Quantum teleportation

Ryuhei Mori

Tokyo Institute of Technology

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Time evolution of a system

Time evolution of a system is represented by a map from a state to a state.

$$T : \text{The set of states} \rightarrow \text{the set of states.}$$

$$pT(\rho_1) + (1-p)T(\rho_2) = T(p\rho_1 + (1-p)\rho_2)$$

for any density matrices ρ_1, ρ_2 and $p \in [0,1]$.

$$T : \mathcal{H}(V) \rightarrow \mathcal{H}(W) \text{ must be linear (a proof is needed).}$$

Schrödinger picture and Heisenberg picture

$T^\dagger$: The set of binary measurements $\rightarrow$ the set of binary measurements.

$$\langle T(\rho), P \rangle = \langle \rho, T^\dagger(P) \rangle$$

for any $\rho \in \mathcal{H}(V)$ and $P \in \mathcal{H}(W)$. $T^\dagger$ is an adjoint map of T.

$$\langle T_3(T_2(T_1(\rho))), P \rangle = \langle T_2(T_1(\rho)), T_3^\dagger(P) \rangle$$
$$= \langle T_1(\rho), T_2^\dagger(T_3^\dagger(P)) \rangle = \langle \rho, T_1^\dagger(T_2^\dagger(T_3^\dagger(P))) \rangle$$

No-cloning theorem

$$|0\rangle \langle 0| \longmapsto |0\rangle \langle 0| \otimes |0\rangle \langle 0|$$
$$|1\rangle \langle 1| \longmapsto |1\rangle \langle 1| \otimes |1\rangle \langle 1|$$

From the linearlity,

$$\frac{1}{2}(|0\rangle \langle 0| + |1\rangle \langle 1|) \longmapsto \frac{1}{2}(|0\rangle \langle 0| \otimes |0\rangle \langle 0| + |1\rangle \langle 1| \otimes |1\rangle \langle 1|)$$

This is not equal to

$$\frac{1}{2}(|0\rangle \langle 0| + |1\rangle \langle 1|) \otimes \frac{1}{2}(|0\rangle \langle 0| + |1\rangle \langle 1|).$$

Axioms for quantum channel

$T \colon \mathcal{H}(V) \to \mathcal{H}(W)$.

1. Trace-preserving: $\text{Tr}(T(\rho)) = \text{Tr}(\rho)$.

2. Positive : $T(\rho) \succeq 0$ for any $\rho \succeq 0$.

3. Completely positive: $\text{id} \otimes T$ is positive.

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Positive but not completely positive 1/2

$T \in \mathcal{L}(\mathcal{H}(\mathbb{C}^2))$: transposition according to $\{|0\rangle, |1\rangle\}$.

The transposition is obviously trace-preserving.

The transposition is positive.

Proof.
For any $A \succeq 0$ and $|\psi\rangle \in \mathbb{C}^2$,

$$\langle\psi| T(A) |\psi\rangle = \langle\psi| A^T |\psi\rangle = \langle\psi| A^* |\psi\rangle = (\langle\psi|^* A |\psi\rangle^*)^* \geq 0 \qquad \square$$

Positive but not completely positive 2/2

But, the transposition is not completely positive.

Proof.

For $|\Phi\rangle := \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$,

$$
(\mathrm{id}_{\mathcal{H}(\mathbb{C}^2)} \otimes T)(|\Phi\rangle\langle\Phi|)
$$

$$
= (\mathrm{id}_{\mathcal{H}(\mathbb{C}^2)} \otimes T)\left(\frac{1}{2}\Big(|0\rangle\langle 0| \otimes |0\rangle\langle 0| + |0\rangle\langle 1| \otimes |0\rangle\langle 1| \right.
$$

$$
\left. + |1\rangle\langle 0| \otimes |1\rangle\langle 0| + |1\rangle\langle 1| \otimes |1\rangle\langle 1| \Big) \right)
$$

$$
= \frac{1}{2}\Big(|0\rangle\langle 0| \otimes |0\rangle\langle 0| + |0\rangle\langle 1| \otimes |1\rangle\langle 0|
$$

$$
+ |1\rangle\langle 0| \otimes |0\rangle\langle 1| + |1\rangle\langle 1| \otimes |1\rangle\langle 1| \Big)
$$

$$
|00\rangle \mapsto |00\rangle \qquad |01\rangle \mapsto |10\rangle \qquad |10\rangle \mapsto |01\rangle \qquad |11\rangle \mapsto |11\rangle
$$

Hence, $|01\rangle - |10\rangle \mapsto |10\rangle - |01\rangle$. $(\mathrm{id}_{\mathcal{H}(\mathbb{C}^2)} \otimes T)(|\Phi\rangle\langle\Phi|)$ is not positive semidefinite. $\qquad\square$
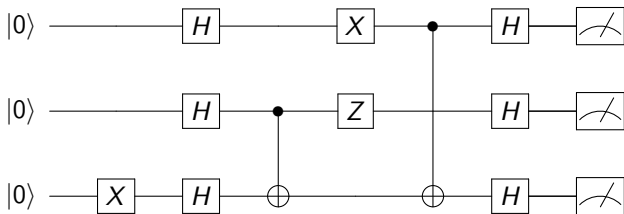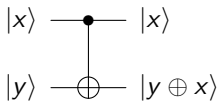
Unitary operations

$$\rho \longmapsto U\rho U^{\dagger}.$$

1. Trace-preserving: $\mathrm{Tr}(U\rho U^{\dagger}) = \mathrm{Tr}(\rho)$.

2. Completely positive: $(\mathrm{id} \otimes T)(\rho) = (I \otimes U)\rho(I \otimes U^{\dagger})$.

In the most of quantum computing, only pure states and unitary operations are used.
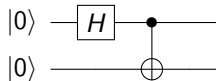
Quantum circuit

Controlled not

$$|x\rangle \longrightarrow \bullet \longrightarrow |x\rangle$$

$$|y\rangle \longrightarrow \oplus \longrightarrow |y \oplus x\rangle$$

$$\text{CNOT} \, |x\rangle \, |y\rangle \longmapsto |x\rangle \, |y \oplus x\rangle$$

$$\text{CNOT} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

Bell states and quantum circuit

$$|0\rangle |0\rangle \longmapsto \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) |0\rangle = \frac{1}{\sqrt{2}}(|0\rangle |0\rangle + |1\rangle |0\rangle)$$

$$\longmapsto \frac{1}{\sqrt{2}}(|0\rangle |0\rangle + |1\rangle |1\rangle)$$

$$|x\rangle |y\rangle \longmapsto \frac{1}{\sqrt{2}}(|0\rangle + (-1)^x |1\rangle) |y\rangle = \frac{1}{\sqrt{2}}(|0\rangle |y\rangle + (-1)^x |1\rangle |y\rangle)$$

$$\longmapsto \frac{1}{\sqrt{2}}(|0\rangle |y\rangle + (-1)^x |1\rangle |\bar{y}\rangle).$$

Conditional density operator

A probability of outcome of local measurement in a joint system is

$$P(a, b) = \text{Tr}(\rho_{V \otimes W}(P_a \otimes Q_b)).$$

$$P(a \mid b) = \frac{1}{P(b)}\text{Tr}(\rho_{V \otimes W}(P_a \otimes Q_b)) = \frac{1}{P(b)}\text{Tr}(\text{Tr}_W(\rho_{V \otimes W}(I_V \otimes Q_b))P_a).$$

$$\rho_{V|Q_b} := \frac{1}{P(b)}\text{Tr}_W(\rho_{V \otimes W}(I_V \otimes Q_b)).$$

For $\rho_{V \otimes W} = |\varphi\rangle_{V \otimes W} \langle\varphi|_{V \otimes W}$ and $Q_b = |\psi_b\rangle_W \langle\psi_b|_W$,

$$\text{Tr}_W(\rho_{V \otimes W}(I \otimes Q_b)) = \text{Tr}_W(|\varphi\rangle_{V \otimes W} \langle\varphi|_{V \otimes W} (I_V \otimes |\psi_b\rangle_W \langle\psi_b|_W))$$

Conditional density operator for pure state

For $\rho = |\varphi\rangle_{V \otimes W} \langle\varphi|_{V \otimes W}$ and $Q_b = |\psi_b\rangle_W \langle\psi_b|_W$,

$$\text{Tr}_W(\rho_{V \otimes W}(I_V \otimes Q_b)) = \text{Tr}_W(|\varphi\rangle_{V \otimes W} \langle\varphi|_{V \otimes W}(I \otimes |\psi_b\rangle_W \langle\psi_b|_W))$$

From an expression $|\varphi\rangle_{V \otimes W} = \sum_{i,j} \varphi_{i,j} |i\rangle_V |\psi_j\rangle_W$,

$$\text{Tr}_W(|\varphi\rangle_{V \otimes W} \langle\varphi|_{V \otimes W}(I_V \otimes |\psi_b\rangle_W \langle\psi_b|_W))$$

$$= \text{Tr}_W \left(\sum_{i,j,k,l} \varphi_{i,j} \varphi_{k,l}^* |i\rangle_V |\psi_j\rangle_W \langle k|_V \langle\psi_l|_W (I_V \otimes |\psi_b\rangle_W \langle\psi_b|_W) \right)$$

$$= \text{Tr}_W \left(\sum_{i,j,k,l} \varphi_{i,j} \varphi_{k,l}^* |i\rangle_V \langle k|_V \otimes |\psi_j\rangle_W \langle\psi_l|_W (I_V \otimes |\psi_b\rangle_W \langle\psi_b|_W) \right)$$

$$= \sum_{i,j,k,l} \varphi_{i,j} \varphi_{k,l}^* |i\rangle_V \langle k|_V \, \text{Tr} \left(|\psi_j\rangle_W \langle\psi_l|_W |\psi_b\rangle_W \langle\psi_b|_W \right)$$

$$= \sum_{i,k} \varphi_{i,b} \varphi_{k,b}^* |i\rangle_V \langle k|_V = \left(\sum_i \varphi_{i,b} |i\rangle_V \right) \left(\sum_k \varphi_{k,b}^* \langle k|_V \right)$$

$$|\varphi\rangle_{V \otimes W} = \sum_{i,j} \varphi_{i,j} |i\rangle_V |\psi_j\rangle_W \longmapsto \frac{1}{P(b)} \sum_i \varphi_{i,b} |i\rangle_V$$
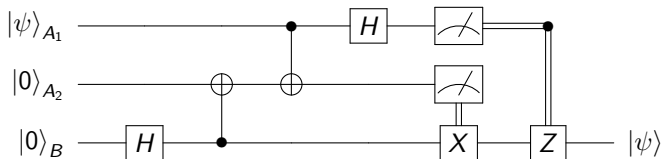
Examples of conditional density operator

For $|\psi\rangle_{V \otimes W} := \sum_{i,j=0}^{1} \alpha_{i,j} |i\rangle_V |j\rangle_W$, we measure the system W by $(|0\rangle \langle 0|, |1\rangle \langle 1|)$.

if the outcome is 0, the state $\frac{1}{\sqrt{|\alpha_{0,0}|^2 + |\alpha_{1,0}|^2}} \sum_{i=0}^{1} \alpha_{i,0} |i\rangle_V$.

if the outcome is 1, the state $\frac{1}{\sqrt{|\alpha_{0,1}|^2 + |\alpha_{1,1}|^2}} \sum_{i=0}^{1} \alpha_{i,1} |i\rangle_V$.

Quantum teleportation

Quantum teleportation

$$|\Phi\rangle_{A_2 \otimes B} = \frac{1}{\sqrt{2}} \left(|0\rangle_{A_2} |0\rangle_B + |1\rangle_{A_2} |1\rangle_B \right)$$

$$(H_{A_1} \otimes I_{A_2} \otimes I_B)(\text{CNOT}_{A_1 A_2} \otimes I_B) |\psi\rangle_{A_1} |\Phi\rangle_{A_2 \otimes B}$$

For $|\psi\rangle_{A_1} = \alpha |0\rangle_{A_1} + \beta |1\rangle_{A_1}$

$$
\begin{aligned}
|\psi\rangle_{A_1} |\Phi\rangle_{A_2 \otimes B} &= \left(\alpha |0\rangle_{A_1} + \beta |1\rangle_{A_1} \right) \frac{1}{\sqrt{2}} \left(|0\rangle_{A_2} |0\rangle_B + |1\rangle_{A_2} |1\rangle_B \right) \\
&\overset{\text{CNOT}_{A_1 A_2}}{\longmapsto} \frac{\alpha}{\sqrt{2}} \left(|0\rangle_{A_1} |0\rangle_{A_2} |0\rangle_B + |0\rangle_{A_1} |1\rangle_{A_2} |1\rangle_B \right) \\
&\quad + \frac{\beta}{\sqrt{2}} \left(|1\rangle_{A_1} |1\rangle_{A_2} |0\rangle_B + |1\rangle_{A_1} |0\rangle_{A_2} |1\rangle_B \right) \\
&\overset{H_{A_1}}{\longmapsto} \frac{\alpha}{2} \left(|000\rangle_{A_1 A_2 B} + |100\rangle_{A_1 A_2 B} + |011\rangle_{A_1 A_2 B} + |111\rangle_{A_1 A_2 B} \right) \\
&\quad + \frac{\beta}{2} \left(|010\rangle_{A_1 A_2 B} - |110\rangle_{A_1 A_2 B} + |001\rangle_{A_1 A_2 B} - |101\rangle_{A_1 A_2 B} \right)
\end{aligned}
$$

Quantum teleportation

$$\frac{\alpha}{2} \left(|000\rangle_{A_1 A_2 B} + |100\rangle_{A_1 A_2 B} + |011\rangle_{A_1 A_2 B} + |111\rangle_{A_1 A_2 B} \right)$$

$$+ \frac{\beta}{2} \left(|010\rangle_{A_1 A_2 B} - |110\rangle_{A_1 A_2 B} + |001\rangle_{A_1 A_2 B} - |101\rangle_{A_1 A_2 B} \right)$$

$$= \frac{1}{2} \Bigg[|00\rangle_{A_1 A_2} \left(\alpha |0\rangle_B + \beta |1\rangle_B \right) + |01\rangle_{A_1 A_2} \left(\alpha |1\rangle_B + \beta |0\rangle_B \right)$$

$$+ |10\rangle_{A_1 A_2} \left(\alpha |0\rangle_B - \beta |1\rangle_B \right) + |11\rangle_{A_1 A_2} \left(\alpha |1\rangle_B - \beta |0\rangle_B \right) \Bigg]$$

According to the measurement outcome of $A_1 A_2$

$$00 \Rightarrow I$$
$$01 \Rightarrow X$$
$$10 \Rightarrow Z$$
$$11 \Rightarrow ZX$$

Assignments

1 Show the state vector $\rho \in H(\mathbb{C}^2)$ of the Bell state

$$\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \in \mathbb{C}^2 \otimes \mathbb{C}^2$$

when $|0\rangle \langle 0|$ is measured at the second system.

2 Show the state vector $\rho \in H(\mathbb{C}^2)$ of the Bell state when $|+\rangle \langle +|$ is measured at the second system.

3 Show the state vector $\rho \in H(\mathbb{C}^2)$ of the Bell state when $|\psi\rangle \langle \psi|$ is measured at the second system where $|\psi\rangle := \alpha |0\rangle + \beta |1\rangle$.