Universality of quantum circuit

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Universality of a quantum circuit

Theorem (Universality of finite gate set)

For any unitary matrix $U \in L(\mathbb{C}^{2^n})$ and $\epsilon > 0$, there is a quantum circuit with X, Y, Z, H, S, T, CNOT gates computing $\widetilde{U}$ satisfying $\|U - \widetilde{U}\| < \epsilon$.

Proof.

1. Any unitary matrix can be decomposed to a product of two-level unitary matrices. Done
2. Any two-level unitary matrix can be decomposed to a product of controlled-unitary gates. Done
3. Any controlled-untary gate can be decomposed to a product of CNOT and arbitrary single-qubit gates.
4. Any single-qubit gate can be approximated by X, Y, Z, H, S and T.

Special unitary group

- $\mathrm{U}(n) :=$ the set of $n \times n$ unitary matrices.

- $\mathrm{SU}(n) :=$
  the set of $n \times n$ unitary matrices U with $\det(U) = 1$.

- $\mathrm{U}(n)$ and $\mathrm{SU}(n)$ are groups.

- For $U \in \mathrm{SU}(n)$ and $V \in \mathrm{U}(n)$, $VUV^\dagger \in \mathrm{SU}(n)$.

- For $V \in \mathrm{U}(n)$ and $W \in \mathrm{U}(n)$, $VWV^\dagger W^\dagger \in \mathrm{SU}(n)$.

- For $U \in \mathrm{U}(n)$, there exists $V \in \mathrm{SU}(n)$ and $\theta \in \mathbb{R}$ such that $U = \mathrm{e}^{i\theta} V$.

Controlled-unitary

Theorem

Any controlled-unitary gate can be decomposed to a product of CNOT and arbitrary single-qubit gates.

Proof.

1. Controlled-U(2) with single controlled qubit.
2. Controlled-SU(2) with n controlled qubits.
3. Controlled-U(2) with n controlled qubits.

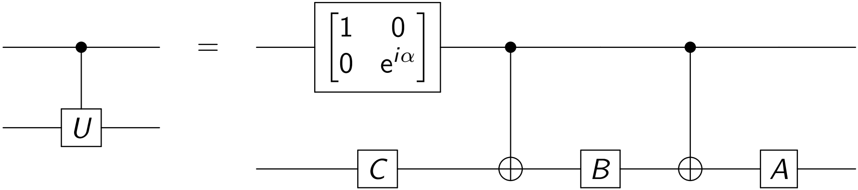
□

Decomposition of single qubit unitary

Lemma

Any single qubit unitary $U \in \mathrm{U}(2)$, there is single qubit unitary matrices A, B, C such that $ABC = I$ and $\mathrm{e}^{i\alpha} AXBXC = U$.

From this lemma,



Decomposition of single qubit unitary

Lemma

Any single qubit unitary $U \in \mathrm{U}(2)$, there is single qubit unitary matrices A, B, C and $\alpha \in \mathbb{R}$ such that $ABC = I$ and $\mathrm{e}^{i\alpha} AXBXC = U$.

Proof.

For any 2×2 unitary matrix, there exist α, β, γ, $\delta \in \mathbb{R}$ such that

$$U = \mathrm{e}^{i\alpha} R_Z(\beta) R_Y(\gamma) R_Z(\delta).$$

Let $A := R_Z(\beta) R_Y(\gamma/2)$, $B := R_Y(-\gamma/2) R_Z(-(\beta+\delta)/2)$, $C := R_Z((\delta-\beta)/2)$. Then, $ABC = I$. Since $R_Y(\theta)X = XR_Y(-\theta)$ and $R_Z(\theta)X = XR_Z(-\theta)$,.

$$\begin{aligned} AXBXC &= R_Z(\beta) R_Y(\gamma/2) X R_Y(-\gamma/2) R_Z(-(\beta+\delta)/2) X R_Z((\delta-\beta)/2) \\ &= R_Z(\beta) R_Y(\gamma/2) R_Y(\gamma/2) R_Z((\beta+\delta)/2) R_Z((\delta-\beta)/2) \\ &= R_Z(\beta) R_Y(\gamma) R_Z(\delta) = \mathrm{e}^{-i\alpha} U. \end{aligned}$$

Decomposition of single qubit unitary

$$
\begin{aligned}
&\mathrm{e}^{i\alpha} R_Z(\beta) R_Y(\gamma) R_Z(\delta) \\
&= \mathrm{e}^{i\alpha} \begin{bmatrix} e^{-i\beta/2} & 0 \\ 0 & \mathrm{e}^{i\beta/2} \end{bmatrix} \begin{bmatrix} \cos(\gamma/2) & -\sin(\gamma/2) \\ \sin(\gamma/2) & \cos(\gamma/2) \end{bmatrix} \begin{bmatrix} e^{-i\delta/2} & 0 \\ 0 & \mathrm{e}^{i\delta/2} \end{bmatrix} \\
&= \mathrm{e}^{i\alpha} \begin{bmatrix} \mathrm{e}^{i(-\beta/2-\delta/2)} \cos(\gamma/2) & -\mathrm{e}^{i(-\beta/2+\delta/2)} \sin(\gamma/2) \\ \mathrm{e}^{i(+\beta/2-\delta/2)} \sin(\gamma/2) & \mathrm{e}^{i(+\beta/2+\delta/2)} \cos(\gamma/2) \end{bmatrix}
\end{aligned}
$$

$$
U = \begin{bmatrix} a & b \\ c & d \end{bmatrix}
$$

$$
|a|^2 + |b|^2 = 1 \qquad |c|^2 + |d|^2 = 1
$$

$$
a^* c + b^* d = 0
$$

$$
\alpha = \frac{1}{2} \arg ad = \frac{1}{2} \arg bc + \frac{\pi}{2} \qquad \beta = \arg a^* c
$$

$$
\gamma = 2 \arccos(|a|) \qquad \delta = \arg c^* d
$$

Controlled-unitary

Theorem

Any controlled-unitary gate can be decomposed to a product of CNOT and arbitrary single-qubit gates.

Proof.

1. Controlled-U(2) with single controlled qubit. Done
2. Controlled-SU(2) with n controlled qubits.
3. Controlled-U(2) with n controlled qubits.

□

Special unitary group and rotation

For a real unit vector $\hat{n} = [n_X \;\; n_Y \;\; n_Z]$, let

$$R_{\hat{n}}(\theta) := \cos\frac{\theta}{2} I - i \sin\frac{\theta}{2}(n_X X + n_Y Y + n_Z Z).$$

For any $U \in \mathrm{U}(2)$, there exist α, $\theta \in \mathbb{R}$ and a real unit three-dimensional vector $\hat{n}$ such that $U = \mathrm{e}^{i\alpha} R_{\hat{n}}(\theta)$.

$U \in \mathrm{U}(2)$ is in $\mathrm{SU}(2)$ iff two eigenvalues of U are in the form $\{\mathrm{e}^{i\theta}, \mathrm{e}^{-i\theta}\}$.

If $\mathrm{Tr}(U) \in \mathbb{R} \setminus \{0\}$, then $U \in \mathrm{SU}(2)$.

$R_{\hat{n}}(\theta) \in \mathrm{SU}(2)$ if $\theta \neq \pm\pi + 2n\pi$. From the continuity, $R_{\hat{n}}(\theta) \in \mathrm{SU}(2)$ for any $\theta \in \mathbb{R}$.

Special unitary group and group commutator

Theorem

For any $U \in \mathrm{SU}(2)$, there exist V, $W \in \mathrm{SU}(2)$ such that $U = VWV^\dagger W^\dagger$.

Proof.

$$R_Z(\theta)R_X(\theta)R_Z(\theta)^\dagger R_X(\theta)^\dagger = R_Z(\theta)R_X(\theta)R_Z(-\theta)R_X(-\theta)$$

$$= \left[\cos\frac{\theta}{2}I - i\sin\frac{\theta}{2}Z\right]\left[\cos\frac{\theta}{2}I - i\sin\frac{\theta}{2}X\right]\left[\cos\frac{\theta}{2}I + i\sin\frac{\theta}{2}Z\right]\Bigl[\cos\frac{\theta}{2}I + i\sin\frac{\theta}{2}$$

$$= \left[\cos^4\frac{\theta}{2} + 2\cos^2\frac{\theta}{2}\sin^2\frac{\theta}{2} - \sin^4\frac{\theta}{2}\right] I + \cdots$$

$$= \left[1 - 2\sin^4\frac{\theta}{2}\right] I + \cdots = R_{\widehat{n}_\theta}(\varphi)$$

$\cos\frac{\varphi}{2} = 1 - 2\sin^4\frac{\theta}{2}$. For some $S \in \mathrm{U}(2)$ and $\varphi \in \mathbb{R}$, $U = SR_{\widehat{n}_\theta}(\varphi)S^\dagger$.
For $V := SR_Z(\theta)S^\dagger$ and $W := SR_X(\theta)S^\dagger$, $U = VWV^\dagger W^\dagger$. □

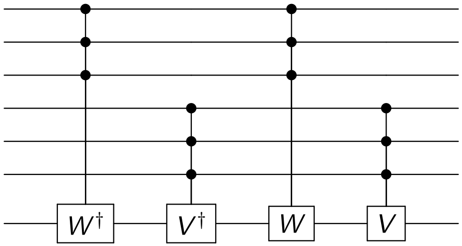
Group commutator and controlled-unitary

Theorem

For any $U \in \mathrm{SU}(2)$, controlled-U gate with n controlled qubits can be realized by $O(n^2)$ CNOT and arbitrary single-qubit gates without ancillas (working qubits).

Proof.

Induction on n. For the group commutator decomposition $U = VWV^\dagger W^\dagger$ using V, $W \in \mathrm{SU}(2)$,



$S_n = 4S_{n/2} = 4^{\log n} S_1 = O(n^2)$. □

Controlled-unitary

Theorem

Any controlled-unitary gate can be decomposed to a product of CNOT and arbitrary single-qubit gates.

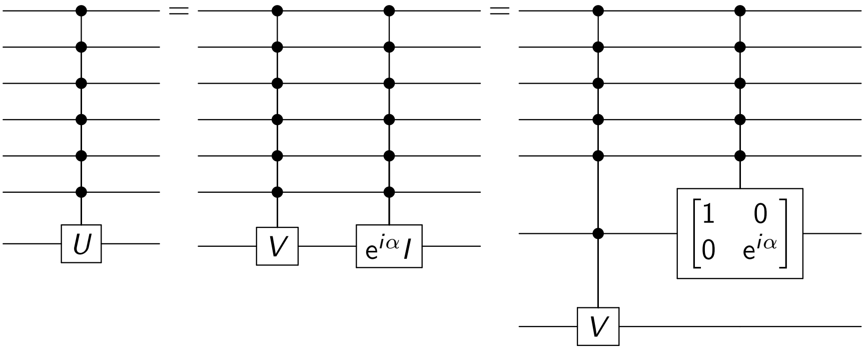
Proof.

1. Controlled-U(2) with single controlled qubit. Done
2. Controlled-SU(2) with n controlled qubits. Done
3. Controlled-U(2) with n controlled qubits.

□

Controlled-$U(2)$ with n controlled qubits

For any $U \in \mathrm{U}(2)$, there exists $V \in \mathrm{SU}(2)$ and $\alpha \in \mathbb{R}$ such that $U = \mathrm{e}^{i\alpha} V$.



$$A_n = S_n + A_{n-1} = O(n^3)$$

Controlled-unitary

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3. Controlled-U(2) with n controlled qubits. Done

$\square$

Universality of a quantum circuit

Theorem (Universality of finite gate set)

For any unitary matrix $U \in L(\mathbb{C}^{2^n})$ *and* $\epsilon > 0$, *there is a quantum circuit with* X, Y, Z, H, S, T, CNOT *gates computing* $\widetilde{U}$ *satisfying* $\|U - \widetilde{U}\| < \epsilon$.

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4. Any single-qubit gate can be approximated by X, Y, Z, H, S and T.

Approximation of a single-qubit gate is sufficient

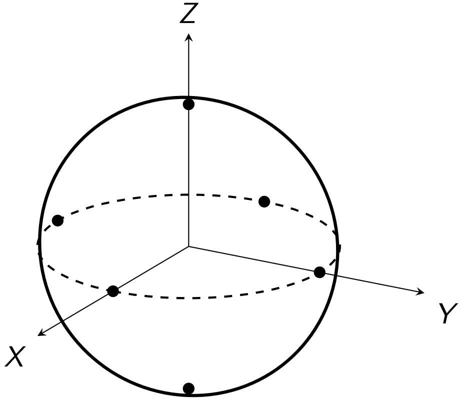
Theorem

Any single-qubit gate can be approximated by X, Y, Z, H, S and T.

Assume that this theorem holds. For $A \in L(\mathbb{C}^d)$, Let $\|A\|$ be the spectral norm, which satisfies $\|UAV\| = \|A\|$ for any unitary matrices U and V. Assume $\|U_i - V_i\| \leq \epsilon$ for $i = 1, \ldots, m$.

$$
\begin{aligned}
&\|U_m U_{m-1} \cdots U_1 - V_m V_{m-1} \cdots V_1\| \\
&= \left\| \sum_{i=1}^{m} (U_m \cdots U_i V_{i-1} \cdots V_1 - U_m \cdots U_{i+1} V_i \cdots V_1) \right\| \\
&\leq \sum_{i=1}^{m} \|U_m \cdots U_i V_{i-1} \cdots V_1 - U_m \cdots U_{i+1} V_i \cdots V_1\| \\
&= \sum_{i=1}^{m} \|U_m \cdots U_{i+1} (U_i - V_i) V_{i-1} \cdots V_1\| = \sum_{i=1}^{m} \|U_i - V_i\| \leq m\epsilon.
\end{aligned}
$$

Universality of X, Y, Z, H, S, T



Universality of X, Y, Z, H, S, T

$T \cong R_Z(\pi/4)$. $HTH \cong R_X(\pi/4)$.

$$
\begin{aligned}
R_Z(\pi/4)R_X(\pi/4) &= \left[\cos\frac{\pi}{8}I - i\sin\frac{\pi}{8}Z\right]\left[\cos\frac{\pi}{8}I - i\sin\frac{\pi}{8}X\right] \\
&= \cos^2\frac{\pi}{8}I - i\sin\frac{\pi}{8}\left[\cos\frac{\pi}{8}(X+Z) + \sin\frac{\pi}{8}Y\right] \\
&=: \cos\frac{\eta}{2}I - i\sin\frac{\eta}{2}\left(n_x X + n_Y Y + n_Z Z\right) \\
&= R_{\widehat{n}}(\eta)
\end{aligned}
$$

where η satisfying $\cos(\eta/2) = \cos^2(\pi/8)$ and $\widehat{n}$ is a unit vector along with $(\cos\frac{\pi}{8}, \sin\frac{\pi}{8}, \cos\frac{\pi}{8})$. Here, η is an irrational multiple of π. $HR_{\widehat{n}}(\eta)H = R_{\widehat{m}}(\eta)$ where $\widehat{m}$ is a unit vector along with $(\cos\frac{\pi}{8}, -\sin\frac{\pi}{8}, \cos\frac{\pi}{8})$.

$U = e^{i\alpha}R_{\widehat{n}}(\beta)R_{\widehat{m}}(\gamma)R_{\widehat{n}}(\delta)$.

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Solovay–Kitaev theorem

Theorem

Assume $\{U_1, \dots, U_k\}$ *generates a dense subset of* $\mathrm{SU}(2)$*. Then, any* $U \in \mathrm{SU}(2)$ *can be approxmiated with error* ϵ *by* $[\log(1/\epsilon)]^c$ *multiplications of* $\{U_1, \dots, U_k\}$*.*