

A single qubit

Ryuhei Mori

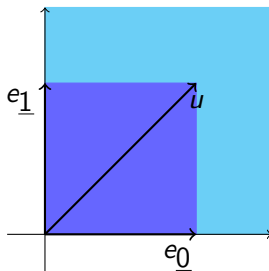
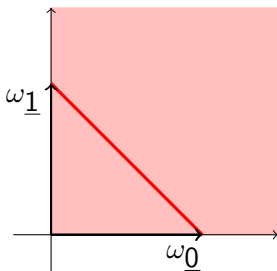
Tokyo Institute of Technology

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A single bit

Let $u := \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

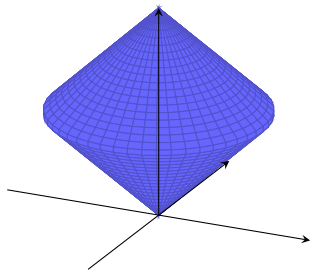
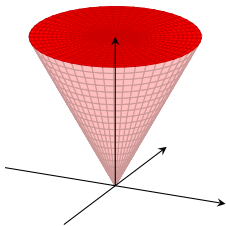
- Set of states = $\{\omega \in \mathbb{R}^2 \mid \omega \in C_{\geq 0}, \langle u, \omega \rangle = 1\}$.
- Set of binary measurements = $\{e \in \mathbb{R}^2 \mid e \in C_{\geq 0}, u - e \in C_{\geq 0}\}$.



A single qubit

Let $u := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\langle e, \omega \rangle := \text{Tr}(e\omega)$.

- Set of states = $\{\omega \in V \mid \omega \in C_{\geq 0}, \langle u, \omega \rangle = 1\}$.
- Set of binary measurements = $\{e \in V \mid e \in C_{\geq 0}, u - e \in C_{\geq 0}\}$.



A single qubit

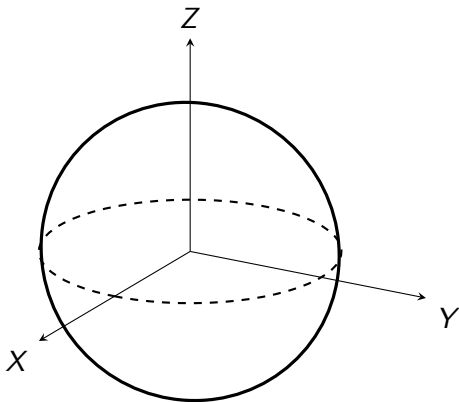
- A qubit can be represented by

$$\rho = \frac{1}{2} (I + r_X X + r_Y Y + r_Z Z)$$

for $[r_X \ r_Y \ r_Z] \in \mathbb{R}^3$ satisfying $r_X^2 + r_Y^2 + r_Z^2 \leq 1$.

- A qubit can be represented by a point $[r_X \ r_Y \ r_Z]$ in a three-dimensional sphere of radius 1.

The Bloch sphere



Complex space and Hermitian operator

- \mathcal{X} : A finite-dimensional inner product space on \mathbb{C} .
- $\mathcal{L}(\mathcal{X})$: A set of linear operators on \mathcal{X} .

For $A \in \mathcal{L}$, an adjoint map A^\dagger of A is a unique operator satisfying

$$\langle v, Aw \rangle = \langle A^\dagger v, w \rangle$$

for any $v, w \in \mathcal{X}$. $H \in \mathcal{L}(\mathcal{X})$ is Hermitian if and only if $H^\dagger = H$.

- $\mathcal{H}(\mathcal{X})$: A set of Hermitian operators on \mathcal{X} .

$\mathcal{L}(\mathcal{X})$ and $\mathcal{H}(\mathcal{X})$ are often regarded as inner product space on \mathbb{C} and \mathbb{R} , respectively for the Hilbert–Schmidt inner product $\langle A, B \rangle = \text{Tr}(A^\dagger B)$.

Spectral decomposition theorem

Definition (Normal operator)

$A \in \mathcal{L}(\mathcal{X})$ is said to be **normal** if $AA^\dagger = A^\dagger A$.

Hermitian matrix ($H^\dagger = H$) and unitary matrix ($UU^\dagger = I$) are normal.

Theorem (Spectral decomposition theorem)

$A \in \mathcal{L}(\mathbb{C}^n)$ is **normal** if and only if there exist orthonormal basis $\{|\psi_j\rangle\}$ of \mathbb{C}^n and complex numbers $\{\lambda_j\}$ such that

$$A = \sum_j \lambda_j |\psi_j\rangle \langle \psi_j|.$$

Pauli matrices

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$$Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \end{bmatrix}$$

-

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \begin{bmatrix} 1 & -1 \end{bmatrix}$$

-

$$Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 \\ i \end{bmatrix} \begin{bmatrix} 1 & -i \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 1 \\ -i \end{bmatrix} \begin{bmatrix} 1 & i \end{bmatrix}$$

Bracket notation

$$|0\rangle := \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

$$|1\rangle := \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\begin{aligned} |+\rangle &:= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \\ &= \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle), \end{aligned}$$

$$\begin{aligned} |-\rangle &:= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \\ &= \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \end{aligned}$$

$$|\psi\rangle = \alpha |0\rangle + \beta |1\rangle = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

for $|\alpha|^2 + |\beta|^2 = 1$.

$$\langle\psi| = |\psi\rangle^\dagger = \alpha^* \langle 0| + \beta^* \langle 1| = [\alpha^* \quad \beta^*]$$

Pauli matrices in bracket notation

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$$Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = |0\rangle\langle 0| - |1\rangle\langle 1|$$

-

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = |+\rangle\langle +| - |-\rangle\langle -|$$

-

$$Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 \\ i \end{bmatrix} \begin{bmatrix} 1 & -i \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 1 \\ -i \end{bmatrix} \begin{bmatrix} 1 & i \end{bmatrix}$$

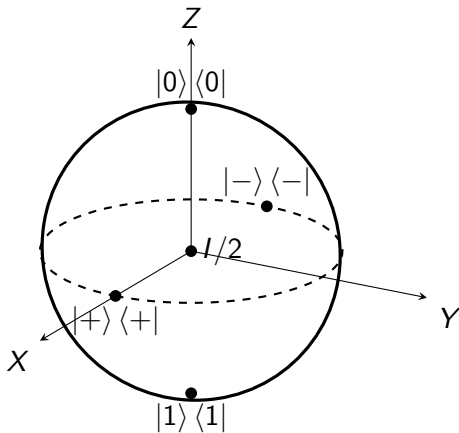
Special states

$$\rho = \frac{1}{2} (I + r_X X + r_Y Y + r_Z Z)$$

$$r_X^2 + r_Y^2 + r_Z^2 \leq 1.$$

| Coordinate | State |
|----------------|---|
| $[0 \ 0 \ 0]$ | $\frac{1}{2}I$ |
| $[1 \ 0 \ 0]$ | $\frac{1}{2}(I + X) = +\rangle \langle + $ |
| $[-1 \ 0 \ 0]$ | $\frac{1}{2}(I - X) = -\rangle \langle - $ |
| $[0 \ 0 \ 1]$ | $\frac{1}{2}(I + Z) = 0\rangle \langle 0 $ |
| $[0 \ 0 \ -1]$ | $\frac{1}{2}(I - Z) = 1\rangle \langle 1 $ |

Special states in the Bloch sphere



Pure states are rank-1 density operators

ρ is a **pure state**

$$\stackrel{\text{def}}{\iff} \rho \neq p\rho_1 + (1-p)\rho_2 \quad \forall p \in [0, 1] \text{ and states } \rho_1 \neq \rho_2.$$

Lemma

A quantum state ρ is a pure state if and only if ρ is **rank-1**.

Proof.

Let the spectral decomposition of ρ be

$$\rho = \sum_j \lambda_j |\psi_j\rangle \langle \psi_j|$$

where $\lambda_j \geq 0$ and $\sum_j \lambda_j = 1$. If ρ is not rank-1, ρ is a convex combination of quantum states $(|\psi_j\rangle \langle \psi_j|)_j$.

$\text{Tr}(\rho |\varphi\rangle \langle \varphi|) = 1$ if and only if $\rho = |\varphi\rangle \langle \varphi|$ since

$\text{Tr}(\rho |\varphi\rangle \langle \varphi|) = \langle \varphi | \rho | \varphi \rangle = \sum_j \lambda_j |\langle \psi_j | \varphi \rangle|^2$. Assume that $\rho = |\varphi\rangle \langle \varphi|$ and $\rho = p\rho_1 + (1-p)\rho_2$. Then, $\text{Tr}(\rho_1 |\varphi\rangle \langle \varphi|) = \text{Tr}(\rho_2 |\varphi\rangle \langle \varphi|) = 1$ that means $\rho_1 = \rho_2 = \rho$. □

Pure states and state vector

Pure state $|\psi\rangle\langle\psi|$ can be represented by a **state vector** $|\psi\rangle \in \mathbb{C}^2$ with $\langle\psi|\psi\rangle = 1$.

$|\psi\rangle$ and $|\varphi\rangle := e^{i\theta} |\psi\rangle$ represent the same state since $|\psi\rangle\langle\psi| = |\varphi\rangle\langle\varphi|$.

Inner product of pure states

- ρ is a qubit pure state with a coordinate $[r_X \ r_Y \ r_Z]$.
- σ is a qubit pure state with a coordinate $[-r_X \ -r_Y \ -r_Z]$.

$$\text{Tr}(\rho\sigma) = \text{Tr}(\rho(I - \rho)) = \text{Tr}(\rho) - \text{Tr}(\rho^2) = 1 - 1 = 0$$

- $\rho = |\psi\rangle \langle\psi|$.
- $\sigma = |\varphi\rangle \langle\varphi|$.

$$\begin{aligned}\text{Tr}(\rho\sigma) &= \text{Tr}(|\psi\rangle \langle\psi| |\varphi\rangle \langle\varphi|) = \langle\psi|\varphi\rangle \text{Tr}(|\psi\rangle \langle\varphi|) \\ &= \langle\psi|\varphi\rangle \langle\varphi|\psi\rangle = |\langle\psi|\varphi\rangle|^2\end{aligned}$$

Single qubit measurement

Set of measurements = $\{(e_1, \dots, e_k) \mid e_1 + \dots + e_k = I, e_j \in C_{\geq 0}$
 $i = 1, 2, \dots, k, k = 1, 2, \dots\}$

If $e_i e_j = \delta_{ij} e_i$, the measurement is called an **orthogonal measurement**.

If $|0\rangle\langle 0|$ is measured by $(|0\rangle\langle 0|, |1\rangle\langle 1|)$, the output is 0 with probability $\text{Tr}(|0\rangle\langle 0| |0\rangle\langle 0|) = |\langle 0|0\rangle|^2 = 1$.

If $|+\rangle\langle +|$ is measured by $(|0\rangle\langle 0|, |1\rangle\langle 1|)$, the output is 0 with probability $\text{Tr}(|0\rangle\langle 0| |+\rangle\langle +|) = |\langle 0|+\rangle|^2 = 1/2$.

If $|\psi\rangle\langle \psi|$ is measured by $(|\varphi_0\rangle\langle \varphi_0|, |\varphi_1\rangle\langle \varphi_1|)$, the output is 0 with probability $\text{Tr}(|\varphi_0\rangle\langle \varphi_0| |\psi\rangle\langle \psi|) = |\langle \varphi_0|\psi\rangle|^2$.

Unitary operation

Unitary operation

$$\rho \mapsto U\rho U^\dagger$$

It is easy to see that

- $\text{Tr}(U\rho U^\dagger) = 1$
- $U\rho U^\dagger \succeq 0$

A pure state $|\psi\rangle$ is mapped to a pure state $U|\psi\rangle$.

U and $e^{i\theta}U$ are physically equivalent.

Examples of unitary operations

- The identity matrix I .
- Pauli matrices X , Y and Z .
- Hadamard matrix $H := \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$
- Product UV of unitary operators U and V .

Multiplications of Pauli matrices

For any unitary matrices U and V , UV is also unitary matrix.

- $XY = iZ$
- $YZ = iX$
- $ZX = iY$

Pauli matrices X on the Bloch sphere

$$\rho = \frac{1}{2} (I + r_X X + r_Y Y + r_Z Z)$$

$$\begin{aligned} X\rho X^\dagger &= X\rho X = \frac{1}{2} (X^2 + r_X X^3 + r_Y XYX + r_Z XZX) \\ &= \frac{1}{2} (I + r_X X - r_Y Y - r_Z Z) \end{aligned}$$

$$[r_X \ r_Y \ r_Z] \xrightarrow{X} [r_X \ -r_Y \ -r_Z]$$

π -rotation with respect to X axis.

Similarly, Y and Z corresponds to π -rotation with respect to Y and Z axes, respectively.

Hadamard matrix

Hadamard matrix H is unitary and Hermitian.

$$\begin{aligned} H &:= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = |+\rangle \langle 0| + |-\rangle \langle 1| \\ &= |0\rangle \langle +| + |1\rangle \langle -| \end{aligned}$$

$$|0\rangle, |1\rangle \xleftrightarrow{H} |+\rangle, |-\rangle$$

$$\begin{aligned} HXH &= H(|+\rangle \langle +| - |-\rangle \langle -|)H \\ &= |0\rangle \langle 0| - |1\rangle \langle 1| = Z \end{aligned}$$

Similarly, $HZH = X$.

$$HYH = H(iXZ)H = iHXHRZH = iZX = -Y$$

Hadamard matrix on the Bloch sphere

$$\rho = \frac{1}{2} (I + r_X X + r_Y Y + r_Z Z)$$

$$\begin{aligned} H\rho H^\dagger &= H\rho H = \frac{1}{2} (H^2 + r_X HXH + r_Y HYH + r_Z HZH) \\ &= \frac{1}{2} (I + r_X Z - r_Y Y + r_Z X) \end{aligned}$$

$$[r_X \ r_Y \ r_Z] \xrightarrow{H} [r_Z \ -r_Y \ r_X]$$

Hadamard operation can be decomposed to $\pi/2$ -rotation with respect to Y axis

$$[r_X \ r_Y \ r_Z] \xrightarrow{R_Y(\pi/2)} [r_Z \ r_Y \ -r_X]$$

and X .

Rotation matrices

$$R_X(\theta) := \cos \frac{\theta}{2} I - i \sin \frac{\theta}{2} X$$

$$R_X(\theta)^\dagger = R_X(-\theta)$$

$$\begin{aligned} R_X(\theta) R_X(\theta)^\dagger &= (\cos \frac{\theta}{2} I - i \sin \frac{\theta}{2} X)(\cos \frac{\theta}{2} I + i \sin \frac{\theta}{2} X) \\ &= \cos^2 \frac{\theta}{2} I + \sin^2 \frac{\theta}{2} X^2 = I \end{aligned}$$

$$R_X(\theta)X = XR_X(\theta), \quad R_X(\theta)Y = YR_X(-\theta), \quad R_X(\theta)Z = ZR_X(-\theta)$$

$$R_X(\theta)R_X(\tau) = R_X(\theta + \tau)$$

$$\begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \xrightarrow{R_X(\theta)} \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \xrightarrow{R_X(\theta)} \begin{bmatrix} 0 & \cos \theta & \sin \theta \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_X(\theta)} \begin{bmatrix} 0 & -\sin \theta & \cos \theta \end{bmatrix}$$

Matrix function

Definition

For $f: \mathbb{C} \rightarrow \mathbb{C}$ and orthonormal basis $(|\psi_j\rangle)_j$ of \mathcal{X} ,

$$f\left(\sum_j \lambda_j |\psi_j\rangle \langle \psi_j|\right) := \sum_j f(\lambda_j) |\psi_j\rangle \langle \psi_j|.$$

For $H \in \mathcal{H}(\mathcal{X})$, $\exp(iH)$ is unitary.

Since the radius of convergence of \exp at 0 is infinity,

$$\exp(A) = \sum_{j=0}^{\infty} \frac{1}{j!} A^j.$$

In general, $\exp(A + B) \neq \exp(A) \exp(B)$.

Assignments

- For $a, b, c, d \in \mathbb{R}$, show $r_I, r_X, r_Y, r_Z \in \mathbb{R}$ satisfying
$$\begin{bmatrix} a & b+ci \\ b-ci & d \end{bmatrix} = r_I I + r_X X + r_Y Y + r_Z Z.$$
- Show a matrix representation of $\exp(-i\frac{\theta}{2}X)$ for $\theta \in \mathbb{R}$.
- [Advanced] For $a_I, a_X, a_Y, a_Z \in \mathbb{R}$, represents

$$\exp(i(a_I I + a_X X + a_Y Y + a_Z Z))$$

by a linear combination of I, X, Y and Z . A summation with infinite number of terms is not allowed.