A single qubit

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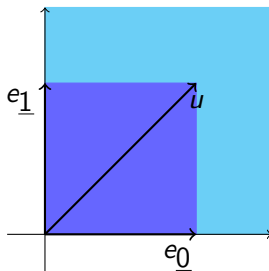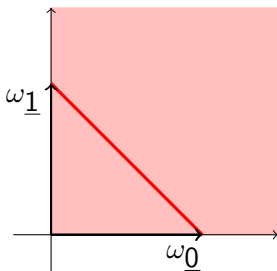
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Let $u := \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

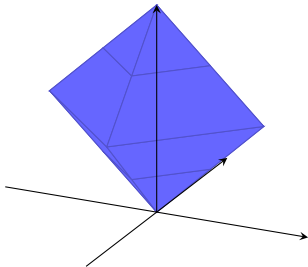- Set of states $= \left\{ \omega \in \mathbb{R}^2 \mid \omega \in C_{\geq 0}, \langle u, \omega \rangle = 1 \right\}$.

- Set of binary measurements $=$
  $\left\{ e \in \mathbb{R}^2 \mid e \in C_{\geq 0}, u - e \in C_{\geq 0} \right\}$.

Let $u := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\langle e, \omega \rangle := \mathsf{Tr}(e\omega)$.

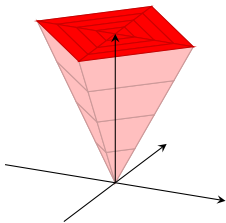- Set of states $= \{\omega \in V \mid \omega \in C_{\succeq 0}, \langle u, \omega \rangle = 1\}$.

- Set of binary measurements $=$
  $\{e \in V \mid e \in C_{\succeq 0}, u - e \in C_{\succeq 0}\}$.
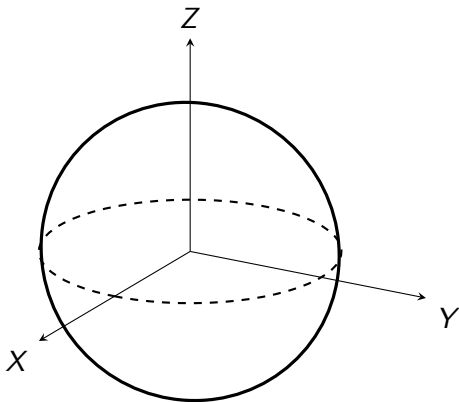
A single qubit

- A qubit can be represented by

$$\rho = \frac{1}{2} \left(I + r_X X + r_Y Y + r_Z Z \right)$$

for $[r_X \ r_Y \ r_Z] \in \mathbb{R}^3$ satisfying $r_X^2 + r_Y^2 + r_Z^2 \leq 1$.

- A qubit can be represented by a point $[r_X \ r_Y \ r_Z]$ in a three-dimensional sphere of radius 1.

The Bloch sphere

Complex space and Hermitian operator

- $\mathcal{X}$: A finite-dimensional inner product space on $\mathbb{C}$.
- $\mathcal{L}(\mathcal{X})$: A set of linear operators on $\mathcal{X}$.

For $A \in \mathcal{L}$, an adjoint map $A^\dagger$ of A is a unique operator satisfying

$$\langle v, Aw \rangle = \langle A^\dagger v, w \rangle$$

for any v, $w \in \mathcal{X}$. $H \in \mathcal{L}(\mathcal{X})$ is Hermitian if and only if $H^\dagger = H$.

- $\mathcal{H}(\mathcal{X})$: A set of Hermitian operators on $\mathcal{X}$.

$\mathcal{L}(\mathcal{X})$ and $\mathcal{H}(\mathcal{X})$ are often regarded as inner product space on $\mathbb{C}$ and $\mathbb{R}$, respectively for the Hilbert–Schmidt inner product $\langle A, B \rangle = \text{Tr}(A^\dagger B)$.

Spectral decomposition theorem

Definition (Normal operator)
$A \in \mathcal{L}(\mathcal{X})$ is said to be normal if $AA^\dagger = A^\dagger A$.

Hermitian matrix ($H^\dagger = H$) and unitary matrix ($UU^\dagger = I$) are normal.

Theorem (Spectral decomposition theorem)
$A \in \mathcal{L}(\mathbb{C}^n)$ is normal if and only if there exist orthonormal basis $\{|\psi_j\rangle\}$ of $\mathbb{C}^n$ and complex numbers $\{\lambda_j\}$ such that

$$A = \sum_j \lambda_j |\psi_j\rangle \langle \psi_j|.$$

Pauli matrices

-
$$Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \end{bmatrix}$$

-
$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \begin{bmatrix} 1 & -1 \end{bmatrix}$$

-
$$Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 \\ i \end{bmatrix} \begin{bmatrix} 1 & -i \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 1 \\ -i \end{bmatrix} \begin{bmatrix} 1 & i \end{bmatrix}$$

Braket notation

$$|0\rangle := \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad\qquad |1\rangle := \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$|+\rangle := \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad\qquad |-\rangle := \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$
$$= \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle), \qquad\qquad = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

$$|\psi\rangle = \alpha |0\rangle + \beta |1\rangle = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

for $|\alpha|^2 + |\beta|^2 = 1$.

$$\langle\psi| = |\psi\rangle^\dagger = \alpha^* \langle 0| + \beta^* \langle 1| = \begin{bmatrix} \alpha^* & \beta^* \end{bmatrix}$$

Pauli matrices in braket notation

-
$$Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = |0\rangle \langle 0| - |1\rangle \langle 1|$$

-
$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = |+\rangle \langle +| - |-\rangle \langle -|$$

-
$$Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 \\ i \end{bmatrix} \begin{bmatrix} 1 & -i \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 1 \\ -i \end{bmatrix} \begin{bmatrix} 1 & i \end{bmatrix}$$
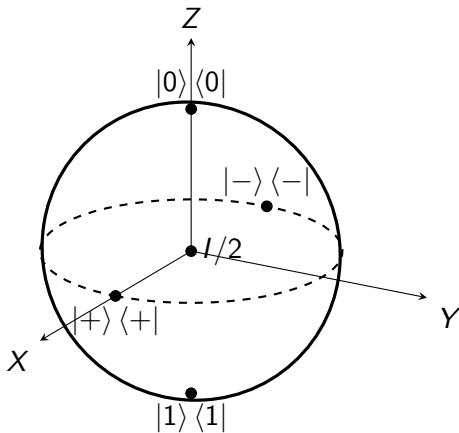
Special states

$$\rho = \frac{1}{2}\left(I + r_X X + r_Y Y + r_Z Z\right)$$

$$r_X^2 + r_Y^2 + r_Z^2 \leq 1.$$

Coordinate	State
[0 0 0]	$\frac{1}{2}I$
[1 0 0]	$\frac{1}{2}(I + X) = \lvert+\rangle\langle+\rvert$
[−1 0 0]	$\frac{1}{2}(I - X) = \lvert-\rangle\langle-\rvert$
[0 0 1]	$\frac{1}{2}(I + Z) = \lvert0\rangle\langle0\rvert$
[0 0 −1]	$\frac{1}{2}(I - Z) = \lvert1\rangle\langle1\rvert$

Special states in the Bloch sphere

Pure states are rank-1 density operators

ρ is a pure state

$$\stackrel{\text{def}}{\Longleftrightarrow} \quad \rho \neq p\rho_1 + (1-p)\rho_2 \quad \forall p \in [0,1] \text{ and states } \rho_1 \neq \rho_2.$$

Lemma

A quantum state ρ is a pure state if and only if ρ is rank-1.

Proof.

Let the spectral decomposition of ρ be

$$\rho = \sum_j \lambda_j \, |\psi_j\rangle \langle\psi_j|$$

where $\lambda_j \geq 0$ and $\sum_j \lambda_j = 1$. If ρ is not rank-1, ρ is a convex combination of quantum states $(|\psi_j\rangle \langle\psi_j|)_j$.
$\text{Tr}(\rho \, |\varphi\rangle \langle\varphi|) = 1$ if and only if $\rho = |\varphi\rangle \langle\varphi|$ since
$\text{Tr}(\rho \, |\varphi\rangle \langle\varphi|) = \langle\varphi| \rho |\varphi\rangle = \sum_j \lambda_j |\langle\psi_j|\varphi\rangle|^2$. Assume that $\rho = |\varphi\rangle \langle\varphi|$ and
$\rho = p\rho_1 + (1-p)\rho_2$. Then, $\text{Tr}(\rho_1 |\varphi\rangle \langle\varphi|) = \text{Tr}(\rho_2 |\varphi\rangle \langle\varphi|) = 1$ that
means $\rho_1 = \rho_2 = \rho$. $\qquad\square$

Pure states and state vector

Pure state $|\psi\rangle \langle\psi|$ can be represented by a state vector $|\psi\rangle \in \mathbb{C}^2$ with $\langle\psi|\psi\rangle = 1$.

$|\psi\rangle$ and $|\varphi\rangle := e^{i\theta} |\psi\rangle$ represent the same state since $|\psi\rangle \langle\psi| = |\varphi\rangle \langle\varphi|$.

Inner product of pure states

- ρ is a qubit pure state with a coordinate $[r_X \ r_Y \ r_Z]$.
- σ is a qubit pure state with a coordinate $[-r_X \ -r_Y \ -r_Z]$.

$$\text{Tr}(\rho\sigma) = \text{Tr}(\rho(I - \rho)) = \text{Tr}(\rho) - \text{Tr}(\rho^2) = 1 - 1 = 0$$

- $\rho = |\psi\rangle \langle\psi|$.
- $\sigma = |\varphi\rangle \langle\varphi|$.

$$\text{Tr}(\rho\sigma) = \text{Tr}(|\psi\rangle \langle\psi| \, |\varphi\rangle \langle\varphi|) = \langle\psi|\varphi\rangle \, \text{Tr}(|\psi\rangle \langle\varphi|)$$
$$= \langle\psi|\varphi\rangle \, \langle\varphi|\psi\rangle = |\langle\psi|\varphi\rangle|^2$$

Single qubit measurement

Set of measurements $= \{(e_1, \ldots, e_k) \mid e_1 + \cdots + e_k = I, e_j \in C_{\succeq 0}$
$$i = 1, 2, \ldots, k, \ k = 1, 2, \ldots \}$$

If $e_i e_j = \delta_{i,j} e_i$, the measurement is called an orthogonal measurement.

If $|0\rangle \langle 0|$ is measured by $(|0\rangle \langle 0|, |1\rangle \langle 1|)$, the output is 0 with probability $\mathsf{Tr}(|0\rangle \langle 0| \, |0\rangle \langle 0|) = |\langle 0|0\rangle|^2 = 1$.

If $|+\rangle \langle +|$ is measured by $(|0\rangle \langle 0|, |1\rangle \langle 1|)$, the output is 0 with probability $\mathsf{Tr}(|0\rangle \langle 0| \, |+\rangle \langle +|) = |\langle 0|+\rangle|^2 = 1/2$.

If $|\psi\rangle \langle \psi|$ is measured by $(|\varphi_0\rangle \langle \varphi_0|, |\varphi_1\rangle \langle \varphi_1|)$, the output is 0 with probability $\mathsf{Tr}(|\varphi_0\rangle \langle \varphi_0| \, |\psi\rangle \langle \psi|) = |\langle \varphi_0|\psi\rangle|^2$.

Unitary operation

Unitary operation

$$\rho \mapsto U\rho U^\dagger$$

It is easy to see that

- $\text{Tr}(U\rho U^\dagger) = 1$
- $U\rho U^\dagger \succeq 0$

A pure state $|\psi\rangle$ is mapped to a pure state $U|\psi\rangle$.

U and $e^{i\theta}U$ are physically equivalent.

Examples of unitary operations

- The identity matrix I.

- Pauli matrices X, Y and Z.

- Hadamard matrix $H := \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

- Product UV of unitary operators U and V.

Multiplications of Pauli matrices

For any unitary matrices U and V, UV is also unitary matrix.

- $XY = iZ$

- $YZ = iX$

- $ZX = iY$

Pauli matrices X on the Bloch sphere

$$\rho = \frac{1}{2} \left(I + r_X X + r_Y Y + r_Z Z \right)$$

$$X\rho X^\dagger = X\rho X = \frac{1}{2} \left(X^2 + r_X X^3 + r_Y XYX + r_Z XZX \right)$$

$$= \frac{1}{2} \left(I + r_X X - r_Y Y - r_Z Z \right)$$

$$[r_X \ r_Y \ r_Z] \overset{X}{\longmapsto} [r_X \ -r_Y \ -r_Z]$$

π-rotation with respect to X axis.

Similarly, Y and Z corresponds to π-rotation with respect to Y and Z axes, respectively.

Hadamard matrix

Hadamard matrix H is unitary and Hermitian.

$$H := \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = |+\rangle \langle 0| + |-\rangle \langle 1|$$
$$= |0\rangle \langle +| + |1\rangle \langle -|$$

$$|0\rangle, |1\rangle \xleftrightarrow{H} |+\rangle, |-\rangle$$

$$HXH = H(|+\rangle \langle +| - |-\rangle \langle -|)H$$
$$= |0\rangle \langle 0| - |1\rangle \langle 1| = Z$$

Similarly, $HZH = X$.
$HYH = H(iXZ)H = iHXHHZH = iZX = -Y$

Hadamard matrix on the Bloch sphere

$$\rho = \frac{1}{2}\left(I + r_X X + r_Y Y + r_Z Z\right)$$

$$H\rho H^\dagger = H\rho H = \frac{1}{2}\left(H^2 + r_X HXH + r_Y HYH + r_Z HZH\right)$$

$$= \frac{1}{2}\left(I + r_X Z - r_Y Y + r_Z X\right)$$

$$[r_X \ r_Y \ r_Z] \overset{H}{\longmapsto} [r_Z \ -r_Y \ r_X]$$

Hadamard operation can be decomposed to $\pi/2$-rotation with respect to Y axis

$$[r_X \ r_Y \ r_Z] \overset{R_Y(\pi/2)}{\longmapsto} [r_Z \ r_Y \ -r_X]$$

and X.

Rotation matrices

$$R_X(\theta) := \cos\frac{\theta}{2}I - i\sin\frac{\theta}{2}X$$

$$R_X(\theta)^\dagger = R_X(-\theta)$$

$$R_X(\theta)R_X(\theta)^\dagger = (\cos\frac{\theta}{2}I - i\sin\frac{\theta}{2}X)(\cos\frac{\theta}{2}I + i\sin\frac{\theta}{2}X)$$

$$= \cos^2\frac{\theta}{2}I + \sin^2\frac{\theta}{2}X^2 = I$$

$$R_X(\theta)X = XR_X(\theta), \quad R_X(\theta)Y = YR_X(-\theta), \quad R_X(\theta)Z = ZR_X(-\theta)$$

$$R_X(\theta)R_X(\tau) = R_X(\theta + \tau)$$

$$[1\ 0\ 0] \overset{R_X(\theta)}{\longmapsto} [1\ 0\ 0]$$

$$[0\ 1\ 0] \overset{R_X(\theta)}{\longmapsto} [0\ \cos\theta\ \sin\theta]$$

$$[0\ 0\ 1] \overset{R_X(\theta)}{\longmapsto} [0\ -\sin\theta\ \cos\theta]$$

Matrix function

Definition
For $f \colon \mathbb{C} \to \mathbb{C}$ and orthonormal basis $(|\psi_j\rangle)_j$ of $\mathcal{X}$,

$$f\left(\sum_j \lambda_j |\psi_j\rangle \langle\psi_j|\right) := \sum_j f(\lambda_j) |\psi_j\rangle \langle\psi_j|.$$

For $H \in \mathcal{H}(\mathcal{X})$, $\exp(iH)$ is unitary.

Since the radius of convergense of exp at 0 is infinity,

$$\exp(A) = \sum_{j=0}^{\infty} \frac{1}{j!} A^j.$$

Assignments

- For $a, b, c, d \in \mathbb{R}$, show $r_I, r_X, r_Y, r_Z \in \mathbb{R}$ satisfying
$$\begin{bmatrix} a & b + ci \\ b - ci & d \end{bmatrix} = r_I I + r_X X + r_Y Y + r_Z Z.$$

- Show a matrix representation of $\exp\left(-i\frac{\theta}{2}X\right)$ for $\theta \in \mathbb{R}$.

- [Advanced] For $a_I, a_X, a_Y, a_Z \in \mathbb{R}$, represents
$$\exp\left(i(a_I I + a_X X + a_Y Y + a_Z Z)\right)$$
by a linear combination of I, X, Y and Z. A summation with infinite number of terms is not allowed.