i) if g_α is convex and bounded above by a constant M for every α in Λ , then g defined by $g(x) = \sup_{\alpha \in \Lambda} g_\alpha(x)$ is also convex, and
- (ii) if g_α is concave and bounded below by a constant M for every α in Λ , then g defined by $g(x) = \inf_{\alpha \in \Lambda} g_\alpha(x)$ is also concave.

Important examples of concave and convex scalar-valued functions include certain kinds of quadratic forms. For the following you should recall that a symmetric $n \times n$ matrix A is called

- **positive semidefinite** if $x'Ax \geq 0$ for any x in \mathbb{R}^n
- **positive definite** if $x'Ax > 0$ for any nonzero x in \mathbb{R}^n
- **negative semidefinite** if $x'Ax \leq 0$ for any x in \mathbb{R}^n
- **negative definite** if $x'Ax < 0$ for any nonzero x in \mathbb{R}^n

It's important to remember (and easy to forget) that symmetry is part of the definition of these properties. Since definiteness is a familiar concept in economics, you will probably remember that A is positive definite (resp., semidefinite) if and only if all its eigenvalues are strictly positive (resp., nonnegative). Analogous statements hold for the negative case. We also have the following fundamental links between convexity, concavity and definiteness: The **quadratic form** $q(x) = x'Ax$ is

- convex if and only if A is positive semidefinite,
- strictly convex if and only if A is positive definite,
- concave if and only if A is negative semidefinite, and
- strictly concave if and only if A is negative definite.

See, for example, [give ref.](#)

2.3.2 Matrix Space

[Add roadmap.](#)

2.3.2.1 Matrix Norms

Consider the vector space \mathbb{R}^n . What structure does it have? We can add and scalar multiply vectors. We can also take the norm $\|x\|$ of any element x , with the norm satisfying certain properties listed above.

Now consider the set of matrices $\mathcal{M}_{n \times k}$. This is *also* a vector space, since we can add any two elements and we can scalar multiply. Moreover, we can impose a norm in order to discuss distance, continuity and convergence. The most commonly used is the **spectral norm**, which, for $A \in \mathcal{M}_{n \times k}$ is defined as

$$\|A\| := \sup\{\|Au\| : u \in \mathbb{R}^k \text{ and } \|u\| = 1\}. \quad (2.18)$$

The spectral norm has the following properties, similar to the norm on \mathbb{R}^n :

- (i) $\|A\| \geq 0$ and $\|A\| = 0 \iff A = 0$,

- (ii) $\|\alpha A\| = |\alpha| \|A\|$ for any scalar α and
- (iii) $\|A + B\| \leq \|A\| + \|B\|$.

EXERCISE 6. Show that $\|A\|$ equals the supremum of $\|Au\|/\|u\|$ over all $u \neq 0$.

It is immediate from the definition of the operator norm that

$$\|Au\| \leq \|A\| \cdot \|u\| \quad \forall u \in U. \quad (2.19)$$

This **submultiplicative property** also extends to products: if A and B are elements of $\mathcal{M}_{n \times k}$ and $\mathcal{M}_{n \times j}$ respectively, then

$$\|AB\| \leq \|A\| \cdot \|B\|. \quad (2.20)$$

To see this, fix $v \in V$. Using (2.19), we have $\|ABv\| \leq \|A\| \cdot \|Bv\| \leq \|A\| \cdot \|B\| \cdot \|v\|$, from which (2.20) easily follows. One implication of (2.20) is that $\|A^i\| \leq \|A\|^i$ for any $i \in \mathbb{N}$ and $A \in \mathcal{M}_{n \times n}$, where A^i is the i -th composition of A with itself.

Now we have a norm on $\mathcal{M}_{n \times k}$, we can introduce convergence. We say that $\{A_i\} \subset \mathcal{M}_{n \times k}$ **converges** to A in $\mathcal{M}_{n \times k}$ if $\|A_n - A\| \rightarrow 0$. We say that $\sum_{i=1}^{\infty} A_i = A$ if the partial sum $\sum_{i=1}^I A_i$ converges to A .

Later we will discuss finite dimensional spaces again (see §11.1.4). At that time we note that all such spaces with metric induced by a norm are, in some senses, equivalent. So the next result is no surprise.

Theorem 2.3.1. $\mathcal{M}_{n \times k}$ is complete under the metric induced by the spectral norm.

2.3.2.2 Geometric Series

The **spectral radius** of A is defined as

$$r(A) = \lim_{k \rightarrow \infty} \|A^k\|^{1/k}.$$

This definition of the spectral radius is sometimes called **Gelfand's formula**. We discuss an equivalent definition below.

EXERCISE 7. Show that $r(A) < 1 \implies \|A^k\| \rightarrow 0$ as $k \rightarrow \infty$.

We can now state the following key result that extends the familiar scalar identity $\sum_{i \geq 0} a^i = 1/(1 - a)$ when $|a| < 1$. In the statement, I is the $n \times n$ identity matrix.

Theorem 2.3.2 (Neumann Series Lemma). *If A is in $\mathcal{M}_{n \times n}$ and $r(A) < 1$, then $I - A$ is nonsingular and*

$$(I - A)^{-1} = \sum_{i=0}^{\infty} A^i.$$

This theorem is central to many of our arguments, so let's step through the proof.

Proof. We first observe that the sequence $B_n := \sum_{i=0}^n A^i$ is Cauchy when $r(A) < 1$. Indeed, using the spectral norm,

$$\|B_k - B_{k+n}\| \leq \left\| \sum_{i=k}^{\infty} A^i \right\| \leq \sum_{i=k}^{\infty} \|A^i\|.$$

The final term will converge to zero in k if $\sum_{i=0}^{\infty} \|A^i\|$ is finite. By the root test for convergence of series, this will be true whenever we have $\limsup_{i \rightarrow \infty} \|A^i\|^{1/i} < 1$. We know this is true by the hypothesis $r(A) < 1$ and Gelfand's formula.

Since $\mathcal{M}_{n \times n}$ is complete under the spectral norm (Theorem 2.3.1), this Cauchy property implies that the limit $\sum_{i=0}^{\infty} A^i$ exists. Moreover, $(I - A) \sum_{i=0}^{\infty} A^i = I$, since

$$\left\| (I - A) \sum_{i=0}^{\infty} A^i - I \right\| = \lim_{n \rightarrow \infty} \left\| (I - A) \sum_{i=0}^n A^i - I \right\| = \lim_{n \rightarrow \infty} \|A^{n+1}\|$$

and the right hand side converges to zero by $r(A) < 1$ and Exercise 7. □

The Neumann series lemma tells us that if $r(A) < 1$, then, for every choice of $b \in V$, the linear system of equations $x = Ax + b$ has the unique solution $x^* \in V$ given by

$$x^* = \sum_{i=0}^{\infty} A^i b.$$

2.3.2.3 Eigenvalues

Extend matrices to linear maps over \mathbb{C}^n , state vector space properties, and then introduce complex eigenvalues and eigenvectors. Discuss complex operators, show connection to $r(A)$.

Let A be an element of $\mathcal{M}_{n \times n}$. A complex scalar λ is called an **eigenvalue** of A if there exists a nonzero vector $e \in \mathbb{R}^n$ such that $Ae = \lambda e$.

The spectral radius satisfies

$$r(A) := \sup\{|\lambda| : \lambda \text{ is an eigenvalue of } A\}. \quad (2.21)$$

Prove in the diagonal case?

EXERCISE 8. Show that, for each $A \in \mathcal{M}_{n \times n}$,

- (i) $\|A\| = r(A'A)^{1/2}$,
- (ii) $r(A) \leq \|A^k\|^{1/k}$ for all $k \in \mathbb{N}$, and
- (iii) $\|A'\| = \|A\|$ and $r(A') = r(A)$.

2.3.3 Linear Systems

Add roadmap. This section needs to be rewritten slightly because we haven't yet formalized stability.

2.3.3.1 Global Stability

Consider the linear system $x_{t+1} = Ax_t + b$ from Example 5.3.2. We saw in example 5.3.8 that global stability holds in the scalar case when $|A| < 1$. This result generalizes to n dimensions as follows.

Proposition 2.3.3. *If $r(A) < 1$, then $Sx = Ax + b$ is globally stable on \mathbb{R}^n with unique steady state*

$$x^* := (I - A)^{-1}b = \sum_{i=0}^{\infty} A^i b \quad (2.22)$$

Here I is identity in $\mathcal{M}_{n \times n}$ and $r(A)$ is the spectral radius of A (see §2.3.2.2).

Proof. If $r(A) < 1$, then, by the Neumann series theorem (p. 38), the map $x \mapsto (I - A)x$ is invertible, implying that $(I - A)x = b$ has exactly one solution in \mathbb{R}^n for every choice of b . Moreover, by the same theorem, we have $(I - A)^{-1} = \sum_{i=0}^{\infty} A^i$. This proves that x^* in (2.22) is a unique steady state for (\mathbb{R}^n, S) when $r(A) < 1$.

To show global stability, fix x_0 and y_0 in \mathbb{R}^n , and observe that, by (5.26) and the definition of the matrix norm (see §2.3.1.2), we have

$$\|A^t x_0 - A^t y_0\| = \|A^t(x_0 - y_0)\| \leq \|A^t\| \cdot \|x_0 - y_0\|$$

Moreover, $\|A^t\| \rightarrow 0$ by Exercise 7 on page 37, so $\|x_t - y_t\| \rightarrow 0$ as $t \rightarrow \infty$. Taking $y_0 = x^*$, we see that $\mathcal{O}(x^*)$ is all of \mathbb{R}^n . \square

There is another view on this argument when A is **diagonalizable**, which means that we can write $A = P^{-1}DP$ for a diagonal matrix $D = \text{diag}(\lambda_1, \dots, \lambda_n)$ and square and nonsingular P . A sufficient condition for this property is that all eigenvalues are distinct.

EXERCISE 9. Show that each column of P is an eigenvector of A and each λ_i is an eigenvalue. Use induction to show that, for all $t \in \mathbb{N}$, we have

$$A^t = P^{-1} \text{diag}(\lambda_1^t, \lambda_2^t, \dots, \lambda_n^t) P. \quad (2.23)$$

The implication of this expression is that the asymptotic properties of $\{A^t\}$ are determined by the size of its eigenvalues.

Applying P to both sides of $x_{t+1} = P^{-1}DPx_t + b$ yields

$$y_{t+1} = Dy_t + c \quad \text{where } y_k := Px_k \text{ and } c := Pb. \quad (2.24)$$

Let's make two important observations. First, analyzing $\{y_t\}$ will tell us all we need to know about $\{x_t\}$, because we can easily transform from any y_t to x_t via $P^{-1}y_t = x_t$. Also, if we find that y_t converges to some point y^* , then, since this transformation is continuous, it must be that $x_t = Py_t$ converges to Py^* .

The second observation is that (2.24) is very easy to analyze because D is diagonal. In fact, we can write it as a decoupled collection of n independent equations, the i -th of which has the form

$$y_{t+1}^{(i)} = \lambda_i y_t^{(i)} + c_i. \quad (2.25)$$

Each of these independent systems evolves in the complex plain \mathbb{C} , since eigenvalues are, in general, complex.

The scalar system (2.25) is globally stable in the complex plane whenever the modulus satisfies $|\lambda_i| < 1$. Hence the vector system (2.24) is globally stable if $\max_i |\lambda_i| < 1$. As $\max_i |\lambda_i| = r(A)$, this is the same result we found in Proposition 2.3.3.

Although we have found our way back to Proposition 2.3.3, the advantage of the diagonalization approach is that we have much more detailed information about the time paths via (2.25).

2.3.3.2 Saddle Path Stability

Consider adding a linear, backward looking, saddle path stable model. Include a phase diagram for both the original system and the conjugate system under the basis.

2.3.3.3 Irreducibility

This needs to be better integrated with earlier discussion on irreducibility.

We saw in Proposition 2.3.3 that $r(A) < 1$ is sufficient for global stability of the system $x_{t+1} = Ax_t + b$. If the system is scalar, then the condition $r(A) < 1$ is the same as $|A| < 1$, and is necessary as well as sufficient (see Exercise 42). But in higher dimensions, this is not always so. It's now natural to ask whether there are important situations in economics and finance where $r(A) < 1$ is necessary as well as sufficient, so $r(A) = 1$ provides an exact dividing line between stability and instability. It turns out that the answer is affirmative. Under a condition called irreducibility, the spectral radius condition becomes necessary and, as we will see later, the irreducibility condition holds naturally in applications such as asset pricing.

Prove that $r(A) < 1$ is necessary when A is irreducible. Give geometric intuition: dominant eigenvalue takes over.

2.3.4 The Samuelson Accelerator

This needs to be rewritten and integrated. An example of a vector-valued discrete time system. The theme is iterating on matrices.

One classic example of a linear discrete time system is the **multiplier-accelerator model** of [Samuelson \(1939\)](#), where aggregate consumption obeys the Keynesian linear specification $c_t = \alpha y_{t-1} + \gamma$ and investment increases with output growth:

$$i_t = \beta(y_{t-1} - y_{t-2}).$$

Letting g be a constant level of government spending and using the accounting identity $y_t = c_t + i_t + g$ yields the second order difference equation

$$y_t = (\alpha + \beta)y_{t-1} - \beta y_{t-2} + g + \gamma. \quad (2.26)$$

Although this is a second order system (i.e., the right hand side contains two lags, as compared to the first order system introduced in (5.25)), we can map (2.26) into the first order framework as follows: Let

$$x_t := \begin{pmatrix} y_t \\ y_{t-1} \end{pmatrix}, \quad A := \begin{pmatrix} \alpha + \beta & -\beta \\ 1 & 0 \end{pmatrix}, \quad \text{and} \quad b := \begin{pmatrix} g + \gamma \\ 0 \end{pmatrix}$$

It is easy to verify that the first entry in the two dimensional system $x_{t+1} = Ax_t + b$ coincides with (2.26).

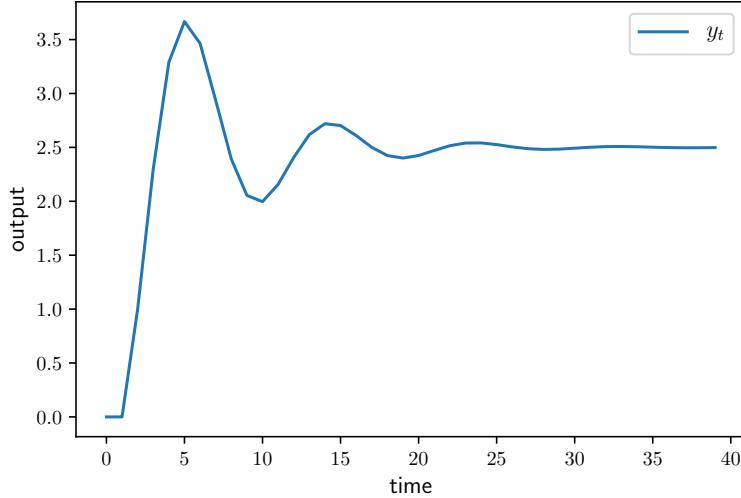


Figure 2.17: Time series of output

To analyze stability we investigate the spectral radius of A . The first step is to calculate its eigenvalues, which solve $\det(A - \lambda I) = 0$. The two solutions are the roots of the quadratic term $\lambda^2 - (\alpha + \beta)\lambda + \beta$, or

$$\lambda_i = \frac{\alpha + \beta \pm \sqrt{(\alpha + \beta)^2 - 4\beta}}{2} \quad i = 1, 2 \quad (2.27)$$

If both are interior to the unit circle, then $r(A) < 1$ and global stability holds.

Figure 2.17 shows a time path generated by the model when $Y_0 = Y_1 = 0$ and the parameters are $\alpha = 0.6$, $\beta = 0.7$ and $G = \gamma = 0.5$. In this case the spectral radius evaluates to 0.837. The figure shows that, for these parameters, global stability takes the form of damped oscillations.

2.3.5 Positivity

This section to be added. Question: when are the geometric series conditions necessary? Also, why is positivity important? Introduce irreducibility as a form of positivity, P-F thm.

2.4 Chapter Notes

References, background, historical notes.

In our discussion of wealth dynamics, it was mentioned that inequality can have subsidiary effects for phenomena such as productivity growth, economic fluctuations, government revenue and fiscal policy. For example, Glaeser et al. (2003) show how inequality alters economic and social outcomes through subversion of institutions. Further analysis of interactions between inequality and political decision making can be found in Acemoglu and Robinson (2002). For discussion of the impact of wealth dispersion on fiscal policy and economic aggregates, see, for example, Brinca et al. (2016) or Ahn et al. (2018).

Robert Gibrat (1904–1980) proposed the law of proportional effect (Gibrat (1931)) and argued that it accounted for many observed size distributions, including those for income, wealth, and firm size. Akhundjanov and Toda (2019) provide a fresh look at Gibrat’s data using modern statistical techniques.

Gabaix et al. (2016) consider inequality dynamics in and out of steady state. Fernández-Villaverde et al. (2019) investigate financial frictions and their impact on the wealth distribution.

For background on modeling pandemics and their interactions with economic variables, see Atkeson (2020) and Atkeson et al. (2020). Guerrieri et al. (2020) consider Keynesian supply shocks and demand shortage in the context of COVID-19. In addition, Glover et al. (2020) study the distributional effects of controlling a pandemic. Krueger et al. (2020) consider the relative costs and benefits of government mandated social distancing and endogenous reallocation of consumption towards non-social goods. Alvarez et al. (2020) examine the trade-off between health and output costs. Berger et al. (2020) study the effect of testing and quarantines. Baker et al. (2020) address the impact of COVID-19 on uncertainty.

Chapter 3

Discrete States

[Add roadmap.](#)

3.1 Discrete Probability

[Add roadmap.](#)

3.1.1 Distributions

Distributions describe allocation of probability mass across sets. For example, the standard normal distribution—the classic bell-shaped curve—allocates probability over \mathbb{R} , with, as is well-known, roughly 95% of total probability mass falling within the interval $(-2, 2)$. The entire set of all probability distributions over \mathbb{R} is large and requires some effort to describe in detail (see Chapter 11). For now we focus on distributions over discrete sets. Such distributions are important in applications, central to computation, and require less investment in mathematical machinery.

The discrete case begins with a countable set X over which we wish to allocate probabilities. Typical elements of X are denoted x, y , etc. The expression \sum_x means $\sum_{x \in X}$ and similarly for \sum_y and so on.

A **distribution** on X is a function φ from X to \mathbb{R}_+ such that $\sum_x \varphi(x) = 1$. For example, if $X = \{1, \dots, n\}$, then a distribution is nonnegative map φ such that $\varphi(1) + \dots + \varphi(n) = 1$. Notice that φ can also be understood as a vector $(\varphi(1), \dots, \varphi(n))$ in this case, and $\varphi(i)$ is often written as φ_i . Distributions over discrete sets are sometimes called **probability mass functions (pmfs)**.

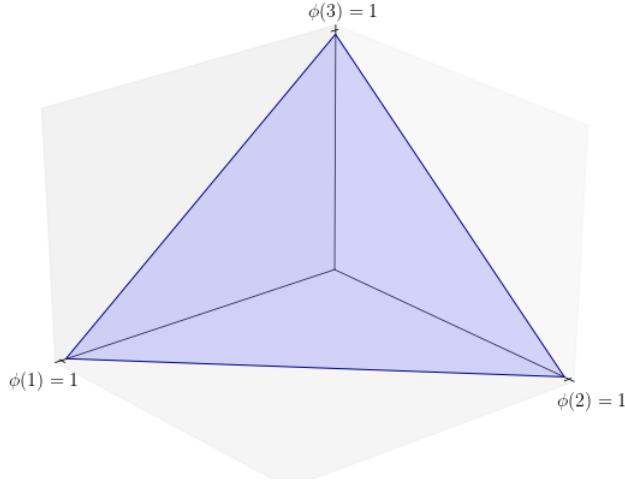


Figure 3.1: If $X = \{1, 2, 3\}$, then $\mathcal{D}(X)$ is the unit simplex in \mathbb{R}^3

The set of all distributions on X is denoted $\mathcal{D}(X)$. As just stated, when X is finite, elements of $\mathcal{D}(X)$ can be viewed either as functions or as vectors. Figure 3.1 provides a visualization when $X = \{1, 2, 3\}$. In this case each φ is identified by the point $(\varphi(1), \varphi(2), \varphi(3))$ in \mathbb{R}^3 . The set of distributions equals the unit simplex in \mathbb{R}^3 , as shown in the figure.¹

Throughout, given $x \in X$, we use the symbol δ_x to represent the element of $\mathcal{D}(X)$ that puts all mass on x . In other words, $\delta_x(y) = \mathbb{1}\{y = x\}$ for all $y \in X$. In Figure 3.1, each δ_i is a vertex of the unit simplex.

An example of a distribution over a discrete infinite set is the **Poisson distribution**, where

$$X = \{0, 1, 2, \dots\} \quad \text{and} \quad \varphi(x) = \frac{\lambda^x e^{-\lambda}}{x!}.$$

A **random variable** X on X is a map from some underlying sample space Ω to X . Letting \mathbb{P} be a probability on Ω , we say that X **has distribution** φ if $\varphi(x) = \mathbb{P}\{X = x\}$ for all x . Such a φ is necessarily in $\mathcal{D}(X)$. The intuitive meaning is that X is randomly assigned to some point in X , and this point is x with probability $\varphi(x)$. In this setting, we also write $X \sim \varphi$.

[A quick discussion of expectations in the discrete case, mainly to fix notation.](#)

¹The **unit simplex** in \mathbb{R}^n is the set of all n -vectors that are nonnegative and sum to one.

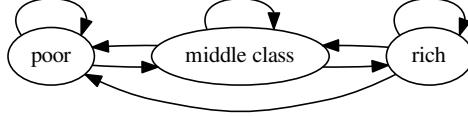


Figure 3.2: A digraph of classes

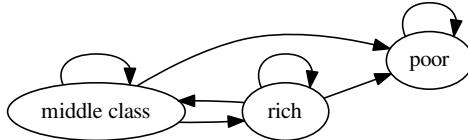


Figure 3.3: An alternative arc list

3.1.2 Finite Markov Chains

Markov chains are a highly practical class of stochastic processes, routinely employed in quantitative modeling. For now we restrict ourselves to discrete time Markov chains taking values in a *finite* nonempty set X called the **state space**. Many workhorse models in economics and finance have finite state spaces.

[Complete roadmap.](#)

3.1.2.1 Graphs

We wish to think about states and transitions between them. One way to represent states and transitions is via graphs. In general, a **directed graph** or **digraph** is a set of **nodes**, which we identify with our state space X , and a collection of **arcs**, which are ordered pairs (x, y) in $X \times X$. Typically, an arc (x, y) is visualized as an arrow from x to y . Examples are given in figures 3.2–3.3, where arcs can be thought of as representing positive possibility of transition over a given unit of time. These figures share the same nodes but have a different set of arcs.

A node y is called **accessible** from another node x if $y = x$ or there exists a sequence of arcs leading from x to y . The graph is called **strongly connected** if y is accessible from x for all $x, y \in X$. This corresponds to the idea that any state can

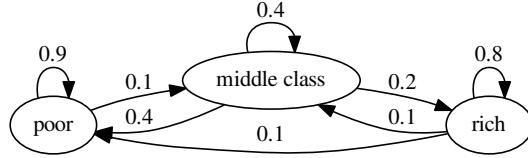


Figure 3.4: A weighted digraph

be reached from any other. Latter we will see how accessibility and connectedness are related to stability of Markov chains.

Example 3.1.1. In figure 3.2, the graph is strongly connected. In contrast, in figure 3.3, rich is not accessible from poor and an absolute poverty trap exists. The graph is not strongly connected.

We can attach numbers to the edges of a digraph, thereby capturing information about transition probabilities. The resulting object is called a **weighted digraph**. See, for example, figure 3.4. Here you can think of the numbers on the arcs as transition probabilities for a household over, say, one year. A rich household has a 10% chance of becoming poor.

3.1.2.2 Kernels and Matrices

To collect all of the transition probabilities associated with a weighted graph, we use a **Markov kernel** on X , which is a function $P: \mathsf{X} \times \mathsf{X} \rightarrow \mathbb{R}_+$ satisfying

$$\sum_y P(x, y) = 1 \text{ for all } x \in \mathsf{X}.$$

The value $P(x, y)$ represents the probability of moving from x to y in one step. We can think of $P(x, \cdot)$ as an element of $\mathcal{D}(\mathsf{X})$ for all $x \in \mathsf{X}$. When in state x , we update to a new state via the distribution $P(x, \cdot)$.

If X has elements x_1, \dots, x_n , we can view P as the matrix

$$P = \begin{pmatrix} P(x_1, x_1) & \cdots & P(x_1, x_n) \\ \vdots & & \vdots \\ P(x_n, x_1) & \cdots & P(x_n, x_n) \end{pmatrix}. \quad (3.1)$$

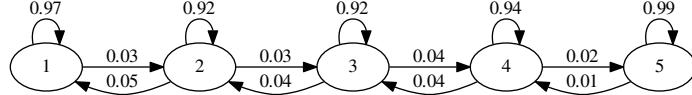


Figure 3.5: Quah's income dynamics as a digraph

Here the distributions are rows, stacked vertically. The resulting matrix is square and nonnegative, with rows that sum to one. Such a matrix is called a **Markov matrix**.

In the current setting, where \mathbf{X} is restricted to be finite, there is a one-to-one correspondence between Markov matrices and Markov kernels, and we freely use the same symbol P for both objects. Later, when we allow \mathbf{X} to become infinite, we will lose the connection to matrices but continue to work with kernels.

The Markov matrix associated with the weighted digraph in Figure 3.4 is

$$P_a := \begin{pmatrix} 0.9 & 0.1 & 0.0 \\ 0.4 & 0.4 & 0.2 \\ 0.1 & 0.1 & 0.8 \end{pmatrix} \quad (3.2)$$

Here `poor`, `middle` and `rich` are understood as states 1, 2 and 3 respectively.

As a more concrete example, consider the Markov model estimated in the international growth dynamics study of [Quah \(1993\)](#). The state is real GDP per capita in a given country relative to the world average. Quah discretizes the possible values to $0-1/4$, $1/4-1/2$, $1/2-1$, $1-2$ and $2-\infty$, calling these states 1 to 5 respectively. The transitions are over a one year period. Estimated one step transition probabilities are represented as a weighted digraph in Figure 3.5. We can also represent them as the Markov matrix

$$P_Q = \begin{pmatrix} 0.97 & 0.03 & 0.00 & 0.00 & 0.00 \\ 0.05 & 0.92 & 0.03 & 0.00 & 0.00 \\ 0.00 & 0.04 & 0.92 & 0.04 & 0.00 \\ 0.00 & 0.00 & 0.04 & 0.94 & 0.02 \\ 0.00 & 0.00 & 0.00 & 0.01 & 0.99 \end{pmatrix} \quad (3.3)$$

The most striking feature of this Markov matrix is strong persistence. Large numbers on the principal diagonal indicate that the state stays constant from period to period with high probability.

As another example, [Benhabib et al. \(2015a\)](#) estimate the following Markov matrix for intergenerational social mobility:

$$P_B := \begin{pmatrix} 0.222 & 0.222 & 0.215 & 0.187 & 0.081 & 0.038 & 0.029 & 0.006 \\ 0.221 & 0.22 & 0.215 & 0.188 & 0.082 & 0.039 & 0.029 & 0.006 \\ 0.207 & 0.209 & 0.21 & 0.194 & 0.09 & 0.046 & 0.036 & 0.008 \\ 0.198 & 0.201 & 0.207 & 0.198 & 0.095 & 0.052 & 0.04 & 0.009 \\ 0.175 & 0.178 & 0.197 & 0.207 & 0.11 & 0.067 & 0.054 & 0.012 \\ 0.182 & 0.184 & 0.2 & 0.205 & 0.106 & 0.062 & 0.05 & 0.011 \\ 0.123 & 0.125 & 0.166 & 0.216 & 0.141 & 0.114 & 0.094 & 0.021 \\ 0.084 & 0.084 & 0.142 & 0.228 & 0.17 & 0.143 & 0.121 & 0.028 \end{pmatrix} \quad (3.4)$$

Here the states are percentiles of the wealth distribution. In particular, with the states represented by $1, 2, \dots, 8$, the corresponding percentiles are

$0\text{--}20\%$, $20\text{--}40\%$, $40\text{--}60\%$, $60\text{--}80\%$, $80\text{--}90\%$, $90\text{--}95\%$, $95\text{--}99\%$, $99\text{--}100\%$

Transition probabilities are estimated from US 2007–2009 Survey of Consumer Finances data and, relative to the highly persistent matrix P_Q , show considerable mobility.

EXERCISE 1. Show that, if P is a Markov matrix, then the spectral radius of P equals 1. (See (2.21) on page 39 for the definition of the spectral radius.)

3.1.2.3 Markov Chains

Let $\{X_t\}_{t \geq 0}$ be a collection of random variables taking values in \mathbb{X} . We say that $\{X_t\}$ is a **Markov chain** on \mathbb{X} if there exists a Markov kernel P on \mathbb{X} such that

$$\mathbb{P}\{X_{t+1} = y \mid X_0, X_1, \dots, X_t\} = P(X_t, y) \quad \text{for all } t \geq 0, y \in \mathbb{X}. \quad (3.5)$$

In this case we say that $\{X_t\}$ is **generated by** P . We call either X_0 or its distribution ψ_0 the **initial condition**, depending on context.

We call any Markov chain $\{X_t\}$ generated by P and having initial condition ψ_0 a **(ψ_0, P) -chain**. If $\psi = \delta_x$, the point mass concentrated at x , then we will say that $\{X_t\}$ is a **(x, P) -chain**.

One way to think about Markov chains is algorithmically: Let P be a Markov kernel and let ψ_0 be an element of $\mathcal{D}(\mathbb{X})$. Now generate $\{X_t\}$ via algorithm 3. The resulting sequence is a (ψ_0, P) -chain.

We can translate this algorithm into a stochastic difference equation for $\{X_t\}$ in the finite state setting $\mathbb{X} = \{x_1, \dots, x_n\}$. Given $x \in \mathbb{X}$ and $u \in (0, 1)$, set

$$F(x, u) := \sum_{i=1}^n y_i \mathbb{1}\{q_{i-1}(x) < u \leq q_i(x)\}$$

```

1 set  $t = 0$  and draw  $X_t$  from  $\psi_0$  ;
2 while  $t < \infty$  do
3   | draw  $X_{t+1}$  from the distribution  $P(X_t, \cdot)$  ;
4   | let  $t = t + 1$  ;
5 end

```

Algorithm 3: Generation of a (ψ_0, P) -chain

where $\{y_1, \dots, y_n\} = \mathbb{X}$ and

$$q_i(x) := \sum_{j=1}^i P(x, y_j) \quad \text{with } q_0 = 0.$$

If, with $U(a, b)$ meaning the uniform distribution on (a, b) , we now take

$$X_{t+1} = F(X_t, U_{t+1}) \quad \text{where } \{U_t\} \stackrel{\text{IID}}{\sim} U(0, 1) \quad (3.6)$$

and X_0 is an independently drawn random variable with distribution ψ_0 on \mathbb{X} , then $\{X_t\}$ is a (ψ_0, P) -chain on \mathbb{X} , as Exercise 2 asks you to show.

EXERCISE 2. Conditional on $X_t = x$, show that, for each i in $1, \dots, n$,

- (i) $X_{t+1} = y_i$ if and only if U_{t+1} lies in the interval $(q_{i-1}(x), q_i(x)]$.
- (ii) This event has probability $P(x, y_i)$.

Conclude that X_{t+1} in (3.6) is a draw from $P(x, \cdot)$.

Figure 3.6 shows two such simulations, both generated using estimated wealth percentile dynamics as given in (3.4). One path starts from $X_0 = 1$ while the other starts from the top state $X_0 = 8$. Notice that the paths rapidly “mix,” in the sense that the difference in initial states has little impact on outcomes after an initial “burn in” period. We’ll talk more about mixing and its connection to stability below.

3.1.2.4 Higher Order Kernels

Given a Markov kernel P on \mathbb{X} , define

$$P^1 := P \quad \text{and} \quad P^{k+1}(x, y) := \sum_z P(x, z)P^k(z, y). \quad (3.7)$$

In terms of matrices, P^k is just the k -th power of the matrix P .

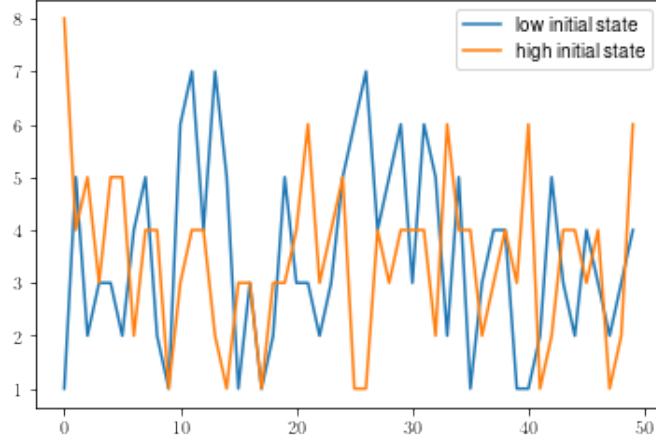


Figure 3.6: Wealth percentile over time

EXERCISE 3. Using induction, prove that P^k is a Markov kernel on X for all $k \in \mathbb{N}$.

In this context, P^k is called the **k -step Markov kernel** corresponding to P . It is important for the following reason: If $\{X_t\}$ is a Markov chain generated by P , then, for any $t, k \in \mathbb{N}$ and $x, y \in \mathsf{X}$, we have

$$P^k(x, y) = \mathbb{P}\{X_{t+k} = y \mid X_t = x\}. \quad (3.8)$$

Equation 3.8 states that P^k provides the k -step transition probabilities, from any starting date t . This claim can be proved by induction. Fix $t \in \mathbb{N}$ and $x, y \in \mathsf{X}$. The claim is true by definition when $k = 1$. Suppose the claim is also true at k and now consider the case $k + 1$. By the law of total probability, we have

$$\mathbb{P}\{X_{t+k+1} = y \mid X_t = x\} = \sum_z \mathbb{P}\{X_{t+k+1} = y \mid X_{t+k} = z\} \mathbb{P}\{X_{t+k} = z \mid X_t = x\}.$$

The induction hypothesis allows us to use (3.8), so the last equation becomes

$$\mathbb{P}\{X_{t+k+1} = y \mid X_t = x\} = \sum_z P(z, y) P^k(x, z) = P^{k+1}(x, y).$$

This completes our proof by induction.

A useful identity for the higher order kernels is

$$P^{j+k}(x, y) = \sum_z P^k(x, z) P^j(z, y) \quad ((x, y) \in \mathsf{X} \times \mathsf{X}) \quad (3.9)$$

which holds for any j, k in \mathbb{N} . This is called the **Chapman–Kolmogorov equation**. To see why it holds, let $X_0 = x$ and let $y \in \mathsf{X}$ be given. Using the law of total probability again, we have

$$\mathbb{P}\{X_{j+k} = y \mid X_0 = x\} = \sum_z \mathbb{P}\{X_{j+k} = y \mid X_0 = x, X_k = z\} \mathbb{P}\{X_k = z \mid X_0 = x\}$$

By the definition of a Markov chain in (3.5), the future and past are independent given the present, so

$$\sum_z \mathbb{P}\{X_{j+k} = y \mid X_0 = x, X_k = z\} = \sum_z \mathbb{P}\{X_{j+k} = y \mid X_k = z\}.$$

As a result of this fact and (3.8), the equation before this one can be rewritten as (3.9), which is what we aimed to show.

3.1.3 Distribution Dynamics

Add roadmap.

3.1.3.1 Updating Marginal Distributions

Let $\{X_t\}$ be a (ψ_0, p) -chain and let ψ_t be the distribution of X_t . Sometimes ψ_t is called the **marginal distribution** in the present setting. There is a simple link between successive marginal distributions of this chain: By the law of total probability,

$$\mathbb{P}\{X_{t+1} = y\} = \sum_x \mathbb{P}\{X_{t+1} = y \mid X_t = x\} \cdot \mathbb{P}\{X_t = x\},$$

so, letting ψ_t be the distribution of X_t for each t ,

$$\psi_{t+1}(y) = \sum_x P(x, y) \psi_t(x) \quad (y \in \mathsf{X}). \quad (3.10)$$

Thus fundamental expression shows us how to update marginal distributions given the Markov kernel P .

When each ψ_t is interpreted as a row vector and P is understood as the Markov matrix (3.1), we can write (3.10) as

$$\psi_{t+1} = \psi_t P. \quad (3.11)$$

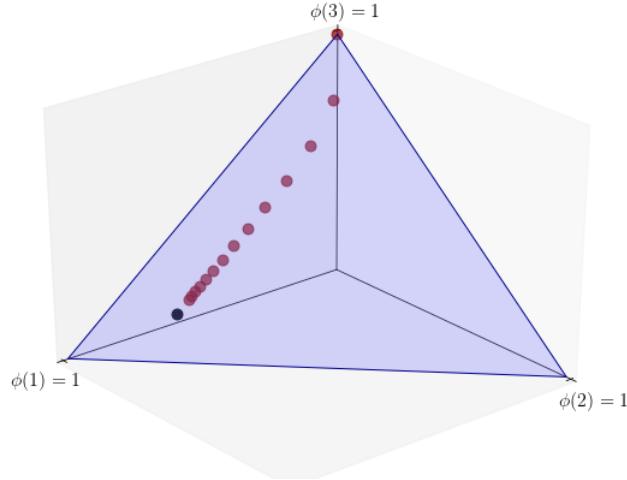


Figure 3.7: A trajectory from $\psi_0 = (0, 0, 1)$

Think of this as a difference equation in distribution space. Iterating backwards, we deduce that

$$\psi_t = \psi_{t-1}P = (\psi_{t-2}P)P = \psi_{t-2}P^2 = \cdots = \psi_0P^t.$$

This provides a formula for marginal distributions as a function of an initial condition and the Markov matrix: given any $\psi_0 \in \mathcal{D}(X)$ and $t \in \mathbb{N}$, the row vector ψ_0P^t is the distribution of X_t given $X_0 \stackrel{d}{=} \psi_0$.

To illustrate these dynamics in distribution space, Figure 3.7 shows the trajectory $\{\psi P_a^t\}$ when $X = \{1, 2, 3\}$, $\psi = (0, 0, 1)$, and P_a is the Markov matrix displayed in (3.2). Motion is towards the black dot that appears to be the limit. This black dot is a “stochastic steady state,” defined and discussed in detail below. Figure 3.8 shows dynamics that start from initial condition $\psi = (0, 1/2, 1/2)$.

The fact that both sequences in these figures converge towards the same point suggests that P_a exhibits some kind of global stability on $\mathcal{D}(X)$. Later we will confirm this using a combination of fixed point theory and probabilistic methods.

Figure 3.9 shows an another view on a distribution sequence, this time generated from Quah’s Markov matrix P_Q . The initial condition was $\psi = (0, 0, 0, 1, 0)$. Each distribution shown was calculated as ψP_Q^t , where P_Q^t is the t -th power of the matrix P_Q .

Notice that the cross-country distribution of income converges towards a bimodal shape, which was one of Quah’s predictions. If the computations are re-run at an alternative initial condition ψ , the last distribution looks much the same, suggesting

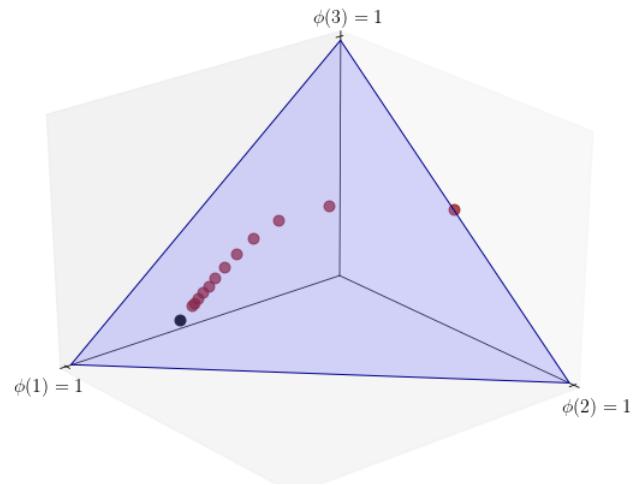
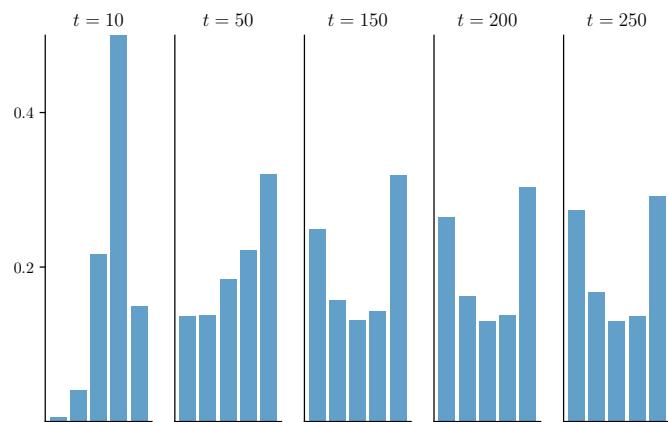
Figure 3.8: A trajectory from $\psi_0 = (0, 1/2, 1/2)$ 

Figure 3.9: Distributions from Quah's Markov kernel

again that we are observing a manifestation of global stability. We confirm this in [add ref](#).

3.1.3.2 Conditional Expectations

Let P be a Markov kernel on X and let h be any function from X to \mathbb{R} . If we regard h as a column vector, P as a matrix, and premultiply h by P , we get

$$(Ph)(x) = \sum_{y \in \mathsf{X}} h(y)P(x, y) \quad (x \in \mathsf{X}). \quad (3.12)$$

This expression has an important interpretation:

$$(Ph)(x) = \mathbb{E}[h(X_{t+1}) | X_t = x] \quad \text{when } \{X_t\} \text{ is generated by } P. \quad (3.13)$$

The idea is clear: $P(x, \cdot)$ is the distribution of X_{t+1} given $X_t = x$, so summing the values of h weighted by the probabilities in $P(x, \cdot)$ gives the conditional expectation on the right hand side of (3.13).

The interpretation in (3.13) extends to powers of P . In particular, we have

$$(P^k h)(x) = \sum_{y \in \mathsf{X}} h(y)P^k(x, y) = \mathbb{E}[h(X_{t+k}) | X_t = x]. \quad (3.14)$$

3.1.3.3 Forecasting Geometric Sums

A task we often confront is computing the expectation of a discounted sum of future values of a random variable that is a function of Markov state. The sum we wish to compute takes the form

$$s(h, x, \beta) := \mathbb{E} \sum_{t \geq 0} \beta^t h(X_t). \quad (3.15)$$

Here β is a constant discount factor, h is some function on X , and $\{X_t\}$ is a Markov chain. One example is the expected present value of a cash flow $\{h(X_t)\}$ that depends on an underlying Markov state $\{X_t\}$ conditional on $X_0 = x$. We provide an application along these lines in §3.1.4.

Lemma 3.1.1. *Fix $h \in \mathbb{R}^\mathsf{X}$. If $\{X_t\}$ is an (x, P) -chain and $\beta \in (0, 1)$, then $s(h, x, \beta)$ is finite and satisfies*

$$s(h, x, \beta) = \sum_{t \geq 0} \beta^t (P^t h)(x) = [(I - \beta P)^{-1} h](x), \quad (3.16)$$

In (3.16), we treat P as a matrix and I is the identity matrix.

Proof. We have

$$\mathbb{E} \sum_{t \geq 0} \beta^t h(X_t) = \sum_{t \geq 0} \beta^t \mathbb{E} h(X_t) = \sum_{t \geq 0} \beta^t (P^t h)(x). \quad (3.17)$$

The last equality in (3.17) follows from (3.14) and the assumption that $\{X_t\}$ is an (x, P) -chain.²

The second step in the proof of Lemma 3.1.1 is to show that $\sum_{t \geq 0} (\beta P)^t = (I - \beta P)^{-1}$. We can use the Neumann Series Lemma (p. 38) applied to the matrix βP . The result is applicable because $r(\beta P) = \beta r(P) = \beta < 1$, as follows from Exercise 1. \square

3.1.3.4 Application: Valuing Consumption Streams

To be added. Take a chain with states 1, 2, 3. Consider the constant chain, always at state 2. Now increase volatility while preserving the mean. How does this affect lifetime utility at a fixed parameterization?

3.1.4 Risk-Neutral Asset Pricing

Dividends depend on finite MCs. Connect to Neumann series. Discuss irreducibility, necessity of conditions.

3.1.5 Asset Pricing with Risk-Aversion

With finite state MCs. Connect to Neumann series.

3.1.5.1 A Two Period Problem

Go back to two periods and show why risk neutrality is not generally appropriate.

3.1.5.2 Infinite Horizon Models with Risk Aversion

Include Arrow–Debreu prices, state price deflators, the Lucas SDF.

²A rigorous proof that \mathbb{E} can pass through the sum to obtain the first equality uses the Dominated Convergence Theorem, which we discuss later (see Theorem 11.3.6).

3.1.6 Joint Distributions

Rewrite this to be more practical. When does knowledge of marginal distributions not suffice? How about the model of S-s inventory dynamics? What is the probability of restocking k times before T ?

One of us is known for reminding his students on a regular basis that an economic model is a probability distribution on a sequence space. Let's clarify this idea in the present context. The full story requires measure theory, which we discuss in Chapter 11. But the key ideas are simple.

The sequence space referenced above is, in this context, the infinite product

$$\mathbb{X}^\infty := \mathbb{X} \times \mathbb{X} \times \mathbb{X} \times \cdots . \quad (3.18)$$

Think of a point in \mathbb{X} as describing the state of the model economy at any given time. An element $\{x_t\}_{t \geq 0}$ of the sequence space \mathbb{X}^∞ is a time series of infinite length. A truncated version was shown in figure 3.6, where the state space \mathbb{X} is $\{1, \dots, 8\}$ and t is truncated at a finite integer for your viewing convenience.³

If an economic model is a probability distribution over a sequence space and (3.18) is the sequence space, then what is the probability distribution? The answer is that this probability distribution corresponds to the joint distribution associated with a particular model. One draw from this joint distribution picks out a full time series $\{x_t\}_{t \geq 0}$ as a realization.

In the present context, models are defined recursively via a kernel p on \mathbb{X} giving transition probabilities, along with an initial condition $\psi \in \mathcal{D}(\mathbb{X})$. The **joint distribution** associated with Markov kernel p and initial condition ψ is defined for finite sequences by

$$q(x_0, x_1, \dots, x_n) = \psi(x_0) \prod_{t=1}^n p(x_{t-1}, x_t). \quad (3.19)$$

This expression q can be obtained by recalling that joint and marginal densities are related by, in generic notation,

$$\text{Prob}(x_t | x_0, x_1, \dots, x_{n-1}) = \frac{\text{Prob}(x_0, x_1, \dots, x_{n-1}, x_n)}{\text{Prob}(x_0, x_1, \dots, x_{n-1})}.$$

³Note that the product space \mathbb{X}^∞ is not just infinite but *uncountably infinite* whenever \mathbb{X} has more than one element (see Cantor's beautiful diagonal argument). This is why measure theory is essential for a rigorous definition.

In our case, rearranging and using the Markov assumption, we get

$$q(x_0, x_1, \dots, x_n) = q(x_0, x_1, \dots, x_{n-1})p(x_{n-1}, x_n).$$

Iterating backwards yields (3.19).

So far we have only defined a distribution over \mathbb{X}^{n+1} , rather than all of \mathbb{X}^∞ . But it turns out that this finite dimensional distribution extends uniquely to a distribution over \mathbb{X}^∞ by well-known theorems by Andrey Kolmogorov (1903–1987) and Cassius Ionescu–Tulcea (1923–), which we review later, in §13.2.1. These theorems provide the formal machinery mapping recursive representations of economic models to sequential ones.

In what follows, we denote the uniquely defined extension by \mathbf{P}_ψ . It is the joint distribution over \mathbb{X}^∞ of the random sequence $\{X_t\}$ when the latter is a (ψ, p) -chain. For finite sequences it agrees with (3.19), as indeed it should do, being an extension. For example,

$$\mathbf{P}_\psi(\{x_0\} \times \{x_1\} \times \cdots \times \{x_n\} \times \mathbb{X} \times \mathbb{X} \times \cdots) = \psi(x_0) \prod_{t=1}^n p(x_{t-1}, x_t)$$

for all $(x_0, x_1, \dots, x_n) \in \mathbb{X}^{n+1}$.

The joint distribution connects to the marginal distributions $\{\psi_t\}$ of $\{X_t\}$ via

$$\mathbf{P}_\psi\{\{x_t\} \in \mathbb{X}^\infty : x_k = x\} = \psi_k(x) = (\psi P^k)(x) \quad (x \in \mathbb{X}, k \in \mathbb{N}).$$

For the joint distribution of an (x, p) -chain we write \mathbf{P}_x .

One might ask why we bother with recursive representations at all. Why not go straight to the joint distribution for any given problem, which encodes all the information we need about the model in question and the sequences that it generates? The answer to that question is that the recursive approach is far more parsimonious, and hence much easier to manipulate and estimate.

3.1.7 Inventory Dynamics in Continuous Time

Exponential arrival times for demand. Treat this as a jump chain and derive the Markov semigroup probabilistically. Use it to represent density flows. (Perhaps skip infinitesimal description.) Show some simulations. Jump times need to be exponential or the Markov property fails!

3.2 Linear Programming

3.2.1 Basic Theory

To be added. Duality here? Farkas Lemma? If so, give geometric intuition.

3.2.2 Shortest Paths

Complete this, connecting to LP.

The **shortest path** problem is a famous topic in dynamic programming that has applications in artificial intelligence, operations research, network design, and other areas. It is also one of the clearest illustrations of a something called Bellman's principle of optimality—a concept that is crucial to what follows in this text.

The aim is to traverse a graph, following arcs (arrows) from one specified node to another at minimum cost. Consider as an example the graph shown in figure 3.10. We wish to travel from node A to node G . Arrows indicate movements we can take, while numbers on the arcs indicate costs of traveling along them. Since this graph is small, we can find the minimum cost path visually. A quick scan of the arcs shows that the minimum attainable cost is 8. Two paths can attain this cost: A, C, F, G , as shown in figure 3.11, and A, D, F, G , as shown in figure 3.12.

3.2.3 Optimal Transport

To be added. Start with the Monge problem. Intro Kantorovich formulation. Explain how convexification and shifting to a higher dimensional space provides regularity and opportunity to use standard methods. But computation is still hard!

3.2.3.1 Monge's Formulation

To be added.

3.2.3.2 The Monge–Kantorovich Problem

Let φ and ψ be elements of $\mathcal{D}(\mathsf{X})$ and $\mathcal{D}(\mathsf{Y})$ respectively. A **coupling** of (φ, ψ) is an element of $\mathcal{D}(\mathsf{X} \times \mathsf{Y})$ marginals φ and ψ . This restriction on marginals means that

$$\sum_{y' \in \mathsf{Y}} \pi(x, y') = \varphi(x) \quad \text{and} \quad \sum_{x' \in \mathsf{X}} \pi(x', y) = \psi(y) \quad (x \in \mathsf{X}, y \in \mathsf{Y}). \quad (3.20)$$

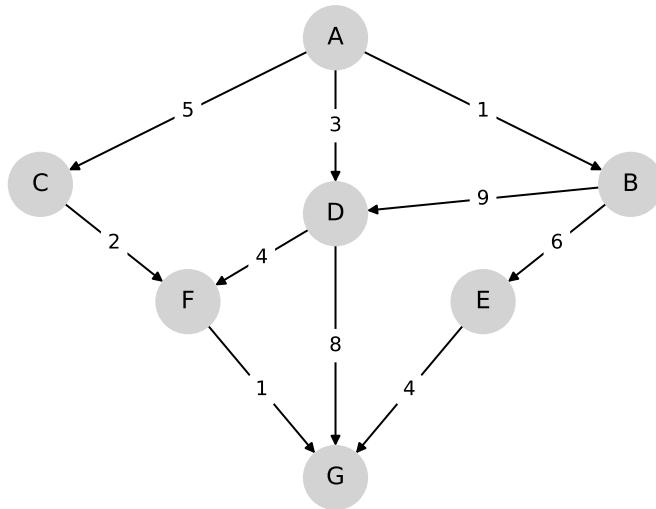


Figure 3.10: Graph for the shortest path problem

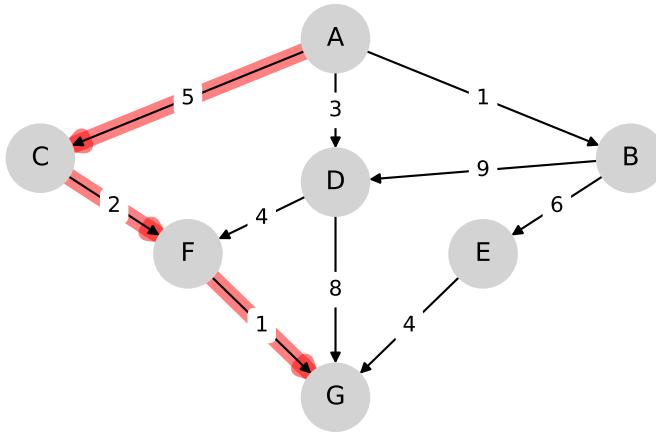


Figure 3.11: Solution 1

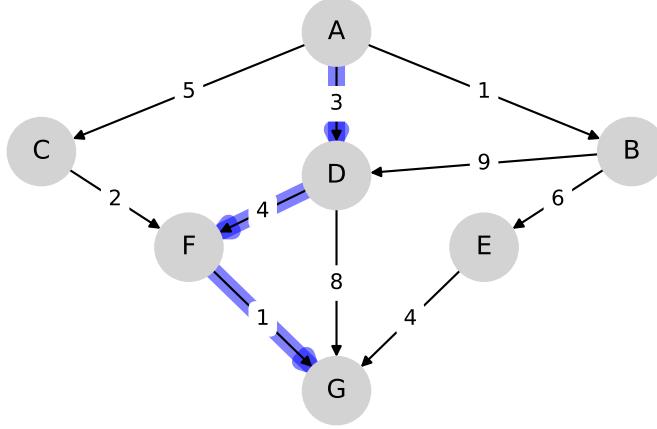


Figure 3.12: Solution 2

Let $\Pi(\varphi, \psi)$ be the set of all **couplings** of ψ and φ . In the present setting, a coupling is also called a **feasible transport plan**.

A **realization** of the coupling π in $\Pi(\varphi, \psi)$ is a pair of random elements (X, Y) such that $X \sim \varphi$ and $Y \sim \psi$. To economize on notation, we also call (X, Y) a **coupling** of (φ, ψ) and write $(X, Y) \in \Pi(\varphi, \psi)$.

Let c be a cost function, which, for now, we take to be a map from $\mathsf{X} \times \mathsf{Y}$ to \mathbb{R}_+ . The **Monge–Kantorovich problem**, also called the **optimal transport problem**, is to solve

$$P(c, \varphi, \psi) := \min_{\pi} \sum_{x,y} c(x, y) \pi(x, y) \quad \text{subject to } \pi \in \Pi(\varphi, \psi). \quad (3.21)$$

Here φ and ψ are given elements of $\mathcal{D}(\mathsf{X})$ and $\mathcal{D}(\mathsf{Y})$. Identifying couplings with their realizations, we can also express (3.21) as the infimum of $\mathbb{E} c(X, Y)$ over all (X, Y) in $\Pi(\varphi, \psi)$. To ensure finiteness of (3.21), we impose

Assumption 3.2.1. There exist functions $c_1 \in \ell_1(\varphi)$ and $c_2 \in \ell_1(\psi)$ such that $c(x, y) \leq c_1(x) + c_2(y)$ on $\mathsf{X} \times \mathsf{Y}$.

The set $\ell_1(\varphi)$ is all $h \in \mathbb{R}^{\mathsf{X}}$ with $\sum_x |h(x)|\varphi(x) < \infty$. The set $\ell_1(\psi)$ is defined symmetrically. (The ℓ_1 notation is consistent with later usage.)

Later, we will prove that a solution to the Monge–Kantorovich problem exists in very general settings, including the one described here (Theorem 12.1.4). We call any such π an **optimal plan**.

3.2.4 Kantorovich Duality

To be added. Add a proof up to min/max? Or a proof by strong duality? Interpretation in terms of prices, free markets. Note the irony of the anti-Marxist message! See Villani’s comments. Connect back to fundamental welfare theorems. Note, however, that computation is still hard. Briefly mention regularization, to be treated later.

3.2.4.1 Statement

Let c be a cost function on $\mathsf{X} \times \mathsf{Y}$. We now state the famous **Kantorovich duality theorem**, named after Russian mathematician Leonid Kantorovich (1912–1986). In the statement, \mathcal{F}_c is all pairs (f, g) in $\ell_1(\varphi) \times \ell_1(\psi)$ such that

$$g(y) \leq c(x, y) + f(x) \text{ on } \mathsf{X} \times \mathsf{Y}. \quad (3.22)$$

Theorem 3.2.1 (Kantorovich Dual Formula, Discrete Case). *If $\varphi \in \mathcal{D}(\mathsf{X})$ and $\psi \in \mathcal{D}(\mathsf{Y})$ satisfy Assumption 3.2.1, then $P(c, \varphi, \psi) = D(c, \varphi, \psi)$, where*

$$D(c, \varphi, \psi) := \max_{(f,g) \in \mathcal{F}_c} \left\{ \sum_y g(y)\psi(y) - \sum_x f(x)\varphi(x) \right\}. \quad (3.23)$$

We restate this result again in a more general setting in Theorem 12.1.5.

For part of a formal proof see `monge.pdf` in references dir. What’s missing is proof that the min-max step is valid. Consider referring to von Neumann’s min-max, although it lacks sufficient generality. Add intuition in terms of both prices and Lagrange multipliers.

3.2.4.2 Connection to Equilibria

Background story and intuition yet to be added. This is one great example of the beautiful and powerful links between optimality and competitive equilibria.

EXERCISE 4. Let X and Y be nonempty sets. Let f and g be real-valued on X and Y such that $\min_{x \in \mathsf{X}} f(x) = \max_{y \in \mathsf{Y}} g(y)$. Prove the following statement: If there exist

$\bar{x} \in \mathsf{X}$ and $\bar{y} \in \mathsf{Y}$ such that $f(\bar{x}) = g(\bar{y})$, then \bar{x} is a minimizer of f on X and \bar{y} is a maximizer of g on Y .

Let c be a given cost function and let φ and ψ be elements of $\mathcal{D}(\mathsf{X})$ and $\mathcal{D}(\mathsf{Y})$ respectively. Let us agree to call functions f and g **prices** if $f \in \ell_1(\varphi)$ and $g \in \ell_1(\psi)$. We say that prices f and g satisfy the **no arbitrage** condition if $g(y) \leq c(x, y) + f(x)$ for all (x, y) in $\mathsf{X} \times \mathsf{Y}$. Given a feasible transport plan $\pi \in \Pi(\varphi, \psi)$, we say that (π, f, g) satisfies the **zero profits** condition if

$$g(y) = c(x, y) + f(x) \quad \text{whenever } \pi(x, y) > 0. \quad (3.24)$$

We call (π, f, g) a **competitive equilibrium** if the no arbitrage and zero profit conditions both hold.

In the next theorem, competitive equilibria and optima are defined with respect to fixed distributions φ and ψ satisfying Assumption 3.2.1, along with a given cost function c .

Theorem 3.2.2. *If (π, f, g) is a competitive equilibrium, then π is an optimal transport plan. Conversely, if π is an optimal transport plan, then there exist prices (f, g) such that (π, f, g) is a competitive equilibrium.*

Proof. Suppose first that (π, f, g) is a competitive equilibrium. Since the equality in (3.24) holds when $\pi(x, y) > 0$, we can multiply both sides by $\pi(x, y)$ and sum over all x, y to obtain

$$\sum_{x,y} c(x, y) \pi(x, y) = \sum_y g(y) \psi(y) - \sum_x f(x) \psi(x). \quad (3.25)$$

From the arbitrage condition we have $(f, g) \in \mathcal{F}_c$, so applying the result of Exercise 4, π attains the minimum in (3.21). In other words, π is an optimal plan.

Regarding the converse, suppose that π is an optimal plan. By Kantorovich's duality theorem, we can obtain $(f, g) \in \mathcal{F}_c$ such that (3.25) holds. Since $(f, g) \in \mathcal{F}_c$ we have $c(x, y) + f(x) - g(y) \geq 0$. From this and (3.25) we see that (3.24) must hold. In other words, (f, g) are prices satisfying the no arbitrage and zero profit conditions, so (π, f, g) is a competitive equilibrium. \square

3.2.5 Mean Field Games

To be added. A first look. Cournot-Nash equilibria in the discrete case. Minimizing potentials and the role of duality. Mention regularization.

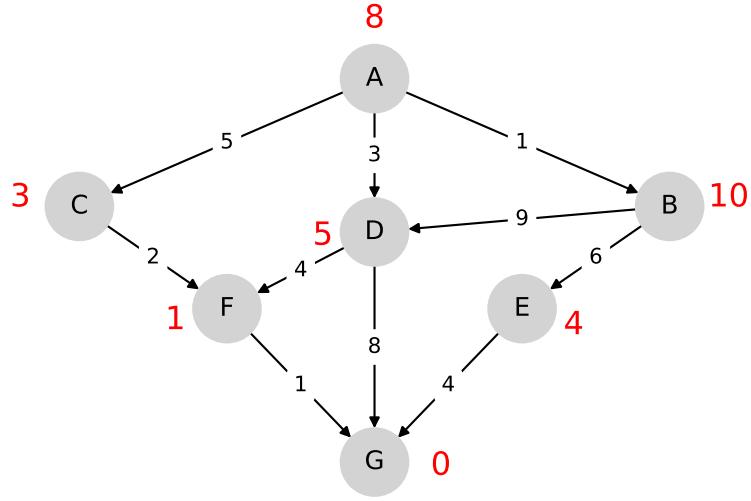


Figure 3.13: The cost-to-go function, with $v(x)$ indicated by red digits at each x

3.3 Discrete Decision Processes

Roadmap to be added.

3.3.1 Shortest Paths Again

Connect the discussion below to the earlier LP treatment.

While solved this small problem by eyeballing the graph, for large graphs we need a systematic solution. To this end, let $v(x)$ denote the **minimum cost-to-go** from node x . That is, $v(x)$ is the total cost of traveling to the final node from x if we take the best route. The function v is usually called the **cost-to-go function** or the **value function**. Its values are shown at each node in figure 3.13. We can represent v in vector form

$$(v(A), v(B), v(C), v(D), v(E), v(F), v(G)) = (8, 10, 3, 5, 4, 1, 0) \quad (3.26)$$

As can be seen clearly from studying Figure 3.13, once v is known, the least cost path can be computed as follows: Start at A and from then on, at node x , move to the node y that solves

$$\min_{y \in \Gamma(x)} \{c(x, y) + v(y)\}. \quad (3.27)$$

Here $\Gamma(x)$ is the set of nodes that can be reached from x in one step and $c(x, y)$ is the cost of traveling from x to y . In other words, to minimize the cost-to-go, we choose the next step to minimize current cost plus remaining cost. Thus, if we know the function v , then finding the best path is almost trivial.

Evidently the cost-to-go function v is key to finding optimal paths. But now another problem arises: how to find v in more complicated cases in which a graph is large and there are many paths. Fortunately, there is an elegant solution to this problem, which is to exploit the equation

$$v(x) = \min_{y \in \Gamma(x)} \{c(x, y) + v(y)\}, \quad (3.28)$$

that holds for every node x in the graph. Take the time to convince yourself that, for our example, the function v satisfies (3.28). You can do this by checking the equality (3.28) at each node in Figure 3.13.

[Complete this discussion.](#)

We return to shortest path problems and apply these methods in §10.3.4.

3.3.2 Optimal Inventories

Rewrite the following since S-s will come later. Better explanation of value of policies through v-N series. Discuss optimality, intro Bellman, explain difference with v-N: the operator is nonlinear. But we can use related ideas. Flag Banach contraction mapping theorem. Give some intuition as to what contractions mean without stating the theorem. Maybe show directly how they imply uniqueness.

In §7.1.7.1 we studied a firm whose inventory behavior followed so-called s, S dynamics, which means that the firm orders inventory infrequently, and only when the amount of inventory on hand falls below some specified level. Let's replicate this in an optimizing model, where the firm chooses its inventory path to maximize profits in each period.

Let inventory for the firm obey the law of motion

$$i_{t+1} = (i_t - D_{t+1})_+ + Sa_t \quad (3.29)$$

Here $\{i_t\}$ is inventory, which is the state process, $\{D_t\}$ is a demand shock and $t_+ := \max\{t, 0\}$. The term a_t is a binary control variable. If $a_t = 1$, then the firm orders amount S . If not the firm orders nothing.

Profits for the firm are, assuming a unit markup on the stocked item,

$$\mathbb{E} \sum_{t \geq 0} \beta^t \pi_t \quad \text{where } \pi_t := \min\{i_t, D_{t+1}\} - ca_t \quad (3.30)$$

We take the minimum because orders in excess of inventory are lost rather than back-filled. The term c is a fixed cost of ordering inventory.

We assume that the firm can stock at most kS items at one time. If we set

$$\Gamma(i) = \begin{cases} \{0, 1\} & \text{if } i \leq (k-1)S \\ \{0\} & \text{otherwise} \end{cases} \quad (3.31)$$

then $\Gamma(i)$ gives the set of feasible choices for a_t when the current inventory state is i .

Assuming IID demand shocks with common probability mass function φ , the Bellman equation for this problem is

$$v(i) = \max_{a \in \Gamma(i)} \left\{ \sum_{d \geq 0} \min\{i, d\} \varphi(d) - ca + \beta \sum_{d \geq 0} v((i-d)_+ + Sa) \varphi(d) \right\} \quad (3.32)$$

over i in

$$\mathsf{X} := \{0, 1, \dots, kS\} \quad (3.33)$$

The function φ is defined on $\{0, 1, \dots\}$. In what follows we take it to be the geometric distribution on that set. The Bellman equation says that optimal value is attained when the firm chooses a to balance current expected profits with the value of a higher inventory next period.

We will solve this problem by value function iteration. The Bellman operator in this context is

$$Tv(i) = \max_{a \in \Gamma(i)} \left\{ \sum_{d \geq 0} \min\{i, d\} \varphi(d) - ca + \beta \sum_{d \geq 0} v((i-d)_+ + Sa) \varphi(d) \right\} \quad (3.34)$$

This operator is a contraction mapping on the set \mathbb{R}^X paired with the supremum norm $\|v\|_\infty := \sup_{i \in \mathsf{X}} |v(i)|$ because, in view of lemma 5.1.17 on page 151, we have, for any v, w in \mathbb{R}^X ,

$$\begin{aligned} |Tv(i) - Tw(i)| &\leq \beta \max_{a \in \Gamma(i)} \left| \sum_{d \geq 0} v((i-d)_+ + Sa) \varphi(d) - \sum_{d \geq 0} w((i-d)_+ + Sa) \varphi(d) \right| \\ &\leq \beta \max_{a \in \Gamma(i)} \sum_{d \geq 0} |v((i-d)_+ + Sa) - w((i-d)_+ + Sa)| \varphi(d) \end{aligned}$$

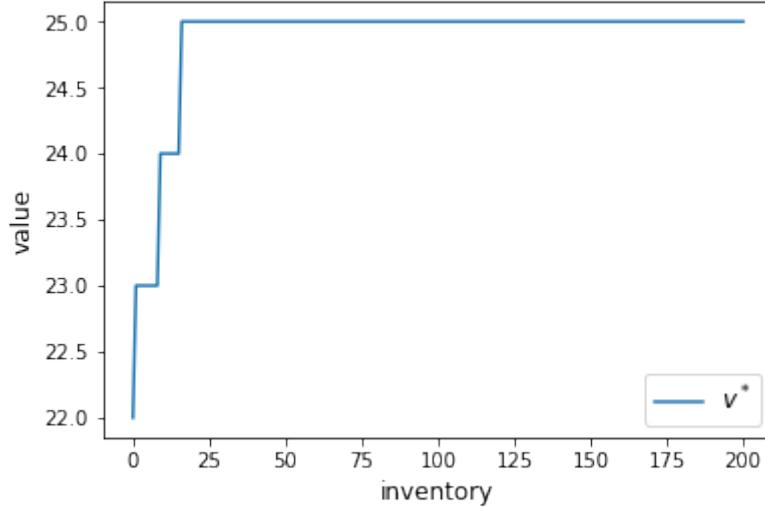


Figure 3.14: The value function for the inventory problem

Since $\sum_{d \geq 0} \varphi(d) = 1$, it follows that, for arbitrary $i \in \mathbb{X}$,

$$|Tv(i)| - Tw(i)| \leq \beta \|v - w\|_\infty$$

Taking the supremum over all $i \in \mathbb{X}$ yields the desired result.

As shown in more detail in §3.3.3 and again in chapter 10, the unique fixed point of T is the optimal value function v^* , which, for each i in \mathbb{X} , gives the maximal amount of lifetime value that can be extracted from that initial condition.

Figure 3.14 exhibits this fixed point—or at least a close approximation, computed by iterating with T starting at $v \equiv 1$ when $S = 100$, $\beta = 0.98$ and $c = k = 2$. The geometric distribution has parameter $p = 0.4$, so that $\varphi(d) = (1 - p)^d p = 0.6^d \times 0.4$. Figure 3.15 shows the optimal policy, obtained by maximizing the right hand side of the Bellman equation:

$$\sigma^*(i) = \operatorname{argmax}_{a \in \Gamma(i)} \left\{ \sum_{d \geq 0} \min\{i, d\} \varphi(d) - ca + \beta \sum_{d \geq 0} v^*((i - d)_+ + Sa) \varphi(d) \right\}$$

There is a value of the state below which the firm always orders and above which the firm never orders, consistent with s, S inventory dynamics. Figure 3.16 shows a simulation of inventory under the optimal policy starting from $i_0 = S$.

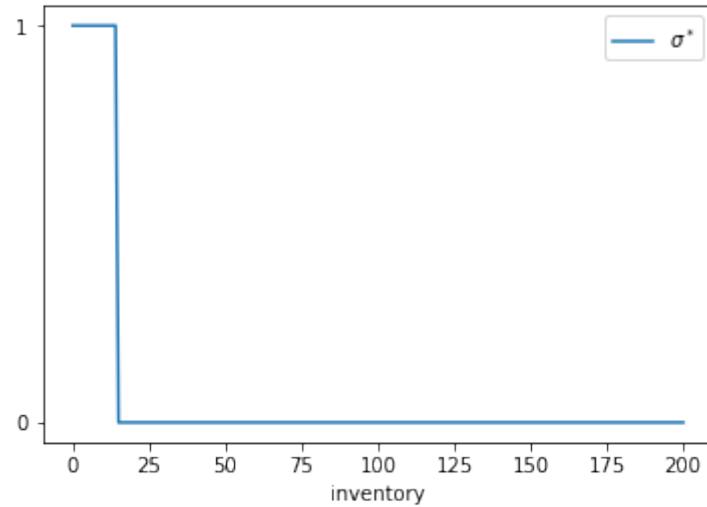


Figure 3.15: The optimal policy for the inventory problem

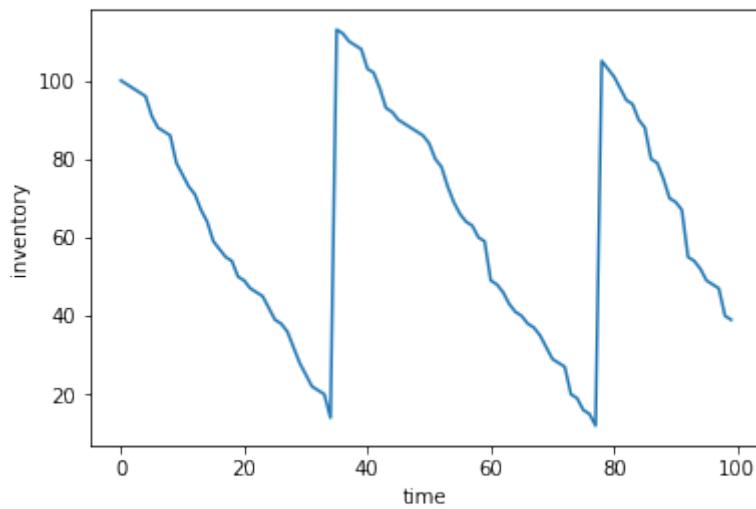


Figure 3.16: Optimal inventory dynamics

3.3.3 Discrete MDPs

Add roadmap. Rewrite so that Banach fpt is anticipated but not used.

3.3.3.1 Finite State Markov Decision Problems

Let's place the last example in a more general framework. A **finite state Markov decision problem** consists of

- (i) a nonempty finite set X called the **state space**,
- (ii) a nonempty finite set A called the **action space**,
- (iii) a **feasible correspondence** Γ from $X \rightarrow A$,
- (iv) a **reward function** $r: G \rightarrow \mathbb{R}$, where

$$G := \{(x, a) \in X \times A : a \in \Gamma(x)\}$$

- (v) a **discount factor** $\beta \in (0, 1)$ and
- (vi) a **stochastic kernel** Π from G to X

The set G is called the set of **feasible state-action pairs**.

Regarding the stochastic kernel Π , previously we defined a stochastic kernel on discrete set X as a family of distributions $\Pi(x, \cdot)$ over X , one for each $x \in X$. Here we are using a natural generalization: A **stochastic kernel** from E to F is a family of distributions $\Pi(e, \cdot)$ over F , one for each e in E . Thus, the stochastic kernel in item (vi) provides us with a distribution over the state space for each feasible state-action pair (x, a) . The next period state x' is selected from this distribution.

The dynamics and reward flow are summarized informally in algorithm 4.

The decision maker's objective is to choose a **state-contingent** action path $\{a_t\}$ that maximizes expected discounted rewards

$$\mathbb{E} \sum_{t \geq 0} \beta^t r(x_t, a_t) \tag{3.35}$$

State contingency means that a_t can be chosen contingent on the current state x_t .

We can map our inventory problem from the previous section into the framework of a finite state Markov decision problem as follows. The state is $x = i$, the state space

```

1 set  $t \leftarrow 0$  and take input  $x_0$  ;
2 while  $t < \infty$  do
3   observe  $x_t$  ;
4   choose action  $a_t$  ;
5   receive  $r(x_t, a_t)$  ;
6   draw  $x_{t+1}$  from  $\Pi(x_t, a_t, \cdot)$  ;
7    $t \leftarrow t + 1$  ;
8 end

```

Algorithm 4: State, actions, and rewards

X is as given in (3.33) and the action space is $A = \{0, 1\}$. The feasible correspondence Γ is as given in (3.31). The reward function is

$$r(x, a) = \sum_{d \geq 0} \min\{x, d\} \varphi(d) - ca$$

The stochastic kernel from the set of feasible state-action pairs G induced by Γ is, in view of (3.29),

$$\Pi(x, a, y) = \mathbb{P}\{(x - D_{t+1})_+ + Sa = y\} \quad (3.36)$$

EXERCISE 5. Write down an expression for the stochastic kernel (3.36) using only x, a, y and the parameters of the model. Continue to assume that the demand shock has geometric distribution with parameter p .

3.3.3.2 Optimality

In this discrete state environment, let us take the opportunity to be explicit about the meaning of optimality. Actions will be governed by policies, which are maps from states to actions. The set of **feasible policies** is

$$\Sigma := \{\sigma \in A^X : \sigma(x) \in \Gamma(x) \text{ for all } x \in X\} \quad (3.38)$$

If we select a particular policy σ from Σ , it is understood that we respond to state x_t with action $a_t := \sigma(x_t)$ at every date t .

What happens if we commit to a policy σ in Σ for the lifespan of the problem? Now the state evolves by drawing x_{t+1} from $\Pi(x_t, \sigma(x_t), \cdot)$ at every point in time. Given

initial condition $x_0 = x$, this process is exactly an (x, Π_σ) -chain for Π_σ defined by

$$\Pi_\sigma(x, y) := \Pi(x, \sigma(x), y) \quad (x, y \in \mathsf{X})$$

In particular, fixing a policy “closes the loop” in the state transition process and sets a given Markov chain for the state.

Under the policy σ , rewards at each point in time are $r(x_t, a_t) = r(x_t, \sigma(x_t))$. If we now introduce the notation

$$r_\sigma(x) := r(x, \sigma(x)) \quad (x \in \mathsf{X})$$

then the expected time t reward is

$$\mathbb{E}[r(x_t, a_t) | x_0 = x] = \mathbb{E}[r_\sigma(x_t) | x_0 = x] = \Pi_\sigma^t r_\sigma(x)$$

The last equality uses our conditional expectation notation (3.14) from page 55.

The lifetime value of following σ starting from state x can now be written as

$$\begin{aligned} v_\sigma(x) &= \mathbb{E} \left[\sum_{t \geq 0} \beta^t r(x_t, \sigma(x_t)) | x_0 = x \right] \\ &= \sum_{t \geq 0} \beta^t \mathbb{E}[r(x_t, \sigma(x_t)) | x_0 = x] = \sum_{t \geq 0} \beta^t (\Pi_\sigma r_\sigma)(x) \end{aligned}$$

or, in vector notation with v_σ and r_σ viewed as column vectors,

$$v_\sigma = \sum_{t \geq 0} \beta^t \Pi_\sigma^t r_\sigma \tag{3.39}$$

If we need to invoke the function v_σ by name, we will call it the **σ -value function**.

The **value function** is then defined as

$$v^*(x) = \sup_{\sigma \in \Sigma} v_\sigma(x) \quad (x \in \mathsf{X}) \tag{3.40}$$

This is consistent with all of our previous usage of the expression “value function.” It is the lifetime value we can extract from each state, conditional on optimal behaviour at each point in time.

A policy $\sigma \in \Sigma$ is called **optimal** if $v_\sigma = v^*$. In other words, a policy is optimal if its lifetime value is maximal at each state. The next proposition justifies the procedure we used to compute an optimal policy in the inventory problem of §3.3.2.

Proposition 3.3.1. *The value function v^* satisfies the Bellman equation*

$$v^*(x) = \max_{a \in \Gamma(x)} \left\{ r(x, a) + \beta \sum_{y \in \mathcal{X}} v^*(y) \Pi(x, a, y) \right\} \quad (3.41)$$

at every $x \in \mathcal{X}$. Moreover, a feasible policy σ is optimal if and only

$$\sigma(x) \in \operatorname{argmax}_{a \in \Gamma(x)} \left\{ r(x, a) + \beta \sum_{y \in \mathcal{X}} v^*(y) \Pi(x, a, y) \right\} \quad (3.42)$$

At least one such policy exists.

A full proof of this result is given in chapter 10.

The last statement, which claims existence of at least one optimal policy, is trivial in this setting—we simply select a point a_x^* from the nonempty set on the right hand side of (3.42) at every x in \mathcal{X} . By the sufficiency of condition (3.42), the resulting policy $\sigma(x) := a_x^*$ is optimal. The same existence claim will be less trivial when we switch to continuous state spaces—but still viable under reasonable conditions.⁴

3.3.3.3 Algorithms

In §3.3.2, to compute the optimal policy, we used **value function iteration**, which requires iterating with the Bellman operator

$$Tv(x) = \max_{a \in \Gamma(x)} \left\{ r(x, a) + \beta \sum_{y \in \mathcal{X}} v(y) \Pi(x, a, y) \right\} \quad (3.43)$$

The general procedure in the present finite state setting is given by algorithm 5.

There is another popular algorithm for computing the optimal policy, called **Howard's policy iteration algorithm**. The technique is described in algorithm 6. In the algorithm, a **v -greedy policy** is defined, for arbitrary $v \in \mathbb{R}^{\mathcal{X}}$, as a policy $\sigma \in \Sigma$ satisfying

$$\sigma(x) \in \operatorname{argmax}_{a \in \Gamma(x)} \left\{ r(x, a) + \beta \sum_{y \in \mathcal{X}} v(y) \Pi(x, a, y) \right\}$$

⁴As an aside, the finite case is perhaps the most important when it comes to dynamic programming because it's what your computer can implement. If you are using a 64 bit machine then your machine can implement 2^{64} different floating point numbers, with a range of around $\pm 10^{-308} \dots 10^{308}$. In the interval $[0, 1]$ there are more than a billion floating point numbers. In almost all cases, this will be ample precision.

```

1 input  $v_0 \in \mathbb{R}^X$ , an initial guess of  $v^*$  ;
2 input  $\tau$ , a tolerance level for error ;
3  $\varepsilon \leftarrow \tau + 1$  ;
4  $n \leftarrow 0$  ;
5 while  $\varepsilon > \tau$  do
6   for  $x \in X$  do
7     |  $v_{n+1}(x) \leftarrow T v_n(x)$  ;
8   end
9    $\varepsilon \leftarrow \|v_n - v_{n+1}\|_\infty$  ;
10   $n \leftarrow n + 1$  ;
11 end
12 return  $v_n$ 
```

Algorithm 5: Value function iteration (finite state space)

for all $x \in X$. A visualization of the algorithm is given in figure 3.17.

```

1 input  $\sigma_0 \in \Sigma$ , an initial guess of  $\sigma^*$  ;
2  $n \leftarrow 0$  ;
3  $\varepsilon \leftarrow 1$  ;
4 while  $\varepsilon > 0$  do
5    $v_n \leftarrow$  the  $\sigma_n$ -value function  $\sum_{t \geq 0} \beta^t \Pi_{\sigma_n}^t r_{\sigma_n}$  ;
6    $\sigma_{n+1} \leftarrow$  the  $v_n$  greedy policy ;
7    $\varepsilon \leftarrow \|\sigma_n - \sigma_{n+1}\|_\infty$  ;
8    $n \leftarrow n + 1$  ;
9 end
10 return  $\sigma_n$ 
```

Algorithm 6: Howard's policy iteration algorithm (finite state space)

One of the attractive features of Howard's policy function method is that, in a finite state setting, it always converges to the exact optimal policy in a finite number of steps. For a proof, see, for example, Puterman (2005) or theorem 10.2.6 of Stachurski (2009). The basic intuition is that the value difference $v_{n+1}(x) - v_n(x)$ is strictly positive at at least one point in the state space when the current policy σ_n is not optimal. In other words, when the policy is not optimal there is always some strict improvement in value. Thus, the sequence of policies $\{\sigma_n\}$ generated by the algorithm does not cycle. Since there are only finitely many policies in Σ , convergence is guaranteed.

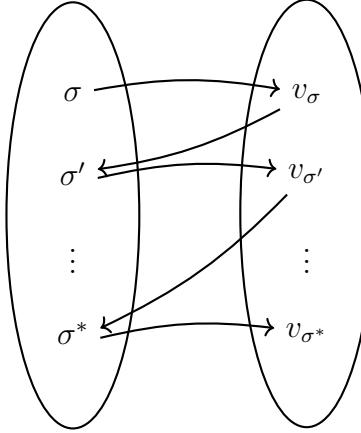


Figure 3.17: Howard's policy function iteration algorithm

3.3.3.4 Computing the Value of a Policy

One of the steps in Howard's policy iteration routine is computing the value v_σ of a given policy σ . We could do this by truncating the sum in (3.39), which implies that

$$v_\sigma \approx \sum_{t=0}^T \beta^t \Pi_\sigma^t r_\sigma \quad (3.44)$$

when T is large.

Another way to compute v_σ is by making use of the operator T_σ defined at $v \in \mathbb{R}^X$ by

$$T_\sigma v(x) = r(x, \sigma(x)) + \beta \sum_{y \in X} v(y) \Pi(x, \sigma(x), y) \quad (3.45)$$

or, in vector notation,

$$T_\sigma v = r_\sigma + \beta \Pi_\sigma v \quad (3.46)$$

The next lemma explains why this operator is important.

Lemma 3.3.2. *For any given σ in Σ , the σ -value function v_σ is the unique fixed point of T_σ in \mathbb{R}^X . Moreover, we have $T_\sigma^n v \rightarrow v_\sigma$ as $n \rightarrow \infty$ for all $v \in \mathbb{R}^X$.*

Proof. Fix σ in Σ . First let us show that v_σ is a fixed point of T_σ : We have, in vector notation,

$$T_\sigma v_\sigma = r_\sigma + \beta \Pi_\sigma \left(\sum_{t \geq 0} \beta^t \Pi_\sigma^t r_\sigma \right) = r_\sigma + \left(\sum_{t \geq 1} \beta^t \Pi_\sigma^t r_\sigma \right) = \sum_{t \geq 0} \beta^t \Pi_\sigma^t r_\sigma$$

which is v_σ . (The passage of Π_σ through the limit associated with the infinite sum is justified here because Π_σ is a linear operator acting on a finite dimensional space, and therefore continuous.)

Moreover, T_σ is a contraction of modulus β on \mathbb{R}^X . Indeed, for any v, w in \mathbb{R}^X we have

$$\begin{aligned}|T_\sigma v(x) - T_\sigma w(x)| &= \beta \left| \sum_y \Pi(x, \sigma(x), y) v(y) - \sum_y \Pi(x, \sigma(x), y) w(y) \right| \\ &\leq \sum_y \Pi(x, \sigma(x), y) \beta |v(y) - w(y)| \leq \beta \|v - w\|_\infty\end{aligned}$$

Taking the supremum over all $x \in X$ yields the desired result. This establishes all claims in the lemma. \square

Another way to think about the result in lemma 3.3.2 uses the Neumann series theorem (see page 38), which, in the present context, states that the linear system

$$v = r_\sigma + \beta \Pi_\sigma v \tag{3.47}$$

has the unique solution

$$v_\sigma = \sum_{t \geq 0} \beta^t \Pi_\sigma^t r_\sigma = (I - \beta \Pi_\sigma)^{-1} r_\sigma \tag{3.48}$$

whenever the spectral radius of $\beta \Pi_\sigma$ is less than one. That this is the case for $\beta \Pi_\sigma$ follows from exercise 1 on page 49. The term on the right of (3.48) gives us a means of computing v_σ by matrix inversion, which works well when X is not large.

3.3.4 Connection to Linear Programming

Add an intuitive discussion.

3.4 Chapter Notes

References, background, historical notes.

Chapter 4

General States

Roadmap to be added.

4.1 Probability on the Continuum

Roadmap to be added.

4.1.1 A Note on Borel Functions

Explain a bit more why we need to consider this now.

Here we give an introduction to Borel functions, with a full treatment deferred to Chapter 11.

When working with functions, we often need to place restrictions on the functions we consider for the problem in question to make sense. For example, it would be embarrassing if our proposed solution method for a given problem involved a Taylor expansion and yet the functions we applied this method to had kinks. In this case the “algorithm” is not well defined. At the same time, some interesting functions do have kinks, so we would not want to rule such functions out if no differentiation is involved.

These thoughts lead us to consider classes of “nice” functions, that are well behaved in one way or another. Linear functions on \mathbb{R} are certainly well behaved, as well as being easy to describe. Polynomial functions are a natural generalization. These, in turn, are a special case of the functions on \mathbb{R} that have derivatives of every order. Such functions are special case of the “smooth” functions, which are those functions having continuous first derivative. The smooth functions are contained in the class of Lipschitz functions, which lie inside the class of everywhere continuous functions.

Need we generalize any more? The answer is “yes!” Sometimes economic variables exhibit jumps. Agents make sudden changes in behavior. This means that we must admit discontinuities. At the same time, we do not wish to stray too far from continuity, so as to avoid dealing with the wildest functions mathematicians can dream up. This naturally leads us to the Borel functions.

Recall from Theorem 5.1.1 that, when M and N are metric spaces and f maps M to N , continuity of f is equivalent to the statement that f pulls open sets back to open sets. In other words, f is continuous on M if and only if $f^{-1}(G)$ is open in M whenever G is open in N . The definition of a **Borel function** (sometimes called a **Borel measurable function**) weakens this restriction by requiring instead that, whenever G is open in N , the preimage $f^{-1}(G)$ lies in a larger class of sets than the open sets called the Borel sets.

The Borel sets are discussed in detail in §11.3.1.1 but we can define them easily enough. The set of **Borel sets** of a metric space M , denoted here by \mathcal{B} , is the smallest collection of subsets of M that contains the open sets and is also closed under the taking of complements and countable unions. In other words, \mathcal{B} contains the open sets, satisfies

- (i) $M \in \mathcal{B}$,
- (ii) $B \in \mathcal{B}$ implies $B^c \in \mathcal{B}$, and
- (iii) if $\{B_n\}_{n \geq 1}$ is a sequence contained in \mathcal{B} , then $\cup_n B_n \in \mathcal{B}$;

and, in addition, if \mathcal{A} is another collection of subsets of M that contains then open sets and satisfies these three properties, then $\mathcal{B} \subset \mathcal{A}$.

There are three reasons why Borel functions are so important to modern analysis. First, in most settings, the set of Borel functions is much larger than the class of continuous functions, precisely because the class of Borel sets is much larger than the class of open sets. For example, it can be shown that, when $M = N = \mathbb{R}$, the class of Borel functions includes not just the continuous functions but also any increasing function, any convex function, any concave function, any linear combination of these kinds of functions, any continuous transformation of any of these functions, and so on.

Second, the class of Borel functions is closed under all the standard arithmetic and limiting operations. Sums and scalar multiples of Borel functions are Borel functions. Pointwise limits, suprema and infima of Borel functions are Borel functions, and so on. This is important for consistency because it means that we will not inadvertently introduce unpleasant functions through standard operations.

Third, when considering integrals such as $\int f(x) dx$ or $\mathbb{E}f(X)$, we often need to admit discontinuous f but, at the same time, cannot admit arbitrary f while still

being confident about basic properties of the integral, such as linearity. It turns out that Borel functions admit a well defined theory of integration with all the usual helpful properties. We will revisit this in Chapter 11.

4.1.2 Distributions

In contrast to the discrete case, the density case arises when probability mass is distributed smoothly over a continuum. Given some nonempty interval X in \mathbb{R}^n , we call a function φ from X to \mathbb{R}_+ a **density** if $\int \varphi(x) dx = 1$. Here the integral is over all of X , and captures the area or volume under the function. We say that X is a **random variable** with density φ on X if the probability that X falls in some subset B of X is equal to $\int_B \varphi(x) dx$, which is the area or volume under the function when the domain is restricted to B . The same statement is represented in symbols by writing

$$\mathbb{P}\{X \in B\} = \int_B \varphi(x) dx$$

We will often want to make statements that are true for both densities and discrete distributions, so it's helpful to have a common framework that contains both cases. This desire leads us to cumulative distribution functions. On \mathbb{R} , a **cumulative distribution function (cdf)** is a function F from \mathbb{R} to $[0, 1]$ such that F is increasing (i.e., nondecreasing), right continuous and satisfies $F(x) \rightarrow 0$ as $x \rightarrow -\infty$ and $F(x) \rightarrow 1$ as $x \rightarrow \infty$. Any density φ on \mathbb{R} has a corresponding cdf, defined by $F(x) = \int_{-\infty}^x \varphi(t) dt$. The density φ can be recovered from F via the Fundamental Theorem of Calculus, which gives $\varphi = F'$. Similarly, any discrete distribution φ on countable set $X \subset \mathbb{R}$ has a corresponding cdf F , defined by $F(x) = \sum_{t \in X, t \leq x} \varphi(t)$.

In what follows, you will see expressions such as $\int g(x)F(dx)$ where F is a cdf. You should interpret this as

$$\int g(x)F(dx) = \mathbb{E}g(X) \text{ where } X \text{ is a random variable drawn from } F \quad (4.1)$$

Here \mathbb{E} is expectation. For example, if g is the identity map (i.e., $g(x) = x$ for all x) then the value of the integral is the mean of the distribution. If $g(x) = x^2$ then the value is the second moment.

We can be more specific about the meaning of the integral in (4.1). If X has a density and $F' = f$, so that f is the density of X , then

$$\int g(x)F(dx) := \int_{-\infty}^{\infty} g(x)f(x) dx \quad (4.2)$$

If F corresponds to a probability mass function φ supported on a countable set X , then

$$\int g(x)F(dx) := \sum_{x \in X} g(x)\varphi(x) \quad (4.3)$$

When we discuss measure and integration in §11.3 we will provide a theoretical framework that covers all of the above in more detail (although the above information will suffice for now).

There are various transformations of random variables and their distributions that will be useful in some settings. For example, if F is a cdf on \mathbb{R} , then the corresponding **characteristic function** is a map from \mathbb{R} to the complex numbers \mathbb{C} defined by

$$\psi(t) := \int \exp(itx)F(dx) := \int \cos(tx)F(dx) + i \int \sin(tx)F(dx) \quad (4.4)$$

This function ψ is well defined on all of \mathbb{R} , since \cos and \sin are bounded functions. Moreover, F can be recovered from ψ . One consequence is that, if we know the characteristic function of a random variable X on \mathbb{R} , defined analogously as $\psi(t) := \mathbb{E} \exp(itX)$, we also know its cdf. Hence, if one states that a random variable X has, say, characteristic function $\psi(t) = \exp(bt - a|t|)$, then this implies pins down the distribution of X . In particular, this characteristic function corresponds to the **Cauchy distribution** with parameters (a, b) , which has the density

$$f(x) = \frac{1}{\pi a} \left[\frac{a^2}{(x - b)^2 + a^2} \right]$$

More details on characteristic functions can be found in [Çinlar \(2011\)](#), §2.2.

4.1.2.1 Some Useful Inequalities

To be completed. This might need to be shifted. Discuss **Jensen's inequality** both for concave and convex functions. Discuss triangle inequality as a special case. Discuss the **Markov inequality** and **Chebychev Inequality**.

4.1.3 Prediction

When we know the distribution of a random variable X but have no other additional information on its outcome, the most commonly used prediction is its **expectation** $\mathbb{E}X$, which was defined in equations (4.1)–(4.3) for the case when the distribution F of X is either discrete or differentiable. These cases suffice for now, although we will

return to expectations in a general setting in §11.3. For now we note that, for random variables X, Y and scalars α, β , the usual properties

- (i) $\mathbb{E}[\alpha X + \beta Y] = \alpha \mathbb{E}X + \beta \mathbb{E}Y$,
- (ii) $X \leq Y$ implies $\mathbb{E}X \leq \mathbb{E}Y$,
- (iii) $\mathbb{E}[\alpha] = \alpha$, and
- (iv) X and Y independent implies $\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$

always hold.

Given p in \mathbb{N} , the **p -th moment** of a random variable X is defined as $\mathbb{E}[X^p]$. If X is nonnegative we will also refer to $\mathbb{E}[X^p]$ as the p -th moment when p is any nonnegative number. In this context, a useful property for nonnegative random variables is the identity

$$\mathbb{E}X^p = \int_0^\infty px^{p-1}\mathbb{P}\{X > x\} dx \text{ for all } p \geq 0. \quad (4.5)$$

(See, for example, Çınlar (2011), p. 63.)

Next let us review prediction based on conditional expectations. Conditional expectations are themselves a cornerstone of economic theory and empirics, since they describe optimal forecasts based on limited information. Here we provide a brief treatment that is sufficient for what follows. (§11.3.4 contains a more formal treatment and proofs.)

Let Y and $\mathcal{G} := \{X_1, \dots, X_k\}$ be (collections of) scalar random variables with finite second moments. Consider the problem of predicting Y given \mathcal{G} . That is, we wish to form a prediction of the value that Y will take once X_1, \dots, X_k are known, without any additional information on the state of the world. Another way to say this is that we seek a (deterministic) function $f: \mathbb{R}^k \rightarrow \mathbb{R}$ such that

$$\hat{Y} := f(X_1, \dots, X_k) \text{ is a good predictor of } Y$$

To find such an f we must of course define what “good” means, and the most common definition in the present context is that **mean squared error** $\mathbb{E}[(\hat{Y} - Y)^2]$ is small. Thus, we have a minimization problem in function space (the set from which f is chosen). Based on projection arguments, the full details of which are deferred to §11.3.4, one can show that there exists an (almost everywhere) unique \hat{f} in the set of Borel measurable functions from \mathbb{R}^k to \mathbb{R} that solves

$$\hat{f} = \operatorname{argmin}_f \mathbb{E}[(Y - f(X_1, \dots, X_k))^2] \quad (4.6)$$

We call the resulting variable

$$\hat{Y} := \hat{f}(X_1, \dots, X_k)$$

the **conditional expectation** of Y given \mathcal{G} . Common alternative notations for \hat{Y} include

$$\mathbb{E}_{\mathcal{G}} Y := \mathbb{E}[Y | \mathcal{G}] := \mathbb{E}[Y | X_1, \dots, X_k]$$

Incidentally, the restriction in the minimization in (4.6) to Borel measurable functions is a weak regularity condition imposed to ensure that f is sufficiently well behaved that the expectation on the right hand side of (4.6) makes sense. The definition of Borel measurability is given in §11.3.1.1.

In the present context, \mathcal{G} is often called an **information set**, which, for our purposes, is just a set of random variables. Also, the following equivalent expressions for conditional expectation are common: In stating the next proposition, we consider Y to be **\mathcal{G} -measurable** if there exists a Borel measurable function f such that $Y = f(X_1, \dots, X_k)$, so Y is perfectly predictable given the data in \mathcal{G} .

Proposition 4.1.1. *Let X and Y be random variables with finite first moment, let α and β be scalars, and let \mathcal{G} and \mathcal{H} be information sets. The following properties hold:*

- (i) $\mathbb{E}_{\mathcal{G}}[\alpha X + \beta Y] = \alpha \mathbb{E}_{\mathcal{G}}X + \beta \mathbb{E}_{\mathcal{G}}Y$.
- (ii) If $\mathcal{G} \subset \mathcal{H}$, then $\mathbb{E}_{\mathcal{G}}[\mathbb{E}_{\mathcal{H}}Y] = \mathbb{E}_{\mathcal{G}}Y$ and $\mathbb{E}[\mathbb{E}_{\mathcal{G}}Y] = \mathbb{E}Y$.
- (iii) If Y is independent of the variables in \mathcal{G} , then $\mathbb{E}_{\mathcal{G}}Y = \mathbb{E}Y$.
- (iv) If Y is \mathcal{G} -measurable, then $\mathbb{E}_{\mathcal{G}}Y = Y$.
- (v) If X is \mathcal{G} -measurable, then $\mathbb{E}_{\mathcal{G}}[XY] = X\mathbb{E}_{\mathcal{G}}Y$.

Property (i) states that the linearity of expectations is preserved under conditioning. Property (ii) is called the **law of iterated expectations**, and is shared by all projections. Property (v) is sometimes called **conditional determinism**, since X can be treated like a constant when it is pinned down by the information set.

The proofs of properties (i)–(v) are deferred to §11.3.4.

If $Y = (Y_1, \dots, Y_m)$ is a vector, then the conditional expectation of this vector is just the vector containing the conditional expectation of each element, similar to ordinary vector expectations. Thus, written as column vectors,

$$\mathbb{E}_{\mathcal{G}} \begin{pmatrix} Y_1 \\ \vdots \\ Y_m \end{pmatrix} = \begin{pmatrix} \mathbb{E}_{\mathcal{G}}Y_1 \\ \vdots \\ \mathbb{E}_{\mathcal{G}}Y_m \end{pmatrix}.$$

4.1.4 Sample Averages

To be added. LLN and CLT in vector space. Brief. Consider, as one application, the LAE for finite state Markov chains. Gives context for vector-valued LLN and CLT.

4.1.5 Distribution and Density Estimation

Create a preliminary discussion of ECDFs, KDEs, problems of estimating densities. Rewrite to include discussion of LAE, some brief surrounding discussion of variance reduction and Rao-Blackwellization.

Rewrite intro. The focus is on estimating CDFs and densities. Wealth dynamics are just an application. Some of the references are for later in the text and need to be directed back to the introduction to wealth dynamics in §2.2.1.

Let's continue our discussion of the model of household wealth with financial risk, as defined in §7.2.6.2. Our aim is to learn more about the distributions of this model in terms of both short and long run dynamics. We turn now to numerical methods, which provide crucial insights when the distributions produced by the model cannot be determined analytically. Since this is true of almost every interesting model, and since distributions have rightfully regained their place at the center of macroeconomic modeling in recent years, we will spend some time considering optimal numerical methods for tracking distributions.

4.1.5.1 Overview

We have in (7.66) a law of motion for the marginal densities of wealth (which coincide with the cross-sectional distributions over the set of households in our model). At the same time, this updating rule involves integrals that are not analytically tractable. Moreover, even if we can easily evaluate ψ_t and are able to apply numerical integration to evaluate

$$\psi_{t+1}(w') = \int \int \varphi(y - zs(w)) \nu(dz) \psi_t(w) dw$$

at some given w' , this tells us the value of ψ_{t+1} at only one point. To obtain a reasonable picture of ψ_{t+1} we would have to evaluate it at many points. Then, if we wanted to continue on to ψ_{t+2} , we would have to apply some form of interpolation or approximation so that updating via integration could proceed.

Apart from the fact that this is time consuming and tedious to code, errors creep into our calculation each time we approximate the distributions, and these errors compound as we iterate forwards.

Fortunately there are faster, more efficient ways to compute ψ_t at any t using Monte Carlo. The starting point is to generate m independent draws of w_t using algorithm 7. Once we have a set of independent draws of a random variable, we can estimate its distribution—or quantities that depend on its distribution—using one of several alternative methods.

Before we go on and describe them, here is a short aside: You might occasionally hear people dismiss Monte Carlo methods as “slow” but this point of view is dated. The main reason Monte Carlo algorithm such as algorithm 7 are thought of by some researchers as slow is that they involve explicit loops, so the routine is not easily **vectorized**; that is, not easily passed out as a single array processing operation to specialized machine code. Loops are indeed slow in most high level computing environments because the interpreter is forced to implement type checking at each arithmetic operation, and because data tends to be more dispersed across memory space.

However, we live in the age of just-in-time (JIT) compilation, where the best scientific computing environments deliver generation of specialized and highly optimized machine code on the fly. With a state of the art JIT compiler, loops such as those in algorithm 7 can easily run as fast as when hand crafted in Fortran or C.

Second, Monte Carlo algorithms tend to be highly parallelizable. For example, algorithm 7 can be parallelized at the outer loop, with one sample path running forward in time along a single thread within a multi-core machine. With JIT engines such as **Numba**, parallelization can be implemented at the same time that the routine is compiled into efficient machine code by indicating the for loop at which point division should occur. GPUs are another option, the potential of which for Monte Carlo simulations is yet to be widely exploited.

Third, Monte Carlo is less sensitive to the curse of dimensionality than popular alternatives such as discretization. So it is precisely for hard, high dimensional problems that Monte Carlo shines.

4.1.5.2 The Empirical Distribution

Given our sample $\{w_t^m\}$ generated by algorithm 7 and our desire to track the marginal distributions $\{\psi_t\}$ generated by the dynamical system (\mathcal{D}, Π) , one option is to compute the empirical distribution

$$F_t^m(x) := \frac{1}{m} \sum_{i=1}^m \mathbb{1}\{w_t^i \leq x\} \quad (4.7)$$

The empirical distribution is an estimator of the cumulative distribution function corresponding to the time t marginal ψ_t , which we denote by Ψ_t . It is an **unbiased**

estimator for Ψ_t , in that its expectation at x is equal to $\Psi_t(x)$. Indeed, given that each w_t^i is, by construction, an independent draw from ψ_t ,

$$\mathbb{E}[F_t^m(x)] = \frac{1}{m} \sum_{i=1}^m \mathbb{E}[\mathbb{1}\{w_t^i \leq x\}] = \frac{1}{m} m \mathbb{P}\{w_t \leq x\} = \Psi_t(x) \quad (4.8)$$

The first equality is by linearity of expectations and the second uses the fact that the expectation of an indicator of an event is equal to its probability (see, e.g., (11.10) on page 336).

In terms of asymptotics, the strong law of large numbers (SLLN) applied to (4.7) yields

$$\lim_{m \rightarrow \infty} F_t^m(x) = \mathbb{E}[\mathbb{1}\{w_t^i \leq x\}] = \mathbb{P}\{w_t \leq x\} = \Psi_t(x) \quad (4.9)$$

with probability one.

While (4.9) already indicates convergence at each point, it turns out that we can state more. Most importantly, it is also true that, with $\|\cdot\|_\infty$ equal to the supremum norm, we have

$$\lim_{m \rightarrow \infty} \|F_t^m - \Psi_t\|_\infty = 0 \quad (4.10)$$

with probability one. In other words, the function F_t^m converges to the function Ψ_t uniformly with probability one. This result is called the Glivenko–Cantelli theorem. It recognizes that the problem is one of computing a function, and that our estimate is therefore a random function. Thus, (4.10) is an example of a law of large numbers in function space—and perhaps the most famous one.

Figure 4.1 shows the estimate F_t^m for our model of wealth dynamics when the initial distribution ψ_0 is set to a point mass at 200. The model in question is the dynamic model for wealth in algorithm 7, previously stated in (7.65) on page 255. The parameters are the same as those used to produce figures 7.10–7.11. The value of m is 5,000 and the values of t are as shown in the plot. The plot indicates that, given our initial condition, probability mass shifts to the left as time goes by.

Incidentally, the initial condition that we used here was a point mass. But our dynamical system (\mathcal{D}, Π) is defined on density space. How does that work? The answer is that this particular operator Π maps any distribution on \mathbb{R}_+ to a density, and thereafter the trajectory evolves in density space. The reason is that, since $w_{t+1} = R_{t+1}s(w_t) + y_{t+1}$ and the conditional distribution $\pi(w, w')$ of w_{t+1} given $w_t = w$ is a density, even if w_t puts mass on points, the distribution of w_{t+1} will be absolutely continuous.¹

¹The formal argument here using measure-theoretic notation is that the distribution of $R_{t+1}s(w_t) + y_{t+1}$ when w_t has arbitrary distribution ψ_t is, by the law of total probability $\psi_{t+1}(B) =$

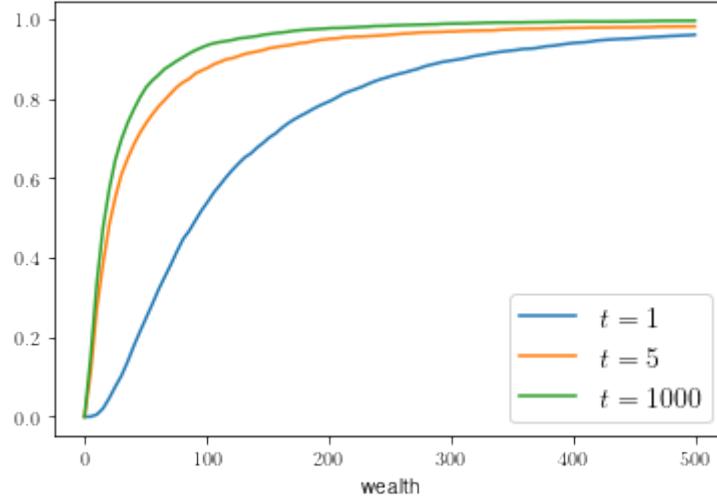


Figure 4.1: The empirical distribution F_m^t for different values of t

How accurate are the estimates in figure 4.1? In particular, how accurate is F_t^m as a measure of the true distribution Ψ_t , given that we have set $m = 5,000$?

One way we can answer this question rigorously is to use the Dvoretzky–Kiefer–Wolfowitz (DKW) inequality, which states that, for all $m \in \mathbb{N}$ and all $\varepsilon > 0$,

$$\mathbb{P} \{ \|F_t^m - \Psi_t\|_\infty > \varepsilon \} \leq 2 \exp(-2m\varepsilon^2) \quad (4.11)$$

From this inequality one can construct $1 - \alpha$ confidence intervals for Ψ_t . Reversing (4.11) gives

$$\mathbb{P} \{ |F_t^m(x) - \Psi_t(x)| \leq \varepsilon \text{ for all } x \} = \mathbb{P} \{ \|F_t^m - \Psi_t\|_\infty \leq \varepsilon \} \geq 1 - 2 \exp(-2m\varepsilon^2)$$

Now setting the bounding term $2 \exp(-2m\varepsilon^2)$ equal to the desired level of confidence α leads to the probability $1 - \alpha$ confidence band

$$F_t^m(x) - c(\alpha, m) \leq \Psi_t(x) \leq F_t^m(x) + c(\alpha, m) \text{ for all } x$$

where

$$c(\alpha, m) := \sqrt{\frac{\ln(\alpha/2)}{2m}}$$

Figure 4.2 shows the empirical distribution and these bands when under the same conditions as before, with $t = 100$, $\alpha = 0.01$ and $m = 5000$. The interpretation is that

$\int \int_B \pi(w, w') dw' \psi_t(dw)$. If B has Lebesgue measure zero then $\int_B \pi(w, w') dw' = 0$, so $\psi_{t+1}(B) = 0$. In particular, ψ_{t+1} is absolutely continuous with respect to Lebesgue measure.

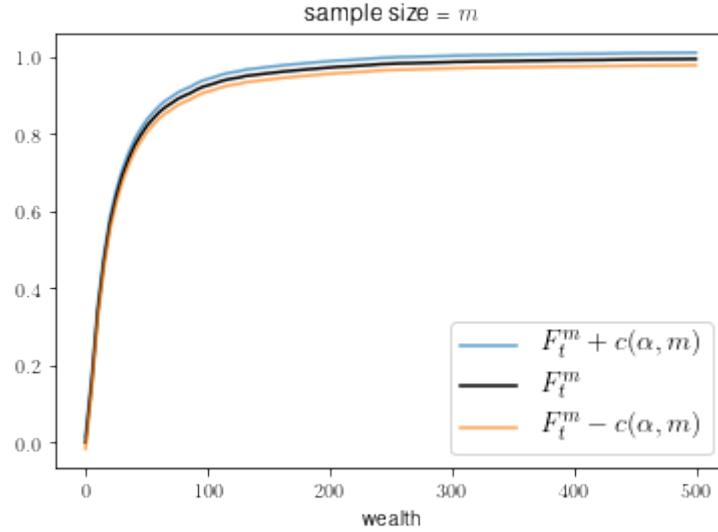


Figure 4.2: DKW bands for the empirical distribution when $\alpha = 0.01$ and $m = 5,000$

the true model distribution Ψ_t lies entirely within these bands with 99% probability.

In fact these bands are pessimistic: the true distribution would most likely be indistinguishable from the empirical distribution if we could plot both. The DKW inequality is necessarily pessimistic, since it has to hold for *any* underlying distribution, and in finite samples rather than asymptotically.

4.1.5.3 Estimating Densities

Empirical distributions have attractive theoretical foundations, are simple to construct and are valid estimators regardless of whether the distribution that we are trying to estimate is absolutely continuous (i.e., can be represented by a density) or not.

In the present setting, however, we know that the target distribution $\psi\Pi^t$ *can* be represented by a density, and this is structure that we would like to exploit. While the present problem is relatively low dimensional, exploiting structure is vital for solving problems in high dimensions and, as such, it's a skill we would like to build. In addition, even for low dimensional problems, using available structure is important when the low dimensional problem is nested in a larger equilibrium problem, or when we try to extract information from the tails of the probability distribution. Finally, density estimates are visually far more informative than estimates of the cumulative distribution. So let's now switch to a density perspective.

Unfortunately there is no natural estimator of densities in a general setting that is on par with the empirical distribution. The reason is that the empirical distribution only

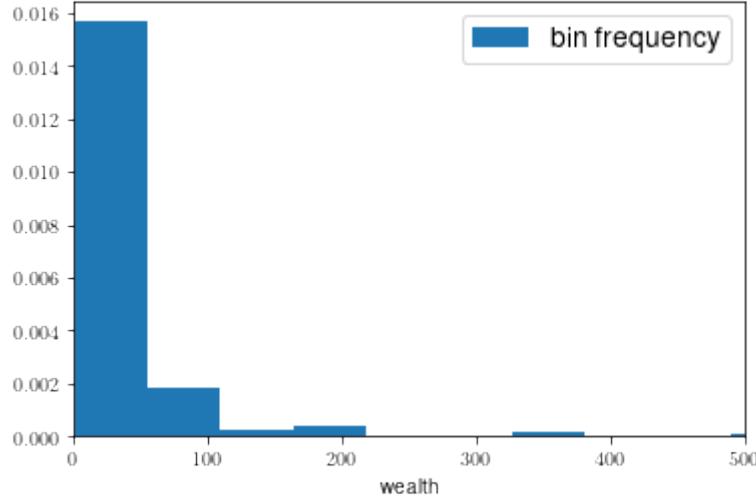


Figure 4.3: A histogram of ψ_t using sample $\{w_t^i\}_{i=1}^m$

reflects the sample, putting $1/m$ point mass on each of the m sampled observations w_t^m , whereas any density estimate must by its nature make statements about probability mass in the *neighborhood* of each observation. This isn't possible without imposing assumptions or some other structure.

One relatively naive option is to pack the data into bins and produce a histogram, as in figure 4.3. Here $t = 100$, $m = 5,000$, the initial condition is a point mass at 200 and parameters are as before. The histogram can be considered as a random function f_t^m estimating the density ψ_t . While this estimate is known to have good asymptotic properties, the rate of convergence is slow. Intuitively, this is because densities are usually smooth, and in this case we are trying to estimate a smooth function with a rough function containing jumps. The estimate will of course be particularly poor where there is little data, and this occurs in the tails, as seen in the right tail in figure 4.3.

An alternative option that does successfully exploit the fact that the target density is likely to be smooth is **nonparametric kernel density estimation**. In essence, this method replaces binning by taking a rescaled density K with total mass $1/m$, creating m instances of this density, centering one on each of the m data points and summing them up. The resulting formula is

$$\hat{f}_t^m(x) = \frac{1}{mh} \sum_{i=1}^m K\left(\frac{x - w_t^i}{h}\right)$$

which is a density in x for any realization of the sample.

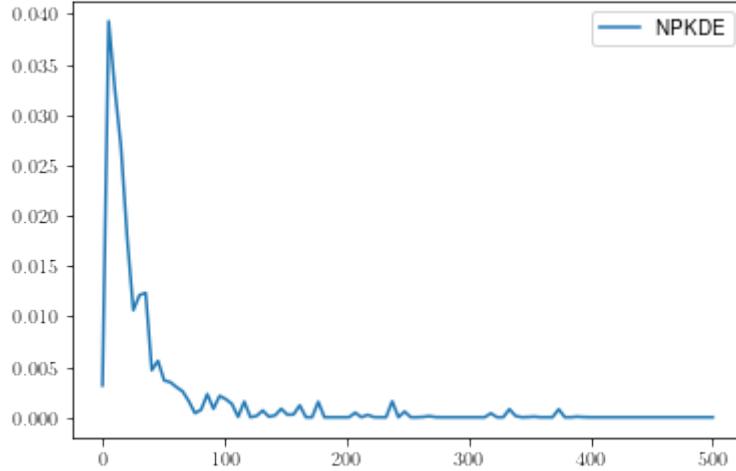


Figure 4.4: A nonparametric kernel density estimate of ψ_t

EXERCISE 1. Prove that $\int f_t^m(x) dx = 1$.²

The free parameter h is the standard deviation of the density K . In this context it is called the **bandwidth** of the estimator. If h is small then each of the “bumps” $(1/h)K((x - w_t^i)/h)$ will be strongly peaked about the data point w_t^i , and the sum—which equals the density estimate—will be relatively irregular. If h is large then each bump will be smooth, and so will the density estimate. The downside of smoothness is that we might be smoothing away actual features of the underlying distribution that we seek to estimate.

Figure 4.4 shows a nonparametric kernel density estimate of ψ_t when $t = 100$, $m = 500$ and, as before, the initial condition is a point mass at 200. The kernel density estimate uses a default implementation of the estimator from a popular statistical and machine learning library called **scikit-learn**. With our relatively small sample size, the estimate is rough and the estimate out in the tail looks spurious. This reflects the fact that while nonparametric kernel density estimators have strong asymptotic properties, there is no guarantee of good estimates in small samples. Even though we have imposed smoothness on our estimate, which is almost certainly helpful in this instance, we don’t actually know how much smoothness to impose. Moreover, the smoothness we impose in a nonparametric kernel density estimate induces bias, unlike the unbiased empirical distribution discussed above.

Fortunately, for the particular problem of estimating the density ψ_t , there is a better method, which is unbiased and produces smoothing needed to estimate the density ψ_t

²Hint: use a change-of-variable argument and the assumption that $\int K(x) dx = 1$.

without introducing a free parameter like the bandwidth. This is the so-called **look ahead estimator**, which is in the present case equal to

$$\ell_t^m(w') := \frac{1}{m} \sum_{i=1}^m \pi(w_{t-1}^i, w') \quad (4.12)$$

There are two noteworthy features of this estimator. First, we are using the conditional density from the model in question, which is in this case given by (7.67) on page 256. Thus the smoothing we are adding to form a density estimate is coming from the actual model, rather than an arbitrary nonparametric kernel. Second, the sample $\{w_{t-1}^i\}$ we insert into the look ahead density estimate is from time $t - 1$, even though the density we wish to estimate is ψ_t . To see why this makes sense, observe that

$$\mathbb{E}[\ell_t^m(w')] = \frac{1}{m} \sum_{i=1}^m \mathbb{E}[\pi(w_{t-1}^i, w')] = \frac{m}{m} \int \pi(w, w') \psi_{t-1}(w) dw = \psi_t(w')$$

Hence $\mathbb{E}[\ell_t^m(w')]$ is an unbiased estimate of ψ_t , the density we are targeting, at the point w' .

From the SLLN, we also have

$$\ell_t^m(w') \rightarrow \mathbb{E}[\pi(w_{t-1}^i, w')] = \psi_t(w') \quad \text{as } m \rightarrow \infty$$

with probability one. Moreover, similar to the Glivenko–Cantelli theorem, we can apply a SLLN in function space to obtain

$$\|\ell_t^m - \psi_t\|_1 \rightarrow 0 \quad \text{as } m \rightarrow \infty$$

with probability one. The rate of convergence can be shown to be faster than that of the nonparametric kernel density estimator too.

Figure 4.5 shows the look ahead estimator applied to the same sample used in figure 4.4. The estimate is considerably smoother. Also of note is the long right hand tail. This is suggestive of relatively high inequality in the wealth distribution, a point that we return to in §2.2.3.2.

4.1.6 Heavy Tails

Add some intro.

In §2.2.3 we considered dispersion of distributions, such as functions that represent inequality. Now we consider another aspect of distributions: the amount of probability

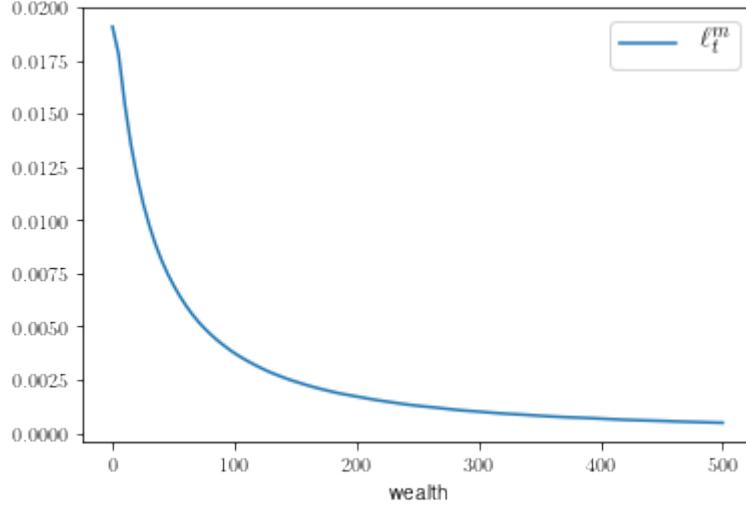


Figure 4.5: The look ahead estimate of ψ_t

mass in their tails. Not surprisingly, this is related to dispersion, and hence to measures such as the Gini coefficient.

As background, we note that most probability distributions used in classical statistics and the natural sciences have either bounded support or light tails. For example, the tails of the normal density go to zero like $\exp(-x^2)$ as $|x| \rightarrow \infty$, which is very fast. When a distribution is light-tailed, draws tend not to deviate too far from the mean. As a result, researchers tend to use rules of thumb such as “outcomes more than three standard deviations from the mean are rarely observed.”

However, many distributions encountered in economics have far more probability mass in the tails than distributions like the normal distribution. With such *heavy-tailed* distributions, what would be regarded as extreme outcomes for someone accustomed to thin tailed distributions occur relatively frequently. Examples of heavy-tailed distributions observed in economic and financial settings include income and wealth distributions (see, e.g., [Pareto \(1896\)](#), [Benhabib and Bisin \(2018\)](#)), firm size distributions ([Axtell \(2001\)](#), [Gabaix \(2016\)](#)), the distribution of returns on holding assets over short time horizons ([Mandelbrot \(1963\)](#), [Rachev \(2003\)](#)), and the distribution of city sizes ([Rozenfeld et al. \(2011\)](#), [Gabaix \(2016\)](#)).

These heavy tails turn out to be important for our understanding economic outcomes and their impact. For example:

- Tail risk has substantial impacts on asset prices (see, e.g., [Kelly and Jiang \(2014\)](#)) that in turn influence investment, saving, and other decisions.³

³[Taleb \(2007\)](#) gives an entertaining account of the impact of tail risk on financial markets.

- The heavy tail in the wealth distribution matters for taxation and redistribution policies, as well as for flow-on effects for productivity growth, business cycles, and political economy (see, e.g., Acemoglu and Robinson (2002), Glaeser et al. (2003), Brinca et al. (2016), Bhandari et al. (2018) or Ahn et al. (2018)).
- The heavy tail in the firm size distribution is connected to market concentration, as well as to the impact of firm-level shocks on the macroeconomy (see, e.g., Carvalho and Grassi (2019)).
- Many standard results in statistics and econometrics break down as tails become heavier. This is especially true for confidence intervals and hypothesis tests, which typically rely on finiteness of second moments. This can fail when tails are heavy. (See, e.g., Rachev (2003), Nishiyama et al. (2004) or Gabaix and Ibragimov (2011).)

4.1.6.1 Visual Comparison

A way to build intuition about the difference between light- and heavy-tailed distributions is to simulate independent draws from these two types of distributions and plot them. Figure 4.6 gives one illustration. It shows 120 independent draws from the Cauchy distribution, with density $f(x) = (\pi(1 + x^2))^{-1}$, in the bottom subfigure. This is a heavy-tailed distribution, as will be discussed in §4.1.6.5, along with an equal number of independent draws from the normal distribution in the top two subfigures.

In the top subfigure, the standard deviation of the normal distribution is 2, and the draws are clustered around the mean. In the middle subfigure, the standard deviation is increased to 25 and, as expected, the amount of dispersion rises. The bottom subfigure, with the Cauchy draws, shows a different pattern: tight clustering around the mean for the great majority of observations, combined with a few sudden large deviations from the mean.

To provide a comparison with some financial data, figure 4.7 shows daily change in the price of Amazon (AMZN) stock for the period 1st January 2015 to 1st November 2019, which equates to daily returns if we set dividends aside. Five of the 1217 observations are more than 5 standard deviations from the mean. Overall, the figure is suggestive of heavy tails, although not to the same degree as the Cauchy distribution in figure 4.6. If, however, one takes tick-by-tick data rather daily data, the heavy-tailedness of the distribution tends to increase. [Add figure—see commented out url.](#)

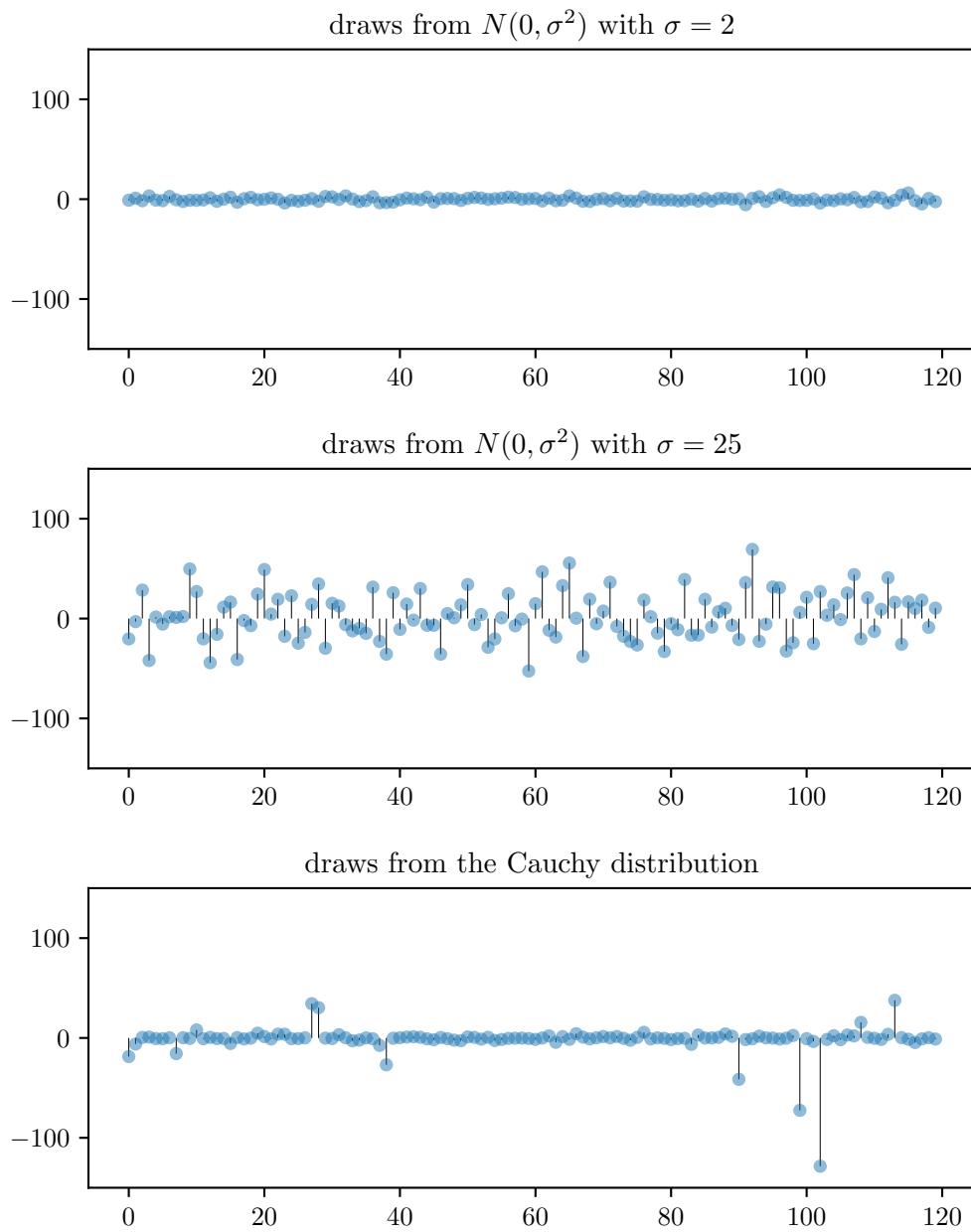


Figure 4.6: 120 independent draws from three distributions

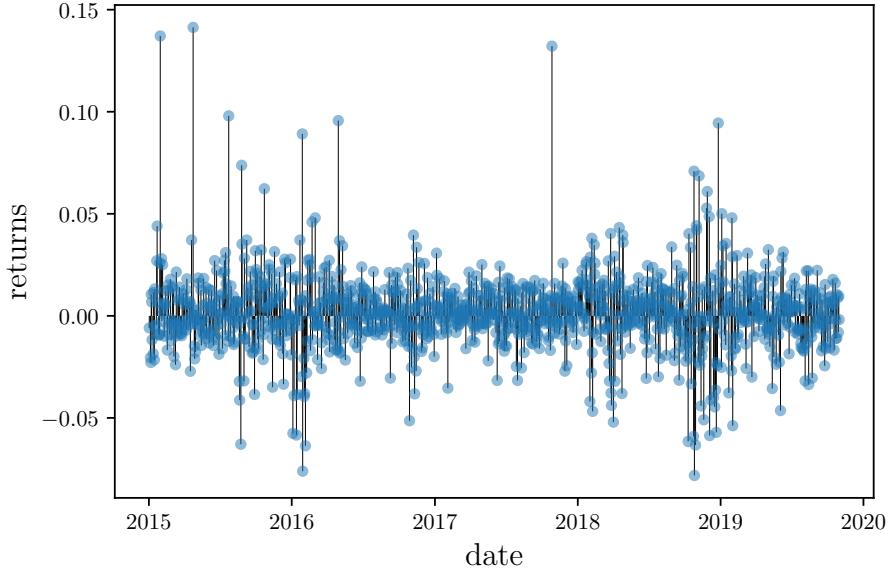


Figure 4.7: Daily returns on AMZN. Source: Yahoo finance

4.1.6.2 Pareto Tails and Rank-Size Plots

It turns out that many heavy-tailed distributions encountered in economics and finance have an extreme form of heaviness called a Pareto tail. We defer a formal definition until §4.1.6.5, but the main idea is that, when X is a random variable with distribution function F , we have, for suitably chosen $\alpha, c > 0$,

$$\mathbb{P}\{X > x\} \approx cx^{-\alpha} \text{ for large } x. \quad (4.13)$$

This means that the probability of exceeding x goes to zero relatively slowly.

(To illustrate why this would be considered slow, consider the exponential distribution. If Y is exponentially distributed with rate parameter $\alpha > 0$, then $\mathbb{P}\{Y > x\} = \exp(-\alpha x)$. The right hand side of this expression goes to zero at an exponential rate, which is much faster than the polynomial rate of the function $x \mapsto cx^{-\alpha}$.)

The rate at which $\mathbb{P}\{X > x\}$ converges to zero is slower for smaller α . Thus, a smaller “tail index” α is associated with a heavier tail.

A common graphical technique for investigating Pareto tails and power laws is the so-called **rank-size plot**. While it offers no formal test, the technique does provide a way to visualize data and some preliminary evidence for or against a Pareto tail.

The starting point for this plot is the following fact: if $\{X_i\}_{i=1}^n$ is a large collection

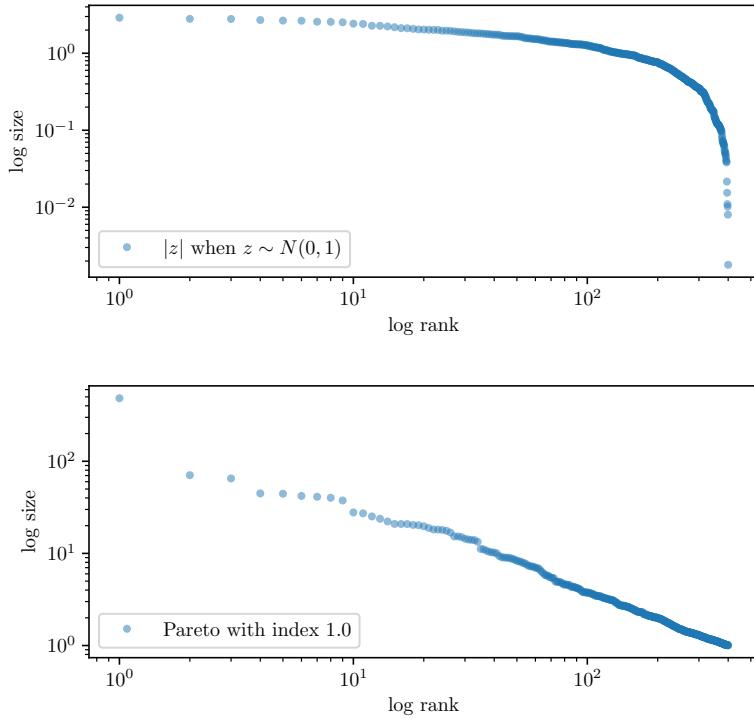


Figure 4.8: Rank-size plots for alternative distributions

of draws from a distribution with Pareto tail as in (4.13) and $X_{(i)}$ is the i -th largest value in the sample, then, for some $d > 0$, we have

$$\ln X_{(i)} \approx d - \alpha \ln i \quad \text{for } i = 1, \dots, n. \quad (4.14)$$

A proof can be found in proposition 1 of [Nishiyama et al. \(2004\)](#). The key implication is that, for a sufficiently large number of draws from a Pareto distribution, plotting size against rank (i.e., $X_{(i)}$ against i) in a log-log plot produces a straight line. For distributions with thinner tails, the data points are typically concave.

Figure 4.8 gives an example, showing the rank-size plot for 400 draws from the folded normal and Pareto distributions. The Pareto sample produces a straight line, as predicted, while the line produced by the other samples is concave.

To give an illustration with observed data, recall that the discussion above flagged the distribution of city sizes to be heavy-tailed. Figure 4.9 shows the city size distribution for the US in 2018, for all cities with population above 100,000. The top left

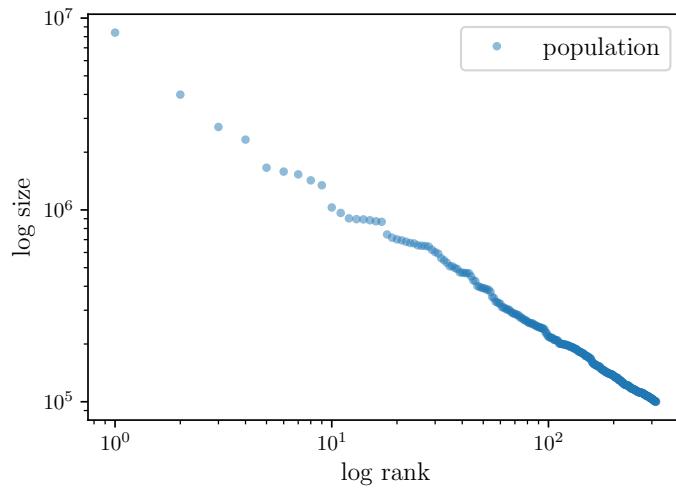


Figure 4.9: Rank-size plot for US city sizes in 2018 (Source: US Census Bureau)

data point is New York City. The rank size plot suggests that a Pareto distribution provides a good fit to the data.

Now Forbes 400, etc. Ask Shu for the plots.

4.1.6.3 What Causes Heavy Tails?

Before finishing this section, we pause to consider why heavy-tailed distributions are prevalent in economics and finance. While there are competing theories for every distribution with this property (income, wealth, firm size, etc.), it is interesting to speculate more broadly about why human behaviour and markets routinely generate these properties. Perhaps some form of self-reinforcing choice behavior plays a role?

For example, consider the number of citations received by a given scientific paper. Papers with many citations have higher visibility. As a result, they are likely to attract additional citations. When we look at the data, it turns out that observed citation counts are consistent with power laws (Redner (1998), Clauset et al. (2009)).⁴

One can imagine similar phenomena at work in other social, economic, and financial systems. Large cities offer low matching frictions for work and other opportunities,

⁴Mathematically, this is connected to the degree distribution of random graphs, which is the distribution formed by counting the number of connections to each node for every node in the network. If (a) the graph is formed by attaching connections to a given set of nodes via some stochastic mechanism and (b) those nodes that already have many attachments are preferred in the selection process, then a power law over the degree distribution often arises. See, for example, Barabási and Albert (1999), which is itself far out in the right hand tail of the distribution of citations!

and hence attract migrants. These migrants further reduce matching frictions, so the attraction is reinforced. Popular books attract new readers through word of mouth or recommendations, and these new readers spread the word again. City sizes and book sales data are also consistent with power laws ([Gabaix and Ioannides \(2004\)](#), [Clauset et al. \(2009\)](#)). Similarly, the fact that returns on assets exhibit power laws in high frequency data raises the possibility that, over short time horizons, buy and sell decisions are often governed by herding behavior (see, e.g., [Lux and Alfarano \(2016\)](#)).⁵

Although we will not cover all of these phenomena, we take from them the message that, while nonlinearities (e.g., self-reinforcing systems) make modeling harder and more technically demanding, they cannot be avoided if we wish accurately to describe certain economic outcomes. We saw some evidence of this even in the simple model of wealth dynamics considered earlier in this chapter: the model with stochastic financial returns generates a more accurate representation of the wealth distribution than the linear version described in §2.2.1.2, due to the compounding effect on wealth of the multiplicative stochastic component $\{r_t\}$. The next section elaborates on this point.

4.1.6.4 Heavy Tails in Income and Wealth

As mentioned above, heavy tails are observed in income and wealth distributions. [A less visual discussion, focusing on evidence in the literature of Pareto tails, size of tail indices.](#)

Pareto tails are indicative a parametric regularity of a kind that we see less often in economics than in fields like physics or chemistry. For this reason they might seem either surprising or spurious to some readers. However, it turns out that Pareto tails are a natural consequence of the model of wealth dynamics presented above, provided that returns to holding assets are stochastic. This can be shown using some striking results due to [Kesten \(1973\)](#) and [Goldie \(1991\)](#). We treat these topics in depth in [add ref.](#)

[Note difference between wealth and income distributions, in terms of Pareto tail index. This matches differences in Gini's discussed above. More discussion can be found in Benhabib and Bisin \(2018\) and de Vries and Toda \(2020\).](#)

The difference between Pareto tail indices in the income and wealth distributions is interesting for a number of reasons, one of them being that it helps us to identify accurate models. A model of consumption and saving might aim to reproduce these regularities. We return to this issue in §7.2.6.

⁵For an overview of power law behavior in economic data see [Gabaix \(2016\)](#). Another nice survey can be found in [Mitzenmacher \(2004\)](#).

4.1.6.5 Classifying Tail Properties

Now we turn to more precise definitions about tail properties. We will focus our attention on the right tails of nonnegative random variables and their distributions. Definitions for left tails are very similar; we omit them to simplify the exposition.

While terminology varies across the literature, we follow Foss et al. (2011) in defining a distribution F on \mathbb{R}_+ to be **heavy-tailed** if

$$\int_0^\infty e^{tx} F(dx) = \infty \text{ for all } t > 0. \quad (4.15)$$

We say that a nonnegative random variable X is **heavy-tailed** if its distribution $F(x) := \mathbb{P}\{X \leq x\}$ is heavy-tailed. This is equivalent to stating that its **moment generating function** $m(t) := \mathbb{E} \exp(tX)$ is infinite for all $t > 0$.

Example 4.1.1. If X is lognormal then its moment generating function is infinite everywhere on $(0, \infty)$. Hence X is heavy-tailed.

A distribution F on \mathbb{R}_+ is called **light-tailed** if it is not heavy-tailed. A nonnegative random variable X is **light-tailed** if its distribution F is light-tailed.

Example 4.1.2. It is immediate from the definition that every random variable with bounded support is light-tailed. (Why?)

Example 4.1.3. If X has the exponential distribution, with cdf $F(x) = 1 - \exp(-\lambda x)$ for some $\lambda > 0$, then its moment generating function is finite whenever $t < \lambda$, taking the form $m(t) = \lambda/(\lambda - t)$. Hence X is light-tailed.

Lemma 4.1.2. *If X is light-tailed, then all of its moments are finite.*

Proof. Pick any $r > 0$. We will show that $\mathbb{E}X^r < \infty$. Since X is light-tailed, there exists a $t > 0$ such that $m(t) = \mathbb{E} \exp(tx) < \infty$. For a sufficiently large constant \bar{x} we have $\exp(tx) \geq x^r$ whenever $x \geq \bar{x}$. As a consequence, with F as the distribution of X , we have

$$\mathbb{E}X^r = \int_0^{\bar{x}} x^r F(dx) + \int_{\bar{x}}^\infty x^r F(dx) \leq \bar{x}^r + m(t) < \infty. \quad \square$$

From the contrapositive of lemma 4.1.2, a sufficient condition for heavy-tailedness is that at least one moment is infinite. This condition is not necessary, however. For example, the lognormal distribution is heavy-tailed but every moment is finite.

One specific class of heavy-tailed distributions has been found repeatedly in economic and social data: power laws. These are distributions that have polynomial decay rates for their tails. More specifically, given $\alpha > 0$, a nonnegative random variable X is said to have a **Pareto tail** with **tail index** α if

$$\lim_{x \rightarrow \infty} x^\alpha \mathbb{P}\{X > x\} = c. \quad (4.16)$$

Evidently (4.16) implies the existence of positive constants b and \bar{x} such that $\mathbb{P}\{X > x\} \geq bx^{-\alpha}$ whenever $x \geq \bar{x}$. The implication is that $\mathbb{P}\{X > x\}$ converges to zero no faster than $x^{-\alpha}$. This polynomial decay rate is much slower than, say, the exponential decay rate of the normal distribution. In some sources, a random variable obeying (4.16) is said to have a **power law tail**.⁶

Example 4.1.4. The **Pareto distribution** on \mathbb{R}_+ has a cdf of the form

$$F(x) = \begin{cases} 1 - (\bar{x}/x)^\alpha & \text{if } x \geq \bar{x} \\ 0 & \text{if } x < \bar{x} \end{cases} \quad (4.17)$$

for some positive constants \bar{x} and α . It is easy to see that if $X \sim F$, then $\mathbb{P}\{X > x\}$ satisfies (4.16). Thus, in line with the terminology, a Pareto distributed random variables have a Pareto tail. Figure 4.10 shows 120 independent draws from Pareto distributions with three choices of α and \bar{x} set to 1.0.

Lemma 4.1.3. *If X has a Pareto tail with tail index α , then $\mathbb{E}[X^r] = \infty$ for all $r \geq \alpha$.*

Proof. Let X have a Pareto tail with tail index α and let F be its cdf. Fix $r \geq \alpha$. As discussed after (4.16), we can take positive constants b and \bar{x} such that $\mathbb{P}\{X > x\} \geq bx^{-\alpha}$ whenever $x \geq \bar{x}$. Using (4.5) we have

$$\mathbb{E}X^r = r \int_0^\infty x^{r-1} \mathbb{P}\{X > x\} dx \geq r \int_0^{\bar{x}} x^{r-1} \mathbb{P}\{X > x\} dx + r \int_{\bar{x}}^\infty x^{r-1} bx^{-\alpha} dx.$$

But $\int_{\bar{x}}^\infty x^{r-\alpha-1} dx = \infty$ whenever $r - \alpha - 1 \geq -1$. Since $r \geq \alpha$, we have $\mathbb{E}X^r = \infty$. \square

⁶The terminology surrounding power laws is far from settled. Sometimes a random variable X satisfying (4.16) is said to have **fat tails**. The tail index α in (4.16) is also referred to as the **Pareto exponent**.

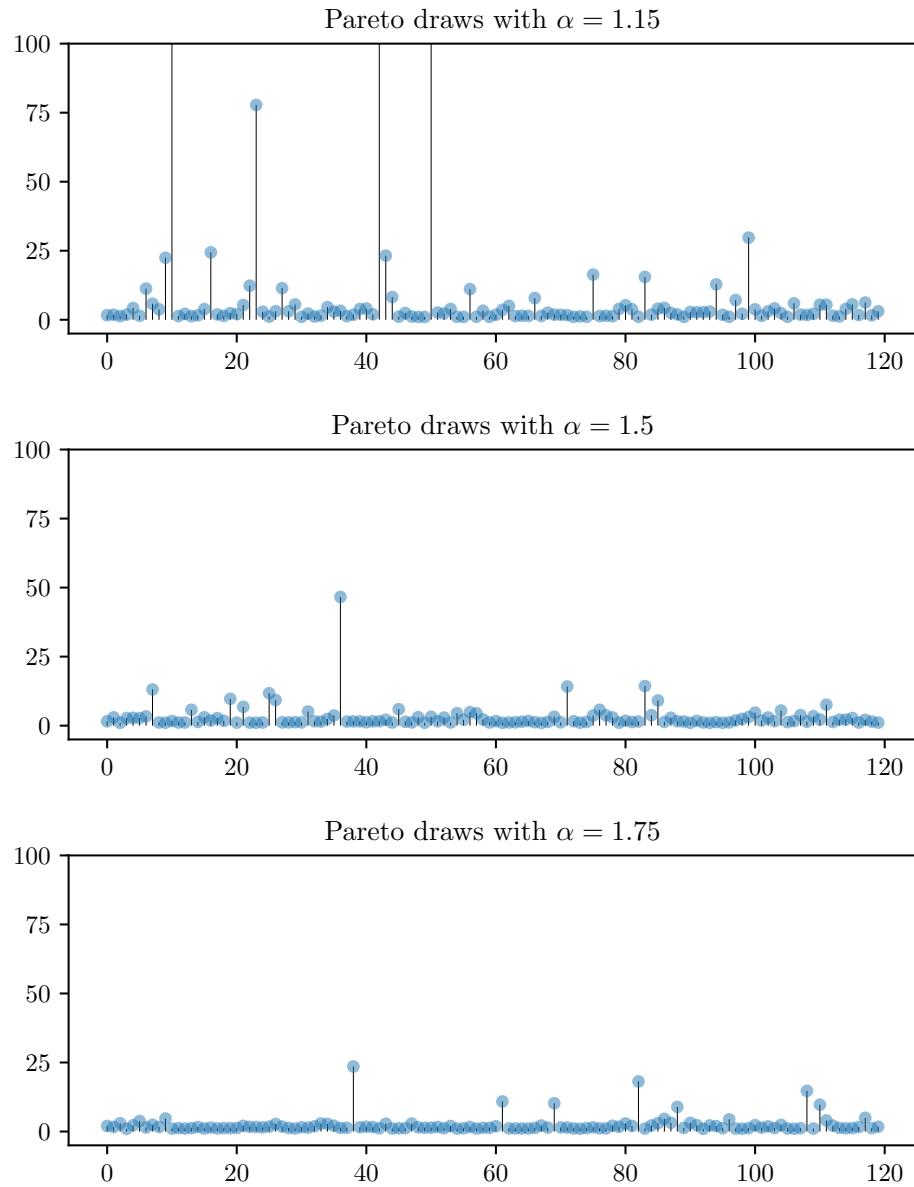


Figure 4.10: 120 independent draws from three Pareto distributions

4.1.6.6 Application: Diversification

To give one example of a setting where heavy tails matter, consider the common idea that diversification reduces risk when comparing portfolios of assets.⁷ It has, however, been pointed out that if the returns distribution of the assets in question have heavy tails, then diversification can have little impact on risk, or even accentuate it. [Ibragimov and Walden \(2007\)](#) give a detailed exposition. We can illustrate the key ideas in a simple setting.

Consider an investor with one dollar of wealth and a choice over n assets with payoffs X_1, \dots, X_n . Suppose that returns on distinct assets are independent and each return has mean μ and variance σ^2 . If the investor puts all wealth in one asset, say, then the expected payoff of the portfolio is μ and the variance is σ^2 . If instead the investor puts share $1/n$ of her wealth in each asset, then the portfolio payoff is $Y_n := (1/n) \sum_{i=1}^n X_i$. The mean is unchanged at μ , while the variance of the portfolio has fallen to σ^2/n . The second statement implies that the standard deviation of the portfolio goes to zero like $n^{-1/2}$. Diversification reduces risk, as expected.

There is, however, one crucial assumption here, which is that the variance is finite. Figure 4.11 illustrates why this matters by showing how simulated outcomes of Y_n vary with n for two different cases: returns with finite variance and infinite variance. In the top subfigure, each X_i is standard normal, so Y_n has distribution $N(0, 1/n)$. The subfigure shows 40 simulated observations of Y_n for a range of n values. For low values of n , the observations are relatively dispersed, while for high values they are concentrated. This is the impact of diversification on risk.

The bottom subfigure repeats the exercise, but now each X_i is t -distributed with 1.2 degrees of freedom (implying infinite variance). Outcomes for Y_n remain dispersed as n increases. The figure illustrates the fact that heavy tails can significantly reduce the rate at which risk falls with diversification.

4.1.6.7 Application: Forecasting

Not surprisingly in light of the above discussion, heavy tails make forecasting the future from historical data more challenging. Just as in the discussion of diversification above, light tails mean finite variances, so if X_1, \dots, X_n are independent draws from such a distribution, then the standard deviation of the sample mean is σ/\sqrt{n} , where σ is the common standard deviation of each X_i . Hence a standard deviation of a sample mean converges to a population mean relatively quickly. Thus, a sample mean of reasonable

⁷See [Markowitz \(1991\)](#) for an overview of Nobel prize winning research in this area. Consider adding a pointer to the quantecon lecture on the Black-Litterman article.

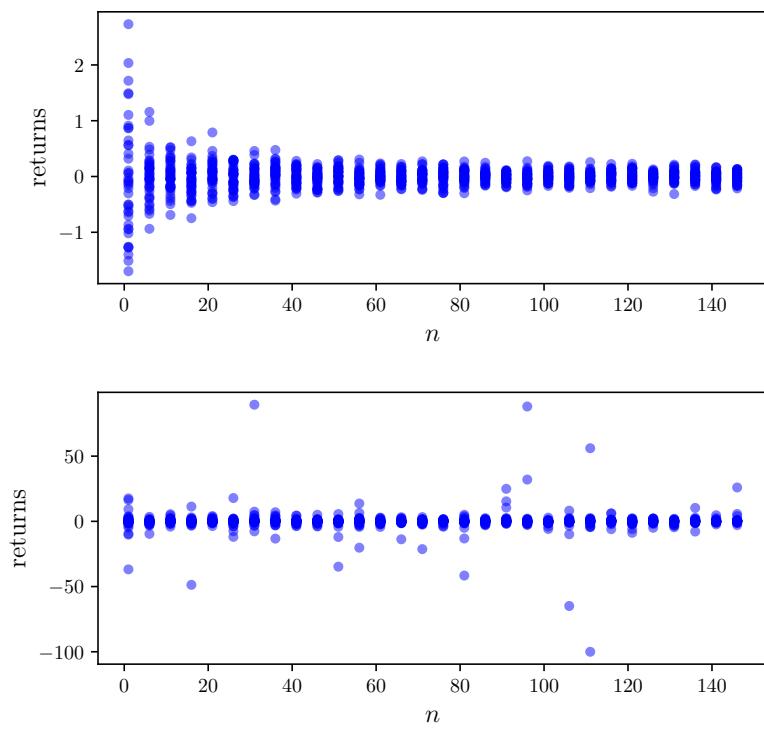


Figure 4.11: Portfolio returns vs portfolio size with light and heavy tails

size will form a good predictor of a new draw from the distribution. But that will not be as true when an underlying distribution is heavy tailed.

4.1.7 Granular Origins of Aggregate Fluctuations

Add a simple version of Gabaix's model, as an illustration of heavy tails and the LLN. Refer also to Nirei's work.

4.2 Linear Dynamics

Next we discuss linear and conditionally linear dynamics. In these settings we can be relatively precise about long run outcomes, even with random components and a continuous state space. At the same time, these workhorse models can generate very flexible outcomes, particularly in high dimensions.

Roadmap to be added.

4.2.1 Martingales

Stochastic models are often pieced together from elementary components, such as IID innovations. Another such building block is martingales. To define them, we need the notion of a **filtration**, which is a sequence of information sets $\{\mathcal{G}_t\}_{t \geq 0}$ increasing in the sense of set inclusion, so that $\mathcal{G}_t \subset \mathcal{G}_{t+1}$ for all t . For example, if $\{\xi_t\}_{t \geq 0}$ is a stochastic process, then the set of information sets $\{\mathcal{G}_t\}_{t \geq 0}$ defined by $\mathcal{G}_t = \{\xi_0, \dots, \xi_t\}$ is a filtration. We call this the **filtration generated by $\{\xi_t\}_{t \geq 0}$** .

A stochastic process $\{\eta_t\}$ is said to be **adapted** to filtration \mathcal{G}_t if η_t is \mathcal{G}_t -measurable for all t .

A stochastic process $\{w_t\}_{t \geq 1}$ taking values in \mathbb{R}^n is called a **martingale** with respect to a filtration $\{\mathcal{G}_t\}$ if it is adapted to $\{\mathcal{G}_t\}$, if $\mathbb{E}\|w_t\|_1 < \infty$ and if

$$\mathbb{E}[w_{t+1} | \mathcal{G}_t] = w_t, \quad \forall t \geq 1$$

In other words, our best forecast of next period's value is the current value.

Example 4.2.1. Consider a scalar **random walk**, which is a sequence $\{w_t\}$ of the form

$$w_t = \sum_{i=1}^t \xi_i, \quad \{\xi_t\} \text{ is IID with } \mathbb{E}[\xi_t] = 0$$

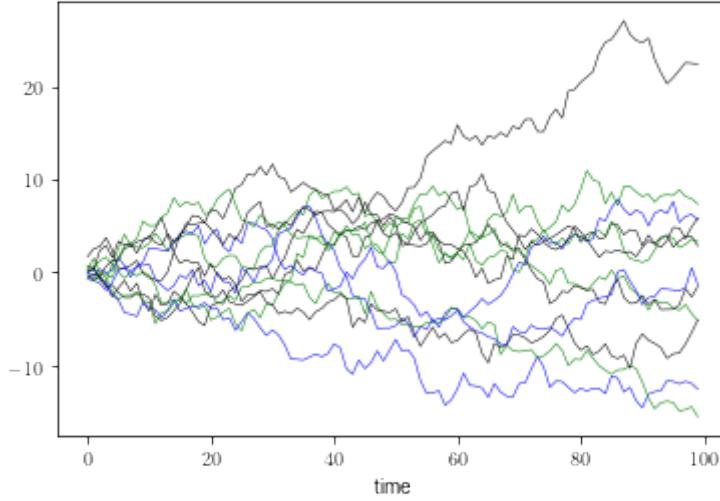


Figure 4.12: Twelve realizations of a random walk

For example, w_t might be a player's wealth over a sequence of fair gambles. Figure 4.12 shows 12 realizations of a random walk when $\{\xi_t\}$ is standard normal.

This process is a martingale with respect to the filtration generated by $\{\xi_t\}$. Observe that

$$\mathbb{E}[w_{t+1} | \mathcal{G}_t] = \mathbb{E}[w_t + \xi_{t+1} | \mathcal{G}_t] = \mathbb{E}[w_t | \mathcal{G}_t] + \mathbb{E}[\xi_{t+1} | \mathcal{G}_t]$$

But $\mathbb{E}[w_t | \mathcal{G}_t] = w_t$ because $w_t = \sum_{i=1}^t \xi_i$ is \mathcal{G}_t -measurable and $\mathbb{E}[\xi_{t+1} | \mathcal{G}_t] = \mathbb{E}[\xi_{t+1}] = 0$ by independence and the zero mean assumption on ξ_{t+1} . The martingale property now follows.

EXERCISE 2. Consider the sequence $\{w_t\}$ defined by

$$w_t = \prod_{i=1}^t \xi_i, \quad \{\xi_t\} \text{ is IID with } \mathbb{E}[\xi_t] = 1$$

Show that this process is a martingale with respect to the filtration generated by $\{\xi_t\}$.

A stochastic process $\{w_t\}_{t \geq 1}$ in \mathbb{R}^n is called a **martingale difference sequence** (or **MDS**) with respect to a filtration $\{\mathcal{G}_t\}$ if $\{w_t\}_{t \geq 1}$ is adapted to $\{\mathcal{G}_t\}$, if $\mathbb{E}\|w_t\|_1 < \infty$ and if

$$\mathbb{E}[w_{t+1} | \mathcal{G}_t] = 0, \quad \forall t \geq 1.$$

For example, if $\{v_t\}$ is a martingale with respect to $\{\mathcal{G}_t\}$ then the first difference

$w_t := v_t - v_{t-1}$ is an MDS with respect to $\{\mathcal{G}_t\}$, since for any t ,

$$\mathbb{E}[w_{t+1} | \mathcal{G}_t] = \mathbb{E}[v_{t+1} - v_t | \mathcal{G}_t] = \mathbb{E}[v_{t+1} | \mathcal{G}_t] - \mathbb{E}[v_t | \mathcal{G}_t] = v_t - v_t = 0$$

An MDS is a generalization of the idea of a zero mean IID sequence, and is often used in economics and related fields to represent the idea of an “unpredictable” sequence. To see that it is a generalization, suppose that $\{w_t\}$ is IID with $\mathbb{E}[w_t] = 0$. Then $\{w_t\}$ is an MDS with respect to the **natural filtration**, which is the filtration generated by itself. This follows from independence, since, with $\mathcal{G}_t = \{w_1, \dots, w_t\}$, we have $\mathbb{E}[w_{t+1} | \mathcal{G}_t] = \mathbb{E}[w_{t+1}]$ for all t . The conclusion follows.

EXERCISE 3. Show that if $\{w_t\}$ is an MDS with respect to some filtration $\{\mathcal{G}_t\}$, then $\mathbb{E}[w_t] = 0$ for all t .

EXERCISE 4. Show that if $\{w_t\}$ is an MDS with respect to $\{\mathcal{G}_t\}$, then w_s and w_t are **orthogonal**, in the sense that $\mathbb{E}[w_s w'_t] = 0$ whenever $s \neq t$.

4.2.2 Vector Autoregressions

Vector autoregressions are routinely deployed in economic applications, particularly in macroeconomics.

The discussion immediately below is of general interest and needs to be edited and shifted to the intro or somewhere else more suitable.

Even if you are completely convinced that economic processes are highly nonlinear and these nonlinearities cannot be ignored, you should still study linear systems. The reason is that many nonlinear models can be mapped into linear systems—typically at the cost of higher dimensionality—thereby opening a new line of analysis or estimation.⁸ Moreover, linear models are often used as building blocks for more complex models, where they simplify some parts of an overall system that might contain nonlinearities.

4.2.2.1 The VAR Model

To begin, let’s replace the deterministic system 5.25 with the **first order vector autoregression**

$$x_{t+1} = Ax_t + b + C\xi_{t+1} \tag{4.18}$$

⁸If you would like to look ahead to an example, consider the operator Π in (7.64). This is a linear operator in function space, even though the underlying system is, in general, nonlinear.

where

- $\mathcal{G}_t = \{x_0, \xi_0, \xi_1, \dots, \xi_t\}$
- $\{\xi_t\}_{t \geq 1}$ is an \mathbb{R}^j -valued MDS with respect to \mathcal{G}_t satisfying

$$\mathbb{E}[\xi_t] = \begin{pmatrix} \mathbb{E}\xi_{1t} \\ \mathbb{E}\xi_{2t} \\ \vdots \\ \mathbb{E}\xi_{jt} \end{pmatrix} = 0 \text{ and } \mathbb{E}[\xi_t \xi_t'] = I$$

Here 0 is a $j \times 1$ vector of zeros and I is the $j \times j$ identity matrix. The zero mean and zero covariance restrictions in fact follow from the MDS assumption—see §4.2.1. The only restriction added in the preceding display is that the second moments along the principle diagonal of $\mathbb{E}[\xi_t \xi_t']$ are unity.

EXERCISE 5. Show that, under the stated assumptions, $\mathbb{E}[x_t \xi_{t+1}'] = 0$.

When we study a system such as (4.18), there are two kinds of questions that usually arise. One is the dynamics of the **sample paths** $\{x_t\}_{t \geq 0}$ across realizations of uncertainty (and, in particular, realizations of the shock sequence $\{\xi_t\}$). The second is the dynamics of the *distributions* of each random vector x_t . We'll start with the second question, at first confining our attention to dynamics of the first two moments:

- $\mu_t := \mathbb{E}[x_t]$ and
- $\Sigma_t := \text{Var}[x_t] := \mathbb{E}[(x_t - \mu_t)(x_t - \mu_t)']$

In doing so we will take x_0 to be a given random vector that is independent of the sequence $\{\xi_t\}$ and has finite first moments μ_0 and Σ_0 . The $n \times n$ matrix Σ_t is called the **variance-covariance matrix** of x_t .

Starting with the vector mean sequence $\{\mu_t\}$, we can take expectations on both sides of (4.18) to obtain

$$\mu_{t+1} = A\mu_t + b \tag{4.19}$$

It's immediate from proposition 2.3.3 on page 39 that for (4.19) we have

$$r(A) < 1 \implies \text{global stability with unique fixed point } \mu^* = \sum_{i=0}^{\infty} A^i b$$

Figure 4.13 shows convergence of the mean (and of the entire distribution) when $n = j = 1$, $A = a$ and the sequence $\{\xi_t\}$ is IID and standard normal.

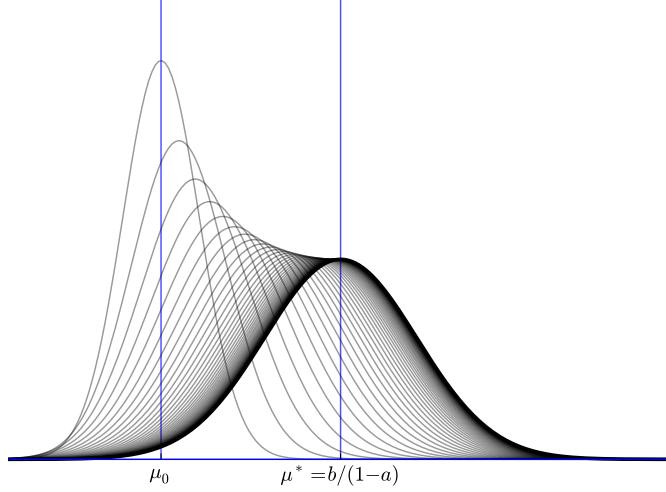


Figure 4.13: Convergence of μ_t to μ^* in the scalar model

Next we seek a law of motion analogous to (4.19) for the matrix sequence $\{\Sigma_t\}$. By definition,

$$\begin{aligned}\Sigma_{t+1} &= \mathbb{E}[(x_{t+1} - \mu_{t+1})(x_{t+1} - \mu_{t+1})'] \\ &= \mathbb{E}[(A(x_t - \mu_t) + C\xi_{t+1})(A(x_t - \mu_t) + C\xi_{t+1})']\end{aligned}$$

Expanding out the last expression and using the fact that

$$\mathbb{E}[A(x_t - \mu_t)\xi'_{t+1}C'] = \mathbb{E}[C\xi_{t+1}(x_t - \mu_t)'A'] = 0$$

(see Exercise 5), we can reduce this to

$$\Sigma_{t+1} = \mathbb{E}[A(x_t - \mu_t)(x_t - \mu_t)'A'] + \mathbb{E}[C\xi_{t+1}\xi'_{t+1}C']$$

More succinctly

$$\Sigma_{t+1} = A\Sigma_t A' + CC'$$

This is a difference equation in matrix space. We can identify it with the dynamical system $(\mathcal{M}_{n \times n}, S)$ where $S(\Sigma) := A\Sigma A' + CC'$. In order to analyze this dynamical system, let's study the slightly more general **discrete Lyapunov equation**

$$\Sigma = A\Sigma A' + M \tag{4.20}$$

where all matrices are in $\mathcal{M}_{n \times n}$ and Σ is the unknown. To this end, we introduce the **Lyapunov operator** associated with A and M , defined on $\mathcal{M}_{n \times n}$ by

$$\ell(\Sigma) = A\Sigma A' + M \quad (4.21)$$

Lemma 4.2.1. *If $r(A) < 1$, then $(\mathcal{M}_{n \times n}, \ell)$ is globally stable*

Proof. By theorem 5.1.10 on page 139, it suffices to show that ℓ^k is a uniform contraction on $(\mathcal{M}_{n \times n}, \|\cdot\|)$ for some $k \in \mathbb{N}$. Iterating with ℓ from arbitrary $\Sigma \in \mathcal{M}_{n \times n}$, we obtain

$$\ell^k(\Sigma) = A^k \Sigma (A^k)' + A^{k-1} M (A^{k-1})' + \cdots + M$$

Hence, for any Σ, T in $\mathcal{M}_{n \times n}$, we have

$$\begin{aligned} \|\ell^k(\Sigma) - \ell^k(T)\| &= \|A^k \Sigma (A^k)' - A^k T (A^k)'\| \\ &= \|A^k (\Sigma - T) (A^k)'\| \\ &\leq \|A^k\| \cdot \|\Sigma - T\| \cdot \|(A^k)'\| \end{aligned}$$

Transposes don't change norms, so $\|(A^k)'\| = \|A^k\|$ and hence $\|\ell^k(\Sigma) - \ell^k(T)\| \leq \|A^k\|^2 \|\Sigma - T\|$. Since $r(A) < 1$, we can find a $k \in \mathbb{N}$ and a constant $\lambda < 1$, both independent of Σ and T , such that $\|\ell^k(\Sigma) - \ell^k(T)\| \leq \lambda \|\Sigma - T\|$. Then ℓ^k is a uniform contraction on $\mathcal{M}_{n \times n}$, as was to be shown. \square

Returning to the dynamics of the second moment of our vector autoregression, it follows from lemma 4.2.1 that for (4.2.1) we see that $r(A) < 1$ implies global stability of $(\mathcal{M}_{n \times n}, S)$ with unique fixed point satisfying $\Sigma^* = A\Sigma^* A' + CC'$. It's notable that the stability conditions for both the first and second moment are identical.

EXERCISE 6. Consider again the Lyapunov operator $\ell(\Sigma) = A\Sigma A' + M$ in the setting where $r(A) < 1$. Show that, if M is positive semidefinite, then the unique fixed point Σ^* is positive semidefinite.⁹ Show that if, in addition, M is positive definite, then so is Σ^* .

4.2.2.2 Application: Dynamics of Log Output

In one of several influential studies, Kydland and Prescott (1980) used the second order stochastic difference equation

$$y_{t+1} = \alpha_1 y_t + \alpha_2 y_{t-1} + \varepsilon_{t+1} \quad (4.22)$$

⁹Lemma 5.3.2 might be useful here.

to estimate and analyze the dynamics of detrended log output. The shock sequence $\{\varepsilon\}$ can be regarded as IID with zero mean and standard deviation σ .

Although this is a second order system, we can map it into the first order system (4.18) using techniques similar to [add ref](#). To begin, let

$$x_t := \begin{pmatrix} y_t \\ y_{t-1} \end{pmatrix}, \quad A := \begin{pmatrix} \alpha_1 & \alpha_2 \\ 1 & 0 \end{pmatrix}, \quad C := \begin{pmatrix} \sigma \\ 0 \end{pmatrix} \quad \text{and} \quad \xi_t := \frac{1}{\sigma} \varepsilon_t \quad (4.23)$$

It is now easy to verify that the first entry in the two dimensional system

$$x_{t+1} = Ax_t + b + C\xi_{t+1}$$

coincides with (4.22).

To analyze stability we investigate the spectral radius of A . The first step is to calculate its eigenvalues, which solve $\det(A - \lambda I) = 0$. The two solutions are, in this case, the roots of the quadratic term $\lambda^2 - \alpha_1\lambda - \alpha_2$, or

$$\lambda_i = \frac{\alpha_1 \pm \sqrt{\alpha_1^2 + 4\alpha_2}}{2} \quad i = 1, 2 \quad (4.24)$$

If both are interior to the unit circle in the complex plane, then $r(A) < 1$ and stability will hold. In the case of [Kydland and Prescott \(1980\)](#), data is quarterly and the estimated values are $\hat{\alpha}_1 = 1.386$ and $\hat{\alpha}_2 = -0.477$. Both eigenvalues are real and both lie inside the unit circle in \mathbb{C} . The spectral radius of A is approximately 0.75. Figure 4.14 shows a simulated time series when $\{\varepsilon_t\}$ is $N(0, \sigma^2)$ with $\sigma = 0.05$. The initial conditions are $y_0 = y_1 = 0$.

4.2.2.3 Application: Price Dynamics

[Mankiw and Reis \(2002\)](#) consider both forward and backward looking models of price formation in a study of New Keynesian models and their forecasting properties. One purely backward looking model they consider is

$$p_{t+1} = \frac{1}{1+\beta}(2p_t - p_{t-1} + \beta m_{t+1}) \quad (4.25)$$

where $\{p_t\}$ is a price level, $\{m_t\}$ is a measure of money supply and β is a positive parameter. As in (4.23), we can reorganize this into a first order VAR via the system

$$x_t := \begin{pmatrix} p_t \\ p_{t-1} \end{pmatrix}, \quad A := \frac{1}{1+\beta} \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}, \quad C := \begin{pmatrix} \beta/(1+\beta) \\ 0 \end{pmatrix} \quad \text{and} \quad \xi_t := m_t$$

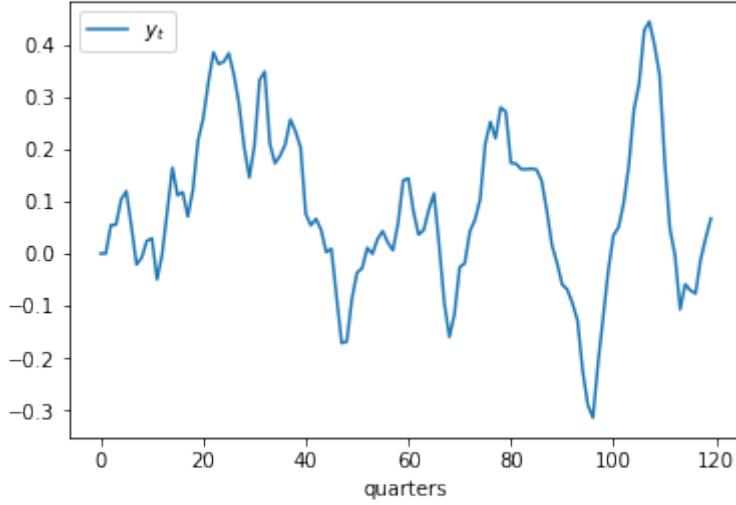


Figure 4.14: Time series of detrended log output

EXERCISE 7. Write down an expression for the spectral radius of A in terms of β . Argue that the stability condition $r(A) < 1$ holds whenever $\beta > 0$.

Figures 4.15–4.16 illustrate dynamics over a 200 and 2,000 period horizons respectively when β is set to 0.05 and $\{\xi_t\}$ is standard normal.

4.2.3 Distributions and Sample Paths

Add roadmap.

4.2.3.1 Distribution Dynamics: The Gaussian Case

In §4.2.2 we studied the dynamics of the first two moments of the vector autoregression

$$x_{t+1} = Ax_t + b + C\xi_{t+1} \quad (4.26)$$

where $\{\xi_t\}_{t \geq 1}$ is an \mathbb{R}^j -valued MDS satisfying $\mathbb{E}[\xi_t \xi_t'] = I$. For example, we found that the time t mean and variance-covariance matrix were given by

- $g^t(\mu_0)$ where $g(\mu) := A\mu + b$ on \mathbb{R}^n and
- $S^t(\Sigma_0)$ where $S(\Sigma) := A'\Sigma A + CC'$ on $\mathcal{M}_{n \times n}$.

These moments (μ_t, Σ_t) tell us something about the distribution x_t , denoted henceforth by ψ_t . In general, ψ_t is a complex, high dimensional object about which the first

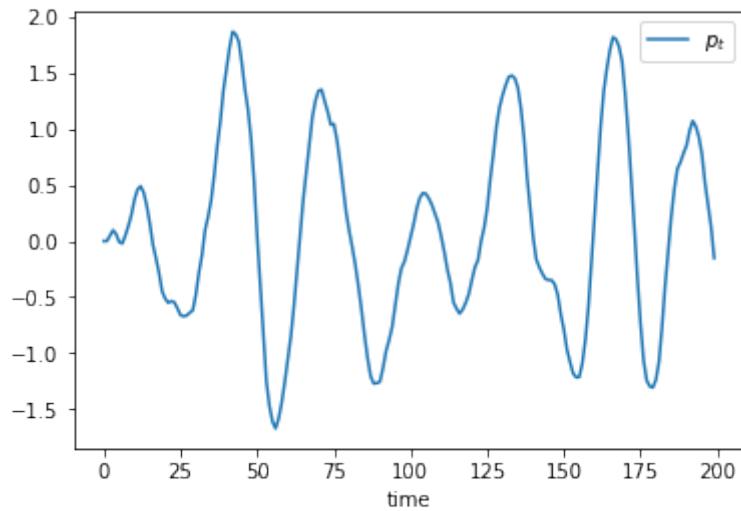


Figure 4.15: Time series of prices

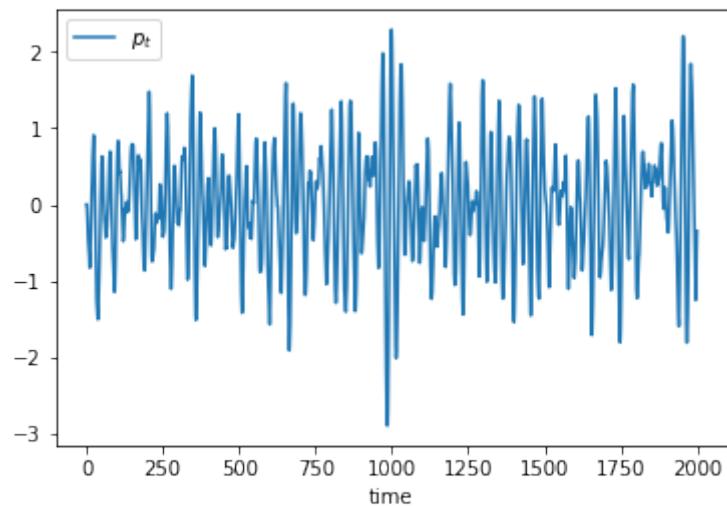


Figure 4.16: Time series of prices, long horizon

two moments provide only limited information. There is, however, one case where we can easily extract the full distribution ψ_t at every point in time from the first two moments: the Gaussian case.

Let's get our definitions straight. First, a scalar random variable z has a (univariate) **standard normal distribution** if it has a density given by

$$\varphi(s) = \sqrt{\frac{1}{2\pi}} \exp\left(\frac{-s^2}{2}\right) \quad (s \in \mathbb{R})$$

In this case we write $z \stackrel{d}{=} N(0, 1)$. Next, we say that scalar random variable x has normal (or Gaussian) distribution $N(\mu, \sigma)$ for some $\mu \in \mathbb{R}$ and $\sigma \geq 0$ if x has the same distribution as $\mu + \sigma z$, for some z with $z \stackrel{d}{=} N(0, 1)$. Note that we allow $\sigma = 0$, in which case x is a point mass on μ , and is often referred to as degenerate. Finally, a random vector x in \mathbb{R}^n is said to be **multivariate Gaussian** with distribution $N(\mu, \Sigma)$ if μ is a vector in \mathbb{R}^n , Σ is a positive semidefinite element of $\mathcal{M}_{n \times n}$ and

$$h'x \stackrel{d}{=} N(h'\mu, h'\Sigma h) \text{ on } \mathbb{R} \text{ for any } h \in \mathbb{R}^n$$

In particular, a random vector is multivariate Gaussian if every linear combination formed from its elements is scalar Gaussian.

Remark 4.2.1. Just because x_1 and x_2 are normally distributed in \mathbb{R} , we cannot claim that $x = (x_1, x_2)$ multivariate Gaussian. A bit of searching will turn up examples where sums of normal random variables fail to be normal.

When Σ is positive definite, one can show that x has an everywhere positive density on \mathbb{R}^n given by

$$\varphi(s) = \det(2\pi\Sigma)^{-1/2} \exp\left(-\frac{1}{2}(s - \mu)' \Sigma^{-1} (s - \mu)\right) \quad (s \in \mathbb{R}^n)$$

In this setting, the distribution spreads probability mass over the entire space \mathbb{R}^n .

Now let's return to the VAR dynamics in (4.26). To shift this model to the Gaussian case we will assume that

$$\{\xi_t\}_{t \geq 1} \stackrel{\text{iid}}{\sim} N(0, I) \quad \text{and} \quad x_0 \stackrel{d}{=} N(\mu_0, \Sigma_0) \tag{4.27}$$

where μ_0 is any vector in \mathbb{R}^j and Σ_0 is any positive semidefinite $j \times j$ matrix. The random vector x_0 is assumed to be independent of $\{\xi_t\}$. Under these Gaussian conditions we have

$$x_t \stackrel{d}{=} N(g^t(\mu_0), S^t(\Sigma_0)) \quad \text{for all } t \geq 0 \tag{4.28}$$

Here the claim that x_t has the first two moments specified in (4.28) has already been verified, while normality can be checked using the definition of multivariate Gaussians and an induction argument.

EXERCISE 8. Confirm this. In doing so, you can exploit the fact that any affine combination of *independent* normal random variables in \mathbb{R} is normal.

Proposition 4.2.2. *Let ψ_t be the distribution of x_t for each $t \geq 0$. If $r(A) < 1$, then under the Gaussian conditions in (4.27), we have*

$$\psi_t \xrightarrow{w} N(\mu^*, \Sigma^*) \quad (t \rightarrow \infty) \quad (4.29)$$

where

- (i) $\mu^* = \sum_{i=0}^{\infty} A^i b$ and
- (ii) Σ^* is the unique fixed point of $\Sigma := A' \Sigma A + C C'$.

Here \xrightarrow{w} means weak convergence of distributions—see §12.1.2 for a discussion.

Proof of proposition 4.2.2. To establish the claim it suffices to show that the characteristic function of the distribution $N(\mu_t, \Sigma_t)$ converges pointwise to that of $N(\mu^*, \Sigma^*)$.¹⁰ In our case, this translates to the claim that, at any fixed $s \in \mathbb{R}^n$,

$$\lim_{t \rightarrow \infty} \exp \left(i s' \mu_t - \frac{1}{2} s' \Sigma_t s \right) = \exp \left(i s' \mu^* - \frac{1}{2} s' \Sigma^* s \right) \quad (4.30)$$

Fixing such an s , to prove (4.30) it suffices to show that

$$s' \mu_t \rightarrow s' \mu^* \quad \text{and} \quad s' \Sigma_t s \rightarrow s' \Sigma^* s \quad \text{in } \mathbb{R} \text{ as } t \rightarrow \infty \quad (4.31)$$

We have already showed that $\mu_t \rightarrow \mu^*$ in norm. From this fact and the Cauchy–Schwarz inequality we have

$$|s' \mu_t - s' \mu^*| = |s' (\mu_t - \mu^*)| \leq \|s\| \cdot \|\mu_t - \mu^*\| \rightarrow 0$$

The proof of the second part of (4.31) is similar. □

Example 4.2.2. Consider the scalar **AR(1)** case, where $\{x_t\}$ is real valued and evolves according to

$$x_{t+1} = ax_t + b + \sigma \varepsilon_{t+1}, \quad \{\varepsilon_t\} \stackrel{\text{iid}}{\sim} N(0, 1) \quad (4.32)$$

¹⁰Cinlar (2011) contains a discussion of the connects between characteristic functions and weak convergence. See, for example, theorem 5.15 of that text.

This is a version of the Gaussian VAR with $A = a$ and the other obvious identifications. The case $|a| < 1$ is known as the **mean-reverting** case, under which the distribution of x_t converges weakly to

$$\psi^* := N\left(\frac{b}{1-a}, \frac{\sigma^2}{1-a^2}\right) \quad (4.33)$$

Since, in this case $r(A) = |a|$, the stable case in the sense of proposition 4.2.2 coincides with the mean-reverting case.

We can translate the results from this section into the language of dynamical systems. Let \mathcal{G} be the set of all Gaussian distributions on \mathbb{R}^n , endowed with the topology of weak convergence (see §12.1.2). Let Π be the operator on \mathcal{G} defined by

$$\psi := N(\mu, \Sigma) \mapsto \psi\Pi := N(g(\mu), S(\Sigma))$$

Then Π updates distributions by one period, in the sense that the marginal distributions $\{\psi_t\}_{t \geq 0}$ of the state process obey $\psi_{t+1} = \psi_t\Pi$. We have written the argument ψ to the left, as in $\psi\Pi$ rather than, say, $\Pi\psi$ or $\Pi(\psi)$, so as to tie in with the notation in §3.1.3.1, and in particular with (3.11) on page 52, where an analogous operation is carried out in a discrete setting.

Proposition 4.2.2 tells us that (\mathcal{G}, Π) is globally stable whenever $r(A) < 1$.

4.2.3.2 Distribution Dynamics: The General Density Case

Now let's drop the Gaussian assumption and see what we can say about dynamics. While we cannot in general provide an analytical expression for the marginal distributions $\{\psi_t\}$, we can provide a law of motion over distributions.

Assumption 4.2.1. The shock process and matrix C satisfy the following conditions.

- (i) $\{\xi_t\}_{t \geq 1}$ is IID on \mathbb{R}^n with density φ and $\mathbb{E} \|\xi_t\|^2 < \infty$.
- (ii) C is $n \times n$ and nonsingular.

By a change of variable argument (theorem 8.1.3 in Stachurski (2009) gives the exact result we want here), when ξ has density φ , the random vector $y = Ax + b + C\xi$ has density

$$\pi(x, y) = \varphi(C^{-1}(y - Ax - b)) |\det C|^{-1} \quad (4.34)$$

This is also the density of x_{t+1} conditional on $x_t = x$. (We could alternatively write this density $\pi(x, y)$ as $\pi(y | x)$, since we are conditioning on x . However, the notation

$\pi(x, y)$ used above is standard in this context—heuristically, $\pi(x, y) dy$ represents the probability of moving from x to y in one unit of time.)

The law of total probability tells us that given arbitrary random variables X and Y with (i) marginal densities $p(x)$ and $p(y)$ and (ii) conditional distribution $p(y | x) :=$ density of Y given $X = x$, the marginal and conditional densities are linked by

$$p(y) = \int p(y | x)p(x) dx \quad (4.35)$$

Applying this rule to our setting, we can link the marginal densities ψ_t and ψ_{t+1} via

$$\psi_{t+1}(y) = |\det C|^{-1} \int \varphi(C^{-1}(y - Ax - b)) \psi_t(x) dx \quad (4.36)$$

If we introduce an operator Π from the set of densities \mathcal{D} on \mathbb{R}^n to itself via

$$(\psi\Pi)(y) = \int \pi(x, y)\psi(x) dx \quad (4.37)$$

then we can express (4.36) more succinctly as

$$\psi_{t+1} = \psi_t\Pi$$

Once again, we have written the argument to the left to tie in with notation such as (3.11) on page 52. Equation (4.2.3.2) is just a continuous state version of the same concept.¹¹

The pair (\mathcal{D}, Π) can be regarded as a dynamical system once we specify a topology on \mathcal{D} .¹² The next result uses the topology of weak convergence (i.e., convergence in distribution) on \mathcal{D} .

Proposition 4.2.3. *If assumption 4.2.1 holds and $r(A) < 1$, then (\mathcal{D}, Π) is globally stable. Moreover, if h is any function such that $\int |h(x)|\psi^*(x) dx$ is finite, then*

$$\mathbb{P} \left\{ \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{t=1}^n h(x_t) = \int h(x)\psi^*(x) dx \right\} = 1 \quad (4.38)$$

Proof. To be added. An n -step contraction argument using some metric that generates

¹¹The practice of writing ψ_t to the left of the Markov operator Π in general state settings is common in the literature (see, e.g., [Meyn and Tweedie \(2009\)](#)).

¹²We should also be precise about what we mean by the “set of densities on \mathbb{R}^n .” In this instance, we mean the set of all nonnegative Borel measurable functions f from \mathbb{R}^n to \mathbb{R} satisfying $\int f(x) dx = 1$. As in other occasions through the text, Borel measurability is a weak regularity requirement imposed to ensure that the functions can in fact be integrated.

the weak topology? \square

Often the only way to track the marginal distributions $\{\psi_t\}$ as they converge to the unique limit indicated by proposition 4.2.3 is via simulation, a topic we discuss in detail in §4.1.5. We can however say something more about the stationary distribution ψ^* . In stating the next result, we take $\{\eta_i\}_{i \geq 0}$ to be a sequence of IID random vectors such that each η_i has the same distribution as $b + C\xi_t$ and let x^* be a random variable defined by

$$x^* = \sum_{i \geq 0} A^i \eta_i \quad (4.39)$$

whenever the series converges.

Proposition 4.2.4. *If assumption 4.2.1 holds and $r(A) < 1$, then the series (4.39) converges absolutely with probability one and the distribution ψ^* of x^* is the unique steady state of the dynamical system (\mathcal{D}, Π) .*

Proof. Suppose that the primitives satisfy assumption 4.2.1 and $r(A) < 1$. Our claim is that $\sum_{i \geq 0} \|A^i \eta_i\|$ is finite with probability one, where $\|\cdot\|$ is the Euclidean norm. For this to be true it suffices that

$$\sum_{i \geq 0} \|A^i\| \|\eta_i\| < \infty \quad (4.40)$$

with probability one. $\mathbb{E} \|\eta_i\|^2$ is finite by assumption and $\sum_{i \geq 0} \|A^i\| < \infty$ because $r(A) < 1$, as can be readily verified from Gelfand's formula. The finiteness of (4.40) now follows from the Kolmogorov three series theorem.

It remains to show that the distribution ψ^* of x^* is a stationary distribution for $x_{t+1} = Ax_t + b + C\xi_{t+1}$. This claim is equivalent to the statement that $x_t \stackrel{d}{=} x^*$ implies $x_{t+1} \stackrel{d}{=} x^*$, which will be verified if we can show that $Ax^* + b + C\xi_{t+1}$ has the same distribution as x^* . To see that this is so, let $\eta := b + C\xi_{t+1}$ and observe that

$$Ax^* + b + C\xi_{t+1} = A(\eta_0 + A\eta_1 + \dots) + \eta = \eta + A\eta_0 + A^2\eta_1 + \dots \stackrel{d}{=} x^* \quad \square$$

4.2.4 Linear-Quadratic Asset Pricing

To be added. See NYU lectures. Use finite dim v-N series results. (Translate from contraction discussion to a discussion of linear maps over matrices. This is a step towards v-N series in abstract Banach space.)

4.2.5 Linear State Space Models

Add roadmap.

4.2.5.1 The Model

Let's now extend the VAR model from (4.18) to the standard **linear state space** model

$$x_{t+1} = Ax_t + b + C\xi_{t+1} \quad (4.41)$$

$$y_t = Gx_t + H\zeta_t \quad (4.42)$$

where

- A is $n \times n$, b is $n \times 1$ and C is $n \times j$.
- G is $k \times n$ and H is $k \times \ell$.
- $\{\xi_t\}$ are IID copies of $j \times 1$ random vector ξ , where $\mathbb{E}\xi = 0$ and $\mathbb{E}\xi\xi' = I$.
- $\{\zeta_t\}$ are IID copies of $\ell \times 1$ random vector ζ , where $\mathbb{E}\zeta = 0$ and $\mathbb{E}\zeta\zeta' = I$.

As usual $\{x_t\}$ is called the **state** process. Its initial condition x_0 is assumed to be independent of $\{\xi_t\}$ and $\{\zeta_t\}$. The $k \times 1$ process $\{y_t\}$ is called the **observation process**. The processes $\{\xi_t\}$ and $\{\zeta_t\}$ are also independent.

Linear state space models are often used in a setting where we envisage imperfect observation of an economic system, either by an econometrician or an agent within a model. We will discuss an example of this form below. In other settings, the linear state space model is simply a convenient extension of the basic VAR model.

Example 4.2.3. The “canonical linear model” of (log) labor earnings discussed in De Nardi et al. (2018) is

$$y_t = x_t + h\zeta_t \quad \text{where } x_{t+1} = \rho x_t + b + c\xi_{t+1}$$

and $\{\xi_t\}$ and $\{\zeta_t\}$ are IID and standard normal in \mathbb{R} . Here h, ρ, b, c are parameters, with $|\rho| < 1$ being a common assumption, so that the state process is mean reverting. This is an example of a linear state space model where both state and observation are scalar. In this context, x_t is called the **persistent component** of labor income, while $\{\zeta_t\}$ is called the **transitory component**.

Example 4.2.4. Consider again the dynamic second order linear model in (4.22), which we reorganized into a first order model

$$x_t := \begin{pmatrix} y_t \\ y_{t-1} \end{pmatrix}, \quad A := \begin{pmatrix} \alpha_1 & \alpha_2 \\ 1 & 0 \end{pmatrix}, \quad C := \begin{pmatrix} \sigma \\ 0 \end{pmatrix} \quad \text{and} \quad \xi_t := \frac{1}{\sigma} \varepsilon_t$$

If we now take $G = (1, 0)'$ and $H = 0$, we extract $\{y_t\}$ from $\{x_t\}$. In this way, the technique for converting a higher order linear model into a first order model and extracting the original stochastic process as one component of the first order model can be accommodated within the linear state space framework.

4.2.5.2 Dynamics

We can easily compute the first two moments of the observation process given our results on the moments of the state process in §4.2.2.1. Recalling that

- $\mu_t = g^t(\mu_0)$ where $g(\mu) := A\mu + b$ on \mathbb{R}^n and
- $\Sigma_t = S^t(\Sigma_0)$ where $S(\Sigma) := A'\Sigma A + CC'$ on $\mathcal{M}_{n \times n}$

we obtain

$$\mathbb{E}y_t = G\mu_t \quad \text{and} \quad \text{Var } y_t = G\Sigma_t G' + HH' \quad (4.43)$$

The evolution of this sequence is determined by $g^t(\mu_0)$ and $S^t(\Sigma_0)$. This is natural because the state process is the driver of dynamics in the linear state space model. We know that if $r(A) < 1$, then $g^t(\mu_0)$ and $S^t(\Sigma_0)$ converge. In particular, we have

$$\mathbb{E}x_t \rightarrow \mu^*, \quad \mathbb{E}y_t \rightarrow G\mu^*, \quad \text{Var } x_t \rightarrow \Sigma^*, \quad \text{and} \quad \text{Var } y_t \rightarrow G\Sigma^*G' + HH' \quad (4.44)$$

as $t \rightarrow \infty$. Here $\mu^* = \sum_{i=0}^{\infty} A^i b$ and Σ^* is the unique steady state of $(\mathcal{M}_{n \times n}, S)$.

The next result is stronger in the sense that it addresses the entire distributions of x_t and y_t .

Theorem 4.2.5. *If $r(A) < 1$ and x^* is as given in (4.39), then $x_t \xrightarrow{w} x^*$ and*

$$y_t \xrightarrow{w} y^* := Gx^* + H\zeta \quad \text{as } t \rightarrow \infty \quad (4.45)$$

Proof of theorem 4.2.5. To be added. (The claim regarding $\{x_t\}$ has already been established, in propositions 4.2.3–4.2.4.) \square

We also have ergodicity results for $\{y_t\}$ that flow on from those established for $\{x_t\}$ in proposition 4.2.3. For example, when $r(A) < 1$

$$\frac{1}{n} \sum_{t=1}^n y_t = \frac{1}{n} \sum_{t=1}^n (Gx_t + H\zeta_t) = G \frac{1}{n} \sum_{t=1}^n x_t + H \frac{1}{n} \sum_{t=1}^n \zeta_t \rightarrow G\mu^*$$

with probability one as $n \rightarrow \infty$. Here we are using the IID strong law of large numbers for $\{\zeta_t\}$.

A common setting for the linear state space model is the Gaussian one, where the assumptions above are supplemented by

Assumption 4.2.2. The random vectors ξ and ζ are multivariate Gaussian.

Provided that x_0 is also Gaussian, the first two moments then pin down the distribution of x_t , which we saw in (4.28), and from that the distribution of y_t :

$$y_t \stackrel{d}{=} N(G\mu_t, G\Sigma_t G' + HH') \quad (4.46)$$

When $r(A) < 1$, the stationary distribution is Gaussian with the moments provided in (4.44).

EXERCISE 9. Use the characteristic function approach found in the proof of Proposition 4.2.2 to show that

$$r(A) < 1 \implies y_t \stackrel{d}{\rightarrow} N(G\mu^*, G\Sigma^* G' + HH')$$

4.2.5.3 Forecasts

At times we wish to forecast geometric sums such as $\mathbb{E}_t \left[\sum_{j=0}^{\infty} \beta^j y_{t+j} \right]$ where β is a positive scalar. For example, if $\{y_t\}$ is a cash flow, then this sum becomes a model of asset price, while if $\{y_t\}$ is money supply, then the sum is a model of the price level.

EXERCISE 10. As a preliminary result, show that $\mathbb{E}_t x_{t+j} = A^j x_t$ and $\mathbb{E}_t y_{t+j} = G A^j x_t$ for all $j \geq 0$.

This leads to the formulas

$$\mathbb{E}_t \left[\sum_{j=0}^{\infty} \beta^j x_{t+j} \right] = [I - \beta A]^{-1} x_t \quad \text{and} \quad \mathbb{E}_t \left[\sum_{j=0}^{\infty} \beta^j y_{t+j} \right] = G[I - \beta A]^{-1} x_t$$

which are valid whenever $r(A) < 1/\beta$.

EXERCISE 11. Establish the validity of these forecasts.

4.2.6 Filtering

Add roadmap, reorganize.

4.2.6.1 Controllability

Consider the following problem, which has a long history in control theory and will emerge in several settings throughout the remainder of the notes.

Suppose that the state vector $\{x_t\}$ obeys

$$x_{t+1} = Ax_t + Bu_t, \quad x_0 \text{ given} \quad (4.47)$$

where $\{u_t\}$ is a m -vector of **controls**. The matrices A and B are $n \times n$ and $n \times m$ respectively. We imagine that an agent chooses the controls $\{u_t\}$ to guide the state $\{x_t\}$ according to some criterion, such as hitting some given target $\bar{x} \in \mathbb{R}^n$.

One of the difficulties here is that the control set is often limited when compared to the size of the state space. To understand the difficulty, suppose that $m < n$, and our aim is to hit \bar{x} in one step. Then we would need to choose a $u_0 \in \mathbb{R}^m$ such that

$$Bu_0 = \bar{x} - Ax_0.$$

However, the range of $u \mapsto Bu$ is equal to the number of linearly independent columns of B , which is at most m . So even though we vary u_0 over all of \mathbb{R}^m , we can at best trace out a linear subspace of \mathbb{R}^n that has dimension m . Since $m < n$, this subspace has measure zero in the whole space.¹³

This discussion implies that we cannot in general reliably hit an arbitrary target \bar{x} in one step. Instead, we will aim to hit it in n steps. The condition that allows us to do this is called controllability. Specifically, the pair (A, B) is called **controllable** if the $n \times nm$ matrix

$$C := (B, AB, A^2B, \dots, A^{n-1}B)$$

is full rank. This corresponds to the statement that C has the maximal number of linearly independent columns. Since $nm > n$, that number is n .

¹³This is a formal way of saying that the amount of “volume” taken up by the subspace, when viewed as a subset of \mathbb{R}^n , is zero. Think of a plane in \mathbb{R}^2 , for example, or a line in \mathbb{R}^2 .

Controllability is the condition we need because, after iterating backwards, x_n can be written as

$$x_n = Bu_{n-1} + ABu_{n-2} + A^2Bu_{n-3} + \cdots + A^{n-1}Bu_0 + A^n x_0.$$

Stacking the control vectors and combining matrices, we can write this as

$$Cu := (B, AB, A^2B, \dots, A^{n-1}B) \begin{pmatrix} u_{n-1} \\ u_{n-2} \\ \vdots \\ u_0 \end{pmatrix} = x_n - A^n x_0 \quad (4.48)$$

The range space of the mapping $u \mapsto Cu$ is all of \mathbb{R}^n if and only if C has n linearly independent columns, since any n linearly independent columns form a basis of \mathbb{R}^n . Thus, we can hit any target $\bar{x} \in \mathbb{R}^n$ precisely when (A, B) is controllable.

4.2.6.2 Observability

A pair (A, G) in $\mathcal{M}_{n \times n} \times \mathcal{M}_{k \times n}$ is called **observable** if (A', G') is controllable. A prime here denotes transpose. To see where this language comes from, consider the following problem from filtering, which turns out to be dual to the controllability problem while also hinting at the deep connections between filtering and control.

Suppose that we have n **observations** y_0, \dots, y_{n-1} of the form $y_t = Gx_t$, where each y_t is a k -vector and G is $k \times n$. The state $\{x_t\}$ evolves according to $x_{t+1} = Ax_t$ where A is $n \times n$. Our aim is to learn the state from the observations. In this effort it is assumed that we know the matrices A and G .

Since we know A , pinning down the state at each point in time comes down to knowing x_0 , since x_t can then be computed as $A^t x_0$. To obtain x_0 from our measurements, we note that $y_t = GA^t x_0$ for each t in $0, \dots, n-1$, leading to the system of equations

$$y = Ox_0 \quad \text{where} \quad y := \begin{pmatrix} y_{n-1} \\ y_{n-2} \\ \vdots \\ y_0 \end{pmatrix} \quad \text{and} \quad O := \begin{pmatrix} GA^{n-1} \\ GA^{n-2} \\ \vdots \\ G \end{pmatrix} \quad (4.49)$$

When does this system of equations have a unique solution x_0 ? The **observation matrix** O is $nk \times n$, so, for each n -vector x , the vector Ox is an $nk \times 1$ vector. Since O has at most n linearly independent columns, the range space of $x \mapsto Ox$ is at most a linear subspace of \mathbb{R}^{nk} of dimension n . Fortunately, we know that the observation y is

a point in this n dimensional subspace, since we must have $y = Ox$ for some $x \in \mathbb{R}^n$. The condition for this x to be uniquely determined is precisely that O has n linearly independent columns.

If (A, G) is observable, then (A', G') is controllable, which means that

$$(G', A'G', (A')^2G', \dots, (A')^{n-1}G')$$

has n linearly independent rows. But this is just O' , so if (A, G) is observable, then O has n linearly independent columns (i.e., is of full column rank), which is exactly the condition we require.

4.2.6.3 The Kalman Filter

To understand the Kalman filter, it is easiest to start with the following static problem. There is a random vector $x \in \mathbb{R}^n$ that we wish to know but cannot observe directly. What we do have is

- (i) a prior belief $p = N(x, \Sigma)$
- (ii) a noisy observation $y = Gx + \zeta$, where $\zeta \stackrel{d}{=} N(0, R)$ on \mathbb{R}^k

Here y is a k vector and G and R have appropriate sizes. The noise term ζ is sampled independent of x . We wish to update our belief on the basis of this observation. We will do so using elementary mathematics that connects conditional to joint probability distributions and that tells us to update $p(x)$ to $p(x | y)$ via

$$p(x | y) = \frac{p(y | x) p(x)}{p(y)} \quad \text{where} \quad p(y) = \int p(y | x) p(x) dx \quad (4.50)$$

(This formula is often called Bayes' rule. Following the usual Bayesian tradition, we are using the generic symbol p to represent marginal and conditional densities but the meaning should be clear from the arguments to these distributions.) The conditional density $p(y | x)$ is obtained from the observation equation in (ii), the marginal density $p(x)$ is the prior in (i). This is a standard calculation in the Bayesian paradigm and the solution is known to be

$$p(x | y) = N(\mu_f, \Sigma_f) \quad (4.51)$$

where

$$\mu_f := \mu + \Sigma G' (G \Sigma G' + R)^{-1} (y - G\mu) \quad (4.52)$$

and

$$\Sigma_f := \Sigma - \Sigma G' (G \Sigma G' + R)^{-1} G \Sigma \quad (4.53)$$

See, for example, [Bishop \(2006\)](#), page 93. This completes our **filtering step**, where we attempt to locate the current state through a noisy observation.

Next, suppose that x will update to $\hat{x} = Ax + \eta$ where A is $n \times n$ and η is drawn from $N(0, Q)$. We wish to forecast \hat{x} on the basis of our current knowledge, which consists of our posterior distribution (4.51). In particular, when $x \stackrel{d}{=} N(\mu_f, \Sigma_f)$, we seek the distribution of $Ax + \eta$, which is

$$N(\mu_p, \Sigma_p) = N(A\mu_f, A\Sigma_f A' + Q) \quad (4.54)$$

More explicitly,

$$\mu_p := A\mu_f + A\Sigma_f G'(G\Sigma_f G' + R)^{-1}(y - G\mu_f) \quad (4.55)$$

and

$$\Sigma_p := A\Sigma_f A' - A\Sigma_f G'(G\Sigma_f G' + R)^{-1}G\Sigma_f A' + Q \quad (4.56)$$

Now consider again the linear state space system under the Gaussian assumption (assumption 4.2.2) and suppose that $\{y_t\}$ is observable while $\{x_t\}$ is not. With the identifications $Q := CC'$ and $R := HH'$, we have obtained a recursive rule for updating beliefs about the unobserved state. Letting p_t be the current belief state $N(\mu_t, \Sigma_t)$ and $p_{t+1} = N(\mu_{t+1}, \Sigma_{t+1})$ be updated in the matter of (4.54), then the distribution dynamics are given by the following laws of motion for first and second moments:

$$\mu_{t+1} = A\mu_t + A\Sigma_t G'(G\Sigma_t G' + R)^{-1}(y_t - G\mu_t)$$

and

$$\Sigma_{t+1} = A\Sigma_t A' - A\Sigma_t G'(G\Sigma_t G' + R)^{-1}G\Sigma_t A' + Q \quad (4.57)$$

The law of motion for the mean $\{\mu_t\}$ is stochastic because it depends on the observation term y_t , which in turn depends on x_t . The variance-covariance matrix evolves deterministically. We can think of it as a difference equation on $\mathcal{M}_{n \times n}$.

In some instances, the variance matrix converges to a steady state level of uncertainty that is independent of initial conditions. This fixed point, when it exists, solves the **discrete time Riccati equation**

$$\Sigma = A\Sigma A' - A\Sigma G'(G\Sigma G' + R)^{-1}G\Sigma A' + Q \quad (4.58)$$

We will discuss conditions under which this equation has a solution in §6.2, when it reappears in the context of optimal control theory.

4.2.7 Uncertainty Traps

As an application of Kalman filtering, let's consider a highly simplified version of the uncertainty traps model of Fajgelbaum et al. (2017) that's also presented in the QuantEcon lecture series. As in the original paper, the model consists of a collection of firms with imperfect knowledge regarding the state of the economy, which varies stochastically over time. Firm owners are risk averse. Each one can be either active or inactive at any given point in time. All have beliefs about the fundamentals expressed as probability distributions, and uncertainty is understood as the degree of dispersion in these distributions.

The output of active entrepreneurs is observable, supplying a noisy signal that helps all agents infer fundamentals. The model exhibits uncertainty traps because of the following feedback loop:

- Being risk averse, entrepreneurs are less active when uncertainty is high.
- Low participation diminishes the flow of information about fundamentals.
- Less information translates to higher uncertainty, further discouraging entrepreneurs from choosing to participate, and so on.

The evolution of the fundamental process $\{x_t\}$ is given by

$$x_{t+1} = \rho x_t + c w_{t+1} \quad (4.59)$$

where $c > 0$, $\{w_t\}$ is IID and standard normal and $0 < \rho < 1$. The random variable x_t is not observable at any time.

There are \bar{M} entrepreneurs. The output of the m -th entrepreneur, conditional on being active in the market at time t , is equal to

$$y_m = x + \varepsilon_m \quad \text{where} \quad \varepsilon_m \sim N(0, \sigma_y^2) \quad (4.60)$$

Here the time subscript has been dropped to simplify notation. Output shocks are independent across time and firms.

All entrepreneurs start with identical beliefs about x_0 . Signals are publicly observable and hence all agents have identical beliefs at each point in time. Beliefs for current x are represented by the normal distribution $N(\mu, \sigma^2)$. Here σ is the standard deviation of beliefs and measures the amount of uncertainty.

Let $\mathbb{M} \subset \{1, \dots, \bar{M}\}$ denote the set of currently active firms and let $M := |\mathbb{M}|$

denote the number of currently active firms. Let

$$y := \frac{1}{M} \sum_{m \in \mathbb{M}} y_m = x + \frac{1}{M} \sum_{m \in \mathbb{M}} \varepsilon_m$$

be average output over the active firms. With this notation and primes for next period values, we can use the Kalman filter to update beliefs:

$$\mu' = \rho \frac{\sigma_y^2 \mu + M \sigma^2 y}{\sigma_y^2 + M \sigma^2} \quad \text{and} \quad \sigma'_2 = \frac{\rho^2}{\frac{1}{\sigma^2} + M \frac{1}{\sigma_y^2}} + c^2 \quad (4.61)$$

EXERCISE 12. Verify the update rules in (4.61) using the results in §4.2.6.3.

EXERCISE 13. Prove that the law of motion for σ^2 in (4.61) is globally stable on $(0, \infty)$ using Proposition 5.3.8.

One difference to the standard Kalman filtering set up is that the law of motion for the variance (or precision) is stochastic, since the number of participating entrepreneurs M is endogenous and, as we now show, fluctuates stochastically.

In particular, to complete the model, we assume that entrepreneurs enter the market in the current period if

$$\mathbb{E}[u(y_m - F_m)] > K \quad (4.62)$$

where $u(y) = \frac{1}{a}(1 - \exp(-ay))$ for some $a > 0$ and the mathematical expectation of $y_m = x + \varepsilon_m$ is based on beliefs $N(\mu, \sigma^2)$ for x . The term F_m is a stochastic but previsible fixed cost, independent across time and firms, while K is a constant reflecting opportunity costs. (The statement that F_m is previsible means that it is realized at the start of the period and treated as a constant in (4.62).)

It follows that entrepreneur m participates in the market when

$$\frac{1}{a} \{1 - \mathbb{E}[\exp(-a(x + \varepsilon_m - F_m))]\} > K \quad (4.63)$$

Using the standard formula for expectations of lognormal random variables, this is equivalent to

$$\psi(\mu, \sigma, F_m) := \frac{1}{a} \left(1 - \exp \left(-a\mu + aF_m + \frac{a^2 (\sigma^2 + \sigma_y^2)}{2} \right) \right) - K > 0 \quad (4.64)$$

Notice that participation is decreasing in uncertainty σ^2 .

Figure 4.17 shows a simulation of the exogenous state process $\{x_t\}$, the mean belief μ_t of x_t , the variance $\{\sigma_t^2\}$ and M_t , the number of active firms at t . When the exogenous state is low, beliefs follow and the number of active firms decreases. This leads to a rise in uncertainty, as fewer firms provide signals that can be used to update beliefs, which further depresses economic activity. Eventually a rise in the exogenous state increases μ sufficiently to allow activity to recover.

4.3 Chapter Notes

References, background, historical notes.

Heavy tails in the income distribution were examined by Nirei and Souma (2007), and again more recently in Aoki and Nirei (2017). Benhabib et al. (2015b) proved that a Pareto tail emerges for the cross-sectional wealth distribution in Bewley models under mild conditions.

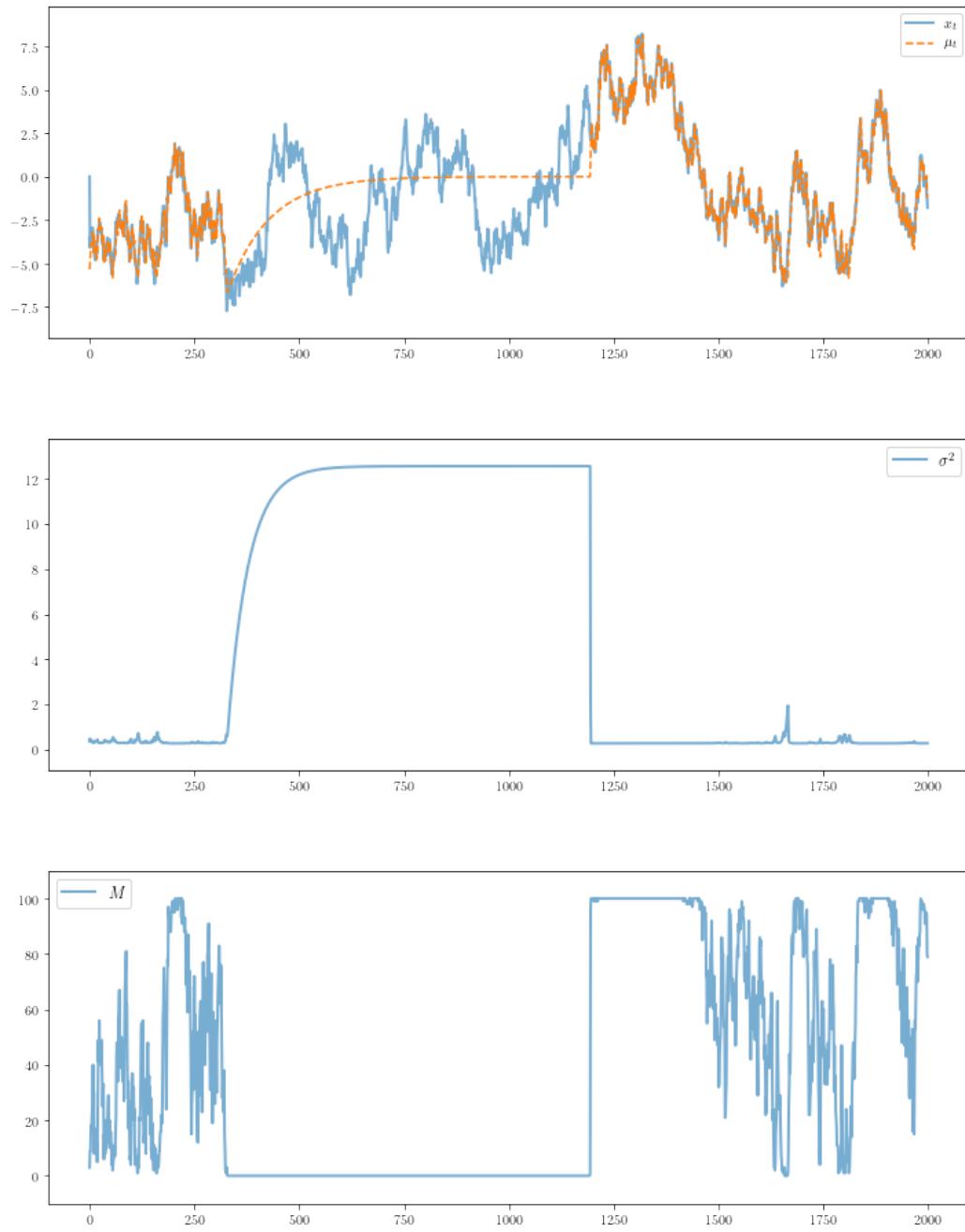


Figure 4.17: Time series of belief parameters and activity

```
1 for  $i$  in 1 to  $m$  do
2   draw  $w$  from the initial condition  $\psi_0$  ;
3   for  $j$  in 1 to  $t$  do
4     draw  $R'$  and  $y'$  from their distributions ;
5     set  $w = R's(w) + y'$  ;
6   end
7   set  $w_t^i = w$  ;
8 end
9 return  $(w_t^1, \dots, w_t^m)$ 
```

Algorithm 7: Draws from the marginal distribution ψ_t

Chapter 5

Useful Abstractions

Add roadmap.

5.1 Analysis

Add roadmap.

5.1.1 Metric Spaces

Mathematicians have built a battery of theorems about solving problems involving vectors in \mathbb{R}^d , including conditions under which solutions to systems of equations exist, conditions for uniqueness, algorithms for computing solutions, and so on. But none of these results directly apply to the problem of solving for, say, the function v^* in (2.15) of page 31 when w and z are continuous variables, since v^* is a function on a continuous domain rather than a vector in Euclidean space.

Fortunately, it is possible to make analogies that tell how, in an abstract sense, functions and vectors behave in similar ways.¹ For example, just as the Euclidean distance $e(x, y)$ between two vectors $x = (x_1, \dots, x_d)$ and $y = (y_1, \dots, y_d)$ in Euclidean space is given by

$$e(x, y) := \|x - y\| := \left\{ \sum_{i=1}^d (x_i - y_i)^2 \right\}^{1/2}$$

¹These analogies were uncovered and formalized in the late 19th and early 20th centuries, by mathematicians such as Jacques Hadamard (1865–1963), Maurice Fréchet (1878–1973), and Stefan Banach (1892–1945).

we can introduce a “distance” $d(g, h)$ between functions g and h defined over, say, an interval $[a, b]$, via

$$d(g, h) := \left\{ \int_a^b (g(t) - h(t))^2 dt \right\}^{1/2} \quad (5.1)$$

One can then show that at least *some* of the results known about Euclidean vector space carry over to our new setting in which functions are the main objects of interest. In particular, we can solve certain equations that have functions as solutions—that is, *functional equations*—by applying methods developed for equations that have vectors as solutions. This is powerful because it leverages the battery of theorems about Euclidean space referred to just above.

The distance (5.1), which is known as L_2 distance, seems natural because its form is similar to Euclidean distance (replacing sums with integrals). But sometimes the L_2 distance leads to conditions or results that aren’t helpful for a problem that we want to tackle. Fortunately, there are other notions of distance between functions we can entertain, allowing us to pick and choose in different applications.

To understand and organize these ideas, the first step is to define an abstract version of Euclidean vector space and Euclidean distance, stripped down to a few basic properties, and see how many results from Euclidean vector space carry over to this abstract setting. These abstract spaces are called *metric spaces*. We’ll use them as a lens to classify and understand different classes of objects, such as functions and distances between them.

Our treatment will be fast paced. More details can be found in, say, [Aliprantis and Border \(1999\)](#) or [Dudley \(2002\)](#).

5.1.1.1 Metrics

To begin, let M be any nonempty set. A function $\rho: M \times M \rightarrow \mathbb{R}$ is called a **metric** on M if, for any $u, v, w \in M$,

$$\begin{aligned} \rho(u, v) &\geq 0, & & \text{(nonnegativity)} \\ \rho(u, v) = 0 &\iff u = v, & & \text{(identifiability)} \\ \rho(u, v) &= \rho(v, u) \text{ and} & & \text{(symmetry)} \\ \rho(u, v) &\leq \rho(u, w) + \rho(w, v). & & \text{(triangle inequality)} \end{aligned}$$

Together, the pair (M, ρ) is called a **metric space**. When the metric is clear from context we refer to the metric space by the symbol M alone.

Example 5.1.1 (Euclidean space). Euclidean vector space \mathbb{R}^d with $\rho(x, y) = \|x - y\|$ is a metric space. You can verify this using the properties of the norm.

Example 5.1.2 (Supremum norm and the space bX). Let X be any set and let bX be all bounded functions from X to \mathbb{R} . For all f, g in bX , let

$$\|f\|_\infty := \sup_{x \in X} |f(x)| \quad \text{and} \quad d_\infty(f, g) := \|f - g\|_\infty.$$

The map $f \mapsto \|f\|_\infty$ is called the **supremum norm** and d_∞ is called the **supremum distance**. The pair (bX, d_∞) is a metric space. The triangle inequality holds because, given f, g, h in bX and $x \in X$, we have (by the triangle inequality in \mathbb{R}),

$$|f(x) - g(x)| \leq |f(x) - h(x)| + |h(x) - g(x)| \leq d_\infty(f, h) + d_\infty(h, g).$$

The right side is an upper bound for the left side, so

$$d_\infty(f, g) \leq d_\infty(f, h) + d_\infty(h, g)$$

as was to be shown.

Example 5.1.3 (The space bcX). Let X be a metric space and let bcX be the set of all continuous functions in bX . Endowed with the distance d_∞ from Example 5.1.2, it is also a metric space. In quantitative modeling, this space is used frequently as a candidate space for solutions to dynamic problems, as will be seen below.

Example 5.1.4 (The space $\ell_p(X)$). This example is both a generalization of ordinary Euclidean space and a special case of Example 5.1.2. Let X be any discrete (i.e., countable) set and fix p with $1 \leq p \leq \infty$. If $p < \infty$, define

$$\|h\|_p := \left\{ \sum_{x \in X} |h(x)|^p \right\}^{1/p} \quad \text{and} \quad d_p(g, h) = \|g - h\|_p.$$

Now define the function space

$$\ell_p(X) := \{h \in \mathbb{R}^X : \|h\|_p < \infty\}.$$

The pair $(\ell_p(X), d_p)$ is a metric space. The triangle inequality has its own name in this setting: the **Minkowski inequality**. The inequality is in turn established via a generalization of the Cauchy–Schwarz inequality called the **Hölder inequality** that states that

$$\|fg\|_1 \leq \|f\|_p \|g\|_q \quad \text{whenever } p, q \in [1, \infty] \text{ with } 1/p + 1/q = 1$$

Note that with $X = \{1, \dots, d\}$ and $p = 2$ this is just Euclidean vector space, since $\mathbb{R}^X = \mathbb{R}^{\{1, \dots, d\}} =: \mathbb{R}^d$ and, for $h \in \mathbb{R}^d$

$$\|h\|_2 = \left\{ \sum_{i=1}^d |h(i)|^2 \right\}^{1/2} = \text{Euclidean norm of vector } h.$$

If $p = +\infty$, set

$$\|h\|_\infty := \sup_{x \in X} |h(x)| \quad \text{and} \quad d_\infty(g, h) = \|g - h\|_\infty.$$

In our current discrete setting this coincides with example 5.1.2.

Example 5.1.5 (Discrete metric). Let M be any nonempty set and consider the **discrete metric** on M given by $\rho(u, v) = \mathbb{1}\{u \neq v\}$. This is a metric on M , as the name suggests. To see that it satisfies the triangle inequality, pick any $u, v, w \in M$. We claim that $\rho(u, v) \leq \rho(u, w) + \rho(w, v)$. If $u = v$, this bound is trivial, so suppose they are distinct. We then need to show that $1 \leq \rho(u, w) + \rho(w, v)$. Suppose to the contrary that $\rho(u, w) + \rho(w, v) = 0$. It follows that $u = w$ and $v = w$. But then $u = v$ — a contradiction.

Given any point $u \in M$, the **ε -ball** around u is the set

$$B_\varepsilon(u) := \{v \in M : \rho(u, v) < \varepsilon\}$$

A **sequence** $\{x_n\}$ in M is a mapping $n \mapsto x_n$ from \mathbb{N} to M . We say that sequence $\{u_n\} \subset M$ **converges to $u \in M$** if

$$\forall \varepsilon > 0, \exists N \in \mathbb{N} \text{ s.t. } n \geq N \implies u_n \in B_\varepsilon(u)$$

Example 5.1.6. Recall that a sequence $\{x_n\}$ in \mathbb{R} converges to $x \in \mathbb{R}$ if, given any $\varepsilon > 0$, there is an $N \in \mathbb{N}$ such that $|x_n - x| < \varepsilon$ for all $n \geq N$. This is equivalent to the statement that $x_n \rightarrow x$ in the metric space (M, ρ) when $M = \mathbb{R}$ and $\rho(x, y) = |x - y|$.

EXERCISE 1. Let ρ be the discrete metric. Show that, for any $u \in M$, there exists an $\varepsilon > 0$ such that $B_\varepsilon(u) = \{u\}$. Show in addition that if $\{u_n\}$ is a sequence in M converging to some point in M , then $\{u_n\}$ is eventually constant.²

²A sequence $\{u_n\}$ is eventually constant if there exists a finite N such that $u_m = u_n$ whenever $n, m \geq N$.

EXERCISE 2. Show that limits in metric spaces are unique. In other words, show that if $u_n \rightarrow u$ and $u_n \rightarrow y$ in a metric space \mathbf{M} , then $u = y$.

As is the case with real numbers, a **subsequence** of a sequence $\{u_n\}$ in a metric space \mathbf{M} is a sequence of the form $\{u_{\sigma(n)}\}$ where σ is a strictly increasing function from \mathbb{N} to itself. You can think of forming a subsequence from a sequence by deleting some of its elements—while still retaining infinitely many.

EXERCISE 3. Prove that a sequence $\{u_n\}$ in a metric space \mathbf{M} converge to a point u in \mathbf{M} if and only if every subsequence of $\{u_n\}$ also converges to u .

Given two metric spaces (\mathbf{M}, ρ) and (\mathbf{N}, τ) , a function $f: \mathbf{M} \rightarrow \mathbf{N}$ is called **continuous at $u \in \mathbf{M}$** if

$$f(u_n) \rightarrow f(u) \text{ in } (\mathbf{N}, \tau) \quad \text{whenever} \quad u_n \rightarrow u \text{ in } (\mathbf{M}, \rho)$$

We call f **continuous on \mathbf{M}** if f is continuous at all $u \in \mathbf{M}$.

EXERCISE 4. Show that every f mapping (\mathbf{M}, ρ) to (\mathbf{N}, τ) is continuous when ρ is the discrete metric.

Example 5.1.7. If \mathbf{M} is Euclidean vector space \mathbb{R}^d with the Euclidean distance and $f(x) = f(x_1, \dots, x_d) = \prod_{i=1}^d x_i$ on \mathbf{M} , then f is continuous at every $x \in \mathbf{M}$. This follows from one of the standard rules concerning real sequences and their interactions with algebraic operations given on page 6. Similar arguments can be applied to many standard functions.

A point $u \in A \subset \mathbf{M}$ is called **interior** to A if there exists an $\varepsilon > 0$ such that $B_\varepsilon(u) \subset A$.

EXERCISE 5. Let $\mathbf{M} = \mathbb{R}$ and let $\rho(x, y) = |x - y|$. Show that 1 is interior to $A := [0, 1)$ but 0 is not. Show that \mathbb{Q} , the set of rational numbers in \mathbb{R} , contains no interior points.

EXERCISE 6. Let \mathbf{M} be arbitrary and let ρ be the discrete metric. Let A be any subset of \mathbf{M} . Show that every point of A is interior to A .

A subset G of \mathbf{M} is called **open** in \mathbf{M} (or just **open**) if every $u \in G$ is interior to G .

Example 5.1.8. By Exercise 6, every subset of a discrete metric space is open.

EXERCISE 7. Let M be any metric space. Show that the ε -ball around u is open for any $u \in M$ and any $\varepsilon > 0$.

EXERCISE 8. Show that

- Arbitrary unions and finite intersections of open sets in M are open in M .
- Arbitrary intersections and finite unions of closed sets in M are closed in M .

The following theorem provides an alternative characterization of continuity. See, for example, Section 2.4 of [Maddox \(1988\)](#).

Theorem 5.1.1. *Let M and N be metric spaces. A function $f: M \rightarrow N$ is continuous on M if and only if $f^{-1}(G)$ is open in M whenever G is open in N .*

A subset F of M is called **closed** if given any sequence $\{u_n\}$ satisfying $u_n \in F$ for all n and $u_n \rightarrow u$ for some $u \in M$, the point u is in F . In other words, F contains the limit points of all convergent sequences that take values in F .

Example 5.1.9. Limits in \mathbb{R} preserve orders, so $a \leq x_n \leq b$ for all $n \in \mathbb{N}$ and $x_n \rightarrow x$ implies $a \leq x \leq b$. Thus, any closed interval $[a, b]$ in \mathbb{R} is closed in the standard (one dimensional Euclidean) metric.

Example 5.1.10. As in the definition on page 130, let X be a metric space and let bX be the set of all continuous functions in bX (see example 5.1.2 for the definition). The set bX is a closed set in bX because uniform limits of continuous functions are continuous.

EXERCISE 9. Let \mathcal{C} denote all continuously differentiable functions f from $[-1, 1]$ to \mathbb{R} . As before let $d_\infty(f, g) = \sup_{x \in S} |f(x) - g(x)|$. The set \mathcal{C} is *not* a closed subset of (bX, d_∞) . To prove this, show that $d_\infty(f_n, f) \rightarrow 0$ as $n \rightarrow \infty$ when

$$f_n(x) := (x^2 + 1/n)^{1/2} \quad \text{and} \quad f(x) := |x|$$

Conclude that (\mathcal{C}, d_∞) is not closed.

Theorem 5.1.2. *Let M be any metric space. A subset G of M is open if and only if G^c is closed.*

Try proving this as an exercise. The full proof can be found in any text on analysis.

5.1.1.2 Equivalence of Metrics

Since there are many metric spaces, and even many metrics that can be defined on a given set, it becomes valuable to collect spaces into groups with similar properties. One way to do this is via the notion of equivalence. Two metrics ρ_1 and ρ_2 on a given space M are said to be **equivalent** if there exist positive constants α, β such that

$$\alpha\rho_1(x, y) \leq \rho_2(x, y) \leq \beta\rho_1(x, y) \quad \text{for all } x, y \in M$$

(Some texts call this property “strong equivalence.”) To see why this might matter, try the following exercises:

EXERCISE 10. Let ρ_1 and ρ_2 be equivalent metrics on M . Verify the following:

- (i) If $\{x_n\}$ is convergent in (M, ρ_1) then the same sequence is convergent in (M, ρ_2) .
- (ii) If a sequence is Cauchy in (M, ρ_1) then the same is true in (M, ρ_2) .
- (iii) If $C \subset M$ has property P in (M, ρ_1) then C has property P in (M, ρ_2) , when $P =$ open, closed, bounded, precompact or compact.
- (iv) If $f: M \rightarrow \mathbb{R}$ is continuous on (M, ρ_1) then f is continuous on (M, ρ_2) .

In view of Exercise 10, when ρ_1 and ρ_2 are equivalent on M , the metric spaces (M, ρ_1) and (M, ρ_2) have very similar properties. This can be very helpful when multiple metrics are in play. One particularly important case is equivalence of metrics on some versions of $\ell_p(X)$, as defined on page 130:

Theorem 5.1.3. *If X is finite, then, whenever $1 \leq p, q \leq \infty$, the metrics d_p and d_q are equivalent on $\ell_p(X) = \ell_q(X)$.*

(The fact that $\ell_p(X) = \ell_q(X)$ when X is finite is immediate from the definition of these spaces.) Theorem 5.1.3 is due to a more general result on norms over finite dimensional spaces, which will be discussed later in the text (see Theorem 11.1.5).

5.1.1.3 Completeness

Analogous to the case of scalar sequences, a sequence $\{u_n\} \subset M$ is called **Cauchy** if, given any $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that $n, m \geq N$ implies $\rho(u_n, u_m) < \varepsilon$.

EXERCISE 11. Show that if $M = \mathbb{R}$, $\rho(u, v) = |u - v|$ and $u_n = 1/n$, then $\{u_n\}$ is Cauchy.

A metric space (M, ρ) is called **complete** if every Cauchy sequence in M converges to some point in M . Loosely speaking, under completeness, sequences that “look convergent” do in fact converge to some point in the space. This turns out to be vital for various aspects of fixed point theory and for the success of many iterative algorithms. Hence the completeness property is highly valued, and we will work almost exclusively with complete metric spaces.

One famous result about the real line with the standard distance is that \mathbb{R} is complete. The completeness of \mathbb{R} is sometimes stated as a theorem but it’s essentially an axiom. You can think of completeness of \mathbb{R} as part of its definition. Once the irrational numbers are mixed in with the rational numbers, there are “no more gaps” in the real line.

Example 5.1.11. The following spaces are complete:

- (i) The space $(\ell_p(X), d_p)$ is complete for every countable X and every $p \in [1, \infty]$.
- (ii) As a special case of the last result, ordinary Euclidean space is complete.
- (iii) The spaces (bX, d_∞) and (bcX, d_∞) are both complete.

Here $(\mathbb{R}^n, \|\cdot\|) = (\mathbb{R}^n, \rho)$ where ρ is Euclidean distance. The proof of these claims uses completeness of \mathbb{R} and can be found in any textbook on analysis.

EXERCISE 12. Let $M = (0, 1)$ and $\rho(u, y) = |u - y|$. Show (M, ρ) is not complete.

EXERCISE 13. Given two equivalent metric spaces, show that completeness of one implies completeness of the other.

EXERCISE 14. Let (M, ρ) be complete and $C \subset M$ be closed. Show that (C, ρ) is a complete metric space.

5.1.1.4 Bounded Sets and Compact Sets

A set D in M is called **bounded** if there exists a finite K such that $d(u, v) \leq K$ whenever $u, v \in D$.

EXERCISE 15. Show that a subset D of M is bounded if and only if there exists an ε -ball $B_\varepsilon(u)$ such that $u \in D$ and $D \subset B_\varepsilon(u)$.

A subset K of M is called **precompact** in M if every sequence in K has a subsequence converging to some point in M . The set K is called **compact** if, in addition,

the limit points always lie in K . Thus, K is compact if and only if K is closed and precompact.

EXERCISE 16. Prove that every precompact subset of a metric space is bounded.

The converse is not true. For example, consider the set $b\mathbb{R}$ with the supremum distance (example 5.1.2). Let f_n be the normal density with variance 1 and mean n for each n in \mathbb{N} . The set $\{f_n\}_{n \in \mathbb{N}}$ is bounded, since $d_\infty(f_n, f_m) \leq 1$ for all n, m . But it is not precompact. For example, the sequence $\{f_n\}_{n \in \mathbb{N}}$ has not convergent subsequence. Indeed, every pair of distinct points f_n, f_m in the sequence has $d_\infty(f_n, f_m) = 1$.

Nonetheless, there is one very important setting where boundedness does imply precompactness, an issue that we return to in Theorem 5.1.4 below.

One way to understand (pre)compactness is as a generalization of finiteness. In fact, these two concepts coincide when ρ is the discrete metric. To see this, note that, under this metric, any sequence in M taking only distinct values has no convergent subsequence. (Why?) Hence, if K is infinite, then K is not compact (or even precompact). Conversely, if K is finite, then every sequence has a constant subsequence. So K is compact.

Here are two other ways that precompactness resembles finiteness:

EXERCISE 17. Show that (i) every subset of a precompact subset of M is also precompact in M , and (ii) the union of finitely many precompact (resp., compact) subsets of M is also precompact (resp., compact) in M

The following is a fundamental result about bounded sets in \mathbb{R}^d .

Theorem 5.1.4 (Bolzano–Weierstrass). *A subset of Euclidean space \mathbb{R}^d is bounded if and only if it is precompact.*

Another way to state this is that, in \mathbb{R}^d , the compact sets are precisely those that are both closed and bounded. In infinite dimensional space, however, this equivalence no longer holds.

Example 5.1.12. Consider the space $(\ell_p(X), d_p)$ defined on page 130. Let $p = 1$, let $X = \{x_1, x_2, \dots\}$ be countably infinite, and consider the sequence $\{h_n\} \subset \ell_1(X)$ defined by $h_n(x) = \mathbb{1}\{x = x_n\}$. For any $m \neq n$ we have

$$d_1(h_n, h_m) = \sum_x |h_n(x) - h_m(x)| = 2$$

Clearly $\{h_n\}$ has convergent subsequence in $\ell_1(X)$. In particular, $\{h_n\}$ is bounded but not precompact.

The next theorem gives one more way that compactness and finiteness are related.

Theorem 5.1.5 (Alexandrov–Urysohn). *A subset K of a metric space M is compact if and only if every open cover of K has a finite subcover.*

The meaning here is that, if $\{\mathcal{G}\}$ is any collection of open sets such that K is contained in the union of these sets, then there exists a finite subcollection of sets in $\{\mathcal{G}\}$ such that K lies in the union of this subcollection. A proof of Theorem 5.1.5 can be found in §2.3 of [Dudley \(2002\)](#).

A final point for this section is that continuity and compactness have important interactions. Perhaps the most important is that continuous functions carry compact sets to compact sets:

Lemma 5.1.6. *If f is a continuous function from one metric space M to another metric space N , then $f(K)$ is compact in N whenever K is compact in M .*

An elegant proof can be constructed using Theorem 5.1.5 combined with the characterization of continuity in Theorem 5.1.1.

5.1.2 Fixed Points

Let M be any set and let S be a function defined on M . In this section we often write Su instead of $S(u)$ to represent the image of u under S . We say that S is a **self-map** on M if Su is in M for all u in M . In other words, S maps M to itself. A point u in M is called a **fixed point** of S if $Su = u$. Figure 5.1 shows an example where $M = [0, 2]$ and the function S is defined by $S = g$ where $g(x) = 2.125/(1 + x^{-4})$. Fixed points are numbers $x \in [0, 2]$ where g meets the 45 degree line. In this case there are three.

The next lemma is helpful in locating fixed points.

Lemma 5.1.7. *Let S be a self-map on metric space M . If $S^t u \rightarrow u^*$ for some pair $u, u^* \in M$ and S is continuous at u^* , then u^* is a fixed point of S .*

Proof. Assume the hypotheses of Lemma 5.1.7 and let $u_t := S^t u$. Let $\bar{u} := Su^*$. By continuity we have $Su_t \rightarrow Su^* = \bar{u}$. But $\{Su_t\}$ is just $\{u_t\}$ with the first element omitted, so, given that $u_t \rightarrow u^*$, we must have $Su_t \rightarrow u^*$. Since limits are unique, we now have $\bar{u} = u^*$, implying $u^* = Su^*$. \square

The very best scenario in terms of stability of S on M is when S has a unique fixed point u^* in M and $S^n u \rightarrow u^*$ as $n \rightarrow \infty$ for every u in M . Anticipating terminology used to discuss dynamic models in §5.3, any such self-map S is called **globally stable** on M . The map g studied in Figure 2.2 has this property when $M = (0, \infty)$.

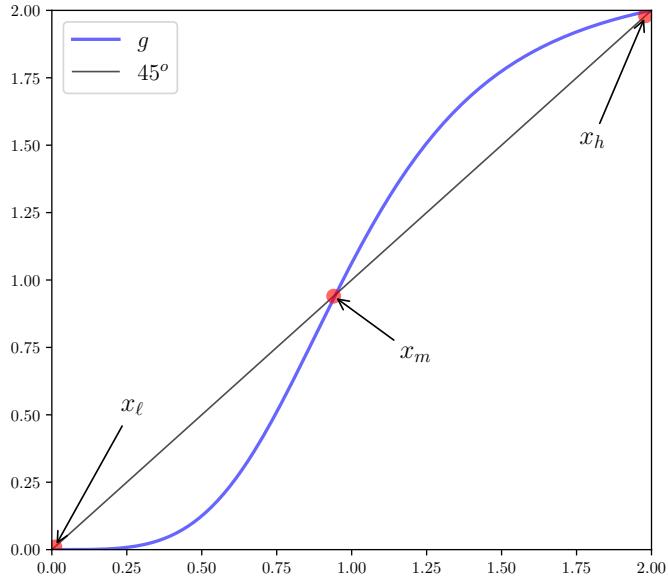


Figure 5.1: Graph and fixed points of $g(x) = 2.125/(1 + x^{-4})$

5.1.2.1 Brouwer's Theorem

One fixed point theorem routinely used in economics, due to Luitzen Egbertus Jan Brouwer (1881–1966), is as follows:

Theorem 5.1.8 (Brouwer). *If C is a convex compact subset of \mathbb{R}^d and S is a continuous self-map on C , then S has at least one fixed point in C .*

For a proof, see, for example, [add cite](#). The proof is not trivial. An alternative way to build intuition is to sketch the case where $C = [0, 1]$ in \mathbb{R} , and fixed points are intersections between the function and the 45 degree line.

The disadvantage of Theorem 5.1.8 is that it provides only existence, without uniqueness or any form of stability. The main advantage of the theorem is that its conditions are relatively weak.

There are many useful extensions to Brouwer's theorem, such as one for set-valued functions due to [Kakutani \(1941\)](#).³ Another, due to [Schawder \(1930\)](#), extends Brouwer's result to infinite dimensional spaces.

³See, for example, a rather famous little paper due to [Nash \(1950\)](#).

5.1.2.2 Contractions

One of the most important approaches fixed point theory is via contractive maps. Below, a self-map S on metric space (M, ρ) is called

- **nonexpansive** if $\rho(Su, Sv) \leq \rho(u, v)$ for all u, v in M ,
- **strictly contracting** if $\rho(Su, Sv) < \rho(u, v)$ for all distinct u, v in M ,
- **uniformly contracting** or **a contraction of modulus λ** if there exists a $\lambda < 1$ such that

$$\rho(Su, Sv) \leq \lambda \rho(u, v) \quad \text{for all } u, v \in M \quad (5.2)$$

EXERCISE 18. Show that every strictly contracting self-map S on M has at most one fixed point. Is the same true if S is only nonexpansive?

EXERCISE 19. Prove: If S is nonexpansive on M , then S is continuous on M .

Theorem 5.1.9 (Banach's contraction mapping theorem). *Let M be a complete metric space and let self-map S be a contraction of modulus λ on M . Then S is globally stable on M with unique fixed point u^* satisfying*

$$\rho(S^n u, u^*) \leq \lambda^n \rho(u, u^*) \quad \text{for all } n \in \mathbb{N}$$

This fundamental result has many practical implications. For a proof, see, for example, [Aliprantis and Border \(1999\)](#), theorem 3.36, or any other text that treats metric spaces. The main line of argument runs as follows: First one exploits the contraction condition to show that, for any initial condition u , the sequence $\{S^n u\}$ is Cauchy. By completeness, we then have existence of a point u^* such that $S^n u \rightarrow u^*$. The fact that u^* is a fixed point of S now follows from Lemma 5.1.7 and Exercise 19. Uniqueness is implied by Exercise 18.

For most of the conclusions of Banach's contraction mapping theorem, it suffices that the k -th composition S^k is a uniform contraction on M for some k . The next result gives details.

Theorem 5.1.10. *Let M be a complete metric space. If S is a self-map on M and S^k is a uniform contraction on M for some $k \in \mathbb{N}$, then S is globally stable on M .*

EXERCISE 20. Prove Theorem 5.1.10. (Hint: Theorem 5.1.9 is self-improving! It implies this seemingly stronger result.)

If S is strictly contracting, then trajectories slow down, in the sense that each step is smaller than the last:

$$s_{n+1} := \rho(S^{n+1}u, S^n u) = \rho(SS^n u, SS^{n-1} u) < \rho(S^n u, S^{n-1} u) =: s_n$$

Unlike uniform contractivity (see page 139), however, strict contractivity is not enough for existence of a fixed point, even when M is complete.⁴ The problem is that, although the step size decreases, it might not decrease fast enough to force convergence.

If, however, some other force prevents divergence, such as compactness of the state space, then stability returns. Here's a typical example:

Proposition 5.1.11. *If M is a compact metric space and S is strictly contracting, then S is globally stable on M .*

For a proof see, for example, pp. 145–146 of Stachurski (2003).

5.1.3 Order

Add roadmap.

5.1.3.1 Suprema and Infima on the Real Line

Points in \mathbb{R} are ordered by the standard relation \leqslant . An **upper bound** of a subset A of \mathbb{R} is any u such that $a \leqslant u$ for all $a \in A$. Write $U(A)$ for the set of all upper bounds of A . If $s \in U(A)$ and $s \leqslant u$ for all $u \in U(A)$, then s is called the **supremum** of A and we write $s = \sup A$. At most one such supremum s exists. (Why?) If s is in $U(A)$ then the following are equivalent:

- (i) $s = \sup A$
- (ii) for all $\varepsilon > 0$, there exists a point $a \in A$ with $a > s - \varepsilon$

EXERCISE 21. Prove the last claim. Prove also that $\sup(0, 1] = 1$ and $\sup(0, 1) = 1$.

Theorem 5.1.12. *Every nonempty subset of \mathbb{R} which is bounded above has a supremum in \mathbb{R} .*

This is *equivalent* to the axiom that every Cauchy sequence in \mathbb{R} converges (and also to theorem 2.1.1 above). If A is not bounded above, then it is conventional to set $\sup A := \infty$. With this convention,

⁴For example, consider $(M, \rho) = (\mathbb{R}_+, |\cdot|)$ and $Su = u + \exp(-u)$.

EXERCISE 22. Prove: If $A \subset B$, then $\sup A \leq \sup B$.

For $A \subset \mathbb{R}$ a **lower bound** of A is any number l such that $l \leq a$ for all $a \in A$. If $i \in \mathbb{R}$ is a lower bound for A and also satisfies $i \geq l$ for every lower bound l of A , then i is called the **infimum** of A and we write $i = \inf A$. At most one such i exists, and every nonempty subset of \mathbb{R} bounded from below has an infimum.

5.1.3.2 Partial Orders

In addition to the usual order \leq on \mathbb{R} , there's also a natural notion of order that we can apply to \mathbb{R}^d : a vector $x = (x_1, \dots, x_d)$ is dominated by another vector $y = (y_1, \dots, y_d)$ if $x_i \leq y_i$ for all i . In this case we write $x \leq y$. This notion of order is called the **pointwise order** on \mathbb{R}^d and it has many uses. Rather than study its properties directly, however, we will first shift to a more general setting. In particular, just as we extracted the key features of Euclidean *distance* to try to leverage results about Euclidean space in more general settings, we apply the axiomatic method to generalize pointwise *order*.

To this end, we define a **partial order** on nonempty set P to be a relation \preceq on $P \times P$ satisfying, for any x, y, z in P ,

$$\begin{aligned} x \preceq x, & & & \text{(reflexivity)} \\ x \preceq y \text{ and } y \preceq x \text{ implies } x = y \text{ and} & & & \text{(antisymmetry)} \\ x \preceq y \text{ and } y \preceq z \text{ implies } x \preceq z & & & \text{(transitivity)} \end{aligned}$$

Example 5.1.13. The usual order \leq on \mathbb{R} is obviously a partial order.

EXERCISE 23. Let P be any set and consider the relation induced by equality, so that $x \preceq y$ if and only if $x = y$. Show that this relation is a partial order on P .

EXERCISE 24. Let M be any set and let $\wp(M)$ be the set of all subsets of M . Show that \subset is a partial order on $\wp(M)$.

Example 5.1.14. Let X be any set. For f, g in \mathbb{R}^X , we write

$$f \leq g \text{ if } f(x) \leq g(x) \text{ for all } x \in X.$$

This relation \leq on \mathbb{R}^X is called the **pointwise partial order** on \mathbb{R}^X and we will use it extensively. The axioms in the definition of a partial order are easy to check.

When paired with a partial order \preceq , the set P (or the pair (P, \preceq)) is called a **partially ordered set**. Some authors abbreviate to **poset**.

Our next step is to generalize the concepts concerning bounds, suprema and infima, which were discussed for the case $\mathsf{P} \subset \mathbb{R}$ in §5.1.3.1, to the present abstract setting. You will notice that the definitions are often identical, apart from a switch from the concrete order \leq on \mathbb{R} to the abstract partial order \preceq on a partially ordered set.

First, given a subset E of a partially ordered set P , we call $u \in \mathsf{P}$ an **upper bound** of E in P if $e \preceq u$ whenever $e \in E$. We write $U(E)$ for the set of upper bounds of E . If there exists an $s \in \mathsf{P}$ such that

- (i) $s \in U(E)$ and
- (ii) $s \preceq u$ whenever $u \in U(E)$,

then s is called the **supremum** of E in P .

EXERCISE 25. Show that a subset E of P can have at most one supremum.

EXERCISE 26. Let K be a finite constant and let \mathcal{G} be a collection of functions in $b\mathsf{X}$ that lies inside $B_K(0) = \{f \in b\mathsf{X} : \|f\|_\infty \leq K\}$. Endow $b\mathsf{X}$ with the pointwise partial order \leq . Show that the infimum and supremum of \mathcal{G} in $(b\mathsf{X}, \leq)$ are, respectively,

$$\check{g}(x) := \inf_{g \in \mathcal{G}} g(x) \quad \text{and} \quad \hat{g}(x) := \sup_{g \in \mathcal{G}} g(x) \quad (x \in \mathsf{X}) \quad (5.3)$$

These definitions are not circular because, in (5.3), the sup and inf on the right hand side are taken in \mathbb{R} , following the rules in §5.1.3.1.

Similarly, $\ell \in \mathsf{P}$ is called a **lower bound** of E in P if $\ell \preceq e$ for every $e \in E$. If there exists an $i \in \mathsf{P}$ such that

- (i) i is a lower bound of E and
- (ii) $\ell \preceq i$ whenever ℓ is a lower bound of E ,

then i is called the **infimum** of E in P .

A subset B of a partially ordered set (P, \preceq) is called

- **increasing** if $x \in B$ and $x \preceq y$ implies $y \in B$.
- **decreasing** if $x \in B$ and $y \preceq x$ implies $y \in B$.

Example 5.1.15. When \mathbb{R}^d is paired with the pointwise order \leqslant , its **positive cone**

$$\mathbb{R}_+^d := \{x \in \mathbb{R}^d : x \geqslant 0\}$$

is an increasing subset of $(\mathbb{R}^d, \leqslant)$.

You will be familiar with the notation $[a, b]$ as a representation of all $x \in \mathbb{R}$ such that $a \leqslant x \leqslant b$. This notation is extended to a partially ordered set (P, \preceq) as follows: Given points a and b in P , the **order interval** $[a, b]$ is defined as all $x \in P$ such that $a \preceq x \preceq b$.

Example 5.1.16. If g, h are functions in bcX for some metric space X , then $[g, h]$ is all continuous functions f in \mathbb{R}^X such that $g(x) \leqslant f(x) \leqslant h(x)$ for all x in X .

Given two partially ordered sets (P, \preceq) and (Q, \trianglelefteq) , a function S from P to Q is called **isotone** if

$$x \preceq y \implies Sx \trianglelefteq Sy \quad (5.4)$$

and **antitone** if $x \preceq y$ implies $Sy \trianglelefteq Sx$. If $P = Q = \mathbb{R}$ and \preceq and \trianglelefteq are both equal to \leqslant , the standard order on \mathbb{R} , then isotonicity reduces to the usual notion of an **increasing function** (i.e., nondecreasing function), and we will use the terms “increasing” and “isotone” interchangeably in this setting.⁵ Similarly, real-valued antitone functions on \mathbb{R} will also be called **decreasing functions**.

In addition, if $S = g$ maps $A \subset \mathbb{R}$ into \mathbb{R} , then we will call g

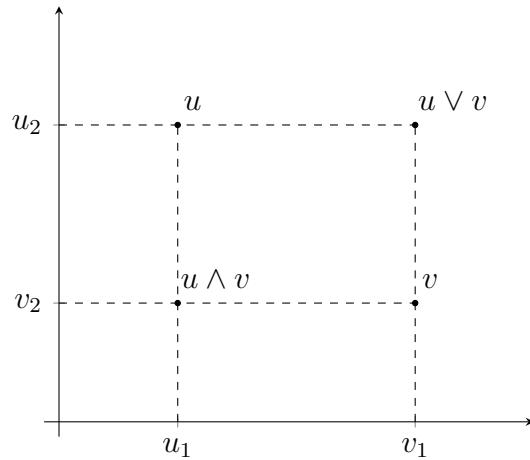
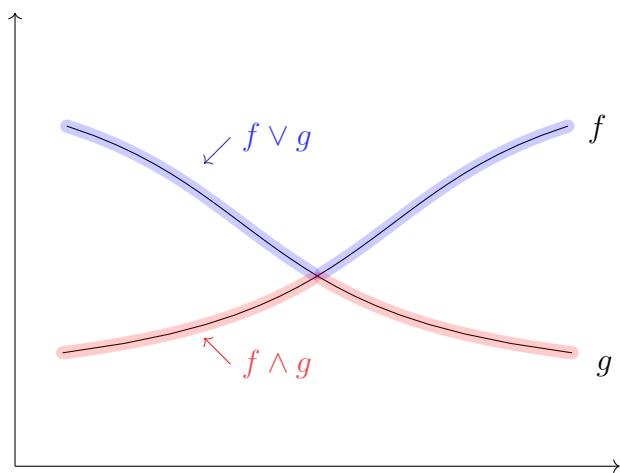
- **strictly increasing** if $x < y$ implies $g(x) < g(y)$, and
- **strictly decreasing** if $x < y$ implies $g(x) > g(y)$.

Given u and v in P , the supremum of the two point set $\{u, v\}$, when it exists, is also called **join** of u and v , and is written $u \vee v$. The infimum of $\{u, v\}$, when it exists, is also called **meet** of u and v , and is written $u \wedge v$. Figure 5.2 provides a visualization when $P = \mathbb{R}^2$. Figure 5.3 when $P = bX$. It illustrates $f \wedge g$ and $f \vee g$ when f and g are functions defined on an interval in \mathbb{R} . In both cases, \preceq is the usual pointwise partial order.

Suprema and infima do not always exist.

Example 5.1.17. Consider $P = \mathbb{R}$ with the usual order, where $E = \mathbb{R}_+$ has no upper bounds in P and hence no supremum.

⁵Other common terms for isotone in the literature include “monotone increasing,” “monotone” and “order-preserving.”

Figure 5.2: The points $u \vee v$ and $u \wedge v$ in \mathbb{R}^2 Figure 5.3: Functions $f \vee g$ and $f \wedge g$ when defined on a subset of \mathbb{R}

If (P, \preceq) has the property that every *finite* subset of P has both a supremum and an infimum then (P, \preceq) is called a **lattice**. If *every* subset of P has both a supremum and an infimum, then (P, \preceq) is called a **complete lattice**.

Obviously, in a lattice, $u \vee v$ and $u \wedge v$ both exist in P .

Example 5.1.18. The set bcX of continuous bounded functions on metric space X is a lattice when endowed with the pointwise partial order, since the pointwise supremum and pointwise infimum of any finite collection of continuous functions are both continuous. On the other hand, the set of continuously differentiable functions on $[-1, 1]$ is not a lattice under the pointwise partial order. For example, the supremum of $\{x \mapsto x, x \mapsto -x\}$ is $x \mapsto |x|$.

EXERCISE 27. Following on from example 5.1.18, show by counterexample that bcX is not a complete lattice.

If L is a subset of P such that $u \wedge v \in L$ and $u \vee v \in L$ whenever $u, v \in L$, then L is called a **sublattice** of (P, \preceq) .

EXERCISE 28. Prove: If P is the positive cone of \mathbb{R}^d , then both P and the interior of P are sublattices of \mathbb{R}^d .

When partial orders live on metric spaces, we want them to play well with that metric. This is the idea behind the following definition: A partial order \preceq on metric space (M, ρ) is called **closed** if

$$x_n \rightarrow x, y_n \rightarrow y \text{ and } x_n \preceq y_n \text{ for all } n \in \mathbb{N} \implies x \preceq y \quad (5.5)$$

For example, on \mathbb{R}^d with Euclidean distance, the pointwise partial order is closed. This is because convergence in Euclidean distance implies pointwise convergence, and hence convergence in \mathbb{R} for all components of the vectors ($x_n^i \rightarrow x^i$ for all i , etc.). Scalar limits preserve the usual order \leqslant in \mathbb{R} , as mentioned above.

5.1.3.3 Order-Theoretic Fixed Point Methods

We begin with an order-theoretic fixed point result that has applications in many fields. Like Brouwer's theorem, it provides existence but not uniqueness or convergence.

Theorem 5.1.13 (Knaster–Tarski). *If (P, \preceq) is a complete lattice and S is an isotone self-map on P , then the set of fixed points of S is also a complete lattice.*

Since complete lattices are posets and hence, by definition, nonempty, one implication is that at least one fixed point exists. Since complete lattices have both a supremum and an infimum, another implication is the existence of both a least fixed point and a greatest fixed point. To get a sense of why theorem 5.1.13 is true, let $P = [0, 1]$ and try to draw an increasing function S that maps P to itself and yet fails to touch the 45 degree line. You will find that it's impossible.

Here's a partial proof of the theorem that shows existence of a fixed point. Note carefully where we use the fact that (P, \preceq) is a complete lattice, as well as the properties of a partial order: Let U be all $u \in P$ such that $u \preceq Su$. This set is nonempty because $i = \inf P$ must satisfy $i \preceq Si$. Let $\bar{u} = \sup U$. We claim that \bar{u} is a fixed point, for which it suffices to obtain the two inequalities $\bar{u} \preceq S\bar{u}$ and $S\bar{u} \preceq \bar{u}$. Regarding the first inequality, given any $u \in U$, we have $u \preceq \bar{u}$, by definition of \bar{u} , and hence $Su \preceq S\bar{u}$. But $u \preceq Su$, so $u \preceq S\bar{u}$. It follows that $S\bar{u}$ is an upper bound of U . As \bar{u} is a least upper bound, we have $\bar{u} \preceq S\bar{u}$. To complete the argument, we use the last inequality to obtain $S\bar{u} \preceq SS\bar{u}$, which implies that $S\bar{u} \in U$. Since \bar{u} is an upper bound of U , this yields $S\bar{u} \preceq \bar{u}$. The proof is done.

A limitation of the Knaster–Tarski fixed point theorem is that it says nothing about uniqueness. For example, since a closed interval $[a, b]$ in \mathbb{R} is a complete lattice for any $a < b$, figure 5.1 shows that uniqueness does not hold under the conditions of the theorem. We need additional properties that control the shape of the self-map.

A path forward is suggested by our investigation of the Solow model in chapter 1, and in particular Figure 2.2 on p. 9. It is clear from the figure that we will have only one fixed point if the self-map is concave, starts above the 45 degree line and is eventually below it. Moreover, the fact that, in the Solow model, capital stock always converges to the unique fixed point can be carried over to a more general result. The following is one such theorem, proved in a very general setting (which we discuss later) by Du (1990).⁶ In stating it we write

$$x \ll y \iff y - x \text{ is a strictly positive vector}$$

and $x \gg y$ if $y - x$ is strictly positive. Another way to phrase this is that $y - x$ lies in the interior of the positive cone.

Theorem 5.1.14 (Du). *Let $[a, b]$ be a nonempty order interval in \mathbb{R}^d and let S be an isotone map from $[a, b]$ to \mathbb{R}^d . If either*

- (i) *S is concave, $Sa \gg a$ and $Sb \ll b$, or*
- (ii) *S is convex, $Sa \geq a$ and $Sb \leq b$*

⁶ Aside from the original reference, a proof can be found in corollary 2.1.1 of Zhang (2012).

then S has a unique fixed point \bar{x} in $[a, b]$ and, moreover, there exist constants $M > 0$ and $\lambda \in (0, 1)$ such that $\|S^n x - \bar{x}\| \leq \lambda^n M$ whenever $x \in [a, b]$ and $n \in \mathbb{N}$.

From this theorem, we can derive the following corollary. The corollary is nontrivial because the domain of the function is a sublattice, which need not be an order interval. In particular, it can be unbounded.

Corollary 5.1.15. *Let L be a sublattice of \mathbb{R}^d and let S be a self-map on L with the following properties:*

- (i) *For all $x \in L$, there exists a point $a \in L$ with $a \leq x$ and $Sa \gg a$.*
- (ii) *For all $x \in L$, there exists a point $b \in L$ with $b \geq x$ and $Sb \ll b$.*

If S is also isotone and either concave or convex, then S is globally stable on L .

Proof. Pick any $x \in L$. By assumption, we can choose a, b in L with $a \leq x \leq b$, $Sa \gg a$ and $Sb \ll b$. It follows from Theorem 5.1.14 that S has a fixed point \bar{x} in $[a, b]$.

Now let v be any other point in L . We claim that $S^n v \rightarrow \bar{x}$ as $n \rightarrow \infty$. To see this, we use the stated hypotheses to choose c, d in L such that

$$c \leq a \wedge v, \quad d \geq b \vee v, \quad Sc \gg c \quad \text{and} \quad Sd \ll d$$

Applying Theorem 5.1.14 again, the map S is globally stable on $[c, d]$. Both \bar{x} and v lie in $[c, d]$ and \bar{x} is a fixed point of S . Therefore $S^n v \rightarrow \bar{x}$ as required.

Since every point in L converges to \bar{x} under iteration of S , it is also clear that \bar{x} is the only fixed point of S in all of L . \square

5.1.4 Orders over Distributions

Distributions are objects that decision makers naturally have preferences over. For example, a speculator has ideas about probability distributions of returns of prospective investments, perhaps preferring distributions that offer high expected returns with low risk. A planner might have preferences over the cross-sectional distributions of consumption and wealth. In this section, we discuss common methods for ordering distributions and their relationships with each other.

5.1.4.1 Stochastic Dominance

Let \mathbf{X} equal \mathbb{R} or an interval in \mathbb{R} and let \mathcal{F} be the set of all distributions on \mathbf{X} , each one represented by a cumulative distribution function F . For F and G in \mathcal{F} we say that G **first order stochastically dominates** F if

$$\int u(x)F(dx) \leq \int u(x)G(dx) \text{ for every increasing function } u \text{ in } b\mathbf{X}$$

In this case we write $F \preceq_F G$.

Example 5.1.19. If x and y are points in \mathbb{R} with $x \leq y$, then $\delta_x \preceq_F \delta_y$, since, for any increasing function u we have $u(x) \leq u(y)$.

EXERCISE 29. Let Y be a random variable on \mathbb{R} with distribution F . Let m be a nonnegative constant and let G be the distribution of $Y + m$. Show that F is stochastically dominated by G .

We say that G **second order stochastically dominates** F if

$$\int u(x)F(dx) \leq \int u(x)G(dx) \text{ for every concave increasing function } u \text{ in } b\mathbf{X}$$

In this case we write $F \preceq_S G$. If sometimes we simply refer to stochastic dominance without explicitly stating the order, then the understanding is that we mean first order stochastic dominance.

In the preceding definitions, u is increasing in the usual sense: $x \leq y$ implies $u(x) \leq u(y)$ where \leq is the standard order. Boundedness is required only to ensure that the integral is well defined and finite.

The definition of first order stochastic dominance can be understood as follows: For a each member of a class of agents who observe outcomes taking values in set \mathbf{X} , who prefer more to less in the sense that $x \preceq y$ implies $u(x) \leq u(y)$, and who rank \mathbf{X} -valued gambles in terms of expected utility, drawing from G rather than F improves expected utility when G is stochastically dominant.

When testing first order stochastic dominance, one can show it is sufficient to restrict attention to increasing functions $u \in b\mathbf{X}$ that take the form $u(x) = \mathbb{1}\{a < x\}$ for some $a \in \mathbf{X}$. Recalling the interpretation of the integral given in (4.1), this leads to the statement that $F \preceq_F G$ if and only if $1 - F(a) \leq 1 - G(a)$ for all $a \in \mathbf{X}$, or

$$F \preceq_F G \iff G(x) \leq F(x) \quad \text{for all } x \in \mathbf{X} \tag{5.6}$$

EXERCISE 30. Show that the relation \preceq_F yields a partial order on \mathcal{F} .

Discuss second order SD in the same context. Antisymmetry should fail. Example?

EXERCISE 31. Consider the set \mathcal{F} paired with the metric

$$d_\infty(F, G) = \sup_{x \in X} |F(x) - G(x)|$$

In this context, the metric d_∞ is usually called the **Kolmogorov distance**. Show that \preceq_F is a *closed* partial order on (\mathcal{F}, d_∞) .

5.1.4.2 Montone Likelihood Ratios

Here is a property that implies first order stochastic dominance: Two distributions with positive densities f and g on an interval I contained in \mathbb{R} are said to have a **monotone likelihood ratio** if f/g is increasing on I ; that is, if

$$x, x' \in I \text{ and } x \leq x' \implies \frac{f(x)}{g(x)} \leq \frac{f(x')}{g(x')} \quad (5.7)$$

Example 5.1.20. The exponential density is $p(x, \lambda) = \lambda e^{-\lambda x}$ on \mathbb{R}_+ , where λ is a positive constant. Taking the ratio of two exponential densities with $\lambda_1 \leq \lambda_2$, we have

$$\frac{p(x, \lambda_1)}{p(x, \lambda_2)} = \frac{\lambda_1}{\lambda_2} \exp((\lambda_2 - \lambda_1)x)$$

The right hand side is increasing in x , so the monotone likelihood ratio property holds.

Proposition 5.1.16. *If f and g have the monotone likelihood ratio property on I , then the distribution represented by f first order stochastically dominates that represented by g .*

Proof. Let $a := \inf I$ and $b := \sup I$. (These values can be infinite.) Writing the monotone likelihood ratio property as

$$x \leq x' \implies f(x)g(x') \leq f(x')g(x) \quad (5.8)$$

and integrating with respect to x from a to x' gives

$$F(x')g(x') \leq f(x')G(x')$$

On the other hand, integrating (5.8) with respect to x' from x to b gives

$$f(x)[1 - G(x)] \leq [1 - F(x)]g(x)$$

Setting $x = x' = y$ in the last two inequalities yields

$$\frac{1 - G(y)}{1 - F(y)} \leq \frac{g(y)}{f(y)} \leq \frac{G(y)}{F(y)}$$

This implies $F(y) \leq G(y)$ for arbitrary y , so $G \preceq_F F$. □

5.1.4.3 Mean-Preserving Spreads

Finally, we introduce the notion of a mean-preserving spread. For a given distribution φ , we say that ψ is a **mean-preserving spread** of φ if there exists a pair of random variables (Y, Z) such that

$$\mathbb{E}[Z | Y] = 0, \quad Y \stackrel{d}{=} \varphi \quad \text{and} \quad Y + Z \stackrel{d}{=} \psi$$

In other words ψ is a mean-preserving spread of φ if it adds noise without changing the mean.

5.1.5 Optimization

A number m contained in a subset A of \mathbb{R} is called the **maximum** of A and we write $m = \max A$ if $a \leq m$ for every $a \in A$. It is called the **minimum** of A if $a \geq m$ for every $a \in A$. For finite subsets of \mathbb{R} , maxima and minima clearly exist. For infinite collections the same is not true. For example, the set $(0, 1)$ has neither a maximum nor a minimum.

EXERCISE 32. Prove the following two statements:

- (i) If $s = \sup A$ and $s \in A$, then $s = \max A$.
- (ii) If $i = \inf A$ and $i \in A$, then $i = \min A$.

EXERCISE 33. Show that, if A is a closed and bounded subset of \mathbb{R} , then A has both a maximum and a minimum.

Given an arbitrary set D and a function $f: D \rightarrow \mathbb{R}$, define

$$\sup_{x \in D} f(x) := \sup\{f(x) : x \in D\} \quad \text{and} \quad \max_{x \in D} f(x) := \max\{f(x) : x \in D\}$$

whenever the latter exists. A point $x^* \in D$ is called a

- **maximizer** of f on D if $x^* \in D$ and $f(x^*) \geq f(x)$ for all $x \in D$, and a
- **minimizer** of f on D if $x^* \in D$ and $f(x^*) \leq f(x)$ for all $x \in D$.

Equivalently, $x^* \in D$ is a maximizer of f on D if $f(x^*) = \max_{x \in D} f(x)$, and a minimizer if $f(x^*) = \min_{x \in D} f(x)$.

The following lemma will be handy for dynamic programming:

Lemma 5.1.17. *Let D be any set. If f and g are bounded functions in \mathbb{R}^D , then*

$$|\sup_{x \in D} f(x) - \sup_{x \in D} g(x)| \leq \sup_{x \in D} |f(x) - g(x)|$$

EXERCISE 34. Prove lemma 5.1.17. Show that, if the maxima exist, then we can replace sup with max in lemma 5.1.17 and the statement is still true.

5.1.5.1 The Weierstrass Theorem

When do maxima and minima exist? There are a variety of approaches to this issue, depending on the structure of the problem. In this section we treat one of the most fundamental, usually attributed to the German mathematician Karl Weierstrass (1815–1897).

A function f from a metric space X to \mathbb{R} is called **lower semicontinuous** on X if, for each x in X and each X -valued sequence $\{x_n\}$ converging to x , the function satisfies $f(x) \leq \liminf f(x_n)$. If, instead, $f(x) \geq \limsup f(x_n)$ always holds, then f is called **upper semicontinuous**.

EXERCISE 35. Show that

- (i) $f: X \rightarrow \mathbb{R}$ is continuous if and only if it is both lower and upper semicontinuous.
- (ii) $f: X \rightarrow \mathbb{R}$ is lower semicontinuous on X if and only if, for each $\alpha \in \mathbb{R}$, the sublevel set $C_\alpha := \{x \in X : f(x) \leq \alpha\}$ is closed.

Theorem 5.1.18 (Extreme Value Theorem, Weierstrass). *Let K be a compact subset of a metric space M and let $f: K \rightarrow \mathbb{R}$.*

- (i) If f is lower semicontinuous, then f has a minimizer on K .
- (ii) If f is upper semicontinuous, then f has a maximizer on K .

It follows that if f is a continuous function from K to \mathbb{R} , then f has both a maximizer and a minimizer in K .

EXERCISE 36. Prove this last statement concerning continuous functions using Lemma 5.1.6.

5.1.5.2 Parametric Continuity

A frequent question in optimization is whether or not continuity passes from primitives to solutions. The most commonly used theorem in this domain is **Berge's theorem of the maximum**. In stating it, we take A and X to be metric spaces and Γ to be a **correspondence** from X to A . This means that $\Gamma(x)$ is a subset of A for every $x \in X$.

Now let q be a real valued function on $G := \{(x, a) \in X \times A : a \in \Gamma(x)\}$ and set

$$v(x) := \max_{a \in \Gamma(x)} q(x, a) \quad (x \in X) \tag{5.9}$$

whenever the maximum is well defined. The following theorem is proved in [Aliprantis and Border \(1999\)](#).

Theorem 5.1.19 (Berge). *If Γ is continuous and compact valued on X and q is continuous on G , then v is well defined and continuous on X . Moreover, there exists a Borel measurable function $\sigma: X \rightarrow A$ such that $q(x, \sigma(x)) = v(x)$ for all x . If the maximizer in (5.9) is unique at each x , then σ is continuous.*

Continuity of correspondences like Γ has not been defined here. It will suffice for us to know the following result.

Lemma 5.1.20. *Let X and A be subsets of finite dimensional Euclidean space. Let g and h be a pair of functions from X to A and let*

$$\Gamma(x) = \{a \in A : g(x) \leq a \leq h(x)\}.$$

If g and h are both continuous on X , then Γ is a continuous and compact-valued correspondence from X to A .

A proof can be found in [Stachurski \(2009\)](#), Lemma B.1.1.

5.1.5.3 Convexity and Optimization

For convex and concave functions, local optimizers are global optimizers. To state this more clearly, recall that if $C \subset \mathbb{R}^d$ and f is a real-valued function on C , then $u^* \in C$ is a **local minimizer** of f on C if there exists an open set G in C such that $u^* \in G$ and $f(u^*) \leq f(u)$ whenever $u \in G$. A **local maximizer** is defined analogously.

EXERCISE 37. Show that, if $C \subset \mathbb{R}^d$ is convex, f is convex and u^* is a local minimizer of f on C , then u^* is a minimizer of f on C .

Similarly, if f is concave on C , then any local maximizer is a global maximizer.

One result we will need later is optimization of a quadratic form. In particular, suppose that we wish to solve

$$v(x) = \min_u \{u'Qu + (Ax + Bu)'P(Ax + Bu)\} \quad (5.10)$$

when P is positive semidefinite and $n \times n$, Q is positive definite and $m \times m$, and A is $n \times n$ and B is $n \times m$.

Lemma 5.1.21. *The minimizer of v in (5.10) is*

$$u^* := -(Q + B'PB)^{-1}B'PAx \quad (5.11)$$

and the minimized value v satisfies

$$v(x) = x'\tilde{P}x \quad \text{where } \tilde{P} := A'PA - A'PB(Q + B'PB)^{-1}B'PA. \quad (5.12)$$

EXERCISE 38. Confirm the claims in Lemma 5.1.21 using matrix algebra and the following two facts from matrix calculus:

$$\frac{d}{du}a'u = a \quad \text{and} \quad \frac{d}{du}u'Hu = (H + H')u. \quad (5.13)$$

5.1.6 Entropic Regularization

Add a discussion from Comp Opt Trans, Ch 4 and connect back to optimal transport and mean field games. (The discussion there is also interesting for its treatment of the Hilbert metric on the projective cone – see Remark 4.2. This could be useful for finite Markov chain contractivity.)

5.1.7 Min-Max

Discussion and motivation to be added. Give connections to Kantorovich duality. Note that the result holds for compact subsets of Banach spaces, if we shift Brouwer to Schauder.

Let P_X and P_Y represent two players in a bilateral zero-sum game. Let X and Y be convex compact subsets of finite dimensional Euclidean space. Let τ be a transfer function on $X \times Y$, with $\tau(x, y)$ understood as a payment from P_X to P_Y when P_X chooses $x \in X$ and P_Y chooses $y \in Y$.

Theorem 5.1.22 (von-Neumann Min-Max Theorem). *If τ is continuous, $x \mapsto \tau(x, y)$ is convex for all $y \in Y$ and $y \mapsto \tau(x, y)$ is concave for all $x \in X$, then*

$$\inf_x \sup_y \tau(x, y) = \sup_y \inf_x \tau(x, y). \quad (5.14)$$

One way to think about this is as follows. Suppose that P_X chooses first, selecting $x \in X$. Now P_Y responds by making the transfer as large as possible, obtaining $\sup_y \tau(x, y)$. Regardless of which x was chosen, P_Y can do better than a setting where she was forced to go first, selecting y and receiving $\inf_x \tau(x, y)$. That is to say, $\inf_x \sup_y \tau(x, y) \geq \sup_y \inf_x \tau(x, y)$.

EXERCISE 39. Provide a more mathematical proof of this inequality.

Proof of Theorem 5.1.22. From the preceding discussion, we only need to show that \leq holds in (5.14). We will do so under the additional assumption that the convexity and concavity in the statement of Theorem 5.1.22 is strict.⁷

First note that the reaction functions of P_X and P_Y , given by

$$\sigma_X(y) := \operatorname{argmin}_x \tau(x, y) \quad \text{and} \quad \sigma_Y(x) := \operatorname{argmax}_y \tau(x, y),$$

are well-defined by Weierstrass's theorem. In fact, the uniqueness generated by strict convexity and concavity combined with Berge's theorem of the maximum (p. 152) implies that both reaction functions are continuous.

Now consider the function $s := \sigma_Y \circ \sigma_X$. The point $s(y)$ arises when P_Y chooses y , P_X responds with $\sigma_X(y)$ and then P_Y adjusts to $\sigma_Y(\sigma_X(y))$. Will this process ever stop?

⁷The more general proof requires extra steps without providing a significant amount of additional intuition (see, e.g., Theorem 2.5 of Guillen and McCann (2010)).

The answer is yes: since \mathbb{Y} is compact and convex, and since s is continuous, a y_0 with $s(y_0) = y_0$ exists by Brouwer's fixed point theorem. With $x_0 := \sigma_X(y_0)$ we have

$$\inf_x \sup_y \tau(x, y) \leq \sup_y \tau(x_0, y) = \tau(x_0, y_0) = \inf_x \tau(x, y_0) \leq \sup_y \inf_x \tau(x, y).$$

This completes the proof of Theorem 5.1.22. \square

5.2 Applications of Fixed Point Theory

Add roadmap.

5.2.1 Asset Pricing Revisited

Stick to the finite case. Note connection to Neumann series lemma, which is stronger in this case.

5.2.2 Bellman Operators

Revisit Bellman operators we have met so far. These are nonlinear operators so Neumann doesn't apply.

5.2.3 Job Search

This was originally in the introduction as a motivating example. Rewrite in line with its new location. Show the contraction property.

Next let's consider a model of job search due to McCall (1970) that has heavily influenced economists' way of thinking about labor markets. To understand unemployment, McCall modeled the decision problem of unemployed agents directly, in terms of factors such as current and likely future wages, a worker's impatience, and the availability of unemployment compensation. To solve the decision problem he used dynamic programming.

To set up the problem, consider a worker who is currently unemployed and receives in each period one job offer at wage w_t . Upon receiving each offer, he or she (let's say she) has two choices: Either accept the offer and work permanently at constant wage w_t or reject the offer, receive unemployment compensation c , and reconsider next period.

The wage sequence $\{w_t\}$ is assumed to be IID with common density q supported on \mathbb{R}_+ .

Suppose as a first step that the worker enters the workforce at $t = 1$, lives for two periods and maximizes

$$v_1(w_1) := \max\{y_1 + \beta \mathbb{E}y_2\} \quad \text{where } y_j := \text{is income at time } j.$$

Income y_j is either wage income or unemployment compensation. The constant β lies in $(0, 1)$ and represents discounting of future payoffs relative to current payoffs. The smaller is β , the more the worker discounts the future. The value $v_1(w_1)$, which is maximal lifetime expected rewards, depends on the current offer w_1 but not on w_2 . This is because that draw is unpredictable and in the first period we only know the mathematical expectation y_2 of next period's income.

The worker's options are either to accept w_1 or to reject it and receive unemployment compensation c , and then, in the second period, choose the maximum of w_2 and c . If we consider the value that results, we see that

$$v_1(w_1) = \max\{w_1 + \beta w_1, c + \beta \mathbb{E} \max\{c, w_2\}\}. \quad (5.15)$$

Now let's suppose that the agent works in period $t = 0$ as well, entering the workforce at that point in time and maximizing

$$v_0(w_0) := \max\{y_0 + \beta \mathbb{E}y_1 + \beta^2 \mathbb{E}y_2\}.$$

The value of accepting the current offer w_0 is $w_0 + \beta w_0 + \beta^2 w_0$. The expected value of rejecting and waiting—sometimes called the **continuation value**—is unemployment compensation c and then, after discounting by β , choosing optimally at $t = 1$ and $t = 2$. The value of choosing optimally at $t = 1$ and $t = 2$ has already been calculated: it is $v_1(w_1)$, as given in (5.15). Thus,

$$\text{continuation value} = c + \beta \mathbb{E}v_1(w_1).$$

Since total value $v_0(w_0)$ is the maximum of the value of these two options,

$$v_0(w_0) = \max\{w_0 + \beta w_0 + \beta^2 w_0, c + \beta \mathbb{E}v_1(w_1)\}. \quad (5.16)$$

Notice that we have a recursive relationship between v_0 and v_1 . This is analogous to equation (3.28) that links current and next period cost-to-go in the shortest path problem.

In fact we already had a recursive relationship between current and next period

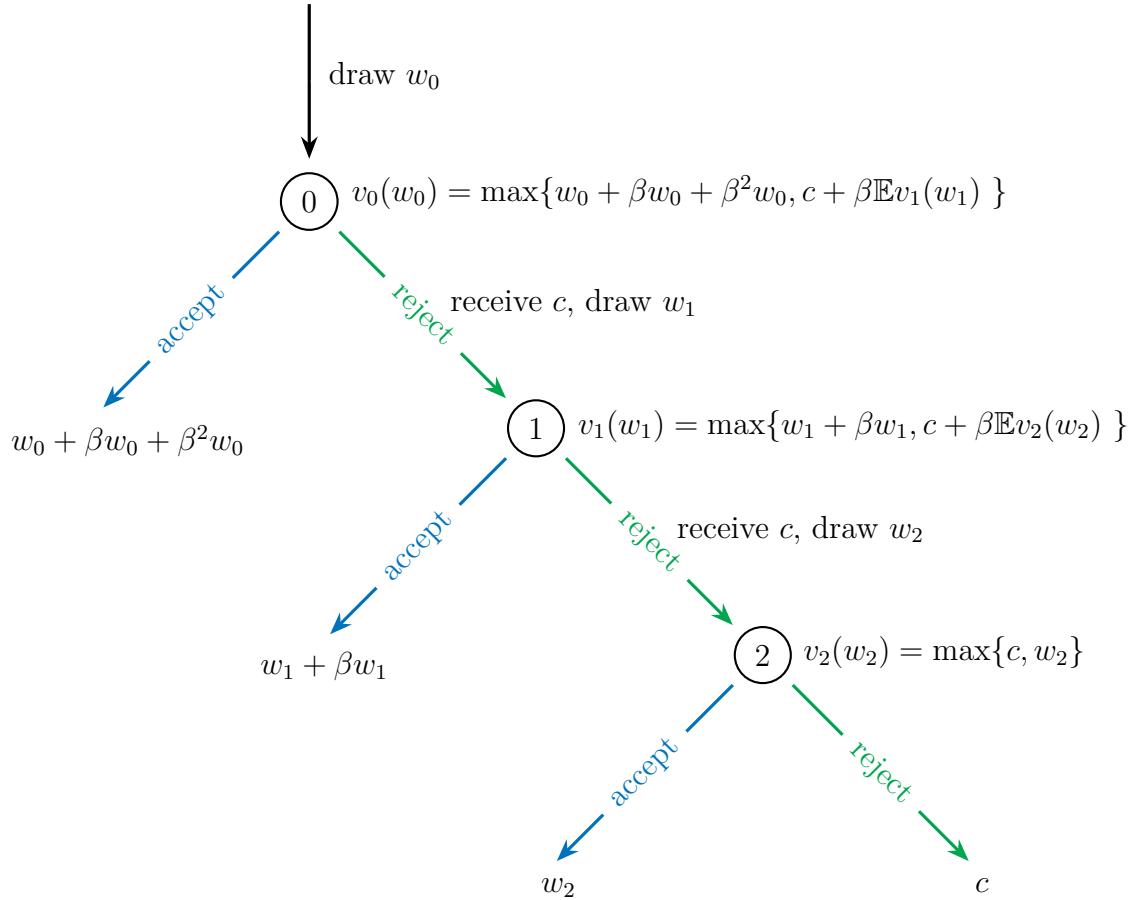


Figure 5.4: Decision tree for the job seeker

lifetime value in (5.15). In particular, since $\max\{c, w_2\} = v_2(w_2)$, the maximal lifetime income of an agent from the viewpoint of $t = 2$, equation (5.15) can alternately be written as a recursive expression linking v_1 and v_2 :

$$v_1(w_1) = \max \{w_1 + \beta w_1, c + \beta \mathbb{E}v_2(w_2)\}. \quad (5.17)$$

These recursions are also examples of Bellman equations. We'll see a new and more sophisticated example in the next section.

5.2.3.1 Infinite Lives

Now let's suppose that the worker is infinitely lived and aims to maximize the expected discounted sum

$$\mathbb{E} \sum_{t=0}^{\infty} \beta^t y_t, \quad (5.18)$$

where y_t is earnings (from either wages or unemployment compensation) at time t . Now the trade-off is as follows: Waiting for a good offer is costly, since the future is discounted. Accepting early is costly too, since offers better than the current one will eventually arrive with probability one. To determine optimal behavior in the face of this trade-off we use dynamic programming.

As we saw above, dynamic programming is a two step procedure that

- (i) assigns values to states and
- (ii) deduces optimal actions given those values

States in the shortest path problem were locations, or nodes in the graph. States for the job search problem are employment and unemployment. Current choice (accept or reject) not only affects current reward (one period wage or unemployment compensation) but also the next period state.

To trade off current and future rewards optimally, then, we need to compare (a) the current payoffs we get from our two choices with (b) the states that those choices will lead to and the maximum value that can be extracted from those states. (This is precisely what (5.16) does and it is also akin to the choice being made in equation (3.27) of the shortest path problem, although in that setting we were minimizing cost rather than maximizing value.)

But how do we calculate the “maximum amount of value” that can be extracted from each of the two states? Consider first the maximal lifetime value of being employed with wage w . This case is easy because once the worker is employed she is employed forever and has no remaining choices to exercise. Her lifetime value is

$$w + \beta w + \beta^2 w + \dots = \frac{w}{1 - \beta}. \quad (5.19)$$

What about the maximum lifetime value that can be realized when entering the current period unemployed but with wage offer w in hand? Denote this (as yet unknown) value by $v^*(w)$. Think of v^* as a *function* that assigns to each possible wage w the maximal lifetime value that can be obtained with that offer in hand. We call v^* the **value**

function. A crucial observation is that this function v^* should satisfy the recursion

$$v^*(w) = \max \left\{ \frac{w}{1-\beta}, c + \beta \int v^*(w') q(w') dw' \right\} \quad (5.20)$$

at every $w \in \mathbb{R}_+$. This is a version of a Bellman equation similar to (3.28) and, in particular, to (5.16).

Later we will prove that v^* satisfies (5.20). For now consider the following intuition: The first term inside the max operation is (5.19), the *stopping value*, corresponding to the lifetime payoff from accepting current offer w . The second term inside the max operation is the continuation value, which is the current value of the lifetime payoff from rejecting the current offer and then behaving optimally thereafter. The best choice in the present period is just the largest of these two alternatives.

Now, if we optimize and pick the best of these alternatives, then, since our current choice is optimal and our next period value is calculated based on optimal future choice, we should obtain maximal lifetime value from today, given current offer w . But this is precisely $v^*(w)$, which is why the left side of (5.20) equals the right.

If you found that reasoning less than fully convincing, don't worry. We'll be working through many, many examples in the coming pages, as well as all the theory. The line of thinking associated with dynamic programming will eventually become second nature.

In terms of techniques, think of (5.20) as an equation to be solved for a *function* v^* that will then allow us to make optimal choices in the manner described above. One thing that might be new to you here is that the unknown object in this equation is not a number or a vector but rather an entire function that records a value of $v^*(w)$ for every possible w . It's also nonlinear. We also have to ask ourselves whether a solution even exists, and whether there could be many valid solutions.

To answer these questions we use fixed point theory. The connections between fixed point theory and dynamic programming are deep and genuinely beautiful—one of the many joys of learning dynamic programming. We'll draw out these connections and then return to this specific problem later.

5.2.4 Consumption Streams Revisited

Illustrate contraction mappings. Later, when the Neumann series lemma is given in a Banach space, circle back to this application and use it as an illustration.

A standard model of the value of a finite consumption stream c_0, c_1, \dots, c_n for an individual with discount factor β and one period von Neumann-Morgenstern utility

function u is

$$V_0 := \mathbb{E} \sum_{t=0}^n \beta^t u(c_t) \quad (5.21)$$

As long as $\mathbb{E}u(c_t)$ is finite for each t , V_0 is well defined. We can pass the expectation through the sum in (5.21) so that we are then just summing a finite set of numbers.

It is often convenient to work with infinite horizon models. Working with the infinite limit can, perhaps surprisingly, simplify things.⁸ But more sophisticated mathematics is sometimes required to verify existence the sum on the right side of (5.21) and to compute it when $n = +\infty$. To illustrate, consider an infinite horizon version of (5.21), where an agent standing at time t looks to an infinite future and values a consumption stream c_t, c_{t+1}, \dots via

$$V_t := \mathbb{E}_t \sum_{i \geq 0} \beta^i u(c_{t+i}) \quad (5.22)$$

Here \mathbb{E}_t is a mathematical expectation conditional on time t information.

We will take a recursive approach, writing

$$V_t = u(c_t) + \mathbb{E}_t \sum_{i \geq 1} \beta^i u(c_{t+i}) = u(c_t) + \beta \mathbb{E}_t \mathbb{E}_{t+1} \sum_{i \geq 0} \beta^i u(c_{t+1+i})$$

In the last equality we used the law of iterated expectations (see §4.1.3) to insert \mathbb{E}_{t+1} . Applying the definition (5.22) again, we obtain the recursion

$$V_t = u(c_t) + \beta \mathbb{E}_t V_{t+1} \quad (5.23)$$

To ensure that value is finite and unravel the recursion in (5.23), we need some assumptions. We will assume that u is continuous and increasing on \mathbb{R}_+ and that, for each t , we have $c_t = g(x_t)$, where g is a fixed function and $\{x_t\}$ is a state process obeying $x_{t+1} = ax_t + b + c\xi_{t+1}$. Here a, b and c are constants and $\{\xi_t\}$ is IID. We suppose that the function g is continuous, nonnegative and bounded.

At this point we are going to guess that there exists a continuous, bounded, real-valued function v on \mathbb{R} such that, if we take $V_t = v(x_t)$ for each t , then the sequence $\{V_t\}$ satisfies (5.23). We will then verify the guess.

Let $bc\mathbb{R}$ be the set of all continuous bounded function on \mathbb{R} and suppose that there exists a $v \in bc\mathbb{R}$ such that

$$v(x) = u(g(x)) + \beta \int v(ax + b + cz) \varphi(dz) \quad (5.24)$$

⁸Another example of this appears when we compare finite horizon and infinite horizon LQ control models in §6.2.

where φ is the common distribution of each ξ_t . If we evaluate this at x_t , recalling that $c_t = g(x_t)$, we get

$$v(x_t) = u(c_t) + \beta \int v(ax_t + b + cz)\varphi(dz) = u(c_t) + \beta \mathbb{E}_t v(x_{t+1})$$

But with $V_t = v(x_t)$ for each t , this is just (5.23). So we have shown that, if there does exist a function v in $bc\mathbb{R}$ satisfying (5.24), and if we know how to compute it, then we have a solution to the recursive valuation problem (5.23), and hence the equivalent and original expression for the value in (5.22).

Complete contraction mapping argument, with full details.

5.2.5 Speculation and Commodity Prices

The commodity pricing model. A contraction.

5.2.6 Production Chains

Describe the model. Connect back to optimal transport duality. Use Tarski to get equilibrium prices. But it doesn't give us uniqueness. We might hope to use something like the concave operator approach discussed in §5.1.3.3. Unfortunately, that was stated for vectors, and here the unknown object is a function on $[0, 1]$. Fortunately, it turns out that the results for concave operators obtained in §5.1.3.3 can be extended to a more general setting and then applied to XXX. We will put this aside for now, circling back to in YYY.

5.2.7 Coase's Theory of the Firm

Look up Hal Varian's NY Times article, perhaps quote from it. Derive the first order condition that recovers Coase's insight.

5.2.8 Epstein–Zin Preferences

Valuing a consumption stream in a finite Markov setting. Use concave monotone operator theory to obtain fixed point of the Koopmans operator recursion. Use simplified Borovička and Stachurski (2020). Relate to current literature such as Kaplan and Violante (2014).

5.3 Semidynamical Systems

Add roadmap. The following needs to be rewritten slightly because we have already seen instances of global stability. See, for example, §2.3.3.

5.3.1 Formalities

Add roadmap.

5.3.1.1 Definition and Properties

Let \mathbb{T} be either \mathbb{R}_+ or \mathbb{Z}_+ . A **semidynamical system** is a pair $(\mathbf{M}, (S_t)_{t \in \mathbb{T}})$, where $\mathbf{M} = (\mathbf{M}, \rho)$ is a metric space⁹ and $(S_t) = (S_t)_{t \in \mathbb{T}}$ is a family of maps from \mathbf{M} to itself such that

- (i) $S_{s+t} = S_s \circ S_t$ for all $s, t \in \mathbb{T}$ and
- (ii) $S_0 = I$, where I is the **identity map** on \mathbf{M} (i.e., $Iu = u$ for all u in \mathbf{M}).

In this context \mathbf{M} is called the **state space**. For a given $u \in \mathbf{M}$, the point $S_t u$ is understood as the “state of the system” at time t , having started at state u . If $\mathbb{T} = \mathbb{R}_+$ then $(\mathbf{M}, (S_t)_{t \in \mathbb{T}})$ is called a **continuous time** semidynamical system. If \mathbb{T} is the nonnegative integers then it is called a **discrete time** semidynamical system.

The symbol \circ in part (i) above refers to composition of functions. Sometimes we omit this symbol, along with parentheses around arguments, so that, for given u in \mathbf{M} , property (i) can be expressed as $S_{s+t}u = S_s S_t u$ for all $s, t \in \mathbb{T}$ and $u \in \mathbf{M}$. Property (i) is also called the **semigroup** property of (S_t) , since it implies that this family of operators is a semigroup under composition. This is the source of the term “semidynamical system.”

Example 5.3.1. Let $x(t)$ be the balance of an account that pays a continuously compounded interest rate r , so that $x'(t) = rx(t)$ where x' is the time derivative. Given initial balance $x(0) = x_0$, the solution to this linear differential equation is $x(t) = e^{rt}x_0$. If we introduce the family of maps (S_t) by $S_t x_0 = e^{rt}x_0$, then $(\mathbb{R}, (S_t))$ is a continuous time semidynamical system such that $S_t x_0$ is the time t balance given initial balance x_0 . To check that the definition of a semidynamical system is satisfied, we observe that each S_t maps \mathbb{R} to itself, that $S_0 x = e^0 x = x$, so S_0 is the identity, and that, for all $s, t \geq 0$ and real x ,

$$S_{s+t} x = e^{r(s+t)} x = e^{rs} e^{rt} x = S_s S_t x.$$

⁹We can also take \mathbf{M} to be a Hausdorff topological space (see §12.1.1). None of the definitions from this section need to be modified.

The trajectory $t \mapsto xe^{rt}$ of x is the balance of the account at each point in time.

The discrete time case is rather simple: the semigroup (S_t) is described by a single self-map S on \mathbf{M} . From there we define $S_0 = I$ and, recursively, $S_n = SS_{n-1}$ for each $n \in \mathbb{N}$. In other words, S_n is just S^n , or n compositions of S with itself. The semigroup property is then satisfied. The corresponding difference equation is $u_{t+1} = Su_t$. To simplify notation, we usually refer to (\mathbf{M}, S) as the semidynamical system in the discrete time case, rather than $(\mathbf{M}, (S_n))$.¹⁰

Example 5.3.2. Consider the vector linear difference equation

$$x_{t+1} = Ax_t + b, \quad (5.25)$$

where x_t is an $n \times 1$ column vector of **state variables**, A is $n \times n$ and b is $n \times 1$. This difference equation in \mathbb{R}^n corresponds to a discrete time semidynamical system (\mathbb{R}^n, S) when $Sx = Ax + b$. For this particular case, we can also obtain an explicit representation of $S_t = S^t$. You can verify by induction that

$$S^t x_0 := A^t x_0 + \sum_{i=0}^{t-1} A^i b \quad (5.26)$$

for all $t \in \mathbb{N}$.¹¹ The vector $S_t x_0 := S^t x_0$ is the time t state x_t given the initial condition x_0 . We can also confirm this by working backwards to obtain

$$x_t = Ax_{t-1} + b = A(Ax_{t-2} + b) + b = A^2 x_{t-2} + Ab + b = \dots = S_t x_0.$$

Example 5.3.3. Let $g(k) = sf(k) + (1 - \delta)k$, as in the Solow–Swan growth model described in section 2.1.3. Since g maps \mathbb{R}_+ to itself, the pair (\mathbb{R}_+, g) is a discrete time semidynamical system when \mathbb{R}_+ has its usual topology.

Example 5.3.4. If S maps u to $2u$, then $([0, 1], S)$ is *not* a semidynamical system because S is not a self-mapping on $[0, 1]$.

For each $u \in \mathbf{M}$, the function $t \mapsto S_t u$ from \mathbb{T} to \mathbf{M} is called **trajectory** of $u \in \mathbf{M}$. We will also call it a *time series*. The starting point u is called the **initial condition**. In the discrete time case, $S_t u = S^t u$ is called the **t -th iterate of u under S** .

¹⁰We are working with time homogeneous discrete time systems here, which initially appears to exclude models such as $x_{t+1} = g(x_t, t)$, due to the explicit time dependence in the update rule. However, if we write $y_{t+1} = y_t + 1$ and $z_t = (x_t, y_t)$, then the time dependent model can be shifted to the time homogeneous model $z_{t+1} = G(z_t)$ with $G(z) = G(x, y) = (g(x, y), y + 1)$.

¹¹Evidently (5.26) holds at $t = 1$. Now suppose it holds at $t - 1$ and compute $S^t x_0$ via $SS^{t-1} x_0$. You should obtain the right hand side of (5.26).

Given semidynamical system $(M, (S_t))$, a subset C of M is called **invariant** under (S_t) if S_t is a self-mapping on C for all $t \in \mathbb{T}$. In the discrete time case, it is enough to check that $S_1 = S$ maps C to itself.

A **stationary point** or **steady state** of a semidynamical system $(M, (S_t))$ is a point u^* in M such that $S_t u^* = u^*$ for all t . Evidently, if u^* is stationary and a trajectory reaches u^* at some finite time t , then it stays there forever. For a discrete time semidynamical system, stationarity just means that u^* is a fixed point of $S = S_1$.

Example 5.3.5. For (M, g) with $g(k) := sAk^\alpha + (1 - \delta)k$, $M = (0, \infty)$, $0 < s, \alpha, \delta < 1$ and $0 < A$, the point k^* in Figure 2.2 is a steady state.

5.3.2 Stability

Add roadmap.

5.3.2.1 Local Stability

Given a steady state u^* of semidynamical system $(M, (S_t))$, the **stable set** of u^* is

$$\mathcal{O}(u^*) := \{u \in M : S_t u \rightarrow u^* \text{ as } t \rightarrow \infty\}$$

This set is nonempty. (Why?) The steady state u^* called **locally stable** or an **attractor** if there exists a neighborhood N of u^* such that $\mathcal{O}(u^*) \subset N$.

Example 5.3.6. Consider the one dimensional continuous time system $x'(t) = ax(t) + b$, where a and b are constants. You can check that $x(t) = (x_0 + b/a) \exp(ta) - b/a$ is a solution by differentiating it with respect to t . The corresponding semidynamical system is $(\mathbb{R}, (S_t))$ with $S_t x_0 = (x_0 + b/a) \exp(ta) - b/a$. If $a < 0$, then $S_t x_0 \rightarrow -b/a$ as $t \rightarrow \infty$ regardless of the initial condition x_0 . Hence $x^* := -b/a$ is an attractor with $\mathcal{O}(x^*) = \mathbb{R}$.

EXERCISE 40. Check that $(\mathbb{R}, (S_t))$ defined in Example 5.3.6 satisfies the definition of a semidynamical system.

Lemma 5.3.1 (Hartman–Grobman). *If S is a continuously differentiable self map on the interior of $M \subset \mathbb{R}^d$ and u^* is an interior point, then u^* is locally stable whenever $\|S'u^*\| < 1$, where $S'u^*$ is the Jacobian evaluated at u^* .*

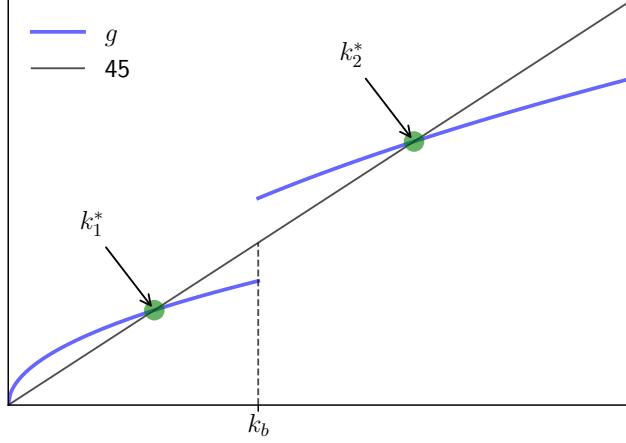


Figure 5.5: The threshold model $g(k) = sA(k)k^\alpha + (1 - \delta)k$ when $k_1^* < k_b < k_2^*$

The intuition is that, in a neighborhood of u^* , we have

$$Su \approx Lu := Su^* + S'(u^*)(u - u^*) = u^* + S'(u^*)(u - u^*).$$

If $\|S'u^*\| < 1$, then L is a uniform contraction with unique fixed point u^* . The formal proof uses a local topological conjugacy between S and L (see, e.g., [Add cite](#)).

Example 5.3.7. In the spirit of [Azariadis and Drazen \(1990\)](#), let's modify the Solow-Swan model to $k_{t+1} = g(k_t)$ with positive productivity levels $A_1 < A_2$ and

$$g(k) = sA(k)k^\alpha + (1 - \delta)k \quad \text{where} \quad A(k) = \begin{cases} A_1 & \text{if } 0 < k < k_b \\ A_2 & \text{if } k_b \leq k. \end{cases} \quad (5.27)$$

Figure 5.5 shows one parameterization, with two locally stable steady states on $(0, \infty)$.

EXERCISE 41. Show that any positive fixed point of (5.27) is locally stable.

5.3.2.2 Global Stability

Semidynamical system $(\mathbb{M}, (S_t))$ is called **globally stable** if

- (i) the system has a unique stationary point u^* in \mathbb{M} and
- (ii) $\mathcal{O}(u^*) = \mathbb{M}$.

Global stability is an important concept that we shall apply progressively to more and more complicated models.

To avoid tedious notation in the discrete time case, we will say that a self-map S on \mathbf{M} is **globally stable** on \mathbf{M} if (\mathbf{M}, S) is globally stable.

Remark 5.3.1. The uniqueness requirement in the definition of global stability is redundant and requires no separate verification. Indeed, suppose that $(\mathbf{M}, (S_t))$ has a steady state u^* and $\mathcal{O}(u^*) = \mathbf{M}$. If v^* is another steady state, then $S_t v^*$ converges to both v^* and u^* . Since limits are unique in a metric space, we see that $v^* = u^*$.

Example 5.3.8. Let $n = 1$ and $Sx = ax + b$ for scalars a and b . If $a \neq 1$, then g has a unique fixed point $x^* = b/(1 - a)$. Moreover, iterating backwards via

$$x_t = ax_{t-1} + b = a^2x_{t-2} + ab + b = a^3x_{t-3} + a^2b + ab + b = \dots$$

eventually yields $x_t = a^t x_0 + b \sum_{i=0}^{t-1} a^i$, which converges to $b/(1 - a)$ whenever $|a| < 1$. In other words, the dynamical system (\mathbb{R}, S) is globally stable whenever $|a| < 1$.

EXERCISE 42. Show that, for $Sx = ax + b$, the condition $|a| < 1$ is necessary as well as sufficient for global stability.

Example 5.3.9. The Solow–Swan growth model (2.3) is globally stable when $f(k) = Ak^\alpha$ and all parameters are strictly positive. In particular, if $g(k) = Ak^\alpha + (1 - \delta)k$ where $0 < \alpha, \delta < 1$ and $A > 0$, then g is globally stable on $(0, \infty)$ with unique fixed point $k^* := ((sA)/\delta)^{1/(1-\alpha)}$. We proved this in §2.1.3.

This example shows why it is important to clarify the state space when defining a dynamic model. For example, while g is globally stable on $\mathbf{M} = (0, \infty)$, the same is not true when $\mathbf{M} = \mathbb{R}_+$. In this case, both k^* and 0 are steady states of g .

Example 5.3.10. Let S be a uniformly contracting self-map on \mathbf{M} . Then S is globally stable on \mathbf{M} by Theorem 5.1.9.

Example 5.3.11. Let \mathbf{M} be any metric space and let I be the identity on \mathbf{M} . Then (\mathbf{M}, I) is a discrete time semidynamical system. Every point is stationary, so global stability fails whenever \mathbf{M} has more than one point.

We shall repeatedly refer to the following result, the proof of which is an exercise.

Lemma 5.3.2. *Let $(\mathbf{M}, (S)_{t \in \mathbb{T}})$ be a globally stable semidynamical system with steady state u^* . If $F \subset \mathbf{M}$ is a nonempty, closed and invariant, then u^* is in F .*

5.3.2.3 From Higher to Lower Order Maps

The next fact about discrete time systems will be useful when we study high dimensional models.

Lemma 5.3.3. *Let S be a self-map on \mathbf{M} . Suppose that, for some $i \in \mathbb{N}$, the map S^i is globally stable on \mathbf{M} with steady state u^* . If, in addition, S is continuous at u^* , then S is also globally stable on \mathbf{M} , with the same steady state.*

Proof. Let S and i satisfy the stated hypotheses. Fix $u \in \mathbf{M}$. We claim that $S^t(u) \rightarrow u^*$. To verify this, observe that any integer m can be expressed as $ni + j$ for some integer n and some j in $0, \dots, i - 1$. Let U be a neighborhood of u^* and, for each such j , choose N_j such that

$$S^{ni+j}u = S^{ni}S^j u \in U \text{ whenever } n \geq N_j$$

With $N = \max_j N_j$, we have $S^t u \in U$ whenever $t \geq Ni$, so $S^t u \rightarrow u^*$ is established.

Since S is continuous at u^* , Lemma 5.1.7 implies that u^* is a fixed point of S . Since u was arbitrary, (\mathbf{M}, S) is globally stable (with uniqueness implied by remark 5.3.1). \square

EXERCISE 43. Prove Theorem 5.1.10 on page 139 using Lemma 5.3.3.

For the case of continuous time, there is a valuable result similar to Lemma 5.3.3, although it requires a little more structure. To start, let $(\mathbf{M}, (S_t))$ be a semidynamical system and consider a uniform Lipschitz property under which there exists a continuous function k such that

$$d(S_t u, S_t v) \leq k(t) d(u, v) \text{ for all } u, v \text{ in } \mathbf{M} \text{ and } t \in \mathbb{T}. \quad (5.28)$$

Lemma 5.3.4. *Let (S_t) have the uniformly Lipschitz property stated above. If there is an $s > 0$ such that S_s is globally stable on \mathbf{M} , then $(\mathbf{M}, (S_t))$ is globally stable with the same steady state.*

Proof. Let $s > 0$ be as in the statement of the lemma and let u^* be the unique fixed point of S_s . For any given $t > 0$ and any $n \in \mathbb{N}$ we have

$$d(S_t u^*, u^*) = d(S_t S_{ns} u^*, u^*) = d(S_{sn} S_t u^*, u^*) = d(S_s^n S_t u^*, u^*).$$

The right hand side tends to zero as $n \rightarrow \infty$ by stability of S_s , so u^* is stationary for S_t . To prove stability, we fix $u \in \mathbf{M}$ and $\varepsilon > 0$. Let $K := \max_{0 \leq r \leq s} k(r)$, which is finite by continuity of k . Choose $N \in \mathbb{N}$ such that $d(S_{ns} u, u^*) < \varepsilon/K$ whenever $n \geq N$. Set

$T := Ns$. For $t \geq T$ we have $t = ns + r$ for some $n \geq N$ and r in $[0, s]$. This, the semigroup property and the Lipschitz condition allow us to write

$$d(S_t u, u^*) = d(S_r S_{ns} u, u^*) = d(S_r S_{ns} u, S_r u^*) \leq K d(S_{ns} u, u^*) < \varepsilon.$$

Hence $S_t u \rightarrow u^*$ and $(M, (S_t))$ is globally stable. \square

5.3.3 Topological Conjugacy

One of the most fruitful pursuits in mathematics is identification and classification of objects that are similar. For example, metric spaces M and N are called **homeomorphic** if there exists a continuous bijection τ from M to N such that τ^{-1} is also continuous (on N).¹² In this setting, the function τ is called a **homeomorphism**. When such a homeomorphism exists, we know that G is open in M if and only if $\tau(G)$ is open in N , that K is compact in N if and only if $\tau^{-1}(K)$ is compact in M and so on.¹³ This is useful because now we have two different ways to show that a given subset of M (or N) is compact (or open, closed, etc.). Having two angles of attack is better than having one.

EXERCISE 44. Show that, in the homeomorphic setting described above, K is compact in N if and only if $\tau^{-1}(K)$ is compact in M .

For semidynamical systems, the most useful notion of similarity is topological conjugacy. Two semidynamical systems $(M, (S_t))$ and $(N, (T_t))$ are called **topologically conjugate** if there exists a homeomorphism τ from M to N such that

$$S_t = \tau^{-1} \circ T_t \circ \tau \text{ for all } t \in \mathbb{T}. \quad (5.29)$$

For a discrete time pair (M, S) and (N, T) , we need only check that $S = \tau^{-1} \circ T \circ \tau$, since then

$$S^2 = (\tau^{-1} \circ T \circ \tau) \circ (\tau^{-1} \circ T \circ \tau) = \tau^{-1} \circ T^2 \circ \tau$$

and so on. Figure 5.6 illustrates this case.

Example 5.3.12. Matrices A and B in $\mathcal{M}_{n \times n}$ are called **similar** if there exists an invertible P in $\mathcal{M}_{n \times n}$ such that $A = P^{-1}BP$. If we think of each $E \in \mathcal{M}_{n \times n}$ as a map $x \mapsto Ex$, then the semidynamical systems (\mathbb{R}^n, A) and (\mathbb{R}^n, B) are topologically conjugate whenever A and B are similar.

¹²The same definition applies when M and N are topological spaces. See §12.1.1 of the appendix.

¹³Recall that a function f from M to N is continuous if and only if $f^{-1}(G)$ is open in M whenever G is open in N , and that a subset K of a metric space is compact if and only if every open cover can be reduced to a finite subcover.

$$\begin{array}{ccc}
 M & \xrightarrow{S} & M \\
 \tau \downarrow & & \uparrow \tau^{-1} \\
 N & \xrightarrow{T} & N
 \end{array}$$

Figure 5.6: Topological conjugacy between (M, S) and (N, T)

Here's why topological conjugacy is important:

Theorem 5.3.5. *Let $(M, (S_t))$ and $(N, (T_t))$ be two semidynamical systems that topologically conjugate under a homeomorphism τ . In this setting:*

- (i) *If u^* is a steady state of $(M, (S_t))$, then $\tau(u^*)$ is a steady state of $(N, (T_t))$.*
- (ii) *If u^* is globally stable in $(M, (S_t))$, then $\tau(u^*)$ is globally stable in $(N, (T_t))$.*

Proof. Regarding part (i), if u^* is a steady state of $(M, (S_t))$, then, for any $t \in \mathbb{T}$, we have $\tau(S_t u^*) = \tau(u^*)$, and hence $T_t \tau(u^*) = \tau(u^*)$. In particular, $\tau(u^*)$ is a steady state of $(N, (T_t))$.

Regarding part (ii), suppose that u^* is globally stable in $(M, (S_t))$. We have already established that $\tau(u^*)$ is a steady state of $(N, (T_t))$. To show global stability, pick any $v \in N$ and consider the trajectory $t \mapsto T_t v$. Rearranging the definition of conjugacy gives $T_t v = \tau(S_t u)$ where $u := \tau^{-1}(v)$. Since $S_t u \rightarrow u^*$ and τ is continuous, we have $T_t v \rightarrow \tau(u^*)$, as was to be shown. \square

Consider the linear system $Sx = Ax + b$ analyzed in §2.3.3. When A is diagonalizable with $A = P^{-1}DP$, this system is topologically conjugate to the simple decoupled system in (2.24), which is much easier to analyze.

EXERCISE 45. Show that (\mathbb{R}^n, S) is topologically conjugate to (\mathbb{C}^n, T) under the homeomorphism P , where $Ty = Dy + c$ with $c := Pb$.

EXERCISE 46. Let g be defined on $(0, \infty)$ by $g(k) = Ask^\alpha$ where A, s, α are positive constants. Let h be defined on \mathbb{R} by $h(x) = \ln A + \ln s + \alpha x$ (i.e., h is its log-linearization). Show that $((0, \infty), g)$ and (\mathbb{R}, h) are topologically conjugate under the homeomorphism $\tau = \ln$.

5.3.4 Monotonicity in Discrete Time

We found monotonicity to be a useful tool for analyzing the stability of the Solow–Swan growth model (see, e.g., example 5.3.9). This is not an isolated case. Let’s now look more systematically at connections between monotonicity and dynamics. Because we want to apply our results in diverse settings. We’ll proceed relatively abstractly.

We shall study semidynamical systems on a metric space M . In addition, let \preceq be a closed partial order on M . As indicated in (5.5) on page 145, the statement that \preceq is closed on M means that the partial order is preserved under limits.

5.3.4.1 Monotone Trajectories

Lemma 5.3.6. *If $M \subset \mathbb{R}$ and S is increasing on M , then every trajectory is monotone.*

This means that, for any trajectory $\{u_t\}$, we have either $u_t \leq u_{t+1}$ for all t or $u_t \geq u_{t+1}$ for all t . Thus, increasing maps generate monotone sequences.

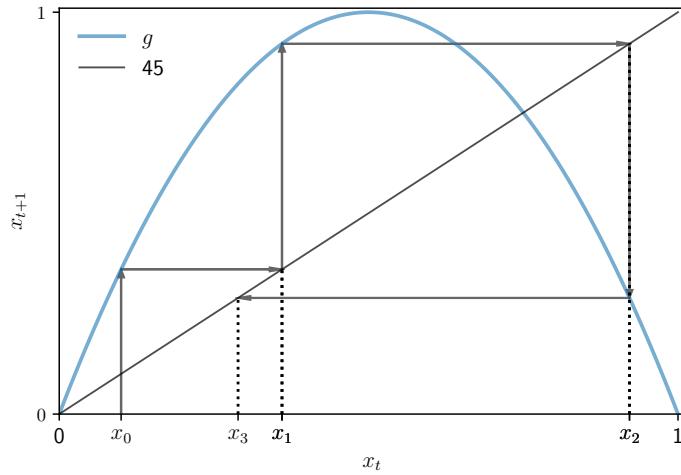
Proof. Pick any $u \in M$ and observe that either $Su \leq u$ or $u \leq Su$. In the first case, since S is increasing, the fact that $Su \leq u$ implies $SSu \leq Su$. Chaining these two inequalities gives $S^2u \leq Su \leq u$. Continuing in the same way shows that S^tu is decreasing in t . The proof of the second case is similar. \square

Lemma 5.3.6 is quite significant. For example, if S is increasing and M is closed and bounded, then every trajectory generated by S must converge to some point in M (recall Theorem 2.1.1).

EXERCISE 47. Prove or find a counterexample: The statement in Lemma 5.3.6 remains true if the word “increasing” is replaced by “decreasing.”

Figure 5.7 gives an example of a one dimensional discrete time model where trajectories are *not* always monotone. The system is $([0, 1], g)$ where $g(x) = 4x(1 - x)$. It is well known that, for most initial conditions, this model generates “chaotic” trajectories, which fluctuate rather than converging. The start of one trajectory, with points $\{x_0, x_1, x_2, x_3\}$, is shown in the figure.

Interestingly, as will be discussed in §5.3.7, these kinds of fluctuations are not possible in one-dimensional continuous time systems. This means that the class of discrete time models is, in some senses, richer than the class of continuous time models. This is neither good nor bad. It simply means that discrete time models can capture more phenomena, while continuous time models have more regularity to exploit when conducting analysis.

Figure 5.7: The quadratic map $g(x) = 4x(1 - x)$

5.3.4.2 Parametric Monotonicity for Steady States

A major concern in economic modeling is whether a change in a parameter shifts an endogenous outcome up or down. Perhaps the parameter enters into a central bank decision rule for pegging a particular interest rate, and we wish to know whether increasing that parameter will increase or decrease steady state inflation. By providing sufficient conditions for monotone shifts in fixed points, results in this section can help tackle such questions.

Given two self-maps S and T on \mathbf{M} , we write $S \preceq T$ if $Su \preceq Tu$ for every $u \in \mathbf{M}$ and say that the function T **dominates** the function S .

One might assume that, in a setting where T dominates S , the fixed points of T will be larger. This can hold, as in figure 5.8, but it can also fail, as in figure 5.9. One difference between these two scenarios is that, in the case of figure 5.8, the map T is globally stable. This leads us to our next result.

Proposition 5.3.7. *Let S and T be self-maps on \mathbf{M} . If $S \preceq T$ and, in addition, T is isotone and globally stable on \mathbf{M} , then its unique fixed point dominates any fixed point of S .*

Proof of proposition 5.3.7. Assume the conditions of the proposition and let u_T be the unique fixed point of T . Since $S \preceq T$, we have $u_S = Sus \preceq Tus$. Applying T to both sides of this inequality and using isotonicity of T and transitivity of \preceq gives $u_S \preceq T^2u_S$.

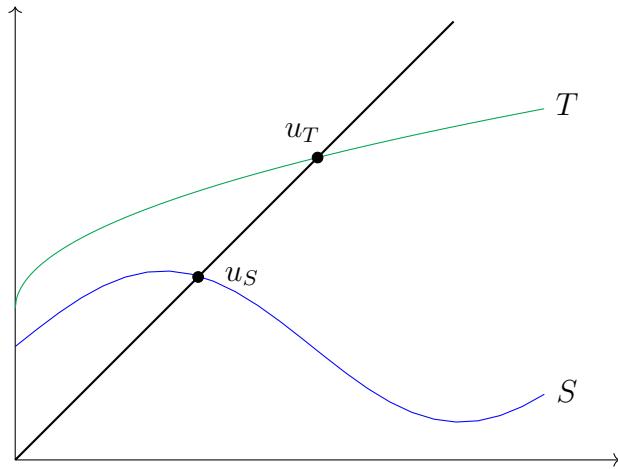


Figure 5.8: Ordered fixed points when global stability holds

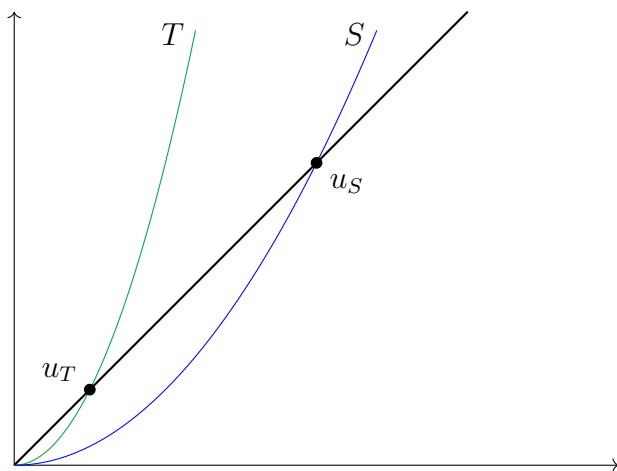


Figure 5.9: Reverse-ordered fixed points when global stability fails

Continuing in this fashion yields $u_S \preceq T^t u_S$ for all t . Taking the limit in t and using the fact that \preceq is closed under limits gives $u_S \preceq u_T$. \square

Proposition 5.3.7 will be applied many times in the remainder of the notes.

As an application of Proposition 5.3.7, consider again the Solow–Swan growth model $k_{t+1} = g(k_t) := sf(k_t) + (1 - \delta)k_t$. We saw in §5.3.1.1 that if $f(k) = Ak^\alpha$ where $A > 0$ and $\alpha \in (0, 1)$, then $((0, \infty), g)$ is globally stable. Clearly $k \mapsto g(k)$ is isotone on $(0, \infty)$. If we now increase, say, the savings rate s , then g will be shifted up everywhere, implying, via Proposition 5.3.7, that the fixed point will also rise. Exercise 48 asks you to step through the details.

EXERCISE 48. Let $g(k) = sAk^\alpha + (1 - \delta)k$ where all parameters are strictly positive, $\alpha \in (0, 1)$ and $\delta \leq 1$. Let $k^*(s, A, \alpha, \delta)$ be the unique fixed point of g in $(0, \infty)$. Without using the expression we derived for k^* previously, show that

- (i) $k^*(s, A, \alpha, \delta)$ is increasing in s and A .
- (ii) $k^*(s, A, \alpha, \delta)$ is decreasing in δ .

Figure 5.10 helps illustrate the results of Exercise 48. The top left sub-figure shows the default parameterization, with $A = 2.0$, $s = \alpha = 0.3$ and $\delta = 0.4$. The other sub-figures show how the steady state changes as parameters shift from that default.

5.3.4.3 From Monotonicity to Stability

Apart from helping us order fixed points, monotonicity is also useful for establishing stability. For example, here is a generalization of the stability results for the Solow–Swan model that we obtained in §5.3.1.1. It uses a very useful fixed point result for isotone concave maps described in §11.1.5 of the appendix.

Proposition 5.3.8. *Let g be a function from $(0, \infty)$ to itself with the following two properties:*

- (i) *For each $k > 0$, there is an $x \leq k$ such that $g(x) > x$.*
- (ii) *For each $k > 0$, there is a $y \geq k$ such that $g(y) < y$.*

If g is also increasing and concave, then $((0, \infty), g)$ is globally stable.

Proof. This follows directly from corollary 5.1.15 on page 147. \square

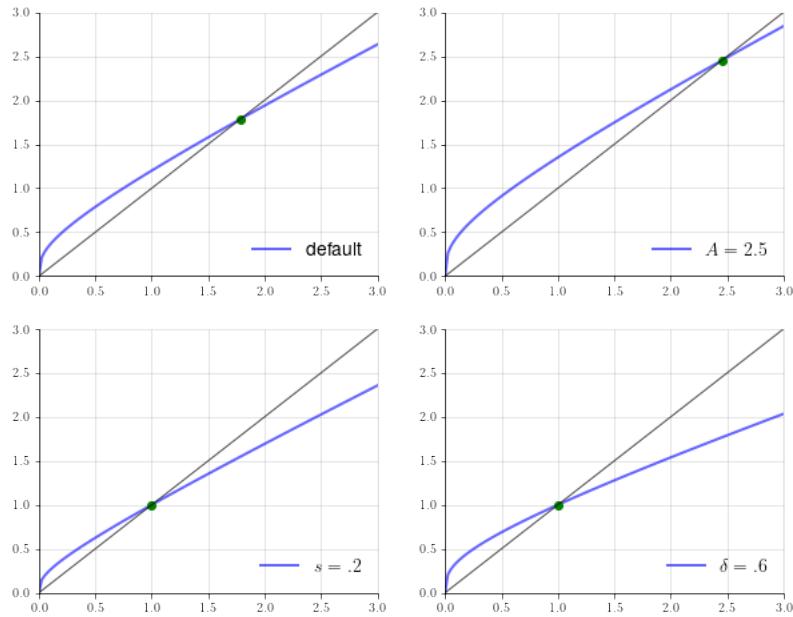


Figure 5.10: Deviations from the default $A = 2.0$, $s = \alpha = 0.3$ and $\delta = 0.4$

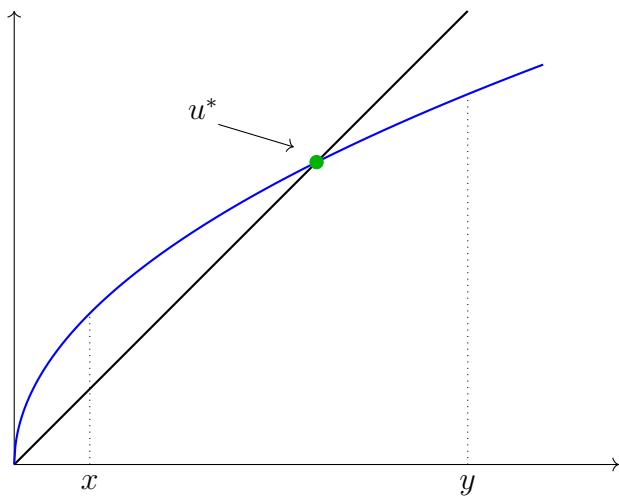


Figure 5.11: Global stability induced by increasing concave functions

Figure 5.11 helps illustrate Proposition 5.3.8. Under iteration all trajectories converge to the unique fixed point u^* .

Using Proposition 5.3.8, we can weaken the conditions of our last stability result for the Solow–Swan model, which required a specific functional form for the production function f (see example 5.3.9).

Proposition 5.3.9. *If $g(k) = sf(k) + (1 - \delta)k$ where $0 < s, \delta < 1$ and f is a strictly positive increasing concave function on $(0, \infty)$ satisfying*

$$f'(k) \rightarrow \infty \text{ as } k \rightarrow 0 \quad \text{and} \quad f'(k) \rightarrow 0 \text{ as } k \rightarrow \infty,$$

then g is globally stable on $(0, \infty)$.

EXERCISE 49. Show that the conditions of Proposition 5.3.9 imply those of Proposition 5.3.8.

As an exercise, revisit the continuous time Solow–Swan model. A steady state is a solution to $sf(k) = \delta k$. This can be rewritten as a fixed point problem that uses Proposition 5.3.8.

5.3.5 Linear Differential Equations

Roadmap to be added.

Add roadmap

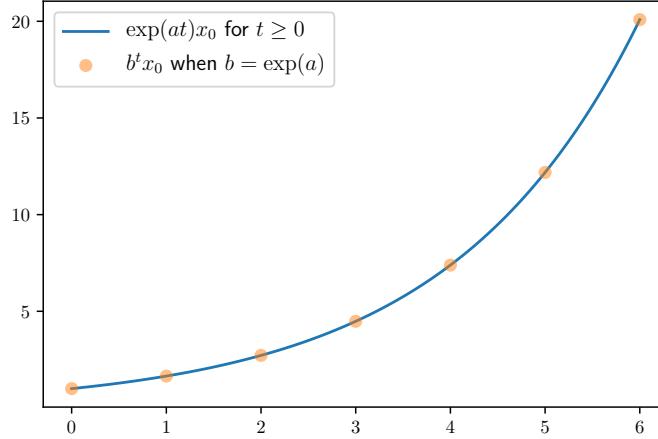
5.3.5.1 Exponentials

Consider the scalar initial value problem

$$x'(t) = ax(t) \quad \text{with} \quad x(0) = x_0 \text{ given.} \quad (5.30)$$

The solution is $x(t) = e^{at}x_0$. You can easily verify that this function $t \mapsto x(t)$ satisfies (5.30) at every $t \geq 0$. As we saw in Example 5.3.1, the pair $(\mathbb{R}, (S_t))$ with $S_t u = e^{at}u$ forms a semidynamical system.

This system is closely related to the discrete time system $x_{t+1} = e^a x_t$. Indeed, if we start at x_0 , then the t -th iterate is $e^{at}x_0$, so the solutions agree at integer times. We can think of the continuous time system extrapolating between the points of the discrete time system. Figure 5.12 illustrates with $a = 1/2$ and $x_0 = 1$.

Figure 5.12: The exponential function $t \mapsto e^{at}$

The exponential function $a \mapsto e^a$ on \mathbb{R} can be defined as $e^a = \sum_{k \geq 0} a^k / k!$. The same formula can be used to define exponentiation of matrices. In other words, $a \mapsto e^a$ extends to a map from $\mathcal{M}_{n \times n}$ to itself via

$$e^A := \sum_{k \geq 0} \frac{A^k}{k!}.$$

EXERCISE 50. Show that the series $\sum_{k \geq 0} \frac{A^k}{k!}$ converges for every A in $\mathcal{M}_{n \times n}$.

EXERCISE 51. Show that the derivative of e^{tA} with respect to t is Ae^{tA} .

In the last exercise and below, differentiation is element by element.

Lemma 5.3.10. *If $A, B \in \mathcal{M}_{n \times n}$ satisfy $AB = BA$, then $e^{A+B} = e^A e^B$.*

The proof of Lemma 5.3.10 uses the definition of the exponential and the binomial formula. We omit the details.

EXERCISE 52. Using Lemma 5.3.10, show that, for any A in $\mathcal{M}_{n \times n}$, the matrix e^A is always invertible, with inverse e^{-A} .

EXERCISE 53. Let A in $\mathcal{M}_{n \times n}$ be diagonalizable, with $A = P^{-1}DP$. Show that $e^A = P^{-1}e^D P$.

5.3.5.2 Vector Valued Initial Value Problems

In the problem below, A is in $\mathcal{M}_{n \times n}$ and $x'(t)$ and $x(t)$ are understood as column vectors.

Theorem 5.3.11. *The unique solution of the n -dimensional initial value problem*

$$x'(t) = Ax(t) \quad \text{with} \quad x(0) = x_0 \in \mathbb{R}^n \text{ given.} \quad (5.31)$$

is the map $t \mapsto x(t)$ given by

$$x(t) = e^{tA}x_0, \quad t \geq 0. \quad (5.32)$$

The fact that (5.32) solves (5.31) is immediate from Exercise 51. Uniqueness is an exercise. [Add a solution](#).

If we write $S_t = e^{tA}$, then $(\mathcal{M}_{n \times n}, (S_t))$ is a semidynamical system where the semigroup of maps generates the flow in (5.32). The proof of the semigroup property is the same as in Example 5.3.1.

5.3.5.3 Dynamics

The dynamics of the solution $x(t) = e^{tA}x_0$ are easiest to analyze when A is diagonalizable, satisfying $A = P^{-1}DP$. To see this, consider the alternative semigroup

$$y(t) = e^{tD}y_0 \quad (t \geq 0) \quad (5.33)$$

EXERCISE 54. Let $S_t x := e^{tA}x$ and $T_t y := e^{tD}y$. Show that, when $A = P^{-1}DP$, the semidynamical systems $(\mathbb{R}^n, (S_t))$ and $(\mathbb{C}^n, (T_t))$ are topologically conjugate under the homeomorphism P .

The system (5.33) is much easier to analyze than the original. To see this, observe that, since $D = \text{diag}(\lambda_1, \dots, \lambda_n)$ is a diagonal matrix storing the eigenvalues of A , we can verify from the definition that $e^{tD} = \text{diag}(e^{t\lambda_1}, \dots, e^{t\lambda_n})$. Hence (5.33) can be decoupled into the individual series

$$y^{(i)}(t) = e^{t\lambda_i}y_0^{(i)} \quad (t \geq 0) \quad (5.34)$$

Complete the analysis. Break eigenvalues into real and imaginary parts. Explain dynamics, oscillations. Relate to Theorem 2 of Hirsh and Smale (1974) on p. 136. Note that diagonalization is not required.

5.3.6 Nonlinear Differential Equations

[Add roadmap](#)

5.3.6.1 Weighted Supremum Norms

Let X be a metric space and let κ be a **weighting function**, which is a map from X to \mathbb{R} such that $\kappa \geq 1$. For any $v: \mathsf{X} \rightarrow \mathbb{R}$, we define the **κ -weighted supremum norm** to be

$$\|v\|_\kappa := \sup_{x \in \mathsf{X}} \frac{|v(x)|}{\kappa(x)}. \quad (5.35)$$

If $\kappa \equiv 1$, then this is just the ordinary supremum norm from §5.1.1.1. If $\|v\|_\kappa < \infty$ then we say that v is **κ -bounded**. Letting $b_\kappa \mathsf{X}$ be the set of all κ -bounded functions on X , we introduce the metric

$$d_\kappa(v, w) := \|v - w\|_\kappa \quad (v, w \in b_\kappa \mathsf{X}).$$

With a small amount of effort, you will be able to confirm that $(b_\kappa \mathsf{X}, d_\kappa)$ is a metric space.

EXERCISE 55. Prove that $v, w \in b_\kappa \mathsf{X}$ and $\alpha, \beta \in \mathbb{R}$ implies $\alpha v + \beta w \in b_\kappa \mathsf{X}$.

A nice fact about d_κ -convergence is that it implies pointwise convergence. Precisely, if (w_n) is a sequence in $b_\kappa \mathsf{X}$ and $\|w_n - w\|_\kappa \rightarrow 0$ for some $w \in b_\kappa \mathsf{X}$, then $w_n(x) \rightarrow w(x)$ for every $x \in \mathsf{X}$. To show this, we pick $x \in \mathsf{X}$ and observe that

$$|w_n(x) - w(x)| \leq \|w_n - w\|_\kappa \kappa(x).$$

The claim follows directly.

EXERCISE 56. Show that if (w_n) is a d_κ -convergent sequence in $b_\kappa \mathsf{X}$ with $w_n \leq w \in b_\kappa \mathsf{X}$ for all $n \in \mathbb{N}$, then $\lim w_n \leq w$.

The space $(b_\kappa \mathsf{X}, d_\kappa)$ would not much use to us if it failed to be complete. Fortunately this is not the case:

Theorem 5.3.12. *The space $(b_\kappa \mathsf{X}, d_\kappa)$ is a complete metric space.*

Proof. Let (v_n) be a Cauchy sequence in $(b_\kappa \mathsf{X}, d_\kappa)$. This implies that (v_n/κ) is Cauchy in $(b \mathsf{X}, d_\infty)$. Since that space is complete, there is a $\hat{v} \in b \mathsf{X}$ with $\|v_n/\kappa - \hat{v}\|_\infty \rightarrow 0$. We claim that $\hat{v} \kappa \in b_\kappa \mathsf{X}$ and $\|v_n - \hat{v} \kappa\|_\kappa \rightarrow 0$, in which case the completeness of $(b_\kappa \mathsf{X}, d_\kappa)$

is established. That $\hat{v}\kappa \in b_\kappa \mathbb{X}$ follows from boundedness of \hat{v} . Moreover, $\|v_n - \hat{v}\kappa\|_\kappa$ is equal to $\|v_n/\kappa - \hat{v}\|_\infty$, and the latter converges to zero by hypotheses. \square

Let $b_\kappa c\mathbb{X}$ be the continuous functions in $b_\kappa \mathbb{X}$.

Theorem 5.3.13. *If κ is continuous, then $(b_\kappa c\mathbb{X}, d_\kappa)$ is complete.*

Proof. Let κ be continuous. It suffices to show that $b_\kappa c\mathbb{X}$ is closed in $(b_\kappa \mathbb{X}, d_\kappa)$. To see that this is so, take $\{v_n\}$ in $b_\kappa c\mathbb{X}$ with $d_\kappa(v_n, v) \rightarrow 0$ for some v in $b_\kappa c\mathbb{X}$. We need to show that v is continuous. To see that this is so, observe that $\|v_n - v\|_\kappa \rightarrow 0$ can also be expressed as $\|v_n/\kappa - v/\kappa\|_\infty \rightarrow 0$. Since ratios and products preserve continuity, and since uniform limits of continuous functions are continuous, we see that v/κ is continuous. But then $\kappa(v/\kappa) = v$ is continuous too. \square

5.3.6.2 Existence and Uniqueness

Consider the autonomous initial value problem (IVP)

$$x'(t) = f(x(t)) \text{ for } t \geq 0 \text{ with } x(0) = x_0 \text{ given,} \quad (5.36)$$

where $f: \mathbb{R} \rightarrow \mathbb{R}$. A **solution** to this IVP is a function $t \mapsto x(t)$ over $t \geq 0$ satisfying (5.36). The first question we need to ask is whether a unique solution exists.

The answer is not always affirmative. For example, if $f(x) = x^2$ and $x_0 = 1$, then $x(t) = 1/(1-t)$ solves (5.36) but only for $t < 1$. As $t \rightarrow 1$, the time path for $x(t)$ explodes. Hence, one restriction we need to impose is that the rate of growth is not too large in absolute value. Another is some degree of smoothness on f , without which uniqueness can be violated.

EXERCISE 57. Let $f(x) = |x|^{1/2}$ and $x_0 = 0$. Note that f is continuous but not smooth. One solution to this IVP is $x(t) = 0$ for all t . Another can be constructed by fixing arbitrary $s > 0$ and then setting $x(t) = 0$ for $t \leq s$ and $x(t) = (t-s)^2/4$ for $t > s$. Confirm this and plot the two solutions.

We can restrict the growth rate associated with f and impose smoothness through a Lipschitz condition. The condition we will use is a relatively strict one that has the benefit of strong implications and a theorem with a beautiful proof.¹⁴ It is usually called Picard's theorem, after the French mathematician Émile Picard (1856–1941). We will work through the proof partly for its own interest and partly because similar lines of argument will be used repeatedly in the book.

¹⁴Many variations are available in the literature ([add cites](#)).

Recall that f is called Lipschitz on \mathbb{R} if there exists a $k < \infty$ with

$$|f(x) - f(y)| \leq k|x - y| \quad (x, y \in \mathbb{R}). \quad (5.37)$$

To each such k , we introduce the weighting function $\kappa(t) = e^{-2kt}$.

Theorem 5.3.14. *If f satisfies (5.37), then the initial value problem (5.36) has exactly one solution in the function space $b_\kappa c \mathbb{R}_+$.*

Proof. Let f satisfy the stated conditions and let $\mathcal{C} := b_\kappa c \mathbb{R}_+$. By the Fundamental Theorem of Calculus, a function $x \in \mathcal{C}$ solves the initial value problem if and only if it satisfies $x(t) = x_0 + \int_0^t f(x(s)) \, ds$ for all $t \geq 0$. An equivalent way to state this is to say that x is a fixed point of the operator A on \mathcal{C} defined at $x \in \mathcal{C}$ by

$$(Ax)(t) = x_0 + \int_0^t f(x(s)) \, ds \quad (t \geq 0). \quad (5.38)$$

In light of these facts, and given the completeness of $\mathcal{C} = b_\kappa c \mathbb{R}_+$ under the κ -weighted supremum norm (Theorem 5.3.13), it suffices to show that A is uniformly contracting self-map on \mathcal{C} .

Regarding the self-map property, pick any $x \in \mathcal{C}$. By the triangle inequality and the Lipschitz property we have $|f(x)| = |f(x) - f(0) + f(0)| \leq k|x| + |f(0)|$. It follows that

$$|(Ax)(t)| \leq |x_0| + \int_0^t |f(x(s))| \, ds \leq |x_0| + k \int_0^t |x(s)| \, ds + k|f(0)|t.$$

Hence, for some finite M ,

$$|(Ax)(t)| \leq |x_0| + k M \|x\|_\kappa \int_0^t e^{-2ks} \, ds + k|f(0)|t.$$

From here a small amount of effort will convince you that Ax is κ -bounded. It is also continuous, since $\int_0^t f(x(s)) \, ds$ is differentiable in t .

Regarding the contraction property, for given $x, y \in \mathcal{C}$ we have

$$\begin{aligned} |Ax(t) - Ay(t)| &= \left| \int_0^t f(x(s)) \, ds - \int_0^t f(y(s)) \, ds \right| \\ &\leq \int_0^t |f(x(s)) - f(y(s))| \, ds \\ &\leq k \int_0^t |x(s) - y(s)| \, ds \leq k \|x - y\|_\kappa \int_0^t e^{-2ks} \, ds. \end{aligned}$$

Solving the integral now gives

$$|Ax(t) - Ay(t)| \leq \frac{1}{2} \|x - y\|_\kappa \kappa(t) \quad (5.39)$$

for arbitrary t . Dividing through by $\kappa(t)$ and taking the supremum shows that A is contraction with respect to d_κ . \square

5.3.6.3 Connection to Semidynamical Systems

To be added. The semigroup property of solutions. See p. 174 of Hirsh and Smale (1974). Some comments on stability. No periodic solutions in one dimension.

5.3.7 Monotonicity in Continuous Time

To be added. One dimensional models generate monotone trajectories. For higher dimensions, discuss monotone semigroups and monotone trajectories. Sufficient conditions via the KM conditions. Implications for stability. Parametric monotonicity

See http://www.hamilton.ie/ollie/Downloads/PiecewiseMonotone_HAM.pdf

5.4 Chapter Notes

References, background, historical notes.

Chapter 6

Some Optimization Problems

In this chapter we treat some of the most common and fundamental classes of dynamic programming problems in economics. Our focus is on solving these problems rather than on producing a general theory of optimality. That second task is left to Chapter 10.

6.1 Optimal Stopping

[add roadmap]

6.1.1 Job Search Revisited

[roadmap]

6.1.1.1 The Job Search Bellman Operator

Consider again the infinite horizon job search problem of [McCall \(1970\)](#) described in §5.2.3, where a currently unemployed agent seeks to maximize expected discounted lifetime earnings $\mathbb{E} \sum_{t=0}^{\infty} \beta^t y_t$. In each period the agent observes an employment opportunity with associated wage offer w_t and chooses whether to accept or reject. Wage offers are IID and drawn from distribution φ . As in §5.2.3, acceptance entails working forever at w_t , with resulting lifetime value $w_t/(1 - \beta)$, while rejection leads to unemployment compensation $c \geq 0$ and a new wage offer next period.

Repeating (5.20) on page 159, the value function v^* , which records the maximal value that can be extracted from any given state w , satisfies a nonlinear recursion called the **Bellman equation**:

$$v^*(w) = \max \left\{ \frac{w}{1-\beta}, c + \beta \int v^*(w') \varphi(dw') \right\} \quad (w \in \mathbb{R}_+) \quad (6.1)$$

The first term on the right hand side is the value of accepting, while the second is the value of waiting until the next period—often called the **continuation value**. The optimal policy for the unemployed agent is: given current offer w , choose between these two option by picking the highest value. With 1 interpreted as accept, zero as reject and σ^* as the optimal policy, we can write this as

$$\sigma^*(w) = \mathbb{1} \left\{ \frac{w}{1-\beta} \geq c + \beta \int v^*(w') \varphi(dw') \right\} \quad (w \in \mathbb{R}_+) \quad (6.2)$$

In chapter 10 we will prove that this is indeed the optimal policy. For now let's concentrate on obtaining it and analyzing it.

To calculate the optimal policy in (6.2), we need to evaluate the right hand side of this expression, which means that we need to know the value function. (In fact we only need to know the expectation $\int v^*(w') \varphi(dw')$ and there's a way to obtain this directly, without having to first calculate v^* . But let's put this point aside until §6.1.2 and take a more traditional approach. The traditional approach gives insight and will in any case be needed as we mix in new assumptions.)

The technique most often used to solve the Bellman equation for the value function is to introduce an operator T , referred to as the **Bellman operator**, such that any fixed point of T solves the Bellman equation and vice versa. This is true by construction for T defined by $v \mapsto Tv$,

$$(Tv)(w) := Tv(w) := \max \left\{ \frac{w}{1-\beta}, c + \beta \int v(w') \varphi(dw') \right\} \quad (6.3)$$

If we can show that T has a unique fixed point in some class of functions, then the Bellman equation will have a unique solution in that same set. We will make life easier for ourselves in this respect by assuming bounded wage offers:

Assumption 6.1.1. There exists an $M \in \mathbb{R}_+$ such that $\int_0^M \varphi(dw) = 1$.

Assumption 6.1.1 helps because it allows us to set up a convenient function space for T to act on. (It isn't actually essential, as we'll see in §6.1.2.)

6.1.1.2 Case 1: Continuous Wage Draws

Suppose first that the wage offer distribution φ can be represented by a density q supported on all of $[0, M]$. In this case a wage offer can be any value in this interval and hence v^* needs to be defined on $[0, M]$. This leads us to seek a fixed point of T in $\mathcal{C} := c[0, M]$, the class of continuous functions on $[0, M]$.¹ This space is paired with the supremum norm $\|g\|_\infty := \sup_{x \in [0, M]} |g(x)|$.

In this setting, we have the following result:

Proposition 6.1.1. *If assumption 6.1.1 holds, then T is a contraction of modulus β on \mathcal{C} . In particular,*

- (i) *T has a unique fixed point in \mathcal{C} ,*
- (ii) *that fixed point is equal to the value function v^* and*
- (iii) *if $v \in \mathcal{C}$, then $\|T^n v - v^*\|_\infty \leq O(\beta^n)$.*

The last claim is existence of a finite constant K , possibly depending on v , such that

$$\sup_{0 \leq w \leq M} |T^n v(w) - v^*(w)| := \|T^n v - v^*\|_\infty \leq K \beta^n$$

holds for all $n \in \mathbb{N}$.

Claim (ii) will be proved later, in chapter 10, for a broad class of problems that include this one. The proof is easier and cleaner in a more abstract setting.

Given (ii) and Banach's contraction mapping theorem (page 139), claims (i) and (iii) will be established if we can show that, with d_∞ defined on \mathcal{C} by $d_\infty(v, v') := \|v - v'\|_\infty$,

- (a) T is a contraction of modulus β on (\mathcal{C}, d_∞) and
- (b) (\mathcal{C}, d_∞) is a complete metric space.

Part (b) follows directly from example 11.1.8 on page 318, which states the well known fact that, for any metric space X , the class of continuous bounded real-valued functions on X paired with the supremum norm is a Banach space. In our case, all functions in

¹Why restrict attention to the *continuous* functions here? While it's not essential, we have chosen to do so in this instance because when we inspect T defined in (6.3), we can see that Tv will be continuous whenever v is continuous. So T will be invariant on (i.e., map back into) the set $c[0, M]$. In general, it is a good idea to take the space that T acts on to be as small as possible, by looking at the definition of T and seeing what properties it will preserve (e.g., continuity, convexity, concavity, monotonicity, etc.) and then choosing a function class that only contains such functions. The reason this is a good idea is that it reduces the size of the space over which we must search for the function v^* , as well as providing structure that we can potentially exploit in our numerical algorithms.

\mathcal{C} are continuous by assumption and bounded because continuous functions defined on compact sets are bounded (see, e.g., theorem 5.1.18 on page 151).

Returning to part (a), this can be established using the elementary bound

$$|\alpha \vee x - \alpha \vee y| \leq |x - y| \quad (\alpha, x, y \in \mathbb{R}) \quad (6.4)$$

Here $a \vee b = \max\{a, b\}$. You can check (6.4) by sketching it on a line (or by appealing to lemma 5.1.17 on page 151).

To see the connection between (6.4) and the contraction claim in part (a), take any f, g in \mathcal{C} and fix any $w \in [0, M]$. The bound in (6.4) gives

$$\begin{aligned} |Tf(w) - Tg(w)| &\leq \left| c + \beta \int f(w') q(w') dw' - \left(c + \beta \int g(w') q(w') dw' \right) \right| \\ &= \beta \left| \int [f(w') - g(w')] q(w') dw' \right| \end{aligned}$$

Applying the triangle inequality for integrals (or Jensen's inequality, if you prefer), we obtain

$$|Tf(w) - Tg(w)| \leq \beta \int |f(w') - g(w')| q(w') dw' \leq \beta \|f - g\|_\infty$$

Taking the supremum over all w on the left hand side of this expression leads to

$$\|Tf - Tg\|_\infty \leq \beta \|f - g\|_\infty$$

Since f, g were arbitrary elements of \mathcal{C} , the contraction claim is established.

6.1.1.3 Case 2: Discrete Wage Draws

Now let's drop the density assumption on wage offers and suppose instead that the distribution is discrete:

Assumption 6.1.2. The wage distribution φ is supported on $W := \{w_1, w_2, \dots, w_m\}$ with probabilities $q(w)$, $w \in W$.

These are the only possible states, so v^* need only be defined on these points. Hence we should adjust the candidate space that the Bellman operator acts on. In particular, we define T on \mathbb{R}^W by

$$Tv(w) = \max \left\{ \frac{w}{1 - \beta}, c + \beta \sum_{w \in W} v(w) q(w) \right\} \quad (w \in W) \quad (6.5)$$

We pair \mathbb{R}^W with the d_∞ distance $d_\infty(f, g) = \max_{w \in W} |f(w) - g(w)|$. This yields a complete metric space by example 11.1.7 on page 318 or Theorem 11.1.7 on page 321. We now have the following result, analogous to Proposition 6.1.1:

Proposition 6.1.2. *If assumption 6.1.2 holds, then T is a contraction of modulus β on \mathbb{R}^W . In particular,*

- (i) *T has a unique fixed point in \mathbb{R}^W ,*
- (ii) *that fixed point is equal to the value function v^* and*
- (iii) *if $v \in \mathbb{R}^W$, then $\|T^n v - v^*\|_\infty \leq O(\beta^n)$.*

EXERCISE 1. Prove that T is a contraction of modulus β on (\mathbb{R}^W, d_∞) .

6.1.1.4 Computation

It follows from Proposition 6.1.2 that to compute the optimal policy we can use **value function iteration**, which means starting with arbitrary $v \in \mathbb{R}^W$ and then iterating with T until $v_k := T^k v$ is a good approximation to v^* . We then insert it into (6.2) and obtain an approximate optimal policy. In other words,

$$\sigma_k(w) := \mathbb{1} \left\{ \frac{w}{1-\beta} \geq c + \beta \int v_k(w') \varphi(dw') \right\} \quad (6.6)$$

is approximately optimal when v_k is close to v^* .

While $T^k v$ never exactly attains v^* in most cases, we can obtain a close approximation by monitoring the distance between successive iterates, waiting until they become small. In doing so, we can make σ_k in (6.6) arbitrarily close to σ^* . (Further discussion of this point, including error bounds, is given in §10.2.5.)

The iteration procedure is straightforward to implement on a computer. Figure 6.1 shows a sequence of iterates $\{T^k v\}$ when $v \equiv 100$, $c = 10$ and φ is the binomial distribution $\text{Bin}(n, p)$ with $n = 40$ and $p = 0.5$. Iterates 0, 1 and 2 are shown, in addition to iterate 100. As we will see, iterate 100 is essentially indistinguishable from the limit, and the figure indicates convergence to this approximate limit. Figure 6.2 shows convergence to the same approximate limit from a different initial condition, as expected from the theory.

Figure 6.3 shows the result of a more concerted effort to compute the limit. Here we terminated iteration when the d_∞ distance between successive iterates dropped below 10^{-6} . The approximate value function v^* is plotted, along with the stopping reward

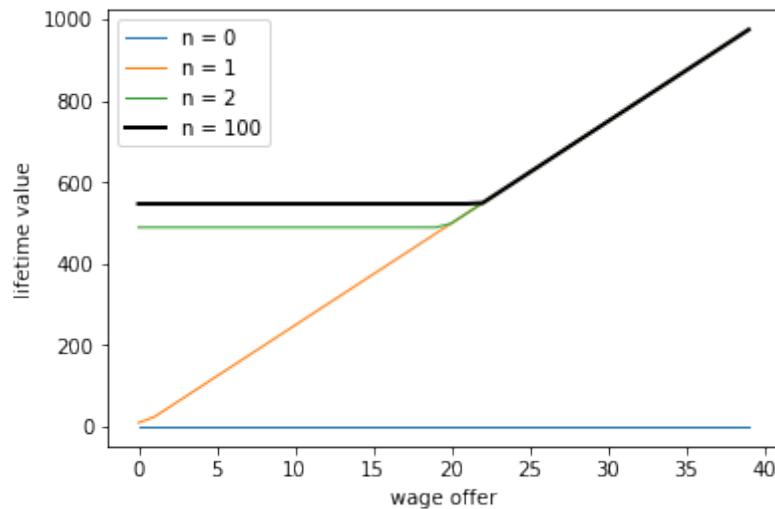
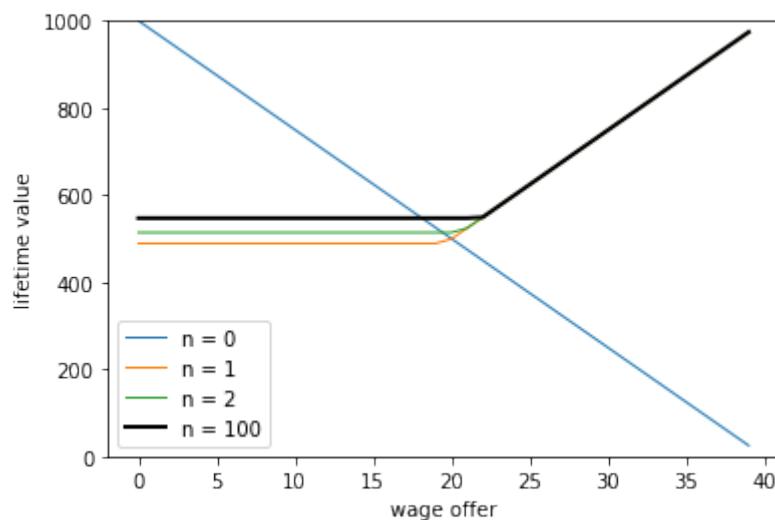


Figure 6.1: A sequence of iterates of the Bellman operator

Figure 6.2: Iterates of T from a different initial condition

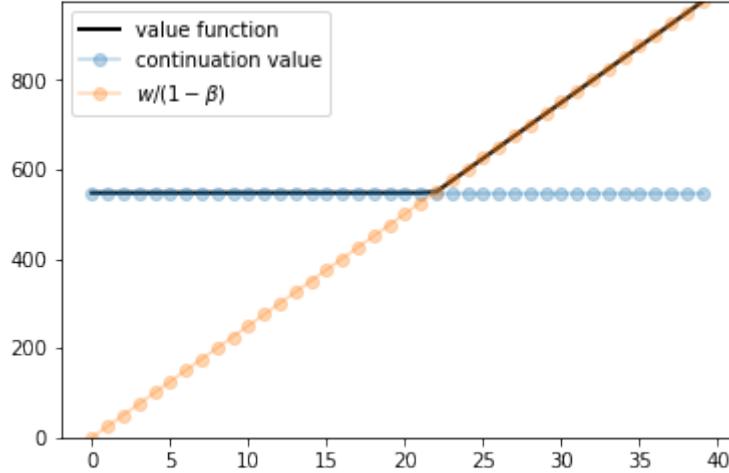


Figure 6.3: The approximate value function for job search

$w/(1 - \beta)$ and the continuation value $c + \beta \sum_w v^*(w)q(w)$. As expected, the value function is the pointwise supremum of these two functions. It appears that the agent chooses to accept an offer only when that offer exceeds some value close to 22. We will confirm this from another direction in §6.1.2.

6.1.2 Rearranging the Bellman Equation

We went to some effort to compute the value function in the previous section. It turns out that most of this effort is not necessary: there is a more straightforward approach for this specific job search problem. Nonetheless, our previous efforts are not wasted: they will be useful in verifying that our new approach is sound.

6.1.2.1 Continuation Values

Recall that a function v satisfies the Bellman equation if

$$v(w) := \max \left\{ \frac{w}{1 - \beta}, c + \beta \int v(w') \varphi(dw') \right\} \quad (6.7)$$

for all w in its support. Taking v as given, consider the term

$$h := c + \beta \int v(w') \varphi(dw') \quad (6.8)$$

We can use h to eliminate the function v from (6.7). To do so we insert h on the right hand side, replace w with w' in (6.7), take expectations, multiply by β and add c to obtain

$$h = c + \beta \int \max \left\{ \frac{w'}{1-\beta}, h \right\} \varphi(dw') \quad (6.9)$$

This is a nonlinear equation in h , the solution of which, henceforth denoted h^* , is, in view of (6.8), the continuation value of our problem. If we can obtain h^* , we have essentially solved the dynamic programming problem, since the optimal policy can be written as

$$\sigma^*(w) = \mathbb{1} \left\{ \frac{w}{1-\beta} \geq h^* \right\} \quad (w \in \mathbb{R}_+) \quad (6.10)$$

Another way to write the optimal policy is

$$\sigma^*(w) = \mathbb{1} \{ w \geq w^* \} \quad \text{where } w^* := (1-\beta)h^* \quad (6.11)$$

The term w^* in (6.11) is called the **reservation wage**. Equation (6.11) states that value maximization requires accepting an offer if and only if it exceeds the reservation wage. This is convenient because w^* provides a scalar summary of the solution to the problem.

In order to solve (6.9), we introduce the mapping

$$g(h) = c + \beta \int \max \left\{ \frac{w'}{1-\beta}, h \right\} \varphi(dw') \quad (6.12)$$

It is constructed such that any solution to (6.9) is a fixed point of g and vice versa. For the function g to be real valued we only need

Assumption 6.1.3. The distribution φ has finite first moment.

With this weak restriction on the wage distribution we have the results listed in Exercise 2.

EXERCISE 2. Confirm that g is a well defined map from \mathbb{R}_+ to itself whenever assumption 6.1.3 is satisfied.² Show that g is a contraction map on \mathbb{R}_+ under the usual Euclidean distance. Conclude that g has a unique fixed point in \mathbb{R}_+ , which is the unique solution to (6.10) in this set.

While the theory works smoothly under assumption 6.1.3, when we turn to computation it is somewhat easier to work with the case where wages are bounded, since

²Hint: For nonnegative numbers a, b we have $a \vee b \leq a + b$.

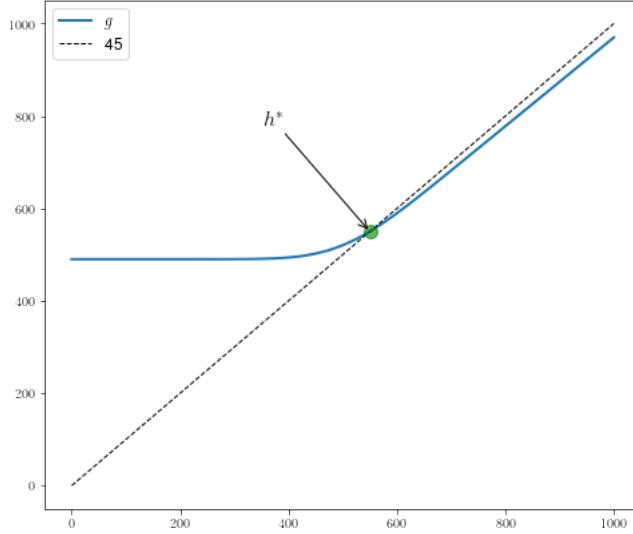


Figure 6.4: The continuation value computed directly

it provides a clear upper limit for the interval in which we can expect to find the continuation value:

EXERCISE 3. Suppose that $\mathbb{P}\{w_t \leq M\} = 1$ for some positive constant M . Confirm that g maps $[0, K]$ to itself, where

$$K := \frac{\max\{M, c\}}{1 - \beta}$$

Conclude that g has a fixed point in $[0, K]$, which is the unique fixed point of g in \mathbb{R}_+ .³

Figure 6.4 shows the function g using the discrete wage offer distribution and parameters as adopted previously. The unique fixed point is h^* . In view of the results in Exercise 2, this value can be computed by iterating with g on any initial condition in \mathbb{R}_+ . Doing so produces a value of around 550. The reservation wage w^* is then calculated as $w^* = (1 - \beta)h^* \approx 21.99$.

EXERCISE 4. As a computational exercise, compare the value function v^* computed via

$$v^*(w) = \max \left\{ \frac{w}{1 - \beta}, h^* \right\}$$

with our previous result, shown in figure 6.3. You should find them essentially identical.

³Lemma 5.3.2 can be applied here.

6.1.2.2 Parametric Monotonicity

How does our solution vary with parameters? In terms of monotonicity, one way to answer this is to appeal to proposition 5.3.7 on page 171. This result tells us that, since g is an increasing contraction mapping on \mathbb{R}_+ , any parameter that shifts up the function g in (6.12) pointwise on \mathbb{R}_+ also shifts its fixed point up.

EXERCISE 5. Show that both the continuation value h^* and the reservation wage w^* are increasing in unemployment compensation c . Is this what you would expect?

EXERCISE 6. Let τ be the first passage time to employment for an unemployed agent. That is,

$$\tau := \inf\{t \geq 0 : \sigma^*(w_t) = 1\}$$

Construct an argument demonstrating that the mean first passage time $\mathbb{E}\tau$ increases with unemployment compensation c .

Now let us consider the wage distribution and how shifts in this distribution might affect the reservation wage. First, it seems likely that a shift to a “more favorable” wage distribution would tend to increase the reservation wage, since the agent can expect better offers. One natural way to order the set of wage distributions in terms of “favorability” is first order stochastic dominance \preceq_F . Below we show that the reservation wage increases as the offer distribution increases in the sense of stochastic dominance.

To state our result, let φ and ψ be two wage distributions on \mathbb{R}_+ with finite first moment and let w_φ^* and w_ψ^* be the associated reservation wages. To simplify matters we suppose that both distributions are supported on $[0, M]$. We then have the following monotonicity result:

Lemma 6.1.3. *If ψ first order stochastically dominates φ , then $w_\varphi^* \leq w_\psi^*$.*

Proof. Let ψ and φ have the stated properties and let h_φ^* and h_ψ^* be the associated continuation values. It suffices to show that $h_\varphi^* \leq h_\psi^*$.

In view of proposition 5.3.7 on page 171, It is enough to show that the function g in (6.12) increases pointwise in \preceq_F , or, in this case, that

$$\int \max\left\{\frac{w'}{1-\beta}, h\right\} \varphi(dw') \leq \int \max\left\{\frac{w'}{1-\beta}, h\right\} \psi(dw')$$

for any given $h \geq 0$. Since $w' \mapsto \max\{w'/(1-\beta), h\}$ is bounded and increasing on $[0, M]$, this inequality follows directly from the definition of stochastic dominance. \square

One more subtle monotonicity result for this model concerns the volatility of the wage process and its impact on the reservation wage and welfare. Intuitively, greater volatility in wages is attractive to the agent, and encourages more patience. Agents are more inclined to wait because the option value of waiting is larger. In the next lemma, we formalize this idea using the notion of mean-preserving spreads, introduced on page 150.

Lemma 6.1.4. *If ψ is a mean-preserving spread of φ , then $w_\varphi^* \leq w_\psi^*$.*

Proof. Let ψ and φ have the stated properties and let h_φ^* and h_ψ^* be the associated continuation values. It suffices to show that $h_\varphi^* \leq h_\psi^*$.

In view of proposition 5.3.7 on page 171, it is enough to show that, under the stated assumptions, the function g in (6.12) increases pointwise with the mean-preserving spread, or, equivalently

$$\int \max \left\{ \frac{w'}{1-\beta}, h \right\} \varphi(dw') \leq \int \max \left\{ \frac{w'}{1-\beta}, h \right\} \psi(dw')$$

for all $h \geq 0$.

To see that this is so, observe that, by the definition of a mean-preserving spread, there exists a pair (w', Z) such that $\mathbb{E}[Z | w'] = 0$, $w' \stackrel{d}{=} \varphi$ and $w' + Z \stackrel{d}{=} \psi$. By this fact and the law of iterated expectations,

$$\begin{aligned} \int \max \left\{ \frac{w'}{1-\beta}, h \right\} \psi(dw') &= \mathbb{E} \left[\max \left\{ \frac{w' + Z}{1-\beta}, h \right\} \right] \\ &= \mathbb{E} \left[\mathbb{E} \left[\max \left\{ \frac{w' + Z}{1-\beta}, h \right\} \mid w' \right] \right] \end{aligned}$$

An application of Jensen's inequality now produces

$$\int \max \left\{ \frac{w'}{1-\beta}, h \right\} \psi(dw') \geq \mathbb{E} \max \left\{ \frac{\mathbb{E}[w' + Z | w']}{1-\beta}, h \right\}$$

Using $\mathbb{E}[w' | w'] = w'$ and $\mathbb{E}[Z | w'] = 0$ leads to the desired inequality. \square

6.1.3 Learning the Offer Distribution

[roadmap]

6.1.3.1 The Model

Next let's consider the variation of the McCall job search model presented in section 6.6 of Ljungqvist and Sargent (2012), which in turn draws on ideas suggested in the original paper by McCall (1970). The framework is as in §6.1.1.2, apart from the fact that the density q is unknown to the worker. Instead, we envisage a situation where the agent learns about q by starting with a prior belief and then successively updating beliefs based on wage offers that he or she observes.

The precise structure of information is as follows: The worker knows there are two possible distributions, F and G , with densities f and g on \mathbb{R}_+ . At the start of time, nature selects q to be either f or g , the wage distribution from which the entire sequence $\{w_t\}$ will be drawn. This choice is not observed by the worker, who puts prior probability π_0 on f being chosen. Thus, the worker's initial guess of q is

$$q_0(w) := \pi_0 f(w) + (1 - \pi_0) g(w)$$

Beliefs subsequently update according to Bayes' rule, which tells us that the agent, having observed w_{t+1} updates π_t to π_{t+1} via

$$\pi_{t+1} = \frac{f(w_{t+1})\pi_t}{f(w_{t+1})\pi_t + g(w_{t+1})(1 - \pi_t)} \quad (6.13)$$

In more intuitive notation, this is

$$\mathbb{P}\{q = f \mid w_{t+1}\} = \frac{\mathbb{P}\{w_{t+1} \mid q = f\}\mathbb{P}\{q = f\}}{\mathbb{P}\{w_{t+1}\}}$$

combined with the law of total probability to obtain the denominator:

$$\mathbb{P}\{w_{t+1}\} = \sum_{\psi \in \{f,g\}} \mathbb{P}\{w_{t+1} \mid q = \psi\}\mathbb{P}\{q = \psi\}$$

The fact that (6.13) is recursive allows us to progress to a recursive solution method for obtaining the optimal policy. Dropping time subscripts, let

$$q_\pi := \pi f + (1 - \pi)g$$

represent the estimate of the wage offer distribution given belief π and let

$$\kappa(w, \pi) := \frac{\pi f(w)}{\pi f(w) + (1 - \pi)g(w)}$$

In particular, $\kappa(w, \pi)$ is the updated value π' of π having observed draw w .

The value function v^* , which, in this context, describes the maximal lifetime value that can be extracted from a given state conditional on currently being unemployed, satisfies the Bellman equation

$$v^*(w, \pi) = \max \left\{ \frac{w}{1 - \beta}, c + \beta \int v^*(w', \kappa(w', \pi)) q_\pi(w') dw' \right\} \quad (6.14)$$

Note that the current belief π is a state variable, since it affects the worker's perception of probabilities for future rewards. It is in fact known as the current **belief state**. An optimal policy observes the current state (w_t, π_t) , inserts it into the right hand side of (6.14), and selects the largest of the two options.

We could now use analysis to try to discern the implications of the Bellman equation (6.14) or implement it on a machine and iterate. However, as in §6.1.2, there is a way to reduce dimensionality here that leads to greater efficiency.

6.1.3.2 An Efficient Solution Method

To begin, let $w^*(\pi)$ be the reservation wage at belief state π , which is the wage level at which the worker is indifferent between accepting and rejecting. In other words, $w^*(\pi)$ is the value of w at which the two choices on the right-hand side of (6.14) have equal value:

$$\frac{w^*(\pi)}{1 - \beta} = c + \beta \int v^*(w', \kappa(w', \pi)) q_\pi(w') dw' \quad (6.15)$$

If we combine (6.14) and (6.15) we can obtain

$$v^*(w, \pi) = \max \left\{ \frac{w}{1 - \beta}, \frac{w^*(\pi)}{1 - \beta} \right\} \quad (6.16)$$

If we then take (6.15) and combine it with (6.16), we find that

$$w^*(\pi) = (1 - \beta)c + \beta \int \max \{w', w^*[\kappa(w', \pi)]\} q_\pi(w') dw' \quad (6.17)$$

Equation (6.17) can be understood as a functional equation where w^* is the unknown function. The solution w^* to (6.17) is the object that we wish to compute.

In order to do so, we proceed in the usual way, by introducing an operator such that solutions to the functional equation (6.17) are fixed points of the operator. This

leads us to introduce Q mapping $\omega \mapsto Q\omega$ via

$$(Q\omega)(\pi) = (1 - \beta)c + \beta \int \max \{w', \omega[\kappa(w', \pi)]\} q_\pi(w') dw' \quad (6.18)$$

Comparing (6.17) and (6.18), we see that the set of fixed points of Q exactly coincides with the set of solutions to the reservation wage functional equation.

For the remainder of this section, let $\mathcal{C} := bc(0, 1)$, the set of bounded continuous real-valued functions on $(0, 1)$, paired with the supremum norm. We also assume that the densities f and g are everywhere positive on a subset of $[0, M]$ and zero elsewhere.

Proposition 6.1.5. *The operator Q is a contraction of modulus β on \mathcal{C}*

Proof of proposition 6.1.5. This proposition includes the claim that Q is a self-mapping on \mathcal{C} , which requires some effort to check. To this end, pick any $\omega \in \mathcal{C}$ and consider the function $Q\omega$ defined by (6.18). To see that this function is bounded, observe that, by the triangle inequality and the fact that q_π is a density,

$$(Q\omega)(\pi) \leq (1 - \beta)c + \beta \max\{M, \|\omega\|_\infty\} \quad (6.19)$$

The right hand side does not depend on π so $Q\omega$ is bounded as claimed.

To show that $Q\omega$ is continuous, it suffices to show that we can pass the limit through the integral in $Q\omega$, in the sense that if $\{\pi_n\}$ is a sequence converging to $\pi \in (0, 1)$, then

$$\int \max \{w', \omega[\kappa(w', \pi_n)]\} q_{\pi_n}(w') dw' \rightarrow \int \max \{w', \omega[\kappa(w', \pi)]\} q_\pi(w') dw'$$

For fixed w' , both $\kappa(w', \pi)$ and $q_\pi(w')$ are continuous in π , so, in view of the Dominated Convergence Theorem (page 339), it suffices to show that

$$H_n(w') := \max \{w', \omega[\kappa(w', \pi_n)]\} q_{\pi_n}(w')$$

satisfies $\sup_n |H_n(w')| \leq H(w')$ for some $H: [0, M] \rightarrow \mathbb{R}$ with $\int H(w') dw' < \infty$. Such an H does indeed exist: one suitable choice is

$$H(w') := \max \{M, \|\omega\|_\infty\} (f(w') + g(w'))$$

Next let's establish the contraction property. Fix $\omega, \varphi \in \mathcal{C}$. The triangle inequality for integrals tells us that, for any fixed $\pi \in (0, 1)$,

$$|(Q\omega)(\pi) - (Q\varphi)(\pi)| \leq \beta \int |\max \{w', \omega[\kappa(w', \pi)]\} - \max \{w', \varphi[\kappa(w', \pi)]\}| q_\pi(w') dw'$$

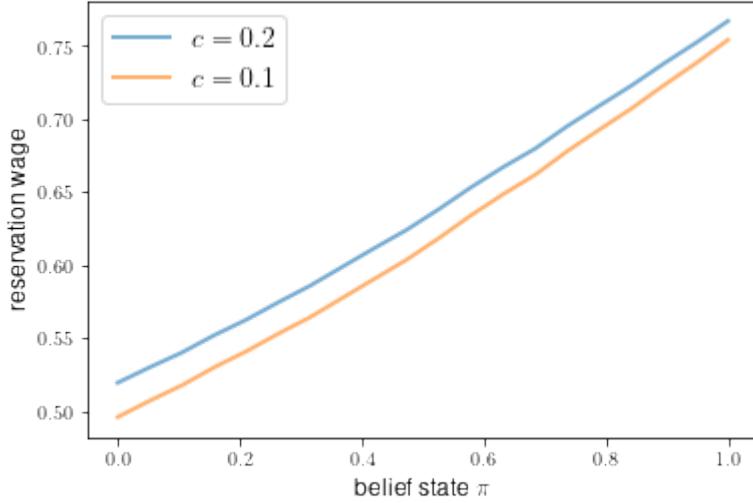


Figure 6.5: Reservation wage as a function of beliefs

Combining this inequality and the bound (6.4) on page 185 yields

$$|(Q\omega)(\pi) - (Q\varphi)(\pi)| \leq \beta \int |\omega[\kappa(w', \pi)] - \varphi[\kappa(w', \pi)]| q_\pi(w') dw' \leq \beta \|\omega - \varphi\|_\infty$$

Taking the supremum over π now gives us

$$\|Q\omega - Q\varphi\|_\infty \leq \beta \|\omega - \varphi\|_\infty$$

□

We have shown that Q is a contraction of modulus β on the complete metric space $(\mathcal{C}, \|\cdot\|_\infty)$. It follows that a unique solution w^* to the reservation wage functional equation exists in \mathcal{C} and $Q^k \omega \rightarrow w^*$ uniformly as $k \rightarrow \infty$, for any $\omega \in \mathcal{C}$.

6.1.3.3 Monotonicity

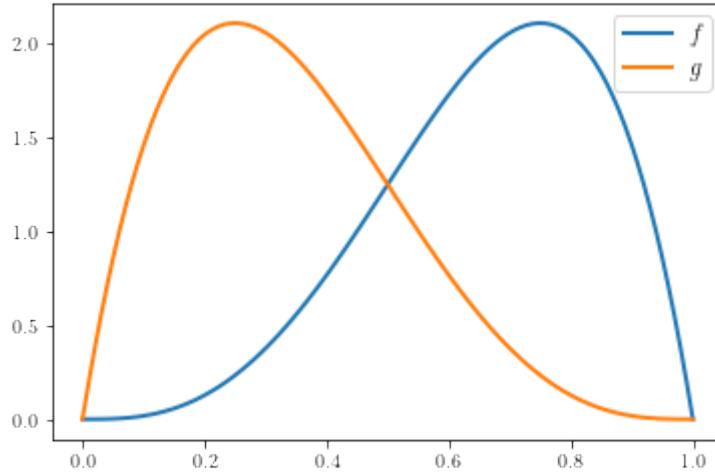
Figure 6.5 shows plots of the (approximate) solution w^* , the reservation wage, as a function of π , the belief state. The two densities here are

$$f = \text{Beta}(4, 2) \quad \text{and} \quad g = \text{Beta}(2, 4) \tag{6.20}$$

The are shown in figure 6.6. The other parameters are $c =$ either 0.1 or 0.2 and $\beta = 0.95$.

Note that the reservation wage function w^*

- (a) shifts upwards when c increases and

Figure 6.6: The two unknown densities f and g

(b) is monotonically increasing in π .

The result (a) coincides with our intuition and matches with our discussion of monotonicity in §6.1.2.2.

EXERCISE 7. Prove that (a) always holds.⁴

Result (b) is also intuitive: From figure 6.6, the density f is likely to lead to better draws, so as our belief shifts toward f , our reservation wage should increase.

Can we prove this result? If so, what conditions are required on f and g ? The next proposition provides one answer.

Proposition 6.1.6. *If f and g have the monotone likelihood ratio property, then w^* is increasing in π .*

Proof. Let $i\mathcal{C}$ be the set of (weakly) increasing functions in \mathcal{C} . This is a closed subset of \mathcal{C} . To see this, let $\{\omega_n\}$ be a sequence in $i\mathcal{C}$ such that $\|\omega_n - \omega\|_\infty \rightarrow 0$ for some $\omega \in \mathcal{C}$. The function ω is increasing because, for $x, x' \in (0, 1)$ with $x \leq x'$, we have $\omega_n(x) \leq \omega(x')$ for all n , and hence, taking the limit, $\omega(x) \leq \omega(x')$. Hence, by lemma 5.3.2 on page 166, it suffices to show that $Q\omega$ is in $i\mathcal{C}$ whenever $\omega \in i\mathcal{C}$. So pick any $\omega \in i\mathcal{C}$.

⁴Hint: Review Exercise 5 and the discussion surrounding it.

Since Q maps \mathcal{C} to itself, we need only show that $Q\omega$ is increasing. For this it suffices to show that, with

$$h(w', \pi) := \omega \left[\frac{\pi f(w')}{\pi f(w') + (1 - \pi)g(w')} \right]$$

the function

$$\pi \mapsto \int \max \{w', h(w', \pi)\} q_\pi(w') dw'$$

is increasing. This will be true if we can establish that

- (i) h is increasing in both π and w' and
- (ii) $\pi \mapsto q_\pi$ is isotone with respect to \preceq_F .

To see that (i) holds, write h as

$$h(w', \pi) = \omega \left[\frac{1}{1 + [(1 - \pi)/\pi][g(w')/f(w')]} \right]$$

Recalling that ω is assumed to be increasing, it is clear that this expression is increasing in π . Also, f and g are assumed to have the monotone likelihood ratio property, which means that $g(w')/f(w')$ is decreasing in w' , and hence $h(w', \pi)$ is increasing in w' . Thus, condition (i) is established.

Condition (ii) follows from proposition 5.1.16 on page 149, along with the result of Exercise 8. \square

EXERCISE 8. Let F and G be two distributions on \mathbb{R} with $G \preceq_F F$. Let H_α be the convex combination defined by

$$H_\alpha := \alpha F + (1 - \alpha)G \quad (0 \leq \alpha \leq 1)$$

Show that $\alpha \leq \beta$ implies $H_\alpha \preceq_F H_\beta$.

EXERCISE 9. Show that f and g in (6.20) have the monotone likelihood ratio property. Hint: the Gamma function is increasing over the interval $[2, 4]$.

6.1.4 Entry and Exit

Add a basic exit problem for a firm, a la Hopenhayn (1992). Then an entry-exit problem. Refer to §7.1.6. We are endogenizing choices.

6.1.5 Optimal Harvesting

Consider a firm that owns a timber plantation with total biomass s_t at time t . The current price of timber per unit of biomass is p_t . Assume that $\{p_t\}$ is IID with common distribution φ on $[0, \bar{p}]$. Biomass evolves according to $s_{t+1} = q(s_t)$ when not harvested and $s_{t+1} = 0$ when harvested. The function q is an increasing self-mapping on $[0, \bar{q}]$. For now we assume that $q(0) > 0$, so the plantation regenerates after each harvest.

The cost of harvesting is m . The cost of maintaining the plantation is for one period rather than harvesting is c . The firm is risk neutral and discounts the future at rate $\beta < 1$. The firm seeks to maximize expected discounted profits. Its decision at each point in time is simply to harvest or not harvest. When making its decision at time t , the firm observes price p_t and, should it decide to harvest, sells at that same price. If not then it waits to the next period, where a new price is drawn, biomass updates and the process repeats.

EXERCISE 10. Express the profit maximization problem as a dynamic program. In particular, write down the Bellman equation.

EXERCISE 11. Assume that s_t and p_t are integer valued. Biomass s_t takes values in the integers between 0 and $\bar{q} \in \mathbb{N}$, while φ puts all mass on the integers between 0 and $\bar{p} \in \mathbb{N}$. Write down the Bellman operator. Show that this operator is a contraction of modulus β on (\mathbb{R}^X, d_∞) , where

$$X := \{0, 1, \dots, \bar{q}\} \times \{0, 1, \dots, \bar{p}\}$$

EXERCISE 12. Suppose now that, in addition to the assumptions in exercise 11, we have $q(0) = 0$. In other words, the plantation never regenerates, so harvesting terminates the decision process. The dynamic program is now an optimal stopping problem. Express the Bellman operator as the maximum of two terms, one of which is the stopping value (i.e., harvesting) and the other is the continuation value.

Is there a way that you can reduce dimensionality by seeing the continuation value directly? If so, write down a Bellman like equation for continuation value and show that the corresponding operator has a unique fixed point. For reference, see the discussion on computing the continuation value in §11.4.2.

EXERCISE 13. Consider again the setting of exercise 12. If you can, continue the argument by showing that, in addition, the fixed point that represents continuation

value decreases when maintenance cost c increases. Use this fact to show that, should we observe two firms, both starting from the same biomass and facing the same price sequence but with different maintenance cost, the firm with higher maintenance costs harvest no later than (i.e., before or at the same time as) the firm with low maintenance costs.

6.2 LQ Problems

[roadmap]

6.2.1 Linear Control Systems

[roadmap]

6.2.1.1 Dynamics

Linear quadratic dynamic programming problems (sometimes called LQ control problems) are a special class of dynamic decision problems in which dynamics are linear and rewards are quadratic. These restrictive assumptions facilitate tractability even in high dimensions.

While we vary rewards slightly through this section in order to accommodate different time horizons, the dynamics will always be

$$x_{t+1} = Ax_t + Bu_t + C\xi_{t+1} \quad (6.23)$$

with x_0 given. As in our discussion of controllability §4.2.6.1,

- the state $\{x_t\}$ takes values in \mathbb{R}^n ,
- the control sequence $\{u_t\}$ takes values in \mathbb{R}^m ,
- the matrices A and B are $n \times n$ and $n \times m$ respectively, while
- C is $n \times j$ and $\{\xi_t\}$ is IID satisfying $\mathbb{E}\xi_t = 0$ and $\mathbb{E}\xi_t\xi_t' = I$.

A decision maker chooses a control sequence $\{u_t\}$ to guide the state $\{x_t\}$ but transitions are buffeted by shocks $\{\xi_t\}$. Figure 6.7 provides a visualization.

For example, consider the law of motion for wealth

$$w_{t+1} = (1+r)(w_t - c_t) + y_{t+1}$$

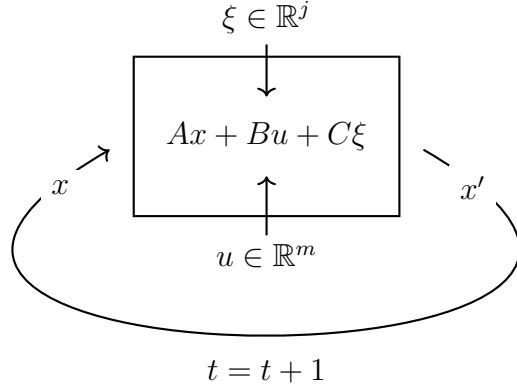


Figure 6.7: State dynamics for MDPs

that we saw previously in §7.2.1. For the moment we will assume that $y_t = \mu + \sigma \xi_t$ where $\{\xi_t\}$ is IID and standard normal in \mathbb{R} . Let's also introduce the control $u_t := c_t - \bar{c}$ where \bar{c} is some “ideal” level of consumption. (You might object that there no such thing, but we have to pay a price for pressing our problem into the LQ framework.) Then

$$w_{t+1} = (1+r)(w_t - u_t - \bar{c}) + \mu + \sigma \xi_{t+1} \quad (6.24)$$

To write (6.24) in the form of equation (6.23), consider

$$\begin{pmatrix} w_{t+1} \\ 1 \end{pmatrix} = \begin{pmatrix} 1+r & -(1+r)\bar{c} + \mu \\ 0 & 1 \end{pmatrix} \begin{pmatrix} w_t \\ 1 \end{pmatrix} + \begin{pmatrix} -(1+r) \\ 0 \end{pmatrix} u_t + \begin{pmatrix} \sigma \\ 0 \end{pmatrix} \xi_{t+1}$$

The first row is equivalent to (6.24). Moreover, the model is now linear and can be written in the form of (6.23) by setting $x_t = (w_t, 1)'$ along with

$$A := \begin{pmatrix} 1+r & -(1+r)\bar{c} + \mu \\ 0 & 1 \end{pmatrix}, \quad B := \begin{pmatrix} -(1+r) \\ 0 \end{pmatrix} \quad \text{and} \quad C := \begin{pmatrix} \sigma \\ 0 \end{pmatrix}$$

6.2.1.2 Rewards

In the LQ model we will aim to *minimize* a flow of losses, where current loss is given by the quadratic expression

$$x_t' R x_t + u_t' Q u_t \quad (6.25)$$

Here

- R is $n \times n$ and positive semidefinite.

- Q is $m \times m$ and positive definite.

Example 6.2.1. Consider the household with budget constraint (6.24). Recalling that $x_t = (w_t, 1)'$ and $u_t := c_t - \bar{c}$, a typical choice of R and Q would be

$$Q := 1 \quad \text{and} \quad R := \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

This leads to current loss

$$x_t' Rx_t + u_t' Qu_t = u_t^2 = (c_t - \bar{c})^2$$

In particular, the household's current loss is the squared deviation of consumption from the ideal level \bar{c} .

6.2.2 Finite Horizon Optimality

Most of this book adopts a infinite horizon setting for a dynamic program. This setting is technically challenging because we cannot use backward induction. At the same time, infinite horizon problems are simple in the sense that, in most cases, optimal policies are time invariant. Time doesn't matter because, at any given point in time, the agent still faces an infinite future.

In some instances, however, we specifically wish to inject time into a model and generate time dependent policies. Studies of life cycle savings and consumption are one example, since savings behavior differs across stages of life. In this section we look at LQ problems in a finite horizon setting.

6.2.2.1 Theory

Assuming terminal time $T \in \mathbb{N}$, the problem is to choose a sequence of controls u_0, \dots, u_{T-1} to minimize

$$\mathbb{E} \left\{ \sum_{t=0}^{T-1} \beta^t (x_t' Rx_t + u_t' Qu_t) + \beta^T x_T' R_f x_T \right\} \quad (6.26)$$

subject to the law of motion (6.23) and initial state x_0 . Here $\beta \in (0, 1]$ is the time discount factor, while $x' R_f x$ gives terminal loss associated with state x . The matrix R_f is assumed to be $n \times n$ positive semidefinite. Notice that we allow $\beta = 1$ to include the undiscounted case. If the initial condition is random then we require it to be independent of the shock sequence ξ_1, \dots, ξ_T .

To solve the finite horizon LQ problem we use a dynamic programming strategy based on backwards induction. In this process it is helpful to introduce the notation $J_T(x) := x'R_fx$ and then consider the problem of the controller in the second to last period (i.e., the last period in which the decision maker acts). In particular, let the time be $T - 1$, and suppose that the state is x_{T-1} . The controller takes x_{T-1} as given—since it can't be changed at this point—and trades off current and final losses by solving

$$\min_u \{x'_{T-1}Rx_{T-1} + u'Qu + \beta \mathbb{E} J_T(Ax_{T-1} + Bu + C\xi_T)\} \quad (6.27)$$

Let $J_{T-1}(x)$ be the minimum value attained when the current state is x :

$$J_{T-1}(x) = \min_u \{x'Rx + u'Qu + \beta \mathbb{E} J_T(Ax + Bu + C\xi_T)\} \quad (6.28)$$

Stepping back to time $T - 2$, the function J_{T-1} now plays a role analogous to that played by the terminal loss $J_T(x) = x'R_fx$ for the decision maker at $T - 1$, in the sense that $J_{T-1}(x)$ summarizes the future loss associated with moving to state x . Once again, the controller chooses u to trade off current loss against future loss, solving

$$\min_u \{x'_{T-2}Rx_{T-2} + u'Qu + \beta \mathbb{E} J_{T-1}(Ax_{T-2} + Bu + C\xi_{T-1})\} \quad (6.29)$$

Letting

$$J_{T-2}(x) = \min_u \{x'Rx + u'Qu + \beta \mathbb{E} J_{T-1}(Ax + Bu + C\xi_{T-1})\} \quad (6.30)$$

the pattern for backwards induction is now clear. We calculate the **cost-to-go functions** $\{J_t\}$ recursively via

$$J_{t-1}(x) = \min_u \{x'Rx + u'Qu + \beta \mathbb{E} J_t(Ax + Bu + C\xi_t)\} \quad \text{and} \quad J_T(x) = x'R_fx \quad (6.31)$$

The function J_t represents the total cost-to-go from time t when the controller behaves optimally. It is analogous to the concept of a value function apart from the fact that we are minimizing. The equations given above correspond to the Bellman equation from dynamic programming theory specialized to the finite horizon LQ problem.

Lemma 6.2.1. *Each J_t has the form $J_t(x) = x'P_tx + d_t$ where P_t is a $n \times n$ matrix and d_t is a scalar that does not depend on x .*

Proof. This is true for $t = T$ with $P_T := R_f$ and $d_T = 0$. Suppose now that it is true at some $t \leq T$. We then have, for arbitrary $x \in \mathbb{R}^n$,

$$J_{t-1}(x) = \min_u \{x'Rx + u'Qu + \beta \mathbb{E} (Ax + Bu + C\xi_t)'P_t(Ax + Bu + C\xi_t) + \beta d_t\}$$

To obtain the minimizer, we use lemma 5.1.21 on page 153, which gives

$$u = -(Q + \beta B' P_t B)^{-1} \beta B' P_t A x \quad (6.32)$$

Plugging this back into our objective function and rearranging yields

$$J_{t-1}(x) = x' P_{t-1} x + d_{t-1} \quad (6.33)$$

where

$$P_{t-1} = R - \beta^2 A' P_t B (Q + \beta B' P_t B)^{-1} B' P_t A + \beta A' P_t A \quad (6.34)$$

and

$$d_{t-1} = \beta(d_t + \text{trace}(C' P_t C)) \quad (6.35)$$

□

EXERCISE 14. Verify the details of these calculations.

With lemma 6.2.1 we obtain an algorithm for computing the cost-to-go functions $\{J_t\}$ via the sequences $\{P_t\}$ and $\{d_t\}$, as shown in algorithm 8.

```

1  $t \leftarrow T$  ;
2  $P_t \leftarrow R_f$  ;
3  $d_t \leftarrow 0$  ;
4 while  $t > 0$  do
5    $P_{t-1} \leftarrow R - \beta^2 A' P_t B (Q + \beta B' P_t B)^{-1} B' P_t A + \beta A' P_t A$  ;
6    $d_{t-1} \leftarrow \beta(d_t + \text{trace}(C' P_t C))$  ;
7    $t \leftarrow t - 1$ 
8 end
9 return  $\{P_t, d_t\}_{t=0}^T$ 
```

Algorithm 8: Computing the cost-to-go functions in finite horizon LQ

Once we have the cost-to-go functions in hand, we can proceed forward from an initial condition x_0 . At each point in time t , we choose the minimizing control using the cost-to-go function, which, recalling (6.32), takes the form

$$u_t = -F_t x_t \quad \text{where} \quad F_t := (Q + \beta B' P_{t+1} B)^{-1} \beta B' P_{t+1} A \quad (6.36)$$

Then the state updates and we repeat. The resulting sequence of controls solves our finite horizon LQ problem.

Rephrasing this more concisely, the sequence u_0, \dots, u_{T-1} given by

$$u_t = -F_t x_t \quad \text{with} \quad x_{t+1} = (A - BF_t)x_t + C\xi_{t+1} \quad (6.37)$$

for $t = 0, \dots, T-1$ attains the minimum of (6.26) subject to our constraints.

6.2.2.2 A Life Cycle Problem

Early Keynesian models assumed that households have a constant marginal propensity to consume from current income, but data contradicts this. In response, a number of economists including Milton Friedman and Franco Modigliani built models based on a consumer's preference for an intertemporally smooth consumption stream (see, e.g., [Friedman \(1956\)](#) or [Modigliani and Brumberg \(1954\)](#)).

To illustrate the key ideas, consider the wealth dynamics given in (6.24), which we saw can be expressed as

$$x_{t+1} = Ax_t + Bu_t + C\xi_{t+1} \quad \text{with} \quad x_t = \begin{pmatrix} w_t \\ 1 \end{pmatrix} \quad \text{and} \quad u_t = c_t - \bar{c}$$

where

$$A := \begin{pmatrix} 1+r & -(1+r)\bar{c} + \mu \\ 0 & 1 \end{pmatrix}, \quad B := \begin{pmatrix} -(1+r) \\ 0 \end{pmatrix} \quad \text{and} \quad C := \begin{pmatrix} \sigma \\ 0 \end{pmatrix}$$

To convert this into a finite horizon problem we set the objective to

$$\mathbb{E} \left\{ \sum_{t=0}^{T-1} \beta^t (c_t - \bar{c})^2 + \beta^T q w_T^2 \right\} \quad (6.38)$$

Here q is a large positive constant, the role of which is to induce the consumer to target zero debt at the end of her life. (Without such a constraint, the optimal choice is to choose $c_t = \bar{c}$ in each period, letting assets adjust accordingly.)

To match with this state and control, the objective function (6.38) can be written in quadratic form by setting

$$Q := 1, \quad R := \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \text{and} \quad R_f := \begin{pmatrix} q & 0 \\ 0 & 0 \end{pmatrix}$$

Now that we have the matrices A , B , C , Q , R and R_f , we can either calculate the cost-to-go functions and optimal controls directly or use existing code such as that

found in the QuantEcon libraries. Details can be found in the corresponding Python and Julia lectures

- <https://lectures.quantecon.org/py/lqcontrol.html>
- <https://lectures.quantecon.org/jl/lqcontrol.html>

Figure 6.8 gives an illustration of the dynamics via simulation once the optimal controls have been obtained. Here we set $r = 0.05$, $\beta = 1/(1+r)$, $\bar{c} = 2$, $\mu = 1$, $\sigma = 0.25$, $T = 45$ and $q = 10^6$. The shocks $\{y_t\}$ were taken to be IID and standard normal.

The top panel shows the time path of consumption c_t and income y_t in the simulation. As anticipated by the discussion on consumption smoothing, the time path of consumption is much smoother than that for income. Note however, that it becomes more irregular towards the end of the agent's life when the zero final asset requirement impinges more on consumption choices.

The second panel in the figure shows that the time path of assets w_t is closely correlated with cumulative unanticipated income, where the latter is defined as $z_t := \sum_{j=0}^t \sigma w_j$. A key message is that unanticipated windfall gains are saved rather than consumed, while unanticipated negative shocks are met by reducing assets. (Again, this relationship breaks down towards the end of life due to the zero final asset requirement)

6.2.3 Infinite Horizon Problems

[roadmap]

6.2.3.1 Theory

Next we consider an infinite horizon problem with dynamics the same as above and objective function

$$\mathbb{E} \left\{ \sum_{t=0}^{\infty} \beta^t (x'_t R x_t + u'_t Q u_t) \right\} \quad (6.39)$$

Now optimal policies depend on time only if time itself is a component of the state vector x_t . In other words, the time dependent policy sequence $\{F_t\}$ in (6.37) is replaced by a fixed matrix F such that $u_t = -Fx_t$ for all t .

In the infinite horizon case the value function is also time invariant. For LQ problems, this translates to the statement that P_t and d_t discussed in lemma 6.2.1 are also constant. The stationary matrix P is the solution to the discrete time algebraic **Riccati equation**

$$P = R - (\beta B' P A)' (Q + \beta B' P B)^{-1} (\beta B' P A) + \beta A' P A \quad (6.40)$$

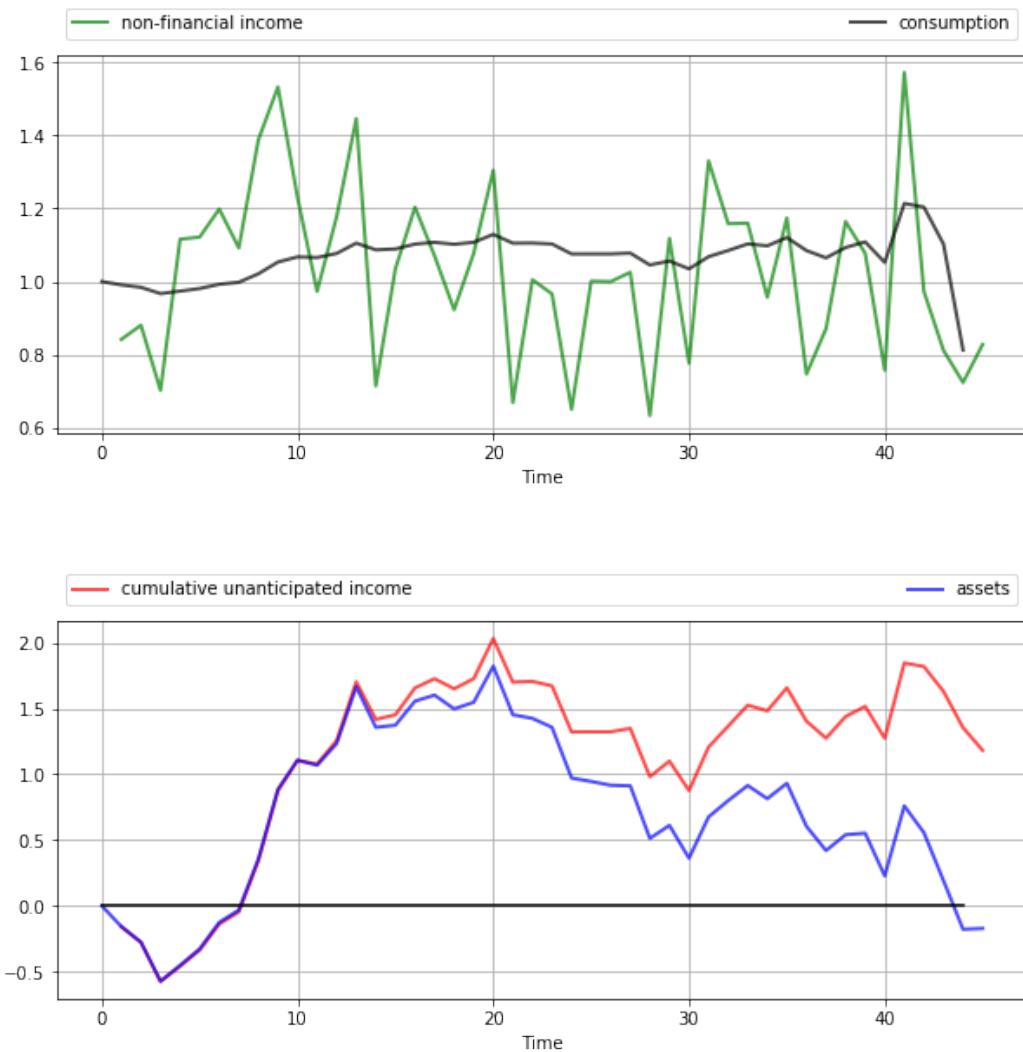


Figure 6.8: Consumption and assets in the life cycle problem

Equation (6.40), which can be understood as the stationary version of (6.34), is also called the **LQ Bellman equation**. The map that sends a given P into the right-hand side of (6.40) is called the **LQ Bellman operator**. The stationary optimal policy for this model is

$$u = -Fx \quad \text{where} \quad F = (Q + \beta B'PB)^{-1}(\beta B'PA) \quad (6.41)$$

Equation 6.41 follows from exactly the same reasoning that led us to the finite horizon version F_t in (6.36).

The sequence $\{d_t\}$ from (6.35) is replaced by the constant value

$$d := \text{trace}(C'PC) \frac{\beta}{1 - \beta} \quad (6.42)$$

The state evolves according to the time-homogeneous process

$$x_{t+1} = (A - BF)x_t + C\xi_{t+1} \quad (6.43)$$

Notice that the only significant computational difficulty here is solving the Riccati equation (6.40). Remaining objects such as F and d are easily calculated once we have P in hand. The following result addresses computation of P . In stating it we take N to be a matrix such that $N'N = R$. Also, let \mathcal{R} be the self-mapping on $\mathcal{M}_{n \times n}$ defined by

$$\mathcal{R}(P) := R - (\beta B'PA)'(Q + \beta B'PB)^{-1}(\beta B'PA) + \beta A'PA$$

and let \mathcal{M}_P be the set of positive definite matrices in $\mathcal{M}_{n \times n}$.

Theorem 6.2.2. *If (A, B) is controllable and (A, N) is observable, then*

- (i) $(\mathcal{M}_P, \mathcal{R})$ is globally stable
- (ii) If P^* is the unique fixed point of \mathcal{R} in \mathcal{M}_P , then

$$u = -F^*x \quad \text{where} \quad F^* := (Q + \beta B'P^*B)^{-1}(\beta B'P^*A)$$

is the unique optimal policy.

Proof. To be added. □

Remark 6.2.1. Linear quadratic control problems of the class discussed above have a special property called **certainty equivalence**. By this we mean that the optimal policy F is not affected by the parameters in C , which specify the shock process. This can be confirmed by inspecting (6.41). In other words, we can ignore uncertainty when solving for optimal behavior, and plug it back in when examining optimal state dynamics

6.2.4 Investment with Adjustment Costs

[rewrite]

Consider a monopolist facing stochastic inverse demand function

$$p_t = a_0 - a_1 q_t + z_t$$

where q_t is output, p_t is price and the demand shock z_t follows

$$z_{t+1} = \rho z_t + \sigma \eta_{t+1}, \quad \{\eta_t\} \stackrel{\text{IID}}{\sim} N(0, 1)$$

The monopolist chooses $\{q_t\}$ to maximize the expected present value of current and future profits

$$\mathbb{E} \sum_{t=0}^{\infty} \beta^t \pi_t \tag{6.44}$$

where current profits are given by

$$\pi_t := p_t q_t - c q_t - \gamma (q_{t+1} - q_t)^2$$

Here $\gamma (q_{t+1} - q_t)^2$ represents adjustment costs associated with changing production scale, parameterized by γ , and c is unit cost of current production.

One way to start thinking about the problem is to consider what would happen if $\gamma = 0$. Without adjustment costs there is no intertemporal trade-off. The monopolist should choose output to maximize current profit in each period, setting

$$\bar{q}_t := \frac{a_0 - c + z_t}{2a_1}$$

For other γ , we might expect that

- if γ is close to zero, then q_t will track the time path of \bar{q}_t relatively closely
- if γ is larger, then q_t will be smoother than \bar{q}_t , as the monopolist seeks to avoid adjustment costs

This intuition turns out to be correct. To show it we set up the Bellman equation, which trades off current profits and future value. In particular, the value function v^* , which measures firm value when the production path is chosen optimally, can be shown to satisfy

$$v^*(q, z) = \max_{q'} \{(p - c)q - \gamma (q' - q)^2 + \beta \mathbb{E}_z v^*(q', z')\} \tag{6.45}$$

Here $p = a_0 - a_1 q + z$, as determined by the inverse demand curve, and primes denote next period values. Later we'll confirm the validity of (6.45), use it to calculate the

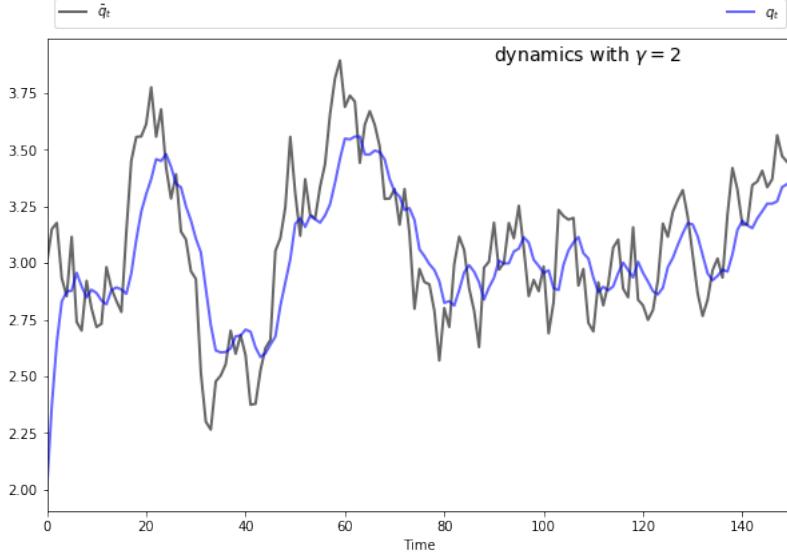


Figure 6.9: Output with adjustment costs when $\gamma = 2$

optimal quantity produced in each period given initial conditions and a demand shock sequence $\{z_t\}$, and confirm the conjectures given above. For now you can see the output of our calculations in figures 6.9–6.10, each of which shows a time path for both \bar{q}_t and optimal output q_t . In the second figure, γ is increased by a factor of 5 and the time series for output is significantly smoother.

Consider the monopolist with adjustment costs that we studied in §6.2.4, with inverse demand curve $p_t = a_0 - a_1 q_t + z_t$, where q_t is output, p_t is price and the demand shock z_t follows

$$z_{t+1} = \rho z_t + \sigma \eta_{t+1}, \quad \{\eta_t\} \stackrel{\text{IID}}{\sim} N(0, 1)$$

As stated previously in (6.44), the monopolist chooses $\{q_t\}$ to maximize

$$\mathbb{E} \sum_{t=0}^{\infty} \beta^t \pi_t \quad \text{where } \pi_t := p_t q_t - c q_t - \gamma (q_{t+1} - q_t)^2$$

Our challenge is to (a) convert this into a minimization problem, (b) write current payoff in the quadratic form (6.25) and (c) simultaneously ensure that the state and control obey linear dynamics (i.e., can be expressed in the form of (6.23)).

As a first step, let us modify the rewards of the firm to

$$\mathbb{E} \sum_{t=0}^{\infty} \beta^t (\pi_t - a_1 \bar{q}_t^2) \quad \text{where } \bar{q}_t := \frac{a_0 - c + z_t}{2a_1} \tag{6.46}$$

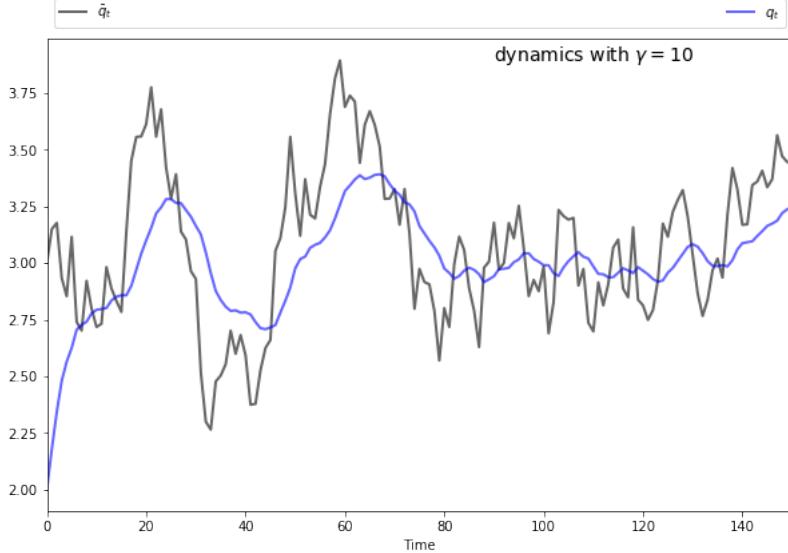


Figure 6.10: Output with adjustment costs when $\gamma = 10$

While such a modification alters lifetime value, the optimal production sequence $\{q_t\}$ will be identical, since

$$\mathbb{E} \sum_{t=0}^{\infty} \beta^t (\pi_t - a_1 \bar{q}_t^2) = \mathbb{E} \sum_{t=0}^{\infty} \beta^t \pi_t - a_1 \mathbb{E} \sum_{t=0}^{\infty} \beta^t \bar{q}_t^2$$

and the second term on the right does not depend on $\{q_t\}$. Moreover, with $u_t := q_{t+1} - q_t$, you will be able to confirm that

$$\pi_t - a_1 \bar{q}_t^2 = -a_1 (q_t - \bar{q}_t)^2 - \gamma u_t^2$$

which is already quadratic. Finally, switching to a minimization problem requires us to multiply by -1 , so the current loss is

$$\ell_t := a_1 (q_t - \bar{q}_t)^2 + \gamma u_t^2 \quad (6.47)$$

It remains to set up dynamics as linear in state and control, in order to fit with the canonical model (6.23). To this end we take $x_t = (\bar{q}_t, q_t, 1)'$ as our state. After setting $m_0 := (a_0 - c)/2a_1$ and $m_1 := 1/2a_1$, we can write $\bar{q}_t = m_0 + m_1 z_t$, and then, with some manipulation

$$\bar{q}_{t+1} = m_0(1 - \rho) + \rho \bar{q}_t + m_1 \sigma \xi_{t+1} \quad (6.48)$$

By our definition of u_t , the dynamics of q_t are $q_{t+1} = q_t + u_t$.

With these observations we can write the dynamic component of the LQ system as $x_{t+1} = Ax_t + Bu_t + C\xi_{t+1}$ when

$$A = \begin{pmatrix} \rho & 0 & m_0(1-\rho) \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} m_1\sigma \\ 0 \\ 0 \end{pmatrix}$$

EXERCISE 15. Complete the LQ specification of the adjustment cost model by expressing (6.47) in the form of (6.25) by suitable choice of R and Q .

6.2.5 Robust Control

[Follow QE lecture]

6.3 Chapter Notes

References, background, historical notes.

Chapter 7

Markov Dynamics

[Add roadmap](#)

7.1 Discrete Time, Discrete State

[Add roadmap](#)

7.1.1 Markov Kernels and Operators

Let's generalize our discussion of finite state Markov chains from Chapter 3 to countable states. We use X to represent the state space. Countability means that X is finite or in one-to-one correspondence with \mathbb{N} .

Notation is similar to §3.1.2. The set of distributions on X is denoted by $\mathcal{D}(X)$ and contains all $\varphi \in \mathbb{R}^X$ such that $\varphi \geq 0$ and $\sum_x \varphi(x) = 1$. (Throughout this chapter, \leq is the pointwise order when applied to functions.)

We will need a distance on $\mathcal{D}(X)$ when we discuss convergence. One option is to embed $\mathcal{D}(X)$ in $(\ell_1(X), d_1)$, as defined on page 130. Thus, we take

$$\|h\|_1 := \sum_x |h(x)| \quad \text{and} \quad \ell_1(X) := \{h \in \mathbb{R}^X : \|h\|_1 < \infty\}.$$

The **positive cone** of the space $\ell_1(X)$ is the set $\ell_1^+(X) := \{h \in \ell_1(X) : h \geq 0\}$. Note that $\mathcal{D}(X) = \{\varphi \in \ell_1^+(X) : \|\varphi\|_1 = 1\}$. In other words, $\mathcal{D}(X)$ is the intersection of the positive cone and unit sphere of $\ell_1(X)$. It turns out that this perspective is convenient for much of our analysis.

7.1.1.1 Markov Kernels

As in §3.1.2, a **Markov kernel** on X is a function $P: \mathsf{X} \times \mathsf{X} \rightarrow \mathbb{R}_+$ satisfying

$$\sum_y P(x, y) = 1 \text{ for all } x \in \mathsf{X}.$$

The definition of a Markov chain $\{X_t\}$ generated by P is the same as the finite case (see §3.1.2.3).

Example 7.1.1. Consider a single server queue, where X_t is the number of people currently in line, ξ_{t+1} is the number of arrivals during period t and η_{t+1} is the number of people who are served during t . We suppose that these two sequences are IID. The dynamics for the queue can be expressed as

$$X_{t+1} = X_t + \xi_{t+1} - \eta_{t+1} \mathbb{1}\{X_t > 0\}. \quad (7.1)$$

We cannot take a finite state space for this model because, without unpleasant assumptions, the size of the queue exceeds any finite bound with positive probability. Hence, for our state space we choose \mathbb{Z}_+ . The Markov kernel for the chain $\{X_t\}$ is

$$P(x, y) = \mathbb{P}\{x + \xi_{t+1} - \eta_{t+1} \mathbb{1}\{x > 0\} = y\} \quad (x, y \in \mathsf{X}).$$

A crucial question for queues such as this is stability. Will the size of the queue diverge to $+\infty$? We return to this topic below.

7.1.1.2 Linear Operators

Since our state space can now be infinite, we need to replace matrix operations with their immediate generalization: linear operators. Let's clarify some terminology before we get going.

Let V be either $\ell_1(\mathsf{X})$ or $b\mathsf{X}$, where $b\mathsf{X}$ is all $h \in \mathbb{R}^\mathsf{X}$ such that $\|h\|_\infty := \sup_x |h(x)| < \infty$. An operator L on V is called **linear** if for any $f, g \in \mathsf{V}$ and $\alpha, \beta \in \mathbb{R}$, we have $L(\alpha f + \beta g) = \alpha Lf + \beta Lg$. The operator L is called **positive** if $Lg \geq 0$ whenever $g \geq 0$ in the usual pointwise sense. Positive linear operators are a generalization of matrices with nonnegative elements.

For each f in $\ell_1(\mathsf{X})$ and $h \in b\mathsf{X}$, we adopt the *dual pair* notation

$$\langle f, h \rangle := \sum_x h(x)f(x). \quad (7.2)$$

If X is finite this is just the standard inner product. The notion of dual pairs is used widely and we will see it in more abstract settings later on.¹ A linear operator L^* mapping $b\mathsf{X}$ to itself is called **adjoint** to a linear operator L on $\ell_1(\mathsf{X})$ if

$$\langle h, Lg \rangle = \langle g, L^*h \rangle \quad (g \in \ell_1(\mathsf{X}), h \in b\mathsf{X}). \quad (7.3)$$

In the finite case, when these operators are matrices, (7.3) just says that L^* is the transpose of L .

EXERCISE 1. Show that each linear operator L on $\ell_1(\mathsf{X})$ has exactly one adjoint.

7.1.1.3 Kernels and Operators

When studying the finite case, we saw that, if the kernel P is viewed as a matrix and ψ is the marginal distribution of the current state, represented as a row vector, then ψP is the marginal distribution of the next period state. To extend these ideas to the case of countable X we rephrase in terms of operators:

Given a Markov kernel P on X , we introduce an operator P mapping $\mathcal{D}(\mathsf{X})$ into itself by

$$\psi \mapsto \psi P, \quad (\psi P)(y) = \sum_x p(x, y)\psi(x) \quad (y \in \mathsf{X}). \quad (7.4)$$

Analogous to (3.10) on p. 52, this operator takes a distribution ψ and updates it one period, to the distribution ψP for the next period state. In what follows, P is called the **Markov operator** associated with the Markov kernel P . We use the same symbol because they are in one-to-one correspondence, as clarified below.

We write ψP rather than $P\psi$ because the Markov operator generalizes the action of a matrix P sending row vector ψ into updated row vector ψP in the finite case. Hence it's natural that we replicate notation.²

EXERCISE 2. Show that ψP is in $\mathcal{D}(\mathsf{X})$ whenever $\psi \in \mathcal{D}(\mathsf{X})$.

EXERCISE 3. Let $P(x, y) = \delta_x(y)$. Show that the associated Markov operator is the identity map I on $\mathcal{D}(\mathsf{X})$.

¹Some background to the expression (7.2) is that the space $b\mathsf{X}$ is **dual** to $\ell_1(\mathsf{X})$, in the sense that, for each continuous linear functional F on $\ell_1(\mathsf{X})$, there exists a $h \in b\mathsf{X}$ such that $F(\psi) = \langle h, \psi \rangle := \sum_x h(x)\psi(x)$ for all $\psi \in \ell_1(\mathsf{X})$. This duality, which is sometimes referred to as the Riesz Representation Theorem, is the source of the notation adopted in (7.2).

²This is standard in the literature on Markov chains—see, for example, Meyn and Tweedie (2009).

Exercise 2 implies that the pair $(\mathcal{D}(\mathsf{X}), P)$ forms a dynamical system. Based on the definition of the operator in (7.4), we can rewrite the fundamental link between marginal distributions as $\psi_{t+1} = \psi_t P$, which is identical to (3.11). By the same logic that was used in the finite state case, the trajectory $\{\psi P^t\}$ of ψ under P is the sequence of marginal distributions for a Markov chain $\{X_t\}$ on X generated by the Markov kernel P and initial condition ψ .

If P is any Markov kernel, then the k -step kernel P^k is defined as before by (3.7) on p. 50. It gives the k -step transition probabilities. Fortunately, if P is the Markov operator corresponding to P , then its k -th power P^k , obtained by k compositions of P with itself, is the Markov operator corresponding to the k -step kernel. Put differently,

$$(\psi P^k)(y) = \sum_x P^k(x, y)\psi(x) \quad (\psi \in \mathcal{D}(\mathsf{X}), y \in \mathsf{X}). \quad (7.5)$$

To see that (7.5) holds, observe that it is true at $k = 1$, and, moreover, if it is true at k , then at $k + 1$ we have, by the induction hypothesis,

$$(\psi P^{k+1})(y) = ((\psi P^k)P)(y) = \sum_z P(z, y) \left[\sum_x P^k(x, z)\psi(x) \right]$$

Reordering sums and using the definition of P^{k+1} in (3.7), the right hand side can be written as

$$\sum_x \left[\sum_z P^k(x, z)P(z, y) \right] \psi(x) = \sum_x P^{k+1}(x, y)\psi(x).$$

This completes the proof of the claim.

EXERCISE 4. Consider $\psi \mapsto \psi P$ as an operator on the larger space $\ell_1(\mathsf{X})$. Prove that ψP is in $\ell_1(\mathsf{X})$ whenever $\psi \in \ell_1(\mathsf{X})$.

Let's consider another operator that arises here, again denoted by P . Given $h \in b\mathsf{X}$, we define as

$$(Ph)(x) = \sum_{y \in \mathsf{X}} h(y)p(x, y) \quad (h \in b\mathsf{X}, x \in \mathsf{X}). \quad (7.6)$$

The interpretation of this operator is

$$(Ph)(x) = \mathbb{E}[h(X_{t+1}) \mid X_t = x]. \quad (7.7)$$

when $\{X_t\}$ is generated by p . These expressions are generalizations of (3.12)–(3.13) from §3.1.3.2.

Although we use the same symbol P yet again, now the operator former acts to the

right, unlike the Markov operator, which acts to the left. We call $h \mapsto Ph$ the **adjoint Markov operator** (or just **adjoint**) associated with the kernel P .

EXERCISE 5. Show that the adjoint operator maps $b\mathbb{X}$ into itself.

Notice that we can recover a Markov kernel P from its Markov operator P , using

$$P(x, y) = (\delta_x P)(y) \quad (x, y \in \mathbb{X}). \quad (7.8)$$

This is just (7.4) with ψ specialized to δ_x .

In fact we could start with an operator P and use it to define a Markov kernel via (7.8), and hence a Markov chain. But what properties does the operator P need to have in order for the left hand side of (7.8) to be a Markov kernel? The next section answers this question. (It can be skimmed on first pass.)

7.1.1.4 Markov Operators

Let \mathbb{M} be the set of all positive linear operators on $\ell_1(\mathbb{X})$ that are invariant on $\mathcal{D}(\mathbb{X})$. Arguments are written to the left, so, for $P \in \mathbb{M}$, the last statement means that $\varphi P \in \mathcal{D}(\mathbb{X})$ when $\varphi \in \mathcal{D}(\mathbb{X})$. For reasons that will become clear, we call \mathbb{M} the class of **Markov operators** on \mathbb{X} ³

Let \mathbb{M}^* be the set of all adjoints of \mathbb{M} . We use P to refer to both the operator in \mathbb{M} and its adjoint, but the adjoint acts to the right on elements of $b\mathbb{X}$. The adjoint relationship is therefore written as

$$\langle h, gP \rangle = \langle g, Ph \rangle \quad (g \in \ell_1(\mathbb{X}), h \in b\mathbb{X}). \quad (7.9)$$

The sets \mathbb{M} and \mathbb{M}^* are in one-to-one correspondence via (7.9).

Theorem 7.1.1. *The set of all Markov operators P on $\ell_1(\mathbb{X})$ and the set of all Markov kernels on \mathbb{X} are uniquely paired by the relationship*

$$P(x, y) := (\delta_x P)(y) \quad (x, y \in \mathbb{X}). \quad (7.10)$$

Moreover, this pair satisfies

$$\sum_x \sum_y h(y) P(x, y) g(x) = \langle h, gP \rangle \quad (g \in \ell_1(\mathbb{X}), h \in b\mathbb{X}). \quad (7.11)$$

³In fact this terminology is standard in some fields of mathematics (see, e.g., §3.1 of Lasota (1994)).

Theorem 7.1.1 is awkward to read because we have anticipated its conclusion, representing the kernel and both operators with the same symbol. Nonetheless, it is valuable because it means we can freely introduce a Markov model through either a kernel, a Markov operator, or its adjoint.

The proof is a series of exercises. Given the uniqueness of adjoints, we sometimes call elements of \mathbb{M}^* **Markov operators** as well, although we will call them adjoints if it's necessary to distinguish. The corresponding Markov kernel is always identified via (7.11).

Two special cases of (7.11) we use frequently, obtained by using the duality in (7.9) and setting $h = \delta_y$ and then $g = \delta_x$, are

$$(gP)(y) = \sum_x P(x, y)g(x) \quad \text{and} \quad (Ph)(x) = \sum_y P(x, y)h(y)$$

These equations confirm that our terminology in this section is consistent with our previous usage. For example, the second equation is the same as (3.12).

EXERCISE 6. Show that \mathbb{M}^* is equal to the set of all positive linear operators on $b\mathbb{X}$ satisfying $P\mathbb{1} = \mathbb{1}$.

EXERCISE 7. Let P be a positive linear operator on $\ell_1(\mathbb{X})$. Show that P is a Markov operator if and only if P is norm preserving on the positive cone, which is to say that $g \geq 0 \implies \|gP\|_1 = \|g\|_1$

EXERCISE 8. Show that every $P \in \mathbb{M}$ is nonexpansive: for any $P \in \mathbb{M}$ and any $f, g \in \mathcal{D}(\mathbb{X})$, we have

$$\|fP - gP\|_1 \leq \|f - g\|_1. \tag{7.12}$$

(One immediate implication is that P is continuous on $(\ell_1(\mathbb{X}), d_1)$).

EXERCISE 9. Let \mathbb{K} be the set of all Markov kernels on \mathbb{X} and consider the mapping $\tau: \mathbb{K} \rightarrow \mathbb{M}$ that takes a kernel $P \in \mathbb{K}$ and maps it into the operator P on $\ell_1(\mathbb{X})$ defined by

$$(gP)(y) = \sum_x P(x, y)g(x) \quad (g \in \ell_1(\mathbb{X}), y \in \mathbb{X}). \tag{7.13}$$

(Operator and kernel are distinguished by the objects they act on.) Prove that (i) τ is a well defined map from \mathbb{K} to \mathbb{M} , (ii) τ is onto, (iii) τ is one-to-one, and (iv) if P is in \mathbb{M} , then $\tau^{-1}P$ is the kernel $P \in \mathbb{K}$ defined by $P(x, y) = (\delta_x P)(y)$.

EXERCISE 10. Let P be a Markov kernel and let P be the Markov operator defined by (7.13). Show that (7.11) holds.

The answer to this exercise needs to be tidied and checked.

This needs to be tidied and checked.

One helpful application of the adjoint relationship (7.9) is in showing that the interpretation in (7.7) extends to powers of P . In particular,

$$(P^k h)(x) = \mathbb{E}[h(X_{t+k}) | X_t = x]. \quad (7.14)$$

To see this, note that, from (7.9),

$$(P^k h)(x) = \langle P^k h, \delta_x \rangle = \langle h, \delta_x P^k \rangle = \sum_y h(y) (\delta_x P^k)(y) = \sum_y P^k(x, y) h(y),$$

where the last step is by (7.5). Thus, we are summing outcomes evaluated by h , weighted by k -step transition probabilities, conditional on starting at x . This matches the right hand side of (7.14).

7.1.2 Stationarity

As we saw in §7.1.1, a Markov model on X can be described by a Markov operator P , which is a linear operator on $\ell_1(\mathsf{X})$ that carries $\mathcal{D}(\mathsf{X})$ into itself (see, e.g., Exercise 7.). From there we can recover a Markov kernel P and an adjoint via (7.10) and (7.9).

We can regard $(\mathcal{D}(\mathsf{X}), P)$ as a semidynamical system whose trajectories $\{\psi P^t\}$ trace out sequences of marginal distributions for (ψ, p) -chains.

Complete roadmap.

7.1.2.1 Stationary Distributions

Let P be a Markov operator on X . A distribution $\psi^* \in \mathcal{D}(\mathsf{X})$ is called **stationary** for P if it is a fixed point; that is, if $\psi^* = \psi^* P$. The distribution ψ^* is called stationary for a Markov kernel P if it is stationary for its Markov operator. In other words,

$$\psi^*(y) = \sum_{x \in \mathsf{X}} P(x, y) \psi^*(x) \quad \text{for all } y \in \mathsf{X}.$$

Since P updates distributions, the implication of stationarity of ψ^* for a Markov chain $\{X_t\}$ is that if X_t has distribution ψ^* then so does X_{t+1} , and hence X_{t+j} for all j .

Example 7.1.2. Suppose that workers are hired at rate α and fired at rate β . Thus, they transition between unemployment and employment according to

$$P_w = \begin{pmatrix} 1 - \alpha & \alpha \\ \beta & 1 - \beta \end{pmatrix} \quad (7.15)$$

Assuming that both parameters are positive, show by direct calculation that $\psi = (\beta/(\alpha + \beta), \alpha/(\alpha + \beta))$ is stationary for P_w .

Example 7.1.3. The black dot in Figures 3.7–3.8 is a stationary distributions for the Markov matrix P_a displayed in (3.2), page 48. We computed it using methods described in §7.1.2.5.

In Figures 3.7–3.8, trajectories converge towards the stationary distribution, suggesting a form of stability. We investigate this stability further in §7.1.4, but note for now that not all stationary distributions have this property. The next example illustrates.

EXERCISE 11. Let X be any countable set and let $P(x, y) = \mathbb{1}\{x = y\}$. Evidently P is a Markov kernel on X . Describe the corresponding Markov operator. Show that every distribution in $\mathcal{D}(X)$ is stationary for this operator.

By definition, ψ^* is stationary for a Markov operator P if and only if ψ^* is a steady state of the dynamical system $(\mathcal{D}(X), P)$. Another way to characterize stationarity is

$$\langle Ph, \psi^* \rangle = \langle h, \psi^* \rangle \quad \text{for all } h \in bX. \quad (7.16)$$

EXERCISE 12. Using (7.9), show that (7.16) is necessary and sufficient for stationarity of ψ^* .

EXERCISE 13. Provide an alternative proof that stationarity implies (7.16) using the law of iterated expectations.

7.1.2.2 Process Stationarity

Connect with strict sense stationarity.

7.1.2.3 Existence

In the finite setting we have the following powerful theorem.

Theorem 7.1.2 (Krylov–Bogolyubov I). *If X is finite then every Markov operator on X has at least one stationary distribution.*

Proof. Let P be a Markov operator and let X be finite. Then $\ell_1(\mathsf{X}) = \mathbb{R}^\mathsf{X}$. (Why?) Suppose in particular that X has d elements, so we can identify functions in \mathbb{R}^X with vectors in \mathbb{R}^d and $\mathcal{D}(\mathsf{X})$ with the unit simplex in \mathbb{R}^d . The set $\mathcal{D}(\mathsf{X})$ is a closed, bounded and convex subset of \mathbb{R}^d that P maps into itself. As a linear matrix operation, the map $\psi \mapsto \psi P$ is continuous. Existence of a fixed point now follows from Brouwer's fixed point theorem (p. 138). \square

When X is infinite, not every Markov kernel has a stationary distribution.

EXERCISE 14. Show that, if $\mathsf{X} = \mathbb{Z}$ and $P(n, m) = \mathbb{1}\{m = n + 1\}$ for all $n, m \in \mathbb{Z}$, then P has no stationary distribution.

Thus, when X is not finite, we need to impose restrictions on p in order to ensure existence of a stationary distribution. These restrictions must rule out the divergence of probability mass observed in Exercise 14.

For a Markov operator P , let \mathcal{L} be defined by

$$\mathcal{L} := P - I, \quad (\mathcal{L}h)(x) = \sum_y P(x, y)h(y) - h(x). \quad (7.17)$$

Here I is the identity map and h is either in $b\mathsf{X}$ or a nonnegative function on X . The operator \mathcal{L} is called the **generator** of P . We say that P exhibits **constant drift** to $C \subset \mathsf{X}$ if there exists a function $v: \mathsf{X} \rightarrow \mathbb{R}_+$, a finite constant d and a $\lambda > 0$ such that

$$\mathcal{L}v \leq d\mathbb{1}_C - \lambda. \quad (7.18)$$

The idea is that $Pv(x) \leq v(x) - \lambda$ when $x \notin C$, so that the “value” of the state under v decreases, at least in expectation. This cannot happen forever because $v \geq 0$, so the state must return to C . This is sufficient to prevent divergence of probability mass, yielding

Theorem 7.1.3 (Krylov–Bogolyubov II). *If a Markov operator P exhibits constant drift to a finite set, then P has at least one stationary distribution.*

We will prove a more general version of this result in §13.2.3.

EXERCISE 15. Show that Theorem 7.1.2 is a special case of Theorem 7.1.3.

7.1.2.4 Application: A Single Server Queue

Recall Example 7.1.1. Intuitively, stability will hold as long as the average number of departures (through service) exceeds the average number of arrivals. For now let's test whether or not a stationary distribution exists. Let $a := \mathbb{E}\xi_t$ and $b := \mathbb{E}\eta_t$. Observe that, on the state space $\mathbf{X} = \mathbb{Z}_+$ and with $v(x) = x$, we have

$$(\mathcal{L}v)(x) = \mathbb{E}[X_{t+1} - X_t | X_t = x] = a - b\mathbb{1}\{x > 0\} = b\mathbb{1}\{x = 0\} - (b - a)$$

Matching our intuition, we see that $a < b$ implies condition (7.18) is satisfied with $C = \{0\}$, $\lambda = b - a$ and $d = b$.

7.1.2.5 Computation

Let's consider how to compute stationary distributions, focusing for now on the case where \mathbf{X} is finite. The characterizing equation $\psi^*P = \psi^*$ is, in this case, a finite set of linear equations we might hope to solve directly for the unknown vector ψ^* . There are, however, some problems with this idea. For example, $\psi^* = 0$ is a trivial solution that we need to exclude.

To force our solution to be in $\mathcal{D}(\mathbf{X})$ we can proceed as follows: Suppose our state has n elements and note that row vector $\psi \in \mathcal{D}(\mathbf{X})$ is stationary if and only if $\psi(I - P) = 0$, where I is the $n \times n$ identity matrix. Let $\mathbb{1}_n$ be the $1 \times n$ row vector $(1, \dots, 1)$. Let $\mathbb{1}_{n \times n}$ be the $n \times n$ matrix of ones.

EXERCISE 16. Show that an element ψ of $\mathcal{D}(\mathbf{X})$, interpreted as a row vector, is stationary for P if and only if

$$\mathbb{1}_n = \psi(I - P + \mathbb{1}_{n \times n}) \tag{7.19}$$

Taking the transpose of (7.19) we get $(I - P + \mathbb{1}_{n \times n})'\psi' = \mathbb{1}'_n$. This is a linear system of the form $Ax = b$, which can be solved for $x = A^{-1}b$ if A is invertible. There is no guarantee of this however, as it depends on the properties of P .

EXERCISE 17. Give a counterexample to the claim that $(I - P + \mathbb{1}_{n \times n})$ is always nonsingular when P is a Markov matrix.

The procedure outlined above works in settings where P has a unique stationary distribution. There are more sophisticated algorithms for computing all of the stationary distributions of a Markov chain, regardless of whether there are one or many. (The QuantEcon libraries have efficient implementations of this type.)

7.1.3 Ergodicity

Add a roadmap. Discuss significance for simulation, econometrics, stats, interpretation of stationarity distributions.

7.1.3.1 Harmonic Functions

Let P be a Markov operator. A function $h \in \mathbb{R}^X$ is called **harmonic** for P if $Ph = h$ on X . In other words, harmonic functions are fixed points of the expectation operator $h \mapsto Ph$. Harmonic functions generate *martingales*, since, if h is harmonic and $\{X_t\}$ is a Markov chain generated by P , then

$$\mathbb{E}[h(X_{t+1}) | X_t] = (Ph)(X_t) = h(X_t).$$

We discuss martingales more in later chapters. For now, our main interest in harmonic functions is their relationship with ergodicity, to be explained in §7.1.3.2.

EXERCISE 18. Let P be a Markov operator on X . Show that every constant function in bX is harmonic for P .

Example 7.1.4. Let $\{X_t\}$ be a random walk on \mathbb{Z} , with $X_{t+1} = X_t + \xi_{t+1}$ for all t . Here $\{\xi_t\}$ is IID on \mathbb{Z} with $\mathbb{E}\xi_t = 0$. If $h(x) = x$, then h is harmonic, since

$$\mathbb{E}[h(X_{t+1}) | X_t] = \mathbb{E}[X_{t+1} | X_t] = \mathbb{E}[X_t + \xi_{t+1}] = X_t = h(X_t).$$

There is a connection between harmonic functions and absorbing sets. To state it, recalled that, given a Markov kernel P on X , a set A is called **absorbing** for P if

$$\sum_{y \in A} P(x, y) = 1 \quad \text{for all } x \in A. \tag{7.20}$$

One example is $\{\text{poor}\}$ in the kernel displayed in Figure 7.1. Likewise, $\{\text{middle, rich}\}$ is absorbing.

Example 7.1.5. If A and A^c are both absorbing for P , then $\mathbb{1}_A$ is harmonic, since

$$(P\mathbb{1}_A)(x) = \sum_y P(x, y)\mathbb{1}_A(y) = \sum_{y \in A} P(x, y) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \in A^c \end{cases}$$

In other words, $P\mathbb{1}_A = \mathbb{1}_A$.

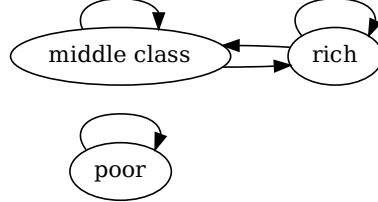


Figure 7.1: A poverty trap

7.1.3.2 Definition and Implications

Let P be a Markov operator on X . We know that every constant function in $b\mathsf{X}$ is harmonic for P . We call P **ergodic** if this is the only possibility. In other words, P is ergodic if the only harmonic functions in $b\mathsf{X}$ are the constant functions.

Example 7.1.6. Let $P(x, y) = \varphi(y)$ for all x, y in X , where φ is some fixed distribution in $\mathcal{D}(\mathsf{X})$. This kernel generates an IID chain, since the next period distribution has no dependence on the current state. This kernel is ergodic, since, if $h \in b\mathsf{X}$ is harmonic, we have

$$h(x) = (Ph)(x) = \sum_y p(x, y)h(y) = \sum_y \varphi(y)h(y)$$

for all $x \in \mathsf{X}$. Hence, h is constant.

Example 7.1.7. Consider again the worker model with hiring and firing in Example 7.1.2. The statement $P_w h = h$ becomes, in this simple case,

$$\begin{pmatrix} 1 - \alpha & \alpha \\ \beta & 1 - \beta \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}$$

The first row is $(1 - \alpha)h_1 + \alpha h_2 = h_1$, or $\alpha h_1 = \alpha h_2$. Thus, P_w is ergodic whenever $\alpha > 0$. By a similar argument, the same statement is true for β .

Example 7.1.8. Let P be any Markov kernel. It is immediate from Example 7.1.5 that if X can be partitioned into two nontrivial absorbing sets, then P is not ergodic. Hence, the poverty trap model in Figure 7.1 is not ergodic. Similarly, if $\alpha = \beta = 0$ in the matrix P_w discussed in Example 7.1.7, then $P_w = I$ and each state is a disjoint absorbing set. Ergodicity fails.

Ergodicity is important for us due to the following deep and fundamental theorem.

Theorem 7.1.4. *Let P be a Markov operator on \mathbb{X} . If P has stationary distribution ψ^* , then the following statements are equivalent:*

(i) P is ergodic.

(ii) For any initial condition $\psi \in \mathcal{D}(\mathbb{X})$ and any (ψ, P) -chain $\{X_t\}$, we have

$$\mathbb{P} \left\{ \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{t=0}^{n-1} h(X_t) = \sum_{x \in \mathbb{X}} h(x) \psi^*(x) \right\} = 1 \quad (7.21)$$

whenever $\sum_x |h(x)| \psi^*(x) < \infty$.

If either and hence both of these statements are true, then ψ^* is the stationary distribution of P in $\mathcal{D}(\mathbb{X})$.

Example 7.1.9. Recall the IID case from Example 7.1.6. We showed that this kernel P is ergodic. By Theorem 7.1.4, the convergence in (7.21) holds with $\psi^* = \varphi$. This is consistent with the Strong Law of Large Numbers for IID sequences.

The uniqueness claim in Theorem 7.1.4 is obvious: if ψ_1 and ψ_2 are two distinct stationary distributions, then they differ at some point y . With $h = \delta_y$ in (7.21), this says that a sequence of random variables converges with probability one to two distinct points. But this is clearly impossible.

A proof of the equivalence claim in Theorem 7.1.4 can be deduced from Proposition 17.1.4, and Theorems 17.1.5 and 17.1.7 of Meyn and Tweedie (2009).⁴ We omit the proof but try to provide a sense of the connection between (i) and (ii) through the following examples.

Example 7.1.10. Let $\mathbb{X} = \{1, 2\}$ and $P = I$, the identity matrix. Markov chains generated by P are constant. Since every $h \in \mathbb{R}^2$ satisfies $Ph = h$, we see that P is not ergodic. This means that the Law of Large Numbers result in (7.21) fails. But this is exactly what we would expect, since a constant chain $\{X_t\}$ implies $\frac{1}{n} \sum_{t=0}^{n-1} X_t = X_0$ for all n . In particular, if X_0 is drawn from a nondegenerate distribution, then the sample mean does not converge to any constant value.

Example 7.1.11. Consider again poverty trap model in Figure 7.1. Say that $h(x)$ is earnings in state x , and that $h(\text{poor}) = 1$, $h(\text{middle}) = 2$ and $h(\text{rich}) = 3$. Households that start with $X_0 = \text{poor}$ will always be poor, so $\frac{1}{n} \sum_{t=0}^{n-1} h(X_t) = 1$ for all n . Households that start with X_0 in <{\text{middle}, \text{rich}} remain in this absorbing set forever, so $\frac{1}{n} \sum_{t=0}^{n-1} h(X_t) \geq 2$ for all n . In particular, the limit of the sum depends on the

⁴A generalization is given in Theorem 3.1 of Kamihigashi and Stachurski (2016), where an extended notion of ergodicity is provided.

initial condition. This violates part (ii) of Theorem 7.1.4, which states that the limit is independent of the distribution of X_0 .

Intuitively, the Large of Large Numbers, which works for IID sequences, can still function with some degree of dependence between observations. However, in Examples 7.1.10–7.1.11, dependence lasts forever, a phenomenon that is sometimes referred to as **path dependence**. The Large of Large Numbers cannot survive this high degree of dependence and ergodicity fails.

Conversely, the Large of Large Numbers can extend to dependent sequences, as long as that dependence dies out in the limit. For example, consider again Figure 3.2 on page 46. This graph is strongly connected and every state is accessible from every other. If we imagine a large population of households simulated independently following these dynamics, initially rich households will be poor at some future point in time, while those that are now poor will have an opportunity to be rich. The different households “mix” across the state space over time. Since initial conditions do not determine long run outcomes, we can hope that ergodicity holds. In fact we will see in §7.1.3.3 that this is so, because the graph is strongly connected.

Ergodicity has many valuable implications. For example, when it holds, ergodicity provides a new *interpretation* for the stationary distribution: Using (7.21) with $h(x) = \mathbb{1}\{x = y\}$, we have

$$\frac{1}{n} \sum_{t=1}^n \mathbb{1}\{X_t = y\} \rightarrow \sum_{x \in \mathcal{X}} \mathbb{1}\{x = y\} \psi^*(x) = \psi^*(y). \quad (7.22)$$

Turning this around, we find that

$$\psi^*(y) \approx \text{the fraction of time that } \{X_t\} \text{ spends in state } y.$$

Another reason that ergodicity is important is that Theorem 7.1.4 explicitly tells us that the convergence in (7.21) occurs for any initial condition. This gives us a means of computing expectations under the stationary distribution (or probabilities under the stationary distribution via (7.22)) without knowing it a priori, via simulation. Simulation is increasingly important in quantitative modeling because it can effectively handle problems that are both nonlinear and high dimensional.

A key question now is: when does ergodicity hold? We will see that it holds under straightforward “mixing” conditions.

7.1.3.3 Irreducibility

In §7.1.3.2 we discussed how initial conditions can persist and hence inhibit ergodicity. One way to rule this out is to require irreducibility. To state the definition of this property, let p be a Markov kernel on X and let x and y be elements of X . We say that y is **accessible** from x if either $x = y$ or there exists a $k \in \mathbb{N}$ such that $p^k(x, y) > 0$. In addition, p is called **irreducible** if, for any pair of states x, y , state y is accessible from x .

And equivalent way to state this is that the digraph induced by p is strongly connected (recall the definitions in §3.1.2.2).

Examples include

- the kernel represented in the digraph in Figure 3.2 on page 46.
- The Markov kernel Π_Q estimated by Quah, since the digraph in figure 3.5 is clearly strongly connected.

On the other hand, the digraph in figure 3.3 is not strongly connected, since neither rich nor middle is accessible from poor, and hence the corresponding Markov kernel is not irreducible. The Markov kernel generating figure 7.1 also fails to be irreducible.

Irreducibility provides enough mixing to induce ergodicity when a stationary distribution exists. The next result makes this clear.

Proposition 7.1.5. *If P is irreducible and admits a stationary distribution, then P is ergodic.*

The general argument can be found in Meyn and Tweedie (2009).⁵ The proof is simple when X is finite, for in this case, given a harmonic function $h \in b\mathsf{X}$, there exists a maximizer x of h on X . Let $h(x) = m$ and, seeking a contradiction, suppose there exists a y in X with $h(y) < m$. Use irreducibility to choose a $k \in \mathbb{N}$ such that $p^k(x, y) > 0$. Since h is harmonic, we have $h = P^k h$, and hence

$$m = h(x) = \sum_z p^k(x, z)h(z) = p^k(x, y)h(y) + \sum_{z \neq y} p^k(x, z)h(z) < m.$$

This contradiction shows that h is constant. We can now conclude that p is ergodic.

EXERCISE 19. Prove: If $\psi \in \mathcal{D}(\mathsf{X})$ is stationary for some irreducible P , then ψ is supported on all of X .

⁵Every Markov chain with countable state space and at least one stationary distribution is positive Harris. See, in particular, Proposition 8.1.4 of Meyn and Tweedie (2009). Every positive Harris Markov chain is ergodic, by Theorem 17.1.7 of the same reference.

7.1.4 Global Stability

Add roadmap.

7.1.4.1 Definitions and Intuition

A Markov operator P is called **globally stable** if the dynamical system $(\mathcal{D}(\mathbf{X}), P)$ is globally stable (see page 5.3.2.2 for the definition). A Markov kernel p is called globally stable if its Markov operator has this property.

Intuitively, to obtain global stability, we need the effect of initial conditions to die out. Following our discussion of ergodicity, one might then hope that irreducibility is enough for global stability. But this is not true, even in the finite state case.

To see why, suppose for example that $\mathbf{X} = \{0, 1\}$ and consider

$$P_d = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad (7.23)$$

It is clear that P_d is irreducible, and a stationary distribution exists by finiteness of \mathbf{X} . Hence P_d is ergodic. But P_d is not globally stable. Indeed, if $\{X_t\}$ is generated by P_d and starts at δ_0 , then $\{X_t\}$ will visit state 1 on odd dates and state 0 on even dates. That is, $P_d^t \delta_0 = \delta_{t \bmod 2}$. In particular, $P_d^t \delta_0$ does not converge.

The issue with P_d is that, even though the chain traverses the whole state space, the distribution of X_t will affect that of X_{t+j} for all j due to periodicity. This causes stability to fail.

One strict but nonetheless useful set of conditions for global stability is as follows:

Theorem 7.1.6. *If \mathbf{X} is finite and p is everywhere positive, then p is globally stable.*

Example 7.1.12. The matrix P_B defined on page 49 is everywhere positive, and hence globally stable.

We will prove Theorem 7.1.6 using a contraction argument that has the added advantage of establishing existence of a stationary distribution.⁶ Later, we will see how everywhere positivity can be relaxed into more natural conditions.

The proof will be build on several lemmas and exercises. The first one is

⁶Of course, in the present finite state environment we already know that a stationary distribution exists, from Theorem 7.1.2. But, while the proof of Brouwer's fixed point theorem is long and nontrivial, the contraction argument used here is relatively straightforward and self-contained.

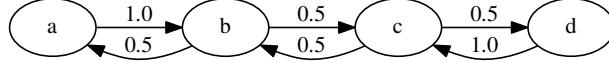


Figure 7.2: Periodicity

EXERCISE 20. Show that, if $\varphi, \psi \in \mathcal{D}(\mathsf{X})$ and $\varphi \neq \psi$, then we can find a pair $x, x' \in \mathsf{X}$ such that $\varphi(x) > \psi(x)$ and $\varphi(x') < \psi(x')$.

EXERCISE 21. Let p be a Markov kernel on X . Show that, if $p(x, y) > 0$ for all x, y in X , then P is strictly contracting on $\mathcal{D}(\mathsf{X})$, in the sense that

$$\|\varphi P - \psi P\|_1 < \|\varphi - \psi\|_1 \quad \text{whenever } \varphi \neq \psi \quad (7.24)$$

Hint: Use Exercise 20 and the strict triangle inequality on page 7.

Proof of Theorem 7.1.6. In view of Exercise 21, the dynamical system $(\mathcal{D}(\mathsf{X}), P^k)$ is strictly contracting. Moreover, in the present setting, $\mathcal{D}(\mathsf{X})$ is compact (see the proof of Theorem 7.1.2). Global stability now follows from Proposition 5.1.11 on page 140. \square

7.1.4.2 Aperiodicity

The everywhere positivity condition used in Theorem 7.1.6 is a strict way of ensuring irreducibility and ruling out the kind of periodicity exhibited by P_d in (7.23). Let's now look at a weaker condition. To state it, let p be a Markov kernel on X . A state $x \in \mathsf{X}$ is called **aperiodic** under p if there exists an $n \in \mathbb{N}$ such that, for all $k \geq n$, we have $p^k(x, x) > 0$. A Markov kernel p on X is called **aperiodic** if every state in X is aperiodic under p . The next lemma is obvious but useful nonetheless.

Lemma 7.1.7. *A Markov kernel p is aperiodic if, for each $x \in \mathsf{X}$, we have $p(x, x) > 0$.*

For example, Quah's Markov kernel P_Q is aperiodic, since, upon starting at any state x , we return to x in the next period with positive probability. Conversely, the Markov kernel represented by the digraph in figure 7.2 is *periodic*, since a chain that starts at some given state x returns with positive probability only at even dates.

The following lemma shows how mixing is strengthened when aperiodicity is combined with irreducibility.

Lemma 7.1.8. *If p is aperiodic and irreducible, then, for all $x, y \in \mathsf{X}$, there exists an $n \in \mathbb{N}$ such that $p^k(x, y) > 0$ whenever $k \geq n$.*

Proof. Let X and p have the stated properties. Pick any x and y in X . Since p is irreducible, there exists a $j \in \mathbb{N}$ such that $p^j(x, y) > 0$. Since P is aperiodic, we can find an $m \in \mathbb{N}$ such that $\ell \geq m$ implies $p^\ell(y, y) > 0$. Picking $\ell \geq m$ and applying the Chapman–Kolmogorov equation, we have

$$p^{j+\ell}(x, y) = \sum_{z \in \mathsf{X}} p^j(x, z)p^\ell(z, y) \geq p^j(x, y)p^\ell(y, y) > 0$$

Thus, with $n = j + m$, we have $p^k(x, y) > 0$ whenever $k \geq n$. \square

7.1.4.3 Aperiodicity and Stability: The Finite Case

We are now ready to state and prove a famous stability result for finite state Markov chains:

Theorem 7.1.9. *If X is finite and p is a Markov kernel on X , the following statements are equivalent:*

- (i) p is aperiodic and irreducible on X
- (ii) There exists a $k \in \mathbb{N}$ such that $p^k(x, y) > 0$ for all x, y in X

If one and hence both of these conditions hold, then p is globally stable.

The key to the stability part of Theorem 7.1.9 is the following observation.

EXERCISE 22. Prove that, if P is any Markov operator and P^k is globally stable for some $k \in \mathbb{N}$, then P is globally stable.

We now complete the proof of Theorem 7.1.9.

Proof of Theorem 7.1.9. First, let us show that (i) implies (ii). By lemma 7.1.8, given any $x, y \in \mathsf{X}$, there exists an $n(x, y) \in \mathbb{N}$ such that $p^i(x, y) > 0$ whenever $i \geq n(x, y)$. Setting $k := \max n(x, y)$ over all (x, y) pairs yields strict positivity of p^k on $\mathsf{X} \times \mathsf{X}$.

To show the reverse implication, suppose that, for some $k \in \mathbb{N}$ we have $p^k > 0$. Note that $p^{k+j} > 0$ for all $j \geq 0$, since, for any given x, y , the Chapman–Kolmogorov relation implies

$$p^{k+j}(x, y) = \sum_{z \in \mathsf{X}} p^j(x, z)p^k(z, y) \geq \min_{s \in \mathsf{X}} p^k(s, y) \sum_{z \in \mathsf{X}} p^j(x, z) = \min_{s \in \mathsf{X}} p^k(s, y)$$

It is now clear that p is both irreducible and aperiodic. Irreducibility is immediate from strict positivity of p^k . Aperiodicity follows from $p^{k+j}(x, x)$ for all $j \geq 0$.

Regarding global stability, suppose that condition (ii) holds. By Exercise 22, it suffices to show that P^k is globally stable. This is true by positivity of p^k and Theorem 7.1.6. \square

Example 7.1.13. Consider again Quah's matrix P_Q , defined on page 48. While not everywhere positive, it is globally stable, being both aperiodic and irreducible.

7.1.4.4 Necessary and Sufficient Conditions

Theorem 7.1.9 is important and appears in many texts on Markov chains. It is, however, both possible and useful to provide conditions for stability that are weaker than those used in Theorem 7.1.9. The main result of this section provides conditions that are not just sufficient but also necessary when the state space is finite.

Theorem 7.1.10. *If X is finite and p is a Markov kernel on X , then the following statements are equivalent:*

- (i) *For any $x, x' \in X$, there exists a $k \in \mathbb{N}$ and a $y \in X$ such that both $p^k(x, y)$ and $P^k(x', y)$ are strictly positive.*
- (ii) *The kernel p is globally stable.*

The most interesting implication here is (i) \implies (ii). The argument is that, when (i) holds, initial conditions don't matter, since two independent chains starting in different locations will have the opportunity to meet up. From this we can show that the marginal distributions of these two chains become increasingly similar. This is convergence. Existence is already given by Theorem 7.1.2 and uniqueness requires no separate proof (see Remark 5.3.1 on page 166).

Example 7.1.14. Let $X = \{0, 1\}$ and let $p(i, 1) = 1$ for $i = 0, 1$. This chain goes immediately to state 1 and stays there. It is not irreducible, but it is certainly globally stable. Condition (i) of Theorem 7.1.10 holds with $k = y = 1$.

The proof we provide for Theorem 7.1.10 using coupling methods, which are an important topic in their own right. Coupling arguments will be used again later in the text. However, in order to maintain flow, the proof is shifted to the appendix (see §19.1).

7.1.5 Drift and Minorization

Theorem 7.1.9 is valuable when the state space is finite, but many applications of interest have an infinite state space. Here even the existence of a stationary distribution is not certain, so irreducibility and aperiodicity are clearly not sufficient for global stability. Most importantly, we need conditions that ensure probability mass does not diverge away from the center (i.e., from some small set on which the kernel exhibits a sufficient degree of mixing). These ideas are formalized below.

7.1.5.1 Minorization

A Markov kernel p is said to be **minorized** on $C \subset \mathsf{X}$ if there exists an $\varepsilon > 0$, $\nu \in \mathcal{D}(\mathsf{X})$ and an $m \in \mathbb{N}$ such that

$$p^m(x, y) \geq \varepsilon \nu(y) \quad \text{for all } x \in C \text{ and all } y \in \mathsf{X}.$$

EXERCISE 23. Let $\mathsf{X} = \{0, 1\}$ and let $p(x, y) = \mathbb{1}\{x = y\}$. Show that p is not minorized on X .

The next lemma provides a relatively straightforward way to test for smallness.

Lemma 7.1.11. *If C is finite and there exists a $z \in \mathsf{X}$ and $m \in \mathbb{N}$ such that $p^m(x, z) > 0$ for all $x \in C$, then p is minorized on C .*

Proof. Let $\nu = \delta_z$, the point mass on z , and set ε to $\min_{x \in C} p^m(x, z)$. Since C is finite, $\varepsilon > 0$. Moreover, for any $x \in C$, we have $p^m(x, y) \geq \varepsilon \mathbb{1}\{y = z\} = \varepsilon \nu(y)$. \square

One way to think about minorization over a set C is as follows. Suppose two independent Markov chains start from arbitrary initial conditions and then enter C at the same time. This two chains now have a positive probability of meeting after m steps (equal to ε in the proof of Lemma 7.1.11). This is related to ergodicity because the ability of chains to meet up implies mixing; initial conditions do not fix long run outcomes. Our next result builds on this idea.

7.1.5.2 Drift and Minorization

We can now state our main result for this section.

Theorem 7.1.12 (Foster–Lyapunov). *Let p be an irreducible Markov kernel on \mathbb{X} . If p exhibits constant drift to a small set, then p is ergodic. If, in addition, p aperiodic, then p is globally stable.*

We prove Theorem 7.1.12 in §13.3.4.⁷

As an example, recall the single server queue model in §7.1.2.4, where the number of people in the queue evolves according to $X_{t+1} = X_t + \xi_{t+1} - \eta_{t+1}\mathbb{1}\{X_t > 0\}$. To simplify the analysis, we suppose that $\{\xi_t\}$ and $\{\eta_t\}$ are IID, independent of each other, and both put positive probability on the integers 0 and 1. We found in §7.1.2.4 that a stationary distribution exists whenever $\mathbb{E}\xi_t < \mathbb{E}\eta_t$. We now show that global stability holds.

Recall that the chain exhibits constant drift to $C = \{0\}$. The kernel p is minorized on C , as can be seen from Lemma 7.1.11 and $p(0, 0) > 0$. As ξ_t and η_t are independent and both put positive probability on 0 and 1, the queue can, over one unit of time, go up by one state, stay constant, or go down by one state (when the state is strictly positive). Hence p is aperiodic and irreducible. Theorem 7.1.12 now implies that p is globally stable.

Add a figure showing the stationary distribution, computed by simulation. Use the look-ahead method. Consider adding a brief discussion on Rao-Blackwellization and pointing ahead to later treatment.

7.1.6 Firm Dynamics with Entry and Exit

Add a first look at how heavy tails can arise in the stationary distribution from proportional growth. See Córdoba (2008) and references within. But focus on simplest case, state space is $\mathbb{Z}_{\geq 0}$. Give some discussion of how this might connect to firm growth and city growth. Gibrat's law.

7.1.6.1 Dynamics

The model, with fixed threshold. Ergodicity. Include the countably infinite case, where we get a proper Pareto tail.

7.1.6.2 Heavy-Tails

Specialize to Gibrat's law. Illustrate by simulation.

⁷See also Brémaud (1999) or Theorem 11.0.1 of Meyn and Tweedie (2009).

7.1.7 Inventory Dynamics

One possibility is to take this material and integrating it into the theory above. Don't do so if it messes up the exposition of theory too much though.

7.1.7.1 S-s Inventory Dynamics

Let X_t denote the inventory of a given product within a given firm. When the inventory of the firm runs low it places an order for replacement stock. In particular, when inventory falls below some positive constant s , the firm orders S units where S is an integer greater than s . Stochastic demand D_{t+1} arrives at the end of period t , and the state then updates as

$$X_{t+1} = \begin{cases} (X_t - D_{t+1})^+ & \text{if } X_t > s \\ (S - D_{t+1})^+ & \text{if } X_t \leq s \end{cases} \quad (7.25)$$

As usual, $x^+ := \max\{x, 0\}$. The implication here is that inventories are not backfilled. When demand exceeds inventory the inventory becomes zero and excess orders are lost. We suppose that $\{D_t\}$ is IID and has the geometric distribution, so that, in particular, $\mathbb{P}\{D_t = d\} = (1-p)^d p$ for all $d \in \mathbb{N}$, where $p \in (0, 1)$ is a parameter.

This process forms a Markov chain on the integers $\mathbb{X} := \{0, 1, \dots, S\}$. The transition probabilities are

$$p(x, y) = \begin{cases} \mathbb{P}\{(x - D_{t+1})^+ = y\} & \text{if } x > s \text{ and} \\ \mathbb{P}\{(S - D_{t+1})^+ = y\} & \text{if } x \leq s. \end{cases}$$

The state space is finite, so a stationary distribution ψ^* exists by Theorem 7.1.2. Regarding stability, consider condition (i) in Theorem 7.1.10. The condition will be satisfied with $k = 1$ and $y = 0$ if $p(x, 0) > 0$ for any $x \in \mathbb{X}$. For this to hold, it suffices that $\mathbb{P}\{D_{t+1} \geq S\} > 0$. Given that demand is geometrically distributed, the last inequality is valid. Hence p is globally stable.

7.1.7.2 Inventory with Backfills

Let X_t denote the inventory of a firm at time t in one particular product. Dynamics are similar to those discussed in §7.1.7.1 with one distinction: demand that exceeds current inventory is backfilled and then resupplied when new stock is on hand. The law of motion for inventory is therefore

$$X_{t+1} = X_t - D_{t+1} + S \mathbb{1}\{X_t \leq s\} \quad (7.26)$$

We take $\{D_t\}$ to be IID and supported on all of \mathbb{Z}_+ . For example, D_t might be geometric or Poisson distributed.

A suitable state space is $\mathbf{X} = \{\dots, -1, 0, 1, \dots, \bar{x}\}$, where $\bar{x} := s + S$. For the Markov kernel we have

$$\Pi(x, y) = \mathbb{P}\{x - D_{t+1} + S\mathbb{1}\{x \leq s\} = y\} = \varphi(x + S\mathbb{1}\{x \leq s\} - y),$$

where φ is the probability mass function for the Poisson distribution.

Intuitively, inventory will not diverge to $-\infty$ if average demand $\bar{D} := \mathbb{E}[D_t]$ does not exceed the restock value S . To test whether this condition is enough for the existence of a stationary distribution, consider the function $V(x) = -x$, which is obviously norm-like on \mathbf{X} . The left hand side of (7.18) is

$$\mathbb{E}[-X_{t+1} + X_t \mid X_t = x] = \mathbb{E}[D_{t+1}] - S\mathbb{1}\{x \leq s\} = \bar{D} - S + S\mathbb{1}\{x > s\}$$

Since $C := \{x \in \mathbf{X} : x > s\} = \{x \in \mathbb{N} : s < x \leq S\}$ is finite, condition (7.18) is satisfied whenever $\bar{D} < S$.

The Markov kernel Π is irreducible. To see this, suppose first that visiting the state \bar{x} is always possible. If the chain visits \bar{x} at t , the next period state is chosen as $\bar{x} - D_{t+1}$ and, given that the demand shock is supported on all of \mathbb{Z}_+ , each point in \mathbf{X} is chosen with positive probability. Thus, we need only show that, starting from any point x , visiting the maximal state \bar{x} is possible eventually.

The state \bar{x} is visited next period with positive probability when the current state X_t is s , since $\mathbb{P}\{D_{t+1} = 0\}$ is positive. In turn, the state s can be reached in one step whenever X_t is in the no-restock region $N := \{s, s+1, \dots, \bar{x}\}$. Finally, even if X_t is not in N , it can reach N eventually—if, say, demand is zero for a sufficiently long time.

The kernel Π is also aperiodic, as can be seen from the sufficient condition on page 229. In the no-restock region N , the chain stays still if demand is zero, which happens with positive probability. In the restock region N^c , the chain stays still if demand equals S , which also happens with positive probability.

Example 7.1.15. Consider the inventory model with backfilled orders shown in (7.26). We have already shown via Foster's theorem that the corresponding Markov kernel is bounded in probability. Suppose that the demand shock $\{D_t\}$ has, say, the geometric distribution. Then it hits any nonnegative integer with positive probability. If we now fix $x \in \mathbf{X}$ then clearly we can choose a sequence of demand realizations d_1, \dots, d_j to drive the state to S , after which demand shocks of zero will see the state remain there. In other words, [to be completed].

7.2 Iterated Function Systems

Add roadmap.

7.2.1 Distribution Dynamics

Add roadmap.

7.2.1.1 Markov Models in Vector Space

Let's consider a **first order Markov process** with **state space** $X \subset \mathbb{R}^k$ defined by

$$X_{t+1} = F(X_t, \xi_{t+1}) \quad (7.27)$$

where $\{\xi_t\}_{t \geq 1}$ is an IID sequence of random vectors taking values in a subset E of \mathbb{R}^j . We are generalizing the basic linear VAR model in (4.18) on page 104 by allowing F to be nonlinear. Now we only require it to be Borel measurable function from $X \times E$ to X , which is a minimal regularity condition.⁸

We also assume that the initial condition X_0 is independent of the shock process $\{\xi_t\}$. This seemingly minor point will be important because it implies independence of X_t and ξ_{t+1} for all t . This holds because X_t is a function only of X_0 and ξ_1, \dots, ξ_t . Indeed,

$$\begin{aligned} X_1 &= F(X_0, \xi_1) \\ X_2 &= F(F(X_0, \xi_1), \xi_2) \\ X_3 &= F(F(F(X_0, \xi_1), \xi_2), \xi_3) \end{aligned}$$

and so on.

While our first order Markov process (7.27) generalizes the linear continuous state model, it also extends the finite state Markov chain environment discussed in §3.1.2 by allowing the state space to be a continuum. In particular, compare our new continuous state model (7.27) with the stochastic recursive sequence representation of a Markov chain in (3.6), page 50.

⁸Borel measurability of the function F is required to ensure our system is well behaved enough to be able to compute expectations of state variables and other related objects. A detailed treatment is given in §11.3.1.1 but for now just note that for Borel measurability it would suffice that F is continuous or, if discontinuous, has at most countably many points in its domain where continuity fails.

Let's look at some examples of nonlinear Markov models that can be represented by (7.27).

Example 7.2.1. Consider again the Solow–Swan model growth model discussed in §5.3, where capital stock evolves on $(0, \infty)$ via $k_{t+1} = sf(k_t) + (1 - \delta)k_t$. Here we are assuming that $f(k) > 0$ when $k > 0$ and $s > 0$, $0 < \delta \leq 1$. Suppose we know introduce a positive stochastic productivity term $\{z_t\}$ multiplying output, so that

$$k_{t+1} = sz_{t+1}f(k_t) + (1 - \delta)k_t \quad \text{where } \{z_t\} \stackrel{\text{IID}}{\sim} \varphi \text{ on } (0, \infty)$$

This is a first order Markov process with

- state variable k_t taking values in state space $X = (0, \infty)$,
- shock space $E = (0, \infty)$ and
- law of motion $F(k, z) := szf(k) + (1 - \delta)k$,

Example 7.2.2. In macroeconomic applications it is common to assume that the aggregate productivity shock process $\{z_t\}$ is correlated. So let us take the stochastic Solow–Swan model in example 7.2.1 and replace the IID assumption on $\{z_t\}$ with the log AR1 assumption $z_t = \exp(y_t)$ where $y_{t+1} = ay_t + b + c\xi_{t+1}$ with $\{\xi_t\}$ IID and $N(0, 1)$. To accommodate this extended Solow–Swan model within our first order Markov framework, we use

- state vector $X_t := (k_t, y_t)$ taking values in state space space $X = (0, \infty) \times \mathbb{R}$,
- law of motion

$$F(x, \xi) = F\left(\begin{pmatrix} k \\ y \end{pmatrix}, \xi\right) = \begin{pmatrix} s \exp(ay + b + c\xi)f(k) + (1 - \delta)k \\ ay + b + c\xi \end{pmatrix} \quad (7.28)$$

7.2.1.2 Linking Marginal Distributions

In both the linear VAR case and the discrete Markov case, we constructed laws of motion that link successive marginal distributions for the state (see, e.g. (4.36) on page 114 and (3.10) on page 52). We wish to do the same for our nonlinear first order Markov model $X_{t+1} = F(X_t, \xi_{t+1})$.

In the VAR case our marginal distributions were represented by densities but let us start here with cumulative distribution functions, which are somewhat less intuitive but have the advantage that they can represent both absolutely continuous distributions (i.e., densities) and distributions with some positive mass on individual points.

To this end, let Ψ_t represent the CDF of the state vector X_t generated by our nonlinear model

$$X_{t+1} = F(X_t, \xi_{t+1}), \quad \{\xi_t\}_{t \geq 1} \stackrel{\text{IID}}{\sim} \Phi \quad (7.29)$$

We are using Φ to represent the CDF of the shock vector ξ_t . We have

$$\mathbb{P}\{X_{t+1} \leq y\} = \mathbb{E}\mathbb{1}\{F(X_t, \xi_{t+1}) \leq y\} = \int \int \mathbb{1}\{F(x, z) \leq y\} \Phi(dz) \Psi_t(dx)$$

The last inequality uses independence of X_t and ξ_{t+1} , since, in this case, the joint CDF of (X_t, ξ_{t+1}) is just the product of the marginals.

We can now see that the marginals $\{\Psi_t\}$ of the state process are linked by the recursion

$$\Psi_{t+1}(y) = \int \Pi(x, y) \Psi_t(dx) \quad (y \in \mathbb{X}) \quad (7.30)$$

where

$$\Pi(x, y) := \int \mathbb{1}\{F(x, z) \leq y\} \Phi(dz) = \mathbb{P}\{F(x, \xi_{t+1}) \leq y\} \quad (7.31)$$

is the conditional distribution of the next period state given the current state. Paralleling our definitions for the discrete Markov case, the object Π is called the **stochastic kernel** for our model.

When convenient we can think of the updating in (7.30) as the action of an operator Π on the set of CDFs over \mathbb{X} and express it as

$$\Psi_{t+1} = \Psi_t \Pi \quad (7.32)$$

Consistent with earlier notation and terminology, we can view Π as a mapping

$$(\Psi \Pi)(y) = \int \Pi(x, y) \Psi(dx) \quad (y \in \mathbb{X}) \quad (7.33)$$

acting in distribution space. If $\mathcal{P}(\mathbb{X})$ is the set of all distributions on \mathbb{X} , then $(\mathcal{P}(\mathbb{X}), \Pi)$ forms a dynamical system. We will call Π the (left) **Markov operator** associated with the Markov process (7.29).

A **stationary distribution** is a fixed point of Π in $\mathcal{P}(\mathbb{X})$.

Example 7.2.3. Returning to the stochastic Solow–Swan model growth model in example 7.2.1, the marginal distributions $\{\Psi_t\}$ of capital $\{k_t\}$ obey the recursion

$$\Psi_{t+1}(k') = \int \Pi(k, k') \Psi_t(dk) \quad (k > 0) \quad (7.34)$$

with

$$\Pi(k, k') = \mathbb{P}\{s\xi_{t+1}f(k) + (1 - \delta)k \leq k'\} \quad (7.35)$$

If Φ is the CDF of ξ_{t+1} , we can rewrite this as

$$\Pi(k, k') = \Phi\left(\frac{k' - (1 - \delta)k}{sf(k)}\right) \quad (7.36)$$

A distribution Ψ^* is stationary for this model if it satisfies

$$\Psi^*(k') = \int \Phi\left(\frac{k' - (1 - \delta)k}{sf(k)}\right) \Psi^*(dk) \quad (k > 0)$$

7.2.1.3 The Density Case

While the marginal CDF law of motion (7.32) is satisfactory from a theoretical perspective, when we can work with densities it is usually more convenient to do so. Whether it is possible to work in a density setting or not depends on whether the marginal distributions generated by the model are absolutely continuous. This might not be the case even when ξ_t has a density, as you can easily see by taking our law of motion to be $F \equiv 0$.

The key condition is that, for every $x \in \mathbb{X}$, the distribution of the conditional next period state $Y = F(x, \xi_{t+1})$ is absolutely continuous. If this is the case then we can represent its distribution by a density, which we denote by $\pi(x, y)$. The conditional density π is called the **density stochastic kernel** for our model.

If we do have this representation, then the marginal distributions of $\{X_t\}$ all have densities, henceforth denoted $\{\psi_t\}$, and these densities are linked by

$$\psi_{t+1}(y) = \int \pi(x, y) \psi_t(x) dx \quad (7.37)$$

The result in (7.37) can be obtained by differentiating (7.30) with respect to y and using the fact that $\Psi_t(dx) = \psi_t(x) dx$ or by following the logic that led to (4.37) on page 114.

One way to check whether the distribution of $Y = F(x, \xi_{t+1})$ is absolutely continuous is to obtain its CDF and test whether or not it is differentiable.

Example 7.2.4. Let's go back to example 7.2.3, where we found that the CDF of the next period state given the current state is

$$\Pi(k, k') = \Phi\left(\frac{k' - (1 - \delta)k}{sf(k)}\right) \quad (7.38)$$

If Φ , the shock distribution, is differentiable with density φ , then we can differentiate (7.38) with respect to k' to obtain

$$\pi(k, k') = \varphi \left(\frac{k' - (1 - \delta)k}{sf(k)} \right) \frac{1}{sf(k)} \quad (7.39)$$

The marginal densities of capital stock obey (7.37) with this choice of π .

7.2.1.4 A Useful Scalar Model

Let's conclude this section by looking at a relatively generic nonnegative scalar model that combines both additive and multiplicative shock components. In particular,

$$X_{t+1} = \zeta_{t+1}g(X_t) + \eta_{t+1} \quad (7.40)$$

where

- (i) g is a Borel measurable function from \mathbb{R}_+ to itself,
- (ii) $\{\zeta_t\}$ are IID copies of an \mathbb{R}_+ -valued random variable ζ with density ν and
- (iii) $\{\eta_t\}$ are IID copies of an \mathbb{R}_+ -valued random η , independent of $\{\zeta_t\}$ and with density φ .

A number of models of income and wealth we study below can be fitted into this formulation. At present our only aim is to obtain an expression for the conditional density $\pi(x, y)$ that represents the stochastic kernel of the model.

In particular, we seek the conditional density of X_{t+1} given $X_t = x$, which is equal to the density of the random variable $Y = \zeta g(x) + \eta$ when ζ and η are drawn independently from ν and φ respectively. Recall that if U is a random variable with density φ_U and $Y = f(U)$ where f is continuously differentiable and strictly monotone, then the density of Y is

$$\varphi_Y(y) = \varphi_U(f^{-1}(y)) \left| \frac{df^{-1}(y)}{dy} \right| \quad (7.41)$$

(This is a standard change-of-variable argument. See, e.g, [Walsh \(2012\)](#), proposition 3.24.)

It follows that the density of $Y = \zeta g(x) + \eta$ given $\zeta = z$ is $y \mapsto \varphi(y - zg(x))$ (Here we take $\varphi(u) = 0$ whenever $u \leq 0$.) Invoking a law of total probability argument, the distribution of Y after dropping conditioning on $\zeta = z$ is the density obtained by

summing over all possible z , weighted by their probabilities:

$$\pi(x, y) = \int \varphi(y - zg(x))\nu(dz) \quad (7.42)$$

Analogous to (7.32), the marginal densities of $\{X_t\}$ correspond to the iterates of Π over the set of densities \mathcal{D} on \mathbb{R}_+ , where

$$(\psi\Pi)(y) = \int \pi(x, y)\psi(x) dx \quad (7.43)$$

This is a Markov operator acting on the space of density over \mathbb{X} .

7.2.2 Random Coefficient Models

In this section we specialize to a class of models that linear apart from the fact that the multiplicative component is stochastic. Such processes are known as **Kesten processes**, or random coefficient models. They take the form

$$X_{t+1} = A_{t+1}X_t + \eta_{t+1}. \quad (7.44)$$

Here $\{X_t\}_{t \geq 0}$ is an $n \times 1$ state vector process and $\{A_t, \eta_t\}_{t \geq 1}$ is a stochastic process where $\{A_t\}$ takes values in $\mathcal{M}_{n \times n}$ and $\{\eta_t\}$ takes values in \mathbb{R}^n . They are named for the German–American mathematician Harry Kesten (1931–). We will see that even simple versions of (7.44) generate interesting and useful dynamics, including heavy tails in their asymptotic distributions.

[roadmap]

7.2.2.1 Stability

The stochastic kernel corresponding to (7.44), referred to below as the Kesten stochastic kernel, is, in CDF format,

$$\Pi(x, y) = \mathbb{P}\{A_{t+1}x + \eta_{t+1} \leq y\} \quad (y \in \mathbb{R}^n) \quad (7.45)$$

As usual, we can identify this kernel with the Markov operator $(\Psi\Pi)(y) = \int \Pi(x, y)\Psi(dx)$, which, when iterated, produces sequences of marginal distributions $\{\Psi_t\}$ for the Kesten sequence (7.44).

Our analysis of heavy tails associated with the Kesten process (7.44) concerns the stationary distribution generated by the model. Hence the first task at hand is to

determine when such a distribution exists. Here we state a stability condition for the IID case that is both necessary and sufficient for existence of a stationary solution.⁹

In the next assumption $\ln^+ x := \max\{\ln x, 0\}$.

Assumption 7.2.1. The process $\{(A_t, \eta_t)\}_{t \geq 1}$ in (7.44) is a sequence of IID draws from joint distribution φ and satisfies

$$\mathbb{E} \ln^+ \|A_t\| < \infty \quad \text{and} \quad \mathbb{E} \ln^+ \|\eta_t\| < \infty \quad (7.46)$$

In addition, if V is a linear subspace of \mathbb{R}^n satisfying $\mathbb{P}\{A_t x + \eta_t \in V\} = 1$ for all $x \in V$, then $V = \mathbb{R}^n$.

The first two conditions are basic regularity conditions, while the third says that the process does not get caught in any subspace of dimension less than n .

Example 7.2.5. Assumption 7.2.1 is satisfied if $\{A_t, \eta_t\}_{t \geq 1}$ is Gaussian and η_t is non-degenerate in the sense of having a positive density on all of \mathbb{R}^n . The weak moment inequalities in (7.46) hold because this distribution has light tails. For the third restriction, suppose to the contrary that there exists a linear subspace V not equal to \mathbb{R}^n that satisfies $\mathbb{P}\{A_t x + \eta_t \in V\} = 1$ whenever $x \in V$. Since the only n dimensional subspace in \mathbb{R}^n is \mathbb{R}^n , we see that V has dimension $k < n$, in which case it has zero measure in \mathbb{R}^n . At the same time, $\mathbb{P}\{\eta_t \in V\} = 1$ must hold because every subspace contains zero. In other words, η_t concentrates all mass in a measure zero subset of \mathbb{R}^n . This contradicts the nondegenerate assumption.

Let

$$L_n := \frac{1}{n} \mathbb{E} \ln \|A_1 \cdots A_n\| \quad (n \in \mathbb{N}) \quad (7.47)$$

As shown below, stability of the Kesten process (7.44) requires that L_n is negative for some n . One way to understand this is to consider the scalar case, where

$$L_n = \frac{1}{n} \mathbb{E} \ln |A_1 \cdots A_n| = \frac{1}{n} \mathbb{E} \sum_{i=1}^n \ln |A_i| = \mathbb{E} \ln |A_t| \quad (7.48)$$

A negative value means that the coefficient $|A_t|$ is small on average, which generates mean reversion.

Theorem 7.2.1. *If assumption 7.2.1 holds then the following statements are equivalent:*

⁹We follow Kesten (1973) and Diaconis and Freedman (1999) in adopting an IID assumption for $\{A_t, \eta_t\}_{t \geq 1}$ but more general settings have been studied in the literature. For example, in the scalar case, Brandt (1986) adopts the weaker assumption that $\{A_t, \eta_t\}$ is stationary and ergodic.

(i) There exists an $n \in \mathbb{N}$ such that $L_n < 0$.

(ii) The random sequence

$$x^* := \sum_{j=1}^{\infty} \prod_{i=1}^{j-1} A_i \eta_j \quad (7.49)$$

converges absolutely with probability one.

If either of these conditions is true, then the random variable x^* in (7.49) is well defined. Its distribution Ψ^* is a stationary distribution for the Kesten process, and $(\mathcal{P}(\mathbb{R}^n), \Pi)$ is globally stable.

In (7.49), it is understood that $\prod_{i=1}^0 A_i = 1$. The stochastic kernel Π in theorem 7.2.1 is the Kesten stochastic kernel, and the statement that Ψ^* is stationary for Π is equivalent to: if (A, η) is an independent draw from φ , then $x^* \stackrel{d}{=} Ax^* + \eta$. In the statement concerning stability, the topology on $\mathcal{P}(X)$ is that of weak convergence (i.e., convergence in distribution).

To help us understand the conditions in theorem 7.2.1, consider the vector autoregression model (4.18) on page 104, which can be expressed as

$$X_{t+1} = AX_t + \eta_{t+1} \quad \text{when } \eta_{t+1} := b + C\xi_{t+1}$$

Regarding condition (i) in theorem 7.2.1, the exponent L_n translates to

$$\frac{1}{n} \mathbb{E} \ln \|A_1 \cdots A_n\| = \frac{1}{n} \ln \|A^n\| = \ln \left\{ \|A^n\|^{\frac{1}{n}} \right\}$$

By Gelfand's formula, we have $\|A^n\|^{\frac{1}{n}} \rightarrow r(A)$ as $n \rightarrow \infty$, where $r(A)$ is the spectral radius of the matrix A . In particular, if the spectral radius is strictly less than one, then $L_n < 0$ for sufficiently large n , and condition (i) of theorem 7.2.1 is satisfied. This suggests we should view condition (i) as a generalization of the restriction $r(A) < 1$ that is central to stability of linear dynamic systems.

A full proof of theorem 7.2.1 can be found in Bougerol and Picard (1992). Here we prove just sufficiency of (i) for convergence of the random sum in the definition of x^* , and only in the scalar case. Then condition (i) reduces to $\mathbb{E} \ln |A_t| < 0$, as discussed above. We will also assume that $\mathbb{E} \ln |\eta_t| < \infty$. On the other hand, we will replace the assumption that $\{A_t, \eta_t\}$ is IID with the weaker assumption that $\{A_t, \eta_t\}$ is stationary and ergodic. Our proof follows Brandt (1986).

Let $S_t := \sum_{j=1}^t \prod_{i=1}^{j-1} A_i \eta_j$. By Cauchy's root criterion, the partial sums $\{S_t\}$

converge absolutely to a finite number whenever $\limsup_{j \rightarrow \infty} H_j < 1$, where

$$H_j := \left(\prod_{i=1}^{j-1} |A_i \eta_j| \right)^{1/(j-1)} = \exp \left(\frac{1}{j-1} \sum_{i=1}^{j-1} \ln |A_i| + \frac{\ln |\eta_j|}{j-1} \right)$$

From stationarity, ergodicity and the conditions on $\ln |\eta_t|$ and $\ln |A_t|$, the sequence $(1/j) \sum_{i=1}^j \ln |A_i|$ converges to a negative constant and $\ln |\eta_j|/(j-1)$ converges to zero with probability one.¹⁰ It follows that, with probability one,

$$\limsup_{j \rightarrow \infty} \left(\frac{1}{j-1} \sum_{i=1}^{j-1} \ln |A_i| + \frac{\ln |\eta_j|}{j-1} \right) \leq \limsup_{j \rightarrow \infty} \frac{1}{j-1} \sum_{i=1}^{j-1} \ln |A_i| + \limsup_{j \rightarrow \infty} \frac{\ln |\eta_j|}{j-1} < 0$$

Hence $\limsup_{j \rightarrow \infty} H_j < 1$.

Example 7.2.6. Consider the GARCH(1, 1) volatility process

$$\sigma_{t+1}^2 = \alpha_0 + \sigma_t^2(\alpha_1 \xi_{t+1}^2 + \beta) \quad (7.50)$$

where $\{\xi_t\}$ is IID with $\mathbb{E} \xi_t^2 = 1$ and all parameters are positive. This model is common in financial settings, where time series such as asset returns exhibit time varying volatility. In this case η_t is a constant and, recalling (7.48), stability will hold when $\mathbb{E} \ln(\alpha_1 \xi_{t+1}^2 + \beta) < 0$. A sufficient condition often used in the literature is $\alpha_1 + \beta < 1$. This suffices because, by Jensen's inequality,

$$\mathbb{E} \ln(\alpha_1 \xi_{t+1}^2 + \beta) \leq \ln \mathbb{E}(\alpha_1 \xi_{t+1}^2 + \beta) = \ln(\alpha_1 + \beta) \quad (7.51)$$

7.2.2.2 Kesten Processes and Power Laws

A striking result due to [Kesten \(1973\)](#) shows that the Kesten process can generate power laws relatively easily. In discussing it, we will focus on the nonnegative scalar case, where

$$X_{t+1} = A_{t+1} X_t + \eta_{t+1}, \quad \{A_t\} \text{ and } \{\eta_t\} \text{ both nonnegative and scalar} \quad (7.52)$$

We provide a version of Kesten's theorem with slightly strengthened assumptions to avoid unnecessary complications. The theorem has been used to establish a power

¹⁰We use the fact that $z_n/n \rightarrow 0$ with probability one as $n \rightarrow \infty$ whenever $\{z_n\}$ is stationary and ergodic with finite mean. This holds because if $s_n := \sum_{j=1}^n z_j$, then $z_n/n = s_n/n - [(n-1)/n](s_{n-1}/(n-1))$. The two sample means converge to the same number with probability one by stationarity and ergodicity.

law in the tail of income and wealth distributions for a variety of dynamic models (see [Nirei and Souma \(2007\)](#), [Nirei \(2009\)](#), [Benhabib et al. \(2011, 2015a,b\)](#) and [Benhabib et al. \(2016\)](#).)

Assumption 7.2.2. The following conditions hold:

- (i) The random variable A_t is positive with probability one and nonarithmetic.¹¹
- (ii) There exists a positive constant α such that

$$\mathbb{E}A^\alpha = 1, \quad \mathbb{E}\eta^\alpha < \infty, \quad \text{and} \quad \mathbb{E}[A^\alpha \ln^+ A] < \infty$$

- (iii) $\mathbb{P}\{Ax + \eta = x\} < 1$ for all $x \in \mathbb{R}_+$.

We can now state the main result of this section, a proof of which can be found in that of theorem 2.4.4 of [Buraczewski et al. \(2016\)](#).

Theorem 7.2.2. *If assumption 7.2.2 holds, then there exists a random variable x^* on \mathbb{R}_+ such that*

$$x^* \stackrel{d}{=} A_t x^* + \eta_{t+1} \quad \text{and} \quad \lim_{x \rightarrow \infty} x^\alpha \mathbb{P}\{x^* > x\} = c \quad (7.53)$$

for some $c > 0$, where α is the constant in assumption 7.2.2.

The proof of theorem 7.2.2 is contains many steps and we omit the details here. However, some intuition can be gained from the following exercise, which is based on the discussion in [Gabaix \(2009\)](#).

Consider the scalar Kesten process (7.52) under the assumptions of theorem 7.2.2. The distribution of the random variable x^* in (7.53) is a stationary distribution of (7.52). The influence of $\{\eta_t\}$ in $X_{t+1} = A_{t+1}X_t + \eta_{t+1}$ is insignificant when x is large, and since this is the region we are interested in, let us set $\eta_t \equiv 0$ and consider the stationary density of $X_{t+1} = A_{t+1}X_t$ when $\{A_t\}$ has density φ on \mathbb{R}_+ . By the change-of-variable argument (7.41) on page 240, the conditional density of $y := X_{t+1}$ given $X_t = x$ is $\pi(x, y) = \varphi(y/x)/x$, and hence the stationary density, denoted here by ψ^* , satisfies

$$\psi^*(y) = \int \pi(x, y)\psi^*(x) dx = \int \varphi\left(\frac{y}{x}\right) \frac{1}{x} \psi^*(x) dx \quad (7.54)$$

Taking α to be the positive constant in assumption 7.2.2, we conjecture a solution of the form $\psi^*(x) = kx^{-\alpha-1}$ for some constant k , which is the density of the Pareto

¹¹A random variable is **arithmetic** if it concentrates probability mass on a set of the form $t\mathbb{Z}$ for some $t > 0$ and **nonarithmetic** otherwise.

distribution with exponent α . Inserting this conjecture into (7.54) and rearranging gives

$$1 = \int \varphi\left(\frac{y}{x}\right) \frac{1}{x} \left(\frac{y}{x}\right)^{\alpha+1} dx = \int \varphi(t) t^\alpha dt$$

The term on the right does indeed equal 1, by the definition of φ and assumption 7.2.2.

7.2.3 Stochastic Stability

Let's now consider the sequences of distributions generated by these models and their asymptotic properties. We will treat the general case (7.43), corresponding to the model in (7.40). The mapping Π from densities to densities described in (7.43) yields a dynamical system (\mathcal{D}, Π) when \mathcal{D} is the set of all densities.

Of course, we need a metric to impose on \mathcal{D} . There are also some formal issues such as proper specification of the concept of a density, so that \mathcal{D} is well defined. Having clarified \mathcal{D} , we should also make sure that Π maps \mathcal{D} into itself. We can then turn to fixed point theorems in order to analyze stability and steady states. A complete treatment of these issues requires some understanding of L_p spaces (see §11.3.5 for a definition). For now, let's state a result that covers our present needs and defer the proof. The metric we impose on \mathcal{D} is the L_1 metric

$$d_1(\varphi, \psi) := \int |\varphi(z) - \psi(z)| dz \quad (7.55)$$

Now consider the dynamical system (\mathcal{D}, Π) where Π is given by (7.66). Does a steady state exist? If so, is this steady state locally or globally stable? The last scenario is attractive because it implies a firm prediction for the model: the wealth distribution converges to this steady state, and if the system has been in motion for a while then it should already be close.

The next proposition will be useful. It is proved in §13.3.

Proposition 7.2.3. *Let Π be as specified in (7.43) and let $\{X_t\}_{t \geq 0}$ be defined on \mathbb{R}_+ by (7.40). If*

- (i) *the density φ of η has finite first moment and is positive everywhere on \mathbb{R}_+ and*
- (ii) *there exist positive constants L and λ such that $\lambda < 1$ and*

$$\mathbb{E}\zeta g(x) \leq \lambda x + L \quad (x \geq 0) \quad (7.56)$$

then (\mathcal{D}, Π) is globally stable, with unique stationary density ψ^ . Moreover, if h is any*

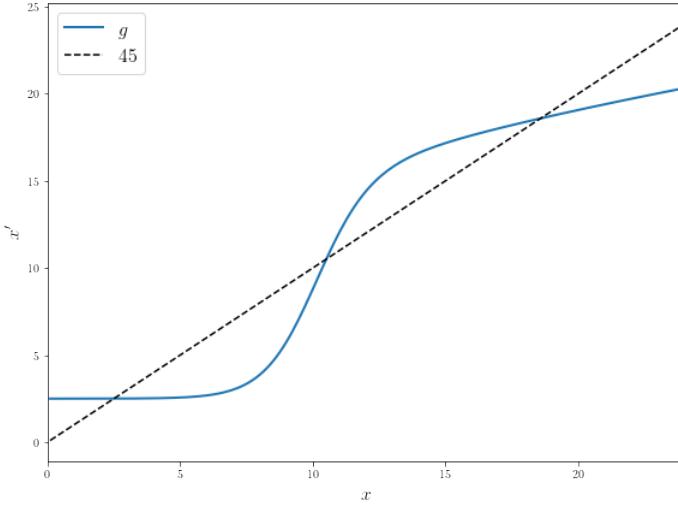


Figure 7.3: Dynamics with a degenerate shock and multiple fixed points

Borel measurable function satisfying $\int |h(x)|\psi^*(x) dx < \infty$, then

$$\frac{1}{n} \sum_{t=1}^n h(X_t) \rightarrow \int h(x)\psi^*(x) dx \quad (7.57)$$

with probability one as $n \rightarrow \infty$.

The first moment restriction in condition (i) is a basic regularity condition. Positivity of the density is a stronger assumption that we use here to generate mixing. To see why mixing matters, consider figure 7.3. This represents an extreme case, where ζ and η are entirely degenerate, concentrating all mass on 1.0 and 0.0 respectively, while g has multiple fixed points. If we allow point masses to be considered as densities, then global stability clearly fails, since point masses at the fixed points will stay constant.

Even if ζ and η are permitted to have densities, when these densities have small supports, paths starting near the lowest fixed point will not attain the neighborhood of the highest fixed point and vice versa. Figure 7.4 illustrates this. The multiplicative shock ζ is held at unity while η is supported on an interval $[a, b]$. Even if the shock sequence $\{\eta_t\}$ was to constantly attain its highest possible value b , a path starting at zero could never attain 25. Similarly, even if the shock sequence $\{\eta_t\}$ was to constantly attain its lowest possible value a , a path starting at 25 could never attain 0. Simulated time paths from the model that support this claim are shown in figure 7.5.

Everywhere positivity of the density φ of η on its support \mathbb{R}_+ is clearly sufficient

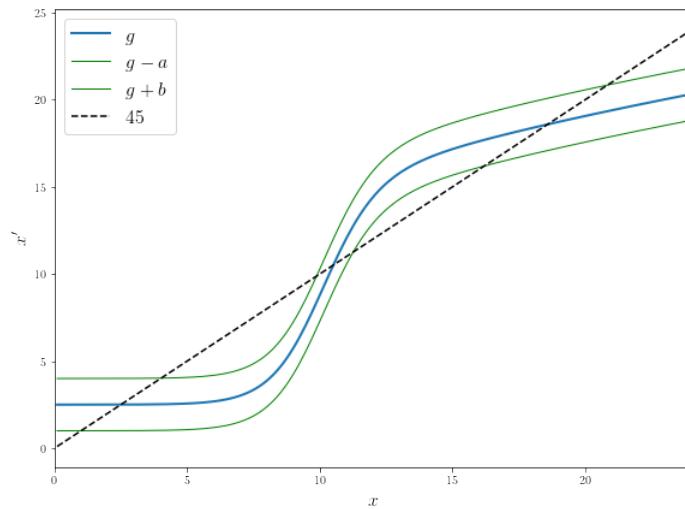


Figure 7.4: Dynamics with small shocks

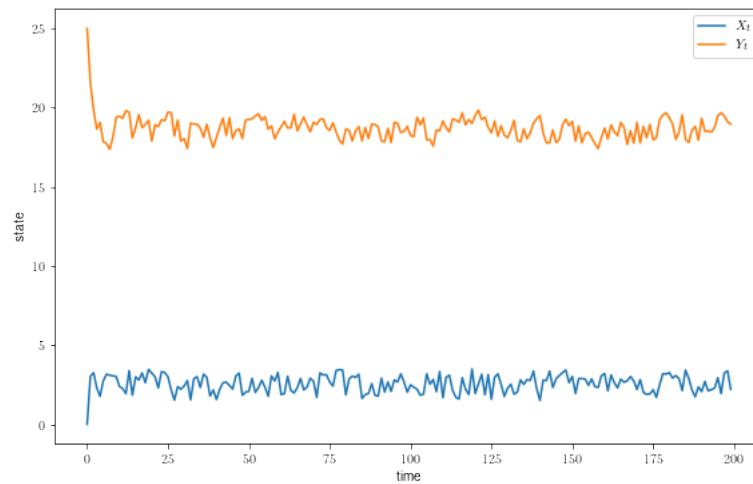


Figure 7.5: Time series with small shocks

to escape these kinds of traps, which is why we impose condition (i).¹²

Condition (ii) has a more straightforward role: It prevents $\{X_t\}$ from diverging to $+\infty$. When it holds we have

$$\mathbb{E}[X_{t+1} | X_t] = \mathbb{E}[\zeta_{t+1}g(X_t) + \eta_{t+1} | X_t] = \mathbb{E}[\zeta]g(X_t) + \mathbb{E}[\eta] \leq \lambda X_t + K$$

where $K := L + \mathbb{E}[\eta]$. Taking expectations of both sides and using the law of iterated expectations gives

$$m_{t+1} \leq \lambda m_t + K$$

where m_t is the mean of X_t for each t . Given that $\lambda < 1$, the mean of X_t is bounded by $K/(1 - \lambda)$. It turns out that, when the mean is bounded in this way, the distributions are prevented from diverging.

Together, the mixing provided by condition (i) and the bounds on probability mass provided by condition (ii) are exactly what we need for stability of this model.

Applied to the wealth process with stochastic financial returns, we see that

Lemma 7.2.4. *The dynamical system (\mathcal{D}, Π) corresponding to the wealth process (7.65) is globally stable whenever*

- (a) *The density of labor income is everywhere positive and has finite first moment.*
- (b) *Average savings from current wealth satisfies*

$$\mathbb{E}[R]s(w) \leq \lambda w + L \text{ for some } \lambda < 1 \text{ and } L < \infty \quad (7.58)$$

Restriction (b) says that expected gross return on post-consumption wealth as a fraction of current wealth is less than 1 when wealth is sufficiently large. This is clearest when (7.58) is expressed as

$$\frac{\mathbb{E}[R]s(w)}{w} \leq \lambda + \frac{L}{w}$$

7.2.4 Analysis of a Stationary Distribution

What is the purpose of this section? State it more clearly or eliminate or move it.

Now let's impose the conditions of lemma 7.2.4, so that a globally attracting stationary distribution ψ^* exists. Since this is the unique (stochastic) steady state and we converge to it regardless of our starting point, ψ^* is a natural focal point for prediction and analysis. In this section we compute it and analyze its properties.

¹²It isn't necessary, however, and we investigate weaker conditions later on.

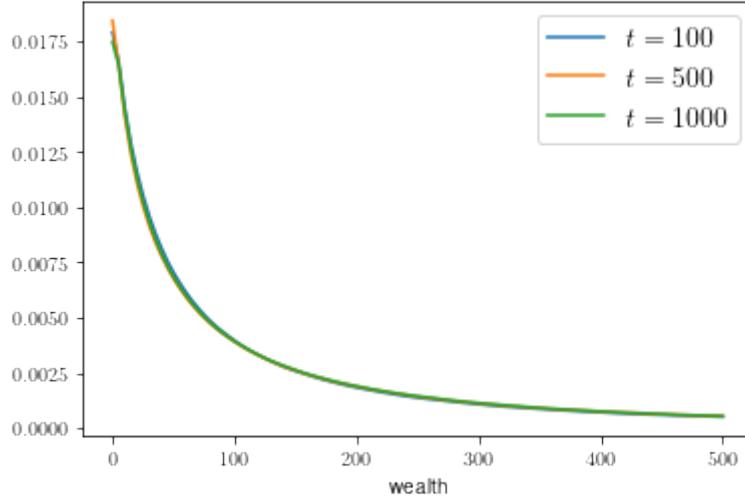


Figure 7.6: Look ahead estimates of ψ_t for different values of t

7.2.4.1 Estimating a Stationary Density

One way to estimate the stationary density is to exploit the fact that $\psi_t \rightarrow \psi^*$ under the stated conditions, and estimate ψ_t for some large t . Of course one has the issue of choosing a suitable value of t , but in the present case it is clear that convergence is quite rapid. We already saw this in figure 4.1, and figure 7.6 gives a further illustration, using the look ahead estimator. Estimates for dates greater than $t = 100$ are essentially overlaid, so ψ_t with t close to 100 is already an excellent estimator.

There is also a dedicated look ahead estimator of the stationary density when it exists—as it does for the default parameterization we are using, since shocks are lognormal and

$$\mathbb{E}R_{t+1}s(w) \leq (1+r)s_0w = (1+0.1)0.6w = 0.66w$$

The estimator in question is

$$\ell_n^*(w') := \frac{1}{n} \sum_{t=1}^n \pi(w_t, w') \quad (7.59)$$

Although this estimator looks similar to the cross-sectional estimator ℓ_t^m , there is a crucial difference: the sample is a single time series $\{w_t\}$ generated by simulation. In terms of wealth dynamics, you can think of this as a simulation of a single household through time, rather than a simulation of a cross-section at a particular point in time.

Why does (7.59) work? The important observation here is that, under the stated

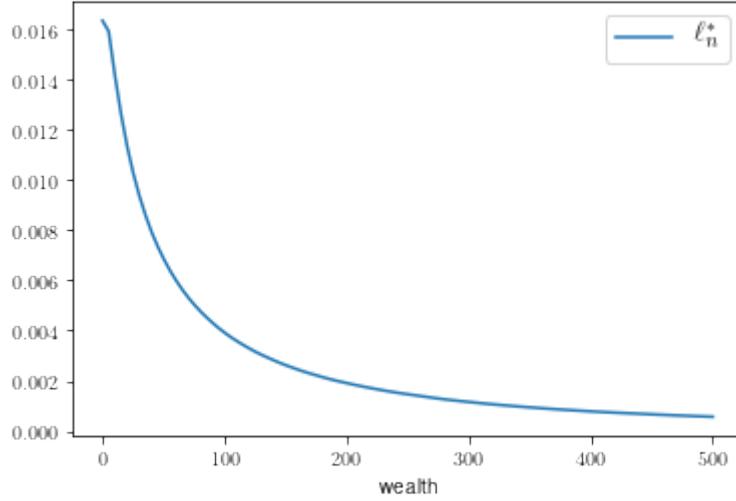


Figure 7.7: The stationary density look ahead estimator of the wealth distribution

stability conditions, the implications of proposition 7.2.3 on page 246 are valid. In particular, $(1/n) \sum_{t=1}^n h(X_t) \rightarrow \int h(x)\psi^*(x) dx$ holds whenever the right hand side is finite. Applying this to (7.59) at a given point w' in its domain, we obtain

$$\ell_n^*(w') = \frac{1}{n} \sum_{t=1}^n \pi(w_t, w') \rightarrow \int \pi(w, w') \psi^*(w) dw = \psi^*(w')$$

with probability one as $n \rightarrow \infty$. In particular, $\ell_n^*(w')$ is consistent for $\psi^*(w')$.

As was the case for the cross-sectional look ahead estimator, it is possible to extend this result to function space, obtaining

$$\|\ell_n^* - \psi^*\|_1 \rightarrow 0 \quad \text{as } n \rightarrow \infty \quad \text{with probability one}$$

The look ahead estimate ℓ_n^* is shown figure 7.7 for the same parameters used previously and with $n = 2,000$. Not surprisingly, we obtain a similar picture to the densities in figure 7.6.

Which estimator is better? On a theoretical level, the dedicated stationary density estimator ℓ_n^* appears preferable to using the cross-sectional estimator ℓ_m^t with large t , since it incorporates every observation of wealth that we generate. In contrast, when computing ℓ_m^t , we generate $(t - 1) \times m$ observations and discard all but m of them.

On the other hand, the cross-sectional estimator has one large advantage in terms of numerical estimation: each path that we generate—one path corresponding to the time series of one household—is independent of the others in terms of the sequence of

tasks implemented by the machine. This matters because these independent tasks are fully parallelizable. How much that buys us depends on a variety of factors, but it can be decisive in applications where the machine can execute many threads and the paths are long enough to offset the overhead of parallelization.

7.2.5 Goldie's Theorem

Put in a simplified version of Goldie's theorem: eventually close to linear dynamics with random coefficient. Apply to a basic entry-exit model. Consider changing title to "Goldie's Theorem and Firm Dynamics".

7.2.6 The Evolution of Wealth

Provides a more formal complement to the discussion of wealth dynamics in the introduction. Claims were made about convergence and stationarity. Verify them, perhaps in a slightly more general setting. Exogenous savings is assumed. Explain that this is so we can focus more clearly on dynamics. Connect back to the discussion of Goldie's theorem. Savings can be zero near the origin and then linear.

Also, somewhere in this section, repeat Lorenz/Gini/Rank-size simulations in introduction, but now with Pareto distributed income. Calibrate this distribution using results from Benhabib and Bisin (2018). Now, show how we can replicate smaller tail index for wealth relative to income (match joint distribution) if and only if we adopt stochastic returns on assets, as promised in §4.1.6.4.

But note also that we can also generate Pareto tails even if we don't assume Pareto income. This is interesting because trying to obtain a Pareto wealth distribution by using a Pareto income distribution is almost like assuming your conclusions. Using the Kesten–Goldie theory, we show that Pareto tails are a very likely outcome regardless of the distribution of wealth.

7.2.6.1 A Simple Model

Next let us consider a population of households with wealth distributed on \mathbb{R}_+ such that for all households, wealth $\{w_t\}$ obeys

$$w_{t+1} = (1 + r_{t+1})(w_t - c_t) + y_{t+1} \quad (7.60)$$

as in (2.8) on page 15.

While one of our aims is to solve the dynamic programming problem posed in §2.2.4 and pin down consumption as a function of the state variables, for now let's fix consumption behavior and focus on dynamics. Thus, throughout this section, we take $w_t - c_t = s(w_t)$ where s is a real valued Borel measurable function (a weak regularity condition—see footnote 8 on page 236) on \mathbb{R}_+ satisfying $0 \leq s(w) \leq w$ for all $w \in \mathbb{R}_+$,

In our first stage of analysis we will simplify the specifications of the labor income and returns processes given in (2.13) on page 29 as follows:

- $r_{t+1} = r$, a positive constant and
- y_{t+1} is IID with common density φ .

The full system for evolution of wealth is therefore

$$w_{t+1} = (1 + r)s(w_t) + y_{t+1}, \quad \{y_t\} \stackrel{\text{IID}}{\sim} \varphi. \quad (7.61)$$

We'll take the labor income process to be **idiosyncratic**, which means that each household i receives its own *independent* draw y_{t+1}^i from φ . We could also consider **aggregate** shocks, which affect all households, but let's put them aside for now. We continue to omit the superscript i for brevity when doing so causes no confusion.

Our objective in this section is to track the **wealth distribution**, which, at each point in time t , is the cross-sectional distribution over current wealth of each household. Let's denote this sequence of distributions by $\{\psi_t\}_{t \geq 0}$. In order to effectively track the sequence, we will seek a recursive relationship between these densities.

As a first step, it will be convenient to take current wealth w_t as equal to some fixed value w and try to compute next period's wealth distribution ψ_{t+1} . Observe that, even if all households have current wealth w , next period's wealth w_{t+1} will have a nondegenerate distribution—specifically, the distribution of the random variable $Y := (1 + r)s(w) + y_{t+1}$.

The assumption that we are dealing with an infinite number of households matters here. In particular, we are using the exact distribution of $(1 + r)s(w) + y_{t+1}$ with the understanding that this object represents the limit of the empirical distribution of a sample

$$\{Y_i\}_{i=1}^n = \{(1 + r)s(w) + y_{t+1}^i\}_{i=1}^n$$

where each Y_i is understood as the wealth of the i -th household and each y_{t+1}^i is drawn independently from the common distribution φ . The limit of this empirical distribution as $n \rightarrow \infty$ is precisely the distribution of the random variable $Y := (1 + r)s(w) + y_{t+1}$ when y_{t+1} is drawn from φ . We discuss empirical distributions and their limits in more detail in §4.1.5.

Returning to the problem of computing the distribution of $Y := (1 + r)s(w) + y_{t+1}$ when y_{t+1} is drawn from φ , we can use (7.41) to find that the density of w_{t+1} given $w_t = w$ is

$$\pi(w, w') := \varphi(w' - (1 + r)s(w)) \quad (7.62)$$

In (7.62), we understand that $\varphi(z) = 0$ whenever $z \leq 0$. Applying (7.37) or invoking a law of total probability argument (recall (4.35) on page 114) now gives

$$\psi_{t+1}(w') = \int \varphi(w' - (1 + r)s(w))\psi_t(w) dw \quad (7.63)$$

This is the recursive relationship between successive wealth distributions we have been seeking.

We can view (7.63) as a difference equation in densities. If we introduce the Markov operator Π from densities to densities defined by

$$(\psi\Pi)(w') = \int \varphi(w' - (1 + r)s(w))\psi(w) dw \quad (7.64)$$

(7.63) becomes $\psi_{t+1} = \psi_t\Pi$. If \mathcal{D} is the set of all densities on \mathbb{R}_+ , then (Π, \mathcal{D}) is a dynamical system such that its trajectories trace out time paths for the wealth distribution, given some initial wealth distribution ψ_0 .

We will analyze the trajectories of (Π, \mathcal{D}) in stages over the next few sections, but for now one way to obtain insight is by looking at a stochastic version of a 45 degree diagram. The idea is to represent the transition probabilities given in (7.62) as a contour plot, with w on the horizontal axis and w' on the vertical. Such a diagram is shown in figure 7.8, where each y_t is $LN(\mu_y, \sigma_y^2)$, in the sense that $y_t = \exp(\mu_y + \sigma_y z)$ for some standard normal variate z , while the savings function takes the form

$$s(w) = \mathbb{1}\{w > \bar{w}\}s_0w \quad (w \geq 0)$$

Here \bar{w} and s_0 are positive parameters. The interpretation is that the households save nothing until wealth is above \bar{w} . For $w > \bar{w}$, the households save a constant fraction s_0 . The parameters are $r = 0.1$, $\mu_y = \sigma_y = 1.5$, $\bar{w} = 1.0$ and $s_0 = 0.6$.

In figure 7.8, a vertical line segment from point w on the horizontal axis corresponds to the function $w' \mapsto \pi(w, w')$. It therefore gives the transition probabilities in one step, with high values indicating a likely transition point. For low values of w , such as $w = 1$, most of the probabilities mass is above the 45 degree line, implying that next period w' is likely to be higher. When w is large (e.g., $w = 10$), most of the probabilities mass is *below* the 45 degree line, implying that next period w' is likely to be *lower*.

This suggests that the dynamical system (Π, \mathcal{D}) is stable in some sense. Poor house-

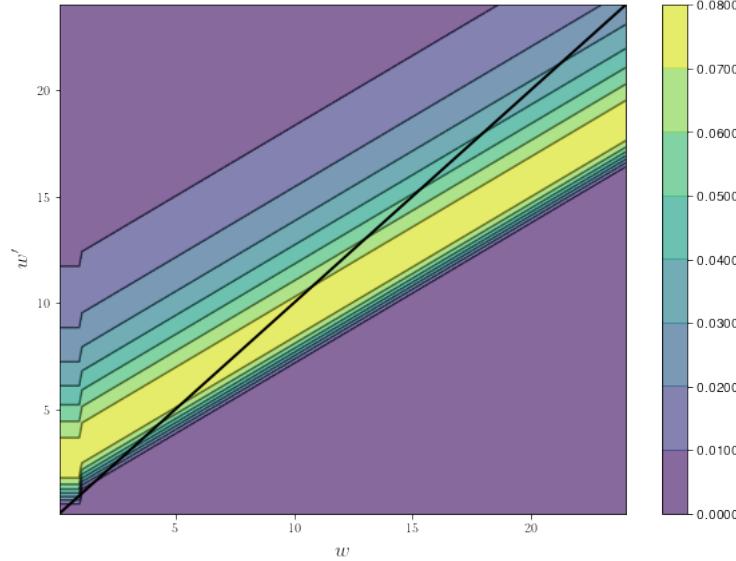


Figure 7.8: Stochastic 45 degree diagram for wealth dynamics

holds tend to gain wealth and rich households tend to lose wealth, so the distribution of wealth neither collapses to zero nor diverges to infinity.

Figure 7.9 shows a different perspective, tracking wealth of an individual household with labor income $\{y_t\}$ drawn IID according to its distribution and parameter values unchanged from figure 7.8. Consistent with our discussion of the dynamics implied by figure 7.8, wealth of the household neither diverges to infinity nor collapses to zero. There are some large spikes in wealth, but these are transitory and driven by the relatively thick right tail of the lognormal labor income distribution.

7.2.6.2 Adding Financial Income Risk

Let's consider again the law of motion for wealth (7.61) but now with the following modification: the constant gross rate of return $R := 1 + r$ on wealth is replaced with an IID process $\{R_t\}$ with distribution ν . As we will see below, this modification substantially improves the fit of the wealth distribution model to the data.

The model is otherwise the same, so that wealth evolves according to

$$w_{t+1} = R_{t+1}s(w_t) + y_{t+1} \quad (7.65)$$

As in §7.2.6.1, shocks will be idiosyncratic in the sense of having independent realizations across households, so we should more correctly write $w_{t+1}^i = R_{t+1}^i s(w_t^i) + y_{t+1}^i$ for evolution of wealth in the i -th household. But let's continue to omit this superscript

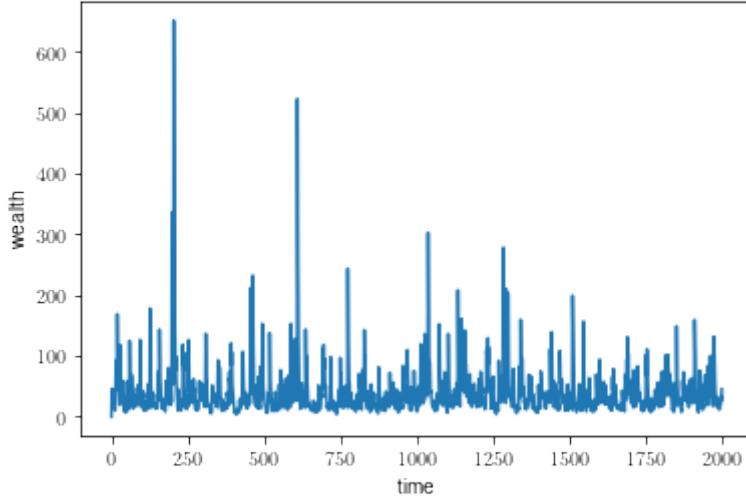


Figure 7.9: Time series of wealth for one household

with the understanding that (7.65) is the law of motion for each household, with its own idiosyncratic shock process $\{R_t^i, y_t^i\}$.

Equation (7.65) is a version of (7.40) on page 240, where we obtained the expression (7.42) for the conditional density of the next period state given the current state. Specializing to (7.65) and assuming that the density of R_t is ν yields our new law of motion for the wealth distribution:

$$(\psi\Pi)(w') = \int \int \varphi(w' - zs(w))\nu(dz)\psi(w) dw \quad (7.66)$$

Prior to a more formal analysis of dynamics, we can again obtain some idea of how the state will evolve by viewing the 45 degree diagram, which in this case is a plot of the conditional density

$$\pi(w, w') := \int \varphi(w' - zs(w))\nu(z) dz \quad (7.67)$$

Such a plot is given in figure 7.10, with the interpretation of the axes and contours being analogous to figure 7.8. In this case, the parameters remain the same as for that figure except that $R_t = (1 + r)\zeta_t$ where each ζ_t is $LN(\mu_r, \sigma_r^2)$. We set $\sigma_r = 1.0$ and $\mu_r = -0.5$, which implies a unit mean. Thus, the mean of R_t is still $1 + r$, but now there is positive variance.

For low values of w we see that the density $w' \mapsto \pi(w, w')$ is similar to the case of non-stochastic returns from figure 7.8. This makes sense because stochastic returns will

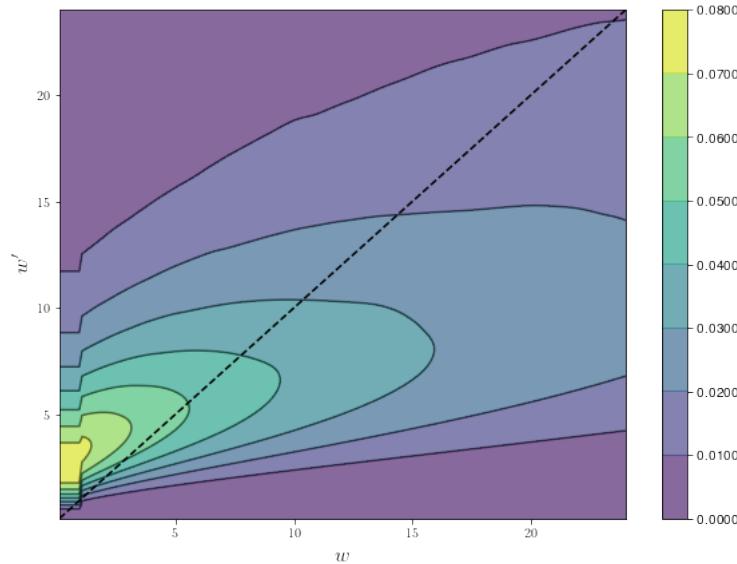


Figure 7.10: Stochastic 45 degree diagram for wealth dynamics

have little effect when wealth is low. However, for high w , the density $w' \mapsto \pi(w, w')$ is considerably more dispersed.

A time series for a single household with stochastic returns on wealth is shown in figure 7.11. Relative to the case of non-stochastic returns shown in figure 7.9, we see larger spikes in wealth. This will impact the cross-sectional distribution of wealth in important ways, as discussed below.

7.3 Chapter Notes

[References, background, historical notes.](#)

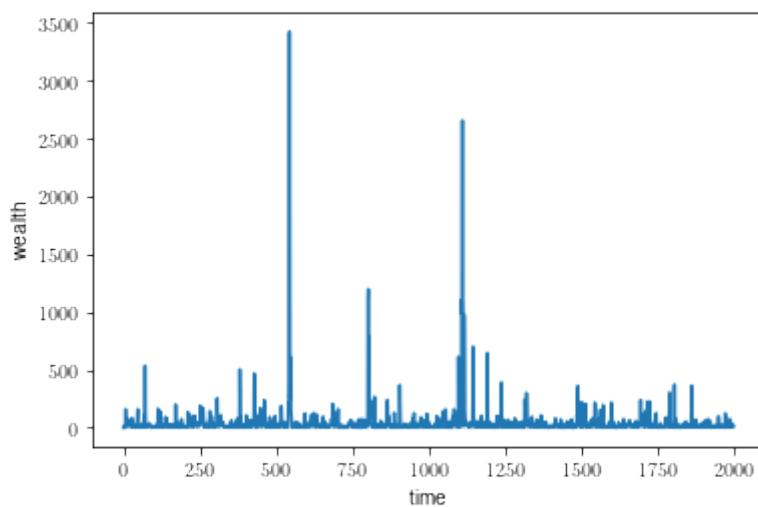


Figure 7.11: Time series of wealth for one household

Chapter 8

Markov Chains in Continuous Time

To be added. Restrict attention to the finite state case in order to get the simplest pairing between transition functions and infinitesimal description. Keep this chapter concrete. Applications include new inventions, productivity processes, inventory models.

8.1 Markov Dynamics

Suggested introduction to the ideas: We have learned that, in discrete time, a Markov model is defined by a positive linear operator P that is invariant on $\mathcal{D}(X)$, the space of distributions on X . As a consequence, iterates of P that start in $\mathcal{D}(X)$ trace out a sequence of distributions. These distributions follow the linear difference equation $\psi_{t+1} = \psi_t P$ and trace out the trajectory of a Markov model from an initial condition $\psi_0 \in \mathcal{D}(X)$. Each ψ_t represent the marginal distributions of a Markov chain over X . When X is finite, P is just a matrix with nonnegative entries and unitary row sums.

If a discrete time Markov model is a linear difference equation that evolves in distribution space, then a continuous time Markov model must be a linear differential equation that evolves in distribution space. For such a model, distributions evolve according to $\psi'(t) = \psi(t)Q$ for some matrix Q . The matrix Q must have structure because its trajectories need to remain within the space of distributions. Once we specify this structure, we obtain the full class of continuous time Markov processes on a finite state.

8.1.1 Markov Semigroups

A **Markov semigroup** $\{p_t\}$ on X is a family of Markov kernels on X such that p_0 is the identity and

$$p_{j+k}(x, y) = \sum_z p_k(x, z)p_j(z, y) \quad ((x, y) \in \mathsf{X} \times \mathsf{X}) \quad (8.1)$$

This is the **Chapman–Kolmogorov equation** in another guise. It is also the source of the term “semigroup.” In fact, if P_t is the Markov operator corresponding to p_t for each t , then $(\mathcal{D}(\mathsf{X}), \{P_t\}_{t \geq 0})$ is a continuous time semidynamical system. For a given distribution $\varphi \in \mathcal{D}(\mathsf{X})$, we understand $\varphi_t := \varphi P_t$ as the marginal distribution of a continuous time Markov process $\{X_t\}$ on X , and the orbit $t \mapsto \varphi P_t$ is the flow of distributions over time.

Over the remainder of the chapter we will show how to construct the Markov process $\{X_t\}$ corresponding to a Markov semigroup, and how to simulate it.

8.1.2 Example: Inventory Dynamics

Orders arrive at an exponential rate λ . Each order is for D units, where D is a nonnegative integer.

Obtain the Markov semigroup by conditioning.

Run a simulation.

8.1.3 Infintessimal Generators

Introduce the concept of a Q matrix and show how to reconstruct the semigroup. Show the reverse process and the exact connection.

Connect with linear ODEs in Euclidean space.

Provide some more examples. Discuss algorithms and simulation.

8.2 Asymptotics

8.2.1 Stationarity

Notions of stationarity and ergodicity extend or carry over naturally from the discrete time case. For example, a distribution $\psi^* \in \mathcal{D}(\mathsf{X})$ is called **stationary** for a Markov semigroup $\{p_t\}$ if it is stationary for each kernel p_t .

[Discuss existence and uniqueness. Connect to infinitesimal generator.]

8.2.2 Ergodicity

8.2.3 Applications

New inventions, productivity processes, inventory models.

8.3 Chapter Notes

References, background, historical notes.

Chapter 9

Optimal Saving

[roadmap]

9.1 Optimal Saving and Consumption

[roadmap]

9.1.1 An Optimal Saving Model

[roadmap]

9.1.1.1 Consumption and the Law of Motion

An decision maker chooses a consumption path $\{c_t\}$ to maximize

$$\mathbb{E} \left[\sum_{t=0}^{\infty} \beta^t u(c_t) \right] \quad (9.1)$$

where u is a utility function and β is a discount factor satisfying $0 < \beta < 1$. Maximization is subject to a law of motion of the form

$$x_{t+1} = g(x_t, c_t, \xi_{t+1}) \text{ with } x_0 \text{ given and } \{\xi_t\} \stackrel{\text{IID}}{\sim} \varphi \quad (9.2)$$

In addition, consumption is required to satisfy the **feasibility constraint**

$$c_t \in \Gamma(x_t) \subset \mathbb{R}_+ \quad (9.3)$$

In this context, x_t is called the **state** variable and we allow it to take values in some arbitrary metric space \mathbf{X} . It can be thought of as assets and other variables that constrain consumption and determine evolution of the state. The shock sequence $\{\xi_t\}$ takes values in metric space E . Typically the state and **innovation process** $\{\xi_t\}$ will be scalar or vector valued and the topology is the ordinary Euclidean topology. Another common case is where these spaces are discrete, in which case the topology is the discrete topology.

Admissible consumption paths are also required to be **adapted** to the history $\mathcal{F}_t := \{x_j\}_{j \leq t}$, so that c_t is a function of past and present realizations of the state. In particular, c_t may not depend on outcomes yet to be observed.

Assumption 9.1.1. The following conditions hold:

- (i) The utility function u is a continuous, strictly concave and strictly increasing function from \mathbb{R}_+ to $\{-\infty\} \cup \mathbb{R}$.
- (ii) The function g is everywhere continuous.
- (iii) The correspondence Γ is nonempty, compact valued and continuous.

The statement that Γ is compact valued simply means that $\Gamma(x)$ is closed and bounded in \mathbb{R}_+ for each x . Continuity of correspondences is defined in §5.1.3.1. For now it suffices to state that Γ will be continuous if it takes the form $\Gamma(x) = [0, \tau(x)]$ where τ is a continuous real valued function on \mathbf{X} .

We refer to the model $(\beta, u, g, \varphi, \Gamma)$ described in (9.1)–(9.3) as the **generic optimal saving model**.

9.1.1.2 Example: Optimal Growth

One interpretation of the generic optimal saving model $(\beta, u, g, \varphi, \Gamma)$ is the optimal growth model set out in [Brock and Mirman \(1972\)](#), where a representative agent owns an amount $k_t \in \mathbb{R}_+$ of capital and uses it to produce output via

$$y_t := f(k_t, z_t) \tag{9.4}$$

Here f is called the **production function** and $\{z_t\}$ is an **exogenous productivity process**. Some of current output is invested and some is consumed. Consumption is chosen to maximize $\mathbb{E}[\sum_{t=0}^{\infty} \beta^t u(c_t)]$. The invested portion becomes next period's capital stock k_{t+1} . In particular, the resource constraint is

$$0 \leq k_{t+1} + c_t \leq y_t \tag{9.5}$$

This combined with the production function leads to the law of motion for capital

$$k_{t+1} = f(k_t, z_t) - c_t \quad (9.6)$$

We are assuming here that the second inequality in (9.5) always binds, as it will in any optimal path given the assumptions on agent preferences we make below. We assume further that the productivity takes values in a set Z and obeys

$$z_{t+1} = G(z_t, \varepsilon_{t+1}) \quad (9.7)$$

where $\{\varepsilon_t\}$ is IID.

This model maps to the generic setting if we set $x_t := (k_t, z_t)$, which takes values in state space $X := \mathbb{R}_+ \times Z$, and $\xi_t = \varepsilon_t$. The general law of motion (9.2) is then determined by (9.6) and (9.7). The constraint $\Gamma(x_t)$ on consumption becomes $[0, f(k_t, z_t)]$, implying that $0 \leq c_t \leq f(k_t, z_t)$. The distribution φ in $(\beta, u, g, \varphi, \Gamma)$ is the distribution of ε_t .

[What do the conditions in assumption 9.1.1 imply for BM?]

Remark 9.1.1. Another interpretation of the model just presented is that k_t as a stock of a renewable resource and c_t as the harvest of a single agent who interacts with this resource.

Remark 9.1.2. While (9.6) is similar to our formulation of capital dynamics in the Solow–Swan growth model, the undepreciated capital term $(1 - \delta)k_t$ is missing. However, depreciation can still be included in the present set up, by folding it into the production function after a change of timing. In particular, imagine that depreciation occurs between the start of the period and production, so that output is $\tilde{f}((1 - \delta)k_t, z_t)$. Now set $f(k_t, z_t) := \tilde{f}((1 - \delta)k_t, z_t)$.

9.1.1.3 Example: Household Wealth

Another interpretation of the generic optimal saving model $(\beta, u, g, \varphi, \Gamma)$ set out in (9.1)–(9.3) is the model of household wealth dynamics

$$w_{t+1} = (1 + r_{t+1})(w_t - c_t) + y_{t+1} \quad (9.8)$$

we first considered in §2.2.4. Here w_t is household assets at t , c_t is consumption, y_{t+1} is non-financial income received and $r_{t+1} > 0$ is the rate of return on financial assets. Consumption is chosen to maximize $\mathbb{E} [\sum_{t=0}^{\infty} \beta^t u(c_t)]$.

Assume as before that both labor income and the interest rate are functions $y_t = y(z_t, \eta_t)$ and $r_t = r(z_t, \zeta_t)$ of some underlying state process $\{z_t\}$. This model maps

to the generic optimal saving model $(\beta, u, g, \varphi, \Gamma)$ if we set $x_t := (w_t, z_t)$, which takes values in state space $\mathbf{X} := \mathbb{R}_+ \times \mathbb{Z}$. Suppose that $\{z_t\}$ obeys

$$z_{t+1} = G(z_t, \varepsilon_{t+1}) \quad (9.9)$$

For the shock process in the general law of motion (9.2) we can take ξ_t to be the vector of innovations. That is, $\xi_t = (\eta_t, \zeta_t, \varepsilon_t)$. The general law of motion g in (9.2) is then

$$g((w, z), c, \xi) = \begin{pmatrix} (1 + r(z, \zeta))(w - c) + y(z, \eta) \\ G(z, \varepsilon) \end{pmatrix}$$

If we constrain wealth to be nonnegative, then it is natural to set $\Gamma(x_t) = [0, w_t]$, so that $0 \leq c_t \leq w_t$. The distribution φ in $(\beta, u, g, \varphi, \Gamma)$ is the joint distribution of $(\eta_t, \zeta_t, \varepsilon_t)$.

[What do the conditions in assumption 9.1.1 imply for BM?]

9.1.2 Policies and Optimality

Let's continue our analysis of the generic optimal saving model $(\beta, u, g, \varphi, \Gamma)$.

[roadmap]

9.1.2.1 Policy Functions

The statement in §9.1.1.1 that consumption choices should be adapted to the state process means that, at each point in time t , we have

$$c_t = \sigma_t(x_0, x_1, \dots, x_t)$$

for some suitable function σ_t . The function σ_t is called a **policy function**, being a map from past and present observables into current action. In what follows we focus exclusively on **stationary Markov policies**, which are time-invariant maps from the *current state* x_t into a current action c_t . These policies are called Markov policies because, as we shall see, they generate first order Markov processes for the state.

Remark 9.1.3. The current focus on stationary Markov policies is in fact the same approach that we've taken in all previous infinite horizon applications. The preceding paragraph simply reinforces and clarifies this point. As proved in §10.1.3.4, for all of the infinite horizon dynamic programming problems we consider, the optimal policy is a stationary Markov policy. In other words, the current state provides a sufficient statistic for the history in terms of making an optimal decision today. This is intuitive:

given that we always look towards an infinite future and discount at the same rate, we should make the same decision if we visit a given state x at different points in time.

In our context, a stationary Markov policy is a function σ mapping X to \mathbb{R}_+ , understood as a mapping from states into actions:

$$c_t = \sigma(x_t) \quad \text{for all } t \geq 0$$

In what follows, we will call σ a **feasible consumption policy** if it is Borel measurable and satisfies

$$\sigma(x) \in \Gamma(x) \quad \text{for all } x \in \mathsf{X} \tag{9.10}$$

In essence, a feasible consumption policy is a stationary Markov policy that respects the resource constraint. Borel measurability is used only to ensure that we can compute expectations involving this function. The set of all feasible consumption policies will be denoted by Σ .

Each $\sigma \in \Sigma$ determines a continuous state Markov process $\{x_t\}$ via

$$x_{t+1} = g(x_t, \sigma(x_t), \xi_{t+1}) \tag{9.11}$$

This is the time path for the state when we choose and stick with policy σ for all time.

If we fix a particular policy $\sigma \in \Sigma$, take the state process (9.11) generated by σ and insert it into the objective function we get

$$v_\sigma(x_0) := \mathbb{E} \left[\sum_{t=0}^{\infty} \beta^t u(\sigma(x_t)) \right] \tag{9.12}$$

This is the total expected present value of following policy σ forever, given initial state x_0 . We call v_σ the **σ -value function**.

Now that we have a well defined notion of value, we can turn to optimality.

9.1.2.2 Optimality

The aim is to select a policy that maximizes lifetime value. In particular, a consumption policy σ^* is called **optimal** if it is feasible and

$$v_{\sigma^*}(x) = v^*(x) \quad \text{for all } x \in \mathsf{X}, \quad \text{where } v^*(x) := \sup_{\sigma \in \Sigma} v_\sigma(x)$$

The function v^* is called the **value function**.

In line with our previous applications, the first step in solving our problem is to pin down the value function. In doing so we exploit the fact that, in most settings, the value function satisfies the Bellman equation, which now takes the form

$$v(x) = \max_{c \in \Gamma(x)} \left\{ u(c) + \beta \int v(g(x, c, z)) \varphi(dz) \right\} \quad (x \in \mathbb{X}) \quad (9.13)$$

This is a functional equation in v , and we can restate the foregoing by saying that, in most settings, v^* is a solution to (9.13), and in fact the only solution within a significant class of functions. Exactly what that class is depends on the primitives and will be addressed in stages below.

As for the other Bellman equations we have met, the intuition behind (9.13) is that maximal value from a given state can be obtained by optimally trading off current reward from a given action versus expected discounted future value of the state resulting from that action.

The Bellman equation is important because it gives us more information about the value function. The value function is important because it leads us to optimal policies, as shown in the following proposition.

Proposition 9.1.1. *Let $(\beta, u, f, \varphi, \Gamma)$ be a generic optimal saving model in which the conditions of assumption 9.1.1 hold. If, in addition, u is bounded, then*

- (i) *v^* is the unique solution to the Bellman equation (9.13) in $b\mathbb{X}$.*
- (ii) *A feasible consumption policy σ is optimal if and only if*

$$\sigma(x) \in \operatorname{argmax}_{c \in \Gamma(x)} \left\{ u(c) + \beta \int v^*(g(x, c, z)) \varphi(dz) \right\} \quad \text{for all } x \in \mathbb{X} \quad (9.14)$$

- (iii) *At least one such policy exists.*

Proof. Part (i) of proposition 9.1.1 will be established in greater generality in theorem 10.2.3 on page 304. The characterization of optimality in part (ii) is just a restatement of Bellman's principle of optimality in the context of optimal saving, the validity of which is established in a general setting in §10.1.3.3. Part (iii) will be proved in greater generality in §10.3.1.1 (see also lemma 10.1.1 on page 299). \square

There is one rather strict assumption in proposition 9.1.1: the boundedness restriction on u . In practice most utility functions in common use are unbounded. We revisit this point after proposition 9.1.2.

Given a real valued function v on X , we say that $\sigma \in \Sigma$ is **v -greedy** if

$$\sigma(x) \in \operatorname{argmax}_{c \in \Gamma(x)} \left\{ u(c) + \beta \int v(g(x, c, z)) \varphi(dz) \right\} \quad (9.15)$$

for all $x \in \mathsf{X}$. This parallels the definition of greedy we gave for finite state Markov decision processes in §3.3.3.

With the terminology of greedy policies in hand, we can restate part (ii) of proposition 9.1.1 by saying that a feasible policy is optimal if and only if it is v^* -greedy. This is exactly Bellman's principle of optimality.

We started with one optimization problem—choosing an optimal consumption path c_0, c_1, \dots to maximize expected discounted lifetime utility—and ended up with another one—finding a greedy policy from the value function. Are we actually better off? The answer is: yes! Finding a greedy policy is a scalar optimization problem performed for each state x , whereas as our previous optimization problem was infinite dimensional. High dimensionality is the mountain we must climb in all hard optimization problems and here we have used the recursive structure inherent in this problem to map a route up to the top.

Of course this claim that we are better off is contingent on us being able to learn what the value function is, so that we can compute v^* -greedy policies. In general we can use value function iteration via the Bellman operator

$$Tv(x) = \max_{c \in \Gamma(x)} \left\{ u(c) + \beta \int v(g(x, c, z)) \varphi(dz) \right\} \quad (9.16)$$

For example, we have the following result:

Proposition 9.1.2. *If the conditions of proposition 9.1.1 hold, then T defined in (9.16) is a contraction of modulus β on $(bc\mathsf{X}, d_\infty)$. Its unique fixed point in $bc\mathsf{X}$ is the value function v^* .*

We will prove a generalization of proposition 9.1.2 in §10.3.1.1. Nevertheless, much of the proof is outsourced via citations, so let's work through some details.

Proof of proposition 9.1.2. First we need to confirm that T is a self-map on $bc\mathsf{X}$.

Clearly Tv is bounded on X whenever $v \in bc\mathsf{X}$, since, for any such v and any feasible pair c, x , the triangle inequality and monotonicity of expectations gives

$$\left| u(c) + \beta \int v(g(x, c, z)) \varphi(dz) \right| \leq \|u\|_\infty + \beta \|v\|_\infty$$

To see that T preserves continuity, pick $v \in bc\mathbf{X}$ and observe that, by Berge's theorem of the maximum (see theorem 5.1.19 on page 152), the function Tv will be continuous whenever

$$q(x, c) := u(c) + \beta \int v(g(x, c, z)) \varphi(dz) \quad (9.17)$$

is continuous on $\mathsf{G} := \{(x, c) \in \mathbf{X} \times \mathbb{R}_+ : c \in \Gamma(x)\}$. Since u is already assumed to be continuous and since sums and scalar multiples of continuous functions are continuous, it suffices to show that, for any given (x, c) in G ,

$$\int v(g(x_n, c_n, z)) \varphi(dz) \rightarrow \int v(g(x, c, z)) \varphi(dz) \quad (9.18)$$

as $n \rightarrow \infty$ whenever (x_n, c_n) converges to (x, c) .

As v and f are continuous by assumption, we have $v(g(x_n, c_n, z)) \rightarrow v(g(x, c, z))$ for any given z . To pass the limit through the integral, yielding (9.18), we can appeal to the dominated convergence theorem (page 339), with the constant $\|v\|_\infty$ as the dominating function.

Finally, we need to show that T is a contraction. To this end, let v and w be elements of $bc\mathbf{X}$. Fix $x \in \mathbf{X}$. By the sup inequality in lemma 5.1.17 (page 151) and the triangle inequality, we have

$$\begin{aligned} |Tv(x) - Tw(x)| &\leq \max_{c \in \Gamma(x)} \beta \left| \int v(g(x, c, z)) \varphi(dz) - \int w(g(x, c, z)) \varphi(dz) \right| \\ &\leq \max_{c \in \Gamma(x)} \beta \int |v(g(x, c, z)) - w(g(x, c, z))| \varphi(dz) \end{aligned}$$

The last term is dominated by $\beta \|v - w\|_\infty$ due to monotonicity of expectations. Taking the supremum over all $x \in \mathbf{X}$ now gives

$$\|Tv - Tw\|_\infty \leq \beta \|v - w\|_\infty \quad \square$$

Remark 9.1.4. Like proposition 9.1.1, proposition 9.1.2 suffers from one significant shortcoming: The utility function is assumed to be bounded. In practice almost all of the utility functions used in applied studies are unbounded—either above, below, or both. In the unbounded setting proposition 9.1.2 breaks down because T is no longer a self-mapping on $bc\mathbf{X}$. Instead its images are unbounded and the supremum norm is in general infinite (i.e., not defined and not a norm). Later we shift our focus to dynamic programming problems with unbounded rewards. We will find there is unfortunately no one overarching theory that can handle all cases of interest. Instead we must work on a case-by-case basis, exploiting whatever structure we can find in a

given application.

9.1.3 Problems with Analytical Solutions

For a small subset of optimal saving problems, both the optimal policy and the value function have known analytical solutions. Although these models are limited and simplistic, they are helpful for building intuition and testing numerical algorithms. Let's look at some of the best known cases.

9.1.3.1 Cake Eating with Interest

Suppose we wish to choose a consumption path to maximize

$$\sum_{t=0}^{\infty} \beta^t u(c_t) \quad (9.19)$$

with

$$u(c) := \frac{c^{1-\gamma}}{1-\gamma} \quad (\gamma > 0, \gamma \neq 1) \quad (9.20)$$

and subject to $0 \leq c_t \leq w_t$ and

$$w_{t+1} = R(w_t - c_t) \quad (9.21)$$

Here R is a gross interest rate, c_t denotes the consumption at period t , and w_t is period t wealth level. The agent is endowed with w_0 at time zero and nothing thereafter. We assume throughout that $\beta R^{1-\gamma} < 1$.

As a dynamic programming problem, this set up can be mapped to the generic optimal saving model $(\beta, u, g, \varphi, \Gamma)$ set out in (9.1)–(9.3): take $x_t = w_t$, $g(x, c, \xi) = R(x - c)$ and $\Gamma(x) = [0, x]$. The shock distribution φ is degenerate.

It turns out that, for the CRRA problem, the optimal consumption policy is linear in w . That is,

$$\text{there exists a constant } \theta \text{ such that } c^*(w) = \theta w \text{ is the optimal policy} \quad (9.22)$$

In the remainder of this section we verify this claim and, at the same time, seek the value of the constant θ .

First, observe that if (9.22) holds, then

$$w_t = R^t(1 - \theta)^t w \quad \text{when } w_0 = w$$

and hence the value function v^* satisfies

$$\begin{aligned} v^*(w) &= \sum_t \beta^t u(\theta w_t) = \sum_t \beta^t u(\theta R^t (1 - \theta)^t w) \\ &= \sum_t \beta^t (\theta R^t (1 - \theta)^t)^{1-\gamma} u(w) = \frac{\theta^{1-\gamma}}{1 - \beta (R(1 - \theta))^{1-\gamma}} u(w) \end{aligned}$$

Our conjecture is that the linear policy $c^*(w) = \theta w$ satisfies the Bellman equation with the value function as given above. Under this conjecture, the Bellman equation takes the form

$$v^*(w) = \max_c \left\{ \frac{c^{1-\gamma}}{1-\gamma} + \beta \cdot \frac{\theta^{1-\gamma}}{1 - \beta (R(1 - \theta))^{1-\gamma}} \cdot \frac{(R(w - c))^{1-\gamma}}{1-\gamma} \right\} \quad (9.23)$$

Taking the derivative with respect to c yields the first-order condition

$$c^{-\gamma} + \beta m (R(w - c))^{-\gamma} (-R) = 0 \quad \text{when } m := \frac{\theta^{1-\gamma}}{1 - \beta (R(1 - \theta))^{1-\gamma}}$$

It then follows that $c^{-\gamma} = \beta m R^{1-\gamma} (w - c)^{-\gamma}$. Substituting the optimal policy $c^*(w) = \theta w$ into this equality gives us

$$(\theta w)^{-\gamma} = \frac{\beta R^{1-\gamma} \theta^{1-\gamma}}{1 - \beta (R(1 - \theta))^{1-\gamma}} (1 - \theta)^{-\gamma} w^{-\gamma}$$

Now solving the above equality for θ yields

$$\theta = 1 - (\beta R^{1-\gamma})^{1/\gamma} \quad (9.24)$$

In this connection, given any initial wealth w , the value function becomes

$$\begin{aligned} v^*(w) &= \frac{\theta^{1-\gamma}}{1 - \beta (R(1 - \theta))^{1-\gamma}} u(w) \\ &= \frac{\left(1 - (\beta R^{1-\gamma})^{1/\gamma}\right)^{1-\gamma}}{1 - \beta R^{1-\gamma} (\beta R^{1-\gamma})^{\frac{1-\gamma}{\gamma}}} u(w) = \theta^{-\gamma} u(w) \end{aligned}$$

It is not difficult to verify that $v^*(w) = \theta^{-\gamma} u(w)$ solves the Bellman equation (9.23) for any w .

9.1.3.2 Log Utility and Cobb–Douglas Production

Next let's review a well known example treated in [Brock and Mirman \(1972\)](#) and [Ljungqvist and Sargent \(2012\)](#), section 3.1.2, where the utility function u is given by $u(c) = \ln c$ and the production function has the Cobb–Douglas form

$$f(k) = Ak^\alpha, \quad 0 < A, \quad 0 < \alpha < 1$$

Let $\{z_t\}$ be a lognormal IID sequence, so that $\ln z_t \stackrel{d}{=} N(\mu, \sigma^2)$ for some $\mu \in \mathbb{R}$ and $\sigma > 0$. The state can be chosen to be

$$y_{t+1} = f(y_t - c_t)z_{t+1} = A(y_t - c_t)^\alpha z_{t+1}$$

The agent maximizes

$$\mathbb{E} \sum_{t \geq 0} \beta^t \ln c_t$$

EXERCISE 1. Conjecture that the optimal policy is linear in income y , so that there exists a positive constant θ such that $c^*(y) = \theta y$ is optimal. Following the steps in §9.1.3.1,

- (i) find the value of θ ,
- (ii) obtain an expression for the value function and
- (iii) confirm that the value function satisfies the Bellman equation.

9.1.4 CRRA Utility and Stochastic Financial Returns

[roadmap]

9.1.4.1 Model and Solution

[Toda \(2018\)](#) studies a more sophisticated version of the optimal saving problem with CRRA preferences analyzed in §9.1.3.1, where the agent maximizes

$$\mathbb{E} \sum_{t=0}^{\infty} \beta(z_t)^t u(c_t) \tag{9.25}$$

with, as before,

$$u(c) := \frac{c^{1-\gamma}}{1-\gamma} \quad (\gamma > 0, \gamma \neq 1) \tag{9.26}$$

Notice that the discount factor is now state dependent, depending on an exogenous state process $\{z_t\}$.

The second generalization relative to the model studied in §9.1.3.1 is that the gross interest rate $R(z_t)$ is stochastic. In particular, the law of motion for wealth is

$$w_{t+1} = R(z_t)(w_t - c_t) \quad (9.27)$$

where $\{z_t\}$ is a Markov chain on finite set Z with stochastic kernel Π . He assumes that

$$\beta(z) > 0 \text{ and } R(z) > 0 \text{ for all } z \in Z$$

The stochastic kernel Π is assumed to be everywhere positive. (It suffices in fact that Π is irreducible but positivity makes the proofs easier.)

The Bellman equation is now

$$v(w, z) = \max_{0 \leq c \leq w} \left\{ u(c) + \beta(z) \sum_{z' \in Z} v[R(z)(w - c), z'] \Pi(z, z') \right\} \quad (9.28)$$

for all $(w, z) \in X := \mathbb{R}_+ \times Z$. Let K be the square matrix defined by

$$K(z, z') = \beta(z)R(z)^{1-\gamma}\Pi(z, z') \quad ((z, z') \in Z \times Z) \quad (9.29)$$

[Toda \(2018\)](#) shows that if the spectral radius of K satisfies $r(K) < 1$, then the optimal saving problem stated above has the optimal policy

$$c^*(w, z) = g^*(z)^{-1/\gamma}w \quad ((w, z) \in X) \quad (9.30)$$

and the value function satisfies

$$v^*(w, z) = g^*(z) \frac{w^{1-\gamma}}{1-\gamma} \quad ((w, z) \in X) \quad (9.31)$$

where $g^* := (g^*(z))_{z \in Z}$ is the smallest strictly positive vector in \mathbb{R}^Z satisfying

$$g(z) = \{1 + [Kg(z)]^{1/\gamma}\}^\gamma \quad (z \in Z) \quad (9.32)$$

Here $Kg(z)$ is the product $\sum_{z'} g(z')K(z, z')$. [Toda \(2018\)](#) proves existence of such a vector under the condition $r(K) < 1$ and also shows necessity of this same condition for existence of a solution.

EXERCISE 2. Confirm that, if $\beta(z) \equiv \beta$ and $R(z) \equiv R$ for positive constants R and β , then the optimal policy and value function found in (9.30) and (9.31) reduce to

those obtained in §9.1.3.1.

EXERCISE 3. Continuing Exercise 2, show that we have $r(K) < 1$ if and only if $\beta R^{1-\gamma} < 1$ when β and R are constant.

Let's establish the result in Toda (2018) that there exists a strictly positive vector satisfying (9.32) under the condition $r(K)$. We show in addition that there is only one strictly positive vector satisfying (9.32) when $r(K) < 1$, so the “smallest” qualification above (9.32) can be dropped.

We adopt the standard pointwise partial order \leqslant on $\mathbb{R}^{\mathbb{Z}}$ (see page 141 for background). Recall that an operator T mapping $\mathbb{R}^{\mathbb{Z}}$ to itself is called isotone if $g \leqslant h$ implies $Tg \leqslant Th$. In addition, we write $g \ll h$ if $g(z) < h(z)$ for all z and $g \gg h$ if $h \ll g$.

To analyze (9.32), let ψ be the scalar map defined by

$$\psi(t) := (1 + t^{1/\gamma})^{\gamma} \quad (t \geq 0) \quad (9.33)$$

Consider the operator S mapping the interior of the positive cone of $\mathbb{R}^{\mathbb{Z}}$, which is

$$C = \mathbb{R}^{\mathbb{Z}}_+ = \{g \in \mathbb{R}^{\mathbb{Z}} : g \geq 0\}$$

to itself via $S = \psi \circ K$, or, more explicitly,

$$Sg(z) = \psi(Kg(z)) \quad (g \in C) \quad (9.34)$$

At this point you should be able to confirm that S maps C to itself and that $g \in C$ solves (9.32) if and only if it is a fixed point of S . Our aim is to prove the following:

Proposition 9.1.3. *If $r(K) < 1$, then (C, S) is globally stable, with unique fixed point $g^* \gg 0$ satisfying*

$$\lim_{n \rightarrow \infty} \|S^n g - g^*\| = 0 \text{ whenever } g \in C \quad (9.35)$$

The norm in (9.35) can be any norm on $\mathbb{R}^{\mathbb{Z}}$.¹

We will prove proposition 9.1.3 using corollary 5.1.15 from the appendix, which requires that, in the present setting

- (i) C is a sublattice of $\mathbb{R}^{\mathbb{Z}}$

¹Recall that on finite dimensional vector space, all norms are equivalent (see theorem 11.1.5 on page 321). Hence, if the convergence in (9.35) holds for some norm, then it holds for all.

- (ii) For all $g \in C$, there exists a point $\check{g} \in C$ with $\check{g} \leq g$ and $S\check{g} \gg \check{g}$.
- (iii) For all $g \in C$, there exists a point $\hat{g} \in C$ with $\hat{g} \geq g$ and $S\hat{g} \ll \hat{g}$.
- (iv) S is isotone.
- (v) S is either concave or convex.

EXERCISE 4. Show that ψ defined in (9.33) is convex on \mathbb{R}_+ whenever $0 < \gamma \leq 1$ and concave on \mathbb{R}_+ whenever $\gamma > 1$.

EXERCISE 5. Using the results in exercise 4, show that $S = \psi \circ K$ is convex on C whenever $0 < \gamma \leq 1$ and concave on C whenever $\gamma > 1$. (The definition of concavity and convexity of operators is given in §11.1.5.)

EXERCISE 6. Show that S is isotone on C and $S0 \gg 0$.

Let e be the **dominant eigenvector** of the everywhere positive matrix K in the sense of the Perron–Frobenius theorem. In particular, $e \gg 0$ and $Ke = \lambda e$ when $\lambda := r(K)$. In this context, λ is also called the **dominant eigenvalue**.

EXERCISE 7. Let α be a positive constant and let $\mathbb{1}$ be a vector of ones. Show that

$$\alpha e \gg \left(\frac{1}{1 - \lambda^{1/\gamma}} \right)^\gamma \mathbb{1} \implies S(\alpha e) \ll \alpha e \quad (9.36)$$

Proof of Proposition 9.1.3. In view of Exercises 4–6, to complete the proof of Proposition 9.1.3, we need only show that, for each $g \in C$, there exists a $\hat{g} \geq g$ such that $S\hat{g} \ll \hat{g}$.

So fix such a g and choose α such that (a) the bound in (9.36) holds (which is possible because $e \gg 0$ and $\lambda < 1$) and (b) $\alpha e \geq g$. For $\hat{g} := \alpha e$, we have $\hat{g} \geq g$ and

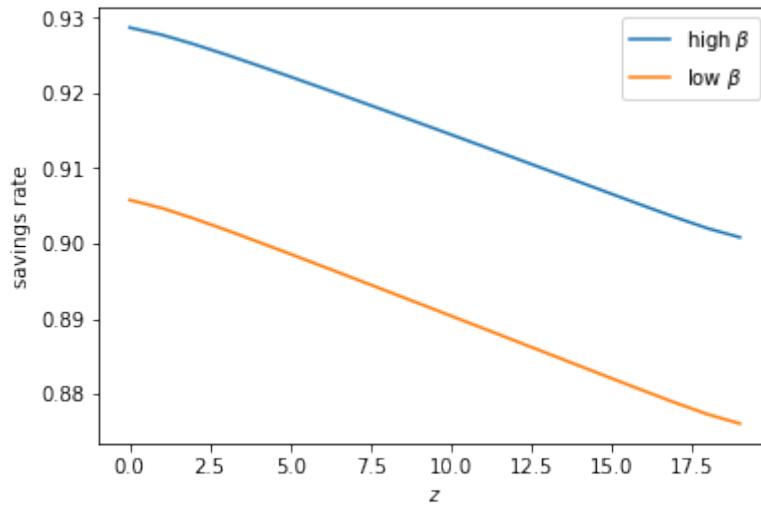
$$S\hat{g} = S(\alpha e) \ll \alpha e =: \hat{g}$$

the proof of Proposition 9.1.3 is now done. □

9.1.4.2 Parametric Monotonicity Results

Let's investigate how the optimal saving rate

$$s(z) := 1 - g^*(z)^{-1/\gamma} \quad (z \in \mathbb{Z}) \quad (9.37)$$

Figure 9.1: Savings rate as a function of z

varies with parameters. Figure 9.1 shows the saving rate at two different specifications β and $\hat{\beta}$ of the state dependent discount function. They satisfy $\beta \leq \hat{\beta}$ and the saving rate associated with $\hat{\beta}$ is higher. This is intuitive: A household who is more patient in every state should save more in every state.

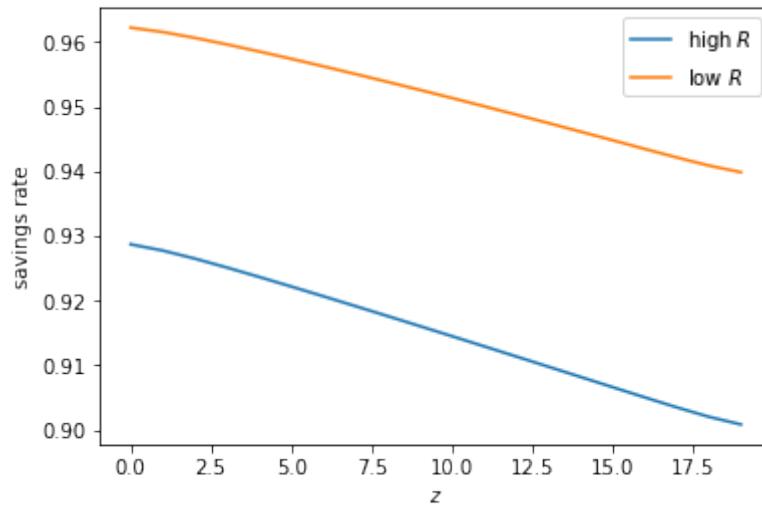
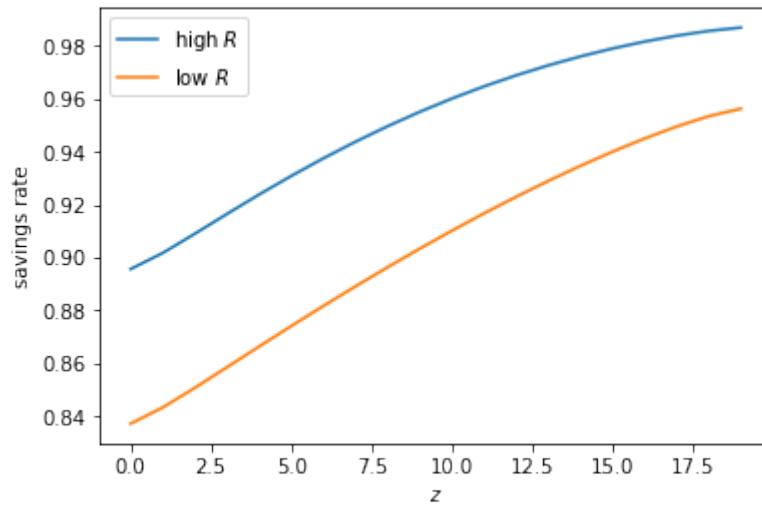
Figures 9.2 and 9.2 show the saving rate at two different specifications R and \hat{R} of the state dependent interest rate. They satisfy $R \leq \hat{R}$. In the first case, where $\gamma = 4$, a higher interest rate leads to *lower saving* in every state. In the second case, where $\gamma = 0.5$, a higher interest rate leads to *higher saving* in every state.

What is the intuition behind this outcome? In general, a higher return on wealth in an optimal saving model encourages both saving and consumption. Savings is encouraged by the increase in the return on savings (the substitution effect). Consumption is boosted by the rise in lifetime wealth associated with an increase in returns to savings (the income effect). As a result, the impact on the saving rate is ambiguous.

In the present, setting, however, increased savings must be funneled to future income through risky savings, due to the stochastic interest rate. Hence high risk aversion depresses the attractiveness of saving. In particular, high γ causes the income effect to dominate the substitution effect. When γ is low, the reverse is true.

The next two exercises ask you to formalize these ideas.

EXERCISE 8. Let β and $\hat{\beta}$ be strictly positive elements of \mathbb{R}^Z with $\beta \leq \hat{\beta}$. Let $s(z) = 1 - g^*(z)^{-1/\gamma}$ be the optimal saving rate under β and let \hat{s} be the optimal saving rate under $\hat{\beta}$. Show that $s \leq \hat{s}$ on Z , stating any additional conditions as you require them.

Figure 9.2: Saving rate as a function of z when $\gamma = 4$ Figure 9.3: Saving rate as a function of z when $\gamma = 0.5$

EXERCISE 9. Let R and \hat{R} be strictly positive elements of $\mathbb{R}^{\mathbb{Z}}$ with $R \leq \hat{R}$. Let s and \hat{s} be the optimal saving rate under R and \hat{R} respectively. Show that

- (i) $0 < \gamma < 1$ implies $s \leq \hat{s}$ and
- (ii) $1 < \gamma$ implies $\hat{s} \leq s$.

State any additional conditions as you require them.

9.2 A Model with Independent Shocks

Let's think about how analysis can proceed when we do not have access to analytical solutions. As a starting point, we will consider a simplified model where shocks are IID. Doing so will allow us investigate some key ideas with minimal distractions. These ideas can then be extended on a case-by-case basis.

9.2.1 Analysis

[roadmap]

9.2.1.1 A Change of State

Consider an optimal growth model where shocks are multiplicative and IID. In particular,

$$y_t = f(k_t)z_t = k_{t+1} + c_t \quad (9.38)$$

where $\{z_t\}$ is positive and IID with common distribution φ . We can express the law of motion (9.6) in terms of $\{y_t\}$ rather than (k_t, z_t) . In particular, by (9.38), we have

$$y_{t+1} = f(y_t - c_t)z_{t+1} \quad (9.39)$$

Although $\{z_t\}$ still appears in this expression, it only enters as a future value, and since the process is IID by assumption, z_t gives no help in predicting it. As a result, $\{y_t\}$ becomes the state process, in the sense that it is a sufficient statistic for choosing optimal consumption.²

What we have described is clearly a version of the generic optimal growth model set out in §9.1.1.1. It is described by the tuple $(\beta, u, f, \varphi, \Gamma)$. Throughout this section we will maintain

²In §10.3.1 we give an overview of so-called **Markov decision processes**, of which the optimal growth model is one example. This discussion will further clarify how to select the state.

Assumption 9.2.1. The production function f is continuous, concave and strictly increasing. It satisfies $f(0) = 0$ and $f(k) \rightarrow \infty$ as $k \rightarrow \infty$. The utility function u satisfies the restrictions in assumption 9.1.1 and, in addition, is bounded.

The Bellman equation is now

$$v(y) = \max_{0 \leq c \leq y} \left\{ u(c) + \beta \int v(f(y - c)z) \varphi(dz) \right\} \quad (9.40)$$

for every $y \in \mathbb{Y}$. The corresponding Bellman operator T is defined by

$$Tv(y) = \max_{0 \leq c \leq y} \left\{ u(c) + \beta \int v(f(y - c)z) \varphi(dz) \right\} \quad (9.41)$$

We view it as a map on the set of $bc\mathbb{Y}$, the continuous bounded real valued functions on \mathbb{Y} . This works well because

Proposition 9.2.1. *T defined in (9.41) is a contraction of modulus β on $bc\mathbb{Y}$ paired with the supremum distance $d_\infty(f, g) = \|f - g\|_\infty$ whenever assumption 9.2.1 holds.*

Proof. This follows directly from proposition 9.1.2. □

9.2.1.2 Optimality

Let's now look at existence and uniqueness of optimal policies in the IID shock setting. The following is a special case of proposition 9.1.1, apart from the additional claim of uniqueness and continuity of the optimal policy.

Proposition 9.2.2. *Let $(\beta, u, f, \varphi, \Gamma)$ be a one sector optimal growth model with IID shocks. If the conditions of assumption 9.2.1, then*

- (i) v^* is the unique fixed point of the Bellman operator T in the set $bc\mathbb{Y}$.
- (ii) v^* is the unique solution to the Bellman equation (9.40) in the set $bc\mathbb{Y}$.
- (iii) A feasible consumption policy σ is optimal if and only if

$$\sigma(y) \in \operatorname{argmax}_{0 \leq c \leq y} \left\{ u(c) + \beta \int v^*(f(y - c)z) \varphi(dz) \right\} \quad (y \in \mathbb{Y}) \quad (9.42)$$

- (iv) Exactly one such policy exists in Σ and that policy is continuous.

Continuity and uniqueness of the optimal policy, as well as some auxillary results, are established in the exercises below.

EXERCISE 10. Let \mathcal{C} be the set of increasing concave functions in $bc\mathbb{Y}$. Show that, under the conditions of assumption 9.2.1, the operator T maps \mathcal{C} into itself.

EXERCISE 11. Use the results of Exercise 10 and Lemma 5.3.2 on page 166 to establish that, under the conditions of assumption 9.2.1, the value function v^* is concave and increasing.

EXERCISE 12. Show the uniqueness component of part (iv) of Proposition 9.2.2.³

9.2.1.3 Value Function Iteration

To compute the value function we can implement a value function iteration algorithm similar to that for the finite case on page 73. There is a significant difference, however: in that algorithm, each element of the sequence of candidate value functions could be represented by a finite array (because the state space is finite). In the present setting, in contrast, the state space is a continuum.

To see why this is problematic, consider iterating with the Bellman operator T as defined in (9.41). We could start with a function v on \mathbb{Y} that can be implemented on a computer, such as $v \equiv 1$, or $v(z) = z$. But the next function Tv will not in general be a simple function that we can easily represent or store on a machine. This causes problems because next we want to compute T^2v , which requires evaluation of the function Tv , and so on. The key issue here is that the set of y that we wish to evaluate Tv on is infinite.

One common solution is to discretize in some suitable manner, reducing the problem to one that exists on a finite state space. This is not unreasonable and, in our one dimensional setting, a discretized solution method will be fast enough. On the other hand, discretization is highly susceptible to the curse of dimensionality (see §12.2.1 for a discussion), which means that we might need to rewrite from scratch if we decide to add more features.

A better approach is to use interpolation over a grid. In particular, each candidate value function v produced by iterating with T is represented by a set of values $v(y_0), \dots, v(y_{n-1})$ on a finite grid y_0, \dots, y_{n-1} , and then implemented as a function through interpolation. Below we use $I\{y_i, v(y_i)\}$ to symbolize this interpolated function. Figure 9.4 shows the case of piecewise linear interpolation.

³Hint: The sum of a strictly concave function and a concave function is strictly concave. Now use the fact that any optimal policy satisfies Bellman's principle of optimality.

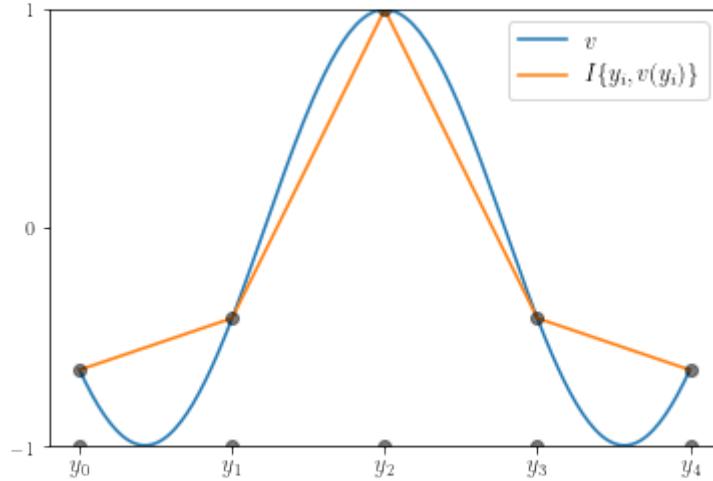


Figure 9.4: Approximation by piecewise linear interpolation

We also need to evaluate the integral on the right hand side of the value function. In algorithm 9, this is performed by Monte Carlo, exploiting the fact that, for IID draws $\{z_j\}$ from φ , we have, by the strong law of large numbers,

$$\frac{1}{m} \sum_{j=1}^m v(f(y - c)z_j) \rightarrow \int v(f(y - c)z)\varphi(dz) \quad (m \rightarrow \infty) \quad (9.43)$$

with probability one. In effect, we are replacing the true distribution φ with an empirical distribution generated by the sample $\{z_j\}$. Although we could use other forms of numerical integration here, the rate of convergence in (9.43) is invariant to the dimension of the integral, so Monte Carlo serves well when more challenging problems are treated.

The routine in algorithm 9 is sometimes called **fitted value function iteration**. We discuss it more formally in §12.2.3.

9.2.2 The Euler Equation

[roadmap]

9.2.2.1 Overview

The **Euler equation** is a first-order condition for optimality that can be used for analysis and computation. In the simple IID setting presented in §9.2.1, it takes the

```

1 draw  $\{z_j\} \stackrel{\text{iid}}{\sim} \varphi$  ;
2 input grid  $G_n := \{y_i\}_{i=0}^{n-1} \subset \mathbb{Y}$  ;
3 input  $\{v_0(y_i)\}_{i=0}^{n-1}$ , an initial guess of  $v^*$  evaluated on  $G_n$  ;
4 input error tolerance  $\tau$  ;
5  $\varepsilon \leftarrow \tau + 1$  ;
6  $k \leftarrow 0$  ;
7 while  $\varepsilon > \tau$  do
8    $v_k \leftarrow I\{y_i, v_k(y_i)\}$  ; // map grid values into interpolated function
9   for  $i \in \{0, \dots, n-1\}$  do
10    |  $v_{k+1}(y_i) \leftarrow \max_{0 \leq c \leq y_i} \left\{ u(c) + \beta \frac{1}{m} \sum_{j=1}^m v_k(f(y_i - c) z_j) \right\}$  ;
11   end
12    $\varepsilon \leftarrow \max_i |v_k(y_i) - v_{k+1}(y_i)|$  ;
13    $k \leftarrow k + 1$  ;
14 end
15 return  $v_k$ 

```

Algorithm 9: Fitted value function iteration

form

$$u'(c_t) = \beta \mathbb{E}_t u'(c_{t+1}) f'(k_{t+1}) z_{t+1} \quad (9.44)$$

and is valid when the following conditions hold:

Assumption 9.2.2. The maps f and u are strictly increasing, continuously differentiable and strictly concave. In addition,

$$\lim_{k \rightarrow 0} f'(k) = \infty \quad \text{and } f(0) = 0$$

and

$$u(0) = 0, \quad \lim_{c \rightarrow 0} u'(c) = \infty \quad \text{and } \lim_{c \rightarrow \infty} u'(c) = 0$$

The limits above are often called the **Inada conditions** and are used to ensure that the optimal choice of c at each y is **interior**, in the sense that $0 < c < y$ whenever $0 < y$. The restrictions above are strong and will fail for certain applications. But they provide a simple setting where we can make sharp observations with relatively simple proofs. Various deviations will later be considered.

EXERCISE 13. Confirm that the policy obtained in §9.1.3.1 satisfies the Euler equation.

EXERCISE 14. Confirm that the policy obtained in §9.1.3.2 satisfies the Euler equation.

The Euler equation is typically understood as a necessary condition for optimality of a consumption-saving path. However it is often sufficient, and studying it leads to new insights on optimal behaviour and computational methods. These ideas are investigated below.

9.2.2.2 The Envelope Condition

As a first step towards understanding the Euler equation, we present a differential characterization of greedy policies that is of interest in its own right.

Proposition 9.2.3. *Let v be an increasing concave function in $b\mathbb{R}_+$ and let σ be the unique v -greedy policy in Σ . If the conditions of assumption 9.2.2 hold, then*

- (i) σ is strictly increasing and interior, while
- (ii) Tv is strictly concave, strictly increasing, continuously differentiable and satisfies

$$(T v)' = u' \circ \sigma \quad \text{on } (0, \infty) \tag{9.45}$$

Here's an obvious but important corollary:

Corollary 9.2.4. *Let c^* be the unique optimal policy for $(\beta, u, f, \varphi, \Gamma)$ and let v^* be the value function. If the conditions of assumption 9.2.2 hold, then*

- (i) c^* is strictly increasing and interior, while
- (ii) v^* is strictly concave, strictly increasing, continuously differentiable and satisfies

$$(v^*)' = u' \circ c^* \quad \text{on } (0, \infty) \tag{9.46}$$

Proposition 9.2.3 and its corollary have been presented in many forms and the latter is often called the **envelope condition** due to its relationship with the envelope theorem. Proposition 9.2.3 itself is somewhat surprising, since v does not have to be differentiable. Hence, the Bellman operator is smoothing, in the sense that images of some nonsmooth functions are smooth.

Early results along the lines of proposition 9.2.3 were established by [Mirman and Zilcha \(1975\)](#) and [Benveniste and Scheinkman \(1979\)](#). A detailed proof of proposition 9.2.3 is given in Section 12.1 of [Stachurski \(2009\)](#). Here we sketch only the key ideas leading to (9.45).

To start, let \mathcal{V} be all strictly concave, continuously differentiable v mapping \mathbb{R}_+ to itself and satisfying $v(0) = 0$ and $v'(y) > u'(y)$ whenever $y > 0$. Pick any $v \in \mathcal{V}$ and let $\sigma \in \Sigma$ be v -greedy. Evaluating the right hand side of the Bellman operator at its maximum gives

$$Tv(y) = u(\sigma(y)) + \beta \int v(f(y - \sigma(y))z)\varphi(dz) \quad (9.47)$$

for each $y \geq 0$. Loosely speaking, the envelope theorem says that we can differentiate $Tv(y)$ with respect to y in (9.47) while ignoring the impact of y on the maximizer $\sigma(y)$. This produces

$$(Tv)'(y) = \beta \int (v)'(f(y - \sigma(y))z)f'(y - \sigma(y))z\varphi(dz) \quad (9.48)$$

At the same time, the first order condition from the Bellman equation evaluated at the optimum yields

$$u'(\sigma(y)) = \beta \int (v)'(f(y - \sigma(y))z)f'(y - \sigma(y))z\varphi(dz) \quad (9.49)$$

Combining this with (9.48), we arrive at the envelope condition (9.45).

9.2.2.3 Mapping Values to Policies

The envelope relationship $(Tv)' = u' \circ \sigma$ suggests that marginal utility u' provides a bridge between greedy policies and values. Let's clarify this idea, as the details will be helpful in the optimality and computational results treated below. Throughout this section assumption 9.2.2 is in force.

First, let

$$m(y) := (u')^{-1}(y) \quad (y > 0)$$

This is just a notational convenience. From assumption 9.2.2, one can show that m is a continuous and strictly decreasing bijection from $(0, \infty)$ to itself.

Second, let \mathcal{V} be as previously defined: all strictly concave, continuously differentiable v mapping \mathbb{R}_+ to itself and satisfying $v(0) = 0$ and $v'(y) > u'(y)$ whenever $y > 0$. For $v \in \mathcal{V}$ let Mv be defined by

$$(Mv)(y) = \begin{cases} m(v'(y)) & \text{if } y > 0 \\ 0 & \text{if } y = 0 \end{cases} \quad (9.50)$$

Let \mathcal{C} be the set of all a continuous, strictly increasing functions on \mathbb{R}_+ satisfying

$0 < \sigma(y) < y$. In other words, \mathcal{C} is all continuous, strictly increasing and interior policy functions for consumption. By continuity, each σ in \mathcal{C} satisfies $\sigma(0) = 0$.

Lemma 9.2.5. *The map M is a bijection from \mathcal{V} to \mathcal{C} , with inverse M^{-1} mapping $\sigma \in \mathcal{C}$ into v defined by*

$$v(y) := (M^{-1}\sigma) = \int_0^y u'(\sigma(x))dx \text{ with } v(0) = 0 \quad (9.51)$$

Moreover, for every increasing concave function in $bc\mathbb{R}_+$, the function MTv is the unique v -greedy policy.

Proof. First observe that, for fixed $v \in \mathcal{V}$, the derivative v' is a continuous, strictly decreasing function. Hence $Mv = m \circ v'$ is strictly increasing and continuous. Moreover, interiority holds because v' strictly dominates u' , implying that, when $y > 0$,

$$(Mv)(y) = m(v'(y)) < m(u'(y)) = y \quad (9.52)$$

In particular, $\sigma(y) := (Mv)(y)$ is an element of \mathcal{C} .

To see that each $\sigma \in \mathcal{C}$ has a preimage $v \in \mathcal{V}$ with $Mv = \sigma$, fix any $\sigma \in \mathcal{C}$ and let v be given by (9.51). An application of the Fundamental Theorem of Calculus yields $v \in \mathcal{V}$ and $Mv = \sigma$. It is also true that M is one-to-one on \mathcal{V} . To see this, suppose that v and w are elements of \mathcal{V} satisfying $Mv = Mw$. Then $v(0) = w(0) = 0$ and $v' = w'$ on $(0, \infty)$. The Fundamental Theorem of Calculus then implies that $v = w$ on \mathbb{R}_+ .

Finally, given $v \in \mathcal{V}$, let σ be the unique v -greedy policy. The claim is that $\sigma = MTv$, or, equivalently, that $u'(\sigma(y)) = (Tv')(y)$ for all $y > 0$. That this statement is true has already been established, in proposition 9.2.3. \square

9.2.2.4 The Euler Equation

As above, let \mathcal{C} be the set of all a continuous, strictly increasing functions on \mathbb{R}_+ satisfying $0 < \sigma(y) < y$. In what follows, we will say that $\sigma \in \mathcal{C}$ **satisfies the Euler equation** if

$$(u' \circ \sigma)(y) = \beta \int (u' \circ \sigma)(f(y - \sigma(y))z) f'(y - \sigma(y)) z \varphi(dz) \quad (9.53)$$

for all $y > 0$.

The Euler equation is closely connected to optimality and also leads to new paths for analysis and computation. To explain these ideas, let us consider the operator K

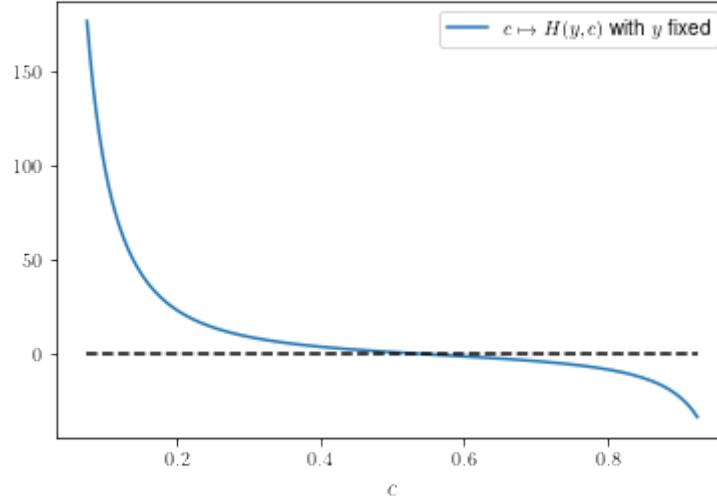


Figure 9.5: Solving for the c that satisfies $H(y, c) = 0$.

from \mathcal{C} to itself defined as follows: for each $\sigma \in \mathcal{C}$ and each $y > 0$, the value $K\sigma(y)$ is the c in $(0, y)$ that solves

$$u'(c) = \beta \int (u' \circ \sigma)(f(y - c)z)f'(y - c)z\varphi(dz) \quad (9.54)$$

We call K the **Coleman–Reffett** operator. It is well defined since, for any $\sigma \in \mathcal{C}$, the right side of (9.54) is continuous and strictly increasing in c on $(0, y)$, and diverges to $+\infty$ as c converges up to y . The left side of (9.54) is continuous and strictly decreasing in c on $(0, y)$ and diverges to $+\infty$ as c converges down to 0. It follows that

$$H(y, c) := u'(c) - \beta \int (u' \circ \sigma)(f(y - c)z)f'(y - c)z\varphi(dz) \quad (9.55)$$

when regarded as a function of c , has exactly one zero (i.e., one point c satisfying $H(y, c) = 0$) in the interval $(0, y)$. Figure shows this situation when $u(c) = c^{1-\gamma}/(1-\gamma)$, $f(k) = Ak^\alpha$, the shock z is lognormal $LN(\mu, \sigma^2)$ and $\sigma y = y/2$. The value y is set to 1.0 and we seek a $c \in (0, y)$ such that $H(y, c) = 0$.⁴

EXERCISE 15. Show that K is isotone on \mathcal{C} , in the sense that if σ_a and σ_b are elements of \mathcal{C} with $\sigma_a \leq \sigma_b$, then $K\sigma_a \leq K\sigma_b$.

The significance of the operator K lies in the following proposition. The proof is

⁴The parameters are $\gamma = 2$, $A = 1$, $\alpha = 0.3$, $\mu = 0.5$, $\sigma = 1.0$ and $\beta = 0.96$. The expectation was computed by Monte Carlo.

immediate from the definitions.

Proposition 9.2.6. *A policy σ in \mathcal{C} is a fixed point of K if and only if it satisfies the Euler equation.*

Thus, the Coleman–Reffett operator plays the same role for the optimal policy that the Bellman operator plays for the value function. Here’s a result that makes the relationship between K and T even clearer.

Proposition 9.2.7. *If assumption 9.2.2 holds, then*

$$T = M^{-1} \circ K \circ M \text{ on } \mathcal{V} \quad (9.56)$$

Proof. Since M is bijective as a map between \mathcal{V} and \mathcal{C} , we can equivalently show that $M \circ T = K \circ M$, or, more explicitly, that

$$(MTv)(y) = (KMv)(y) \text{ for any } v \in \mathcal{V} \text{ and any } y \in (0, \infty) \quad (9.57)$$

To establish (9.57), fix $v \in \mathcal{V}$ and $y > 0$. We saw in lemma 9.2.5 that $\sigma := MTv$ is the unique v -greedy policy. This policy necessarily satisfies the first order condition

$$u'(\sigma(y)) = \beta \int v'(f(y - \sigma(y))z)f'(y - \sigma(y))z\varphi(dz) \quad (9.58)$$

On the other hand, $KMv(y)$ is the unique c in $(0, y)$ that solves

$$\begin{aligned} u'(c) &= \beta \int (u' \circ (Mv))(f(y - c)z)f'(y - c)z\varphi(dz) \\ &= \beta \int (u' \circ ((u')^{-1} \circ v'))(f(y - c)z)f'(y - c)z\varphi(dz) \\ &= \beta \int v'(f(y - c)z)f'(y - c)z\varphi(dz) \end{aligned}$$

In particular, $c = \sigma(y)$. In other words, $KMv(y) = MTv(y)$, as was to be shown. \square

If we can show that the bijection M is also continuous as a map from \mathcal{V} to \mathcal{C} and that its inverse is likewise continuous, then (9.56) implies that (\mathcal{V}, T) and (\mathcal{C}, K) are topologically conjugate. This will be informative about (\mathcal{C}, K) because topologically conjugate dynamical systems have essentially identical properties—and we already have a significant amount of information about the trajectories of T .

The way we will show that M and its inverse are continuous is to cook up a metric on \mathcal{C} that essentially guarantees this result. The metric in question is

$$\rho(\sigma_1, \sigma_2) = d_\infty(M^{-1}\sigma_1, M^{-1}\sigma_2) := \|M^{-1}\sigma_1 - M^{-1}\sigma_2\|_\infty$$

EXERCISE 16. Show that M is continuous as a map from (\mathcal{V}, d_∞) to (\mathcal{C}, ρ) . Show that M^{-1} is continuous as a map from (\mathcal{C}, ρ) to (\mathcal{V}, d_∞) .

We can now state the following result.

Proposition 9.2.8. *If $(\beta, u, f, \varphi, \Gamma)$ is a one sector model such that assumption 9.2.2 holds and c^* is the unique optimal policy, then*

- (i) (\mathcal{C}, K) is globally stable and
- (ii) the unique fixed point of K in \mathcal{C} is c^* .

In particular, $c \in \mathcal{C}$ is optimal if and only it satisfies the Euler equation.

The next exercise asks you to complete the proof of proposition 9.2.8. In doing so you should combine the results we have established above with theorem 5.3.5 on page 169.

EXERCISE 17. Prove Proposition 9.2.8.

9.2.2.5 Properties of the Optimal Policy

Next let's consider how the parameters affect optimal consumption. One parameter that has an unambiguous affect is the discount factor, as the next exercise asks you to prove.

EXERCISE 18. Let $\beta_a \leq \beta_b$ be two discount factors in $(0, 1)$. Let σ_a and σ_b be the respective optimal consumption policies. Show that, in the setting of proposition 9.2.8, we have $\sigma_b \leq \sigma_a$. Provide intuition.

[To be added: show FOSD ordering of the paths and possibly the stationary distribution. (State conditions for global stability and refer to paper for proofs.)]

9.3 The Income Fluctuation Problem

[Add roadmap. The intention is to give a comprehensive set of results. Idea: The Euler eq methods in Qingyin et al must map through conjugacy arguments to contraction results in value function space. Trace through the logic.]

9.3.1 Adding Non-Financial Income

Returning to the dynamics that we looked at in §9.1.1.3, wealth obeys

$$w_{t+1} = R_{t+1}(w_t - c_t) + y_{t+1} \quad (9.59)$$

where y_{t+1} is non-financial income received at the end of period t and $R_{t+1} > 0$ is gross returns on financial assets. As before, the agent seeks to maximize $\mathbb{E} \sum_{t=0}^{\infty} \beta^t u(c_t)$. She is constrained by $c_t \geq 0$ and $w_t \geq 0$ for all t . Both labor income and the interest rate are functions $y_t = y(z_t, \xi_t)$ and $R_t = R(z_t, \zeta_t)$. The exogenous Markov state process $\{z_t\}$ obeys

$$z_{t+1} = G(z_t, \varepsilon_{t+1}) \quad (9.60)$$

where $\{\varepsilon_t\}$ is IID. The Bellman equation is

$$v(w, z) = \max_{0 \leq c \leq w} \{u(c) + \beta \mathbb{E}_z v[R(z', \xi')(w - c) + y(z', \zeta'), z']\} \quad (9.61)$$

The corresponding Bellman operator is defined at v by

$$Tv(w, z) = \max_{0 \leq c \leq w} \{u(c) + \beta \mathbb{E}_z v[R(z', \xi')(w - c) + y(z', \zeta'), z']\} \quad (9.62)$$

Assumption 9.3.1. The utility function is continuous and strictly increasing, The maps y , R and G are everywhere continuous.

9.3.2 Bounded Rewards

Let's suppose first that the utility function is bounded. In that case we can consider T as a map on the set of $bc\mathbf{X}$, the continuous bounded real valued functions on \mathbf{X} .

Lemma 9.3.1. *The Bellman operator T defined in (9.62) is a self-mapping on $bc\mathbf{X}$.*

9.3.3 CRRA Preferences

In this spirit, let us look in detail at the household problem considered in Benhabib et al. (2015b) and see how we can tackle it. This problem is of interest because it uses the CRRA utility function

$$u(c) = \frac{c^{1-\gamma} - 1}{1 - \gamma} \quad (c \geq 0, \gamma > 0) \quad (9.63)$$

which is (a) routinely adopted in quantitative studies and (b) unbounded.

Use the plan factorization method without full proofs.

9.3.4 State Dependence in Returns and Discounting

Present the model of [Ma et al. \(2020\)](#).

9.3.5 The Wealth Distribution Revisited

9.3.5.1 Stationarity

Present results in [Ma et al. \(2020\)](#).

9.3.5.2 Pareto Tails

[Discuss Pareto tails. By simulation. Cite [Benhabib et al. \(2015b\)](#) and [Stachurski and Toda \(2019\)](#).]

Part II

Advanced Methods

Chapter 10

Dynamic Programming: Theory

Now we turn to complete proofs of Bellman's principle of optimality and other key aspects of dynamic programming theory. We do so in an abstract setting that builds on the framework in [Bertsekas \(2018\)](#). This setting admits standard dynamic programming problems as discussed in, say, [Lucas and Stokey \(1989\)](#), [Rust \(1996\)](#), or [Puterman \(2005\)](#), as well as the various recursive preference models, robust control methods and other more sophisticated preference features adopted within economics and finance in recent years.

10.1 Planning Problems

We will study an abstract dynamic program with Bellman equation

$$v(x) = \max_{a \in \Gamma(x)} Q(x, a, v). \quad (10.1)$$

Here x is the state, a is an action, Γ is a feasible correspondence and Q is an abstraction of the usual components of a Bellman equation. A wide range of dynamic programs can be expressed in this way, from simple to exotic. Our first step is to clarify the objects in (10.1).

10.1.1 Markov Decision Processes

In what follows, a **Markov decision process** (MDP) is

- (i) a set X called the **state space**,
- (ii) a set A called the **action space**,

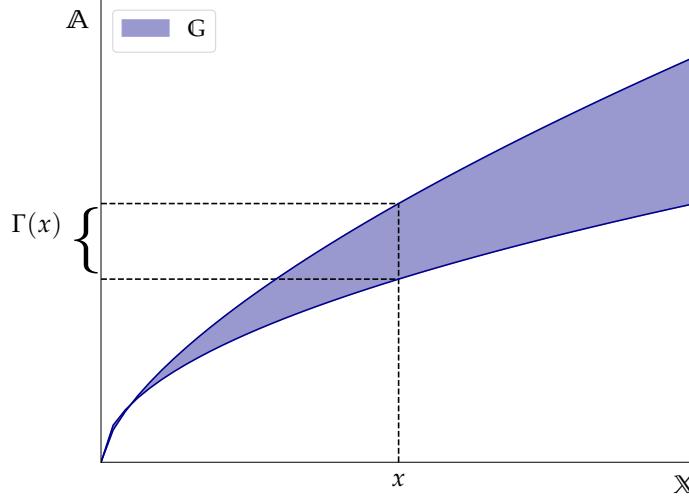


Figure 10.1: Feasible correspondence and feasible state-action pairs

- (iii) a nonempty correspondence Γ from X to A called the **feasible correspondence**, which in turn defines

$$G := \text{graph } \Gamma = \{(x, a) \in X \times A : a \in \Gamma(x)\}$$

the set of **feasible state-action pairs**

- (iv) a subset \mathcal{V} of \mathbb{R}^X called the set of **candidate value functions**, and
- (v) a **state-action aggregator**, which is a map

$$Q: G \times \mathcal{V} \rightarrow \{-\infty\} \cup \mathbb{R}$$

satisfying the **monotonicity condition**

$$v \leq v' \implies Q(x, a, v) \leq Q(x, a, v') \quad (10.2)$$

for every $v, v' \in \mathcal{V}$ and every $(x, a) \in G$.

We think of $\Gamma(x)$ as all actions available to the controller in state x . Figure 10.1 shows an illustration of one possible correspondence Γ when $A = X = \mathbb{R}_+$, along with G , the resulting set of feasible state-action pairs.

The interpretation of the aggregator is:

$Q(x, a, v)$ = total lifetime rewards, contingent on current action a , current state x and the use of v to evaluate future states.

In other words, $Q(x, a, v)$ corresponds to the right hand side of the Bellman equation—the function that we maximize over when choosing an optimal action. Of course optimality is contingent on inserting the correct function v into $Q(x, a, v)$, and locating and calculating this v will be one of our major concerns.

The order on the left side of (10.2) is the usual pointwise partial order for functions. The monotonicity restriction is natural: relatively to v , if rewards are at least as high v' in every future state, then the total rewards we can extract under v' should be at least as high.

Example 10.1.1. Consider the job search problem of §6.1.1. We can map it to the present framework by taking the state to be the wage w and the action to be $a \in \{0, 1\}$, where $a = 1$ means accept the current job offer and $a = 0$ means reject. \mathcal{X} is the support of the wage distribution φ and $\Gamma(x) = \{0, 1\}$ for every $x \in \mathcal{X}$. Given a candidate value function $v \in \mathcal{V} = \mathbb{R}^{\mathcal{X}}$, the aggregator is

$$Q(w, a, v) = a \frac{w}{1 - \beta} + (1 - a) \left[c + \beta \int v(w') \varphi(w') dw' \right]$$

Maximizing the right had side over $a \in 0, 1$ is equivalent to choosing the largest of the two values (accept or continue), so the Bellman equation associated with this problem is the same as (6.1) on page 183. Note that the restriction in (10.2) holds.¹

Example 10.1.2. Consider the same problem as example 10.1.1 but now suppose that w_t obeys a Markov chain with Markov kernel P on countable set \mathcal{X} . The aggregator then becomes

$$Q(w, a, v) = a \frac{w}{1 - \beta} + (1 - a) \left[c + \beta \sum_{w'} v(w') P(w, w') \right]$$

Evidently (10.2) is satisfied.

Example 10.1.3. In the optimal savings model with IID shocks from §9.2.1, the state is $y \in \mathbb{R}_+$, the action is $c \in \Gamma(y) := [0, y]$ and the aggregator is

$$Q(y, c, v) = u(c) + \beta \int v(f(y - c)z) \varphi(dz)$$

¹We are using monotonicity of integrals here: larger integrand implies larger integral. This monotonicity does hold, as we'll see in §11.3.2.

Maximizing over $c \in \Gamma(y)$ yields the Bellman equation (9.40) on page 279. The monotonicity condition (10.2) holds because expectations are monotone.

Example 10.1.4. In the optimal savings problem of §2.2.4, the state is (w, z) , where w is current assets and z is the current value of the exogenous state process. The action a is current consumption c . Let A equal \mathbb{R}_+ and let $X = \mathbb{R}_+ \times Z$, where the latter is a set in which z takes values. The feasible correspondence is $\Gamma(x) = [0, x]$. The aggregator is

$$Q((w, z), c, v) = u(c) + \beta \mathbb{E}_z v((1 + r(z', \xi'))(w - c) + y(z', \zeta'), z').$$

The monotonicity condition (10.2) holds.

Before we continue, a quick remark on terminology. What we have called a “Markov decision process” is more general than most other authors, who typically focus on the additively separable case. That special case is treated by us in §10.3.1.

10.1.2 Policy Functions

As we have seen in a range of applications, rather than attempting to fix the entire sequence of actions the controller should take from time zero onward, a more natural approach is to choose plans that specify actions contingent on the evolution of the state. For Markov decision problems, these contingency plans take the form of mappings from current state to current action and are referred to as **policies**.

To respect the structure of the problem, we need to restrict actions at each state to lie in the constraint set. We also require some regularity properties, to make sure that our decision problem will be well defined. To state these properties, let Σ be a family of maps from X to A , where each $\sigma \in \Sigma$ satisfies

feasibility: the action $\sigma(x)$ is in $\Gamma(x)$ for all $x \in X$,

value consistency: the map $w(x) := Q(x, \sigma(x), v)$ is in \mathcal{V} for all $v \in \mathcal{V}$ and

value uniqueness: there is one and only one $w \in \mathcal{V}$ with

$$w(x) = Q(x, \sigma(x), w) \text{ for all } x \in X.$$

To economize on terminology, let us agree to call such families of policies **regular**. We now step through these conditions to understand their implications. We will show that these are minimal conditions required for the dynamic program to make sense.

Feasibility just imposes the constraint Γ . Figure 10.2 shows one such policy σ .

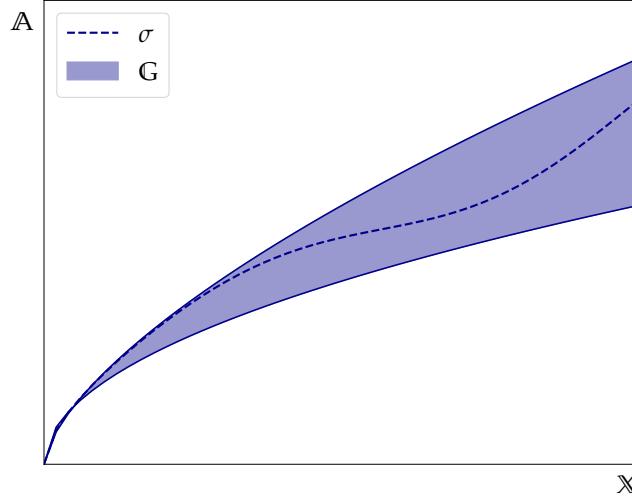


Figure 10.2: The action $\sigma(x)$ lies in $\Gamma(x)$ for all x

Value consistency and uniqueness are joint conditions on Σ and the rest of the objects that define the MDP. Consistency allows us to work exclusively in the class \mathcal{V} of candidate value functions and is typically easy to verify.

Example 10.1.5. If X and A are finite, then one usually takes \mathcal{V} to be all of \mathbb{R}^X and Σ be all σ in A^X satisfying feasibility. Value consistency obviously holds.

The value uniqueness condition is deeper and more subtle. Let's now examine its meaning and look at some examples.

Solutions to MDPs are policies that generate lifetime value as great or greater than any other candidate. To locate them we need to know the “lifetime value” associated with any given policy. In other words, to obtain the maximizer in the MDP, we need to know how to evaluate the objective function, which associates each policy $\sigma \in \Sigma$ to its lifetime value. This association is implemented in the present setting by mapping each $\sigma \in \Sigma$ to the $v_\sigma \in \mathcal{V}$ that satisfies

$$v_\sigma(x) = Q(x, \sigma(x), v_\sigma) \quad \text{for all } x \in X \quad (10.3)$$

Any such function v_σ is called a **σ -value function**. The uniqueness condition above says precisely that, for each σ in Σ , there exists exactly one σ -value function in \mathcal{V} .

Of course, for these ideas to make sense, we still need to convince ourselves that $v_\sigma(x)$ does in fact represent the lifetime value of following policy σ now and forever,

starting from current state x ! To do so, let's look at some examples:

Example 10.1.6. Consider the optimal growth model with IID shocks from §9.2.1. The lifetime value of following consumption policy $\sigma \in \Sigma$ is

$$v_\sigma(y_0) = \mathbb{E} \sum_{t \geq 0} \beta^t u(\sigma(y_t)) = u(\sigma(y_0)) + \mathbb{E} \left[\sum_{t \geq 1} \beta^t u(\sigma(y_t)) \mid y_0 \right]$$

Here $\{y_t\}$ starts at y_0 and then follows $y_{t+1} = f(y_t - \sigma(y_t))z_{t+1}$. The law of iterated expectations then gives

$$v_\sigma(y_0) = u(\sigma(y_0)) + \beta \mathbb{E} \left[\mathbb{E} \left[\sum_{t \geq 1} \beta^{t-1} u(\sigma(y_t)) \mid y_1 \right] \right] = u(\sigma(y_0)) + \beta \mathbb{E} v_\sigma(y_1).$$

Expanding out the last expression and writing y for y_0 yields

$$v_\sigma(y) = u(\sigma(y)) + \beta \int v_\sigma(f(y - \sigma(y))z) \varphi(dz).$$

This is exactly (10.3) in the case of the IID optimal growth model. Thus, a function v_σ satisfying (10.3) returns the lifetime value of following σ from each state.

Example 10.1.7. For the finite state Markov decision process we covered in §3.3.3, the function v_σ representing lifetime value was shown to satisfy the geometric sum $v_\sigma = \sum_{t \geq 0} \beta^t \Pi_\sigma^t r_\sigma$. We also saw in Lemma 3.3.2 on page 74 that v_σ satisfies the recursive representation $v_\sigma = r_\sigma + \beta \Pi_\sigma v_\sigma$. This is (10.3) in the context of finite state Markov decision process.

In summary, the value uniqueness component of our definition of a regular policy class says that (10.3) always has a unique solution, and hence, to each candidate policy, we can associate an unambiguous notion of lifetime value.

When does value uniqueness hold? We will check uniqueness in different ways depending on the application, but here is one example:

Example 10.1.8. For the finite state decision process in Example 10.1.7, we set $\mathcal{V} = \mathbb{R}^X$ and define Σ to be all functions σ from X to A with $\sigma(x) \in \Gamma(x)$ for all $x \in X$. Uniqueness requires that, for any $\sigma \in \Sigma$, there is exactly one v_σ in \mathbb{R}^X such that $v_\sigma = r_\sigma + \beta \Pi_\sigma v_\sigma$. This is true by Lemma 3.3.2 on page 74.

10.1.3 Optimality: Definitions and Concepts

Add roadmap.

10.1.3.1 Optimal Policies and Value Functions

Let Σ be a regular class of policies for the MDP defined above. A policy σ^* is called **optimal** if $\sigma^* \in \Sigma$ and

$$v_{\sigma^*}(x) \geq v_\sigma(x) \quad \text{for all } \sigma \in \Sigma \text{ and all } x \in \mathcal{X}.$$

Thus, an optimal policy is a policy that generates maximal lifetime value from every possible state.

Closely related to optimal policies are value functions. The **value function** associated with our planning problem is the function v^* defined at $x \in \mathcal{X}$ by

$$v^*(x) = \sup_{\sigma \in \Sigma} v_\sigma(x). \quad (10.4)$$

At this stage we are not claiming that v^* lies in \mathcal{V} , or even that v^* is finite. Evidently, a candidate policy σ^* is optimal if and only if

$$v_{\sigma^*}(x) = v^*(x) \quad \text{for all } x \in \mathcal{X}$$

A primary goal of the theory of dynamic programming is to find conditions under which an optimal policy exists and provide properties that characterize such policies. We'll turn to these topics momentarily.

10.1.3.2 Greedy Policies

Given v in \mathcal{V} , we say that a policy $\sigma \in \Sigma$ is **v -greedy** if it satisfies

$$Q(x, \sigma(x), v) = \max_{a \in \Gamma(x)} Q(x, a, v) \quad \text{for all } x \in \mathcal{X} \quad (10.5)$$

One way to understand this is to think of a v -greedy policy as a policy that treats v as the correct value function and sets all actions accordingly.

Greedy policies are relatively easy to compute. Unless $\Gamma(x)$ is very high dimensional, solving (10.5) is typically straightforward (and certainly easier than trying to directly solve the problem (10.4), since Σ is in general far larger than $\Gamma(x)$). This observation is salient because, under certain conditions, solving the overall problem (10.4) reduces to computing a v -greedy policy with the right choice of v . That choice is the value function v^* . Intuitively, v^* assigns the “correct” value to each state, in the sense of maximal lifetime value the controller can extract, so using v^* to calculate greedy policies leads to the optimal outcome.

These ideas are formalized in §10.1.3.3.

Here's a useful result concerning greedy policies. In stating the result, we assume that X and A are metric spaces. The feasible set G inherits the product topology.

Lemma 10.1.1. *If, for some $v \in \mathcal{V}$, the function $(x, a) \mapsto Q(x, a, v)$ is real valued and continuous on G and Γ is continuous and compact valued on X , then there exists a Borel measurable function $\sigma: X \rightarrow A$ such that*

- (i) $\sigma(x)$ maximizes $Q(x, a, v)$ over $\Gamma(x)$ for all $x \in X$ and
- (ii) the function $w(x) = Q(x, \sigma(x), v)$ is continuous on X .

Moreover, if, for each $x \in X$, the value $\sigma(x)$ is the unique maximizer of $Q(x, a, v)$ over $\Gamma(x)$, then σ is continuous on X .

Lemma 10.1.1 is a direct consequence of Berge's Theorem (page 152). If Σ is all Borel measurable functions from X to A that satisfy the feasibility constraint, as is often the case, then the policy σ in Lemma 10.1.1 is v -greedy.

10.1.3.3 Bellman's Principle of Optimality

A function $v \in \mathcal{V}$ is said to satisfy the **Bellman equation** if

$$v(x) = \max_{a \in \Gamma(x)} Q(x, a, v) \quad \text{for all } x \in X. \quad (10.6)$$

The definition requires that, for each x in X , the maximum on the right hand side of (10.6) exists and that this maximum is equal to $v(x)$.

There are many circumstances under which there is exactly one function in \mathcal{V} that satisfies the Bellman equation and that function is the value function. This is a natural idea, since, if we insert v^* into both the left and right hand sides of (10.6), the right hand side is, for each state, the value we obtain if we act optimally now and extract maximal value in the future, which should be equal to the left hand side—the maximal value we can obtain today.

A closely related concept is **Bellman's principle of optimality**, which states that

$$\sigma \text{ is optimal if and only if } \sigma \text{ is } v^*\text{-greedy}$$

In particular, when v^* is known and Bellman's principle of optimality is valid, we can calculate an optimal policy by taking

$$\sigma^*(x) \in \operatorname{argmax}_{a \in \Gamma(x)} Q(x, a, v^*)$$

at each x . The intuition was stated above and we have seen the same principle in numerous applications.

10.1.3.4 Optimality of Stationary Markov Policies

Explain the focus on deterministic stationary Markov policies. Give references to suitable results.

10.1.4 Two Operators

To help us understand the relationship between the concepts listed above, we introduce two operators. First, given a class of feasible policies Σ , we defined for each $\sigma \in \Sigma$ the **σ -value operator** as the mapping $T_\sigma: \mathcal{V} \rightarrow \mathcal{V}$ where

$$T_\sigma v(x) = Q(x, \sigma(x), v) \quad (x \in \mathbb{X}) \quad (10.7)$$

The σ -value operator T_σ is constructed so that, in \mathcal{V} , fixed points of T_σ coincide with σ -value functions—that is, with solutions to (10.3).

Lemma 10.1.2. *If Σ is regular, then T_σ is an isotone self-map on \mathcal{V} . It has a unique fixed point in \mathcal{V} , equal to the σ -value function v_σ .*

Proof. The claim that T_σ maps \mathcal{V} to itself follows from value consistency of σ . The uniqueness component of the definition of a regular policy class in §10.1.2 says precisely that, for each σ in Σ , the operator T_σ has exactly one fixed point in \mathcal{V} . By definition this is v_σ . Finally, T_σ is isotone with respect to the pointwise partial order on \mathcal{V} , since, by the monotonicity restriction (10.2), $v \leq v'$ implies $Q(x, \sigma(x), v) \leq Q(x, \sigma(x), v')$ for all $x \in \mathbb{X}$. In other words, $v \leq v'$ implies $T_\sigma v \leq T_\sigma v'$. \square

Next we introduce a second operator, called the **Bellman operator**, which we define in our setting as the mapping T on \mathcal{V} such that

$$Tv(x) = \sup_{a \in \Gamma(x)} Q(x, a, v) \quad (10.8)$$

Observe that

- (i) Any solution to the Bellman equation is a fixed point of T .
- (ii) A fixed point v of T in \mathcal{V} is a solution to the Bellman equation provided that the supremum on the right hand side of (10.8) can be replaced with max at every x .

Greedy policies can now be characterized as follows:

$$\sigma \text{ is } v\text{-greedy} \iff T\mathbf{v} = T_\sigma \mathbf{v} \quad (10.9)$$

For arbitrary candidate policies the equality in (10.9) fails but we still have

$$T_\sigma \mathbf{v} \leqslant T\mathbf{v} \quad \text{for all } \mathbf{v} \in \mathcal{V} \quad (10.10)$$

At this stage we cannot say whether T maps \mathcal{V} to itself. Instead we regard T as a map from \mathcal{V} into the set of functions from \mathbf{X} to $\mathbb{R} \cup \{-\infty, +\infty\}$.

With the definitions of T and T_σ in hand, we can convert some of our tasks into fixed point problems, for which many useful results exist.

10.2 Optimality Results

[Roadmap]

Our approach in this section will be to obtain sufficient conditions under which (i) the regularity conditions on the policies in §10.1.2 hold, so the problem is well defined, and (ii) Bellman's principle of optimality is valid. Once those tasks are complete, obtaining an optimal policy reducing to finding the value function, plus one small additional calculation (i.e., finding the v^* -greedy policy). Our attention then turns to ways to compute the value function.

10.2.1 When Does Bellman's Principle Hold?

Add roadmap.

10.2.1.1 The Value Function Must Satisfy the Bellman Equation

Our first result of this section describes the exact relationship between Bellman's principle of optimality and the Bellman equation.

Theorem 10.2.1. *Let Σ be regular. If v^* lies in \mathcal{V} and at least one v^* -greedy policy exists, then the following statements are equivalent:*

- (i) v^* satisfies the Bellman equation.
- (ii) The set of optimal policies is nonempty and Bellman's principle of optimality holds.

Proof. Suppose first that v^* lies in \mathcal{V} and satisfies the Bellman equation. By the definition of greedy policies,

$$\sigma \text{ is } v^*\text{-greedy} \iff Q(x, \sigma(x), v^*) = \max_{a \in \Gamma(x)} Q(x, a, v^*), \quad \forall x \in \mathsf{X}$$

Since v^* satisfies the Bellman equation, we then have

$$\sigma \text{ is } v^*\text{-greedy} \iff Q(x, \sigma(x), v^*) = v^*(x), \quad \forall x \in \mathsf{X}$$

But, by uniqueness of σ -value functions, the right hand side is equivalent to the statement that $v^* = v_\sigma$. Hence, by this chain of logic and the definition of optimality,

$$\sigma \text{ is } v^*\text{-greedy} \iff v^* = v_\sigma \iff \sigma \text{ is optimal} \tag{10.11}$$

In other words, Bellman's principle of optimality holds. Moreover, the statement of Theorem 10.2.1 assures us that at least one v^* -greedy policy exists. Since Bellman's principle of optimality holds, each such policy is optimal, so the set of optimal policies is nonempty.

Suppose, on the other hand, that at least one optimal policy exists, and that Bellman's principle of optimality is valid. Seeking a contradiction, let us assume that v^* fails to satisfy the Bellman equation. Let σ be a v^* -greedy policy.

Because σ is v^* -greedy, we have $Q(x, \sigma(x), v^*) = \max_{a \in \Gamma(x)} Q(x, a, v^*)$ at every x . Combining this with the fact that v^* does not satisfy the Bellman equation, there must exist an $x \in \mathsf{X}$ such that $Q(x, \sigma(x), v^*)$ and $v^*(x)$ are distinct. Hence $v^* \neq v_\sigma$. Since, clearly, $v^* \geq v_\sigma$ pointwise on X , there must be some $x \in \mathsf{X}$ such that $v_\sigma(x) < v^*(x)$. In particular, σ is not optimal. But, given our hypothesis that σ is v^* -greedy, this contradicts Bellman's principle of optimality. The contradiction leads us to conclude that v^* satisfies the Bellman equation after all. \square

10.2.1.2 General Conditions

The result in the last section says that Bellman's principle of optimality will be valid whenever v^* satisfies the Bellman equation. So when is this true? Here is one rather general set of conditions. Later we'll look at different restrictions on primitives that imply the conditions of the theorem.

Theorem 10.2.2. *Let Σ be regular and suppose, in addition, that*

- (i) *T has at least one fixed point \bar{v} in \mathcal{V} ,*
- (ii) *there exists at least one \bar{v} -greedy policy in Σ , and*

(iii) for all $\sigma \in \Sigma$ and all $x \in \mathsf{X}$,

$$\limsup_{k \rightarrow \infty} T_\sigma^k \bar{v}(x) \geq v_\sigma(x) \quad (10.12)$$

then $\bar{v} = v^*$ and v^* is the unique solution to the Bellman equation in \mathcal{V} .

Proof. Let the conditions of the theorem hold and let \bar{v} be a fixed point of T in \mathcal{V} . We claim that $\bar{v} = v^*$. To see this, let $\sigma \in \Sigma$ be \bar{v} -greedy (existence is by assumption). For this policy we have $\bar{v} = T\bar{v} = T_\sigma\bar{v}$. By uniqueness of σ -value functions, we know that v_σ is the only fixed point of T_σ in \mathcal{V} , so the last chain of equalities yields $\bar{v} = v_\sigma$. In which case $\bar{v} \leq v^*$, since, by definition, $v_\sigma \leq v^*$ for any $\sigma \in \Sigma$.

To check the reverse inequality, fix $\sigma \in \Sigma$ and note that, by the definition of T , we must have $\bar{v} = T\bar{v} \geq T_\sigma\bar{v}$. Iterating on the inequality $\bar{v} \geq T_\sigma\bar{v}$ and using the monotonicity of T_σ , we obtain $\bar{v} \geq T_\sigma^k \bar{v}$ for all $k \in \mathbb{N}$. Now (10.12) combined with the fact that the pointwise order is closed under pointwise limits yields $\bar{v} \geq v_\sigma$. Since σ was an arbitrary choice from Σ , it follows that $\bar{v} \geq v^*$. Therefore $\bar{v} = v^*$.

Since \bar{v} was an arbitrary fixed point of T in \mathcal{V} , we have now shown that every fixed point of T in \mathcal{V} is equal to v^* . Since we are assuming that at least one such fixed point exists, we see that v^* is the unique fixed point of T in \mathcal{V} .

Finally, v^* is a solution to the Bellman equation in \mathcal{V} because $v^* = \bar{v}$ and \bar{v} has at least one greedy policy, so

$$v^*(x) = \sup_{a \in \Gamma(x)} Q(x, a, v^*) = \max_{a \in \Gamma(x)} Q(x, a, v^*)$$

Moreover, v^* is the only solution to the Bellman equation in \mathcal{V} , since any other solution would also be a fixed point of T and v^* is the only fixed point of T in \mathcal{V} . \square

10.2.1.3 A Sufficient Condition

We need practical conditions to test when the assumptions of Theorem 10.2.2 hold. Given the wide variety of problems to which dynamic programming is applied, there is no one set of sufficient conditions that covers all cases of interest while also being easy to test in practice. However, the result in this section is quite close to that goal.

Assumption 10.2.1. Let Σ be a class of feasible policies. Let ρ be a metric on \mathcal{V} such that convergence with respect to ρ implies pointwise convergence.² The MDP is such that, under this metric,

²That is, if $\{v_n\}$ is a sequence in \mathcal{V} with $\rho(v_n, v) \rightarrow 0$ as $n \rightarrow \infty$ for some $v \in \mathcal{V}$, then $v_n(x) \rightarrow v(x)$ for all $x \in \mathsf{X}$.

- (i) The map T_σ is globally stable on \mathcal{V} for all σ in Σ .
- (ii) There exists a subset $\hat{\mathcal{V}}$ of \mathcal{V} such that
 - (a) To each $v \in \hat{\mathcal{V}}$ there corresponds at least one v -greedy policy in Σ .
 - (b) T is a globally stable self-mapping on $\hat{\mathcal{V}}$.

Settings where conditions (i)-(ii) of Assumption 10.2.1 hold are described below. Notice that the class of policies Σ was not required to be regular. However, regularity is a direct consequence of condition (i), given feasibility.³

The significance of Assumption 10.2.1 stems from the next theorem. It describes an ideal outcome, where optimal policies exist, Bellman's principle of optimality is valid, and the value function can be calculated by successive approximations with T .

Theorem 10.2.3. *If Assumption 10.2.1 holds, then*

- (i) v^* lies in $\hat{\mathcal{V}}$ and is the unique solution to the Bellman equation in \mathcal{V} ,
- (ii) $T^n v \rightarrow v^*$ as $n \rightarrow \infty$ for all $v \in \hat{\mathcal{V}}$, and
- (iii) Bellman's principle of optimality is valid and at least one optimal policy exists.

Proof. Claim (i) holds because the conditions of Theorem 10.2.2 are all satisfied: T has at least one fixed point in \mathcal{V} because T has a fixed point \bar{v} in $\hat{\mathcal{V}}$ and $\hat{\mathcal{V}} \subset \mathcal{V}$. Moreover, Assumption 10.2.1 assures us that there exists at least one \bar{v} -greedy policy in Σ . Finally, given $\sigma \in \Sigma$ and any $x \in \mathbf{X}$, the convergence in Condition (10.12) holds because T_σ is globally stable in a setting that implies pointwise convergence, yielding

$$\limsup_{k \rightarrow \infty} T_\sigma^k \bar{v}(x) = \lim_{k \rightarrow \infty} T_\sigma^k \bar{v}(x) = v_\sigma(x)$$

Claim (ii) holds because the conditions of Theorem 10.2.2 are satisfied, so v^* is the unique fixed point of T in \mathcal{V} , and because T is globally stable on $\hat{\mathcal{V}}$.

Claim (iii) follows from Theorem 10.2.1, given that v^* lies in $\hat{\mathcal{V}}$ and each v in $\hat{\mathcal{V}}$ has at least one v -greedy policy. \square

³For any feasible σ , the fact that T_σ maps \mathcal{V} to itself means that σ is value consistent. Fixed points of T_σ coincide with σ -value functions, so global stability of T_σ on \mathcal{V} implies that there is exactly one σ -value function v_σ in \mathcal{V} for each $\sigma \in \Sigma$. Hence, under this assumption, the class of policies Σ is always regular.

10.2.2 Uniform Contractions

Of the optimality results we have seen so far, Theorem 10.2.3 is the most useful in applications. Let's now look at a class of settings where the conditions of Theorem 10.2.3 are all satisfied.

10.2.2.1 The Finite Contractive Case

Here we look at convergence in finite dimensional settings where ρ is the metric d_∞ induced by the supremum norm $\|\cdot\|_\infty$ on \mathbb{R}^X . Let X and A be finite,

- Σ be all $\sigma \in X^A$ satisfying the feasibility condition $\sigma(x) \in \Gamma(x)$ for all x and
- $\mathcal{V} = \mathbb{R}^X$ with distance defined by the supremum norm $\|\cdot\|_\infty$.

In this setting we have the following result:

Proposition 10.2.4. *If there exists a $\beta < 1$ such that*

$$|Q(x, a, v) - Q(x, a, v')| \leq \beta \|v - v'\|_\infty \quad \text{for all } (x, a) \in G \quad (10.13)$$

then the conditions of Assumption 10.2.1 hold with $\hat{\mathcal{V}} := \mathcal{V}$.

Proof. To see that T_σ is globally stable on \mathcal{V} , fix $\sigma \in \Sigma$ and let v and v' be elements of \mathcal{V} . By (10.13) we have

$$|T_\sigma v(x) - T_\sigma v'(x)| = |Q(x, \sigma(x), v) - Q(x, \sigma(x), v')| \leq \beta \|v - v'\|_\infty$$

for every $x \in X$. Taking the supremum over the left hand side proves that T_σ is a contraction of modulus β . Since (\mathbb{R}^X, d_∞) is complete, it follows from Banach's contraction mapping theorem that T_σ is globally stable on \mathcal{V} .

Similarly, fixing $x \in X$ and applying (10.13) and the sup inequality in Lemma 5.1.17 (page 151), we have

$$|Tv(x) - Tv'(x)| \leq \max_{a \in \Gamma(x)} |Q(x, a, v) - Q(x, a, v')| \leq \beta \|v - v'\|_\infty$$

Taking the supremum over the left hand side shows that T is also contracting on $\hat{\mathcal{V}} := \mathcal{V}$. Finally, existence of greedy policies is trivial when A is finite. We have now checked that all the conditions of Assumption 10.2.1 are satisfied. \square

[Link back to \(and forwards from\) finite state MDP problem in §3.3.3.](#) In particular, prove Proposition 3.3.1.

10.2.2.2 The Bounded Continuous Contractive Case

Now we move up one level of generality, by assuming that X and A are metric spaces. Recall that $bm\mathsf{X}$ be the Borel measurable functions in $b\mathsf{X}$ and let $bc\mathsf{X}$ be the continuous functions in $b\mathsf{X}$. We pair $b\mathsf{X}$ and its subsets with d_∞ , the distance induced by the supremum norm. Throughout this section we set

- $\Sigma :=$ the set of Borel measurable functions from X to A satisfying feasibility and
- $\mathcal{V} := bm\mathsf{X}$.

As a basic regularity condition, we suppose that $(x, a) \mapsto Q(x, a, v)$ is Borel measurable on G for all $v \in bm\mathsf{X}$.

Proposition 10.2.5. *Let Γ be continuous and compact valued. If, in addition,*

- (i) *the map $(x, a) \mapsto Q(x, a, v)$ is*
 - (a) *continuous on G for all $v \in bc\mathsf{X}$ and*
 - (b) *bounded on G for at least one $v \in bc\mathsf{X}$, and*
- (ii) *there exists a $\beta < 1$ such that*

$$|Q(x, a, v) - Q(x, a, v')| \leq \beta \|v - v'\|_\infty \quad \text{for all } (x, a) \in \mathsf{G} \quad (10.14)$$

then the conditions of Assumption 10.2.1 hold with $\hat{\mathcal{V}} := bc\mathsf{X}$.

Proof. First we need to show that T_σ is a self map on $\mathcal{V} = bm\mathsf{X}$. So fix σ in Σ and $v \in \mathcal{V}$. Observe first that, by part (a) of condition (i), we can take \bar{v} to be such that $x \mapsto Q(x, \sigma(x), \bar{v})$ is bounded by some constant M . Then, by condition (ii) and the triangle inequality, we have

$$|T_\sigma v(x)| \leq |Q(x, \sigma(x), v) - Q(x, \sigma(x), \bar{v})| + |Q(x, \sigma(x), \bar{v})| \leq \|v - \bar{v}\|_\infty + M.$$

Hence $T_\sigma v$ is bounded. In addition, $T_\sigma v$ is Borel measurable because Σ contains only Borel measurable functions and compositions of Borel measurable functions are Borel measurable. We have now confirmed that T_σ is a self-map on \mathcal{V} . It is also true that T_σ is globally stable on \mathcal{V} , with the proof being essentially identical to the proof of the same statement in Proposition 10.2.4.

With $\hat{\mathcal{V}} := bc\mathsf{X}$, we also need to show that T is a contractive self-map on $\hat{\mathcal{V}}$. To see that T is a self-mapping on $bc\mathsf{X}$, pick any $v \in \mathcal{V}$ and any $x \in \mathsf{X}$. Let \bar{v} be as above,

so $|Q(x, a, \bar{v})| \leq M$. By (10.14) and the sup inequality in Lemma 5.1.17 (page 151), we have

$$|Tv(x)| \leq \left| \sup_{a \in \Gamma(x)} Q(x, a, v) - \sup_{a \in \Gamma(x)} Q(x, a, \bar{v}) \right| + M \leq \|v - v'\|_\infty + M.$$

Hence Tv is bounded in x . In addition, T is continuous by Lemma 10.1.1. In particular, T is a self-map on $\hat{\mathcal{V}} = bc\mathbb{X}$.

To see that T is globally stable on $\hat{\mathcal{V}}$, let v and v' be elements of $\hat{\mathcal{V}}$. Fix $x \in \mathbb{X}$. Using (10.14) and the sup inequality in Lemma 5.1.17 again, we have

$$|Tv(x) - Tv'(x)| \leq \sup_{a \in \Gamma(x)} |Q(x, a, v) - Q(x, a, v')| \leq \beta \|v - v'\|.$$

Taking the supremum over the left hand side verifies contractivity, which in turn implies global stability by Banach's contraction mapping theorem. Finally, each $v \in \hat{\mathcal{V}}$ has at least one greedy policy by Lemma 10.1.1. The conditions of Assumption 10.2.1 are now verified. \square

10.2.3 Weighted Contractions

To be added.

10.2.4 Convexity and Concavity

This section is about how concavity or convexity of Q can imply global stability. To be added.

10.2.5 Review of Algorithms

Treat value function iteration, policy iteration, optimistic policy iteration.

10.3 Some Special Cases

Add roadmap.

10.3.1 Additively Separable MDPs

In this section we treat a traditional class of MDPs called **additively separable MDPs** (MDPs). The key restriction is that the state-action aggregator Q is the sum of two terms, one for current rewards and the other representing future rewards. Most applications treated in economics use this paradigm. While assumptions about intertemporal preferences are relatively restrictive, MDPs are, for the most part, highly tractable.

10.3.1.1 Structure

A (discounted, infinite horizon) **Markov decision process** is a special case of an MDP where the state-action aggregator has the form

$$Q(x, a, v) = r(x, a) + \beta \int v(F(x, a, z))\varphi(dz) \quad (10.15)$$

whenever $(x, a) \in \mathcal{G}$ and $v \in \mathcal{V}$. Here

- (i) $r: \mathcal{G} \rightarrow \{-\infty\} \cup \mathbb{R}$ is a **reward function**,
- (ii) $\beta \in (0, 1)$ is a **discount factor**,
- (iii) φ is a distribution on a metric space Z and
- (iv) F is a **transition function** from $\mathcal{G} \times Z$ to X .

The state and action spaces are assumed to be metric spaces. We focus here on the classical case where rewards are bounded. In particular,

$$\text{there exists an } M < \infty \text{ such that } |r(x, a)| \leq M \text{ for all } (x, a) \in \mathcal{G}. \quad (10.16)$$

We also assume that r and F are Borel measurable functions. Then we can take

- \mathcal{V} to be bmX , the set of bounded Borel measurable functions from X to \mathbb{R} and
- Σ to be all Borel measurable $\sigma: X \rightarrow A$ satisfying $\sigma(x) \in \Gamma(x)$ for all $x \in X$.

We need to check that the feasibility and consistency requirements in (i)–(ii) on page 295 are satisfied. The first, which is feasibility of policies, was assumed directly just above. The second requirement is that $w(x) := Q(x, \sigma(x), v)$ is an element of \mathcal{V} for all $v \in \mathcal{V}$. Translated to the current case, this requires that if v is bounded and Borel measurable on X and σ is feasible and Borel measurable, then

$$T_\sigma v(x) = r(x, \sigma(x)) + \beta \int v(x')\Pi(x, \sigma(x), dx')$$

is bounded and Borel measurable.

$$|T_\sigma v(x)| \leq |r(x, \sigma(x))| + \beta \left| \int v(x') \Pi(x, \sigma(x), dx') \right| \leq M + \beta \|v\|_\infty \quad (10.17)$$

(We made use of the triangle inequality for integrals in the last step above.) In particular, T_σ maps \mathcal{V} to itself, which is equivalent to the consistency requirement mentioned above.

Here \mathcal{V} is assumed to be a subset of the Borel measurable functions in \mathbb{R}^X so that the integral in (10.15) makes sense. (The integral takes values in $\{-\infty\} \cup \mathbb{R}$ because $Q(x, a, v)$ is already assumed to have this property.)

Uniqueness of σ -value functions holds because $bm\mathsf{X}$ is a complete metric space when paired with the supremum norm and, under the boundedness assumption (10.16),

Lemma 10.3.1. *The operator T_σ is a contraction of modulus β on $bm\mathsf{X}$.*

Proof. Fix σ in Σ . We have already shown that T_σ maps $\mathcal{V} = bm\mathsf{X}$ to itself. In addition, for any v, w in $bm\mathsf{X}$ we have

$$\begin{aligned} |T_\sigma v(x) - T_\sigma w(x)| &= \beta \left| \int v(x') \Pi(x, \sigma(x), dx') - \int w(x') \Pi(x, \sigma(x), dx') \right| \\ &\leq \int \Pi(x, \sigma(x), dx') \beta |v(x') - w(x')| \leq \beta \|v - w\|_\infty \end{aligned}$$

Taking the supremum over all $x \in \mathsf{X}$ yields the desired result. \square

To obtain optimality results we will make use of proposition 10.2.5. To apply this proposition we need some additional structure. The standard conditions are as follows:

Assumption 10.3.1. In addition to the boundedness condition 10.16,

- (i) the reward function r is continuous on G ,
- (ii) Γ is continuous and compact valued and
- (iii) the Markov kernel Π has the **Feller property**.

Here the statement that Π has the Feller property means that

$$\Pi h(x, a) := \int h(x') \Pi(x, a, dx')$$

is continuous on G whenever $h \in bc\mathsf{X}$.

The next result is a relatively simple implication of proposition 10.2.5.

Proposition 10.3.2. *If assumption 10.3.1 holds, then*

- (i) *T is a contraction of modulus β on $bc\mathbb{X}$.*
- (ii) *The value function v^* is the unique solution of the Bellman equation in $bc\mathbb{X}$ and $T^n v \rightarrow v^*$ uniformly as $n \rightarrow \infty$ for every $v \in bc\mathbb{X}$.*
- (iii) *A policy σ in Σ is optimal if and only if*

$$\sigma(x) \in \operatorname{argmax}_{a \in \Gamma(x)} \left\{ r(x, a) + \beta \int v^*(x') \Pi(x, \sigma(x), dx') \right\} \quad \text{for all } x \in \mathbb{X} \quad (10.18)$$

and at least one such policy exists.

Remark 10.3.1. An important special case is the general finite state Markov decision problem discussed in §3.3.3. The conditions in assumption 10.3.1 are all satisfied trivially in this setting when we set the metric on \mathbf{A} and \mathbb{X} to be the discrete metric (implying that all functions are continuous). Hence proposition 10.3.2 fully covers optimality for that section.

Proof of proposition 10.3.2. All of the claims in proposition 10.3.2 will be verified if we can check the conditions of proposition 10.2.5. The nontrivial claims are (a) that $(x, a) \mapsto Q(x, a, v)$ is continuous on \mathbb{G} for all $v \in bc\mathbb{X}$ and bounded on \mathbb{G} for at least one $v \in bc\mathbb{X}$, and (b) that the contraction condition (10.14) holds.

Regarding (a), if we fix any $v \in bc\mathbb{X}$, then

$$Q(x, a, v) = r(x, a) + \beta \int v(x') \Pi(x, a, dx')$$

is clearly bounded by $M + \beta \|v\|_\infty$. It is also continuous by continuity of r and the Feller property of Π .

Regarding (b), the contraction condition (10.14) holds because, given v, w in $bm\mathbb{X}$,

$$\begin{aligned} |Q(x, a, v) - Q(x, a, w)| &\leq \beta \left| \int v(x') \Pi(x, a, dx') - \int w(x') \Pi(x, a, dx') \right| \\ &\leq \beta \int |v(x') - w(x')| \Pi(x, a, dx') \end{aligned}$$

which is dominated by $\beta \|v - w\|_\infty$. □

[add more examples]

10.3.2 Optimal Stopping and Search

The MDP section summarizes what the general theory implies for MDPs. Do the same here for optimal stopping. (Note that optimal stopping problems can be cast as MDPs but a direct treatment is clearer.)

What general things can we say?

When do we have threshold rules?

Include Bandit problems, or put in own section?

Also read tsitsiklis and van roy paper in to-read/optimal-stopping.

10.3.3 Recursive Utility

10.3.4 Shortest Paths

It's a motivating example. Treat a simple case.

10.4 Properties and Extensions

Add roadmap. Here we return to generic MDPs.

10.4.1 Properties of the Value Function

When is the value function increasing? When is it concave?

We treat a class of feasible policies Σ and an MDP where Assumption 10.2.1 on page 303 holds. In this setting we know (by Theorem 10.2.3) that v^* lies in $\hat{\mathcal{V}}$ and is the unique solution to the Bellman equation in \mathcal{V} .

10.4.1.1 Monotonicity

We seek sufficient conditions for the value function to be increasing in the state. We take X to be partially ordered by \preceq and let $i\hat{\mathcal{V}}$ be the increasing functions in $\hat{\mathcal{V}}$.

Assumption 10.4.1. The correspondence Γ satisfies $\Gamma(x) \subset \Gamma(x')$ whenever $x \leq x'$. In addition, the map $x \mapsto Q(x, a, v)$ is increasing for all $v \in i\hat{\mathcal{V}}$, in the sense that

$$x \preceq x' \text{ implies } Q(x, a, v) \leq Q(x', a, v) \text{ for all } a \in \Gamma(x').$$

Proposition 10.4.1. *If the conditions of Assumption 10.4.1 hold, then v^* is increasing.*

Proof. By Lemma 5.3.2 on page 166, it suffices to show that T is invariant on $i\hat{\mathcal{V}}$, since T is, by assumption, globally stable on $\hat{\mathcal{V}}$ and, in addition, $i\hat{\mathcal{V}}$ is a closed subset of $\hat{\mathcal{V}}$.⁴ To see that this holds, pick any $v \in i\hat{\mathcal{V}}$ and fix x and x' with $x \preceq x'$. Since T is already invariant on $\hat{\mathcal{V}}$, we need only show that Tv is increasing. But this must be so, since, by Assumption 10.4.1,

$$\sup_{a \in \Gamma(x)} Q(x, a, v) \leq \sup_{a \in \Gamma(x')} Q(x, a, v) \leq \sup_{a \in \Gamma(x')} Q(x', a, v).$$

Hence $Tv(x) \leq Tv(x')$ and T is invariant on $i\hat{\mathcal{V}}$. □

[Add some examples](#)

10.4.1.2 Concavity

Next we seek sufficient conditions for the value function to be concave. In this section we assume that X and A are convex subsets of finite dimensional Euclidean space.

Assumption 10.4.2. The set of feasible state-action pairs G is convex and $(x, a) \mapsto Q(x, a, v)$ is concave on G whenever v is concave on X .

The convexity requirement on G in Assumption 10.4.2 is equivalent to the statement that, for all x, x' in X , all $a \in \Gamma(x)$ all $a' \in \Gamma(x')$ and all $\lambda \in [0, 1]$, we have

$$\lambda a + (1 - \lambda)a' \in \Gamma(\lambda x + (1 - \lambda)x').$$

By taking $x = x'$, we see that each set $\Gamma(x)$ is convex in A .

Proposition 10.4.2. *If the conditions of Assumption 10.4.1 hold, then v^* is concave.*

Proof. Let $c\hat{\mathcal{V}}$ be the concave functions in $\hat{\mathcal{V}}$. By a similar argument to the one used in the proof of Proposition 10.4.1, it suffices to show that T is invariant on $c\hat{\mathcal{V}}$. To this end, fix v in $c\hat{\mathcal{V}}$, λ in $[0, 1]$ and $x_0, x_1 \in X$. Let a_i satisfy $Tv(x_i) = Q(x_i, a_i, v)$ for each i . Let $x_\lambda = \lambda x_0 + (1 - \lambda)x_1$ and $a_\lambda = \lambda a_0 + (1 - \lambda)a_1$. By convexity of G , we know that a_λ lies in $\Gamma(x_\lambda)$, which gives

$$Tv(x_\lambda) \geq Q(x_\lambda, a_\lambda, v) \geq \lambda Q(x_0, a_0, v) + (1 - \lambda)Q(x_1, a_1, v).$$

⁴Recall our assumption that the convergence in the metric on $\hat{\mathcal{V}}$ implies pointwise convergence. So if $\{v_n\}$ is a sequence in $i\hat{\mathcal{V}}$ converging to some $v \in \hat{\mathcal{V}}$ and x and x' satisfy $x \preceq x'$, then taking the limit in $v_n(x) \leq v_n(x')$ gives $v(x) \leq v(x')$. Hence $v \in i\hat{\mathcal{V}}$.

The right hand side is $\lambda T v(x_0) + (1 - \lambda) T v(x_1)$, which proves concavity of $T v$. Hence T is invariant on $c\hat{\mathcal{V}}$, and the claim in Proposition 10.4.2 holds. \square

[Add examples](#)

10.4.2 Properties of Policies

As in §10.4.1, we treat a class of feasible policies Σ and an MDP where Assumption 10.2.1 on page 303 holds. We are interested in conditions under which optimal policies are unique, continuous and/or increasing. Throughout this section, we assume that X and A are convex subsets of finite dimensional Euclidean space. These sets are ordered by the usual pointwise partial order over vectors.

10.4.2.1 Uniqueness and Continuity

Since Assumption 10.2.1 is in force, we know that at least one optimal policy exists in Σ . The question we ask now is, when is it unique? Not surprisingly, uniqueness can be obtained with a form of strict concavity.

Assumption 10.4.3. Assumption 10.4.2 is satisfied and, in addition, $a \mapsto Q(x, a, v)$ is strictly concave on $\Gamma(x)$ for all x in X and all concave v in X .

Proposition 10.4.3. *If Assumption 10.4.3 holds, then the optimal policy is unique.*

Proof. Let Assumption 10.4.3 hold. An optimal policy exists, since we are assuming the conditions of Theorem 10.2.3. Thus, only uniqueness needs to be shown. By the same theorem, a policy is optimal if and only if it is v^* -greedy. So it suffices to show that there cannot be two such policies in Σ . To see this, observe that v^* is concave on X by Proposition 10.4.2. Hence, under Assumption 10.4.2, the map $a \mapsto Q(x, a, v^*)$ is strictly concave at each x . Strictly concave functions have unique maximizers, so the v^* -greedy policy is unique in Σ . \square

Proposition 10.4.4. *If Assumption 10.4.3 holds, Γ is continuous and compact-valued and $(x, a) \mapsto Q(x, a, v)$ is continuous on \mathbb{G} at all v in $\hat{\mathcal{V}}$, then the unique optimal policy is continuous.*

Proof. This follows directly from Proposition 10.4.3 and Lemma 10.1.1 on page 299. \square

[Add examples](#). Consider adding some examples where continuity fails.

10.4.2.2 Monotonicity

When is the optimal policy increasing? Strictly increasing differences: When is the derivative of the RHS of the Bellman equation with respect to the action increasing in the state variable? See Jensen's 2007 paper, theorem 4. Increasing diffs without differentiability. This is useful for the income fluctuation problem, where the value function is concave but not differentiable.

10.4.2.3 Concavity and Convexity

See Jensen's paper Monotone Comparative Statics in Ordered Vector Spaces, available on his web site.

10.4.3 Plan Factorizations

[to be added]

Chapter 11

Some Measure and Functional Analysis

Elementary probability starts with discrete problems like counting balls in urns. Then densities are introduced and results like the law of total probability are stated twice: once for discrete probability mass functions and once for densities. But there is a nicer, more unified and more powerful way to study all these problems, using a branch of mathematics called measure theory.

This chapter gives a fast-paced introduction to measure. We will see that the theory is valuable not only for probability theory but also for significant parts of functional analysis, mainly because measure theory also provides a modern, rigorous and flexible framework for understanding integration.

11.1 Normed Vector Spaces

To be added.

11.1.1 Abstract Vector Spaces

Abstract vector space is, as the name suggests, an abstraction of Euclidean space. Formally, a **vector space** (also called a linear space) is a nonempty set V with a notion of addition (a map $+$ from $V \times V$ to V) and scalar multiplication (a map \cdot from $\mathbb{R} \times V$ to V) such that for all $u, v, w \in V$ and $\alpha, \beta \in \mathbb{R}$,

- (i) $u + (v + w) = (u + v) + w$

- (ii) $u + v = v + u$
- (iii) there exists an element $0 \in V$ s.t. $u + 0 = u$ for all $u \in V$
- (iv) for all $u \in V$, there exists a $v \in V$ such that $u + v = 0$
- (v) $\alpha \cdot (\beta \cdot u) = (\alpha \cdot \beta) \cdot u$
- (vi) $1 \cdot u = u$
- (vii) $\alpha \cdot (u + v) = \alpha \cdot u + \alpha \cdot v$
- (viii) $(\alpha + \beta) \cdot u = \alpha \cdot u + \beta \cdot u$

The classic example is Euclidean space \mathbb{R}^d with usual notions of addition and scalar multiplication discussed in §2.3.1. The element 0 in point (iii) is the d -vector of zeros. Usually \cdot is not shown. All of the axioms are satisfied under this identification. (It would be shocking if this wasn't true, since Euclidean space serves as the model for these axioms!)

Example 11.1.1. Consider \mathbb{R}^X , the set of real-valued functions on nonempty set X . This is a vector space when paired with the usual notions of addition and scalar multiplication of functions (defined on page 4). The zero element is $f \equiv 0$. The axioms above are easily verified, following as they do from the basic field properties of \mathbb{R} .

As stated above, Euclidean space \mathbb{R}^d with the usual notions of addition and scalar multiplication is a vector space. But Euclidean space is also a special case of example 11.1.1, corresponding to the setting where X is finite. Let's state this clearly for the record:

Lemma 11.1.1. *If X is a finite set containing d elements, then*

$$\mathbb{R}^X \ni h = (h(x_1), \dots, h(x_d)) \longleftrightarrow \begin{pmatrix} h_1 \\ \vdots \\ h_d \end{pmatrix} \in \mathbb{R}^d \quad (11.1)$$

is a one-to-one correspondence between \mathbb{R}^d and the function space \mathbb{R}^X .

On the left hand side, the function h is identified by the set of values that it takes on X . Henceforth we regard \mathbb{R}^X and \mathbb{R}^d as the same set expressed in different ways whenever X has d elements.

Returning to the general case, all vector spaces we exploit in these notes are either (a) the vector space \mathbb{R}^X defined in example 11.1.1 for some special choice of X , or (b) some subset of \mathbb{R}^X where the condition for inclusion in the subset interacts nicely with

addition and scalar multiplication. To clarify this point, recall that a **linear subspace** of vector space V is a set U such that

$$\alpha, \beta \in \mathbb{R} \text{ and } u, v \in U \implies \alpha u + \beta v \in U \quad (11.2)$$

The proof of the following result is an easy exercise.

Proposition 11.1.2. *If V is a vector space when paired with $(+, \cdot)$ and U is a linear subspace of V , then U itself is a vector space when paired with the same notion of addition and scalar multiplication.*

The beauty of this result is that, now we know \mathbb{R}^X is a vector space, to check whether or not $U \subset \mathbb{R}^X$ is a vector space we just need to test whether (11.2) holds.

Example 11.1.2. The set $b\mathbb{X}$ of bounded functions in \mathbb{R}^X is a vector space. Indeed, if f and g are bounded on X , then so is $\alpha f + \beta g$ for any scalars α and β , as follows from the triangle inequality.

Example 11.1.3. The set $bc\mathbb{X}$ of continuous functions in $b\mathbb{X}$ is a vector space, since continuity is preserved under addition and scalar multiplication (and hence the condition in (11.2) holds).

A **linear combination** of vectors u_1, \dots, u_k in V is a vector $y = \alpha_1 u_1 + \dots + \alpha_k u_k$ where $\alpha_1, \dots, \alpha_k$ are scalars. The set of all (by definition, finite) linear combinations of elements of a subset X of V is called the **span** of X , denoted by $\text{span}(X)$. For example, the span of the canonical basis vectors $\{e_1, \dots, e_d\}$ in \mathbb{R}^d is equal to all of \mathbb{R}^d .

A family of vectors $X \subset V$ is called **linearly independent** if

$$\alpha_1 u_1 + \dots + \alpha_k u_k = 0 \text{ implies } \alpha_1 = \dots = \alpha_k = 0$$

Given a linear subspace S of V , a finite subcollection $b_1, \dots, b_k \in S$ form a **basis** of S if, for all $u \in S$, there exist unique scalars $\alpha_1, \dots, \alpha_k$ such that $u = \sum_{i=1}^k \alpha_i b_i$.

The definitions of **convexity** of subsets of a vector space V , as well as **convexity** and **concavity** of functions is identical to the case of $V = \mathbb{R}^n$, as given in §2.3.1.4.

Example 11.1.4. Consider the set $\mathcal{P}(X)$ of distributions on a discrete set X , as defined in §3.1.2. (A distribution is a $\varphi \in \mathbb{R}^X$ such that $\varphi(x) \geq 0$ for all $x \in X$ and $\sum_x \varphi(x) = 1$.) This is a convex subset of \mathbb{R}^X .

Example 11.1.5. Let X be any set and recall from §5.1.3 that the order interval $[g, h]$ in \mathbb{R}^X defined by $g, h \in \mathbb{R}^X$ is all $f \in \mathbb{R}^X$ such that $g \leq f \leq h$. (We are using the pointwise order.) The order interval $[g, h]$ is a convex subset of \mathbb{R}^X .

11.1.2 Norms on Vector Space

In many of the metric spaces that we've studied, such as the ℓ_p space of 5.1.4, the metric is generated by a **norm**, which, in an abstract sense, is a map $\|\cdot\|$ from the underlying space V to \mathbb{R}_+ satisfying, for each u, v in M and each $\alpha \in \mathbb{R}$,

$$\begin{aligned} \|u\| &\geq 0 && \text{(nonnegativity)} \\ \|u\| = 0 &\iff u = 0, && \text{(positive definiteness)} \\ \|\alpha u\| &= |\alpha| \|u\| \text{ and} && \text{(positive homogeneity)} \\ \|u + v\| &\leq \|u\| + \|v\|. && \text{(triangle inequality)} \end{aligned}$$

Of course, for this definition to make sense, we need M to have several things working for us. For identifiability, we need M to have a “zero element.” In Euclidean space this is the zero vector, while in function spaces such as bX and ℓ_p this is the function everywhere equal to zero. For homogeneity and the triangle inequality we need well defined notions of scalar multiplication and addition respectively. Abstract spaces with this structure are called linear spaces.

If we endow a vector space V with a norm $\|\cdot\|$ defined on V (see the definition above), then $(V, \|\cdot\|)$ is collectively called a **normed linear space**. If the metric induced by the norm in the sense of $d(u, v) = \|u - v\|$ is complete, then $(V, \|\cdot\|)$ is called a **Banach space**.

Example 11.1.6. Euclidean vector space \mathbb{R}^d with $\rho(x, y) = \|x - y\|$ is a Banach space (see the discussion of completeness on page 135).

Example 11.1.7. Recall that in example 5.1.2 on page 130, we imposed a distance on f, g in bX via

$$d_\infty(f, g) := \|f - g\|_\infty \quad \text{where} \quad \|f\|_\infty := \sup_{x \in X} |f(x)|$$

The pair $(bX, \|\cdot\|_\infty)$ forms a Banach space. The completeness of this space is inherited from the completeness of \mathbb{R} . See, for example, section 3.2 of Aliprantis and Border (1999).

Example 11.1.8. We previously discussed the fact that bcX is a closed subset of bX , and that closed subsets of complete metric spaces are complete. Hence $(bcX, \|\cdot\|_\infty)$ forms a Banach space.

Example 11.1.9. Recall that in example 5.1.4 on page 130 we defined the map

$$\|h\|_p := \left\{ \sum_{x \in X} |h(x)|^p \right\}^{1/p}$$

on $\ell_p X := \{h \in \mathbb{R}^X : \|h\|_p < \infty\}$. The pair $(\ell_p X, \|\cdot\|_p)$ is a Banach space. We'll discuss this again in a more general setting in §11.3.5.

11.1.3 Linear Operators

Analogous to the finite dimensional case (see §2.3.1.2), a map A from one vector space U to another vector space V is called **linear** if

$$A(\alpha u + \beta v) = \alpha A(u) + \beta A(v) \quad \text{for all } \alpha, \beta \in \mathbb{R} \text{ and all } u, v \text{ in } U$$

In this context, A is usually called a **linear operator**. For example, a matrix $A \in \mathcal{M}_{n \times k}$ is a linear operator from \mathbb{R}^k to \mathbb{R}^n when identified with the map $x \mapsto Ax$. (Here, as before, $\mathcal{M}_{n \times k}$ denotes the set of all $n \times k$ real matrices.)

If U and V are normed linear spaces, then the **operator norm** of A is defined as

$$\|A\| := \sup\{\|Au\| : \|u\| = 1\} \tag{11.3}$$

When $\|A\|$ is finite, A is called a **bounded linear operator**. The set of all bounded linear operators from U to V will be denoted $L(U, V)$. If $U = V$ then we write $L(U)$. The operator norm is in fact a *norm* on $L(U, V)$, since for all $A, B \in L(U, V)$ and all $\alpha \in \mathbb{R}$,

- (i) $\|A\| \geq 0$ and $\|A\| = 0 \iff A = 0$
- (ii) $\|\alpha A\| = |\alpha| \|A\|$ for any scalar α
- (iii) $\|A + B\| \leq \|A\| + \|B\|$

The proofs are left as an exercise.

As discussed in §2.3.1.2, when $U = \mathbb{R}^k$, $V = \mathbb{R}^n$ and A is a matrix, $\|A\|$ is often called the **spectral norm** of A . As in the finite dimensional case,

- we can alternatively define $\|A\|$ as the supremum of $\|Au\|/\|u\|$ over all $u \neq 0$.
- The **submultiplicative property** holds, in that, if A and B are elements of $L(U, V)$ and $L(V, W)$ respectively, where U , V and W are vector spaces, then $\|A \circ B\| \leq \|A\| \|B\|$.

- Iteratively applying the submultiplicative property gives $\|A^j\| \leq \|A\|^j$ for any $j \in \mathbb{N}$ and $A \in L(U, U)$, where A^j is the j -th composition of A with itself.

Once we have a norm on $L(U, V)$, we have an induced metric given by $d(A, B) = \|A - B\|$, and $L(U, V)$ will be a Banach space whenever this metric is complete.

Theorem 11.1.3. *If V is a Banach space, then $L(U, V)$ is also a Banach space.*

For example, $\mathcal{M}_{n \times k}$ endowed with the operator (i.e., spectral) norm is a Banach space, since \mathbb{R}^n is a Banach space (see page 318).

Let V be a normed linear space and let A be an element of $L(V)$. A complex scalar λ is called an **eigenvalue** of $A \in L(V)$ if there exists a nonzero vector e such that $Ae = \lambda e$. The **spectrum** of A , typically denoted $\sigma(A)$, is the set of all scalar λ such that $\lambda I - A$ fails to be bijective on V . Any eigenvalue λ lies in $\sigma(A)$ because if $Ae = \lambda e$ for some nonzero e , then $\lambda I - A$ maps e to 0, while also mapping 0 to 0. Hence $\lambda I - A$ is not bijective.

For $A \in L(V)$, the **spectral radius** of A is defined as

$$r(A) := \sup\{|\lambda| : \lambda \in \sigma(A)\}$$

If V is a Banach space, then given an $A \in L(V)$, we have

$$\|A^k\|^{1/k} \rightarrow r(A) \quad \text{as } k \rightarrow \infty$$

This relationship is called **Gelfand's formula**.

Here's an extension of the Neumann series lemma to the infinite dimensional case:

Theorem 11.1.4 (Neumann series lemma in Banach space). *If V is a Banach space, A is an element of $L(V)$ and $r(A) < 1$, then $I - A$ is nonsingular and*

$$(I - A)^{-1} = \sum_{j=0}^{\infty} A^j$$

The proof is almost identical to the finite dimensional case.

11.1.4 Finite Dimensional Vector Space

A vector space V is called **finite dimensional** if, for some $n \in \mathbb{N}$, there exists a set of linearly independent vectors $\{e_i\}_{i=1}^n$ in V such that $\{e_i\}_{i=1}^n$ spans V . As discussed in §11.1.1, the set $\{e_i\}$ is then a basis for V .

For example, \mathbb{R}^n is finite dimensional, with a basis provided by the n canonical basis vectors in \mathbb{R}^n . Similarly, if X is finite, then \mathbb{R}^X is finite dimensional, with dimension equal to the number of elements in X . A basis is provided by the functions f_i defined by $f_i(x) = \mathbb{1}\{x = i\}$. Any $g \in \mathbb{R}^X$ can be expressed as a linear combination of these basis vectors via

$$g(x) = \sum_{i \in X} g(i) \mathbb{1}\{x = i\} = \sum_{i \in X} g(i) f_i(x)$$

In fact, as shown in (11.1) on page 316, \mathbb{R}^X and \mathbb{R}^n are in one-to-one correspondence when X has n elements, and the functions f_i are mapped to the canonical basis vectors under this correspondence.

Theorem 11.1.5. *If V is finite dimensional, then any two norms $\|\cdot\|_a$ and $\|\cdot\|_b$ on V are equivalent, in the sense that there exist positive constants α, β such that*

$$\alpha \|x\|_a \leq \|x\|_b \leq \beta \|x\|_a \quad \text{for all } x \in M$$

Add a cite. xxx

Theorem 11.1.6. *Every linear function from a finite dimensional normed linear space to another normed linear space is continuous.*

See, for example, [Cheney \(2013\)](#), p. 26.

Two normed linear spaces U and V are called **isometrically isomorphic** if there exists a bijective linear map T from U to V such that $\|Tu\| = \|u\|$ for all u in U . For example, when X has n elements, \mathbb{R}^X with the ℓ_p norm and \mathbb{R}^n with the analogous norm $\|u\| = (\sum_{i=1}^n |u_i|^p)^{1/p}$ are isometrically isomorphic under the identification given in (11.1). In this case we are only talking about a relabeling of essentially equivalent objects. In other cases the isomorphic relationship is less trivial.

Theorem 11.1.7. *Every finite dimensional normed vector space is isomorphic to \mathbb{R}^n .*¹

This needs to be revisited. The point that the metric spaces are equivalent has already been made in Theorems 11.1.5. And the isometric isometry discussion above is not pertinent.

To get an immediate sense of why Theorems 11.1.5–11.1.7 are important, consider the Bolzano–Weierstrass theorem (theorem 5.1.4). Since all equivalent metrics induce the same precompact sets and the same bounded sets, *any* metric induced by a norm on a finite dimensional vector space has the property that its precompact and bounded

¹This result assumes that the underlying field of the vector space is \mathbb{R} . It is possible to define vector spaces over different fields, but we do not consider that case in this book.

sets coincide. More generally, the class of open sets, closed sets, bounded sets, compact sets and precompact sets in finite dimensional normed linear space does not depend on the particular norm being used. The next theorem states one of these facts for the record.

Theorem 11.1.8 (Heine–Borel). *A set K in a finite dimensional vector space is compact if and only if it is closed and bounded.*

This result is similar in content to the Bolzano–Weierstrass theorem on page 151, and sometimes goes under the same name.

11.1.5 Ordered Vector Space

Let V be a vector space and let \leqslant be a partial order on V . If \leqslant is compatible with the algebraic structure on V , in the sense that

- (i) $x \leqslant y$ and $\alpha \geqslant 0$ implies $\alpha x \leqslant \alpha y$, and
- (ii) $x \leqslant y$ implies $x + b \leqslant y + b$ for any $b \in V$,

then the pair (V, \leqslant) is called an **ordered vector space**. The set $P := \{x \in V : x \geqslant 0\}$ is called the **positive cone** of V .

Recalling the definition in §5.1.3, a map S from a subset D of an ordered vector space V into V is, as usual, called isotone, if, for any $x, y \in D$ with $x \leqslant y$ we have $Sx \leqslant Sy$. Extending the definition for scalar valued functions, S is called **convex** if

$$S(\lambda x + (1 - \lambda)y) \leqslant \lambda Sx + (1 - \lambda)Sy \text{ whenever } x, y \in D \text{ and } 0 \leqslant \lambda \leqslant 1$$

The map S is called **concave** if

$$\lambda Sx + (1 - \lambda)Sy \leqslant S(\lambda x + (1 - \lambda)y) \text{ whenever } x, y \in D \text{ and } 0 \leqslant \lambda \leqslant 1$$

In these two definitions we are assuming that D is convex, so the convex combination $\lambda x + (1 - \lambda)y$ lies in the domain of S .

When V is any vector space, a nonempty subset C of V is called a **pointed convex cone** (or just a **cone**, depending on the source) in V if

- (i) C is convex,
- (ii) $x \in C$ and $-x \in C$ implies $x = 0$ and
- (iii) $\alpha x \in C$ whenever $x \in C$ and $\alpha \geqslant 0$.

Each cone C on V introduces a partial order \leqslant on V via

$$x \leqslant y \iff y - x \in C \quad (11.4)$$

EXERCISE 1. Show that the relation in (11.4) is indeed a partial order on V that is compatible with the algebraic structure in the sense of (i)–(ii) above, and that C is the positive cone of (V, \leqslant) . Show, in addition that if V is a normed linear space and C is closed in V , then \leqslant is a closed partial order (recalling the definition of the latter on page 145).

EXERCISE 2. Show conversely that if (V, \leqslant) is an ordered vector space, then the positive cone in V is a pointed convex cone.

An ordered vector space is called a **Riesz space** if (V, \leqslant) is also a lattice (see §5.1.3). In any Riesz space (V, \leqslant) , we can define the **absolute value** of an element x of V by setting

$$|x| := x^+ + x^- \quad \text{where} \quad x^+ := x \wedge 0 \text{ and } x^- := (-x) \wedge 0$$

Now let $\|\cdot\|$ be a norm on our Riesz space (V, \leqslant) , so that $(V, \|\cdot\|)$ is a normed linear space. Any norm compatible with the order structure on V , in the sense that $\|x\| \leqslant \|y\|$ whenever $|x| \leqslant |y|$ is called a **lattice norm**, and $(V, \leqslant, \|\cdot\|)$ is called a **normed Riesz space**. If, in addition, $(V, \|\cdot\|)$ is a Banach space, then $(V, \leqslant, \|\cdot\|)$ is called a **Banach lattice**. In some instances we denote it more simply by V . Note that the partial order in any Banach lattice (in fact any normed Riesz space) is always closed (see theorem 15.1 of [Zaanen \(2012\)](#)).

Example 11.1.10. If X is any compact set, then bcX , the set of continuous real valued functions on X with the supremum norm and pointwise partial order, is a Banach lattice. See p. 89 of [Zaanen \(2012\)](#) for details.

If V is a given Banach lattice with positive cone P , we have from the preceding discussion that $x \leqslant y$ iff $y - x \in P$. We write $x \ll y$ if, in addition, $y - x$ is interior to P . The positive cone P is called **solid** if it has nonempty interior. For example, if $V = bcX$, then $0 \ll f$ if and only if f is strictly positive everywhere on X . In particular, the positive cone of bcX is solid.

Let V be a Banach lattice. The following results are from chapter 2 of [Zhang \(2012\)](#), where full proofs are available. The first is a pure existence result:

Theorem 11.1.9. Let a and b be distinct points in V with $a \leq b$ and let S be an isotone map from $[a, b]$ to V . If $S([a, b])$ is precompact in V , then S has minimal and maximal fixed points \check{x} and \hat{x} in $[a, b]$ and

$$a \leq S^n a \leq \check{x} \leq S^n b \leq \hat{x}, \quad \forall n \in \mathbb{N}$$

Proof. See theorem 2.1.1 of Zhang (2012). \square

Now refer back to results from zhang on convex/concave operators discussed in §5.1.3.3, just state that the continue to hold when the domain is a Banach lattice. Need to redefine \gg and \ll .

11.2 Inner Product Space

[roadmap]

11.2.1 Inner Products

An **inner product** on a vector space V is a map $\langle \cdot, \cdot \rangle$ from $V \times V$ to \mathbb{R} such that, for any x, y, z in V and $\alpha, \beta \in \mathbb{R}$,

- (i) $\langle \alpha x + \beta y, z \rangle = \alpha \langle x, z \rangle + \beta \langle y, z \rangle$
- (ii) $\langle x, y \rangle = \langle y, x \rangle$ and
- (iii) $\langle x, x \rangle \geq 0$ and $\langle x, x \rangle = 0 \implies x = 0$.

One classic example of an inner product on a vector space is the usual notion of inner product on Euclidean vector space, as discussed in §2.3.1.

Any inner product $\langle \cdot, \cdot \rangle$ defines a norm on V via

$$\|x\| := \sqrt{\langle x, x \rangle}$$

If this norm induces a complete metric on V , then the pair $(V, \langle \cdot, \cdot \rangle)$ is called a **Hilbert space**.

Example 11.2.1. Ordinary Euclidean space with the usual inner product is a Hilbert space.

Example 11.2.2. Let X be any countable set and let $\ell_2(X)$ be all $h \in \mathbb{R}^X$ such that $\sum_x h(x)^2 < \infty$. Then $\langle g, h \rangle := \sum_x g(x)h(x)$ is an inner product on $\ell_2(X)$ under which $\ell_2(X)$ becomes a Hilbert space. We treat a more general version of this result in §11.3.4.

Two vectors x and z in V are said to be **orthogonal**, and we write $x \perp z$, if

$$x, z \in V \quad \text{and} \quad \langle x, z \rangle = 0$$

We call $x \in V$ **orthogonal to S** whenever S is a linear subspace of V and $x \perp z$ for all $z \in S$ and write $x \perp S$.

The **orthogonal complement** of linear subspace S is defined as

$$S^\perp := \{x \in V : x \perp S\}$$

S^\perp is always a linear subspace of V . To see this, fix $x, y \in S^\perp$ and $\alpha, \beta \in \mathbb{R}$. If $z \in S$, then

$$\langle \alpha x + \beta y, z \rangle = \alpha \langle x, z \rangle + \beta \langle y, z \rangle = \alpha \times 0 + \beta \times 0 = 0$$

Hence $\alpha x + \beta y \in S^\perp$, as was to be shown.

EXERCISE 3. Show that, for any subspace $S \subset V$, we have $S \cap S^\perp = \{0\}$.

A set of vectors $\{x_1, \dots, x_k\} \subset V$ is called an **orthogonal set** if $x_i \perp x_j$ whenever $i \neq j$. The **Pythagorean law** states that, if $\{x_1, \dots, x_k\}$ is an orthogonal set, then

$$\|x_1 + \dots + x_k\|^2 = \|x_1\|^2 + \dots + \|x_k\|^2$$

Orthogonality implies independence in the sense that, if $X \subset V$ is an orthogonal set and $0 \notin X$, then X is linearly independent. It's an exercise to check this. Note that the converse is not true, although we have algorithms for converting linearly independent sets to orthogonal sets that span the same space—see the Gram–Schmidt theorem.

Now we turn to orthogonal projection. Many problems in analysis are related to orthogonal projection, including least squares linear regression, conditional expectation, Gram–Schmidt orthogonalization and QR decomposition. The basic problem is this: Given $y \in \mathbb{R}^n$ and subspace S , find closest element of S to y . Formally, we wish to solve for

$$\hat{y} := \underset{z \in S}{\operatorname{argmin}} \|y - z\| \tag{11.5}$$

11.2.2 Orthogonal Projection

[Change this to Hilbert space setting]

Theorem 11.2.1 (The Orthogonal Projection Theorem). *If $y \in \mathbb{R}^d$ and S is a linear subspace of \mathbb{R}^d , then there exists a unique solution to (11.5). The solution \hat{y} is the unique vector in \mathbb{R}^n such that*

- (i) $\hat{y} \in S$
- (ii) $y - \hat{y} \perp S$

The vector \hat{y} is called the **orthogonal projection of y onto S** . To see why properties (i)–(ii) are sufficient for \hat{y} to be a solution, fix $y \in \mathbb{R}^n$ and let S be a linear subspace of \mathbb{R}^n . Let \hat{y} have these properties and let z be any other point in S . We have

$$\|y - z\|^2 = \|(y - \hat{y}) + (\hat{y} - z)\|^2 = \|y - \hat{y}\|^2 + \|\hat{y} - z\|^2$$

and hence $\|y - z\| \geq \|y - \hat{y}\|$, as claimed.

Let's agree to write $P = \text{proj } S$ to indicate that Py represents the projection \hat{y} of y onto S . P is called the **orthogonal projection mapping onto S** . The next theorem collects useful facts concerning its properties.

Theorem 11.2.2. *Let S be a subspace of \mathbb{R}^d and let $P = \text{proj } S$. For any $y \in \mathbb{R}^n$, we have*

- (i) $Py \in S$,
- (ii) $y - Py \perp S$,
- (iii) $\|y\|^2 = \|Py\|^2 + \|y - Py\|^2$,
- (iv) $\|Py\| \leq \|y\|$, and
- (v) $Py = y$ if and only if $y \in S$.

Moreover, if $M = \text{proj } S^\perp$, then

$$Py \perp My \quad \text{and} \quad y = Py + My \quad \text{for all } y \in \mathbb{R}^n$$

An orthogonal set $O \subset \mathbb{R}^n$ is called an **orthonormal set** if $\|u\| = 1$ for all $u \in O$. If S is a linear subspace of \mathbb{R}^n , O is orthonormal in S and $\text{span } O = S$, then O is called an **orthonormal basis** of S . For example, the canonical basis $\{e_1, \dots, e_n\}$ forms an orthonormal basis of \mathbb{R}^n .

Projecting y onto an orthonormal basis $\{u_1, \dots, u_k\}$ of a subspace S has the representation

$$Py = \sum_{i=1}^k \langle y, u_i \rangle u_i, \quad \forall y \in \mathbb{R}^n \tag{11.6}$$

To see this, fix $y \in \mathbb{R}^n$ and let Py be as defined above. Clearly, Py is in S . We claim that $y - Py \perp S$ also holds. It suffices to show that $y - Py$ is orthogonal to any basis

element. This is true because

$$\left\langle y - \sum_{i=1}^k \langle y, u_i \rangle u_i, u_j \right\rangle = \langle y, u_j \rangle - \sum_{i=1}^k \langle y, u_i \rangle \langle u_i, u_j \rangle = 0$$

It's easy to check that if S is any linear subspace of \mathbb{R}^n and $P = \text{proj } S$, then P is a linear function from \mathbb{R}^n to \mathbb{R}^n . It follows (recalling equation (2.17) and the surrounding discussion) that $P = \text{proj } S$ has a unique representation as a matrix. This matrix can be expressed as

$$P = X(X'X)^{-1}X'$$

whenever $X \in \mathcal{M}_{n \times k}$ has the property that its columns form a basis of S . To see this, fix $y \in \mathbb{R}^n$. Our claim is that $Py \in S$ and $y - Py \perp S$. The first claim is true because $Py = X(X'X)^{-1}X'y = Xa$ when $a := (X'X)^{-1}X'y$. The second claim holds because it is equivalent to the statement $y - X(X'X)^{-1}X'y \perp Xb$ for all $b \in \mathbb{R}^k$. The latter is true because if $b \in \mathbb{R}^k$, then

$$(Xb)'[y - X(X'X)^{-1}X'y] = b'[X'y - X'y] = 0$$

As an example, suppose that $U \in \mathcal{M}_{n \times k}$ has orthonormal columns. Let u_i be the i -th column of U for each i , let $S := \text{span } U$ and let $y \in \mathbb{R}^n$. We know that the projection of y onto S is $Py = U(U'U)^{-1}U'y$. Since U has orthonormal columns, we have $U'U = I$. Hence $Py = UU'y = \sum_{i=1}^k \langle u_i, y \rangle u_i$, which recovers our claim in (11.6).

In the above setting, where $P = X(X'X)^{-1}X'$, the matrix $M = I - P$ is sometimes called the **annihilator**. As an exercise, try showing that P and M are both idempotent and symmetric.

11.2.3 Overdetermined Systems

Consider system of equations $Xb = y$. Given X and y , we seek a $b \in \mathbb{R}^k$ satisfying this system. We assume throughout this section that

- (i) X lies in $\mathcal{M}_{n \times k}$ and has linearly independent columns,
- (ii) b is in \mathbb{R}^k and y is in \mathbb{R}^n and
- (iii) $n > k$, so that the system $Xb = y$ is **overdetermined** (i.e., the system has more equations than unknowns).

Intuitively, when the system is overdetermined, we may not be able find a b that satisfies all n equations. We can see this more clearly if we take a geometric view,

beginning with the observation that

$$\text{span}(X) = \{\text{all } Xb \text{ with } b \in \mathbb{R}^K\}$$

Therefore, there exists a b such that $Xb = y$ if and only if $y \in \text{span}(X)$. Given that $n > k$, there is usually no such b , since

- y is an arbitrary point in \mathbb{R}^n ,
- $\text{span}(X)$ has dimension k , and
- a k -dimensional subspace has measure zero in \mathbb{R}^n whenever $k < n$.

Hence the usual approach when dealing with overdetermined systems of equations is to accept that an exact solution may not exist and look instead for an approximate solution. We want this approximate solution to be as “good” as possible, or, in other words, we seek the minimizer

$$\beta := \underset{b \in \mathbb{R}^k}{\operatorname{argmin}} \|y - Xb\| \quad (11.7)$$

The vector $\hat{\beta}$ is called the **least squares** solution to the overdetermined system.

Theorem 11.2.3. *Under our assumptions, the unique minimizer of $\|y - Xb\|$ over $b \in \mathbb{R}^k$ is the vector*

$$\hat{\beta} := (X'X)^{-1}X'y$$

Note that $\hat{\beta}$ is well defined because $X'X$ is nonsingular under our assumption that the columns of X are linearly independent.

Proof of theorem 11.2.3. Note that $X\hat{\beta} = X(X'X)^{-1}X'y = Py$, where Py is the orthogonal projection of y onto $\text{span}(X)$. From the orthogonal projection theorem, we have

$$\|y - Py\| \leq \|y - z\| \text{ for any } z \in \text{span}(X)$$

In other words,

$$\|y - X\hat{\beta}\| \leq \|y - Xb\| \text{ for any } b \in \mathbb{R}^K$$

as was to be shown. □

Next let's discuss regression, which is one application of orthogonal projection. Given pairs $(x, y) \in \mathbb{R}^{K+1}$, consider the problem of choosing $f: \mathbb{R}^K \rightarrow \mathbb{R}$ to minimize the **risk**

$$R(f) := \mathbb{E}[(y - f(x))^2]$$

If \mathbb{E} is unknown we can't compute this directly. If, however, a sample is available, a natural strategy is to replace the risk with the **empirical risk** obtained from the sample, which in the present setting is defined as

$$\min_{f \in \mathcal{F}} \frac{1}{N} \sum_{n=1}^N (y_n - f(x_n))^2$$

Here we'll restrict \mathcal{F} to the class of linear functions (in other words, we're going to focus on linear regression). Dropping $1/N$ in the definition of empirical risk—since positive multiplicative constants don't change minimizers—the problem is

$$\min_{b \in \mathbb{R}^K} \sum_{n=1}^N (y_n - b' x_n)^2$$

Switching to matrix notation, let

$$y := \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{pmatrix}, \quad x_n := \begin{pmatrix} x_{n1} \\ x_{n2} \\ \vdots \\ x_{nK} \end{pmatrix} = n\text{-th obs on all regressors}$$

and

$$X := \begin{pmatrix} x'_1 \\ x'_2 \\ \vdots \\ x'_N \end{pmatrix} := \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1K} \\ x_{21} & x_{22} & \cdots & x_{2K} \\ \vdots & \vdots & & \vdots \\ x_{N1} & x_{N2} & \cdots & x_{NK} \end{pmatrix}$$

We assume throughout that $N > K$ and X is full column rank. With a small amount of effort you will be able to confirm that

$$\|y - Xb\|^2 = \sum_{n=1}^N (y_n - b' x_n)^2$$

Since increasing transforms don't affect minimizers we have

$$\operatorname{argmin}_{b \in \mathbb{R}^K} \sum_{n=1}^N (y_n - b' x_n)^2 = \operatorname{argmin}_{b \in \mathbb{R}^K} \|y - Xb\|$$

We already know how to solve this problem: By our results on overdetermined systems,

the solution is

$$\hat{\beta} := (X'X)^{-1}X'y$$

In the context of linear regression, this vector is sometimes called the **least squares estimator**. That terminology doesn't make much sense in the present context, however, since we never claimed that $\hat{\beta}$ is an estimator of anything in particular. In the present context it is best thought of more simply as the vector that minimizes empirical risk.

Let P and M be the projection and annihilator associated with the matrix X :

$$P := X(X'X)^{-1}X' \quad \text{and} \quad M := I - P$$

The **vector of fitted values** is

$$\hat{y} := X\hat{\beta} = Py$$

The **vector of residuals** is

$$\hat{u} := y - \hat{y} = y - Py = My$$

Applying the orthogonal projection theorem, we obtain

$$\hat{u} \perp \hat{y} \quad \text{and} \quad y = \hat{y} + \hat{u}$$

EXERCISE 4. The following definitions are standard: total sum of squares = $\|y\|^2$, sum of squared residuals = $\|\hat{u}\|^2$, explained sum of squares = $\|\hat{y}\|^2$. Show that the total sum of squares equals the explained sum of squares plus the sum of squared residuals.

11.3 Tools from Integration Theory

Mathematics concerns two kinds of objects: functions and sets. Functions without structure are difficult to manage so we put them into groups. In high school and in calculus, nice functions—the ones that we can manage—are those functions that are continuous or smooth. Working with continuous functions is convenient not only because these functions have attractive properties (think of the beautiful and powerful Mean Value Theorem) but also because they tend to reproduce themselves: Addition, subtraction, multiplication and composition of functions all preserve continuity.

As we move on to larger, more complicated problems, the set of continuous functions turns out to be too small. This is when we have to make the leap from continuous

functions to *measurable functions*. Fortunately, measurable functions are also quite manageable, and, just like continuous functions they tend to reproduce themselves. Let's review the key ideas.

11.3.1 Measure Theory

[roadmap]

11.3.1.1 Measurability

To define measurable functions, we first have to define measurable sets. And to define measurable sets, we need the notion of σ -algebras. To this end, let X be any nonempty set. A collection of subsets \mathcal{A} of X is called a **σ -algebra** on X if

- (i) $X \in \mathcal{A}$,
- (ii) $A \in \mathcal{A}$ implies $A^c \in \mathcal{A}$, and
- (iii) if $\{A_n\}_{n \geq 1}$ is a sequence contained in \mathcal{A} , then $\cup_n A_n \in \mathcal{A}$.

A pair (X, \mathcal{A}) where X is a nonempty set \mathcal{A} is a σ -algebra on X is called a **measurable space**.

Points (ii) and (iii) tell us that \mathcal{A} is “stable” under the taking of complements and unions. By De Morgan’s law $(\cap_n A_n)^c = \cup_n A_n^c$, any σ -algebra is stable under countable intersections too. By (i) and (ii), $\emptyset \in \mathcal{A}$ also holds.

Example 11.3.1. Given any set X and any subset $A \subset X$, the family of sets $\mathcal{A} := \{X, A, A^c, \emptyset\}$ is a σ -algebra on X .

Example 11.3.2. The power set $\wp(X)$ is a σ -algebra on X , as is the pair $\{\emptyset, X\}$.

Example 11.3.3. The set of all circles in \mathbb{R}^2 is not a σ -algebra on \mathbb{R}^2 . Indeed, this family is not stable under the taking of either unions or intersections. For the same reason, the set of all rectangles on \mathbb{R}^2 is not a σ -algebra on \mathbb{R}^2 .

One way to define a σ -algebra is to take a collection \mathcal{C} of subsets of X , and consider the smallest σ -algebra that contains this collection.

Definition 11.3.1. Let \mathcal{C} be any collection of subsets of X . The **σ -algebra generated by \mathcal{C}** is the smallest σ -algebra on X that contains \mathcal{C} , and is denoted by $\sigma(\mathcal{C})$.²

²More precisely, $\sigma(\mathcal{C})$ is the intersection of all σ -algebras on X that contain \mathcal{C} . One can show that $\sigma(\mathcal{C})$ is always a well defined σ -algebra, since the intersection is nonempty (it at least contains $\wp(X)$) and any intersection of σ -algebras is again a σ -algebra.

Now let X be a metric space (or a topological space—see §12.1.1 of the appendix). The family of **Borel sets** on X , denoted by either \mathcal{B} or \mathcal{B}_X depending on whether or not the underlying space is clear, is defined as the σ -algebra generated by the open sets of X . Evidently \mathcal{B} contains not only all the open subsets of X but also all the closed ones. From these sets we can continue taking complements and countable unions and everything we produce must be a Borel set. In fact it turns out that every set we work with in day-to-day analysis is a Borel set.

Given two arbitrary measurable spaces (X, \mathcal{A}) and (Y, \mathcal{B}) , a function f from X to Y is called **$(\mathcal{A}, \mathcal{B})$ -measurable** if

$$f^{-1}(B) \text{ is in } \mathcal{A} \text{ whenever } B \in \mathcal{B}.$$

In other words, measurable functions are those functions that pull measurable sets back to measurable sets. If Y is a metric space and \mathcal{B} is its Borel sets, then we will say that f is **Borel measurable**. It can be shown in this case (see, e.g., [Cinlar \(2011\)](#), Proposition 2.3) that f is Borel measurable if and only if either one of the following apparently weaker conditions are satisfied:

- (i) $f^{-1}(G)$ is in \mathcal{A} whenever G is open in Y
- (ii) Y is a Borel subset of \mathbb{R} and $f^{-1}((-\infty, \alpha))$ is in \mathcal{A} for all $\alpha \in \mathbb{R}$.

From this result it is immediate that every continuous function from X to Y is also Borel measurable.

EXERCISE 5. Let (X, \mathcal{A}) be a measurable space and let B be a subset of X . Consider $\mathbb{1}_B$ as a map from X to \mathbb{R} . Show that $\mathbb{1}_B$ is a Borel measurable function if and only if $B \in \mathcal{A}$.

The definition of Borel measurability is not particularly intuitive, since the class of Borel sets is so large it's difficult to get a sense of what it does and doesn't contain. At the same time, Borel functions are ubiquitous in applied mathematics. Why?

When we solve dynamic programming problems and other problems from functional analysis, we start off by specifying a class of functions within which we hope to find a solution. What class should we use?

The class of continuous functions is the go-to class of “well behaved” functions in elementary mathematics. But while this class is closed under uniform limits (see example 5.1.10), it is not closed under pointwise limits,³ which makes it hard to work

³For example, the pointwise limit of the sequence of functions $\{f_n\}$ given by $f_n(x) = x^n$ on $[0, 1]$ is discontinuous.

with in some instances. On the other hand, the set of Borel functions *is* closed under the taking of pointwise limits:

Lemma 11.3.1. *If (X, \mathcal{A}) is a measurable space and $\{f_n\}$ is a sequence of real valued Borel measurable functions on (X, \mathcal{A}) , then the functions*

$$f := \sup_n f_n, \quad f := \limsup_{n \rightarrow \infty} f_n, \quad \text{and} \quad f := \lim_{n \rightarrow \infty} f_n,$$

are all Borel measurable on (X, \mathcal{A}) whenever they exist. The same is true if we replace sup with inf.

In fact, in our setting, the set of Borel measurable functions is precisely the smallest class of functions that contains the continuous functions and is closed under the taking of pointwise limits (see, e.g., §11.7 of [Kechris \(2012\)](#)).

It is also true that compositions of Borel measurable functions are also Borel measurable, and, when the functions are real-valued, that Borel measurability is preserved under algebraic operations. The next lemma gives one statement of these results:

Lemma 11.3.2. *If (X, \mathcal{A}) is a measurable space, α, β are real scalars and f and g are real-valued Borel measurable functions on (X, \mathcal{A}) , then the functions*

$$\alpha f + \beta g, \quad fg \quad \text{and} \quad f/g \text{ when } g \neq 0$$

are all Borel measurable functions on (X, \mathcal{A}) .

See [Cinlar \(2011\)](#), chapter 1, section 2 for proofs.

11.3.1.2 Measures

Through the theory constructed above, we can identify broad classes of sets and functions that are relatively well behaved (e.g., Borel sets and Borel functions). This opens the way to analyzing how to (a) measure these sets and (b) integrate the functions. The first step is to introduce the notion of a **measure**, which is a map μ from a σ -algebra \mathcal{A} to $[0, \infty]$ satisfying

- (i) $\mu(\emptyset) = 0$ and
- (ii) $\mu(\cup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mu(A_n)$ whenever $\{A_n\} \subset \mathcal{A}$ is disjoint.

Here disjointness of $\{A_n\}$ means that any two distinct sets in this sequence are disjoint.

Example 11.3.4. Let $X = \{x_1, x_2, \dots\}$ be a countable set paired with $\wp(X)$, the set of all its subsets. Define $c: \wp(X) \rightarrow \mathbb{R}_+$ by $c(A) = |A|$, where $|A|$ is the number of elements in A , with $c(A) = \infty$ if A is infinite. Some thought will convince you that c is a measure on $\wp(X)$. This measure is called the **counting measure**.

Example 11.3.5. It can be proved (see, e.g., Dudley (2002), §3.2) that there exists exactly one measure on the Borel subsets of \mathbb{R}^2 that assigns area to rectangles in the usual way (i.e., area is the product of the two sides). Its is called **Lebesgue measure** and often denoted by λ . The measure λ also assigns the usual measures of area to other standard sets, such as circles. Indeed, if C is a circle with radius r , then $\lambda(C) = \pi r^2$. In other words, λ is the measure that gives us the classical notion of area in the plane, as taught in basic geometry.

Example 11.3.6. Example 11.3.5 introduced Lebesgue measure for \mathbb{R}^2 . An analogous version, also called Lebesgue measure and denoted by λ , is defined on the Borel subsets of \mathbb{R}^k for every $k \in \mathbb{N}$. For example, if $k = 1$ and I is an interval in \mathbb{R} , then $\lambda(I)$ is the length of that interval.

Returning to the general case of a measure μ on measurable space (X, \mathcal{A}) , , if $\mu(X) < \infty$, then μ is called **finite**. If $\mu(X) = 1$, then μ is called a **probability measure**. If X is a metric space and $\mathcal{A} = \mathcal{B}$ (the Borel sets), then μ is called a **Borel measure**. If $\mathcal{A} = \mathcal{B}$ and $\mu(X) = 1$, then μ is called a **Borel probability measure**. For a Borel probability measure μ , the value $\mu(B)$ usually is interpreted as the probability that, when a random element of X is selected, that element is in B .

Example 11.3.7. Take the setting of example 11.3.4 but now let the measure be given by $\nu(A) = \sum_{x \in A} p(x)$ instead of $|A|$, where p is a function from X to \mathbb{R}_+ . It's not hard to see that ν defines a measure on $\wp(X)$. If $\sum_{x \in X} p(x) < \infty$ then ν is a finite Borel measure. If the sum equals unity then ν is a Borel probability measure.

A **measure space** is a triple (X, \mathcal{A}, μ) where (X, \mathcal{A}) is a measurable space and μ is a measure on \mathcal{A} . If $\mu(X) = 1$, then the measure space is also called a **probability space**. In this case it is common to write the measure space as $(\Omega, \mathcal{F}, \mathbb{P})$. A **random variable** on probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is an $(\mathcal{F}, \mathcal{B})$ -measurable map X from Ω to \mathbb{R} paired with its Borel sets \mathcal{B} . More generally, given measurable space (E, \mathcal{E}) , an E -valued **random element** on probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is an $(\mathcal{F}, \mathcal{E})$ -measurable map X from Ω to E . The **distribution** of this random element X is the probability measure P defined by

$$P(B) = \mathbb{P}\{\omega \in \Omega : X(\omega) \in B\} \quad (B \in \mathcal{E})$$

Here's a reassuring fact that implies Borel probability measures on standard sets like \mathbb{R} are not strange creatures at all. Rather, they are fundamental objects that we've been using all along.

Theorem 11.3.3 (Lebesgue-Stieltjes representation theorem). *There is a one-to-one correspondence between \mathcal{F} , the set of cumulative distribution functions on \mathbb{R} , and the set of Borel probability measures on \mathbb{R} . For each $F \in \mathcal{F}$, the corresponding probability measure μ satisfies*

$$\mu((a, b]) = F(b) - F(a) \text{ for all } a, b \in \mathbb{X} \text{ with } a < b$$

More generally, we have the interpretation

$$\mu(B) = \text{probability that } x \in B \text{ when } x \text{ is drawn from } F$$

This representation in terms of probability measures is attractive because it assigns probabilities to subsets of \mathbb{X} directly, rather than in the roundabout way that F does, and because measures can be defined in abstract settings that cdfs can't handle. Moreover, from measures we can construct a powerful theory of integration, a topic we turn to in §11.3.2.

11.3.2 Integration

[roadmap]

11.3.2.1 Abstract Integrals

When we study calculus, we learn a basic notion of integration—the “area under the curve” of a given function. The Fundamental Theorem of Calculus helps us find these areas, leading to results such as

$$\int_0^1 x^2 dx = \frac{1}{3}. \quad (11.8)$$

We can also prove more general results, such as linearity of integration. That is,

$$\int_0^1 [\alpha f(x) + \beta g(x)] dx = \alpha \int_0^1 f(x) dx + \beta \int_0^1 g(x) dx \quad (11.9)$$

for continuous functions f, g and scalars α, β .

While these results are elegant and important, we need a far more general notion of integration to make sense of all the problems treated in this text. To get a sense of why this is true, let's go right back to the objective function for the household in (2.12) on page 29. The expectation \mathbb{E} used in this objective is, like all expectations, defined

as a kind integral. But the formal definition is not trivial, because we are integrating over an infinity of shocks (due to the infinite horizon). How does one integrate over an infinite-dimensional space?

Let's answer this question in stages, beginning with an abstract notion of an integral. Our definition extends the elementary notion that an integral is a mapping that assigns numbers to functions (the area under their curve) and has certain monotonicity and linear properties (e.g., (11.9) above). To state the definition, we take (X, \mathcal{A}) to be measurable space and $m\mathcal{A}_+$ to be the set of nonnegative real-valued Borel measurable functions on (X, \mathcal{A}) . We define an **integral** on $m\mathcal{A}_+$ to be a function $I: m\mathcal{A}_+ \rightarrow [0, \infty]$ such that

- (i) $I(f) = 0$ when $f = 0$ everywhere on X ,
- (ii) $f_1 \leq f_2 \leq \dots$ and $\lim_{n \rightarrow \infty} f_n = f$ implies $\lim_{n \rightarrow \infty} I(f_n) = I(f)$, and
- (iii) $\alpha, \beta \geq 0$ and $f, g \in m\mathcal{A}_+$ implies $I(\alpha f + \beta g) = \alpha I(f) + \beta I(g)$.

The limit in (ii) is a pointwise limit, so that $\lim_{n \rightarrow \infty} f_n = f$ means $\lim_{n \rightarrow \infty} f_n(x) = f(x)$ for every $x \in X$.

We can see that the standard Riemann integral taught in high school roughly fits this pattern (e.g., compare (iii) and (11.9) above). However, the details contain some crocodiles.⁴ Fortunately, measure theory comes to the rescue here: we can use measure theory to define an integral obeying (i)–(iii) and implementing the Riemann integral as a special case. This integral is constructed using Lebesgue measure, as introduced in example 11.3.5.

Rather than constructing this integral in isolation, let us state the following important result, proved in chapter 1 of [Cinlar \(2011\)](#). It states that *every* measure on a measurable space creates a unique and well defined integral.

Theorem 11.3.4. *Let (X, \mathcal{A}) be measurable space. There exists a one to one correspondence between the set of measures on (X, \mathcal{A}) and the set of integrals on $m\mathcal{A}_+$. For any measure μ , the corresponding integral I_μ satisfies*

$$I_\mu(\mathbb{1}_B) = \mu(B) \text{ whenever } B \in \mathcal{A} \quad (11.10)$$

The value $I_\mu(f)$ is called the **integral of f under μ** and the following notation is common:

$$I_\mu(f) := \int f \, d\mu := \int f(x) \mu(dx).$$

⁴For example, limiting properties like (ii) are tricky because, for one thing, it is hard to guarantee that the limiting function f in (ii) is even Riemann integrable.

Example 11.3.8. Let \mathcal{B} be the Borel sets on \mathbb{R} and let λ be Lebesgue measure. According to theorem 11.3.4, there exists an integral I_λ that associates to each Borel measurable function from \mathbb{R} to \mathbb{R}_+ a number $I_\lambda(f)$, often written as $\int f d\lambda$ or just $\int f(x) dx$. If f is continuous and supported on an interval $[a, b]$, then $I_\lambda(f)$ equals $\int_a^b f(x) dx$ in the standard Riemann sense (see, e.g., [Cinlar \(2011\)](#), §1.4). For example, in the example in (11.8), with $f(x) = x^2$ on $[0, 1]$ and zero elsewhere, we have $\int f d\lambda = 1/3$.

The integral I_λ introduced in example 11.3.8 is called the the **Lebesgue integral**, and it extends the standard Riemann integral to a larger set of functions (the Borel measurable functions), while at the same time guaranteeing that the attractive properties (i)–(iii) in the definition of the integral will hold.

Equation (11.10) makes sense in this setting because if, say, $f = \mathbb{1}_{[a,b]}$, then

$$I_\lambda(f) = \lambda([a, b]) = b - a,$$

where the first equality is by (11.10) and the second is by the fact that Lebesgue measure assigns length to intervals. The value $b - a$ is also what we would expect for the integral, since it is the area under the curve for this simple function.⁵

Example 11.3.9. Let $X = \mathbb{N}$ and let c be the counting measure from example 11.3.4. Let $x := \{x_n\}$ be any nonnegative sequence. We can view this sequence as a map from X to \mathbb{R}_+ and its integral is

$$\int x dc = \sum_{i=1}^{\infty} x_i. \tag{11.11}$$

That is, ordinary series are just a kind of integral. We prove a generalization of (11.11) in example 11.3.10 below.

Example 11.3.10. As in example 11.3.7, let X be countable, let $p: X \rightarrow \mathbb{R}$ be non-negative with $\sum_{x \in X} p(x) < \infty$ and let ν be the measure on the σ -algebra $\wp(X)$ defined by $\nu(A) = \sum_{x \in A} p(x)$. Then, for any $f: X \rightarrow \mathbb{R}_+$, the integral corresponding to ν is

$$\int f(x) \nu(dx) = \sum_{x \in X} f(x)p(x). \tag{11.12}$$

To see this, suppose first that f is zero off a finite set A contained in X . Then f can

⁵A more general perspective on (11.10) that you might find useful is as follows. Suppose we identify measurable sets with their indicator functions. Then μ already provides us with an “integral” over the indicators in $m\mathcal{A}_+$. The map I_μ extends the reach of this function to all of $m\mathcal{A}_+$.

be written as

$$f(y) = \sum_{x \in A} f(x) \mathbb{1}_{\{x\}}(y) \quad (y \in X).$$

By the linearity property in part (iii) of the definition of the integral and the fact that, by definition, $\int \mathbb{1}_B d\nu = \nu(B)$ for all $B \in \wp(X)$, we have

$$\int f(x) \nu(dx) = \sum_{x \in A} f(x) \nu(\mathbb{1}_{\{x\}}) = \sum_{x \in A} f(x) p(x) = \sum_{x \in X} f(x) p(x).$$

Hence (11.12) is valid. To handle arbitrary f , rather than just functions supported on finite sets, let f be any function from $X = \{x_1, x_2, \dots\}$ to \mathbb{R}_+ and let f_n be defined by $f_n(x_i) = f(x_i) \mathbb{1}\{i \leq n\}$. Each f_n is supported on a finite set, so

$$\int f_n d\nu = \sum_{i \leq n} f_n(x_i) p(x_i) = \sum_{i \leq n} f(x_i) p(x_i).$$

Since $\{f_n\}$ is monotone increasing and converges to f , by part (ii) of the definition of the integral we have

$$\int f(x) \nu(dx) = \lim_{n \rightarrow \infty} \int f_n d\nu = \sum_{i=1}^{\infty} f(x_i) p(x_i).$$

This is another way of writing (11.12).

11.3.2.2 Expectation

If μ is a probability measure and $w: X \rightarrow \mathbb{R}$, then one often writes $\mathbb{E}w(x)$ for the integral of $w(x)$ with respect to μ . That is,

$$\mathbb{E}w(x) = \int w d\mu$$

Here we are thinking of x as a random variable drawn from distribution μ and the integral corresponds to the **expectation** of $w(x)$ under μ .

11.3.2.3 Properties of Integrals

From the properties in theorem 11.3.4 we can deduce additional properties that the integral must satisfy. Before stating them, let us note that the notion of the integral extends to functions that take negative values, as well as just the nonnegative functions in $m\mathcal{A}_+$.

Indeed, if (X, \mathcal{A}, μ) is a measure space and $f \in m\mathcal{A}$ is not necessarily nonnegative, then we can still decompose it into the difference between two nonnegative functions via $f = f^+ - f^-$. Imposing linearity, we now set

$$\int f d\mu := \int f^+ d\mu - \int f^- d\mu.$$

The only risk here is that both terms on the right equal $+\infty$, in which case the integral is not well defined. If both integrals are finite we call f **integrable** with respect to μ .

Now, let us state some general properties. In what follows, we leave (X, \mathcal{A}, μ) fixed and write the integral $I_\mu(f)$ of f under μ as $\int f d\mu$.

First, every integral is isotone (with respect to the pointwise ordering over $m\mathcal{A}$), in the sense that

$$f \leq g \implies \int f d\mu \leq \int g d\mu. \quad (11.13)$$

To see this, observe that $g - f$ is nonnegative (and measurable) and hence $\int(g - f) d\mu$ is well defined and nonnegative. Now, using the linearity in part (iii) of theorem 11.3.4, we have

$$\int g d\mu = \int(g - f + f) d\mu = \int(g - f) d\mu + \int f d\mu \geq \int f d\mu.$$

Next we turn to limiting results. A battery of useful limit theorems exist for the integral we have defined. In our statements of these results, (X, \mathcal{A}, μ) is any measure space and f and f_n are $(\mathcal{A}, \mathcal{B})$ -measurable functions from X to \mathbb{R} for all $n \in \mathbb{N}$.

Theorem 11.3.5 (Fatou's Lemma). *If $f_n \geq 0$ for all n and $f = \liminf f_n$, then*

$$\int f d\mu \leq \liminf \int f_n d\mu.$$

Fatou's Lemma (upgraded here to a theorem) says, in essence, that the integral induced by any measure μ is lower semicontinuous on $m\mathcal{A}_+$.

Theorem 11.3.6. *Let $\lim_{n \rightarrow \infty} f_n = f$ hold μ -almost everywhere on X . If either*

(i) $0 \leq f_n \leq f_{n+1}$ for all $n \in \mathbb{N}$ or

(ii) there exists a $g \in m\mathcal{A}_+$ with $\int g d\mu < \infty$ and $|f_n| \leq g$ for all $n \in \mathbb{N}$, then

$$\lim \int f_n d\mu = \int f d\mu. \quad (11.14)$$

The first implication (i.e., (i) \implies (11.14)) is called the Monotone Convergence Theorem. The second is called the Dominated Convergence Theorem.

11.3.3 Products and Independence

Add a treatment of product sigma algebras. We use the concept of image measures so make sure they are defined. Define independence.

Given a distribution γ on a product space $\times_{i=1}^n \mathsf{X}_i$, paired with the product σ -algebra, we say that $\varphi \in \mathcal{D}(\mathsf{X})$ is the k -th marginal of γ if

$$\varphi(B) = \gamma\{(x_1, \dots, x_n) \in \times_{i=1}^n \mathsf{X}_i : x_k \in B\}.$$

Another way to say this is that φ is the image measure of the k -th projection map on $\times_{i=1}^n \mathsf{X}_i$ under γ .

11.3.4 Information and Conditioning

Include a brief treatment of conditional expectations in terms of projections and proofs of claims in §4.1.3.

11.3.5 L_p Spaces

Note that this is a Banach space — the benefit of the integration theory stated above.

One of the reasons we need the theory of integration developed above is in order to view spaces of integrable functions as normed linear spaces. We know from Lemma 11.3.2 that, given a measurable space $(\mathsf{X}, \mathcal{A})$, the class of Borel measurable functions from X to \mathbb{R} will be closed under addition and scalar multiplication, and hence forms a vector subspace of \mathbb{R}^X . It is, therefore, a vector space in its own right (see Proposition 11.1.2). But we still need a norm.

To this end, let μ be a measure on $(\mathsf{X}, \mathcal{A})$ and let $p \geq 1$. Consider the possibly infinite number

$$\|f\|_p := \int |f|^p d\mu$$

This looks like a norm but it isn't one yet. One issue is that it might be infinite. We can resolve this easily by defining $\mathcal{L}_p(\mathsf{X}, \mathcal{A}, \mu)$ to be the set of all Borel measurable real valued f functions on X such that $\|f\|_p < \infty$. So $\|\cdot\|_p$ is finite on this set by construction.

However, there is still one more problem: We may have $f \neq 0$ and yet $\|f\|_p = 0$. This is because a function that is equal to zero everywhere except a set E such that

$\mu(E) = 0$ has integral zero. Indeed, for such a function f ,

$$\int f \, d\mu = \int \mathbb{1}_E f \, d\mu + \int \mathbb{1}_{E^c} f \, d\mu = 0 + \int \mathbb{1}_{E^c} 0 \, d\mu = 0$$

For example, when $X = \mathbb{R}$ and μ is Lebesgue measure, the function $\mathbb{1}_{\mathbb{Q}}$ integrates to zero.

Apart from that, $\|\cdot\|_p$ has the other properties of a norm on $\mathcal{L}_p(X, \mathcal{A}, \mu)$. For example, the triangle inequality holds as a result of the **Minkowski inequality**, which, in the present setting, states that, for $f, g \in \mathcal{L}_p(X, \mathcal{A}, \mu)$,

$$\left\{ \int |f + g|^p \, d\mu \right\}^{\frac{1}{p}} \leq \left\{ \int |f|^p \, d\mu \right\}^{\frac{1}{p}} + \left\{ \int |g|^p \, d\mu \right\}^{\frac{1}{p}} \quad (11.15)$$

For these reason $\|\cdot\|_p$ is referred to as a **seminorm**.

From our seminorm we can create something approaching a metric via

$$d_p(f, g) = \|f - g\|_p \quad (f, g \in \mathcal{L}_p(X, \mathcal{A}, \mu))$$

It isn't quite a metric because $d_p(f, g) = 0$ does not imply $f = g$, since $\|\cdot\|_p$ is only a seminorm. Typically we refer to d_p as a **pseudometric**.

To turn a seminorm into a norm and a pseudometric into a metric, the usual trick is to regard all points at zero distance from each other as the same point. Formally, we partition the original space into *equivalence classes* of points at zero distance from one another, and consider the set of these classes as a new space. The distance between any two equivalence classes is just the distance between arbitrarily chosen members of each class. This value does not depend on the particular members chosen.

The normed linear space derived from the $\mathcal{L}_p(X, \mathcal{A}, \mu)$ is traditionally denoted $L_p(X, \mathcal{A}, \mu)$. Since two functions in $\mathcal{L}_p(X, \mathcal{A}, \mu)$ are at zero distance if and only if they are equal μ -almost everywhere, the new space $L_p(X, \mathcal{A}, \mu)$ consists precisely of equivalences classes of functions that are equal μ -almost everywhere.

Theorem 11.3.7. *The space $L_p(X, \mathcal{A}, \mu)$ paired with the norm $\|\cdot\|_p$ is a Banach space.*

Scheff  s identity provides a useful quantitative interpretation of d_1 distance between densities: For any densities f and g on (X, \mathcal{A}, μ) , we have

$$\|f - g\|_1 = 2 \times \sup_{B \in \mathcal{A}} \left| \int_B f \, d\mu - \int_B g \, d\mu \right| \quad (11.16)$$

11.4 Applications

[Add roadmap]

11.4.1 Asset Pricing Revisited

11.4.2 Job Search with Correlated Wage Draws

In this section we return to the job search problem of §6.1.1, now dropping the unrealistic IID assumption on wage draws. This will lead us to consider an unbounded wage process and require some measure theory and functional analysis to handle.

11.4.2.1 The Model

A typical model of wage dynamics admits some form of correlation (see, e.g., 4.2.3 on page 116). Often wage dynamics are specified as

$$w_t = \exp(z_t) + \exp(\mu + \sigma\zeta_t), \quad (11.17)$$

where

$$z_{t+1} = \rho z_t + d + s\varepsilon_{t+1} \quad (11.18)$$

for some $\rho \in (-1, 1)$ and $\{\zeta\}_{t \geq 1}$ and $\{\varepsilon\}_{t \geq 1}$ are both IID and standard normal. Thus, wages have a persistent component $\exp(z_t)$ and a transient component, both of which are lognormal.

Otherwise the model is unchanged. The worker can either accept an offer and work permanently at that wage or take unemployment compensation c and wait till next period. The value function satisfies the Bellman equation

$$v(w, z) = \max \left\{ \frac{w}{1 - \beta}, c + \beta \mathbb{E}_z v(w', z') \right\} \quad (11.19)$$

which can be compared with the original Bellman equation (6.1) on page 183. Here \mathbb{E}_z is expectation conditional on z . For an arbitrary function g we could more explicitly write this expectation as

$$\mathbb{E}_z g(w', z') = \int g [\exp(\rho z + d + s\varepsilon) + \exp(\mu + \sigma\zeta), \rho z + d + s\varepsilon] \varphi(d\varepsilon, d\zeta)$$

where z and the parameters are taken as given and φ is the $N(0, I)$ distribution on \mathbb{R}^2 .

A natural next step would be to introduce a Bellman operator corresponding to the Bellman equation (11.19) and proceed to analyze its properties. However, in this setting, just as in §6.1.2, there's a way to reduce dimensionality by refactoring the Bellman equation. This both simplifies analysis and accelerates computation.

As a first step, let $h(z)$ be the continuation value associated with current exogenous state z :

$$h(z) := c + \beta \mathbb{E}_z v(w', z') \quad (11.20)$$

(Here v can be thought of as a candidate value function.) Notice that h is a *function* now, as opposed to the IID setting (6.8) where the continuation value was just a constant. That a functional relationship between z and the continuation value exists is intuitive, since the current state can be used to predict future wages, which in turn determine future value.

Once we have h , the Bellman equation can be written as

$$v(w, z) = \max \left\{ \frac{w}{1 - \beta}, h(z) \right\}$$

Combining this with the definition of h , we see that the continuation value function satisfies

$$h(z) = c + \beta \mathbb{E}_z \max \left\{ \frac{w'}{1 - \beta}, h(z') \right\} \quad (11.21)$$

Note the similarity with (6.9).

The function h is defined on all of \mathbb{R} , since this is the domain of z . If we can obtain the solution h^* to this functional equation, we can use it to act optimally via the policy

$$\sigma^*(w, z) = \mathbb{1} \left\{ \frac{w}{1 - \beta} \geq h^*(z) \right\} \quad (11.22)$$

Put differently, we can stop when the current wage exceeds the reservation wage

$$w^*(z) := h^*(z)(1 - \beta)$$

11.4.2.2 Solving for the Continuation Value

We solve the functional equation (11.21) for h^* by introducing the operator $h \mapsto Qh$ defined by

$$Qh(z) = c + \beta \mathbb{E}_z \max \left\{ \frac{w'}{1 - \beta}, h(z') \right\} \quad (11.23)$$

By construction, any solution to (11.21) is a fixed point of Q and vice versa. But does such a fixed point exist? If we want to use a contraction map approach, in what space should we seek a contraction mapping?

One potential stumbling block is that $\{z_t\}$, being a Gaussian AR(1) process, is unbounded above. Unless we modify this feature, we cannot use a space of bounded functions for the domain of Q , as we did for the Bellman operator T in §6.1.1.2.

EXERCISE 6. Show that, for any real valued function h on \mathbb{R} such that Qh is well defined, we have $Qh(z) \rightarrow \infty$ as $z \rightarrow \infty$.

At this point a typical solution is to truncate the innovations ε and ζ , in order to eliminate the problem in exercise 6 and allow Q to map some space of bounded functions into itself. Another popular option is to simply discretize z . That is, the AR(1) process $\{z_t\}$ is replaced by a finite Markov chain. Both of these are reasonable options but we will try an alternative method under which we can establish a contraction without the need to modify our model.

The method is to pick some $p \geq 1$ and take as our function space the set $L_p(\psi) := L_p(\mathbb{R}, \mathcal{B}, \psi)$, where ψ is the stationary density of the AR(1) process (11.18). In other words, $L_p(\psi)$ is all Borel measurable functions g from \mathbb{R} to itself satisfying $\int |g(x)|^p \psi(x) dx < \infty$. In effect we require that $g(X)$ has finite p -th moment when $X \stackrel{d}{=} \psi$. The distance between two elements f and g of $L_p(\psi)$ is given by

$$d_p(f, g) := \left\{ \int |f(x) - g(x)|^p \psi(x) dx \right\}^{1/p}$$

(See (4.33) on page 113 for the specifics of the stationary density ψ . Since ρ in (11.18) lies in $(-1, 1)$, we know that this stationary density exists. More discussion of L_p spaces is given in §11.3.5.) Since $L_p(\psi)$ is a Banach space, the L_p metric is complete, providing us with a nice setting in which to pursue a contraction argument.

Regarding a suitable value for p , common choices are $p = 1$ and $p = 2$. In essence we are controlling the number of finite moments that we want the solution to have. For now let's keep it at some arbitrary value greater than 1.

Lemma 11.4.1. *Q is a self-mapping on $L_p(\psi)$.*

Proof. To see this, fix $h \in L_p(\psi)$. Since $L_p(\psi)$ is, like any vector space, closed under addition and scalar multiplication, it suffices to show that

$$\kappa(z) := \mathbb{E}_z \max \left\{ \frac{w'}{1 - \beta}, h(z') \right\} \tag{11.24}$$

lies in $L_p(\psi)$. But, by Jensen's inequality and the fact that, for nonnegative numbers, $a \vee b \leq a + b$, we have

$$|\kappa(z)|^p \leq \frac{1}{1-\beta} \mathbb{E}_z [\exp(z') + \exp(\mu + \sigma\zeta) + |h(z')|]^p$$

for any $z \in \mathbb{R}$. Let z_t be a draw from ψ , the preceding inequality and the law of iterated expectations yield

$$\begin{aligned} \mathbb{E}|\kappa(z_t)|^p &\leq \frac{1}{1-\beta} \mathbb{E} \mathbb{E}_{z_t} [(\exp(z_{t+1}) + \exp(\mu + \sigma\zeta_{t+1}) + |h(z_{t+1})|)^p] \\ &= \frac{1}{1-\beta} \mathbb{E} [(\exp(z_{t+1}) + \exp(\mu + \sigma\zeta_{t+1}) + |h(z_{t+1})|)^p] \end{aligned}$$

Applying Minkowski's inequality with respect to the joint distribution $\varphi \times \psi$, we see that $\int |\kappa(z)|^p \psi(z) dz$ will be finite if

- $\int \exp(pz) \psi(z) dz < \infty$
- $\int \exp[p(\mu + \sigma\zeta)] \varphi(\zeta) d\zeta < \infty$ and
- $\int |h(z)|^p \psi(z) dz < \infty$

The first two are true by the properties of the normal distribution and the third is true by assumption. So $Qh \in L_p(\psi)$ as claimed. \square

Proposition 11.4.2. *The operator Q is a contraction of modulus β on $L_p(\psi)$.*

Proof. By Jensen's inequality and (6.4) we have

$$\begin{aligned} |Qg(z) - Qh(z)|^p &\leq \beta^p \mathbb{E}_z \left| \max \left\{ \frac{w'}{1-\beta}, g(z') \right\} - \max \left\{ \frac{w'}{1-\beta}, h(z') \right\} \right|^p \\ &\leq \beta^p \mathbb{E}_z |g(z') - h(z')|^p \end{aligned}$$

Integrating both sides of the previous inequality with respect to ψ gives

$$\begin{aligned} \int |Qg(z) - Qh(z)|^p \psi(z) dz &\leq \beta^p \int \mathbb{E}_z |g(z') - h(z')|^p \psi(z) dz \\ &= \beta^p \int |g(z) - h(z)|^p \psi(z) dz \end{aligned}$$

Raising to the power of $1/p$ now gives

$$\left\{ \int |Qg(z) - Qh(z)|^p \psi(z) dz \right\}^{1/p} \leq \beta \left\{ \int |g(z) - h(z)|^p \psi(z) dz \right\}^{1/p}$$

or

$$\|Qg - Qh\|_p \leq \beta \|g - h\|_p$$

where $\|\cdot\|_p$ is the L_p norm. □

Since $L_p(\psi) := L_p(\mathbb{R}, \psi)$ is a Banach space, it follows from Proposition 11.4.2 and Banach's contraction mapping theorem that Q has a unique fixed point h^* in $L_p(\psi)$. This is the continuation value that we seek.

EXERCISE 7. Let c_a and c_b be two levels of unemployment compensation satisfying $c_a \leq c_b$. Let Q_a and Q_b be the corresponding continuation value operators (see (11.23)) and let h_a and h_b be their respective fixed points. Show that $h_a \leq h_b$ pointwise on \mathbb{R} .⁶

EXERCISE 8. Suppose the agent seeks to maximize lifetime value $\mathbb{E} \sum_{t=0}^{\infty} \beta^t u(y_t)$, where y_t is earnings at time t and u is a utility function. Letting $u(c) = \ln c$, write down the modified Bellman equation and the Q operator (11.23). How does the reservation wage change?

⁶Hint: Use Proposition 5.3.7 on page 171.

Chapter 12

Approximation and Optimization: Topics

[roadmap]

12.1 Optimal Transport

At this stage we just list results without intuition or context. That will come before, and then at some stage be connected to the results given here.

12.1.1 Some Topology

It can be valuable to discuss convergence without using metrics. For example, given a metric space X , we say that $\{\varphi_n\}$ in $\mathcal{D}(\mathsf{X})$ **converges weakly** to $\varphi \in \mathcal{D}(\mathsf{X})$ and write $\varphi_n \xrightarrow{w} \varphi$ if

$$\int g \, d\varphi_n \rightarrow \int g \, d\varphi \text{ as } n \rightarrow \infty \text{ for all } g \in bc\mathsf{X}.$$

For this common notion of convergence, one cannot always construct a metric ρ on $\mathcal{D}(\mathsf{X})$ such that $\varphi_n \xrightarrow{w} \varphi$ if and only if $\rho(\varphi_n, \varphi) \rightarrow 0$. This is one reason why we need to abstract from metric space.

Here is another: Let d be the usual Euclidean distance on \mathbb{R}^n . It is easy to check that if we scale d up by $\alpha > 0$, we get another metric on \mathbb{R}^n , and that the continuum of metrics $\{\alpha d\}_{\alpha>0}$ all generate exact the same open sets, closed sets, continuous functions, convergent sequences, etc. How can we reduce to a more unique description?

12.1.1.1 Topological Spaces

Recall that $\{u_n\}$ converges to u in a metric space M if the sequence $\{u_n\}$ is eventually in any ε -ball containing u . The concept of an ε -ball requires existence of a metric. However, we also have the following characterization of convergence:

EXERCISE 1. Let $\{u_n\}$ be a sequence in a metric space M . Show that $\{u_n\}$ converges to $u \in M$ if and only if, for any open set G containing u , there exists an $N \in \mathbb{N}$ such that $u_n \in G$ whenever $n \geq N$.

Thus, as long as we have a notion of open sets, we will have a concept of convergence (i.e., tails of sequences are contained in open neighborhoods of points). Motivated by these ideas, we define a **topology** on a nonempty set T to be a family $\mathcal{G} \subset \wp(T)$ such that

- (i) \emptyset and T are both contained in \mathcal{G} ,
- (ii) $\{G_i\}_{i \in I} \in \mathcal{G}$ implies $\cap_i G_i \in \mathcal{G}$ whenever I is finite, and
- (iii) $\{G_\alpha\} \subset \mathcal{G}$ implies $\cup_\alpha G_\alpha \in \mathcal{G}$ for any $\{G_\alpha\} \subset \mathcal{G}$.

Elements of \mathcal{G} are referred to as **open sets**. Properties (ii)–(iii) are often stated as: \mathcal{G} is closed under finite intersections and arbitrary unions. The pair (T, \mathcal{G}) is called a **topological space**. Clearly, if $(V, \| \cdot \|)$ is a metric space, then its open sets form a topology on T . A topology is an abstraction of this idea.

Example 12.1.1. In the case of Euclidean space \mathbb{R}^d , the **Euclidean topology** is the family of open sets generated by Euclidean distance. More generally, if $(V, \| \cdot \|)$ is a normed vector space, then the **norm topology** on V is the open sets generated by $d(x, y) = \|x - y\|$.

Example 12.1.2. Given a nonempty set T , the set of all subsets of T forms a topology. This is called the **discrete topology** on T . In this topology, every convergent sequence is eventually constant. (Why?)

If \mathcal{G} is a topology and d is a metric such that its open sets equal \mathcal{G} , then d is said to **metrize** the topology \mathcal{G} . If at least one such metric exists, then the topology \mathcal{G} is called **metrizable**. For example, the Euclidean topology is metrized by $d(x, y) = \|x - y\|$, where $\| \cdot \|$ is any norm on \mathbb{R}^d (see Exercise 10).

EXERCISE 2. The discrete topology on T is metrized by the discrete metric of Example 5.1.5 (p. 131). Generalize this statement by showing that $d(x, y) = \mathbb{1}\{x \neq y\}(w(x) + w(y))$ is a metric that also metrizes the discrete topology.

Given a topology \mathcal{G} on M and a point $u \in M$, we call G a **neighborhood** of u if it is open and contains u .¹ A sequence $\{u_n\}$ in \mathcal{G} is said to **converge** to a point u in M if, given any neighborhood G of u , there exists an $N \in \mathbb{N}$ such that $u_n \in G$ for all $n \geq N$. The point u is called the **limit** of the sequence and we write $u_n \rightarrow u$ or $\lim_{n \rightarrow \infty} u_n = u$.

Here are two more important definitions:

- A function f from one topological space (M, \mathcal{G}) to another space (N, \mathcal{H}) is called **continuous** if, for every $H \in \mathcal{H}$ we have $f^{-1}(H) \in \mathcal{G}$.
- A subset K of a topological space (M, \mathcal{G}) is compact if and only if every open cover of K has a finite subcover.

In both cases, we are adopting characterizations from our results on metric spaces (see the characterization of continuity in (5.1.1) and Theorem 5.1.5 respectively) as our definitions. The value of these characterizations in the current context is, obviously, that they only involve open sets, rather than direct reference to a metric.

Lemma 12.1.1. *Let f be a function topological space (M, \mathcal{G}) to topological space (N, \mathcal{H}) . If f is continuous and $u_n \rightarrow u$ in (M, \mathcal{G}) , then $f(u_n) \rightarrow f(u)$ in (N, \mathcal{H}) .*

Proof. Let f have the stated properties and let $u_n \rightarrow u$ in M . Let H be a neighborhood of $f(u)$. By continuity of f , the set $G := f^{-1}(H)$ is open in M . Since $f(u) \in H$ we have $u \in G$. Hence G is a neighborhood of u , and there exists an $n_0 \in \mathbb{N}$ such that $u_n \in G$ whenever $n \geq n_0$. For these n we have $f(u_n) \in H$, so $f(u_n) \rightarrow f(u)$ in N . \square

Note that the converse is not in general true (see, e.g., Maddox (1988), Section 2.4).

EXERCISE 3. One of the fundamental properties of metric spaces is that sequences have unique limits (see Exercise 2). Show by example that, for topological spaces, the same is not true.

Thus, to add a more structure to a topology \mathcal{G} on M , we can assume that \mathcal{G} is a **Hausdorff topology**, which means that, for each distinct $u, v \in M$, we can find open sets G_u and G_v such that $u \in G_u$, $v \in G_v$ and $G_u \cap G_v = \emptyset$. A topological space with a Hausdorff topology is called a **Hausdorff space**.

EXERCISE 4. Prove that (i) in a Hausdorff space, sequences can have at most one limit, and (ii) every metrizable topology is Hausdorff.

¹In some texts, a neighborhood of u is a subset B of M satisfying $u \in G \subset B$ for some $G \in \mathcal{G}$. This definition would also work for our purposes.

12.1.1.2 Product Topology

Define. Connect to pointwise convergence. Note that products of compact sets are compact. Connect to weak topology. Discuss metrizability.

12.1.1.3 Polish Spaces

Introduce. Valuable for probability.

A **Polish space** is a topological space X that is both separable and completely metrizable. This means that there exists a metric ρ such that ρ metrizes the topology on X and (X, ρ) is complete and separable.

Example 12.1.3. Sometimes it is convenient to take a set such as $(0, 1)$ or $(0, \infty)$ as the state space for a given model. While these sets are not complete under the usual Euclidean metric, they are completely metrizable. In fact any G_δ subset of \mathbb{R}^n has this property. Explain and add reference.

Example 12.1.4. If $\{X_i\}$ is a countable family of Polish spaces, then the product set $\prod_i X_i$ is also Polish when paired with the product topology. This applies to spaces such as the set of all real sequences, or the set of all nonnegative sequences.

Let X be metric space. Recall that $\mathcal{D}(X)$ denotes the set of Borel probability measures on X .

Define the weak topology on $\mathcal{D}(X)$ in terms of the product topology. Comment on terminology.

Let ρ be a metric that generates the topology on X . On $\mathcal{D}(X)$, we introduce the **Fortet–Mourier distance**, defined by

$$M_\rho(\varphi, \psi) := \sup_g \left| \int g \, d\varphi - \int g \, d\psi \right|, \quad (12.1)$$

where the supremum is over all $g \in bcX$ such that $|g| \leq 1$ and $|g(x) - g(y)| \leq \rho(x, y)$ for all $x, y \in X$.

Theorem 12.1.2. If X is Polish, then $\mathcal{D}(X)$ is also Polish under the weak topology. Moreover, if ρ is a complete metric on X , then M_ρ is a complete metric on $\mathcal{D}(X)$ that metrizes weak convergence.

Add a citation from Dudley (2002).

EXERCISE 5. Let X be Polish and pick any $\mu, \nu \in \mathcal{D}(X)$. Prove that, as a consequence of Theorem 12.1.2, we must have

$$\mu = \nu \iff \int h \, d\mu = \int h \, d\nu \text{ for all } h \in L_\rho(X).$$

A subset \mathcal{D}_0 of $\mathcal{D}(X)$ is called **tight** if, for each $\varepsilon > 0$, there exists a compact $K \subset X$ such that $\varphi(K) \geq 1 - \varepsilon$ for all $\varphi \in \mathcal{D}_0$.

Theorem 12.1.3 (Prohorov). *If X is Polish then a subset \mathcal{D}_0 of $\mathcal{D}(X)$ is tight if and only if it is precompact in $(\mathcal{D}(X), d_u)$.*

Add a citation from Dudley (2002).

12.1.2 Distributions and Couplings

Connect this discussion and what follows back to §3.2.3.

Let X and Y be Polish spaces. Let φ and ψ be elements of $\mathcal{D}(X)$ and $\mathcal{D}(Y)$ respectively. A **coupling** of (φ, ψ) is a Borel probability measure π on $X \times Y$ with marginals φ and ψ . (See §11.3.3 for the definition of marginals.) Let $\Pi(\varphi, \psi)$ be the set of all **couplings** of ψ and φ .

A **realization** of the coupling π in $\Pi(\varphi, \psi)$ is a pair of random elements (X, Y) such that $X \sim \varphi$ and $Y \sim \psi$. To economize on notation, we also call (X, Y) a **coupling** of (φ, ψ) and write $(X, Y) \in \Pi(\varphi, \psi)$.

EXERCISE 6. Use Prohorov's theorem to show that, for every $(\varphi, \psi) \in \mathcal{D}(X) \times \mathcal{D}(Y)$, the set $\Pi(\varphi, \psi)$ is compact in $\mathcal{D}(X \times Y)$ under the weak topology.

12.1.3 The Monge–Kantorovich Problem

In what follows, X and Y are Polish spaces. A **cost function** is a Borel measurable map c from $X \times Y$ such that $\inf c > -\infty$. The **Monge–Kantorovich problem**, also called the **optimal transport problem**, is to solve

$$\min_{\pi} \int c \, d\pi \quad \text{subject to } \pi \in \Pi(\varphi, \psi), \tag{12.2}$$

where c is a cost function and φ and ψ are given elements of $\mathcal{D}(\mathbf{X})$ and $\mathcal{D}(\mathbf{Y})$ respectively. Identifying couplings with their realizations, we can also express (12.2) as the infimum of $\mathbb{E} c(X, Y)$ over all (X, Y) in $\Pi(\varphi, \psi)$. Since c is bounded below, the objective in (12.2) is well defined, although it may be infinite. To obtain a finite objective, we impose the following assumptions.

Assumption 12.1.1. There exist functions c_1 on \mathbf{X} and c_2 on \mathbf{Y} such that $c(x, y) \leq c_1(x) + c_2(y)$ and the integrals $\int c_1 d\varphi$ and $\int c_2 d\psi$ are finite.

Theorem 12.1.4 (Optimal Coupling Theorem). *If Assumption 12.1.1 holds and c is lower semicontinuous, then a solution to the optimal transport problem exists.*

The solution $\pi \in \Pi(\varphi, \psi)$ is either called an **optimal coupling** or an **optimal transport plan** depending on the literature (probability vs optimal transport).

Proof of Theorem 12.1.4. Fix $\varphi \in \mathcal{D}(\mathbf{X})$ and $\psi \in \mathcal{D}(\mathbf{Y})$. Let $D := \mathcal{D}(\mathbf{X} \times \mathbf{Y})$. Set $F(\pi) := \int c d\pi$. The function F is real-valued by Assumption 12.1.1. By the Extreme Value Theorem (p. 151), it suffices to show that F is lower semicontinuous and D is compact in the weak topology. As D was already shown to be compact (Exercise 6), we need only verify lower semicontinuity.

To this end, take $\pi_n \xrightarrow{w} \pi$ in D and, in addition, let $\{c_i\}$ be an increasing sequence in $bc(\mathbf{X} \times \mathbf{Y})$ with c_i converging pointwise to c . (Existence is by lower semicontinuity.) Then, for all i ,

$$\pi(c_i) = \lim_n \int c_i d\pi_n = \liminf_n \int c_i d\pi_n \leq \liminf_n \int c d\pi_n.$$

Hence, by the Monotone Convergence Theorem, $\pi(c) \leq \liminf_n \int c d\pi_n$, as was to be shown. \square

12.1.4 Kantorovich Duality

Let c be a cost function on $\mathbf{X} \times \mathbf{Y}$. The **Kantorovich duality theorem** states the following:

Theorem 12.1.5 (Kantorovich Dual Formula). *If c is lower semicontinuous and Assumption 12.1.1 holds for $\varphi \in \mathcal{D}(\mathbf{X})$ and $\psi \in \mathcal{D}(\mathbf{Y})$, then*

$$\min_{\pi \in \Pi(\varphi, \psi)} \int c d\pi = \max_{(f, g) \in \mathcal{F}_c} \left\{ \int f d\varphi - \int g d\psi \right\}. \quad (12.3)$$

where \mathcal{F}_c is all pairs of Borel functions (f, g) on \mathbf{X} and \mathbf{Y} such that $f(y) \leq c(x, y) + g(x)$ on $\mathbf{X} \times \mathbf{Y}$ with $\int f d\varphi < \infty$ and $\int g d\psi < \infty$.

For a full proof see part (iii) of Theorem 5.10 of Villani (2008).

12.1.5 Wasserstein Distance

Consider again the optimal transport problem, now for the special case where $\mathbb{Y} = \mathbb{X}$. We also specialize the cost function. In what follows, we say that c is a **ground distance** if c is lower semicontinuous under the given topology on \mathbb{X} and satisfies the axioms of a metric on \mathbb{X} .

12.1.5.1 Moments and Metrics

Let $\mathcal{D}_1(\mathbb{X}, c)$ be the set of all ψ in $\mathcal{D}(\mathbb{X})$ with finite first moments under the ground distance c . In the present context, **finiteness of first moments** means that there exists an \bar{x} in \mathbb{X} such that

$$m(x_0, \psi) := \int c(\bar{x}, x)\psi(dx) < \infty. \quad (12.4)$$

(If c is bounded, then $\mathcal{D}_1(\mathbb{X}, c) = \mathcal{D}(\mathbb{X})$.)

EXERCISE 7. Fix $\psi \in \mathcal{D}(\mathbb{X})$. Show that if $m(\bar{x}, \psi)$ is finite for some \bar{x} in \mathbb{X} , then $m(x, \psi)$ is finite for all $x \in \mathbb{X}$.

In the next theorem, $\text{Lip}_1(\mathbb{X}, c)$ is the set of Borel measurable c -Lipschitz functions with norm 1 on \mathbb{X} , that is, the set of all $h: \mathbb{X} \rightarrow \mathbb{R}$ with $|h(x) - h(y)| \leq c(x, y)$ for all x, y pairs in \mathbb{X} .

Theorem 12.1.6. *If c is a ground distance, then the minimal transport cost*

$$W_c(\varphi, \psi) := \min_{\pi \in \Pi(\varphi, \psi)} \int c d\pi \quad (12.5)$$

admits the dual representation

$$W_c(\psi, \varphi) = \max_{h \in \text{Lip}_1(\mathbb{X}, c)} \left\{ \int h d\psi - \int h d\varphi \right\}. \quad (12.6)$$

Moreover, W_c is a well-defined metric on $\mathcal{D}_1(\mathbb{X}, c)$. Convergence of φ_n to φ in W_c implies $m(\bar{x}, \varphi_n) \rightarrow m(\bar{x}, \varphi)$.

The metric W_c is called the **Wasserstein distance** generated by c . The Wasserstein distance is also called the Kantorovich–Rubenstein distance, the Kantorovich distance, or the L_1 transportation cost.²

In the special case under study, Kantorovich duality implies that

Complete the connection to Fortet–Mourier distance and weak convergence. Comment: While other popular distances metrize weak convergence of probability measures, W is particularly convenient because Kantorovich duality provides two distinct useful representations of the metric.

12.1.5.2 Completeness

We call a ground distance c on X **regular** if either (X, c) is complete and separable or c is discrete. Clarify what discreteness means. We can include the weighted discrete metrics of Hairer and Mattingly (2011), to get weighted total variation. Attempt to drop separability, since then the two cases are united.

Theorem 12.1.7. *If c is a regular ground distance, then $(D_1(\mathsf{X}, c), W_c)$ is complete.*

Use Chapter 6 of Villani (2008) for the separable case. The discrete case is total variation, which is complete.

12.2 Approximate Fixed Point Methods

As we have seen above, we often need to compute fixed points of contraction mappings where the fixed points are functions on continuous spaces. Let's consider the associated computational issues.

12.2.1 The Curse of Dimensionality

The **curse of dimensionality** refers to the fact that, for many numerical problems, computational time is exponential in the number of dimensions. When seeking to mitigate it, perhaps the most important principle is to exploit any smoothness in the

²We will use the term Wasserstein distance to conform to recent conventions, although Villani (2008) makes the case that the Wasserstein distance should probably be called the Kantorovich distance. We are partial to this argument because of the major contributions that Kantorovich made to mathematical economics and operations research. According to Villani (2008), another early investigator of the Wasserstein distance was Italian statistician Corrado Gini (1884–1965), who gave his name to the Gini coefficient.

objects we need to represent. Smoothness means that a function is well represented by its values on a sparse grid.

As an one example, consider a linear function f from \mathbb{R}^n to \mathbb{R} has the form $f(x) = \langle \alpha, x \rangle$. If we do not know that f is linear and try a discretized representation, we take k gridpoints in each of the n dimensions and store the value of $f(x_i)$ at each of these k^n grid points. Thus, a function that should require n parameters to store now requires $O(k^n)$. The difference explodes as n increases. If, say, $k = 10$, $n = 10$ and each parameter is 64 bits, this requires 80 gigabytes per function just to store the evaluations.

If we now realize that f is linear, we can represent it by the n -vector α , avoiding the curse of dimensionality. And even if f is nonlinear, it might be *locally* linear (e.g., differentiable). We can exploit this smoothness to find a midway point between the linear function result ($O(n)$ representation) and the discretization result (parametric representation increases exponentially).

12.2.2 Approximation and Projection

[roadmap]

12.2.2.1 Approximation with Basis Functions

Suppose we wish to represent function f using an approximation \hat{f} that can be implemented on a machine using a finite number of parameters. This can be formalized by taking a normed linear space V such that $f \in V$ and then selecting \hat{f} from a finite dimensional subspace B of V . Since a finite dimensional subspace can, by definition, be spanned by a finite number of basis elements b_1, \dots, b_n , the approximation will take the form

$$f \approx \hat{f} \quad \text{where} \quad \hat{f}(x) = \sum_{i=1}^n \alpha_i b_i(x)$$

for suitable scalars $\alpha_1, \dots, \alpha_n$.

The most common approach is to fix a sequence of subspaces $\{B_n\}_{n \in \mathbb{N}}$ of V such that

- B_n is spanned by n basis vectors b_1, \dots, b_n and
- $\cup_{n \in \mathbb{N}} B_n$ is dense in V .

Next we introduce a sequence of **approximation operators** A_n such that A_n maps V to B_n . In many cases, for given $f \in V$, the function $A_n f$ will equal the closest element

in B_n to f according to some metric, such as the norm-induced metric on V . This procedure is successful asymptotically, in the sense that $\cup_{n \in \mathbb{N}} B_n$ is dense in V , so for each $\varepsilon > 0$, there exists an $n \in \mathbb{N}$ and a $b \in B_n$ such that b is closer to f than ε .

12.2.2.2 Example: Piecewise Linear Approximation

Consider approximating a continuous function f on an interval $[a, b]$ using a piecewise linear continuous interpolation over grid points

$$G := \{x_i\}_{i=0}^{n-1}, \quad a = x_0 < x_1 < \dots < x_{n-1} = b$$

An illustration is given in figure 9.4. The interval $[a, b]$ is $[-1, 1]$ and we have 5 evenly spaced grid points. The target function is $f(x) = \cos(4x)$. The piecewise linear interpolant is denoted by Lf . It is the unique piecewise linear function that agrees with f at all of the points in G .³ One way to express it is

$$Lf(x) = f(x_i) + \frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i}(x - x_i) \quad \text{on } I_i := [x_i, x_{i+1}]$$

We can understand this procedure in terms of an approximation architecture where the function space V is $c[a, b]$, the set of continuous functions on $[a, b]$, and the n -th finite dimensional subspace B_n is the linear span of the basis functions

$$\ell_i(x) := \begin{cases} (x - x_{i-1})/(x_{i-1} - x_i) & \text{if } x \in I_{i-1} \\ (x_{i+1} - x)/(x_i - x_{i+1}) & \text{if } x \in I_i \text{ and} \\ 0 & \text{otherwise} \end{cases} \quad (12.7)$$

with the obvious modifications for the first and last element ℓ_0 and ℓ_{n-1} . The linear span of these basis functions is precisely the set of continuous piecewise linear functions over the vertices in G . An illustration of some of the basis functions $\{\ell_i\}$ is given in figure 12.1.

The corresponding approximation operator $L = A_n$ can be rewritten in terms of the basis functions via

$$(Lf)(x) = \sum_{i=0}^{n-1} f(x_i) \ell_i(x) \quad (12.8)$$

Moreover, it can be proved that $\cup_{n \in \mathbb{N}} B_n$ is dense in $c[a, b]$ when the latter is endowed with the supremum norm. In other words, any continuous function on $[a, b]$ can be

³Actually Lf is piecewise affine rather than piecewise linear, but let's stick with common usage and say linear.

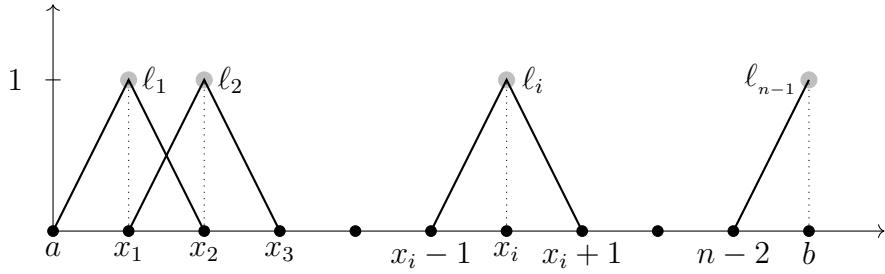


Figure 12.1: A basis for the continuous piecewise linear functions

approximated arbitrarily well by a continuous piecewise linear function.⁴

EXERCISE 8. Confirm that Lf in (12.8) is the piecewise linear interpolant of f on G .

12.2.2.3 Example: Orthogonal Projection

Another well known method of approximation that fits into this framework is orthogonal projection onto a set of basis functions. In this setting, V is not only a normed linear space but also a Hilbert space, and the basis functions b_1, \dots, b_n for a particular basis space B_n are chosen to be orthonormal (rather than just independent).⁵ Moreover, the approximation operator A_n that maps arbitrary elements of V into the basis space is the orthogonal projection map. Hence, for arbitrary $f \in V$,

$$A_n f = \sum_{i=1}^n \langle f, b_i \rangle b_i \quad (12.9)$$

The scalars $\langle f, b_i \rangle$ are called the **generalized Fourier coefficients** of the function f with respect to the basis $\{b_i\}$.

EXERCISE 9. Confirm that the right hand side of (12.9) is the orthogonal projection of f onto B_n when $\{b_1, \dots, b_n\}$ is an orthonormal basis for B_n .

12.2.3 Contractions and Approximation

[roadmap]

⁴The proof uses the **Heine–Cantor theorem**, which states that continuous functions defined on a compact set are uniformly continuous.

⁵If we have a set of n basis functions in a Hilbert space that are independent but not orthogonal, we can always create an orthonormal basis for the same subspace via the Gram–Schmidt procedure.

12.2.3.1 Approximating Successive Iteration

Suppose T is a self-map on a class of functions \mathcal{H} , where each h in \mathcal{H} maps a set X into \mathbb{R} . The set X might be a continuous domain such as a subset of \mathbb{R}^d . Suppose T is a contraction, with unique fixed point h^* satisfying $T^k h \rightarrow h^*$ for all $h \in \mathcal{H}$. The difficulty for numerical computation is that, as discussed in §12.2.1, we cannot in general implement each $T^k h$ on a machine with finite memory.

Given this difficulty, the most common approach is to proceed according to algorithm 10, which can be interpreted as iterating with an approximation \hat{T} to T . The hope is that if \hat{T} is a good approximation of T , then $\{\hat{T}^k h\}$ will converge to a point close to h^* .

```

1 input  $h$ , the initial condition ;
2 while some suitable stopping condition fails do
3   evaluate  $Th$  at a finite number of points in  $X$  ;
4   use this information to produce an approximation  $\hat{Th}$  of  $Th$  ;
5   set  $h \leftarrow \hat{Th}$ 
6 end
7 return  $h$ 
```

Algorithm 10: Iteration with an approximation step

This is of course far from guaranteed. Indeed, even if \hat{T} and T are close in some sense, when we apply \hat{T} to h instead of T we produce an error $d(\hat{Th}, Th)$, where d is a metric on \mathcal{H} . As we continue to iterate, the errors produced at each iteration can compound each other. As a result, $\{\hat{T}^k h\}$ can at times converge to a function at considerable distance from h^* . Even worse, it can fail to converge at all.

To try to control errors, we can apply the following well-known result, which can be found as lemma 2.1 in [Rust \(1997\)](#).

Proposition 12.2.1. *If T and \hat{T} are both contractions of modulus λ on (\mathcal{H}, d) , then their respective fixed points h^* and \hat{h}^* satisfy*

$$d(h^*, \hat{h}^*) \leq \frac{1}{1 - \lambda} d(\hat{T} h^*, h^*)$$

In proposition 12.2.1, the term $d(\hat{T} h^*, h^*)$ can also be written as $d(\hat{T} h^*, Th^*)$, so as long as \hat{T} is a contraction of the same modulus as T and behaves like T at least near h^* , we can expect good performance. But when does \hat{T} retain the contraction property possessed by T ?

To understand when contractivity is preserved, let us think about the approximation step as an application of an approximation operator A , as discussed in §12.2.2.1. (The subscript n of A is omitted because we are at present considering a single generic approximation operator.) The function $\hat{T}h$ can alternatively be written as ATH , and algorithm 10 can be interpreted as iterating with the composition $A \circ T$.

Lemma 12.2.2. *If T is a contraction of modulus λ on (\mathcal{H}, d) and A is nonexpansive on (\mathcal{H}, d) , then $A \circ T$ is a contraction of modulus λ on (\mathcal{H}, d) .*

Proof. For any given h and h' in \mathcal{H} we have

$$d(ATH, ATH') \leq d(Th, Th') \leq \lambda d(h, h')$$

□

Lemma 12.2.2 provides one path to successful approximation when iterating with a contractive map: Pick an approximation architecture such that the corresponding approximation operators are nonexpansive in the same metric with respect to which T is a contraction. We will put these ideas to use in §12.2.3.2 and below.

12.2.3.2 Local Approximation

In what is often called **local approximation** in the context of dynamic programming, we implement the preceding ideas by choosing a set of grid points $\{x_1, \dots, x_n\}$ and a set of basis functions $\{\kappa_1, \dots, \kappa_n\}$ defined on X with the property that

$$\sum_{i=1}^n \kappa_i(x) = 1 \text{ for all } x \in X \quad \text{and} \quad \kappa_j \geq 0 \quad \text{for all } j \in \{1, \dots, n\} \quad (12.10)$$

Elements of a family of basis function satisfying (12.10) are sometimes called **weighting functions** or **kernels**, and an approximation of some function f taking the form

$$A_n f(x) := \sum_{i=1}^n f(x_i) \kappa_i(x) \quad (12.11)$$

is called a **local approximator** or a **kernel averager**. The key idea is that $A_n f(x)$, the approximation to $f(x)$ at $x \in X$, is a *weighted average* of the values $f(x_i)$ of f at the grid points. In general, $\kappa_i(x)$ will be relatively large when x is near x_i , so that $f(x_i)$ has a large weight in determining $A_n f(x)$.

Example 12.2.1. The continuous piecewise linear approximation operator L defined in (12.8) is a kernel averager because the basis functions $\{\ell_i\}$ satisfy $\sum_{i=1}^n \ell_i(x) = 1$ for all $x \in [a, b]$.

For us kernel averagers are particularly interesting because they happen to be non-expansive with respect to the exact distance under which most contraction map results in dynamic programming are obtained:

Lemma 12.2.3. *If A_n is the kernel averager in (12.11) and \mathcal{H} is a set of bounded functions, then A_n is nonexpansive on \mathcal{H} with respect to the supremum distance d_∞ .*

Proof. Pick any $f, g \in \mathcal{H}$. We have

$$\begin{aligned} |Af(x) - Ag(x)| &= \left| \sum_{i=1}^n f(x_i) \kappa_i(x) - \sum_{i=1}^n g(x_i) \kappa_i(x) \right| \\ &\leq \sum_{i=1}^n |f(x_i) - g(x_i)| \kappa_i(x) \\ &\leq \sum_{i=1}^n \|f - g\|_\infty \kappa_i(x) = \|f - g\|_\infty \end{aligned}$$

Taking the supremum over all $x \in \mathbb{X}$ confirms the claim in lemma 12.2.3. \square

The kernel averager can be combined with iteration of an operator to yield a specific implementation of the iterative technique discussed in algorithm 10. The details are given in algorithm 11, which can be expressed mathematically as iteration with the composition map $A_n \circ T$, starting from some initial point h and continuing until successive iterates are sufficiently close together. Combining lemma 12.2.3 with proposition 12.2.1 and lemma 12.2.2, we see that $A_n \circ T$ is a contraction of modulus λ with respect to d_∞ whenever T has this property, and that its unique fixed point h_n^* satisfies

$$d_\infty(h^*, h_n^*) \leq \frac{1}{1-\lambda} d_\infty(A_n T h^*, h^*) = \frac{1}{1-\lambda} d_\infty(A_n h^*, h^*)$$

where h^* is the fixed point of T . Thus, if $A_n h \rightarrow h$ in d_∞ for points of \mathcal{H} , then $d(h^*, h_n^*)$ can be made arbitrarily small.

Moreover, the sequence $(A_n T)^k h$ generated by algorithm 11 converges to h_n^* as $k \rightarrow \infty$, and k can be made arbitrarily large by taking τ sufficiently small.

12.2.3.3 Global Approximation

So-called **global approximation** methods are similar to local approximation methods, with the main difference being that we drop the restrictive assumption (12.10) from the set of basis functions and the function values $\{f(x_i)\}_{i=1}^n$ are replaced with a

```

1 input  $h$ , the initial condition ;
2 input  $\tau$ , a tolerance level for error ;
3  $\varepsilon \leftarrow \tau + 1$  ;
4 while  $\varepsilon > \tau$  do
5   for  $i = 1, \dots, n$  do
6     |  $\alpha_i \leftarrow Th(x_i)$  ;
7   end
8    $h' \leftarrow \sum_{i=1}^n \alpha_i \kappa_i$  ;
9    $\varepsilon \leftarrow d(h', h)$  ;
10   $h \leftarrow h'$  ;
11 end
12 return  $h$ 
```

Algorithm 11: Local approximation corresponding to kernel basis $\{\kappa_i\}$

more generic set of coefficients $\{\theta_j\}_{j=1}^k$. With the basis denoted by $\{b_1, \dots, b_k\}$, the approximations take the form

$$G_\theta f(x) = \sum_{j=1}^k \theta_j b_j(x)$$

Notice that the number of basis elements k has been disassociated from the number of grid points n . In most instances, k will be made smaller, seeking a parsimonious representation.

Parameters are chosen by a technique such as error minimization over sample points formed by a grid $\{x_1, \dots, x_n\}$ and corresponding evaluations $\{f(x_1), \dots, f(x_n)\}$. For example, if we take least squares as our criterion, then the vector θ of coefficients is chosen minimize

$$E(\theta) := \sum_{i=1}^n (G_\theta f(x_i) - f(x_i))^2.$$

If we now take Z to be the $n \times k$ matrix (Z_{ij}) where $Z_{ij} = b_j(x_i)$ and y to be the $n \times 1$ vector $(f(x_1), \dots, f(x_n))'$, then $E(\theta)$ can be expressed as $E(\theta) = \|Z\theta - y\|^2$, where $\|\cdot\|$ is Euclidean norm. By Theorem 11.2.3 (p. 328), the minimizer is

$$\hat{\theta} := (Z'Z)^{-1} Z'y \tag{12.12}$$

whenever Z has full column rank. Algorithm 12 shows how this procedure combines with iteration of an operator T .

```
1 input initial condition  $h$  and tolerance level  $\tau$  ;
2  $\varepsilon \leftarrow \tau + 1$  ;
3 for  $i = 1, \dots, n$  do
4   | for  $j = 1, \dots, k$  do
5     |   |  $Z_{ij} \leftarrow b_j(x_i)$ 
6   | end
7 end
8 while  $\varepsilon > \tau$  do
9   | for  $i = 1, \dots, n$  do
10    |   |  $y_i \leftarrow Th(x_i)$  ;
11    | end
12    |  $\theta \leftarrow (Z'Z)^{-1}Z'y$  ;
13    |  $h' \leftarrow \sum_{j=1}^k \theta_j b_j$  ;
14    |  $\varepsilon \leftarrow d(h', h)$  ;
15    |  $h \leftarrow h'$  ;
16 end
17 return  $h$ 
```

Algorithm 12: Least squares global approximation with basis b_1, \dots, b_k

Chapter 13

General State Markov Processes

Throughout this chapter, X is a Polish (i.e., separable and completely metrizable) space and \mathcal{B} is the Borel sets. We study discrete and continuous time Markov processes taking values in X . As in Chapter 5, the symbol \mathbb{T} denotes either the nonnegative integers or the nonnegative reals. [Complete the roadmap.](#)

13.1 Stochastic Processes

[Add roadmap.](#)

13.1.1 Dynamics and Joint Distributions

[Add roadmap.](#)

13.1.1.1 Stochastic Processes

An X -valued **stochastic process** is a family of X -valued random elements $\{X_t\}$, indexed by $t \in \mathbb{T}$, with each X_t defined on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The point of putting the whole family on a common probability space is that when a particular $\omega \in \Omega$ is realized, it picks out an entire sample path. That is,

$$\omega \mapsto \{X_t(\omega)\} \in X^{\mathbb{T}}. \quad (13.1)$$

In the discrete time case, it can be helpful to think of ω in (13.1) as a seed in a computer simulation, since one seed picks out an entire sample path for the state process determined by a model. We call each realization $\{X_t(\omega)\}$ a **sample path**.

In §4.2.1 we defined filtrations as increasing families of information sets. Repeating the same idea more formally, we define a **filtration** on (Ω, \mathcal{F}) to be a family of σ -algebras $\{\mathcal{G}_t\}_{t \geq 0}$ on Ω , each one contained in \mathcal{F} , satisfying $\mathcal{G}_s \subset \mathcal{G}_t$ for all $s \leq t$. A stochastic process $\{X_t\}$ is said to be **adapted** to the filtration $\{\mathcal{G}_t\}$ if X_t is \mathcal{G}_t -measurable for every t . The meaning is that X_t is a deterministic function of variables in \mathcal{G}_t . For example, if $\{\xi_t\}$ is an IID sequence, then $X_t = \sum_{i=1}^t \xi_i$ is adapted to the filtration associated with the information sets $\mathcal{G}_t := \sigma\{\xi_i, i \leq t\}$.

13.1.1.2 Joint Distributions

We discussed joint distributions informally in §3.1.6, for discrete time Markov chains on countable sets. We now provide a more general view.

Recall from §11.3.1 that the distribution of a random element Y on $(\Omega, \mathcal{F}, \mathbb{P})$ taking values in an arbitrary measurable space (E, \mathcal{E}) is given by $P_Y(B) = \mathbb{P}\{Y \in B\}$. The joint distribution of the stochastic process is precisely the distribution of the sample path map (13.1). The only problem here is that we have not yet specified a measurable space in which the sample path map $\omega \mapsto \{X_t(\omega)\}$ takes values.

Add the canonical construction, brief. The space in which the sample path $X: \omega \mapsto \{X_t(\omega)\}$ takes values is $\mathbf{X}^{\mathbb{T}}$. To build a σ -algebra on this set we take [define the product sigma-algebra].

The joint distribution \mathbf{P} of X is now defined as for any other distribution, that is, by $\mathbf{P}(B) = \mathbb{P}\{X \in B\}$ for B in $\mathcal{B}^{\mathbb{T}}$.

13.1.1.3 The Canonical Process

Sometimes we need to travel in the other direction. Say we have a joint distribution \mathbf{P} over the path space $\mathbf{X}^{\mathbb{T}}$, paired with its product σ -algebra. Can we always create a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and an \mathbf{X} -valued stochastic process defined on this probability space having distribution \mathbf{P} ? The answer is affirmative, which is reassuring because we often wish to reference a process corresponding to a given distribution.

The construction is easy: For the underlying probability space we take $(\Omega, \mathcal{F}, \mathbb{P}) = (\mathbf{X}^{\mathbb{T}}, \mathcal{B}^{\mathbb{T}}, \mathbf{P})$. For the stochastic process $X = \{X_t\}_{t \in \mathbb{T}}$ we take the family of projection maps $\{\pi_t\}_{t \in \mathbb{T}}$. In particular, given $\omega = \{x_t\} \in \mathbf{X}^{\mathbb{T}}$, the random element X_t returns $X_t(\omega) = x_t$. It is straightforward to check that, for arbitrary B in $\mathcal{B}^{\mathbb{T}}$, the set $\{\omega \in \Omega : X(\omega) \in B\}$ is just B . Hence $\mathbb{P}\{X \in B\} = \mathbf{P}(B)$, as required.

13.1.2 Stationarity and Ergodicity

[roadmap]

13.1.2.1 Stationarity

Some stochastic processes are nonstationary. An example is the random walk $X_{t+1} = X_t + \xi_{t+1}$ where X_0 is given and $\{\xi_t\}$ is IID and standard normal. The variance of X_t grows linearly with t , so that, in particular, the distribution changes over time.

In contrast, a stochastic process $\{X_t\}_{t \geq 0}$ is called **stationary** if

$$(X_0, X_1, \dots, X_n) \stackrel{d}{=} (X_k, X_{k+1}, \dots, X_{k+n}) \quad \text{for all } n, k \geq 0 \quad (13.2)$$

One immediate implication is that the marginal distributions do not change, in the sense that X_0 and X_k have the same distribution for any k . In other words, the process is **identically distributed**. But stationarity is of course stronger than the property of being identically distributed, since (13.2) must hold for all $n \geq 0$.

An obvious example of a stationary stochastic process is an IID sequence $\{\xi_t\}_{t \geq 0}$, since joint distributions are just products of marginals. For example, if each ξ_t is standard normal in \mathbb{R} , then, for any nonnegative integers k and n ,

$$(\xi_0, \xi_1, \dots, \xi_n) \stackrel{d}{=} (\xi_k, \xi_{k+1}, \dots, \xi_{k+n}) \stackrel{d}{=} N(0, I)$$

where $N(0, I)$ is the standard normal density in \mathbb{R}^{n+1} .

A more subtle example is the stochastic recursive sequence $\{X_t\}$ analyzed in §7.2.6.2, the joint distribution of which was obtained in (3.19). In general, $\{X_t\}$ is not stationary, since

$$(w_k, w_{k+1}, \dots, w_{k+n}) \stackrel{d}{=} \psi_k(x_k) \prod_{t=1}^n \pi(x_{t+k-1}, x_{t+k}) \quad (13.3)$$

which depends on k through the marginal distribution ψ_k of X_k . If, however, (i) the conditions for stability in lemma 7.2.4 are satisfied, so that a stationary density ψ^* exists, and (ii), the process is started by drawing X_0 from ψ^* , then $\psi_k = \psi^*$ for all k and the time dependence drops out. Now $\{X_t\}$ is stationary.

13.1.2.2 Ergodicity

[roadmap]

Let $\{X_t\}_{t \geq 0}$ be stochastic process taking values in $X \subset \mathbb{R}^d$. Let \mathcal{H} be a family of real-valued functions defined on X . We call $\{X_t\}_{t \geq 0}$ **asymptotically stationary with respect to \mathcal{H}** if there exists a distribution ψ^* on X such that

$$\lim_{t \rightarrow \infty} \mathbb{E}h(X_t) = \int h(x)\psi^*(dx) \quad \text{for every } h \in \mathcal{H} \quad (13.4)$$

For example, if $\{X_t\}$ is stationary, then $\{X_t\}$ is asymptotically stationary with respect to the class of bounded Borel measurable functions on X .

A stochastic process $\{X_t\}_{t \geq 0}$ that is asymptotically stationary with respect to \mathcal{H} will be called **ergodic with respect to \mathcal{H}** if, with probability one,

$$\frac{1}{n} \sum_{t=1}^n h(X_t) \rightarrow \int h(x)\psi^*(dx) \quad \text{for every } h \in \mathcal{H} \quad (13.5)$$

as $t \rightarrow \infty$.

13.1.2.3 Examples

An obvious example is when $\{X_t\}$ is IID. For example, suppose that $\{X_t\}$ is IID and standard normal, and that \mathcal{H} is the class of all polynomials on \mathbb{R} . Since the standard normal density has finite moments of all orders, we have $\mathbb{E}h(X_t) < \infty$ for every $t \geq 0$ and every $h \in \mathcal{H}$. The statement (13.4) is true when $\psi^* = N(0, 1)$. The convergence (13.5) holds by the strong law of large numbers.

More generally, every IID process taking values in \mathbb{R}^d is asymptotically stationary and ergodic when \mathcal{H} is the class of bounded Borel measurable functions from \mathbb{R}^d to \mathbb{R} . Here boundedness is to make sure that the integrals are finite and Borel measurability is a weak regularity condition mentioned above that ensures integrals are well defined—see §11.3.1.1 for more details.

A less trivial example of an asymptotically stationary and ergodic process is stochastic recursive sequence $\{X_t\}$ analyzed in §7.2.6.2. As discussed in §13.1.1.1, this process is not stationary when the initial condition is not the stationary distribution, even if the conditions of proposition 7.2.3 hold. Suppose, however, that we take \mathcal{H} to be the class of bounded Borel measurable functions on \mathbb{R}_+ , say, and assume that the conditions of

proposition 7.2.3 hold. Then, for any given $h \in \mathcal{H}$,

$$\begin{aligned} \left| \mathbb{E}h(X_t) - \int h(x)\psi^*(x) dx \right| &= \left| \int h(x)(\psi_t(x) - \psi^*(x)) dx \right| \\ &\leq \int |h(x)(\psi_t(x) - \psi^*(x))| dx \\ &\leq \sup_x |h(x)| \int |\psi_t(x) - \psi^*(x)| dx \end{aligned}$$

where ψ^* is the stationary distribution of the process $\{X_t\}$, the first inequality is the triangle inequality and the second follows from monotonicity of the integral. The last term converges to zero by proposition 7.2.3, so asymptotic stability is established.

Ergodicity of $\{X_t\}$ with respect to the same family \mathcal{H} is now immediate from (7.57).

13.1.2.4 Remarks

Our definition of ergodicity differs from some textbook treatments, such as Çınlar (2011), where ergodicity is defined in terms of invariant sets. What we call ergodicity is a consequence, obtained via the Birkhoff ergodic theorem. However, there is no general agreement on the meaning of ergodicity. For example, the classic monograph by Meyn and Tweedie (2009) uses a different definition.

For economists, the meaning of ergodicity is that sample path averages are, in the limit, equal to cross sectional averages. To understand this idea, think of the wealth distribution application. Under the conditions of Lemma 7.2.4, a unique stationary density ψ^* exists. If we take a bounded Borel measurable function h on the state space, then, by (7.57), we have

$$\frac{1}{n} \sum_{t=1}^n h(w_t) \rightarrow \int h(w)\psi^*(w) dw$$

with probability one as $n \rightarrow \infty$ for a given sample path $\{w_t\}$. This applies to the wealth process of an individual household, since, by assumption, the wealth dynamics of that household satisfy Lemma 7.2.4. But the density on the right hand side is precisely the (asymptotic) cross sectional distribution of wealth over the set of households in the economy.

Our definition of ergodicity formalizes the economic view of ergodicity.

13.2 Markov Dynamics

13.2.1 Markov Semigroups

Start with a Markov kernel. Then build the semigroup.

13.2.1.1 Markov Kernels

A **Markov kernel** on X is a mapping $P: (x, B) \rightarrow [0, 1]$ with the following properties:

- (i) for each $B \in \mathcal{B}$, the map $x \mapsto P(x, B)$ is \mathcal{B} -measurable, and
- (ii) for each $x \in \mathsf{E}$, the map $B \mapsto P(x, B)$ is a probability measure on \mathcal{B} .

This is a general representation of objects we have met before. The next example is a measure-theoretic version of the CDF representation of Markov kernels constructed in §7.2.1.

Example 13.2.1. Let ξ be a random element on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$, taking values in measurable space $(\mathsf{Z}, \mathcal{Z})$ and having distribution φ . Let F be measurable with respect to the product σ -algebra $\mathcal{B} \otimes \mathcal{Z}$. Then P_F defined by

$$P_F(x, B) = \mathbb{P}\{F(x, \xi) \in B\} = \varphi\{z \in \mathsf{Z} : F(x, z) \in B\} \quad (13.6)$$

is a Markov kernel on X . This is the kernel associated with the stochastic recursion equation $X_{t+1} = F(X_t, \xi_{t+1})$, when $\{\xi_t\}$ is IID copies of ξ .

Sometimes Markov kernels arise as follows: Let $p: \mathsf{X} \rightarrow \mathsf{X}$ be measurable with respect to the product σ -algebra. Let ν be a measure on $(\mathsf{X}, \mathcal{B})$. Then

$$P(x, B) = \int_B p(x, y)\nu(dy) \quad (x \in \mathsf{X}, B \in \mathcal{B})$$

is a Markov kernel on X whenever $\int p(x, y)\nu(dy) = 1$. The underlying object p is, in this case, called the **density kernel**. Examples of this form were presented in §7.2.1.3. See, for example, (7.42) on page 241.

To each Markov kernel P we associate the corresponding **Markov operator** $\varphi \mapsto \varphi P$, defined by

$$(\varphi P)(B) = \int \varphi(dx)P(x, B) \quad (B \in \mathcal{B}). \quad (13.7)$$

This is a generalization of the Markov operator for discrete state Markov chains presented in §3.1.3. We use the same symbol P to refer to both the Markov kernel and the

Markov operator because they are two views on the same object. The Markov kernel can be recovered from the Markov operator via $P(x, B) = (\delta_x)P(B)$.

The n -step Markov kernel is defined for $n \in \mathbb{N}$ by $P^n(x, \cdot) = \delta_x P^n$.

For Borel measurable $h: \mathsf{X} \rightarrow \mathbb{R}$ that is either bounded or nonnegative, we use the notation

$$(Ph)(x) = \int P(x, dy)h(y) \quad (x \in \mathsf{X}). \quad (13.8)$$

As an example, recall P_F from (13.6). In this case we have

$$(\psi P_F)(B) = \int \int \psi(dx)\varphi(dz)\mathbb{1}\{F(x, z)\} \quad (13.9)$$

and

$$(P_F h)(x) = \mathbb{E}h(F(x, \xi)) = \int \varphi(dz)h(F(x, z)). \quad (13.10)$$

EXERCISE 1. Let P be any Markov kernel on X . Verify that φP is in $\mathcal{D}(\mathsf{X})$ whenever $\varphi \in \mathcal{D}(\mathsf{X})$. Prove that $\|Ph\| \leq \|h\|$ for all h in $b\mathcal{B}$, where $\|\cdot\|$ is the supremum norm.

Exercise 1 shows that $h \mapsto Ph$ is Lipschitz continuous on $(b\mathcal{B}, \|\cdot\|)$.

Add duality type results in Çinlar.

13.2.1.2 Markov Semigroups

A **Markov semigroup** is a family of Markov kernels $\{P_t\}_{t \in \mathbb{T}}$ such that

- (i) $P_0(x, B) = \mathbb{1}\{x \in B\}$ for all $x \in \mathsf{X}$ and all $B \in \mathcal{B}$ and
- (ii) $P_{s+t}(x, B) = \int P_s(x, dy)P_t(y, B)$ for all $s, t \in \mathbb{T}$ and all B in \mathcal{B} .

The second equation is called the **Chapman–Kolmogorov** equation. We first met a special case of this equation in §3.1.2.

Connect to semigroups and semidynamical systems. Specialize to discrete time. Illustrate with examples we have seen already.

13.2.2 Construction of the Process

Construction of the process from specified transition kernel. The first step is to build the joint distribution \mathbf{P}_φ corresponding to a given initial condition φ via the Kolmogorov extension theorem. Then the process is constructed via §13.1.1.3. Note that this is only the most basic construction.

13.2.3 Stationary Distributions

[Definition, existence, uniqueness under irreducibility.]

Let $\{P_t\}_{t \in \mathbb{T}}$ be a Markov semigroup on X . A measure $\psi \in \mathcal{D}(\mathsf{X})$ is called **stationary** for $\{P_t\}$ if $\psi P_t = \psi$ for all $t \in \mathbb{T}$. For the discrete time case, where $P_t = P^t$, this is obviously equivalent to the statement that $\psi P = \psi$.

13.2.3.1 Existence

Extend these results so they cover continuous time as well.

A Markov kernel is called **Feller** if $f \mapsto Pf$ is invariant on $bc\mathsf{X}$. In other words, P is Feller if $\int P(x, dy)f(y)$ is bounded and continuous in x whenever $f \in bc\mathsf{X}$. The boundedness property is trivial (recall Exercise 1). so continuity is the binding restriction.

EXERCISE 2. Show that P_F defined in (13.6) is Feller whenever $x \mapsto F(x, z)$ is continuous for all $z \in \mathbb{Z}$.

Theorem 13.2.1 (Krylov–Bogoliubov III). *Let P be a Markov kernel on X . If P has the Feller property, then the following statements are equivalent:*

- (i) P has a stationary distribution.
- (ii) There exists a ψ in $\mathcal{D}(\mathsf{X})$ such that $\{\psi P^t\}$ is tight.

Proof. To be added. Is σ -compactness needed here? □

A common method of establishing tightness of trajectories is to use drift conditions. In stating the next theorem, we use the concept of a **norm-like function**, sometimes called a **Lyapunov function**, which is a nonnegative function v on X such that the sublevel sets $S_\alpha := \{x \in \mathsf{X} : v(x) \leq \alpha\}$ are all precompact.

Example 13.2.2. If $\mathsf{X} = \mathbb{N}$, the metric on X is the discrete metric (see page 131), then $v(n) = n$ is a norm-like function. To see this, observe that

$$S_\alpha = \{x \in \mathsf{X} : v(x) \leq \alpha\} = \{n \in \mathbb{N} : n \leq \alpha\}$$

is finite. Under the discrete metric, subsets of X are precompact if and only if they are finite.

EXERCISE 3. Let $\mathsf{X} = \mathbb{Z}$ paired with the discrete metric. Show that $v(n) = |n|$ is a norm-like function but $v(n) = n$ is not.

Add more examples, to illustrate the concept of norm-like functions. Then, introduce geometric drift, and show directly that this implies tightness. From this deduce existence in the Feller case.

We say that P obeys **geometric drift** with respect to a measurable function $V: \mathsf{X} \rightarrow \mathbb{R}_+$ if there exist constants $L < \infty$ and $\lambda \in (0, 1)$ such that

$$(PV)(x) \leq \lambda V(x) + L, \quad \forall x \in \mathsf{X}.$$

Lemma 13.2.2. *If P obeys geometric drift with respect to a norm-like function, then $\{\delta_x P^t\}$ is tight for any $x \in \mathsf{X}$.*

Add a proof. Connect with existence. Provide examples.

13.2.3.2 Uniqueness

Given a nontrivial measure φ on $(\mathsf{X}, \mathcal{B})$, a Markov kernel P on X is called **φ -irreducible** if, for each $B \in \mathcal{B}$ with $\varphi(B) > 0$, there is an n in \mathbb{N} such that $P^n(x, B) > 0$. In other words, all sets with positive φ -measure get visited eventually.

Theorem 13.2.3. *If P is irreducible, then P has at most one stationary distribution.*

Proof. Add proof or give reference. □

13.3 Stability in Discrete Time

In this section we turn to global stability, extending our discussion from the discrete case in §7.1.4. As before, we call a Markov kernel P **globally stable** if $(\mathcal{D}(\mathsf{X}), P)$ is globally stable (see page 5.3.2.2 for the definition). We focus on two approaches, both of them modern, relatively general, and well suited to economic and financial applications.

13.3.1 Wasserstein Contractions

Add roadmap. Recommend that readers review the notation in §12.1..

13.3.1.1 Contraction and Stability

Let ρ be a cost function on X that is also a complete metric on $\mathsf{X} \times \mathsf{X}$. Note that, while X continues to be Polish, it is not required that ρ metrizes the topology on X . (For example, X need not be separable under ρ .)

Let P be a Markov kernel on X . We say that P **preserves first moments** if P maps $\mathcal{D}_1(\mathsf{X}, \rho)$ into itself.

Example 13.3.1. Consider the wealth dynamics $w_{t+1} = (1 + r_{t+1})(w_t - \sigma(w_t)) + y_{t+1}$ where σ is a consumption function. Let P be the corresponding Markov operator on \mathbb{R}_+ . If r_t and y_t have finite expectation, then P preserves first moments when $\rho = |\cdot|$, since

$$\mathbb{E}w_t < \infty \implies \mathbb{E}|w_{t+1}| = \mathbb{E}w_{t+1} \leq [1 + \mathbb{E}r_{t+1}] \mathbb{E}w_t + \mathbb{E}y_{t+1} < \infty.$$

Let W_ρ be the Wasserstein distance on $\mathcal{D}_1(\mathsf{X}, \rho)$ induced by the distance ρ . We call P a **Wasserstein contraction** if there exists an $\alpha < 1$ such that

$$W_\rho(\varphi P, \psi P) \leq \alpha W_\rho(\varphi, \psi) \text{ for all } \varphi, \psi \in \mathcal{D}_1(\mathsf{X}, \rho)$$

Give the example of kernels with Dobrushin coefficient (see Dobrushin (1956)) less than one.

The importance of Wasserstein contractions stems from the following theorem:

Theorem 13.3.1. *Let P be a Markov kernel on X that preserves first moments. If there exists an $n \in \mathbb{N}$ such that P^n is a Wasserstein contraction, then P is globally stable on $(\mathcal{D}_1(\mathsf{X}), W_\rho)$.*

Proof. The result follows immediately from the extension of Banach's contraction mapping theorem in Theorem 5.1.10, given that $(\mathcal{D}_1(\mathsf{X}), W_\rho)$ is complete and $\varphi \mapsto \varphi P^n$ is a contractive self-map on $\mathcal{D}_1(\mathsf{X})$. \square

It is clear from this discussion that a uniform geometric rate of convergence to the stationary distribution holds when $\alpha_\rho(P) < 1$. In particular, if ψ^* is stationary for P , then

$$W_\rho(\psi P^t, \psi^*) \leq \alpha_\rho(P)^t W_\rho(\psi, \psi^*)$$

for all $t \geq 0$ and all $\varphi \in \mathcal{D}_1(\mathsf{X})$.

The following lemma is helpful in estimating the Dobrushin–Wasserstein coefficient.

Lemma 13.3.2. *If P is a Feller kernel, then*

$$\alpha_\rho(P) = \sup_{x \neq y} \frac{W_\rho(\delta_x P, \delta_y P)}{\rho(x, y)}.$$

Proof. To be added. □

13.3.2 Average Contractions

Consider again the stochastic difference equation $X_{t+1} = F(X_t, \xi_{t+1})$ from Example 13.2.1 on page 368. The sequence $\{\xi_t\}$ is IID with distribution φ . Let P_F be the corresponding Markov kernel, as defined in (13.6). The following useful stability result rests on an average contraction condition.

Theorem 13.3.3. *Let ξ be a draw from φ . If*

- (i) $F(X, \xi)$ has finite first moment whenever X has finite first moment and
- (ii) there exists a positive constant $\alpha < 1$ such that

$$\mathbb{E} \rho(F(x, \xi), F(y, \xi)) \leq \alpha \rho(x, y) \quad \text{for all } x, y \text{ in } \mathsf{X} \times \mathsf{X}, \quad (13.11)$$

then P_F is globally stable on $\mathcal{D}_1(\mathsf{X})$ under the Wasserstein metric.

Proof. We claim that P_F preserves first moments. To this end, fix ψ in $\mathcal{D}_1(\mathsf{X})$ and let X be a draw from ψ . By the first part of our assumptions, $F(X, \xi)$ has finite first moment when ξ is drawn independently from φ . Since $F(X, \xi)$ has law ψP_F , we see that P_F is a self-map on $\mathcal{D}_1(\mathsf{X})$.

Turning to contractivity, pick any ψ, φ in $\mathcal{D}_1(\mathsf{X})$. By Theorem 4.1 of Villani (2008), we can choose a pair (X, Y) in $\Pi(\psi, \varphi)$ that attains $\mathbb{E} \rho(X, Y) = W_\rho(\psi, \varphi)$. For this pair, with ξ as an independent draw from φ and π as the distribution of (X, Y) , we have

$$\mathbb{E} \rho(F(X, \xi), F(Y, \xi)) = \int \int \rho(F(x, z), F(y, z)) \varphi(dz) \pi(dx, dy),$$

Applying (13.11) gives

$$\mathbb{E} \rho(F(X, \xi), F(Y, \xi)) \leq \alpha \int \rho(x, y) \pi(dx, dy) = \alpha W_\rho(\psi, \varphi).$$

But $(F(X, \xi), F(Y, \xi))$ is in the set of couplings $\Pi(\psi P_F, \varphi P_F)$, so, taking the infimum, we have $W_\rho(\psi P_F, \varphi P_F) \leq \alpha W_\rho(\psi, \varphi)$. Since φ and ψ were chosen arbitrarily, we have $\alpha_\rho(P_F) \leq \alpha$. Hence Theorem 13.3.1 applies. □

As an example, consider a Kesten process of the form $X_{t+1} = A_{t+1}X_t + \eta_{t+1}$ where $\{X_t\}$ takes values in \mathbb{R}^n , and $\{(A_t, \eta_t)\}$ is IID, taking values in $\mathcal{M}_{n \times n} \times \mathbb{R}^n$. In Theorem 7.2.1 on page 242 we discussed stability under relatively weak conditions. When the slightly stronger conditions $\mathbb{E}\|A_t\| < 1$ and $\mathbb{E}\|\eta_t\| < \infty$ are satisfied (using matrix and vector norms), we can apply the average contraction result above. Let $\rho(x, y) = \|x - y\|$. For condition (i) we have

$$\mathbb{E}\|X_{t+1}\| = \mathbb{E}\|A_{t+1}X_t + \eta_{t+1}\| \leq \mathbb{E}\|A_{t+1}\| \mathbb{E}\|X_t\| + \mathbb{E}\|\eta_{t+1}\|,$$

where the bound on the right uses the triangle inequality and the submultiplicative property of the matrix norm. So $\mathbb{E}\|X_{t+1}\|$ is finite whenever $\mathbb{E}\|X_t\|$ is finite. Regarding condition (ii) we have, given independent draw (A, η) ,

$$\mathbb{E}\|Ax + \eta - Ay - \eta\| \leq \alpha\|x - y\| \text{ when } \alpha := \mathbb{E}\|A\| < 1.$$

Add example/exercise on partially linear model with reflecting barrier, relate to literature on wealth and income dynamics. Consider adding a log-linear model with metric $\rho(x, y) = |\ln x - \ln y|$.

13.3.3 Doeblin Chains

The modern Wasserstein contraction approach presented above can be used to prove classical results (that are nonetheless important today). As an example, let us now consider Doeblin Chains, named after Wolfgang Doeblin (1915–1940). To do so, consider the distance between two probability measures φ and ψ given by

$$\tau(\varphi, \psi) := \sup_{h \in L} \left| \int h \, d\varphi - \int h \, d\psi \right|, \quad (13.12)$$

where $L := \{h \in b\mathcal{B} : |h(x) - h(y)| \leq 1\}$. This distance is proportional to total variation distance, as the next exercise asks you to confirm.

EXERCISE 4. The total variation distance between φ and ψ in $\mathcal{D}(X)$ is defined as $d_{TV}(\varphi, \psi) := \sup_{|h| \leq 1} \left| \int h \, d\varphi - \int h \, d\psi \right|$. Prove that $d_{TV} = 2\tau$ on $\mathcal{D}(X)$.

The indicator of any Borel set is in L , so $\tau(\varphi_n, \varphi) \rightarrow 0$ implies that $\varphi_n \rightarrow \varphi$ uniformly as a real-valued function on \mathcal{B} . This gives convergence in τ a strong quantitative interpretation. In what follows we ignore the proportional deviation found in Exercise 4 and refer to τ as the **total variation distance**.

We now turn to a striking illustration of the breadth of the Wasserstein metric and Wasserstein contractions. Our first observation is that τ is a special case of (12.6). The pairing is achieved when we take ρ in (12.6) to be the discrete metric on X . That is, $\rho(x, y) = \mathbb{1}\{x \neq y\}$. Then L_ρ , the set of all $h: \mathsf{X} \rightarrow \mathbb{R}$ with $|h(x) - h(y)| \leq \rho(x, y)$ over x, y pairs in X , is exactly L_τ in (13.12).

It now follows from the Kantorovich duality theorem that τ has the alternate representation

$$\tau(\varphi, \psi) = \inf_{(X, Y) \in \Pi(\varphi, \psi)} \mathbb{P}\{X \neq Y\} \quad (13.13)$$

Indeed, this is just (12.5) with $\rho(x, y) = \mathbb{1}\{x \neq y\}$.

Just use the notation for a Wasserstein distance. For example, let τ be the discrete metric and use W_τ . Note that $\mathcal{D}_1(\mathsf{X}) = \mathcal{D}(\mathsf{X})$.

A Markov kernel P on X is said to be **minorized** on $A \subset \mathsf{X}$ if there exists an integer $n \in \mathbb{N}$, an $\varepsilon > 0$ and a $\nu \in \mathcal{D}(\mathsf{X})$ such that

$$P^n(x, B) \geq \varepsilon \nu(B) \text{ for all } x \in A, B \in \mathcal{B}. \quad (13.14)$$

We say that P satisfies **Doeblin's condition** if it is minorized on all of X .

Example 13.3.2. The case of finite state MCs that are irreducible and aperiodic.

Theorem 13.3.4. If P satisfies Doeblin's condition, then P is uniform contraction on $(\mathcal{D}(\mathsf{X}), \tau)$ with modulus $1 - \varepsilon$. In particular, P is globally stable on $\mathcal{D}(\mathsf{X})$.

Rephrase the above in terms of Wasserstein contractions.

Proof. Note that R defined by $R(x, B) = (P(x, B) - \varepsilon \nu(B))/(1 - \varepsilon)$ is a Markov kernel. Let F realize this Markov kernel, in the sense that $F(x, U)$ is a draw from $R(x, \cdot)$ when U is uniform on $[0, 1]$. Give cite for existence. Fix φ, ψ in X . Let (X, Y) be the coupling that attains the infimum in (13.13). Ref existence of optimal coupling. We construct a coupling (X', Y') of $(\varphi P, \psi P)$ as follows.

Let Z be a binary random variable with $\mathbb{P}\{Z = 1\} = \varepsilon$ and let U be uniform on $[0, 1]$. Both are drawn independently. If $Z = 1$, draw $X' \sim \nu$ and set $Y' = X'$. Otherwise, set $X' = F(X, U)$ and $Y' = F(Y, U)$

To see that (X', Y') is a coupling of $(\varphi P, \psi P)$, consider the case of X' . We have

$$\begin{aligned} \mathbb{P}\{X' \in B\} &= \mathbb{P}\{X' \in B \mid Z = 1\} \mathbb{P}\{Z = 1\} + \mathbb{P}\{X' \in B \mid Z = 0\} \mathbb{P}\{Z = 0\} \\ &= \nu(B) + \int R(x, B) \varphi(dx) (1 - \varepsilon) = \int P(x, B) \varphi(dx). \end{aligned}$$

The proof that $Y' \sim \psi P$ is similar.

If either $X = Y$ or $Z = 1$, then $X' = Y'$. This and independence of Z yields

$$\mathbb{P}\{X' \neq Y'\} \leq \mathbb{P}\{Z = 0\}\mathbb{P}\{X \neq Y\}.$$

The right hand side can be written as $(1 - \varepsilon)\tau(\varphi, \psi)$. Since (X', Y') is a coupling of $(\varphi P, \psi P)$, the claim in the theorem holds. \square

13.3.4 Harris Chains

Add a roadmap: Outline basic idea, Hairer and Mattingly's approach, Eberle's approach, its advantages. Supply a proof for Theorem 7.1.12.

13.3.4.1 Definition and Properties

Let P be a Markov kernel on X . We say that P is **Harris recurrent** if it is φ -irreducible and, moreover, for each $B \in \mathcal{B}$ with $\varphi(B) > 0$ and each $x \in \mathsf{X}$ we have

$$\mathbb{P}_x\{X_t \in B \text{ infinitely often}\} = 1. \quad (13.15)$$

In words, nontrivial sets are visited infinitely often from any starting point. If a Harris recurrent kernel P has a stationary distribution then P is called **positive Harris**. Positive Harris chains have many attractive properties related to stability and ergodicity.¹

Rather than studying Harris chains directly, we investigate chains satisfying two related assumptions. We show that the two assumptions below imply not only Harris recurrence but also existence of a stationary distribution and strong forms of stationarity and ergodicity.

Assumption 13.3.1. There exists a measurable function $V: \mathsf{X} \rightarrow \mathbb{R}_+$ such that P satisfies geometric drift with respect to V and P is minorized on $\{x \in \mathsf{X} : V(x) \leq R\}$ for some $R > 2L/(1 - \lambda)$.

Assumption 13.3.1 is a drift condition similar to that used in Lemma 13.2.2. Here V is not required to be norm-like, so the sublevel sets need not be compact. There is, however, a connection, because the minorization restriction typically holds when the sublevel set $C := \{x \in \mathsf{X} : V(x) \leq R\}$ is small. The standard topological notion of smallness is compactness.

¹Harris chains are named after American mathematician Theodore Harris (1919–2005).

13.3.4.2 A Stability Result

[Main theorem and proof. Hide some of these details. Define TV convergence and relate it to the Wasserstein metric. Prove that convergence in d_w below always implies total variance convergence. Later add separate treatment of chains with atoms and Kac's theorem? Some separate treatment of the uniformly ergodic case?]

Let $w: \mathsf{X} \rightarrow [1, \infty)$ be measurable. Define

$$\|h\|_w := \sup_x \frac{|h(x)|}{w(x)}.$$

Let $b_w \mathsf{X}$ be the measurable functions $h: \mathsf{X} \rightarrow \mathbb{R}$ with $\|h\|_w < \infty$. Let $\mathcal{D}_w(\mathsf{X})$ be all $\psi \in \mathcal{D}(\mathsf{X})$ with $\psi(w) < \infty$. Let G_w be all $h \in b_w \mathsf{X}$ with $\|h\|_w \leq 1$. Define

$$d_w(\psi, \varphi) := \sup_{h \in G_w} |\psi(h) - \varphi(h)| \quad (\psi, \varphi \in \mathcal{D}_w(\mathsf{X})).$$

The following lemma is proved in §19.2.

Lemma 13.3.5. *The space $(\mathcal{D}_w(\mathsf{X}), d_w)$ is complete.*

Theorem 13.3.6. *Let P be a stochastic kernel on X . If Assumptions 13.3.1 holds, then there exists a weight function $w \geq 1$ and a $\theta \in (0, 1)$ such that*

$$d_w(\psi P, \varphi P) \leq \theta d_w(\psi, \varphi), \quad \forall \psi, \varphi \in \mathcal{D}_w(\mathsf{X}).$$

In particular, under the stated conditions P is a uniform contraction on the complete metric space $(\mathcal{D}(\mathsf{X}), d_w)$ and hence is globally stable. A proof of Theorem 13.3.6 can be found in §19.2.

To show the value of Theorem 13.3.6, let us use it to establish the stability component of Proposition 7.2.3 on page 246. *Proof to be added.*

13.3.5 Sample Path Properties

LLN. (Outsource CLT results?) Connect to coupling. Verify sample path claims in Proposition 7.2.3 on page 246. Transition to general state space.

Proposition 13.3.7. *If for each $x, x' \in \mathsf{X}$ there exists a successful P -coupling from (x, x') , then every P -harmonic function in $b\mathsf{X}$ is constant.*

Proof. Let P have the stated properties and let h be a P -harmonic function in $b\mathbb{X}$. Pick any $x, x' \in \mathbb{X}$ and let $\{(X_t, X'_t)\}$ be a successful P -coupling from x, x' . Since $\{h(X_t)\}$ and $\{h(X'_t)\}$ are martingales, we have $\mathbb{E}h(X_t) = \mathbb{E}h(X_0) = h(x)$ and $\mathbb{E}h(X'_t) = \mathbb{E}h(X'_0) = h(x')$. Moreover,

$$|h(x) - h(x')| = |\mathbb{E}h(X_t) - \mathbb{E}h(X'_t)| \leq \mathbb{E}|h(X_t) - h(X'_t)| \leq \mathbb{P}\{X_t \neq X'_t\}$$

The right hand side converges to zero in t , so $h(x) = h(x')$. Since x and x' are arbitrary, we conclude that h is constant on \mathbb{X} . \square

13.4 Monotonicity

Next we turn to methods for studying stability of Markov chains based around monotonicity. These methods provide powerful results for a very large range of processes, including standard processes that are not traditionally thought of as monotone.

13.4.1 Monotone Transitions

Review FOSD, give def on monotonicity in its different forms, examples. Transition the following to a general state space. Integrate with previous material. Most will need to be rewritten. Cite Hopenhayn and Prescott (1992) and Hopenhayn (1992).

As before, let \mathbb{X} be a nonempty countable set and let \preceq be a partial order on \mathbb{X} . A Markov kernel P on \mathbb{X} is called **monotone increasing** with respect to \preceq if

$$x \preceq x' \implies P(x, \cdot) \preceq_F P(x', \cdot) \quad (13.16)$$

(Here \preceq_F indicates stochastic dominance, as discussed in §5.1.4. In some instances we will abbreviate this to **increasing**.

Lemma 13.4.1. *For Markov kernel P on \mathbb{X} , the following statements are equivalent:*

- (i) P is monotone increasing.
- (ii) The left Markov operator $\psi \mapsto \psi P$ is isotone on $(\mathcal{D}(\mathbb{X}), \preceq_F)$.
- (iii) The right Markov operator $h \mapsto Ph$ is invariant on $ib\mathbb{X}$.

A Markov kernel P on \mathbb{X} is called *order reversing* if, for any given x and x' in \mathbb{X} with $x \geq x'$, and any independent Q -Markov processes $\{X_t\}$ and $\{X'_t\}$ starting at x and x' respectively, there exists a $t \in \mathbb{N}$ with $\mathbb{P}\{X_t \leq X'_t\} > 0$. In other words, there exists a point in time at which the initial ordering is reversed with positive probability.

Example 13.4.1. Suppose we are studying a model of household wealth dynamics. Informally, the model is order reversing if, for two households receiving idiosyncratic shocks from the same distribution, it is the case that, regardless of the initial ranking of the two households according to wealth, the probability that their relative wealth positions will be reversed at some point in time is strictly positive.

In verifying order reversing, it is clearly sufficient to check the existence of a t with $\mathbb{P}\{X_t \leq X'_t\} > 0$ for an arbitrary pair $x, x' \in \mathbf{X}$ and often this is just as easy.

13.4.2 Stability and Ergodicity

Also to be rewritten.

In the statement of the next theorem, a particular partial order \preceq on \mathbf{X} is in force and P is a Markov kernel on \mathbf{X} .

Theorem 13.4.2. *If P is increasing and order reversing, then P is globally stable if and only P is bounded in probability.*

Proof. Assume that a Markov kernel P is increasing and order reversing. Clearly global stability implies boundedness in probability, so we focus on the reverse implication. To see that this holds, observe that, by lemma A.5 of Kamihigashi and Stachurski (2014), P is order mixing under these assumptions. Moreover, by theorem 13.2.1, at least one stationary distribution exists. Global stability now follows from corollary 3.1 of Kamihigashi and Stachurski (2012). \square

13.5 Continuous Time Models

13.5.1 Infintessimal Generators

Recall from the discussion of Markov chains that the infintessimal description, in terms of Q , is the most parsimonious specification that defines the whole Markov semigroup. Generalize this idea. Focus on the martingale approach of Strook and Vardahan. Focus on examples we have already seen, including Markov chains and deterministic models. Omit the most general cases.

13.5.2 Markov Jump Chains

Or focus on continuous time Markov switching models?

13.5.3 Stability in Continuous Time

Given at least one example that uses the Wasserstein contraction method. Combine this methodology with the Lipschitz extension to discrete stability on page 167. This should work for Markov switching and finite state MCs. Give some general drift result suitable for diffusions.

Chapter 14

Diffusions

Roadmap. A presentation at the level of Lasota and Mackey (1994).

14.1 Brownian Motion and Weiner Measure

A very quick primer, outsourcing most proofs.

14.2 Diffusion Processes

Kolmogorov (Fokker–Plank) equations. Infintessimal generator. Examples, simulation.

14.3 Control Theory

State but don't prove HJB results, focus on examples. Explain viscocity solutions.

Chapter 15

Diffusions

Roadmap. A presentation at the level of Lasota and Mackey (1994).

15.1 Brownian Motion and Weiner Measure

A very quick primer, outsourcing most proofs.

15.2 Diffusion Processes

Kolmogorov (Fokker–Plank) equations. Infintessimal generator. Examples, simulation.

15.3 Control Theory

State but don't prove HJB results, focus on examples. Explain viscocity solutions.

Chapter 16

Industry and Firm Dynamics

This chapter is not really planned yet. Section titles are suggestions. Also consider [Bolton et al. \(2011\)](#).

16.1 The Hopenhayn Entry-Exit Model

See [Hopenhayn \(1992\)](#). Connect to [Hopenhayn and Prescott \(1992\)](#).

16.2 The Melitz Model

16.3 The Carvalho–Grassi Model

See [Carvalho and Grassi \(2019\)](#).

16.4 Markov Perfect Industry Dynamics

Very simple Markov Perfect industry dynamics, as in [Abbring et al. \(2016\)](#).

Chapter 17

General Equilibrium Models

17.1 Bewley–Huggett–Aiyagari Models

Huggett (1993), Aiyagari (1994), Krusell and Smith (1998), Bewley (1986). Start off with closed form solutions.

Toda has an analytical solution to a Huggett model in JEDC, 2017 — perfect for teaching about stationary equilibria in heterogeneous agent models.

17.2 An Example with CARA Utility

17.3 An Example with CRRA Utility

17.4 A Model with Default

Arellano (2008), contraction maps in Aguiar and Amador (2019).

Also consider Merlitz?

17.5 Notes

This chapter is not really planned yet, despite the titles. Papers we could treat include optimal taxation and public finance research such as Golosov et al. (2003), Golosov et al. (2006), Bassetto (2002), Cole and Kocherlakota (2001), Bhandari et al. (2017).

Macrofinance as in Brunnermeier and Sannikov (2014), Cochrane and Piazzesi (2005). Something with robustness or model misspecification, such as Esponda and Pouzo (2016). Sticky prices as in Mankiw and Reis (2002) or Chari et al. (2002). Political economy as in Fernandez and Rodrik (1991). Learning as in Marcet and Sargent (1989). Financial crises as in Schneider and Tornell (2004). More continuous time, as in Kaplan et al. (2018).

Part III

Appendices

Chapter 18

Appendix I: Analysis and Probability

18.1 Testing Compactness

Many of the fixed point results in the appendix required some form of compactness. Now is a good time to discuss this concept in more depth.

One of our tasks is to determine which subsets of infinite dimensional spaces like $\ell_p(\mathbb{X})$ or $cb\mathbb{X}$ are precompact, so that we can check the conditions of theorems that require precompactness or compactness. In this endeavor, the following result is useful:

Theorem 18.1.1. *If (M, d) is a complete metric space, then a subset C of M is precompact in M if and only if it is totally bounded.*

Here **total boundedness** means that, given any $\varepsilon > 0$, there exists a finite set of ε -balls such that C is contained in their union.

Theorem 18.1.2 (Frechét). *A subset \mathcal{G} of $\ell_1(\mathbb{X})$ is precompact if and only if*

- (i) *there exists a finite K such that $|\varphi(x)| \leq K$ for each $\varphi \in \mathcal{G}$ and each $x \in \mathbb{X}$, and*
- (ii) *for each $\varepsilon > 0$, there exists a finite set F such that*

$$\sup_{\varphi \in \mathcal{G}} \sum_{x \in F^c} \varphi(x) < \varepsilon$$

For a proof see [Hanche-Olsen and Holden \(2010\)](#), theorem 4.

Chapter 19

Appendix II: Additional Proofs

[roadmap]

19.1 Proofs from Chapter 3

[roadmap]

19.1.1 Coupling

The following needs to be re-written to fit the new version of Theorem 7.1.9 and connected back to that discussion.

Let's now switch to a probabilistic approach to stability that, while potentially less familiar, is both powerful and elegant. As well as illustrating these methods, which will reappear later in the text, this approach yields conditions for global stability that are weaker than those in Theorem 7.1.9 and often easier to use in applications.

19.1.1.1 Successful Couplings

success of couplings, connection to previous results

The main property required is that the importance of initial conditions declines and, in the limit, is irrelevant for outcomes. Another way to phrase this is that, if we were to start two chains $\{X_t\}$ and $\{X'_t\}$ from distinct points x and x' , these two chains would eventually "mix together," despite their initial differences.

Given $\psi, \psi' \in \mathcal{D}(X)$, let us agree to say that there exists a **successful Π -coupling from (ψ, ψ')** if we can construct a stochastic process $\{(X_t, X'_t)\}$ on $X \times X$ such that

- (i) $\{X_t\}$ is an (ψ, Π) -chain on X ,
- (ii) $\{X'_t\}$ is an (ψ', Π) -chain on X , and
- (iii) $\mathbb{P}\{X_t \neq X'_t\} \rightarrow 0$ as $t \rightarrow \infty$.

In the convergence argument, it will be convenient to use the ℓ_∞ norm instead of the ℓ_1 norm. Thus, the distance between any two elements of $\mathcal{D}(\mathsf{X})$ will be

$$d_\infty(\varphi, \psi) := \|\varphi - \psi\|_\infty = \sup_{x \in \mathsf{X}} |\varphi(x) - \psi(x)| \quad (19.1)$$

We will exploit the following version of the **coupling inequality**, which states that, for elements φ and ψ of $\mathcal{D}(\mathsf{X})$, we have

$$X \stackrel{d}{=} \varphi \text{ and } Y \stackrel{d}{=} \psi \implies \|\varphi - \psi\|_\infty \leq \mathbb{P}\{X \neq Y\} \quad (19.2)$$

The intuition is that, if the probability that X and Y differ is small, then so is the distance between their distributions. To see why (19.2) holds, pick any $x \in \mathsf{X}$ and consider the two identities

- (\star) $\mathbb{P}\{X = x\} = \mathbb{P}\{X = x, X = Y\} + \mathbb{P}\{X = x, X \neq Y\}$
- (\dagger) $\mathbb{P}\{Y = x\} = \mathbb{P}\{Y = x, X = Y\} + \mathbb{P}\{Y = x, X \neq Y\}$

Clearly $\{X = x, X = Y\} = \{Y = x, X = Y\}$, so, subtracting (\dagger) from (\star),

$$\mathbb{P}\{X = x\} - \mathbb{P}\{Y = x\} = \mathbb{P}\{X = x, X \neq Y\} - \mathbb{P}\{Y = x, X \neq Y\}$$

Hence

$$\mathbb{P}\{X = x\} - \mathbb{P}\{Y = x\} \leq \mathbb{P}\{X = x, X \neq Y\} \leq \mathbb{P}\{X \neq Y\}$$

Reversing the roles of X and Y gives

$$|\mathbb{P}\{X = x\} - \mathbb{P}\{Y = x\}| \leq \mathbb{P}\{X \neq Y\}$$

Since x is arbitrary, we have established (19.2).

Next, we construct a *coupling* of two Markov chains, which results in a pair of chains $\{X_t\}, \{X''_t\}$ that *coalesce* whenever they meet, in the sense that

$$X_j = X''_j \text{ at some } j \implies X_t = X''_t \text{ for all } t \geq j \quad (19.3)$$

In addition, while the initial distribution of X_t will be an arbitrary distribution ψ , the initial distribution of X''_t will be ψ^* , the stationary distribution (existence of which

is guaranteed by theorem 7.1.9). Hence, for any given t ,

$$\|\psi\Pi^t - \psi^*\|_\infty = \|\psi\Pi^t - \psi^*\Pi^t\|_\infty \leq \mathbb{P}\{X_t \neq X''_t\} = \mathbb{P}_{\cap_{j \leq t}}\{X_j \neq X''_j\} \quad (19.4)$$

Here the first inequality is the coupling inequality (19.2), while the second crucial inequality is from (19.3): If X and X'' are distinct at t then they cannot have met at any time prior to t by (19.3), so the two events we are computing probabilities over are identical.

It remains to construct Markov chains with the stated properties and then show that the probability of never meeting prior to t goes to zero as $t \rightarrow \infty$ using the conditions in theorem 7.1.10. At this stage you can probably already see, at least on an intuitive level, that condition (ii) in theorem 7.1.10 does imply that separately started chains will meet eventually, so that the probability of never meeting prior to t does indeed go to zero as t gets large.

To formalize this, we begin by constructing the pair $\{X_t\}$, $\{X''_t\}$ such that

- (a) both $\{X_t\}$ and $\{X''_t\}$ are Markov chains generated by Π .
- (b) $\{X_t\}$ has initial condition ψ while $\{X''_t\}$ has initial condition ψ^* .
- (c) Property (19.3) holds.

To build this pair, we first construct $\{X_t\}$ using (3.6) on page 50:

- draw X_0 from ψ and then update via $X_{t+1} = F(X_t, U_{t+1})$

Here F is defined using Π and (3.6), while $\{U_t\}$ is IID and uniform on $[0, 1]$. The result is that $\{X_t\}$ is a (ψ, Π) -chain, as required.

Second, we first construct $\{X'_t\}$ using the same technique but a different initial condition:

- draw X'_0 from ψ^* and then update via $X'_{t+1} = F(X'_t, U'_{t+1})$

Here $\{U'_t\}$ is IID and uniform on $[0, 1]$, and independent of $\{U_t\}$. Third, set

$$\tau := \inf\{t \geq 0 : X_t = X'_t\}$$

which corresponds to the first meeting time of X and X' . Finally, construct $\{X''_t\}$ via

$$X''_{t+1} = \begin{cases} X'_t & \text{if } t \leq \tau \\ X_t & \text{if } t \geq \tau \end{cases} \quad (19.5)$$

Now let's check properties (a)–(c). Property (a) is obvious for $\{X_t\}$ but more subtle for $\{X''_t\}$. The best way to understand it is this: The process $\{X''_t\}$ starts off at X'_0

and is updated using the shock process $\{U'_t\}$, so that it remains equal to $\{X'_t\}$. At the point in time τ , however, we *switch the source of shocks* to $\{U_t\}$, so that the next update is

$$X''_{\tau+1} = F(X''_\tau, U_{\tau+1}) = F(X'_\tau, U_{\tau+1}) = F(X_\tau, U_{\tau+1}) = X_{t+1}$$

From now on, receiving the same shocks $\{U_t\}$ as $\{X_t\}$, the process $\{X''_t\}$ successfully tracks $\{X_t\}$.

At this point it should be clear that $\{X''_t\}$ is just a Markov chain generated by Π , since at each update we apply the update rule

$$X''_{t+1} = F(X''_t, W) \quad \text{where } W \text{ is some independent draw from } U[0, 1]$$

The fact that the source of shocks was switched from $\{U_t\}$ to $\{U'_t\}$ makes no difference to this argument. Hence property (a) holds.

Properties (b) and (c) follow directly from the construction of $\{X_t\}$ and $\{X''_t\}$. We are now in good position to complete the

Proof of theorem 7.1.10. Let X be finite and let Π be any Markov kernel on X . If condition (i) holds, then there is a $y \in X$ and a $k \in \mathbb{N}$ such that $\Pi^k(x, y) > 0$ for all $x \in X$, so condition (ii) clearly holds.

To see that (ii) implies global stability, let ψ be any element of $\mathcal{D}(X)$ and let $\{X_t\}$, $\{X'_t\}$ and $\{X''_t\}$ be as constructed above. To finish our preceding arguments, we only need to show that the term on the right hand side of (19.4) converges to zero. For this it suffices to prove that the probability $\{X_t\}$ and $\{X'_t\}$ never meet is zero, or

$$\mathbb{P} \cap_{j \leq t} \{X_j \neq X'_j\} \rightarrow 0 \quad \text{as } t \rightarrow \infty$$

To this end, recall that, for each x, x' in X , condition (ii) yields a $k(x, x')$ in \mathbb{N} and an $\varepsilon(x, x') > 0$ such that

$$\mathbb{P}\{X_{k(x,x')} = X'_{k(x,x')} \mid X_0 = x, X'_0 = x'\} \geq \varepsilon(x, x')$$

Let $k := \max_{x,x'} k(x, x')$ and let $\varepsilon := \min_{x,x'} \varepsilon(x, x')$. By construction, over any period of length k ; that is, over dates t in $j+1, \dots, j+k$ for any given j , the event $\{X_t = X'_t\}$ occurs at least once with probability greater than ε . This is true regardless of the locations of the two chains $\{X_t\}$ and $\{X'_t\}$ at j .

Let E_n be the event that $X_t = X'_t$ for some integer t satisfying $(n-1)k < t \leq nk$. In other words, if we divide time into a sequence of epochs of length k , starting at date $t = 0$, then E_n is the event that the two chains meet during the n -th epoch. We have

already shown that each of these events has probability no less than ε . Moreover, each of these events is independent. One can prove this independence through the strong Markov property but here there is no need—each epoch E_n depends on the shocks $(U_t, U'_t)_{t \geq 1}$ that are realized in that epoch. Since all shocks are IID, events that depend only on shocks from separate epochs are independent.

In view of this independence, we have

$$\mathbb{P} \cap_{j \leq t} \{X_j \neq X'_j\} \leq \mathbb{P} \cap_{i \leq t/k} E_i^c = \prod_{i \leq t/k} \mathbb{P} E_i^c \leq (1 - \varepsilon)^{t/k}$$

Hence, as $t \rightarrow \infty$, this probability converges to zero, completing our proof of global stability.

To finish the proof of theorem 7.1.10, we only need to show that (iii) implies (i). So let Π be globally stable, let ψ^* be the stationary distribution, and let \bar{y} be an element of \mathbf{X} such that $\psi^*(\bar{y}) > 0$. Fix ε such that $\psi^*(\bar{y}) - \varepsilon > 0$. Pick any $x \in \mathbf{X}$. By global stability, we can choose an $n(x) \in \mathbb{N}$ such that $n \geq n(x)$ implies

$$\max_y |\Pi^n(x, y) - \psi^*(y)| = \|\Pi^n(x, \cdot) - \psi^*\|_\infty = \|\delta_x \Pi^n - \psi^*\|_\infty < \varepsilon$$

In particular, $\Pi^n(x, \bar{y}) > \psi^*(\bar{y}) - \varepsilon > 0$. Setting $k = \max_x n(x)$ produces an integer such that $\Pi^k(x, \bar{y}) > 0$ for all x , which is (i). \square

19.2 Proofs from Chapter 13

Next we aim to prove Theorem 13.3.6 on page 377.

Let $w \geq 1$ be a weighting function. First we establish the claim in Lemma 13.3.5 that $(\mathcal{D}_w(\mathbf{X}), d_w)$ is complete.

Proof. Let $\{\mu_n\}$ be Cauchy in $(\mathcal{D}_w(\mathbf{X}), d_w)$. Define $\hat{\mu}_n$ by $\hat{\mu}_n(h) = \mu_n(hw)$. Evidently $\hat{\mu}_n$ is a finite measure on \mathbf{X} . Moreover, since, for any h with $|h| \leq 1$,

$$|\hat{\mu}_n(h) - \hat{\mu}_m(h)| = |\mu_n(hw) - \mu_m(hw)| \leq d_w(\mu_n, \mu_m),$$

it follows that $d_{TV}(\hat{\mu}_n, \hat{\mu}_m) \leq d_w(\mu_n, \mu_m)$, and, as a result, $\{\hat{\mu}_n\}$ is Cauchy in total variation. Since $(\mathcal{D}(\mathbf{X}), d_{TV})$ is complete, there exists a finite measure $\hat{\mu}$ such that $d_{TV}(\hat{\mu}_n, \hat{\mu}) \rightarrow 0$. Define the set function μ by $\mu(h) = \hat{\mu}(h/w)$. Evidently $\mu \in \mathcal{D}_w(\mathbf{X})$. Moreover, if we fix $h \in G_w$, then $|h/w| \leq 1$, and hence

$$|\mu_n(h) - \mu(h)| = |\hat{\mu}_n(h/w) - \hat{\mu}(h/w)| \leq d_{TV}(\hat{\mu}_n, \hat{\mu}).$$

Therefore $d_w(\mu_n, \mu) \leq d_{TV}(\hat{\mu}_n, \hat{\mu})$, and μ_n converges to μ in d_w . \square

For $h \in b_w \mathbb{X}$, define the seminorm

$$\|h\|_w^* := \sup_{x, x'} \frac{|h(x) - h(x')|}{w(x) + w(x')}.$$

Lemma 19.2.1. *For any $h \in b_w \mathbb{X}$ we have*

- (i) $\|h\|_w^* \leq \|h + c\|_w$ for all $c \in \mathbb{R}$.
- (ii) $\|h\|_w^* = \|h + c_h\|_w$ when $c_h := \inf_x \{w(x)\|h\|_w^* - h(x)\}$.

Proof. Fix $h \in b_w \mathbb{X}$. Observe first that for any x and x' we have $|h(x)| \leq w(x)\|h\|_w$ and $|h(x')| \leq w(x')\|h\|_w$. Adding the inequalities gives $|h(x)| + |h(x')| \leq (w(x) + w(x'))\|h\|_w$. Using the triangle inequality and then taking the supremum gives $\|h\|_w^* \leq \|h\|_w$. Moreover, since $\|h\|_w^* = \|h + c\|_w^*$ clearly holds for any $c \in \mathbb{R}$, we now have

$$\|h\|_w^* = \|h + c\|_w^* \leq \|h + c\|_w, \quad \forall c \in \mathbb{R}.$$

It remains to show that $\|h + c_h\|_w \leq \|h\|_w^*$, where c_h is as defined in the statement of the lemma. To see this, fix $x \in \mathbb{X}$. On one hand,

$$h(x) + c_h \leq h(x) + w(x)\|h\|_w^* - h(x) = w(x)\|h\|_w^*.$$

On the other hand, since $h(x) - h(x') \geq -\|h\|_w^*(w(x) + w(x'))$,

$$h(x) + c_h = \inf_{x'} \{h(x) + w(x')\|h\|_w^* - h(x')\} \geq -w(x)\|h\|_w^*.$$

From the last two displays we conclude that $|h(x) + c_h|/w(x) \leq \|h\|_w^*$, and taking the sup over x gives the desired bound. \square

Let G_w^* be the set of all $h \in b_w \mathbb{X}$ such that $\|h\|_w^* \leq 1$. The next lemma shows that the d_w distance can be computed by taking the supremum over $h \in G_w^*$ rather than G_w .

Lemma 19.2.2. *For any $\mu, \nu \in \mathcal{D}_w(\mathbb{X})$ we have*

$$d_w(\mu, \nu) = \sup_{h \in G_w^*} |\mu(h) - \nu(h)|.$$

Proof. Fix $\mu, \nu \in \mathcal{D}_w(\mathbb{X})$. Let $d_w^*(\mu, \nu) := \sup_{h \in G_w^*} |\mu(h) - \nu(h)|$. The claim is that $d_w^*(\mu, \nu) = d_w(\mu, \nu)$. To see this, pick any $h \in G_w^*$. Letting c_h be as defined in

lemma 19.2.1, we have $\|h + c_h\|_w \leq 1$. Therefore,

$$|\mu(h) - \nu(h)| = |\mu(h) - \nu(h) + c_h - c_h| = |\mu(h + c_h) - \nu(h + c_h)| \leq d_w(\mu, \nu).$$

Taking the supremum over all $h \in G_w^*$ now gives $d_w^*(\mu, \nu) \leq d_w(\mu, \nu)$. To obtain the reverse inequality, note that since $\|h\|_w^* \leq \|h\|_w$ we have $G_w \subset G_w^*$, and hence $d_w(\mu, \nu) \leq d_w^*(\mu, \nu)$. \square

Lemma 19.2.3. *Let $h \in b_w \mathbb{X}$. If $\mu, \nu \in \mathcal{D}_w(\mathbb{X})$, then $|\mu(h) - \nu(h)| \leq \|h\|_w^* d_w(\mu, \nu)$.*

Proof. Let $h \in b_w \mathbb{X}$. The case $\|h\|_w^* = 0$ is clear, because if $\|h\|_w^* = 0$, then h is a constant function, and the left-hand side is zero. Suppose instead that $\|h\|_w^* > 0$. Then we have

$$|\mu(h) - \nu(h)| = \|h\|_w^* \left| \mu\left(\frac{h}{\|h\|_w^*}\right) - \nu\left(\frac{h}{\|h\|_w^*}\right) \right|.$$

Since $h/\|h\|_w^* \in G_w^*$, the right-hand side is dominated by $\|h\|_w^* d_w(\mu, \nu)$. \square

Lemma 19.2.4. *Let $\gamma \geq 0$, and let P be a stochastic kernel. If $\|Ph\|_w^* \leq \gamma \|h\|_w^*$ for all $h \in b_w \mathbb{X}$, then $d_w(\mu P, \nu P) \leq \gamma d_w(\mu, \nu)$ for all $\mu, \nu \in \mathcal{D}_w(\mathbb{X})$.*

Proof. Let $h \in G_w^*$. In view of lemma 19.2.3, we have

$$|(\mu P)(h) - (\nu P)(h)| = |\mu(Ph) - \nu(Ph)| \leq \|Ph\|_w^* d_w(\mu, \nu) \leq \gamma d_w(\mu, \nu).$$

Taking the supremum over all such h establishes the bound. (To apply lemma 19.2.3 we need $Ph \in b_w \mathbb{X}$, but this follows from $\|Ph\|_w^* \leq \gamma \|h\|_w^*$.) \square

Proof of Theorem 13.3.6. Let

$$\lambda_0 := \lambda + \frac{2L}{R}, \quad \sigma := \frac{\alpha}{2L}, \quad \theta := \max \left\{ \frac{2-\alpha}{2}, \frac{2+L\sigma\lambda_0}{2+L\sigma} \right\}, \quad \text{and} \quad w := 1 + \sigma V.$$

Note that $\lambda_0 \in (\lambda, 1)$ by our choice of R . In view of lemma 19.2.4, it suffices to show that $\|Ph\|_w^* \leq \theta \|h\|_w^*$ for all $h \in b_w \mathbb{X}$. For the latter, it is in fact sufficient to show that $\|Ph\|_w^* \leq \theta$ for all $h \in b_w \mathbb{X}$ with $\|h\|_w^* \leq 1$, which in turn will be true if, for arbitrary $x, x' \in \mathbb{X}$ we have

$$|(Ph)(x) - (Ph)(x')| \leq \theta(w(x) + w(x')). \tag{19.6}$$

To see that (19.6) holds, fix any $h \in b_w \mathbb{X}$ with $\|h\|_w^* \leq 1$ and $x, x' \in \mathbb{X}$. Let c_h be the constant defined in lemma 19.2.1, so that $\|h + c_h\|_w \leq 1$, or $|h + c_h| \leq w$. First consider

the case where $V(x) + V(x') \geq R$. By $|h + c_h| \leq 1 + \sigma V$ and the drift condition in assumption 13.3.1, we have

$$\begin{aligned} |(Ph)(x) - (Ph)(x')| &= |(P(h + c_h))(x) - (P(h + c_h))(x')| \\ &\leq (P|h + c_h|)(x) + (P|h + c_h|)(x') \\ &\leq 2 + \sigma(PV)(x) + \sigma(PV)(x') \\ &\leq 2 + \sigma\lambda V(x) + \sigma\lambda V(x') + \sigma 2L \end{aligned}$$

Simple algebra and the assumption $V(x) + V(x') \geq R$ yield

$$\sigma\lambda V(x) + \sigma\lambda V(x') + \sigma 2L \leq \sigma\lambda_0 V(x) + \sigma\lambda_0 V(x').$$

Define $\rho := (2 + L\sigma\lambda_0)/(2 + L\sigma)$. It is easy to check that $\lambda_0 < \rho < 1$. Using $V(x) + V(x') \geq R > L$, we now have

$$\begin{aligned} |(Ph)(x) - (Ph)(x')| &\leq 2 + \sigma\lambda_0[V(x) + V(x')] \\ &= 2 + \sigma \frac{\rho(2 + L\sigma) - 2}{L\sigma} [V(x) + V(x')] \\ &\leq 2\rho + \sigma\rho[V(x) + V(x')]. \end{aligned}$$

From the definition of w , we obtain

$$V(x) + V(x') \geq R \Rightarrow |(Ph)(x) - (Ph)(x')| \leq \rho(w(x) + w(x')). \quad (19.7)$$

It remains to check the case $V(x) + V(x') < R$, which implies that both x and x' lie in C . For $z \in C$, let R be the residual kernel defined by

$$R(z, B) := \frac{1}{1-\alpha} P(z, B) - \frac{\alpha}{1-\alpha} \varphi(B).$$

Using the fact that $R(z, \cdot)$ is a probability measure when $z \in C$, we have

$$\begin{aligned} |(Ph)(x) - (Ph)(x')| &= (1-\alpha)|(Rh)(x) - (Rh)(x')| \\ &= (1-\alpha)|(R(h + c_h))(x) - (R(h + c_h))(x')| \\ &\leq (1-\alpha)(Rw)(x) + (Rw)(x') \\ &= 2(1-\alpha) + (1-\alpha)\sigma[(RV)(x) + \sigma(RV)(x')]. \end{aligned}$$

From the definition of R , we see that $(1-\alpha)(RV)(z) \leq (PV)(z)$, from which we obtain

$$\begin{aligned} |(Ph)(x) - (Ph)(x')| &\leq 2(1-\alpha) + \sigma[(PV)(x) + \sigma(PV)(x')] \\ &\leq 2(1-\alpha) + \sigma\lambda V(x) + \sigma\lambda V(x') + 2\sigma L \\ &= 2(1-\alpha/2) + \lambda[\sigma V(x) + \sigma V(x')]. \\ &\leq 2(1-\alpha/2) + \rho[\sigma V(x) + \sigma V(x')]. \end{aligned}$$

Using the definition of w , this expression can be written as

$$V(x) + V(x') < R \Rightarrow |(Ph)(x) - (Ph)(x')| \leq \max\{1-\alpha/2, \rho\} (w(x) + w(x')). \quad (19.8)$$

Combining (19.7) and (19.8) gives (19.6) for $\theta := \max\{1-\alpha/2, \rho\}$. The proof is done. \square

Chapter 20

Appendix III: Solutions

Solution to Exercise 2. We treat the case where k_0 satisfies $0 < k_0 \leq k^*$. A small amount of algebra shows that $g(k) \geq k$ whenever $k \leq k^*$. Since $k_0 \leq k^*$, we have $g(k_0) \geq k_0$. In other words, $k_1 \geq k_0$. As g is an increasing function, the inequality $k_1 \geq k_0$ implies $g(k_1) \geq g(k_0)$, or, equivalently, $k_2 \geq k_1$. Continuing in this way confirms that $\{k_t\}$ is increasing. To substantiate boundedness, since $k_0 \leq k^*$ and g is increasing, we have $k_1 = g(k_0) \leq g(k^*) = k^*$. Applying g to both sides of this equality gives $k_2 \leq k^*$ and so on. Thus, boundedness is verified.

Solution to Exercise 6. Let

$$a := \sup_{u \neq 0} f(u) \quad \text{where} \quad f(u) := \frac{\|Au\|}{\|u\|} \quad \text{and let } b := \sup_{\|u\|=1} \|Au\|$$

Evidently $a \geq b$ because the supremum is over a larger domain. To see the reverse fix $\varepsilon > 0$ and let u be a nonzero vector such that $f(u) > a - \varepsilon$. Let $\alpha := 1/\|u\|$ and let $u_b := \alpha u$. Then

$$b \geq \|Au_b\| = \frac{\|Au_b\|}{\|u_b\|} = \frac{\|\alpha Au\|}{\|\alpha u\|} = \frac{\alpha \|Au\|}{\alpha \|u\|} = f(u) > a - \varepsilon$$

Since ε was arbitrary we have $b \geq a$.

Solution to Exercise 7. Since $r(A) < 1$, we can find a constant K and an $\varepsilon > 0$ such that $k \geq K$ implies $\|A^k\| < (1 - \varepsilon)^k$. Hence $\|A^k\| \rightarrow 0$.

Solution to Exercise 4. From $\min_{x \in X} f(x) = \max_{y \in Y} g(y)$ we have $f(x) \geq g(y)$ for all $(x, y) \in X \times Y$. Taking (\bar{x}, \bar{y}) with $f(\bar{x}) = g(\bar{y})$, we have $f(\bar{x}) = g(\bar{y}) \leq f(x)$ for

any given $x \in X$. In particular, \bar{x} minimizes f on X . The argument for \bar{y} is similar.

Solution to Exercise 5. The stochastic kernel is

$$\Pi(x, a, y) = \begin{cases} 0 & \text{if } y < Sa \\ (1-p)^x & \text{if } y = Sa \\ (1-p)^{x+Sa-y} p & \text{if } y > Sa \end{cases} \quad (3.37)$$

The middle case is obtained by observing that the next period state hits y when $y = Sa$ if and only if $D_{t+1} \geq x$ and then using the expression for the PMF of the geometric distribution.

Solution to Exercise 3. By the law of iterated expectations, we have $\mathbb{E}[w_t] = \mathbb{E}[\mathbb{E}[w_t | \mathcal{G}_{t-1}]] = \mathbb{E}[0] = 0$.

Solution to Exercise 4. Supposing without loss of generality that $s < t$, we have

$$\mathbb{E}[w_s w'_t] = \mathbb{E}[\mathbb{E}[w_s w'_t | \mathcal{G}_{t-1}]] = \mathbb{E}[w_s \mathbb{E}[w'_t | \mathcal{G}_{t-1}]] = \mathbb{E}[0] = 0$$

Solution to Exercise 7. Computing the modulus of the two eigenvalues leads to $1/(1+\beta)$ in both cases. Hence $r(A) < 1$ whenever $\beta > 0$.

Solution to Exercise 20. Let M be complete, let S be a self-map on M and let S^k be a uniform contraction. Let u^* be the unique fixed point of S^k . Fix $\varepsilon > 0$. We can choose n such that $\rho(S^{nk} S u^*, u^*) < \varepsilon$. Then

$$\rho(SS^{nk} u^*, u^*) = \rho(Su^*, u^*) < \varepsilon.$$

Since ε was arbitrary we have $\rho(Su^*, u^*) = 0$, implying that u^* is a fixed point of S . The proof that $S^n u \rightarrow u^*$ for any u is left to the reader.

Solution to Exercise 30. The claim is that \preceq_F yields a partial order on \mathcal{F} . Reflexivity and transitivity are immediate from the definition. Asymmetry follows from the characterization in (5.6).

Solution to Exercise 36. Continuous functions map compact sets to compact sets (Lemma 5.1.6). Hence $f(K)$ is compact in \mathbb{R} . It is, therefore, closed and bounded, implying the existence of a minimum and a maximum (Exercise 33).

Solution to Exercise 37. Let f and C be as described and let u^* be a local minimizer. Suppose, contrary to the claim in the exercise, that there exists a point u in C such that $f(u) < f(u^*)$. Then, by convexity, for each λ in $[0, 1]$, we have

$$f(\lambda u + (1 - \lambda)u^*) \leq \lambda f(u) + (1 - \lambda)f(u^*) < f(u^*).$$

Taking $\lambda \rightarrow 0$, we can find a point $v := \lambda u + (1 - \lambda)u^*$ arbitrarily close to u^* such that $f(v) < f(u^*)$. This contradicts the definition of a local minimizer.

Solution to Exercise 39. Fix $a \in \mathbb{X}$ and $b \in \mathbb{Y}$. We have $\tau(a, b) \geq \inf_x \tau(x, b)$, so

$$\sup_y \tau(a, y) \geq \sup_y \inf_x \tau(x, y).$$

$$\therefore \inf_x \sup_y \tau(x, y) \geq \sup_y \inf_x \tau(x, y).$$

Solution to Exercise 43. By Banach's contraction mapping theorem, we know that there exists an $i \in \mathbb{N}$ such that (\mathbb{M}, S^i) is globally stable. The claim then follows from Lemma 5.3.3 on page 167 if we can show that S is continuous at the fixed point of S^k . This is clearly true. Indeed, S is continuous everywhere on \mathbb{M} , by Exercise 19.

Solution to Exercise 45. The map P is continuous due to linearity, and a bijection by invertibility. Regarding the conjugacy property, we have, for any $x \in \mathbb{R}^n$,

$$Sx = Ax + b = P^{-1}DPx + b = P^{-1}(DPx + c) = P^{-1}TPx.$$

Hence $S = P^{-1}TP$ on \mathbb{R}^n , as was to be shown.

Solution to Exercise 46. Observe first that τ is a continuous bijection with continuous inverse $\tau^{-1} = \exp$. Moreover, if k is a point in $(0, \infty)$, then

$$\tau(g(k)) = \ln A + \ln s + \alpha \ln k = h(\tau(k))$$

so that $\tau \circ g = h \circ \tau$ on $(0, \infty)$ as required.

Solution to Exercise 47. This is obviously false. For example, if $\mathbb{M} = \mathbb{R}$ and $Sx = -x$, then $\{S^t 1\}$ is periodic.

Solution to Exercise 50. It is enough to prove that the real series $\sum_{k \geq 0} \|A^k\|/(k!)$ converges (see the proof of Theorem 2.3.2 on page 38 for a similar argument). This series is dominated by $\sum_{k \geq 0} \|A\|^k/(k!)$, which converges by the ratio test.

Solution to Exercise 51. If we differentiate e^{tA} with respect to t , we get

$$\frac{d}{dt} e^{tA} = \left(A + t \frac{A^2}{1!} + t^2 \frac{A^3}{2!} + \dots \right) = Ae^{tA}.$$

Solution to Exercise 52. Fix A in $\mathcal{M}_{n \times n}$ and let $B = -A$. Evidently $AB = BA$, so $e^A e^B = e^{A-A} = e^0$. It is easy to check that $e^0 = I$, so $e^A e^{-1} = I$.

Solution to Exercise 53. Given $A = P^{-1}DP$ we have $A^k = P^{-1}D^kP$ for all k , so

$$e^A = \sum_{k \geq 0} \frac{A^k}{k!} = \sum_{k \geq 0} \frac{P^{-1}D^kP}{k!} = P^{-1} \sum_{k \geq 0} \frac{D^k}{k!} P = P^{-1} e^D P.$$

Solution to Exercise 8. For fixed α and any increasing bounded function u , we have

$$\int u dH_\alpha = \alpha \int u dG + \alpha \left(\int u dF - \int u dG \right)$$

By the fact that $G \preceq_F F$ and u is increasing, this expression is increasing in α . Hence $\alpha \leq \beta$ implies $H_\alpha \preceq_F H_\beta$ as claimed.

Solution to Exercise 10. Given the timing of the problem, current profits can be expressed as

$$\pi_t = a_t(p_t s_t - m) - (1 - a_t)c$$

where $a_t = 1$ indicates the decision to harvest, while $a_t = 0$ indicates the opposite. The Bellman equation can be expressed as

$$v(s, p) = \max_{a \in \{0, 1\}} \left\{ a(ps - m) - (1 - a)c + \beta \int v(q(s)(1 - a), p') \varphi(dp') \right\}$$

Another way to write the same thing is

$$v(s, p) = \max \left\{ ps - m + \beta \int v(0, p') \varphi(dp'), -c + \beta \int v(q(s), p') \varphi(dp') \right\}$$

Solution to Exercise 11. The Bellman operator is

$$Tv(s, p) = \max_{a \in \{0, 1\}} \left\{ a(ps - m) - (1 - a)c + \beta \int v(q(s)(1 - a), p') \varphi(dp') \right\}$$

Given $v, w \in \mathbb{R}^X$, we have, by lemma 5.1.17 on page 151 followed by the triangle inequality for integrals,

$$\begin{aligned}|Tv(s, p) - Tw(s, p)| &\leq \beta \max_{a \in \{0, 1\}} \left| \int [v(q(s)(1-a), p') - w(q(s)(1-a), p')] \varphi(p') dp' \right| \\ &\leq \beta \max_{a \in \{0, 1\}} \int |v(q(s)(1-a), p') - w(q(s)(1-a), p')| \varphi(p') dp'\end{aligned}$$

Hence

$$|Tv(s, p) - Tw(s, p)| \leq \beta \|v - w\|_\infty$$

Taking the supremum over all (s, p) in X leads to

$$\|Tv - Tw\|_\infty \leq \beta \|v - w\|_\infty$$

Since v, w were arbitrary elements of \mathbb{R}^X , the contraction claim is established.

Solution to Exercise 12. In this optimal stopping setting the Bellman equation can be written as

$$v(s, p) = \max \left\{ ps - m, -c + \beta \int v(q(s), p') \varphi(dp') \right\}$$

The continuation value is

$$h(s) := -c + \beta \int v(q(s), p') \varphi(dp')$$

This is lower dimensional than v so computing it directly will be advantageous. To this end, we use h to eliminate v from the Bellman equation, first by writing

$$v(s, p) = \max \{ps - m, h(s)\}$$

and then further manipulating to obtain the functional equation

$$h(s) = -c + \beta \int \max \{p's - m, h(s)\} \varphi(dp') \tag{6.21}$$

Next we introduce an operator Q such that fixed points of Q coincide with solutions to (6.21). It takes the form

$$Qh(s) = -c + \beta \int \max \{p's - m, h(s)\} \varphi(dp') \tag{6.22}$$

Let $S = \{0, 1, \dots, \bar{q}\}$. We claim that Q is a contraction mapping on the complete metric space (\mathbb{R}^S, d_∞) . To show this we use Jensen's inequality and the bound (6.4) on page 185 to obtain, for arbitrary $s \in S$ and $g, h \in \mathbb{R}^S$,

$$\begin{aligned} |Qg(s) - Qh(s)| &\leq \beta \int |\max\{p's - m, h(s)\} - \max\{p's - m, g(s)\}| \varphi(dp') \\ &\leq \beta \int |g(s) - h(s)| \varphi(dp') \end{aligned}$$

Hence

$$|Qg(s) - Qh(s)| \leq \beta \|g(s) - h(s)\|_\infty$$

Taking the supremum over s completes the proof.

Solution to Exercise 13. It follows easily from (6.22) that if $c \leq \hat{c}$ and Q and \hat{Q} are the corresponding continuation value operators, then $\hat{Q}h \leq Qh$ for any $h \in \mathbb{R}^S$. Since Q is isotone on \mathbb{R}^S and also globally stable, Proposition 5.3.7 on page 171 implies that, pointwise on S , the fixed point h of Q is larger than the fixed point \hat{h} of \hat{Q} .

Now consider the decision problems of the two firms described in the exercise. The first one has lower maintenance cost and stops when $p_t s_t - m \geq h(s_t)$. The second one has higher maintenance costs and stops when $p_t s_t - m \geq \hat{h}(s_t)$. Since $\hat{h}(s_t) \leq h(s_t)$, we know that if the first firm decides to harvest then the second firm does too. In particular, the firm with higher maintenance costs harvest no later than the firm with low maintenance costs.

Solution to Exercise 15. Set $Q = \gamma$ and

$$R = a_1 \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Solution to Exercise 1. To be added.

Solution to Exercise 4. Fix $g \in \ell_1(\mathsf{X})$. By the triangle inequality for infinite sums and positivity of P , we have

$$\sum_y |(gP)(y)| = \sum_y \left| \sum_x P(x, y)g(x) \right| \leq \sum_y \sum_x P(x, y) |g(x)|.$$

Interchanging the order of summation and using $\sum_y P(x, y) = 1$ yields $\|gP\|_1 \leq \|g\|_1$.

In particular, $\|gP\|_1 < \infty$, so P maps $\ell_1(\mathsf{X})$ to itself.

Solution to Exercise 5. Fix $h \in b\mathsf{X}$. By definition, $|h| \leq K$ for some finite K . Hence, by the triangle inequality, for any $x \in \mathsf{X}$,

$$|(Ph)(x)| = \left| \sum_{y \in \mathsf{X}} h(y)p(x, y) \right| \leq \sum_{y \in \mathsf{X}} |h(y)|p(x, y) \leq K \sum_{y \in \mathsf{X}} p(x, y) = K.$$

Since x was arbitrary, we see that $Ph \in b\mathsf{X}$.

Solution to Exercise 7. Let P be linear and positive. If P is norm preserving on the positive cone, then clearly P maps $\mathcal{D}(\mathsf{X})$ to itself. Conversely, suppose that P is invariant on $\mathcal{D}(\mathsf{X})$. If $g = 0$, then, by linearity, $gP = 0$, in which case the norm preserving property is trivial. If $g \neq 0$, we set $g' := g/\|g\|_1$. Since g is in $\mathcal{D}(\mathsf{X})$, we have $g'P \in \mathcal{D}(Xsf)$. P is norm preserving on the positive cone, since $\|gP\|_1 = \|g\|_1\|g'P\|_1 = \|g\|_1$.

Solution to Exercise 8. Let P be a Markov operator on $\ell_1(\mathsf{X})$ and fix $h \in \ell_1(\mathsf{X})$. We claim that $\|hP\|_1 \leq \|h\|_1$. To see this, let h^+ and h^- be the positive and negative parts of h (see §2.1.1) and observe that

$$\|hP\|_1 = \|(h^+ - h^-)P\|_1 = \|h^+P - h^-P\|_1 \leq \|h^+P\|_1 + \|h^-P\|_1$$

Since P is norm preserving on the positive cone, we conclude that

$$\|hP\|_1 \leq \|h^+\|_1 + \|h^-\|_1 = \|h\|_1$$

To finish the proof of the claim in the exercise, set $h = f - g$ and use $\|hP\|_1 \leq \|h\|_1$ and linearity of P .

Solution to Exercise 9. To prove the first part of Exercise 9, we need to show that P defined in (7.8) is in \mathbb{M} whenever $P \in \mathbb{K}$. Evidently P is linear on $\ell_1(\mathsf{X})$, since, given any pair $f, g \in \ell_1(\mathsf{X})$ and any $\alpha, \beta \in \mathbb{R}$, we have

$$((\alpha f + \beta g)P)(y) = \sum_x P(x, y)(\alpha f + \beta g)(x) = \alpha(fP)(y) + \beta(gP)(y)$$

at each $y \in \mathsf{X}$. It's also clear that $gP \geq 0$ whenever $g \geq 0$. Moreover, if $g \in \ell_1^+(\mathsf{X})$, then

$$\|gP\|_1 = \sum_y (gP)(y) = \sum_x \sum_y \Pi(x, y)g(x) = \sum_x g(x) = \|g\|_1$$

so P is norm preserving on the positive cone.

To show that τ is onto, we need to show that every $P \in \mathbb{M}$ has a $P \in \mathbb{K}$ such that (7.13) holds. To do this, we fix $P \in \mathbb{M}$ and produce a function P via $P(x, y) := (\delta_x P)(y)$, as in (7.8). It is easy to check that $P \in \mathbb{K}$. To show that (7.13) holds, we fix $g \in \ell_1(\mathsf{X})$ and observe that g can be expressed as $g(z) = \sum_x \delta_x(z)g(x)$ for all $z \in \mathsf{X}$. Since P is continuous and linear, we have

$$(gP)(y) = \sum_x (\delta_x P)(y)g(x) = \sum_x P(x, y)g(x).$$

Hence τ is onto as claimed.

Regarding the one-to-one property, let p and q be two distinct Markov kernels and let $P = \tau p$ and $Q = \tau q$. We need to show that $gP \neq gQ$ for some g in $\ell_1(\mathsf{X})$. But this is easy: as P and Q differ at at least one pair (x, y) , we have

$$(\delta_x P)(y) = p(x, y) \neq q(x, y) = (\delta_x Q)(y).$$

Hence, with $g := \delta_x$ we have $gP \neq gQ$, as required.

The last part of the proof, regarding the inverse τ^{-1} has already been established during our discussion of the onto property.

Solution to Exercise 10. Let P , h and g have the stated properties. Then

$$\sum_x \sum_y h(y)P(x, y)g(x) = \sum_y h(y) \sum_x P(x, y)g(x),$$

which is (7.9), with the change in order of summation justified by Fubini's theorem. Fubini's theorem is valid here because

$$\sum_x \sum_y |h(y)P(x, y)g(x)| \leq \|h\|_\infty \sum_x \sum_y P(x, y)g(x) = \|h\|_\infty.$$

Solution to Exercise 11. The corresponding Markov operator P is the identity map on $\ell_1(\mathsf{X})$. Every element of this space is a fixed point of P , and hence every distribution is stationary.

Solution to Exercise 12. Let ψ^* be stationary, so that $\psi^*P = \psi^*$. It is easy to see that (7.9) implies (7.16). Conversely, suppose that (7.16) holds. Then, by (7.9), we have $\langle h, \psi^*P \rangle = \langle h, \psi^* \rangle$ for all $h \in b\mathsf{X}$. Fixing arbitrary $y \in \mathsf{X}$ and Setting $h(x) = \mathbf{1}\{x = y\}$ then gives $(\psi^*P)(y) = \psi^*(y)$. Since y was arbitrary, we conclude that ψ^* is stationarity.

Solution to Exercise 14. Suppose to the contrary that $\psi \in \mathcal{P}(\mathbb{Z})$ is stationary for Π . Then, for any $m \in \mathbb{Z}$,

$$(\psi\Pi)(m) = \sum_n \mathbb{1}\{m = n + 1\}\psi(n) = \psi(m - 1)$$

Since ψ is stationary we have $\psi(m) = \psi(m - 1)$ for all m , so ψ is a constant function. But if ψ is constant and nonnegative, then either $\sum_{n \in \mathbb{Z}} \psi(n) = 0$ or $\sum_{n \in \mathbb{Z}} \psi(n) = \infty$. Either way, $\psi \in \mathcal{P}(\mathbb{Z})$ is contradicted.

Solution to Exercise 19. Let ψ be stationary for P and let P be irreducible. Pick any $y \in \mathsf{X}$ and let x be any point in X such that $\psi(x) > 0$. (Clearly one exists.) By irreducibility, there exists a $k \in \mathbb{N}$ such that $p^k(x, y) > 0$. But then

$$\psi(y) = \sum_z p^k(z, y)\psi(z) \geq p^k(x, y)\psi(x) > 0.$$

Since y was arbitrary, we see that ψ is positive everywhere on X .

Solution to Exercise 21. Let the stated conditions hold, so that $p(x, y) > 0$ for any pair x, y in X , and let φ and ψ be distinct elements of $\mathcal{D}(\mathsf{X})$.

By Exercise 20, there exists a pair $x', x'' \in \mathsf{X}$ such that $\varphi(x') > \psi(x')$ and $\varphi(x'') < \psi(x'')$. Since p is everywhere positive, for any $y \in \mathsf{X}$, we have

$$p(x', y)\varphi(x') > p(x', y)\psi(x') \quad \text{and} \quad p(x'', y)\varphi(x'') < p(x'', y)\psi(x'')$$

By this fact and the strict triangle inequality, we find that

$$\left| \sum_x p(x, y)[\varphi(x) - \psi(x)] \right| < \sum_x |p(x, y)[\varphi(x) - \psi(x)]|$$

Combining this bound with

$$\|\varphi P - \psi P\|_1 = \sum_y \left| \sum_x p(x, y)\varphi(x) - \sum_x p(x, y)\psi(x) \right| = \sum_y \left| \sum_x p(x, y)[\varphi(x) - \psi(x)] \right|$$

yields

$$\begin{aligned}\|\varphi P - \psi P\|_1 &< \sum_y \sum_x |p(x, y)[\varphi(x) - \psi(x)]| \\ &= \sum_y \sum_x p(x, y)|\varphi(x) - \psi(x)| \\ &= \sum_x \sum_y p(x, y)|\varphi(x) - \psi(x)|\end{aligned}$$

Since $\sum_y p(x, y) = 1$ for all x , the last term is just $\|\varphi - \psi\|_1$, completing our proof.

Solution to Exercise 22. This follows from Lemma 5.3.3 on page 167, since P is continuous on $\mathcal{D}(X)$.

Solution to Exercise 8. Since the saving rate is pointwise increasing in the fixed point g^* of the operator S , it suffices to show that the fixed point of S increases with β . To show this is so, let S be the operator under the smaller discount factor function β and let \hat{S} induced by $\hat{\beta}$. In view of Proposition 5.3.7 on page 171, it suffices to show that \hat{S} is isotone, globally stable on C , and has the property $\hat{S}g \geq Sg$ for any $g \in C$.

We already know that \hat{S} is isotone (see, e.g., Exercise 6). To ensure that \hat{S} is globally stable, we assume that

$$\hat{K}(z, z') := \hat{\beta}(z)R(z)^{1-\gamma}\Pi(z, z')$$

satisfies $r(\hat{K}) < 1$. Thus it remains only to show that $\hat{S}g \geq Sg$ for any $g \in C$. This follows directly from the definition $\hat{S} = \psi \circ \hat{K}$ and $S = \psi \circ K$, along with the fact that ψ is isotone and $K \leq \hat{K}$.

Solution to Exercise 9. The proof is very similar to that of exercise 8. In particular, starting with the case $0 < \gamma < 1$, it is enough that the associated K matrices, defined respectively by

$$K(z, z') := \beta(z)R(z)^{1-\gamma}\Pi(z, z') \text{ and } \hat{K}(z, z') := \beta(z)\hat{R}(z)^{1-\gamma}\Pi(z, z')$$

satisfy $r(K) < 1$, $r(\hat{K}) < 1$ and $K \leq \hat{K}$. We assume R and \hat{R} are such that the spectral radius conditions hold. That $K \leq \hat{K}$ when $\gamma \in (0, 1)$ is obvious.

When $\gamma > 1$ we have $1 - \gamma < 0$. The situation then reverses, with $\hat{K} \leq K$. Hence the fixed point of \hat{S} is lower than that of S , and the saving rate correspondingly decreases.

Solution to Exercise 15. Let σ_a and σ_b are elements of \mathcal{C} with $\sigma_a \leq \sigma_b$, $\sigma_a \leq \sigma_b$. To see that $K\sigma_a \leq K\sigma_b$, observe that, for arbitrary $\sigma \in \mathcal{C}$ and fixed $y \in \mathbb{R}_+$, the value $K\sigma(y)$ is the zero of

$$H_\sigma(c) := u'(c) - \beta \int (u' \circ \sigma)(f(y - c)z) f'(y - c) z \varphi(dz)$$

Given our assumptions on the primitives, we have $H_{\sigma_a}(c) \leq H_{\sigma_b}(c)$ for every $c \in (0, y)$. Since H_{σ_b} is pointwise greater, its zero is also larger (see, e.g., figure 9.5 on page 286). In other words, $K\sigma_a(y) \leq K\sigma_b(y)$. Since y was arbitrary, our proof is now done.

Solution to Exercise 17. Let the conditions of the proposition hold and let c^* be the unique optimal policy. We have already seen that T is globally stable on $b\mathbb{R}_+$ under d_∞ and therefore it is globally stable on \mathcal{V} under the same metric. (That T is a self-map on \mathcal{V} follows from proposition 9.2.3.) Moreover, when we endow \mathcal{C} with the metric ρ , T and K are topologically conjugate under the bijection M and so, by theorem 5.3.5 on page 169, the system (\mathcal{C}, K) is also globally stable. Since v^* is the unique fixed point of T in \mathcal{V} , the same theorem implies that the unique fixed point of K in \mathcal{C} is Mv^* . As shown in lemma 9.2.5, $Mv^* = MTv^*$ is the unique v^* -greedy policy, which, by Bellman's principle of optimality, is c^* .

Solution to Exercise 18. Let β_a and β_b be two discount factors with the stated properties. Let K_a and K_b be the respective Coleman–Reffett operators. For $\sigma_b \leq \sigma_a$ to hold, it suffices (by proposition 5.3.7 on page 171) to show that K_a is isotone, globally stable on C , and has the property $K_b\sigma \leq K_a\sigma$ for any $\sigma \in \mathcal{C}$. That K_a is isotone was established in exercise 15. That K_a is globally stable was shown in proposition 9.2.8. To see that $K_b\sigma \leq K_a\sigma$ for all $\sigma \in \mathcal{C}$, fix any such σ and any $y \in (0, \infty)$. Observe that $K_i\sigma(y)$ is the zero of

$$H_i(c) := u'(c) - \beta_i \int (u' \circ \sigma)(f(y - c)z) f'(y - c) z \varphi(dz)$$

for $i = a, b$. Evidently $H_b \leq H_a$ pointwise on $(0, y)$, so the zero of H_b is no larger. In other words, $K_b\sigma(y) \leq K_a\sigma(y)$. Since y was arbitrary, this completes our proof.

The intuition for the result is that larger β means greater patience, which encourages saving and hence reduces consumption.

Solution to Exercise 1. Sufficiency of the condition is obvious: If $\{u_n\}$ is eventually in any open set containing u , then $\{u_n\}$ is eventually in any ε -ball containing u . Regarding necessity, suppose that $\{u_n\}$ converges to $u \in M$ and let G be any open set containing u . As G is open, u is interior to G . Hence there exists an ε -ball containing

u that lies entirely in G . As $\{u_n\}$ is eventually in this ε -ball, it must also eventually be in G .

Solution to Exercise 3. If M is a nonempty set and $\mathcal{G} = \{\emptyset, M\}$, then \mathcal{G} is a topology on M . Under this topology, a given sequence $\{u_n\} \subset M$ converges to every point in M at once.

Solution to Exercise 4. Regarding (i), suppose that a sequence $\{u_n\}$ has two distinct limits u and v . Let G_u and G_v be disjoint neighborhoods of u and v respectively. By the definition of convergence, there exists an $n \in \mathbb{N}$ such that $u_n \in G_u$ and $u_n \in G_v$. Contradiction. Regarding (ii), by definition, any two distinct points u and v in a metric spaces are at positive distance from one another. As a consequence, for sufficiently small ε , we can take ε balls around u and v that do not intersect.

Solution to Exercise 6. Fix $(\varphi, \psi) \in \mathcal{D}(X) \times \mathcal{D}(Y)$ and let $\Pi := \Pi(\varphi, \psi)$. By Prohorov's theorem, it suffices to show that Π is tight and closed in the weak topology. First we prove that Π is tight.

To this end, fix $\varepsilon > 0$ and choose compact sets $K_X \subset X$ and $K_Y \subset Y$ such that $\varphi(K_X^c) \leq \varepsilon/2$ and $\psi(K_Y^c) \leq \varepsilon/2$. Pick any π in Π and let (X, Y) be a realization of π . We have

$$\pi((K_X \times K_Y)^c) = \mathbb{P}\{X \notin K_X \text{ or } Y \notin K_Y\} \leq \mathbb{P}\{X \notin K_X\} + \mathbb{P}\{Y \notin K_Y\} \leq \varepsilon.$$

Since products of compacts are compact in the product topology and π is an arbitrary element of Π , we conclude that Π is tight.

Next we claim that Π is closed. Since $\mathcal{D}(X \times Y)$ is metrizable under the weak topology, it will be enough to show that, given a sequence $\{\pi_n\}$ in Π with $\pi_n \xrightarrow{w} \pi$ for some $\pi \in \mathcal{D}(X \times Y)$, the measure π has marginals φ and ψ . To this end, let μ be the first marginal of π . Fix $h \in bcX$ and consider $(h\mathbb{1})(x, y) := h(x)$, which is in $bc(X \times Y)$. Using the fact that $\pi_n \xrightarrow{w} \pi$, we have

$$\int h d\varphi = \int (h\mathbb{1}) d\pi_n \rightarrow \int (h\mathbb{1}) d\pi = \int h d\mu.$$

Hence $\int h d\mu = \int h d\varphi$ for all $h \in bcX$, and so $\mu = \varphi$ by Exercise 5. A similar argument shows that the second marginal of π is ψ , so $\pi \in \Pi$ and the proof is now complete.

Solution to Exercise 4. Recall that $L := \{h \in b\mathcal{B} : |h(x) - h(y)| \leq 1\}$. Let B be all $h \in b\mathcal{B}$ with $|h| \leq 1$ on X . First we show that $d_{TV} \leq 2\tau$. To this end, fix φ and

ψ in $\mathcal{D}(\mathsf{X})$. Pick $h \in B$. Then $h/2 \in L$, so

$$\left| \int h \, d\varphi - \int h \, d\psi \right| = 2 \left| \int h/2 \, d\varphi - \int h/2 \, d\psi \right| \leq d_\tau(\varphi, \psi).$$

Taking the supremum over B gives $d_{TV} \leq 2d_L$.

The reverse inequality is left to the reader. (Hint: if h is in L , then $2(h - c)$ is in B , where $c := (\inf h + \sup h)/2$.)

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