

# Universality of quantum circuit

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# Universality of a quantum circuit

## Theorem (Universality of finite gate set)

For any unitary matrix  $U \in L(\mathbb{C}^{2^n})$  and  $\epsilon > 0$ , there is a quantum circuit with  $X, Y, Z, H, S, T, \text{CNOT}$  gates computing  $\tilde{U}$  satisfying  $\|U - \tilde{U}\| < \epsilon$ .

## Proof.

- 1 Any unitary matrix can be decomposed to a product of two-level unitary matrices. Done
- 2 Any two-level unitary matrix can be decomposed to a product of controlled-unitary gates. Done
- 3 Any controlled-unitary gate can be decomposed to a product of CNOT and arbitrary single-qubit gates.
- 4 Any single-qubit gate can be approximated by  $X, Y, Z, H, S$  and  $T$ .

## Special unitary group

- $U(n) :=$  the set of  $n \times n$  unitary matrices.
- $SU(n) :=$   
the set of  $n \times n$  unitary matrices  $U$  with  $\det(U) = 1$ .
- $U(n)$  and  $SU(n)$  are groups.
- For  $U \in SU(n)$  and  $V \in U(n)$ ,  $VUV^\dagger \in SU(n)$ .
- For  $V \in U(n)$  and  $W \in U(n)$ ,  $VWV^\dagger W^\dagger \in SU(n)$ .
- For  $U \in U(n)$ , there exists  $V \in SU(n)$  and  $\theta \in \mathbb{R}$  such that  $U = e^{i\theta} V$ .

# Controlled-unitary

## Theorem

*Any controlled-unitary gate can be decomposed to a product of **CNOT** and arbitrary single-qubit gates.*

## Proof.

- 1 Controlled- $U(2)$  with **single** controlled qubit.
- 2 Controlled- $SU(2)$  with  **$n$**  controlled qubits.
- 3 Controlled- $U(2)$  with  **$n$**  controlled qubits.

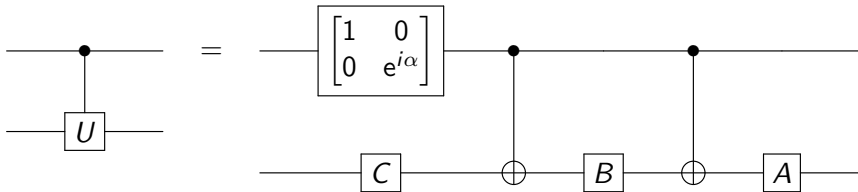


# Decomposition of single qubit unitary

## Lemma

Any single qubit unitary  $U \in \text{U}(2)$ , there is single qubit unitary matrices  $A$ ,  $B$ ,  $C$  such that  $ABC = I$  and  $e^{i\alpha}AXBXC = U$ .

From this lemma,



## Decomposition of single qubit unitary

### Lemma

Any single qubit unitary  $U \in \text{U}(2)$ , there is single qubit unitary matrices  $A, B, C$  and  $\alpha \in \mathbb{R}$  such that  $ABC = I$  and  $e^{i\alpha}AXBXC = U$ .

### Proof.

For any  $U \in \text{U}(2)$ , there exists  $\alpha \in [0, 2\pi)$  and  $V \in \text{SU}(2)$  such that  $U = e^{i\alpha}V$ .

For  $R_Z(\theta) = \begin{bmatrix} e^{-i\frac{\theta}{2}} & 0 \\ 0 & e^{i\frac{\theta}{2}} \end{bmatrix}$ ,  $XR_Z(\theta)XR_Z(-\theta) = R_Z(-2\theta)$ .

For any  $V \in \text{SU}(2)$ , there exists  $\theta \in [0, 2\pi)$  and  $S \in \text{SU}(2)$  such that

$$V = SR_Z(-2\theta)S^\dagger = SXR_Z(\theta)XR_Z(-\theta)S^\dagger.$$

$A = S, B = R_Z(\theta), C = R_Z(-\theta)S^\dagger$  satisfy the conditions. □

# Controlled-unitary

## Theorem

*Any controlled-unitary gate can be decomposed to a product of **CNOT** and arbitrary single-qubit gates.*

## Proof.

- 1 Controlled-**U**(2) with **single** controlled qubit. **Done**
- 2 Controlled-**SU**(2) with  **$n$**  controlled qubits.
- 3 Controlled-**U**(2) with  **$n$**  controlled qubits.



# Group commutator and controlled-unitary

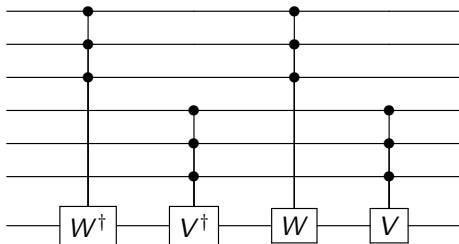
## Theorem

For any  $U \in \text{SU}(2)$ , controlled- $U$  gate with  $n$  controlled qubits can be realized by  $O(n^2)$  CNOT and arbitrary single-qubit gates without ancillas (working qubits).

## Proof.

Induction on  $n$ . For the **group commutator decomposition**

$U = VWV^\dagger W^\dagger$  using  $V = SiXS^\dagger$ ,  $W = SR_Z(\theta)S^\dagger \in \text{SU}(2)$  for some  $\theta \in [0, 2\pi)$  and  $S \in \text{SU}(2)$ .



$$S_n = 4S_{n/2} = 4^{\log n} S_1 = O(n^2).$$





# Controlled-unitary

## Theorem

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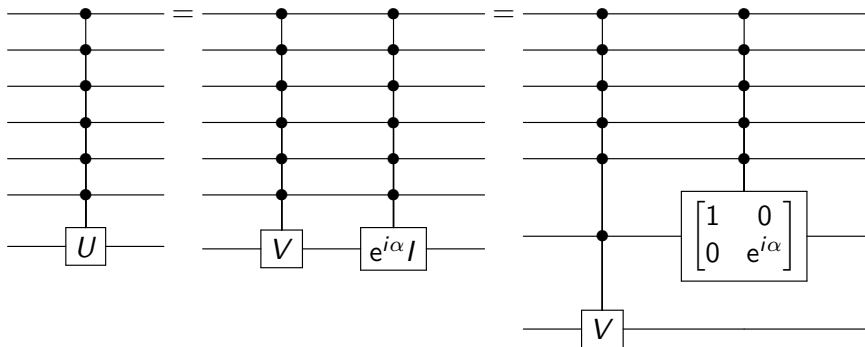
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- 1 Controlled- $U(2)$  with single controlled qubit. Done
- 2 Controlled- $SU(2)$  with  $n$  controlled qubits. Done
- 3 Controlled- $U(2)$  with  $n$  controlled qubits.



## Controlled- $U(2)$ with $n$ controlled qubits

For any  $U \in U(2)$ , there exists  $V \in SU(2)$  and  $\alpha \in \mathbb{R}$  such that  $U = e^{i\alpha} V$ .



$$A_n = S_n + A_{n-1} = O(n^3)$$

# Controlled-unitary

## Theorem

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# Universality of a quantum circuit

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For any unitary matrix  $U \in L(\mathbb{C}^{2^n})$  and  $\epsilon > 0$ , there is a quantum circuit with  $X, Y, Z, H, S, T, \text{CNOT}$  gates computing  $\tilde{U}$  satisfying  $\|U - \tilde{U}\| < \epsilon$ .

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- 4 Any single-qubit gate can be approximated by  $X, Y, Z, H, S$  and  $T$ .

## Approximation of a single-qubit gate is sufficient

### Theorem

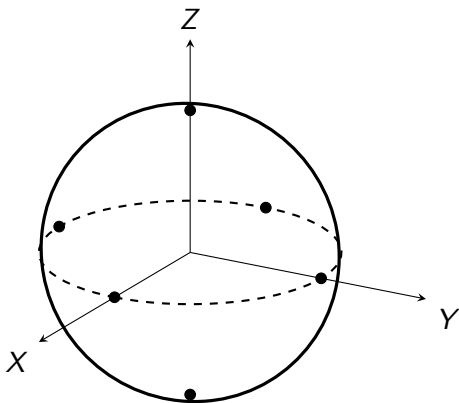
*Any single-qubit gate can be approximated by  $X$ ,  $Y$ ,  $Z$ ,  $H$ ,  $S$  and  $T$ .*

Assume that this theorem holds. For  $A \in L(\mathbb{C}^d)$ , Let  $\|A\|$  be the **spectral norm**, which satisfies  $\|UAV\| = \|A\|$  for any unitary matrices  $U$  and  $V$ .

Assume  $\|U_i - V_i\| \leq \epsilon$  for  $i = 1, \dots, m$ .

$$\begin{aligned} & \|U_m U_{m-1} \cdots U_1 - V_m V_{m-1} \cdots V_1\| \\ &= \left\| \sum_{i=1}^m (U_m \cdots U_i V_{i-1} \cdots V_1 - U_m \cdots U_{i+1} V_i \cdots V_1) \right\| \\ &\leq \sum_{i=1}^m \|U_m \cdots U_i V_{i-1} \cdots V_1 - U_m \cdots U_{i+1} V_i \cdots V_1\| \\ &= \sum_{i=1}^m \|U_m \cdots U_{i+1} (U_i - V_i) V_{i-1} \cdots V_1\| = \sum_{i=1}^m \|U_i - V_i\| \leq m\epsilon. \end{aligned}$$

# Universality of $X, Y, Z, H, S, T$



## Special unitary group and rotation

$$\begin{aligned}\mathrm{SU}(2) \ni U &= \exp\{i(\alpha_X X + \alpha_Y Y + \alpha_Z Z)\} \\&= \sum_{j=0}^{\infty} \frac{i^j}{j!} (\alpha_X X + \alpha_Y Y + \alpha_Z Z)^j \\&= \sum_{j=0}^{\infty} \frac{(-1)^j}{(2j)!} (\alpha_X X + \alpha_Y Y + \alpha_Z Z)^{2j} \\&\quad + i \sum_{j=0}^{\infty} \frac{(-1)^j}{(2j+1)!} (\alpha_X X + \alpha_Y Y + \alpha_Z Z)^{2j+1} \\&= \cos\left(\sqrt{\alpha_X^2 + \alpha_Y^2 + \alpha_Z^2}\right) I \\&\quad + i \sin\left(\sqrt{\alpha_X^2 + \alpha_Y^2 + \alpha_Z^2}\right) \frac{\alpha_X X + \alpha_Y Y + \alpha_Z Z}{\sqrt{\alpha_X^2 + \alpha_Y^2 + \alpha_Z^2}}.\end{aligned}$$

For a real unit vector  $\hat{n} = [n_X \ n_Y \ n_Z]$ , let

$$R_{\hat{n}}(\theta) := \cos \frac{\theta}{2} I - i \sin \frac{\theta}{2} (n_X X + n_Y Y + n_Z Z).$$

For any  $U \in \mathrm{SU}(2)$ , there exist  $\theta \in [0, 2\pi)$  and a real unit three-dimensional vector  $\hat{n}$  such that  $U = R_{\hat{n}}(\theta)$ .

## Universality of $X, Y, Z, H, S, T$

$$T \cong R_Z(\pi/4). \quad HTH \cong R_X(\pi/4).$$

$$\begin{aligned} R_Z(\pi/4)R_X(\pi/4) &= \left[ \cos \frac{\pi}{8} I - i \sin \frac{\pi}{8} Z \right] \left[ \cos \frac{\pi}{8} I - i \sin \frac{\pi}{8} X \right] \\ &= \cos^2 \frac{\pi}{8} I - i \sin \frac{\pi}{8} \left[ \cos \frac{\pi}{8} (X + Z) + \sin \frac{\pi}{8} Y \right] \\ &=: \cos \frac{\eta}{2} I - i \sin \frac{\eta}{2} (n_X X + n_Y Y + n_Z Z) \\ &= R_{\hat{n}}(\eta) \end{aligned}$$

where  $\eta$  satisfying  $\cos(\eta/2) = \cos^2(\pi/8)$  and  $\hat{n}$  is a unit vector along with  $(\cos \frac{\pi}{8}, \sin \frac{\pi}{8}, \cos \frac{\pi}{8})$ . Here,  $\eta$  is an **irrational multiple of  $\pi$** .  $HR_{\hat{n}}(\eta)H = R_{\hat{m}}(\eta)$  where  $\hat{m}$  is a unit vector along with  $(\cos \frac{\pi}{8}, -\sin \frac{\pi}{8}, \cos \frac{\pi}{8})$ .

For any  $U \in \text{SU}(2)$ , there exists  $\beta, \gamma, \delta \in [0, 2\pi)$  such that  $U = R_{\hat{n}}(\beta)R_{\hat{m}}(\gamma)R_{\hat{n}}(\delta)$ .



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## Solovay–Kitaev theorem

### Theorem

*Assume  $\{U_1, \dots, U_k\}$  generates a dense subset of  $SU(2)$ . Then, any  $U \in SU(2)$  can be approximated with error  $\epsilon$  by  $\lceil \log(1/\epsilon) \rceil^c$  multiplications of  $\{U_1, \dots, U_k\}$ .*

## Assignments

- 1 Prove that for any  $U \in \text{SU}(2)$ , there exists  $\beta, \gamma, \delta \in [0, 2\pi)$  such that  $U = R_Z(\beta)R_Y(\gamma)R_Z(\delta)$ .