Solovay–Kitaev theorem

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Solovay–Kitaev theorem

Theorem

Assume that $\{U_1, \dots, U_k\}$ *generates a dense subset of* $\mathrm{SU}(2)$*. Then, any* $U \in \mathrm{SU}(2)$ *can be approxmiated with error* ϵ *by* $[\log(1/\epsilon)]^c$ *multiplications of* $\{U_1, \dots, U_k\}$ *for* $c = \log 5/\log(3/2) \approx 3.97$*.*

Special unitary group

- $\mathrm{U}(n) :=$ the set of $n \times n$ unitary matrices.

- $\mathrm{SU}(n) :=$
  the set of $n \times n$ unitary matrices U with $\det(U) = 1$.

- $\mathrm{U}(n)$ and $\mathrm{SU}(n)$ are groups.

- For $U \in \mathrm{SU}(n)$ and $V \in \mathrm{U}(n)$, $VUV^\dagger \in \mathrm{SU}(n)$.

- For $V \in \mathrm{U}(n)$ and $W \in \mathrm{U}(n)$, $VWV^\dagger W^\dagger \in \mathrm{SU}(n)$.

- For $U \in \mathrm{U}(n)$, there exists $V \in \mathrm{SU}(n)$ and $\theta \in \mathbb{R}$ such that $U = \mathrm{e}^{i\theta} V$.

Special unitary group and rotation

For a real unit vector $\hat{n} = [n_X \;\; n_Y \;\; n_Z]$, let

$$R_{\hat{n}}(\theta) := \cos\frac{\theta}{2} I - i \sin\frac{\theta}{2}(n_X X + n_Y Y + n_Z Z).$$

For any $U \in \mathrm{U}(2)$, there exist α, $\theta \in \mathbb{R}$ and a real unit three-dimensional vector $\hat{n}$ such that $U = \mathrm{e}^{i\alpha} R_{\hat{n}}(\theta)$.

$U \in \mathrm{U}(2)$ is in $\mathrm{SU}(2)$ iff $U = R_{\hat{n}}(\theta)$ for some $\theta \in \mathbb{R}$ and rear unit vector $\widehat{n} \in \mathbb{R}^3$.

Special unitary group and group commutator

Theorem

For any $U \in \mathrm{SU}(2)$, there exist V, $W \in \mathrm{SU}(2)$ such that $U = VWV^\dagger W^\dagger$.

Proof.

From

$$R_z(\theta)(iX)R_z(-\theta)(-iX) = R_z(2\theta)$$

any Z-rotation has the group commutator decomposition. For some unitary S, $U = SR_z(\eta)S^\dagger$ for some $\eta \in \mathbb{R}$. Hence, U has a group commutator decomposition $U = VWV^\dagger W^\dagger$ for $V = SR_z(\eta/2)S^\dagger$ and $W := S(iX)S^\dagger$. $\square$

Special unitary group and group commutator

Theorem

For any $U \in \mathrm{SU}(2)$*, there exist* V*,* $W \in \mathrm{SU}(2)$ *such that* $U = VWV^\dagger W^\dagger$ *for some* V*,* W *satisfying* $\|I - V\| < c_{\mathrm{GC}}\sqrt{\|I - U\|}$ *and* $\|I - W\| < c_{\mathrm{GC}}\sqrt{\|I - U\|}$ *for some constant* $c_{\mathrm{GC}} > 1/\sqrt{2}$*.*

Proof.

$$R_Z(\theta)R_X(\theta)R_Z(\theta)^\dagger R_X(\theta)^\dagger = R_Z(\theta)R_X(\theta)R_Z(-\theta)R_X(-\theta)$$

$$= R_Z(\theta)R_X(\theta)R_Z(-\theta)R_X(-\theta)$$

$$= \left[\cos\frac{\theta}{2} I - i\sin\frac{\theta}{2} Z\right]\left[\cos\frac{\theta}{2} I - i\sin\frac{\theta}{2} X\right]\left[\cos\frac{\theta}{2} I + i\sin\frac{\theta}{2} Z\right]\left[\cos\frac{\theta}{2} I + i\sin\frac{\theta}{2}\right.$$

$$= \left[\cos^4\frac{\theta}{2} + 2\cos^2\frac{\theta}{2}\sin^2\frac{\theta}{2} - \sin^4\frac{\theta}{2}\right] I + \cdots$$

$$= \left[1 - 2\sin^4\frac{\theta}{2}\right] I + \cdots = R_{\hat{n}_\theta}(\varphi)$$

$\cos\frac{\varphi}{2} = 1 - 2\sin^4\frac{\theta}{2}$. For some $S \in \mathrm{U}(2)$ and $\varphi \in \mathbb{R}$, $U = SR_{\hat{n}_\theta}(\varphi)S^\dagger$.
For $V := SR_Z(\theta)S^\dagger$ and $W := SR_X(\theta)S^\dagger$, $U = VWV^\dagger W^\dagger$. $\square$

Rotation matrix and distance

$$\begin{aligned}\|I - R_{\widehat{n}}(\theta)\| &= \left\| \begin{bmatrix} 1 - e^{i\theta/2} & 0 \\ 0 & 1 - e^{-i\theta/2} \end{bmatrix} \right\| \\ &= \left| 1 - e^{i\theta/2} \right| \\ &= 2 \left| \sin \frac{\theta}{4} \right| \end{aligned}$$

For $U \in \mathrm{SU}(2)$, $V, W \in \mathrm{SU}(2)$ satisfying $U = VWV^\dagger W^\dagger$ in the construction

$$\|I - U\| = 2\left|\sin\frac{\varphi}{4}\right| = 2\sqrt{\frac{1-\cos\frac{\varphi}{2}}{2}} = 2\sin^2\frac{\theta}{2} \approx 8\sin^2\frac{\theta}{4} = 2\|I - V\|^2$$

With some constant $c_{\mathrm{GC}} > 1/\sqrt{2}$, $\|I - V\| \leq c_{\mathrm{GC}}\sqrt{\|I - U\|}$.

Solovay–Kitaev algorithm

function $\textsc{Solovay–Kitaev}(U,\ n)$
  if $n = 0$ **then**
    return Basic approximation to U
  end if
  $U_{n-1} \leftarrow \textsc{Solovay–Kitaev}(U,\ n-1)$
  $V, W \leftarrow \textsc{GC–Decompose}(UU_{n-1}^\dagger)$
  $V_{n-1} \leftarrow \textsc{Solovay–Kitaev}(V,\ n-1)$
  $W_{n-1} \leftarrow \textsc{Solovay–Kitaev}(W,\ n-1)$
  return $V_{n-1}W_{n-1}V_{n-1}^\dagger W_{n-1}^\dagger U_{n-1}$.
end function

function $\textsc{GC–Decompose}(\Delta)$
  return $(V,\ W)$ satisfying $VWV^\dagger W^\dagger = \Delta$ with $\|I - V\|, \|I - W\| \leq c_{\mathsf{GC}}\sqrt{\|I - \Delta\|}$.
end function

Analysis

Theorem

If $\|I - V\|, \|I - W\| \leq \delta$, $\|V - \widetilde{V}\|, \|W - \widetilde{W}\| \leq \Delta$

$$\|VWV^\dagger W^\dagger - \widetilde{V}\widetilde{W}\widetilde{V}^\dagger\widetilde{W}^\dagger\| \leq c_{\mathrm{B}}\Delta(\delta + \Delta).$$

From this (surprising) theorem for $\Delta = \epsilon_{n-1}$, $\delta = c_{\mathrm{GC}}\sqrt{\epsilon_{n-1}}$, for $c_{\mathrm{approx}} \approx c_B c_{\mathrm{GC}}$.

$$\ell_n \leq 5\ell_{n-1}$$
$$\epsilon_n \leq c_{\mathrm{approx}}\epsilon_{n-1}^{3/2}$$

Then,

$$\begin{aligned} \ell_n &\leq 5^n \ell_0 \\ c_{\mathrm{approx}}^2 \epsilon_n &\leq c_{\mathrm{approx}}^3 \epsilon_{n-1}^{3/2} = (c_{\mathrm{approx}}^2 \epsilon_{n-1})^{3/2} \\ &\leq (c_{\mathrm{approx}}^2 \epsilon_0)^{(3/2)^n} \end{aligned}$$

If $\epsilon_0 < 1/c_{\mathrm{approx}}^2$, $\ell_n = O\left((\log(1/\epsilon))^{\frac{\log 5}{\log(3/2)}}\right)$.

Proof 1/2

Theorem

If $\|I - V\|, \|I - W\| \leq \delta$, $\|V - \widetilde{V}\|, \|W - \widetilde{W}\| \leq \Delta$

$$\|VWV^\dagger W^\dagger - \widetilde{V}\widetilde{W}\widetilde{V}^\dagger \widetilde{W}^\dagger\| \leq 8\Delta^2 + 8\Delta\delta + 4\Delta\delta^2 + 4\Delta^3 + \Delta^4.$$

Proof.

Let $\Delta_V := \widetilde{V} - V$ and $\Delta_W := \widetilde{W} - W$.

$$\begin{aligned}\widetilde{V}\widetilde{W}\widetilde{V}^\dagger \widetilde{W}^\dagger = {} & VWV^\dagger W^\dagger + \Delta_V WV^\dagger W^\dagger + V\Delta_W V^\dagger W^\dagger \\ & + VW\Delta_V^\dagger W^\dagger + VWV^\dagger \Delta_W^\dagger + O(\Delta^2).\end{aligned}$$

$$\begin{aligned}\|VWV^\dagger W^\dagger - \widetilde{V}\widetilde{W}\widetilde{V}^\dagger \widetilde{W}^\dagger\| \leq {} & \|\Delta_V WV^\dagger W^\dagger + VW\Delta_V^\dagger W^\dagger\| \\ & + \|V\Delta_W V^\dagger W^\dagger + VWV^\dagger \Delta_W^\dagger\| + \binom{4}{2}\Delta^2 + \binom{4}{3}\Delta^3 + \Delta^4.\end{aligned}$$

$\square$

Proof 2/2

Proof.

Let $\delta_W := W - I$.

$$\begin{aligned}\|\Delta_V W V^\dagger W^\dagger + V W \Delta_V^\dagger W^\dagger\| &= \|\Delta_V V^\dagger + V\Delta_V^\dagger + \Delta_V \delta_W V^\dagger + V\Delta_V^\dagger \delta_W^\dagger + \cdots\| \\ &\le \|\Delta_V V^\dagger + V\Delta_V^\dagger\| + 4\Delta\delta + 2\Delta\delta^2\end{aligned}$$

Since V and $V + \Delta_V$ are unitary,

$$\begin{aligned}&(V + \Delta_V)(V + \Delta_V)^\dagger = I \\ \iff\ & VV^\dagger + V\Delta_V^\dagger + \Delta_V V^\dagger + \Delta_V\Delta_V^\dagger = I \\ \iff\ & V\Delta_V^\dagger + \Delta_V V^\dagger + \Delta_V \Delta_V^\dagger = 0\end{aligned}$$

$$\|\Delta_V W V^\dagger W^\dagger + V W \Delta_V^\dagger W^\dagger\| \le \Delta^2 + 4\Delta\delta + 2\Delta\delta^2.$$

$\square$

$c_\mathrm{B} \approx 8$.

Assignments

1. Show a quantum circuit for a controlled-Hadamard gate using arbitrary single-qubit gates and CNOT gates.

2. [Very advanced] By modifying levels of Solovay–Kitaev algorithm in the recursion, can we improve the exponent $c = \log 5 / \log(3/2)$?