Quantum phase estimation

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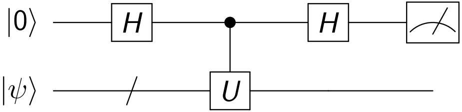
Quantum algorithms

- Quantum phase estimation: Integer factoring.

- Grover search, quantum walk: Unstructured search.

Hadamard test

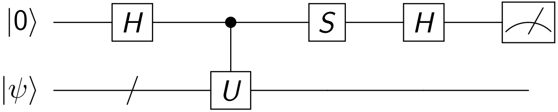


$$
\begin{aligned}
|0\rangle|\psi\rangle &\longmapsto \frac{|0\rangle+|1\rangle}{\sqrt{2}}|\psi\rangle \longmapsto \frac{1}{\sqrt{2}}\left(|0\rangle|\psi\rangle + |1\rangle U|\psi\rangle\right) \\
&\longmapsto \frac{1}{\sqrt{2}}\left(|+\rangle|\psi\rangle + |-\rangle U|\psi\rangle\right) \\
&= \frac{1}{2}\left(|0\rangle\left(|\psi\rangle + U|\psi\rangle\right) + |1\rangle\left(|\psi\rangle - U|\psi\rangle\right)\right).
\end{aligned}
$$

0 is measured with probability $\left\| \frac{|\psi\rangle+U|\psi\rangle}{2} \right\|^2 = \frac{1+\mathrm{Re}(\langle\psi|U|\psi\rangle)}{2}$.

1 is measured with probability $\left\| \frac{|\psi\rangle-U|\psi\rangle}{2} \right\|^2 = \frac{1-\mathrm{Re}(\langle\psi|U|\psi\rangle)}{2}$.

Hadamard test



$$
\begin{aligned}
|0\rangle|\psi\rangle &\longmapsto \frac{|0\rangle+|1\rangle}{\sqrt{2}}|\psi\rangle \longmapsto \frac{1}{\sqrt{2}}\left(|0\rangle|\psi\rangle + i\,|1\rangle\, U|\psi\rangle\right) \\
&\longmapsto \frac{1}{\sqrt{2}}\left(|+\rangle|\psi\rangle + i\,|-\rangle\, U|\psi\rangle\right) \\
&= \frac{1}{2}\left(|0\rangle\left(|\psi\rangle + iU|\psi\rangle\right) + |1\rangle\left(|\psi\rangle - iU|\psi\rangle\right)\right).
\end{aligned}
$$

0 is measured with probability $\left\|\frac{|\psi\rangle+iU|\psi\rangle}{2}\right\|^2 = \frac{1+\mathrm{Im}(\langle\psi|U|\psi\rangle)}{2}$.

1 is measured with probability $\left\|\frac{|\psi\rangle-iU|\psi\rangle}{2}\right\|^2 = \frac{1-\mathrm{Im}(\langle\psi|U|\psi\rangle)}{2}$.

Hadamard test for eigenvector

If $|\psi_\theta\rangle$ is an eigenvector of U for eigenvalue $e^{i\theta}$.

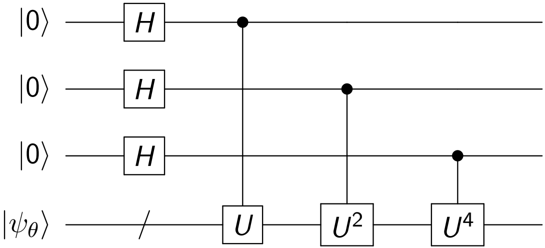
$$\mathrm{Re}(\langle\psi_\theta| U |\psi_\theta\rangle) = \mathrm{Re}(e^{i\theta}) = \cos(\theta)$$
$$\mathrm{Im}(\langle\psi_\theta| U |\psi_\theta\rangle) = \mathrm{Im}(e^{i\theta}) = \sin(\theta)$$

Hence, we can estimate θ. But, this algorithm doesn't work when the eigenvector $|\psi_\theta\rangle$ is not given. For

$$|\psi\rangle := \sum_{j=1}^{N} \alpha_j |\psi_{\theta_j}\rangle$$

$\langle\psi| U |\psi\rangle = \sum_{j=1}^{N} |\alpha_j|^2 e^{i\theta_j}$.

Quantum phase estimation



$$
\begin{aligned}
|0\rangle^{\otimes n} |\psi_\theta\rangle &\longmapsto \frac{1}{\sqrt{2^n}} \sum_{x \in \{0,1\}^n} |x\rangle |\psi_\theta\rangle \\
&\longmapsto \frac{1}{\sqrt{2^n}} \sum_{x \in \{0,1\}^n} e^{i\theta(x_1 + 2x_2 + 2^2 x_3 + \cdots 2^{n-1} x_n)} |x\rangle |\psi_\theta\rangle \\
&= \frac{1}{\sqrt{2^n}} \sum_{x \in \{0,1,2,\ldots,2^n-1\}} e^{i\theta x} |x\rangle |\psi_\theta\rangle
\end{aligned}
$$

Quantum Fourier transform

$$\begin{bmatrix}1\\0\\0\\0\end{bmatrix},\begin{bmatrix}0\\1\\0\\0\end{bmatrix},\begin{bmatrix}0\\0\\1\\0\end{bmatrix},\begin{bmatrix}0\\0\\0\\1\end{bmatrix}\longleftrightarrow\frac{1}{2}\begin{bmatrix}1\\1\\1\\1\end{bmatrix},\frac{1}{2}\begin{bmatrix}1\\\omega\\\omega^2\\\omega^3\end{bmatrix},\frac{1}{2}\begin{bmatrix}1\\\omega^2\\\omega^4\\\omega^6\end{bmatrix},\frac{1}{2}\begin{bmatrix}1\\\omega^3\\\omega^6\\\omega^9\end{bmatrix}$$

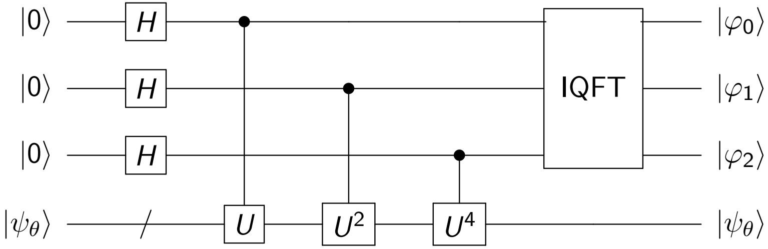
$$|x\rangle\longleftrightarrow|\widehat{x}\rangle:=\frac{1}{\sqrt{N}}\sum_{y=0}^{N-1}\omega_N^{xy}\,|y\rangle$$

where $\omega_N := e^{i\frac{2\pi}{N}}$.

$$U_{\mathrm{QFT}}:=\sum_{x=0}^{N-1}|\widehat{x}\rangle\,\langle x|$$

Hadamard operator is the quantum Fourier transform for $N=2$.

Quantum phase estimation



Assume $\theta = 2\pi \frac{\varphi}{2^n}$ for some integer $\varphi \in \{0, 1, \dots, 2^n - 1\}$.

$$\begin{aligned} &\frac{1}{\sqrt{2^n}} \sum_{x \in \{0,1,2,\dots,2^n-1\}} e^{i \frac{2\pi}{2^n} \varphi x} |x\rangle |\psi_\theta\rangle \\ &= |\widehat{\varphi}\rangle |\psi_\theta\rangle \\ &\longmapsto |\varphi\rangle |\psi_\theta\rangle \end{aligned}$$

Probability of the best approximation

Assume that the eigenvalue is $e^{2\pi i\phi}$. Let $\varphi \in \{0, 1, \ldots, 2^n - 1\}$ be the best n-bit approximation of ϕ, i.e., $|\phi - \frac{\varphi}{2^n}| \leq \frac{1}{2^{n+1}}$.

$$
\begin{aligned}
\Pr(\varphi) &= \frac{1}{2^{2n}} \left| \left(\sum_{x=0}^{2^n-1} e^{-i\frac{2\pi}{2^n}\varphi x} \langle x| \right) \left(\sum_{x=0}^{2^n-1} e^{i2\pi\phi x} |x\rangle \right) \right|^2 \\
&= \frac{1}{2^{2n}} \left| \sum_{x=0}^{2^n-1} e^{2\pi i\left(\phi - \frac{\varphi}{2^n}\right)x} \right|^2 \\
&= \frac{1}{2^{2n}} \left| \frac{1 - e^{2\pi i(2^n\phi - \varphi)}}{1 - e^{2\pi i\left(\phi - \frac{\varphi}{2^n}\right)}} \right|^2 \\
&= \frac{1}{2^{2n}} \frac{\sin^2\left(\pi\left(2^n\phi - \varphi\right)\right)}{\sin^2\left(\pi\left(\phi - \frac{\varphi}{2^n}\right)\right)} \\
&\geq \frac{1}{2^{2n}} \frac{\sin^2\left(\pi\left(2^n\phi - \varphi\right)\right)}{\left(\pi\left(\phi - \frac{\varphi}{2^n}\right)\right)^2} \\
&\geq \frac{1}{2^{2n}} \frac{\left(2\left(2^n\phi - \varphi\right)\right)^2}{\left(\pi\left(\phi - \frac{\varphi}{2^n}\right)\right)^2} = \frac{4}{\pi^2} \approx 0.405
\end{aligned}
$$

Quantum phase estimation for superposition of eigenvectors

$$|\psi\rangle := \sum_{i=1}^{N} \alpha_i |\psi_{\theta_i}\rangle$$

$$\begin{aligned} |0\rangle \left(\sum_{i=1}^{N} \alpha_i |\psi_{\theta_i}\rangle \right) &\longmapsto \sum_{i=1}^{N} \alpha_i |\widehat{\varphi}_i\rangle |\psi_{\theta_i}\rangle \\ &\longmapsto \sum_{i=1}^{N} \alpha_i |\varphi_i\rangle |\psi_{\theta_i}\rangle \end{aligned}$$

Then, φ_i is measured with probability $|\alpha_i|^2$.

Quantum Fourier transform

$$U_{\mathrm{QFT}(N)} := \sum_{x=0}^{N-1} |\widehat{x}\rangle \langle x| \,.$$

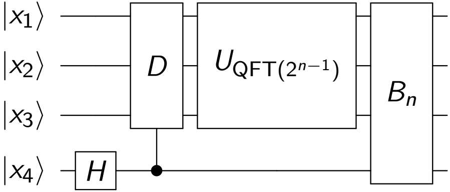
$$\begin{aligned}
U_{\mathrm{QFT}(2^n)} &= \frac{1}{\sqrt{2^n}} \sum_{x=0}^{2^n-1} \sum_{z=0}^{2^n-1} \omega_{2^n}^{xz} |z\rangle \langle x| \\
&= \frac{1}{\sqrt{2^n}} \sum_{a,b\in\{0,1\}} \sum_{x=0}^{2^{n-1}-1} \sum_{z=0}^{2^{n-1}-1} \omega_{2^n}^{(x+2^{n-1}a)(2z+b)} |zb\rangle \langle ax| \\
&= \frac{1}{\sqrt{2^n}} \sum_{a,b\in\{0,1\}} \sum_{x=0}^{2^{n-1}-1} \sum_{z=0}^{2^{n-1}-1} \omega_{2^n}^{2xz+xb+2^{n-1}ab} |zb\rangle \langle ax| \\
&= \frac{1}{\sqrt{2^n}} \sum_{a,b\in\{0,1\}} (-1)^{ab} \sum_{x=0}^{2^{n-1}-1} \sum_{z=0}^{2^{n-1}-1} \omega_{2^n}^{2xz+xb} B_n |bz\rangle \langle ax| \\
&= B_n \frac{1}{\sqrt{2^n}} \sum_{a,b\in\{0,1\}} (-1)^{ab} |b\rangle \langle a| \otimes \sum_{x=0}^{2^{n-1}-1} \sum_{z=0}^{2^{n-1}-1} \omega_{2^n}^{2xz+xb} |z\rangle \langle x|
\end{aligned}$$

Quantum Fourier transform

$$
\begin{aligned}
U_{\mathrm{QFT}(2^n)} &= B_n \frac{1}{\sqrt{2^n}} \sum_{a,b\in\{0,1\}} (-1)^{ab} |b\rangle\langle a| \otimes \sum_{x=0}^{2^{n-1}-1} \sum_{z=0}^{2^{n-1}-1} \omega_{2^n}^{2xz+xb} |z\rangle\langle x| \\
&= B_n \frac{1}{\sqrt{2^{n-1}}} \sum_{b\in\{0,1\}} |b\rangle\langle b| H \otimes \sum_{x=0}^{2^{n-1}-1} \sum_{z=0}^{2^{n-1}-1} \omega_{2^n}^{2xz+xb} |z\rangle\langle x| \\
&= B_n \sum_{b\in\{0,1\}} |b\rangle\langle b| \otimes \sum_{x=0}^{2^{n-1}-1} \omega_{2^n}^{xb} |\widehat{z}\rangle\langle x| \, (H \otimes I_{2^{n-1}}) \\
&= B_n \left(\sum_{b\in\{0,1\}} |b\rangle\langle b| \otimes \sum_{x=0}^{2^{n-1}-1} |\widehat{z}\rangle\langle x| \right) \\
&\quad \cdot \left(\sum_{b\in\{0,1\}} |b\rangle\langle b| \otimes \sum_{x=0}^{2^{n-1}-1} \omega_{2^n}^{xb} |x\rangle\langle x| \right) (H \otimes I_{2^{n-1}}) \\
&= B_n (I_2 \otimes U_{\mathrm{QFT}(2^{n-1})}) \, \Lambda \left(\sum_{x=0}^{2^{n-1}-1} \omega_{2^n}^{x} |x\rangle\langle x| \right) (H \otimes I_{2^{n-1}})
\end{aligned}
$$

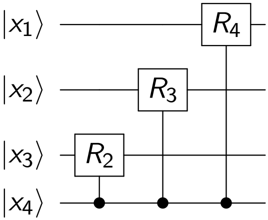
Quantum Fourier transform

$$U_{\mathrm{QFT}(2^n)} = B_n(I_2 \otimes U_{\mathrm{QFT}(2^{n-1})})\, \Lambda \left(\sum_{x=0}^{2^{n-1}-1} \omega_{2^n}^{x} \left|x\right\rangle \left\langle x\right| \right) (H \otimes I_{2^{n-1}})$$



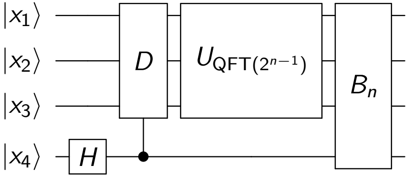
Quantum Fourier transform

$$
\begin{aligned}
\Lambda\left(\sum_{x=0}^{2^{n-1}-1} \omega_{2^n}^{x} \left|x\right\rangle \left\langle x\right|\right) &= \Lambda\left(\sum_{x\in\{0,1\}^n} \omega_{2^n}^{x_1+2x_2+\cdots+2^{n-1}x_n} \left|x\right\rangle \left\langle x\right|\right) \\
&= \Lambda\left(\bigotimes_{i=1}^{n} \sum_{x_i\in\{0,1\}} \omega_{2^n}^{2^{i-1}x_i} \left|x_i\right\rangle \left\langle x_i\right|\right) \\
&= \prod_{i=1}^{n} \Lambda\left(\sum_{x_i\in\{0,1\}} \omega_{2^{n-i+1}}^{x_i} \left|x_i\right\rangle \left\langle x_i\right|\right)
\end{aligned}
$$

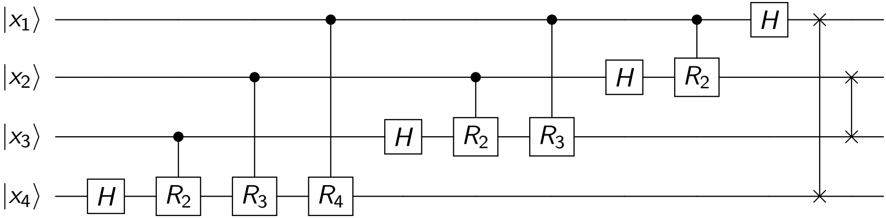


for $R_k := \begin{bmatrix} 1 & 0 \\ 0 & \omega_{2^k} \end{bmatrix}$.

Whole quantum circuit of QFT



Let $R_k := \begin{bmatrix} 1 & 0 \\ 0 & \omega_{2^k} \end{bmatrix}$.



Assignments

1. Show that the Fourier basis $\{|\widehat{x}\rangle\}_{x \in \{0,1,\ldots,N-1\}}$ is orthonormal.