

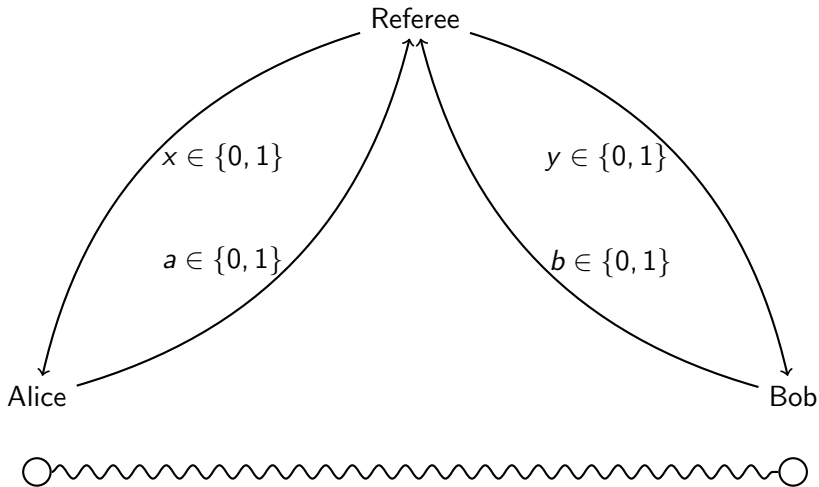
# Nonlocality and Tsirelson's bound

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## Bell test: CHSH game (1964, 1969)



Alice and Bob win iff  $a \oplus b = x \wedge y$ .

## Bell inequality

$a_x$ : Output of Alice for given  $x$ .

$b_y$ : Output of Bob for given  $y$ .

$$a_0 \oplus b_0 = 0$$

$$a_1 \oplus b_0 = 0$$

$$a_0 \oplus b_1 = 0$$

$$a_1 \oplus b_1 = 1$$

By adding all equations, we get  $0 = 1$ , which means there is no solution. Hence, the winning probability 1 cannot be achieved.

Three equalities can be satisfied, so that the largest winning probability is  $3/4$  (Bell inequality or CHSH inequality).

If Alice and Bob share quantum states, then the largest winning probability is  $(2 + \sqrt{2})/4 \approx 0.854$  (Violation of Bell/CHSH inequality)

## Locality (Hidden variable model)

Joint preparation and independent measurements.

Probability distribution  $P(a, b \mid x, y)$  is said to be **local** if

$$P(a, b \mid x, y) = \sum_{\lambda} P(\lambda) P(a \mid x, \lambda) P(b \mid y, \lambda).$$

Quantum physics allow **nonlocal** behaviors.

## Joint probability distribuion

### Lemma

*There exists probability distributions  $P(\lambda)$ ,  $P(a \mid x, \lambda)$  and  $P(b \mid y, \lambda)$  such that*

$$P(a, b \mid x, y) = \sum_{\lambda} P(\lambda) P(a \mid x, \lambda) P(b \mid y, \lambda)$$

*if and only if there exists probability distribution  $q(a_0, a_1, b_0, b_1)$  such that*

$$P(a, b \mid x, y) = \sum_{\substack{a_0, a_1, b_0, b_1 \\ a_x = a, b_y = b}} q(a_0, a_1, b_0, b_1).$$

### Proof.

$$\begin{aligned} (\Rightarrow) \quad q(a_0, a_1, b_0, b_1) &:= \sum_{\lambda} P(\lambda) P(a_0 \mid x = 0, \lambda) P(a_1 \mid x = 1, \lambda) \\ &\quad \cdot P(b_0 \mid y = 0, \lambda) P(b_1 \mid y = 1, \lambda) \end{aligned}$$

$$\begin{aligned} (\Leftarrow) \quad \lambda &= (a_0, a_1, b_0, b_1), \quad P(\lambda) = q(a_0, a_1, b_0, b_1) \\ P(a_x \mid x, (a_0, a_1, b_0, b_1)) &= 1 \end{aligned}$$

## Randomness doesn't help

$$\begin{aligned} & \mathbb{E}_{x,y} [\mathbb{E}_{a_0,a_1,b_0,b_1} [\mathbb{I}\{a_x \oplus b_y = x \wedge y\}]] \\ &= \mathbb{E}_{a_0,a_1,b_0,b_1} [\mathbb{E}_{x,y} [\mathbb{I}\{a_x \oplus b_y = x \wedge y\}]] . \end{aligned}$$

There exists  $a_0^*, a_1^*, b_0^*, b_1^*$  such that

$$\mathbb{E}_{a_0,a_1,b_0,b_1} [\mathbb{E}_{x,y} [\mathbb{I}\{a_x \oplus b_y = x \wedge y\}]] \leq \mathbb{E}_{x,y} [\mathbb{I}\{a_x^* \oplus b_y^* = x \wedge y\}] .$$

## Einstein–Podolsky–Rosen (EPR) paradox (1935)

$$P(a, b \mid x, y) = \sum_{\lambda} P(\lambda) P(a \mid x, \lambda) P(b \mid y, \lambda).$$

$\iff$  there exists a joint distribution of  $(a_0, a_1, b_0, b_1)$ .



In quantum physics,  $a_0, a_1, b_0, b_1$  **cannot exists** simultaneously.



In quantum physics, position and momentum **cannot exists** simultaneously.

## Bell state

Bell state

$$|\Psi\rangle := \frac{1}{\sqrt{2}}(|0\rangle|0\rangle + |1\rangle|1\rangle)$$

Let  $|\psi_\theta\rangle := \cos\theta|0\rangle + \sin\theta|1\rangle$ .

Alice measure this state by  $\{|\psi_{\theta_A}\rangle, |\psi_{\theta_A+\pi/2}\rangle\}$ .

Bob measure this state by  $\{|\psi_{\theta_B}\rangle, |\psi_{\theta_B+\pi/2}\rangle\}$ .

A outcome corresponding to  $|\psi_\theta\rangle|\psi_\tau\rangle$  is obtained with probability

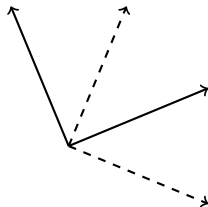
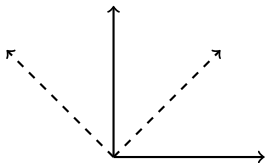
$$\begin{aligned} |\langle\psi_\theta|\langle\psi_\tau|\Psi\rangle|^2 &= \left| \langle\psi_\theta|\langle\psi_\tau| \frac{1}{\sqrt{2}}(|0\rangle|0\rangle + |1\rangle|1\rangle) \right|^2 \\ &= \frac{1}{2} |\cos\theta\cos\tau + \sin\theta\sin\tau|^2 = \frac{1}{2} \cos^2(\theta - \tau). \end{aligned}$$

Another proof:

$$\begin{aligned} \langle\psi_\theta|\langle\psi_\tau|\Psi\rangle &= \text{Tr}(\mathcal{M}(|\psi_\theta\rangle\langle\psi_\tau|)^\dagger \mathcal{M}(|\Psi\rangle\langle\Psi|)) \\ &= \text{Tr}\left(|\psi_\tau\rangle\langle\psi_\theta| \frac{1}{\sqrt{2}}I\right) = \frac{1}{\sqrt{2}} \langle\psi_\theta|\psi_\tau\rangle \end{aligned}$$



## Quantum strategy

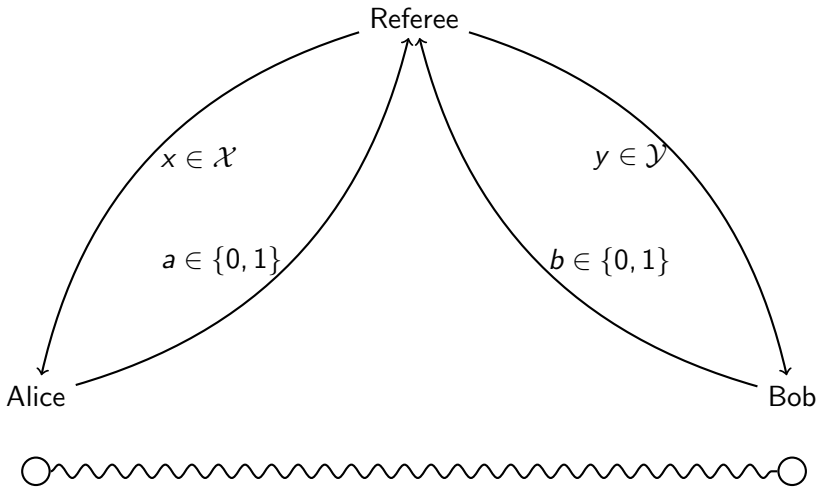


$$\theta_A^{x=0} = 0, \quad \theta_A^{x=1} = \pi/4, \quad \theta_B^{y=0} = \pi/8, \quad \theta_B^{y=1} = -\pi/8$$

For any  $x \in \{0, 1\}, y \in \{0, 1\}$ , the winning probability is

$$\cos^2\left(\frac{\pi}{8}\right) = \frac{2 + \sqrt{2}}{4} \approx 0.854.$$

## XOR game



Alice and Bob win iff  $a \oplus b = f(x, y)$ .

## Tsirelson's theorem

The maximum quantum winning probability is

$$\max: \quad \sum_{x,y} p(x,y) \sum_{\substack{a,b \\ a \oplus b = f(x,y)}} \text{Tr}(\rho(P_a^{(x)} \otimes Q_b^{(y)}))$$

subject to:  $n \in \mathbb{N}$

$$\rho \in \mathcal{H}(\mathbb{C}^n \otimes \mathbb{C}^n)$$

$$\rho \succeq 0$$

$$\text{Tr}(\rho) = 1$$

$$P_a^{(x)}, Q_b^{(y)} \in \mathcal{H}(\mathbb{C}^n) \quad \forall x, y, a, b$$

$$P_a^{(x)} \succeq 0 \quad \forall x, a$$

$$Q_b^{(y)} \succeq 0 \quad \forall y, b$$

$$P_0^{(x)} + P_1^{(x)} = I \quad \forall x$$

$$Q_0^{(y)} + Q_1^{(y)} = I \quad \forall y.$$

## Tsirelson's theorem

The maximum quantum winning probability is

$$\max: \quad \sum_{x,y} p(x,y) \sum_{\substack{a,b \\ a \oplus b = f(x,y)}} \langle \psi | P_a^{(x)} \otimes Q_b^{(y)} | \psi \rangle$$

$$\text{subject to:} \quad n \in \mathbb{N}$$

$$|\psi\rangle \in \mathbb{C}^n \otimes \mathbb{C}^n$$

$$\langle \psi | \psi \rangle = 1$$

$$P_a^{(x)}, Q_b^{(y)} \in \mathcal{H}(\mathbb{C}^n) \quad \forall x, y, a, b$$

$$P_a^{(x)} \succeq 0 \quad \forall x, a$$

$$Q_b^{(y)} \succeq 0 \quad \forall y, b$$

$$P_0^{(x)} + P_1^{(x)} = I \quad \forall x$$

$$Q_0^{(y)} + Q_1^{(y)} = I \quad \forall y.$$

## Binary measurements

### Lemma

$$P_0 \succeq 0, P_1 \succeq 0, P_0 + P_1 = I \iff \exists P, I - P^2 \succeq 0, P_a = \frac{I + (-1)^a P}{2}.$$

### Proof.

$$(\Rightarrow) P = P_0 - P_1.$$

$$(\Leftarrow)$$

$$\begin{aligned} \frac{I + (-1)^a P}{2} &\succeq 0 \\ \frac{I + P}{2} + \frac{I - P}{2} &= I. \end{aligned}$$



## Tsirelson's theorem

By letting  $P^{(x)} := P_0^{(x)} - P_1^{(x)}$  and  $Q^{(y)} := Q_0^{(y)} - Q_1^{(y)}$ , the maximum quantum winning probability is

$$\max: \sum_{x,y} p(x,y) \sum_{\substack{a,b \\ a \oplus b = f(x,y)}} \langle \psi | \frac{I + (-1)^a P^{(x)}}{2} \otimes \frac{I + (-1)^b Q^{(y)}}{2} | \psi \rangle$$

subject to:  $n \in \mathbb{N}$

$$|\psi\rangle \in \mathbb{C}^n \otimes \mathbb{C}^n$$

$$\langle \psi | \psi \rangle = 1$$

$$P^{(x)}, Q^{(y)} \in \mathcal{H}(\mathbb{C}^n) \quad \forall x, y$$

$$I - (P^{(x)})^2 \succeq 0 \quad \forall x$$

$$I - (Q^{(y)})^2 \succeq 0 \quad \forall y$$

## Tsirelson's theorem

The maximum quantum winning probability is

$$\max: \sum_{x,y,a,b} p(x,y) \langle \psi | \frac{I + (-1)^a P^{(x)}}{2} \otimes \frac{I + (-1)^b Q^{(y)}}{2} | \psi \rangle \frac{1 + (-1)^{a+b+f(x,y)}}{2}$$

$$\text{s.t.: } n \in \mathbb{N}$$

$$|\psi\rangle \in \mathbb{C}^n \otimes \mathbb{C}^n$$

$$\langle \psi | \psi \rangle = 1$$

$$P^{(x)}, Q^{(y)} \in \mathcal{H}(\mathbb{C}^n) \quad \forall x, y$$

$$I - (P^{(x)})^2 \succeq 0 \quad \forall x$$

$$I - (Q^{(y)})^2 \succeq 0 \quad \forall y$$

## Tsirelson's theorem

The maximum quantum winning probability is

$$\max: \quad \frac{1}{2} \left( 1 + \sum_{x,y} p(x,y) \langle \psi | P^{(x)} \otimes Q^{(y)} | \psi \rangle (-1)^{f(x,y)} \right)$$

subject to:  $n \in \mathbb{N}$

$$|\psi\rangle \in \mathbb{C}^n \otimes \mathbb{C}^n$$

$$\langle \psi | \psi \rangle = 1$$

$$P^{(x)}, Q^{(y)} \in \mathcal{H}(\mathbb{C}^n) \quad \forall x, y$$

$$I - (P^{(x)})^2 \succeq 0 \quad \forall x$$

$$I - (Q^{(y)})^2 \succeq 0 \quad \forall y$$



# Convexity and extremal points

## Lemma

A set

$$C := \{P \in \mathcal{H}(\mathbb{C}^n) \mid I - P^2 \succeq 0\}$$

is a convex set.  $P \in C$  is *extremal* if and only if  $P^2 = I$ .

## Proof.

If  $P$  has an eigenvalue in  $(-1, +1)$ ,  $P$  can be represented by convex combination of two different points in  $C$ .

Assume  $P$  has an eigenvalue in  $\pm 1$ . Let  $|\psi\rangle$  be an eigenvector of  $P$  for an eigenvalue  $\lambda \in \{-1, +1\}$ . If  $P = pP_0 + (1-p)P_1$  for some  $p \in (0, 1)$  and  $P_0, P_1 \in C$ ,  $p\langle\psi|P_0|\psi\rangle + (1-p)\langle\psi|P_1|\psi\rangle = \lambda$ . This means that  $\langle\psi|P_0|\psi\rangle = \langle\psi|P_1|\psi\rangle = \lambda$ . Hence,  $|\psi\rangle$  is also an eigenvector of  $P_0$  and  $P_1$ . All eigenvectors of  $P$  are also eigenvectors of  $P_0$  and  $P_1$  for the same eigenvalue. Hence,  $P_0 = P_1 = P$ .



## Tsirelson's theorem

The maximum quantum winning probability is

$$\max: \quad \frac{1}{2} \left( 1 + \sum_{x,y} p(x,y) \langle \psi | P^{(x)} \otimes Q^{(y)} | \psi \rangle (-1)^{f(x,y)} \right)$$

subject to:  $n \in \mathbb{N}$

$$|\psi\rangle \in \mathbb{C}^n \otimes \mathbb{C}^n$$

$$\langle \psi | \psi \rangle = 1$$

$$P^{(x)}, Q^{(y)} \in \mathcal{H}(\mathbb{C}^n) \quad \forall x, y$$

$$(P^{(x)})^2 = I \quad \forall x$$

$$(Q^{(y)})^2 = I \quad \forall y$$

Here,  $P^{(x)}$  and  $Q^{(y)}$  are **unitary**. Let

$$|A_x\rangle := (P^{(x)} \otimes I) |\psi\rangle$$

$$|B_y\rangle := (I \otimes Q^{(y)}) |\psi\rangle.$$

Then,  $\langle \psi | P^{(x)} \otimes Q^{(y)} | \psi \rangle = \langle A_x | B_y \rangle$ .

## Tsirelson's theorem

The maximum quantum winning probability is **at most (in fact, equal to)**

$$\max: \quad \frac{1}{2} \left( 1 + \sum_{x,y} p(x,y) \text{Re}(\langle A_x | B_y \rangle) (-1)^{f(x,y)} \right)$$

subject to:  $n \in \mathbb{N}$

$$|A_x\rangle, |B_y\rangle \in \mathbb{C}^n \otimes \mathbb{C}^n \quad \forall x, y$$

$$\langle A_x | A_x \rangle = 1 \quad \forall x$$

$$\langle B_y | B_y \rangle = 1 \quad \forall y$$

## Real vectors

For some orthonormal basis  $\{|e_i\rangle\}$ ,  $|A_x\rangle = \sum_{i=1}^{n^2} \alpha_i^{(x)} |e_i\rangle$ ,  
 $|B_y\rangle = \sum_{i=1}^{n^2} \beta_i^{(y)} |e_i\rangle$ . Let

$$v_x := \begin{bmatrix} \operatorname{Re}(\alpha_1^{(x)}) & \operatorname{Im}(\alpha_1^{(x)}) & \operatorname{Re}(\alpha_2^{(x)}) & \cdots & \operatorname{Im}(\alpha_{n^2}^{(x)}) \end{bmatrix} \in \mathbb{R}^{2n^2}$$

$$w_y := \begin{bmatrix} \operatorname{Re}(\beta_1^{(y)}) & \operatorname{Im}(\beta_1^{(y)}) & \operatorname{Re}(\beta_2^{(y)}) & \cdots & \operatorname{Im}(\beta_{n^2}^{(y)}) \end{bmatrix} \in \mathbb{R}^{2n^2}$$

Then,

$$\langle v_x, v_x \rangle = \langle A_x | A_x \rangle$$

$$\langle w_y, w_y \rangle = \langle B_y | B_y \rangle$$

$$\langle v_x, w_y \rangle = \operatorname{Re}(\langle A_x | B_y \rangle).$$

## Tsirelson's theorem [Tsirelson 1980]

The maximum quantum winning probability is **at most (in fact, equal to)**

$$\begin{aligned} \text{max:} \quad & \frac{1}{2} \left( 1 + \sum_{x,y} p(x,y) \langle v_x, w_y \rangle (-1)^{f(x,y)} \right) \\ \text{subject to:} \quad & n \in \mathbb{N} \\ & v_x \in \mathbb{R}^{2n^2} \quad \forall x \\ & w_y \in \mathbb{R}^{2n^2} \quad \forall y \\ & \|v_x\| = 1 \quad \forall x \\ & \|w_y\| = 1 \quad \forall y \end{aligned}$$

Tsirelson's bound

## SDP for Tsirelson's theorem

The maximum quantum winning probability is **at most (in fact, equal to)**

$$\begin{array}{ll} \text{max:} & \frac{1}{2} \left( 1 + \sum_{x,y} p(x,y) C_{x,y} (-1)^{f(x,y)} \right) \\ \text{subject to:} & C \succeq 0 \\ & C_{z,z} = 1 \quad \forall z \in \mathcal{X} \cup \mathcal{Y} \end{array}$$

## Tsirelson's bound [Tsirelson 1980]

$$\max: \quad \frac{1}{2} \left( 1 + \frac{1}{4} \sum_{x,y} \langle v_x, w_y \rangle (-1)^{(x \wedge y)} \right)$$

$$\begin{aligned} \frac{1}{4} \sum_{x,y} \langle v_x, w_y \rangle (-1)^{(x \wedge y)} &= \frac{1}{4} (\langle v_0, w_0 \rangle + \langle v_0, w_1 \rangle + \langle v_1, w_0 \rangle - \langle v_1, w_1 \rangle) \\ &= \frac{1}{4} (\langle v_0, w_0 + w_1 \rangle + \langle v_1, w_0 - w_1 \rangle) \\ &\leq \frac{1}{4} (\|v_0\| \|w_0 + w_1\| + \|v_1\| \|w_0 - w_1\|) \\ &= \frac{1}{4} (\|w_0 + w_1\| + \|w_0 - w_1\|) \\ &\leq \frac{\sqrt{2}}{4} \sqrt{\|w_0 + w_1\|^2 + \|w_0 - w_1\|^2} \\ &= \frac{\sqrt{2}}{4} \sqrt{2\|w_0\|^2 + 2\|w_1\|^2} = \frac{1}{\sqrt{2}} \end{aligned}$$

# Assignments

① Let

$$T_{2i-1} := \underbrace{I \otimes \cdots \otimes I}_{i-1} \otimes X \otimes \underbrace{Y \otimes \cdots \otimes Y}_{d-i} \in \mathcal{H}(\mathbb{C}^{2^d})$$

$$T_{2i} := \underbrace{I \otimes \cdots \otimes I}_{i-1} \otimes Z \otimes \underbrace{Y \otimes \cdots \otimes Y}_{d-i} \in \mathcal{H}(\mathbb{C}^{2^d})$$

for  $i = 1, \dots, d$ . Show that  $T_i T_j = -T_j T_i$  for all  $1 \leq i < j \leq 2d$ .

② For any  $v, w \in \mathbb{R}^{2d}$  satisfying  $\|v\| = \|w\| = 1$ ,

$$P_v := \sum_{i=1}^{2d} v_i T_i \in \mathcal{H}(\mathbb{C}^{2^d}), \quad Q_w := \sum_{i=1}^{2d} w_i T_i^t \in \mathcal{H}(\mathbb{C}^{2^d})$$

where  $^t$  stands for the transposition. Show that  $P_v^2 = Q_w^2 = I$ .

③ [Advanced] Let  $|\psi\rangle := \frac{1}{2^{d/2}} \sum_{i=1}^{2^d} |i\rangle |i\rangle \in \mathbb{C}^{2^{2d}}$ . Show  $\langle \psi | P_v \otimes Q_w | \psi \rangle = \langle v, w \rangle$  for any  $v, w \in \mathbb{R}^{2d}$  satisfying  $\|v\| = \|w\| = 1$ .