Introduction to quantum thoery: Quantum states and quantum measurements

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What is “Quantum theory”?

Quantum theory

- Physics for microscopic phenomena, e.g., atoms, light.

Why is quantum theory important ?

- Just because it's reality.

- Because it gives more efficient information processing, e.g., quantum factoring algorithm, quantum secret-key sharing, etc.

On this course

We study mathematical foundation of quantum theory.

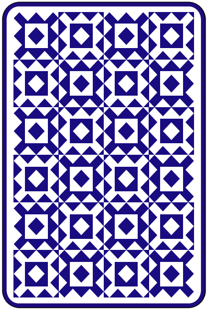
- Mathematical foundation of quantum physics
- Quantum algorithms
- Other quantum information processing, e.g, quantum communication, quantum error-correction.

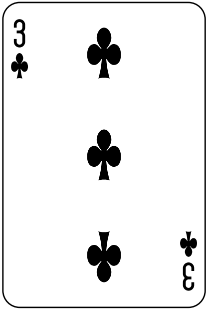
Score

- Assignments: 100%

https://github.com/QuantumComputationQuantumInformation/slides2020

Experimental facts





- ✗ The card was the three of club.
- ✓ When we flip the card, we see the three of club.
- State: The card
- Measurement: Flipping the card
- The number on the card before the flipping the card cannot be defined in quantum theory

States, measurements and distinguishability

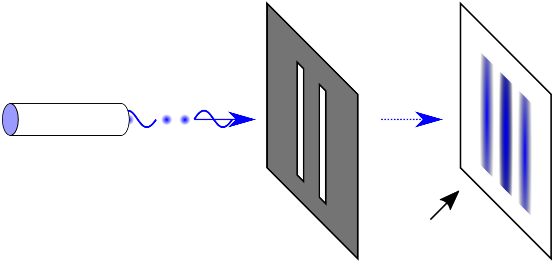
- Two states are equivalent if the probability distributions of outcomes are equal for arbitrary measurement.

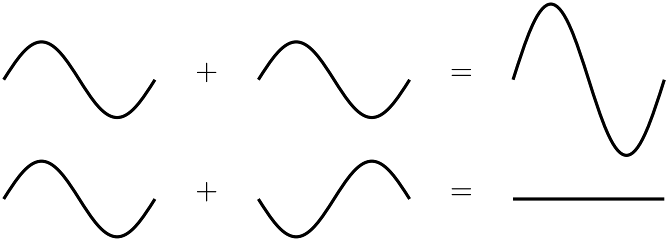
- Two measurements are equivalent if the probability distributions of outcomes are equal for arbitrary states.

- We should not distinguish states (measurements) methematically if we cannot distinguish them physically.

Light is wave (1801)

Thomas Young
(1773 – 1829)

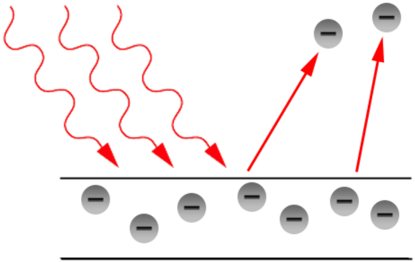




https://en.wikipedia.org/wiki/Young's_interference_experiment

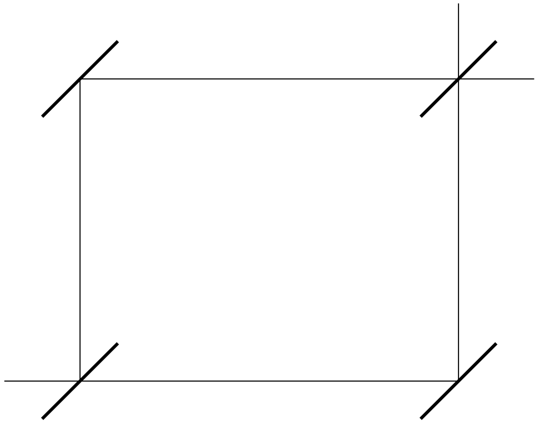
Light is particle (1905)

Albert Einstein
(1879 – 1955)



https://en.wikipedia.org/wiki/File:Photoelectric_effect.png

Mach–Zehnder interferometer



Quantum states and quantum measurements

A single photon $\Rightarrow$ BS1 $\Rightarrow$ BS2 $\Rightarrow$ detection

State	Measurement
A single photon $\Rightarrow$ BS1 $\Rightarrow$ BS2	detection
A single photon $\Rightarrow$ BS1	BS2 $\Rightarrow$ detection
A single photon	BS1 $\Rightarrow$ BS2 $\Rightarrow$ detection

All understandings are valid

Mathematical representations of states and measurements

How "States" and "Measurements" are represented mathematically ?

A table of probabilities of outcome 'YES' for each binary measurment on each state

	Measurement 1	Measurement 2	$\cdots$
State A	p_{A1}	p_{A2}	$\cdots$
State B	p_{B1}	p_{B2}	$\cdots$
$\vdots$			

* The number of states and measurements are not necessarily countable.

Linear space

	Measurement 1	Measurement 2	$\cdots$
State A	p_{A1}	p_{A2}	$\cdots$
State B	p_{B1}	p_{B2}	$\cdots$
State C	$0.7p_{A1} + 0.3p_{B1}$	$0.7p_{A2} + 0.3p_{B2}$	

Assumption

- Probabilistic mixture of states is also state.
- Probabilistic mixture of binary measurement is also binary measurement.

States and measurements can be represented by vectors!

Classical theory

States: $\underline{0}$, $\underline{1}$
Binary measurements: $\underline{0}?$, $\underline{1}?$

$$\underline{0}?(\underline{0}) = 1, \qquad \underline{0}?(\underline{1}) = 0,$$
$$\underline{1}?(\underline{0}) = 0, \qquad \underline{1}?(\underline{1}) = 1$$

	$\underline{0}?$	$\underline{1}?$
$\underline{0}$	1	0
$\underline{1}$	0	1

State and measurement

	$\underline{0}?$	$\underline{1}?$
$\underline{0}$	1	0
$\underline{1}$	0	1

$S := \underline{0}$ with probability p, $\underline{1}$ with probability $1-p$.
S is also regarded as a state.

$$\underline{0}?(S) = p, \qquad \underline{1}?(S) = 1-p.$$

Similarly,
$E_1 := \underline{0}?$ with probability p, $\underline{1}?$ with probability $1-p$.
$E_2 := (\underline{0} \text{ or } \underline{1})?$.
E_1 and E_2 are also regarded as a binary measurement.

Linear space

$$\omega_{\underline{0}} := \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad \omega_{\underline{1}} := \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$e_{\underline{0}} := \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad e_{\underline{1}} := \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\underline{0}?(\underline{0}) = \langle e_{\underline{0}}, \omega_{\underline{0}} \rangle, \qquad \underline{0}?(\underline{1}) = \langle e_{\underline{0}}, \omega_{\underline{1}} \rangle$$

$$\underline{1}?(\underline{0}) = \langle e_{\underline{1}}, \omega_{\underline{0}} \rangle, \qquad \underline{1}?(\underline{1}) = \langle e_{\underline{1}}, \omega_{\underline{1}} \rangle$$

$S := \underline{0}$ with probability p, $\underline{1}$ with probability $1 - p$

$$\omega_S = p\omega_{\underline{0}} + (1-p)\omega_{\underline{1}} = \begin{bmatrix} p \\ 1-p \end{bmatrix}.$$

$$\underline{0}?(S) = p = \langle e_{\underline{0}}, \omega_S \rangle.$$

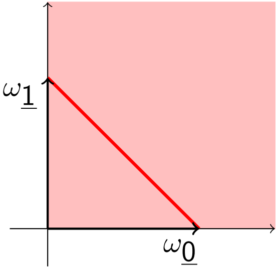
States and measurements in a linear space

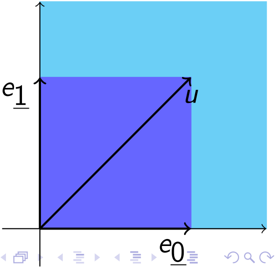
$$\omega_{\underline{0}} := \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad \omega_{\underline{1}} := \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$e_{\underline{0}} := \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad e_{\underline{1}} := \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\text{Set of states} = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \mid x \geq 0, y \geq 0, x + y = 1 \right\}.$$

$$\text{Set of binary measurements} = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \mid x \geq 0, y \geq 0, x \leq 1, y \leq 1 \right\}.$$





State and measurement in a linear space

$$\text{Set of states} = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \mid x \geq 0, y \geq 0, x + y = 1 \right\}.$$

$$\text{Set of binary measurements} = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \mid x \geq 0, y \geq 0, x \leq 1, y \leq 1 \right\}.$$

Let $C_{\geq 0}$ be the set of nonnegative vectors and $u := \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

$$\text{Set of states} = \left\{ \omega \in \mathbb{R}^2 \mid \omega \in C_{\geq 0}, \langle u, \omega \rangle = 1 \right\}.$$

$$\text{Set of binary measurements} = \left\{ e \in \mathbb{R}^2 \mid e \in C_{\geq 0}, u - e \in C_{\geq 0} \right\}.$$

$$\text{Set of measurements} = \{(e_1, \ldots, e_k) \mid e_1 + \cdots + e_k = u,\ e_i \in C_{\geq 0} \\ i = 1, 2, \ldots, k,\ k = 1, 2, \ldots \}$$

Outcome of the measurement $M = (e_1, \ldots, e_k)$ on ω is i with probability $\langle e_i, \omega \rangle$.

Quantum theory

$C_{\geq 0} \subseteq \mathbb{R}^2$: the set of nonnegative vectors, $u := \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

$$\text{Set of states} = \left\{\omega \in \mathbb{R}^2 \mid \omega \in C_{\geq 0}, \langle u, \omega \rangle = 1\right\}.$$

$$\text{Set of binary measurements} = \left\{e \in \mathbb{R}^2 \mid e \in C_{\geq 0}, u - e \in C_{\geq 0}\right\}.$$

$$\text{Set of measurements} = \{(e_1, \dots, e_k) \mid e_1 + \cdots + e_k = u,\ e_i \in C_{\geq 0}\ i = 1, 2, \dots, k,\ k = 1, 2, \dots, \}$$

V : the linear space on $\mathbb{R}$ spanned by 2×2 Hermitian matrices.
$\langle e, \omega \rangle := \mathrm{Tr}(e\omega)$ for $\omega, e \in V$ (Hilbert-Schmidt inner product).
$C_{\succeq 0} \subseteq V$: the set of positive semidefinite matrices, $u := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

$$\text{Set of states} = \{\omega \in V \mid \omega \in C_{\succeq 0}, \langle u, \omega \rangle = 1\}.$$

$$\text{Set of binary measurements} = \{e \in V \mid e \in C_{\succeq 0}, u - e \in C_{\succeq 0}\}.$$

Linear space spanned by 2x2 Hermitian matrices

Basis

$$A := \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, B := \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, C := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, D := \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$

Another choice of basis

$$I := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, X := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, Y := \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, Z := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Both are orthogonal basis.
The second basis (I and Pauli matrices X, Y and Z) has nice properties.

1. $\mathrm{Tr}(I) = 2$. $\mathrm{Tr}(X) = \mathrm{Tr}(Y) = \mathrm{Tr}(Z) = 0$.
2. $X^2 = Y^2 = Z^2 = I$ (X, Y and Z have eigenvalues ± 1).
3. $XY = -YX$, $YZ = -ZY$, $ZX = -XZ$.

Positive semidefinite cone

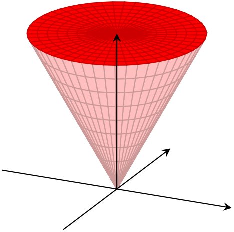
$$I := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, X := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, Y := \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, Z := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

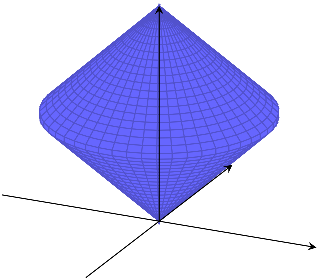
$$\rho = \frac{1}{\sqrt{2}} (a_I I + a_X X + a_Y Y + a_Z Z)$$

$$\begin{aligned}
\lambda_1 \geq 0,\ \lambda_2 \geq 0 \iff& \lambda_1 + \lambda_2 \geq 0,\ \lambda_1 \lambda_2 \geq 0 \\
\iff& \mathrm{Tr}(\rho) \geq 0,\ \mathrm{Tr}(\rho)^2 - \mathrm{Tr}(\rho^2) \geq 0 \\
\iff& a_I \geq 0,\ 2a_I^2 - (a_I^2 + a_X^2 + a_Y^2 + a_Z^2) \geq 0 \\
\iff& a_I \geq 0,\ a_I^2 \geq a_X^2 + a_Y^2 + a_Z^2
\end{aligned}$$

$$\mathrm{Tr}(\rho) = 1 \iff a_I = \frac{1}{\sqrt{2}}$$

Geometry of quantum states and effects





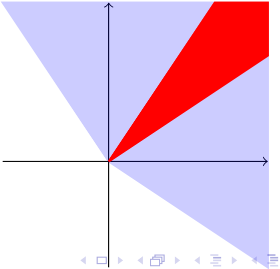
Convex cone and dual cone

$$C \subseteq V \text{ is a convex cone} \iff x + y \in C,\ \lambda x \in C,$$
$$\forall x \in C, y \in C,\ \lambda \geq 0$$

Proper cone: closed, not V, full-dimensional.

$$C^* \subseteq V \text{ is a dual cone of } C$$
$$\iff C^* := \{x \in V \mid \langle x, y\rangle \geq 0,\ \forall y \in C\}$$

$C_{\geq 0}$ and $C_{\succeq 0}$ are self-dual cones.



Generalized probabilistic theories

C : convex cone.
$u \in$ interior of C^*.

$$\text{Set of states} = \{\omega \in V \mid \omega \in C, \langle u, \omega \rangle = 1\}.$$

$$\text{Set of effects} = \{e \in V \mid e \in C^*, u - e \in C^*\}.$$

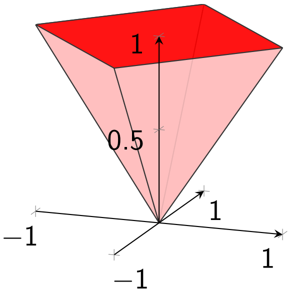
$$\text{Set of measurements} = \{(e_1, \dots, e_k) \mid e_1 + \cdots + e_k = u, e_i \in C^* \\ i = 1, 2, \dots, k, \; k = 1, 2, \dots\}$$

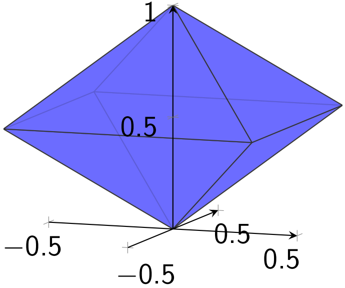
Classical theory
$V = \mathbb{R}^n$, $C = C_{\geq 0}$, $u =$ the all-1 vector.

Quantum theory
$V =$ A set of $n \times n$ Hermitian matrices, $C = C_{\succeq 0}$, $u = I$.

Other theories ?

Toy theory: Gbit





$$\omega_0 = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix}, \qquad \omega_1 = \begin{bmatrix} 0 & 1 & 1 \end{bmatrix},$$
$$\omega_2 = \begin{bmatrix} -1 & 0 & 1 \end{bmatrix}, \qquad \omega_3 = \begin{bmatrix} 0 & -1 & 1 \end{bmatrix}.$$

$$e_0 = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}, \qquad e_1 = \frac{1}{2}\begin{bmatrix} -1 & 1 & 1 \end{bmatrix},$$
$$e_2 = \frac{1}{2}\begin{bmatrix} 1 & -1 & 1 \end{bmatrix}, \qquad e_3 = \frac{1}{2}\begin{bmatrix} -1 & -1 & 1 \end{bmatrix},$$
$$u = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}.$$

Nonlocality

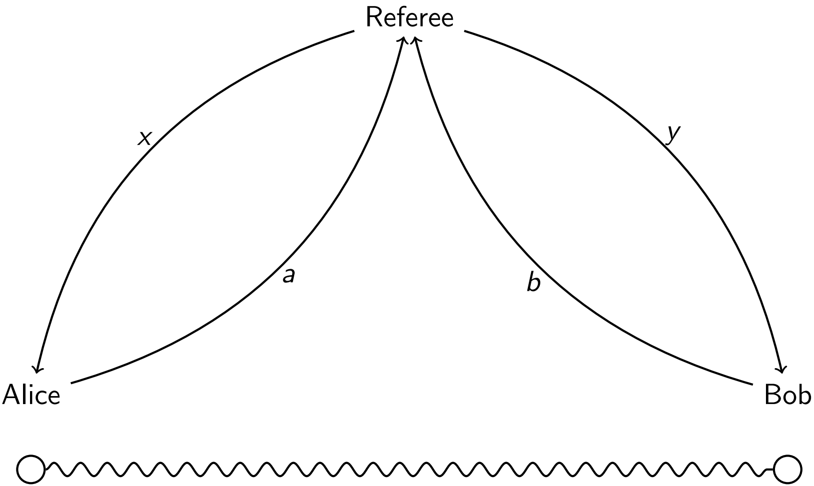
OK, generalized probabilistic theory is quite simple and easy to understand.

But, what is essential difference between classical theory and quantum theory ?

Can classical theory "simulates" or "explains" quantum theory ?

Are Hermitian positive-semidefinite matrices really needed for explaining reality ?

Bell test: CHSH game (1964, 1969)



Alice and Bob win iff $a \oplus b = x \wedge y$.

Bell inequality

a_x: Output of Alice for given x.
b_y: Output of Bbob for given y.

$$\begin{aligned} a_0 \oplus b_0 &= 0 \\ a_1 \oplus b_0 &= 0 \\ a_0 \oplus b_1 &= 0 \\ a_1 \oplus b_1 &= 1 \end{aligned}$$

By adding all equations, we get $0 = 1$, which means there is no solution. Hence, the winning probability 1 cannot be achieved.

Three equalities can be satisfied, so that the largest winning probability is $3/4$ (Bell inequality or CHSH inequality).

If Alice and Bob share quantum states, then the largest winning probability is $(2 + \sqrt{2})/4 \approx 0.854$ (Violation of Bell/CHSH inequality)

Locality (Hidden variable model)

Joint preparation and independent measurements.

Probability distribution $P(a, b \mid x, y)$ is said to be **local** if

$$P(a, b \mid x, y) = \sum_{\lambda} P(\lambda) P(a \mid x, \lambda) P(b \mid y, \lambda).$$

Quantum physics allow nonlocal behaviors.

Summary

- Let V be the linear space spanned by $n \times n$ Hermitian matrices. Let $C_{\succeq 0}$ be the set of PSD matrices. Let I be the $n \times n$ identity matrix. Let $\langle e, \omega \rangle := \mathrm{Tr}(e\omega)$ for $\omega, e \in V$ (Hilbert-Schmidt inner product).

$$\text{Set of states} = \{\omega \in V \mid \omega \in C_{\succeq 0}, \langle u, \omega \rangle = 1\}.$$

$$\text{Set of binary measurements} = \{e \in V \mid e \in C_{\succeq 0}, u - e \in C_{\succeq 0}\}.$$

- Classical theory and quantum theory are special cases of generalized probabilistic theories.

- Violation of Bell (CHSH) inequality show that quantum theory is essentially different from classical theory (Arguments on composite system is needed).

Assignments

1. Show the dimension and one of the basis of the real linear space spanned by $n \times n$ Hermitian matrices.
2. Show that $XY = -YX$, $YZ = -ZY$ and $ZX = -XZ$.
3. Show that the Hilbert–Schmidt inner product satisfies the axioms of inner product.
4. [Advanced] Show that $C_{\succeq 0}$ is a self-dual cone.