# Introduction to quantum thoery: Quantum states and quantum measurements

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# What is “Quantum theory”?

Quantum theory

- Physics for microscopic phenomena, e.g., atoms, light.

Why is quantum theory important ?

- Just because it’s reality.

- Because it gives more efficient information processing, e.g., quantum factoring algorithm, quantum secret-key sharing, etc.

## On this course

We study mathematical foundation of quantum theory.

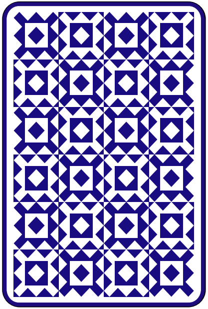
- Mathematical foundation of quantum physics
- Quantum algorithms
- Other quantum information processing, e.g, quantum communication, quantum error-correction.

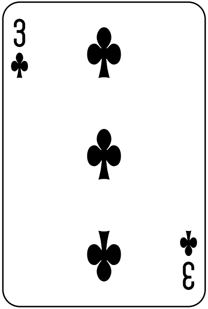
Score

- Assignments: 100%

`https://github.com/QuantumComputationQuantumInformation/slides2020`

# Experimental facts





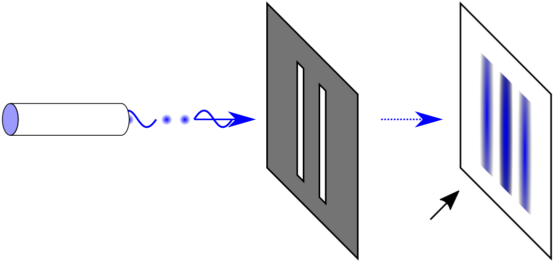

- ✗ The card was the three of club.
- ✓ When we flip the card, we see the three of club.
- State: The card
- Measurement: Flipping the card
- The number on the card before the flipping the card cannot be defined in quantum theory

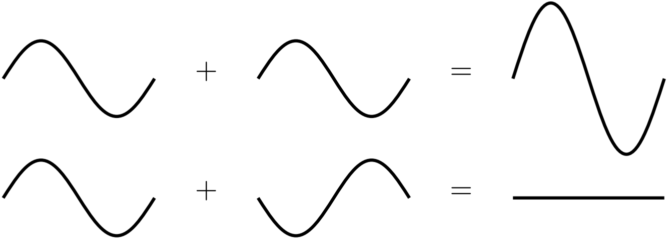
# States, measurements and distinguishability

- Two states are equivalent if the probability distributions of outcomes are equal for arbitrary measurement.
- Two measurements are equivalent if the probability distributions of outcomes are equal for arbitrary states.
- We should not distinguish states (measurements) methematically if we cannot distinguish them physically.

# Light is wave (1801)

Thomas Young
(1773 – 1829)

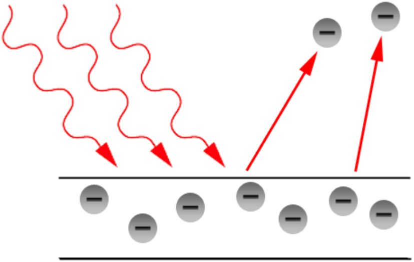




https://en.wikipedia.org/wiki/Young's_interference_experiment

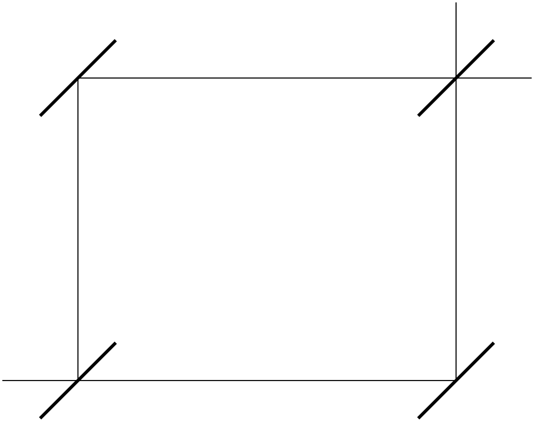
# Light is particle (1905)

Albert Einstein
(1879 – 1955)



https://en.wikipedia.org/wiki/File:Photoelectric_effect.png

# Mach–Zehnder interferometer



# Quantum states and quantum measurements

A single photon $\Rightarrow$ BS1 $\Rightarrow$ BS2 $\Rightarrow$ detection

| State | Measurement |
|---|---|
| A single photon $\Rightarrow$ BS1 $\Rightarrow$ BS2 | detection |
| A single photon $\Rightarrow$ BS1 | BS2 $\Rightarrow$ detection |
| A single photon | BS1 $\Rightarrow$ BS2 $\Rightarrow$ detection |

All understandings are valid

# Mathematical representations of states and measurements

How "States" and "Measurements" are represented mathematically ?

A table of probabilities of outcome 'YES' for each binary measurment on each state

| | Measurement 1 | Measurement 2 | $\cdots$ |
|---|---|---|---|
| State A | $p_{A1}$ | $p_{A2}$ | $\cdots$ |
| State B | $p_{B1}$ | $p_{B2}$ | $\cdots$ |
| $\vdots$ | | | |

* The number of states and measurements are not necessarily countable.

# Linear space

| | Measurement 1 | Measurement 2 | $\cdots$ |
|---|---|---|---|
| State A | $p_{A1}$ | $p_{A2}$ | $\cdots$ |
| State B | $p_{B1}$ | $p_{B2}$ | $\cdots$ |
| State C | $0.7p_{A1} + 0.3p_{B1}$ | $0.7p_{A2} + 0.3p_{B2}$ | |

## Assumption

- Probabilistic mixture of states is also state.
- Probabilistic mixture of binary measurement is also binary measurement.

States and measurements can be represented by vectors!

# Classical theory

States: $\underline{0}$, $\underline{1}$
Binary measurements: $\underline{0}?$, $\underline{1}?$

$$\underline{0}?(\underline{0}) = 1, \qquad \underline{0}?(\underline{1}) = 0,$$
$$\underline{1}?(\underline{0}) = 0, \qquad \underline{1}?(\underline{1}) = 1$$

| | $\underline{0}?$ | $\underline{1}?$ |
|---|---|---|
| $\underline{0}$ | 1 | 0 |
| $\underline{1}$ | 0 | 1 |

## State and measurement

| | $\underline{0}?$ | $\underline{1}?$ |
|---|---|---|
| $\underline{0}$ | 1 | 0 |
| $\underline{1}$ | 0 | 1 |

$S := \underline{0}$ with probability $p$, $\underline{1}$ with probability $1-p$.
$S$ is also regarded as a state.

$$\underline{0}?(S) = p, \qquad\qquad \underline{1}?(S) = 1-p.$$

Similarly,
$E_1 := \underline{0}?$ with probability $p$, $\underline{1}?$ with probability $1-p$.
$E_2 := (\underline{0} \text{ or } \underline{1})?$.
$E_1$ and $E_2$ are also regarded as a binary measurement.

# Linear space

$$\omega_{\underline{0}} := \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad \omega_{\underline{1}} := \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$e_{\underline{0}} := \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad e_{\underline{1}} := \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\underline{0}?(\underline{0}) = \langle e_{\underline{0}}, \omega_{\underline{0}} \rangle, \qquad \underline{0}?(\underline{1}) = \langle e_{\underline{0}}, \omega_{\underline{1}} \rangle$$

$$\underline{1}?(\underline{0}) = \langle e_{\underline{1}}, \omega_{\underline{0}} \rangle, \qquad \underline{1}?(\underline{1}) = \langle e_{\underline{1}}, \omega_{\underline{1}} \rangle$$

$S := \underline{0}$ with probability $p$, $\underline{1}$ with probability $1 - p$

$$\omega_S = p\omega_{\underline{0}} + (1-p)\omega_{\underline{1}} = \begin{bmatrix} p \\ 1-p \end{bmatrix}.$$

$$\underline{0}?(S) = p = \langle e_{\underline{0}}, \omega_S \rangle.$$

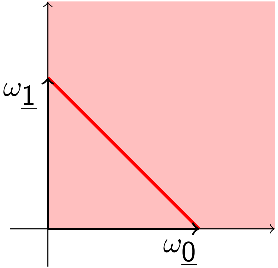
# States and measurements in a linear space

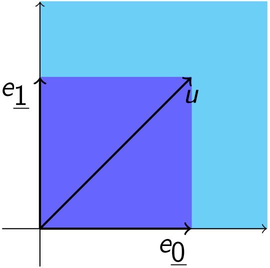
$$\omega_{\underline{0}} := \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad \omega_{\underline{1}} := \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$e_{\underline{0}} := \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad e_{\underline{1}} := \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\text{Set of states} = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \mid x \geq 0, y \geq 0, x + y = 1 \right\}.$$

$$\text{Set of binary measurements} = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \mid x \geq 0, y \geq 0, x \leq 1, y \leq 1 \right\}.$$



# State and measurement in a linear space

$$\text{Set of states} = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \mid x \geq 0, y \geq 0, x + y = 1 \right\}.$$

$$\text{Set of binary measurements} = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \mid x \geq 0, y \geq 0, x \leq 1, y \leq 1 \right\}.$$

Let $C_{\geq 0}$ be the set of nonnegative vectors and $u := \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

$$\text{Set of states} = \left\{ \omega \in \mathbb{R}^2 \mid \omega \in C_{\geq 0}, \langle u, \omega \rangle = 1 \right\}.$$

$$\text{Set of binary measurements} = \left\{ e \in \mathbb{R}^2 \mid e \in C_{\geq 0}, u - e \in C_{\geq 0} \right\}.$$

$$\begin{aligned} \text{Set of measurements} = \{(e_1, \dots, e_k) \mid e_1 + \cdots + e_k = u,\ e_i \in C_{\geq 0} \\ i = 1, 2, \dots, k,\ k = 1, 2, \dots \} \end{aligned}$$

Outcome of the measurement $M = (e_1, \dots, e_k)$ on $\omega$ is $i$ with probability $\langle e_i, \omega \rangle$.

# Quantum theory

$C_{\geq 0} \subseteq \mathbb{R}^2$ : the set of nonnegative vectors, $u := \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

$$\text{Set of states} = \left\{\omega \in \mathbb{R}^2 \mid \omega \in C_{\geq 0}, \langle u, \omega \rangle = 1 \right\}.$$

$$\text{Set of binary measurements} = \left\{e \in \mathbb{R}^2 \mid e \in C_{\geq 0}, u - e \in C_{\geq 0} \right\}.$$

$$\text{Set of measurements} = \{(e_1, \dots, e_k) \mid e_1 + \cdots + e_k = u,\ e_i \in C_{\geq 0} \\ i = 1, 2, \dots, k,\ k = 1, 2, \dots, \}$$

$V$ : the linear space on $\mathbb{R}$ spanned by $2 \times 2$ Hermitian matrices.
$\langle e, \omega \rangle := \mathrm{Tr}(e\omega)$ for $\omega, e \in V$ (Hilbert-Schmidt inner product).
$C_{\succeq 0} \subseteq V$ : the set of positive semidefinite matrices, $u := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

$$\text{Set of states} = \{\omega \in V \mid \omega \in C_{\succeq 0}, \langle u, \omega \rangle = 1\}.$$

$$\text{Set of binary measurements} = \{e \in V \mid e \in C_{\succeq 0}, u - e \in C_{\succeq 0}\}.$$

# Linear space spanned by 2x2 Hermitian matrices

Basis

$$A := \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, B := \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, C := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, D := \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$

Another choice of basis

$$I := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, X := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, Y := \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, Z := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Both are orthogonal basis.
The second basis ($I$ and Pauli matrices $X$, $Y$ and $Z$) has nice properties.

1. $\mathrm{Tr}(I) = 2$. $\mathrm{Tr}(X) = \mathrm{Tr}(Y) = \mathrm{Tr}(Z) = 0$.
2. $X^2 = Y^2 = Z^2 = I$ ($X$, $Y$ and $Z$ have eigenvalues $\pm 1$).
3. $XY = -YX$, $YZ = -ZY$, $ZX = -XZ$.

# Positive semidefinite cone

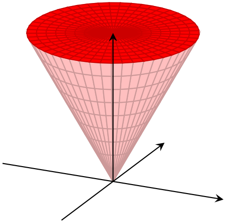
$$I := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, X := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, Y := \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, Z := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

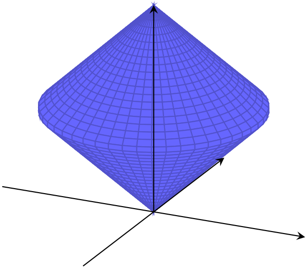
$$\rho = \frac{1}{\sqrt{2}} (a_I I + a_X X + a_Y Y + a_Z Z)$$

$$\begin{aligned} \lambda_1 \geq 0,\ \lambda_2 \geq 0 &\iff \lambda_1 + \lambda_2 \geq 0,\ \lambda_1 \lambda_2 \geq 0 \\ &\iff \mathrm{Tr}(\rho) \geq 0,\ \mathrm{Tr}(\rho)^2 - \mathrm{Tr}(\rho^2) \geq 0 \\ &\iff a_I \geq 0,\ 2a_I^2 - (a_I^2 + a_X^2 + a_Y^2 + a_Z^2) \geq 0 \\ &\iff a_I \geq 0,\ a_I^2 \geq a_X^2 + a_Y^2 + a_Z^2 \end{aligned}$$

$$\mathrm{Tr}(\rho) = 1 \iff a_I = \frac{1}{\sqrt{2}}$$

# Geometry of quantum states and effects





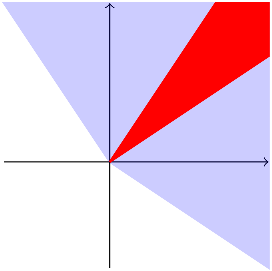
## Convex cone and dual cone

$$C \subseteq V \text{ is a convex cone} \iff x + y \in C,\ \lambda x \in C,$$
$$\forall x \in C, y \in C,\ \lambda \geq 0$$

Proper cone: closed, not $V$, full-dimensional.

$$C^* \subseteq V \text{ is a dual cone of } C$$
$$\iff C^* := \{x \in V \mid \langle x, y \rangle \geq 0,\ \forall y \in C\}$$

$C_{\geq 0}$ and $C_{\succeq 0}$ are self-dual cones.



## Generalized probabilistic theories

$C$ : convex cone.
$u \in$ interior of $C^*$.

$$\text{Set of states} = \{\omega \in V \mid \omega \in C, \langle u, \omega \rangle = 1\}.$$

$$\text{Set of effects} = \{e \in V \mid e \in C^*, u - e \in C^*\}.$$

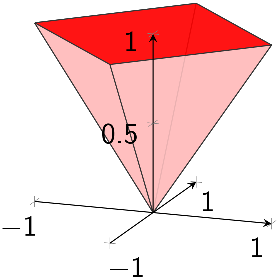
$$\text{Set of measurements} = \{(e_1, \dots, e_k) \mid e_1 + \cdots + e_k = u, e_i \in C^* \\ i = 1, 2, \dots, k,\ k = 1, 2, \dots\}$$

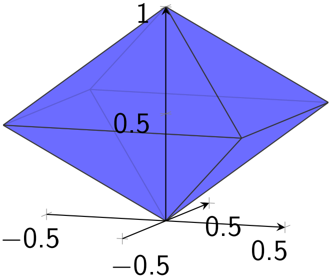
Classical theory
$V = \mathbb{R}^n,\ C = C_{\geq 0},\ u =$ the all-1 vector.

Quantum theory
$V =$ A set of $n \times n$ Hermitian matrices, $C = C_{\succeq 0},\ u = I$.

Other theories ?

## Toy theory: Gbit




$$\omega_0 = \begin{bmatrix}1 & 0 & 1\end{bmatrix}, \qquad \omega_1 = \begin{bmatrix}0 & 1 & 1\end{bmatrix},$$
$$\omega_2 = \begin{bmatrix}-1 & 0 & 1\end{bmatrix}, \qquad \omega_3 = \begin{bmatrix}0 & -1 & 1\end{bmatrix}.$$

$$e_0 = \frac{1}{2}\begin{bmatrix}1 & 1 & 1\end{bmatrix}, \qquad e_1 = \frac{1}{2}\begin{bmatrix}-1 & 1 & 1\end{bmatrix},$$
$$e_2 = \frac{1}{2}\begin{bmatrix}1 & -1 & 1\end{bmatrix}, \qquad e_3 = \frac{1}{2}\begin{bmatrix}-1 & -1 & 1\end{bmatrix},$$
$$u = \begin{bmatrix}0 & 0 & 1\end{bmatrix}.$$

# Nonlocality

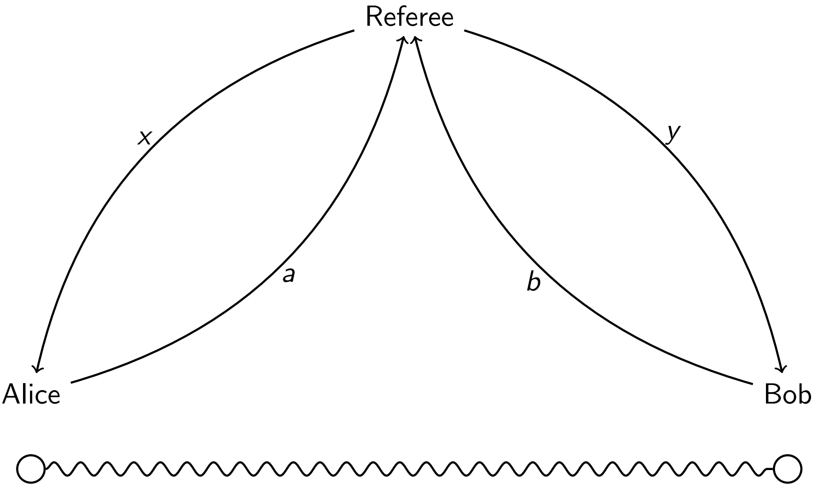
OK, generalized probabilistic theory is quite simple and easy to understand.

But, what is essential difference between classical theory and quantum theory ?

Can classical theory “simulates” or “explains” quantum theory ?

Are Hermitian positive-semidefinite matrices really needed for explaining reality ?

# Bell test: CHSH game (1964, 1969)




Alice and Bob win iff $a \oplus b = x \wedge y$.

# Bell inequality

$a_x$: Output of Alice for given $x$.
$b_y$: Output of Bbob for given $y$.

$$a_0 \oplus b_0 = 0$$
$$a_1 \oplus b_0 = 0$$
$$a_0 \oplus b_1 = 0$$
$$a_1 \oplus b_1 = 1$$

By adding all equations, we get $0 = 1$, which means there is no solution. Hence, the winning probability 1 cannot be achieved.

Three equalities can be satisfied, so that the largest winning probability is $3/4$ (Bell inequality or CHSH inequality).

If Alice and Bob share quantum states, then the largest winning probability is $(2+\sqrt{2})/4 \approx 0.854$ (Violation of Bell/CHSH inequality)

# Locality (Hidden variable model)

Joint preparation and independent measurements.

Probability distribution $P(a, b \mid x, y)$ is said to be **local** if

$$P(a, b \mid x, y) = \sum_{\lambda} P(\lambda) P(a \mid x, \lambda) P(b \mid y, \lambda).$$

Quantum physics allow nonlocal behaviors.

## Summary

- Let $V$ be the linear space spanned by $n \times n$ Hermitian matrices. Let $C_{\succeq 0}$ be the set of PSD matrices. Let $I$ be the $n \times n$ identity matrix. Let $\langle e, \omega \rangle := \mathrm{Tr}(e\omega)$ for $\omega, e \in V$ (Hilbert-Schmidt inner product).

$$\text{Set of states} = \{\omega \in V \mid \omega \in C_{\succeq 0}, \langle u, \omega \rangle = 1\}.$$
$$\text{Set of binary measurements} = \{e \in V \mid e \in C_{\succeq 0}, u - e \in C_{\succeq 0}\}.$$

- Classical theory and quantum theory are special cases of generalized probabilistic theories.

- Violation of Bell (CHSH) inequality show that quantum theory is essentially different from classical theory (Arguments on joint system is needed).

# Assignments

1. Show the dimension and one of the basis of the real linear space spanned by $n \times n$ Hermitian matrices.
2. Show that $XY = -YX$, $YZ = -ZY$ and $ZX = -XZ$.
3. Show that the Hilbert–Schmidt inner product satisfies the axioms of inner product.
4. [Advanced] Show that $C_{\succeq 0}$ is a self-dual cone.