Quantum teleportation and BB84 protocol

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Time evolution of a system

Time evolution of a system is represented by a map from a state to a state.

T : The set of states $\rightarrow$ the set of states.

$$pT(\rho_1) + (1-p)T(\rho_2) = T(p\rho_1 + (1-p)\rho_2)$$

for any density matrices ρ_1, ρ_2 and $p \in [0,1]$.

T must be linear (a proof is needed).

Schrödinger picture and Heisenberg picture

$T^\dagger$: The set of binary measurements $\to$ the set of binary measurements.

$$\langle T(\rho), P\rangle = \langle \rho, T^\dagger(P)\rangle$$

for any $\rho \in \mathcal{H}(V)$ and $P \in \mathcal{H}(W)$. $T^\dagger$ is an adjoint map of T.

$$\begin{aligned} &\langle T_3(T_2(T_1(\rho))), P\rangle = \langle T_2(T_1(\rho)), T_3^\dagger(P)\rangle \\ =&\langle T_1(\rho), T_2^\dagger(T_3^\dagger(P))\rangle = \langle \rho, T_1^\dagger(T_2^\dagger(T_3^\dagger(P)))\rangle \end{aligned}$$

No-cloning theorem

$$|0\rangle\langle 0| \longmapsto |0\rangle\langle 0| \otimes |0\rangle\langle 0|$$
$$|1\rangle\langle 1| \longmapsto |1\rangle\langle 1| \otimes |1\rangle\langle 1|$$

From the linearlity,

$$\frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|) \longmapsto \frac{1}{2}(|0\rangle\langle 0| \otimes |0\rangle\langle 0| + |1\rangle\langle 1| \otimes |1\rangle\langle 1|)$$

This is not equal to

$$\frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|) \otimes \frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|).$$

Axioms for quantum channel

$T\colon \mathcal{H}(V) \to \mathcal{H}(W)$.

1. Trace preserving: $\mathrm{Tr}(T(\rho)) = \mathrm{Tr}(\rho)$.
2. Positive : $T(\rho) \succeq 0$ for any $\rho \succeq 0$.
3. Completely positive: $\mathrm{id} \otimes T$ is positive.

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Unitary operations

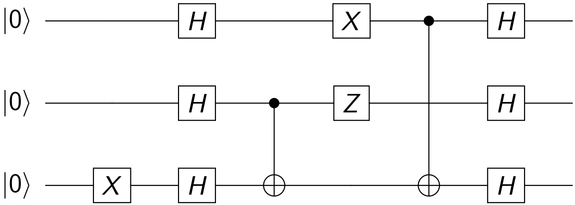
$$\rho \longmapsto U\rho U^\dagger.$$

1. Trace preserving: $\mathrm{Tr}(U\rho U^\dagger) = \mathrm{Tr}(\rho)$.

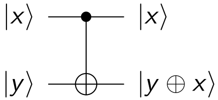
2. Completely positive: $(\mathrm{id} \otimes T)(\rho) = (I \otimes U)\rho(I \otimes U^\dagger)$.

In the most of quantum computing, only pure states and unitary operations are used.

Quantum circuit



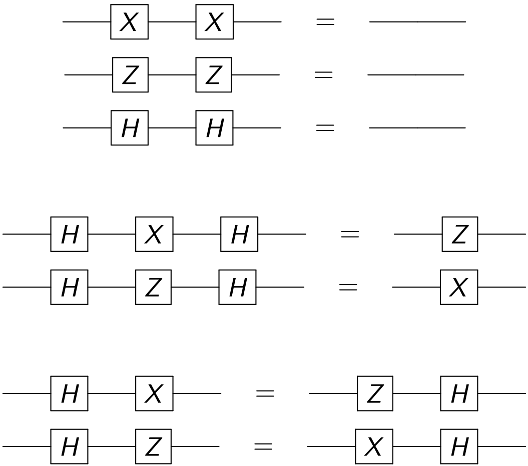
Controlled not



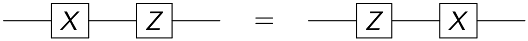
$$\mathrm{CNOT}\,|x\rangle\,|y\rangle \longmapsto |x\rangle\,|y \oplus x\rangle$$

$$\mathrm{CNOT} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

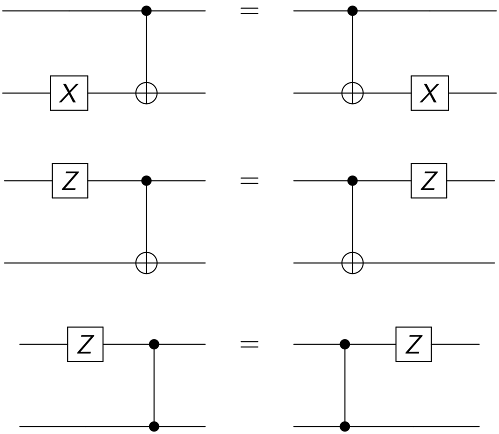
Simplification of quantum circuit 1/5



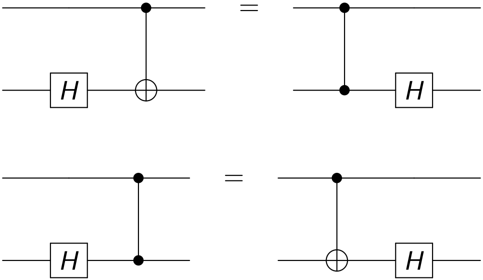
Simplification of quantum circuit 2/5



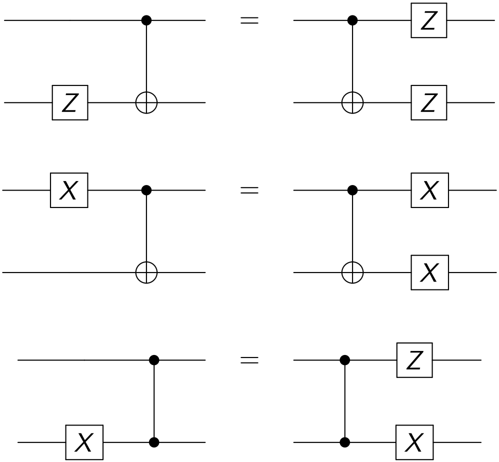
Simplification of quantum circuit 3/5



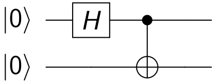
Simplification of quantum circuit 4/5



Simplification of quantum circuit 5/5



Bell states and quantum circuit



$$
\begin{aligned}
|0\rangle\,|0\rangle &\longmapsto \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\,|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle\,|0\rangle + |1\rangle\,|0\rangle) \\
&\longmapsto \frac{1}{\sqrt{2}}(|0\rangle\,|0\rangle + |1\rangle\,|1\rangle)
\end{aligned}
$$

$$
\begin{aligned}
|x\rangle\,|y\rangle &\longmapsto \frac{1}{\sqrt{2}}(|0\rangle + (-1)^x\,|1\rangle)\,|y\rangle = \frac{1}{\sqrt{2}}(|0\rangle\,|y\rangle + (-1)^x\,|1\rangle\,|y\rangle) \\
&\longmapsto \frac{1}{\sqrt{2}}(|0\rangle\,|y\rangle + (-1)^x\,|1\rangle\,|\bar{y}\rangle).
\end{aligned}
$$

Conditional density operator

A probability of outcome of local measurement in a joint system is

$$P(a, b) = \mathrm{Tr}(\rho_{V\otimes W}(P_a \otimes Q_b)).$$

$$P(a \mid b) = \frac{1}{P(b)}\mathrm{Tr}(\rho_{V\otimes W}(P_a \otimes Q_b)) = \frac{1}{P(b)}\mathrm{Tr}(\mathrm{Tr}_W(\rho_{V\otimes W}(I_V \otimes Q_b))P_a).$$

For $\rho_{V\otimes W} = |\varphi\rangle_{V\otimes W}\langle\varphi|_{V\otimes W}$ and $Q_b = |\psi_b\rangle_W \langle\psi_b|_W$,

$$\mathrm{Tr}_W(\rho_{V\otimes W}(I \otimes Q_b)) = \mathrm{Tr}_W(|\varphi\rangle_{V\otimes W}\langle\varphi|_{V\otimes W}(I_V \otimes |\psi_b\rangle_W \langle\psi_b|_W))$$

Conditional density operator for pure state

For $\rho = |\varphi\rangle_{V\otimes W}\langle\varphi|_{V\otimes W}$ and $Q_b = |\psi_b\rangle_W\langle\psi_b|_W$,

$$\mathrm{Tr}_W(\rho_{V\otimes W}(I_V \otimes Q_b)) = \mathrm{Tr}_W(|\varphi\rangle_{V\otimes W}\langle\varphi|_{V\otimes W}(I \otimes |\psi_b\rangle_W\langle\psi_b|_W))$$

From an expression $|\varphi\rangle_{V\otimes W} = \sum_{i,j}\varphi_{i,j}|i\rangle_V|\psi_j\rangle_W$,

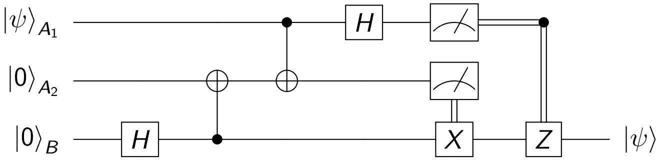
$$\begin{aligned}
&\mathrm{Tr}_W(|\varphi\rangle_{V\otimes W}\langle\varphi|_{V\otimes W}(I_V \otimes |\psi_b\rangle_W\langle\psi_b|_W)) \\
&= \mathrm{Tr}_W\left(\sum_{i,j,k,l}\varphi_{i,j}\varphi^*_{k,l}|i\rangle_V|\psi_j\rangle_W\langle k|_V\langle\psi_l|_W(I_V \otimes |\psi_b\rangle_W\langle\psi_b|_W)\right) \\
&= \mathrm{Tr}_W\left(\sum_{i,j,k,l}\varphi_{i,j}\varphi^*_{k,l}|i\rangle_V\langle k|_V \otimes |\psi_j\rangle_W\langle\psi_l|_W(I_V \otimes |\psi_b\rangle_W\langle\psi_b|_W)\right) \\
&= \sum_{i,j,k,l}\varphi_{i,j}\varphi^*_{k,l}|i\rangle_V\langle k|_V\,\mathrm{Tr}\left(|\psi_j\rangle_W\langle\psi_l|_W|\psi_b\rangle_W\langle\psi_b|_W\right) \\
&= \sum_{i,k}\varphi_{i,b}\varphi^*_{k,b}|i\rangle_V\langle k|_V = \left(\sum_i\varphi_{i,b}|i\rangle_V\right)\left(\sum_k\varphi^*_{k,b}\langle k|_V\right)
\end{aligned}$$

$$|\varphi\rangle_{V\otimes W} = \sum_{i,j}\varphi_{i,j}|i\rangle_V|\psi_j\rangle_W \longmapsto \sum_i\varphi_{i,b}|i\rangle_V$$

Examples of conditional density operator

For $|\psi\rangle_{V\otimes W} := \sum_{i,j=0}^{1} \alpha_{i,j} |i\rangle_V |j\rangle_W$

Quantum teleportation



Quantum teleportation

$$|\Phi\rangle_{A_2\otimes B} = \frac{1}{\sqrt{2}}\left(|0\rangle_{A_2}|0\rangle_B + |1\rangle_{A_2}|1\rangle_B\right)$$

$$(H_{A_1}\otimes I_{A_2}\otimes I_B)(\mathrm{CNOT}_{A_1A_2}\otimes I_B)\,|\psi\rangle_{A_1}|\Phi\rangle_{A_2\otimes B}$$

For $|\psi\rangle_{A_1} = \alpha|0\rangle_{A_1} + \beta|1\rangle_{A_1}$

$$\begin{aligned}
|\psi\rangle_{A_1}|\Phi\rangle_{A_2\otimes B} &= \left(\alpha|0\rangle_{A_1} + \beta|1\rangle_{A_1}\right)\frac{1}{\sqrt{2}}\left(|0\rangle_{A_2}|0\rangle_B + |1\rangle_{A_2}|1\rangle_B\right)\\
&\overset{\mathrm{CNOT}_{A_1A_2}}{\longmapsto} \frac{\alpha}{\sqrt{2}}\left(|0\rangle_{A_1}|0\rangle_{A_2}|0\rangle_B + |0\rangle_{A_1}|1\rangle_{A_2}|1\rangle_B\right)\\
&\quad + \frac{\beta}{\sqrt{2}}\left(|1\rangle_{A_1}|1\rangle_{A_2}|0\rangle_B + |1\rangle_{A_1}|0\rangle_{A_2}|1\rangle_B\right)\\
&\overset{H_{A_1}}{\longmapsto} \frac{\alpha}{2}\left(|000\rangle_{A_1A_2B} + |100\rangle_{A_1A_2B} + |011\rangle_{A_1A_2B} + |111\rangle_{A_1A_2B}\right)\\
&\quad + \frac{\beta}{2}\left(|010\rangle_{A_1A_2B} - |110\rangle_{A_1A_2B} + |001\rangle_{A_1A_2B} - |101\rangle_{A_1A_2B}\right)
\end{aligned}$$

Quantum teleportation

$$
\begin{aligned}
&\frac{\alpha}{2}\left(|000\rangle_{A_1A_2B} + |100\rangle_{A_1A_2B} + |011\rangle_{A_1A_2B} + |111\rangle_{A_1A_2B}\right) \\
&\quad + \frac{\beta}{2}\left(|010\rangle_{A_1A_2B} - |110\rangle_{A_1A_2B} + |001\rangle_{A_1A_2B} - |101\rangle_{A_1A_2B}\right) \\
&= \frac{1}{2}\Bigg[{\color{red}|00\rangle_{A_1A_2}} \left(\alpha\,|0\rangle_B + \beta\,|1\rangle_B\right) + {\color{red}|01\rangle_{A_1A_2}} \left(\alpha\,|1\rangle_B + \beta\,|0\rangle_B\right) \\
&\qquad + {\color{red}|10\rangle_{A_1A_2}} \left(\alpha\,|0\rangle_B - \beta\,|1\rangle_B\right) + {\color{red}|11\rangle_{A_1A_2}} \left(\alpha\,|1\rangle_B - \beta\,|0\rangle_B\right) \Bigg]
\end{aligned}
$$

According to the measurement outcome of A_1A_2

$$
\begin{aligned}
00 &\Rightarrow I \\
01 &\Rightarrow X \\
10 &\Rightarrow Z \\
11 &\Rightarrow ZX
\end{aligned}
$$

Transpose trick

$$|\Phi\rangle := \frac{1}{\sqrt{d}} \sum_{j=0}^{d-1} |j\rangle\,|j\rangle$$

Lemma

$$(A \otimes I)\,|\Phi\rangle = (I \otimes A^T)\,|\Phi\rangle$$

Proof.

$$\mathcal{M}\left((A \otimes I)\,|\Phi\rangle\right) = \mathcal{M}\left(\frac{1}{\sqrt{d}} \sum_{j=0}^{d-1} A\,|j\rangle\,|j\rangle\right)$$

$$= \frac{1}{\sqrt{d}} \sum_{j=0}^{d-1} A\,|j\rangle\,\langle j| = \frac{1}{\sqrt{d}} A \sum_{j=0}^{d-1} |j\rangle\,\langle j| = \frac{1}{\sqrt{d}} \sum_{j=0}^{d-1} |j\rangle\,\langle j|\,A$$

$$\mathcal{M}\left((I \otimes A^T)\,|\Phi\rangle\right) = \frac{1}{\sqrt{d}} \sum_{j=0}^{d-1} |j\rangle\,(A^T\,|j\rangle)^T$$

BB84 protocol

Assignments

1. Show the density matrix $\rho \in H(\mathbb{C}^2)$ of the Bell state

$$\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \in \mathbb{C}^2 \otimes \mathbb{C}^2$$

   when $|0\rangle \langle 0|$ is measured at the second system.

2. Show the density matrix $\rho \in H(\mathbb{C}^2)$ of the Bell state when $|+\rangle \langle +|$ is measured at the second system.

3. Show the density matrix $\rho \in H(\mathbb{C}^2)$ of the Bell state when $|\psi\rangle \langle \psi|$ is measured at the second system where $|\psi\rangle := \alpha |0\rangle + \beta |1\rangle$.