Joint system and entanglement

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9, Oct., 2020

Joint system

- System = Set of states & set of measurements
- Joint system = "Product" of systems.
- Joint system of a system of a coin (two-dimentional classical system) and a system of a dice (six-dimentional classical system) is twelve-dimentional classical system.
- What is a joint system of quantum systems ?

Tensor product of linear spaces

For linear product spaces V and W over a field F (usually $\mathbb{R}$ or $\mathbb{C}$), a tensor space $V \otimes W$ is a linear space spanned by $v \otimes w$ for all $v \in V$, $w \in W$.

- $\forall c \in F, \forall v \in V, \forall w \in W,\ c(v \otimes w) = (cv) \otimes w = v \otimes (cw)$.
- $\forall u, v \in V, \forall w \in W,\ (u + v) \otimes w = u \otimes w + v \otimes w$.
- $\forall v \in V, \forall w, y \in W,\ v \otimes (w + y) = v \otimes w + v \otimes y$.

Let $(e_i)_i$ be an orthonormal basis of V and $(f_j)_j$ be an orthonormal basis of W. Since $v \otimes w = (\sum_i v_i e_i) \otimes (\sum_j w_j f_j) = \sum_{i,j} v_i w_j (e_i \otimes f_j)$
This implies $\dim(V \otimes W) = \dim(V)\dim(W)$.

If V and W are inner product spaces, $V \otimes W$ is also a inner product space defined by

$$\langle v \otimes w, u \otimes y \rangle = \langle v, u \rangle \langle w, y \rangle.$$

Vector representation in tensor product

Let $V := \mathbb{C}^n$, $W := \mathbb{C}^m$.

$$e_i := \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \begin{array}{c} 1 \\ \\ i-1 \\ i \\ i+1 \\ \\ n \end{array} \in \mathbb{C}^n, \qquad f_j := \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \begin{array}{c} 1 \\ \\ j-1 \\ j \\ j+1 \\ \\ m \end{array} \in \mathbb{C}^m$$

$$e_i \otimes f_j = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \begin{array}{c} (1,1) \\ \\ (i,j-1) \\ (i,j) \\ (i,j+1) \\ \\ (n,m) \end{array} \in \mathbb{C}^n \otimes \mathbb{C}^m$$

Vector representation in tensor product

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$$e_i \otimes f_j = \begin{array}{cc} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} & \begin{matrix} (1,1) \\ \\ (i, j-1) \\ (i,j) \\ (i, j+1) \\ \\ (n, m) \end{matrix} \end{array} \in \mathbb{C}^n \otimes \mathbb{C}^m$$

$$v \otimes w = \left(\sum_i v_i e_i \right) \otimes \left(\sum_j w_j f_j \right) = \sum_{i,j} v_i w_j (e_i \otimes f_j)$$

$$= \begin{array}{cc} \begin{bmatrix} \vdots \\ v_i w_j \\ \vdots \end{bmatrix} & \begin{matrix} \\ (i,j) \\ \\ \end{matrix} \end{array} = \begin{bmatrix} v_1 w \\ v_2 w \\ \vdots \\ v_n w \end{bmatrix} \in \mathbb{C}^n \otimes \mathbb{C}^m$$

Linear spaces

- $\mathcal{L}(V, W)$: A linear space spanned by linear maps from a linear space V to a linear space W.
- $\mathcal{L}(V) := \mathcal{L}(V, V)$.
- $\mathcal{H}(V)$: A real linear space spanned by Hermitian operators acting on a complex linear space V.

Tensor product of linear maps

$$\mathcal{L}(V, X) \otimes \mathcal{L}(W, Y) \cong \mathcal{L}(V \otimes W, X \otimes Y)$$

since the both sides are complex linear spaces with dimension

$$\dim(V)\dim(W)\dim(X)\dim(Y).$$

A natural choice of an isomorphism is

$$\begin{aligned} \mathcal{L}(V, X) \otimes \mathcal{L}(W, Y) &\longrightarrow \mathcal{L}(V \otimes W, X \otimes Y) \\ A \otimes B &\longmapsto (v \otimes w \mapsto A(v) \otimes B(w)). \end{aligned}$$

$$A \otimes B = \begin{bmatrix} A_{11}B & A_{12}B & \dots & A_{1m}B \\ A_{21}B & A_{22}B & \dots & A_{2m}B \\ \vdots & \vdots & \vdots & \vdots \\ A_{n1}B & A_{n2}B & \dots & A_{nm}B \end{bmatrix}$$

Tensor product of Hermitian maps

$$\mathcal{H}(V) \otimes \mathcal{H}(W) \cong \mathcal{H}(V \otimes W)$$

since the both sides are real linear spaces with dimension

$$\dim(V)^2 \dim(W)^2.$$

A natural choice of an isomorphism is

$$\begin{aligned} \mathcal{H}(V) \otimes \mathcal{H}(W) &\longrightarrow \mathcal{H}(V \otimes W) \\ A \otimes B &\longmapsto (v \otimes w \mapsto A(v) \otimes B(w)). \end{aligned}$$

Joint quantum system

A quantum system on a complex linear space V:

- Set of states $= \{\omega \in \mathcal{H}(V) \mid \omega \in C_{\succeq 0}, \mathrm{Tr}(\omega) = 1\}$.
- Set of binary measurements $= \{e \in \mathcal{H}(V) \mid e \in C_{\succeq 0}, I - e \in C_{\succeq 0}\}$.

For a quantum systems on V and W, a joint system is a quantum system on $V \otimes W$.

A useful formula.

$$
\begin{aligned}
\mathrm{Tr}(A \otimes B) &= \sum_{i,j} \langle i| \otimes \langle j| \, A \otimes B \, |i\rangle \otimes |j\rangle \\
&= \sum_{i,j} \langle i| \, A \, |i\rangle \langle j| \, B \, |j\rangle \\
&= \mathrm{Tr}(A)\mathrm{Tr}(B)
\end{aligned}
$$

Examples: two-qubit system

Examples of states

- $|0\rangle\langle 0| \otimes |1\rangle\langle 1| = |01\rangle\langle 01|$
- $\frac{1}{2}(|0\rangle\langle 0| \otimes |0\rangle\langle 0| + |0\rangle\langle 0| \otimes |1\rangle\langle 1|) = |0\rangle\langle 0| \otimes \frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|) = |0\rangle\langle 0| \otimes \frac{1}{2}I$.
- $\frac{1}{2}(|1\rangle\langle 1| \otimes |0\rangle\langle 0| + |0\rangle\langle 0| \otimes |1\rangle\langle 1|)$.
- $\frac{1}{2}(|0\rangle\langle 0| \otimes |0\rangle\langle 0| + |0\rangle\langle 1| \otimes |0\rangle\langle 1| + |1\rangle\langle 0| \otimes |1\rangle\langle 0| + |1\rangle\langle 1| \otimes |1\rangle\langle 1|) = |\Phi\rangle\langle\Phi|$ for $|\Phi\rangle := \frac{1}{\sqrt{2}}(|0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle)$.

Separable states & entangled states

A quantum state ρ in a joint system is said to be separable if

$$\rho = \sum_i p_i \rho_1^i \otimes \rho_2^i$$

for some probability distribution p and quantum states $\{\rho_1^i\}$ and $\{\rho_2^i\}$ for subsystems.

If a quantum state is not separable, the state is said to be entangled state.

In general, it is difficult to determine whether given state is separable or entangled.

Lemma

A pure state $|\psi\rangle \in V \otimes W$ *is separable if and only if there exist pure states* $|\varphi\rangle \in V$ *and* $|\phi\rangle \in W$ *such that* $|\psi\rangle = |\varphi\rangle\,|\phi\rangle$.

Proof.

$$
\begin{aligned}
|\psi\rangle\langle\psi| &= \sum_i p_i \rho_i \otimes \sigma_i \\
&= \sum_i p_i \left(\sum_j \lambda_{i,j} |\varphi_{i,j}\rangle\langle\varphi_{i,j}| \right) \otimes \left(\sum_k \gamma_{i,k} |\phi_{i,k}\rangle\langle\phi_{i,k}| \right) \\
&= \sum_\ell q_\ell |\varphi_\ell\rangle\langle\varphi_\ell| \otimes |\phi_\ell\rangle\langle\phi_\ell|
\end{aligned}
$$

$$
\begin{aligned}
1 &= \mathrm{Tr}\left(|\psi\rangle\langle\psi| \left(\sum_i p_i \rho_i \otimes \sigma_i \right) \right) \\
&= \sum_\ell q_\ell \left| \langle\psi| \left(|\varphi_\ell\rangle |\phi_\ell\rangle \right) \right|^2 \iff |\psi\rangle = e^{i\theta_\ell} |\varphi_\ell\rangle |\phi_\ell\rangle \qquad \square
\end{aligned}
$$

Isomorphism between $V \otimes W$ and $\mathcal{L}(W, V)$

We consider isomporphism $\mathcal{M}$ between $V \otimes W$ and $\mathcal{L}(W, V)$ defined by

$$\begin{aligned} \mathcal{M} : V \otimes W &\to \mathcal{L}(W, V) \\ |i\rangle_V |j\rangle_W &\mapsto |i\rangle_V \langle j|_W \end{aligned}$$

where $(|i\rangle_V)_i$ and $(|j\rangle_W)_j$ are orthonormal basis of V and W, respectively.

$$\begin{aligned} &\mathcal{M}\left(|\psi\rangle_V |\varphi\rangle_W\right) \\ &= \mathcal{M}\left(\left(\sum_i \psi_i |i\rangle_V\right) \otimes \left(\sum_j \varphi_j |j\rangle_W\right)\right) \\ &= \sum_{i,j} \psi_i \varphi_j \mathcal{M}\left(|i\rangle_V |j\rangle_W\right) \\ &= \sum_{i,j} \psi_i \varphi_j |i\rangle_V \langle j|_W \\ &= \left(\sum_i \psi_i |i\rangle_V\right) \left(\sum_j \varphi_j \langle j|_W\right) = |\psi\rangle_V \langle \varphi|_W^* \end{aligned}$$

Determine the separability of pure state

$$
\begin{aligned}
|\psi\rangle \in V \otimes W \text{ is separable } &\iff |\psi\rangle = |\varphi\rangle\,|\phi\rangle \text{ for some } |\varphi\rangle \in V, |\phi\rangle \in W \\
&\iff \mathcal{M}(|\psi\rangle) = |\varphi\rangle\,\langle\phi| \text{ for some } |\varphi\rangle \in V, |\phi\rangle \in W \\
&\iff \mathcal{M}(|\psi\rangle) \text{ is rank } 1
\end{aligned}
$$

$$
\begin{aligned}
&\mathcal{M}\left(\frac{1}{\sqrt{2}}\left(|0\rangle\,|0\rangle + |1\rangle\,|1\rangle\right)\right) \\
&= \frac{1}{\sqrt{2}}\left(|0\rangle\,\langle 0| + |1\rangle\,\langle 1|\right) = \frac{1}{\sqrt{2}} I
\end{aligned}
$$

$\frac{1}{\sqrt{2}}\left(|0\rangle\,|0\rangle + |1\rangle\,|1\rangle\right)$ is entangled!

Theorem (Schmidt decomposition)

For any pure state $|\psi\rangle \in V \otimes W$, there exist orthonormal basis $(|v_i\rangle)_i$ of V and $(|w_j\rangle)_j$ of W, and positive real numbers $(\lambda_i)_i$ such that

$$|\psi\rangle = \sum_i \lambda_i \, |v_i\rangle_V \, |w_i\rangle_W \, .$$

Proof.

Let $A := \mathcal{M}(|\psi\rangle)$. By the singular value decomposition,

$$A = \sum_i \lambda_i \, |s_i\rangle_V \, \langle t_i|_W$$

Since $|\psi\rangle = \mathcal{M}^{-1}(A)$,

$$|\psi\rangle = \sum_i \lambda_i \, |s_i\rangle_V \, |t_i\rangle_W^* \, . \qquad \square$$

The number of the terms in the decomposition is called the Schmidt rank.

Assignments

1. Show $A \otimes B \succeq 0$ for any $A \succeq 0$ and $B \succeq 0$.

2. Show the Schmidt decomposition of the following states
   - A $\frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle)$
   - B $\frac{1}{2}(|00\rangle + i\,|01\rangle + |10\rangle + |11\rangle)$
   - C $\frac{1}{2}(|00\rangle + i\,|01\rangle + i\,|10\rangle + |11\rangle)$
   - D $\frac{1}{2}(|00\rangle + i\,|01\rangle + i\,|10\rangle - |11\rangle)$