

Quantum state discrimination and Holevo–Helstrom theorem

Ryuhei Mori

Tokyo Institute of Technology

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Discrimination of probability distributions

Discrimination of probability distributions:

- Input: $x \in \mathcal{X}$ drawn from probability distribution p_0 with probability λ , and from p_1 with probability $1 - \lambda$.
- Output: $i \in \{0, 1\}$ that indicates the probability distribution p_i .

Maximum probability of success: Discrimination of probability distributions

Theorem

The maximum probability of success is equal to

$$\frac{1 + \|\lambda p_0 - (1 - \lambda)p_1\|_1}{2}$$

where $\|a\|_1 := \sum_x |a(x)|$.

Proof.

Let $d: \mathcal{X} \rightarrow \{0, 1\}$ be a discriminator. Assume we get 1 point if succeeds, and loose 1 point if fails. Then, expected point is bounded by

$$\begin{aligned} & \lambda \sum_{x \in \mathcal{X}} p_0(x)(-1)^{d(x)} + (1 - \lambda) \sum_{x \in \mathcal{X}} p_1(x)(-1)^{d(x)+1} \\ &= \sum_{x \in \mathcal{X}} (\lambda p_0(x) - (1 - \lambda)p_1(x))(-1)^{d(x)} \leq \|\lambda p_0 - (1 - \lambda)p_1\|_1. \end{aligned}$$

The equality is achieved by $d(x)$ that is 0 iff $\lambda p_0(x) \geq (1 - \lambda)p_1(x)$. On the other hand, the expected point is $p_{\text{succ}} - p_{\text{fail}} = 2p_{\text{succ}} - 1$. □

Discrimination of quantum states

Discrimination of **quantum** states:

- Input: A **quantum** state ρ_0 is given with probability λ , and ρ_1 is given with probability $1 - \lambda$.
- Output: $i \in \{0, 1\}$ that indicates the given state ρ_i .

Maximum probability of success: Discrimination of quantum states

Theorem (Holevo–Helstrom theorem)

The maximum probability of success is equal to

$$\frac{1 + \|\lambda\rho_0 - (1 - \lambda)\rho_1\|_1}{2}$$

where $\|A\|_1 := \text{Tr}(\sqrt{A^\dagger A})$, which is a sum of the singular values of A .

Proof.

Once a measurement $(P_y)_{y \in \mathcal{Y}}$ is fixed, we get a classical probability distribution $p_0(y) := \text{Tr}(\rho_0 P_y)$ and $p_1(y) := \text{Tr}(\rho_1 P_y)$. In this case, the maximum probability of success is given by

$$\frac{1 + \|\lambda p_0 - (1 - \lambda)p_1\|_1}{2}$$

Hence, it's sufficient to show

$$\max_{(P_y)_{y \in \mathcal{Y}}} \|\lambda p_0 - (1 - \lambda)p_1\|_1 = \|\lambda\rho_0 - (1 - \lambda)\rho_1\|_1.$$

Holevo–Helstrom theorem

$$\begin{aligned}
 \max_{(P_y)_{y \in \mathcal{Y}}} \|\lambda \rho_0 - (1 - \lambda) \rho_1\|_1 &= \max_{(P_y)_{y \in \mathcal{Y}}} \sum_{y \in \mathcal{Y}} |\lambda \rho_0(y) - (1 - \lambda) \rho_1(y)| \\
 &= \max_{(P_y)_{y \in \mathcal{Y}}} \sum_{y \in \mathcal{Y}} |\lambda \text{Tr}(\rho_0 P_y) - (1 - \lambda) \text{Tr}(\rho_1 P_y)| \\
 &= \max_{(P_y)_{y \in \mathcal{Y}}} \sum_{y \in \mathcal{Y}} |\text{Tr}((\lambda \rho_0 - (1 - \lambda) \rho_1) P_y)|
 \end{aligned}$$

Let

$$\lambda \rho_0 - (1 - \lambda) \rho_1 = \sum_{x \in \mathcal{X}} \mu_x |\psi_x\rangle \langle \psi_x|$$

be a spectral decomposition. Then,

$$\begin{aligned}
 \max_{(P_y)_{y \in \mathcal{Y}}} \sum_{y \in \mathcal{Y}} |\text{Tr}((\lambda \rho_0 - (1 - \lambda) \rho_1) P_y)| &= \max_{(P_y)_{y \in \mathcal{Y}}} \sum_{y \in \mathcal{Y}} \left| \sum_{x \in \mathcal{X}} \mu_x \langle \psi_x | P_y | \psi_x \rangle \right| \\
 &\leq \max_{(P_y)_{y \in \mathcal{Y}}} \sum_{y \in \mathcal{Y}} \sum_{x \in \mathcal{X}} |\mu_x| \langle \psi_x | P_y | \psi_x \rangle = \sum_{x \in \mathcal{X}} |\mu_x| = \|\lambda \rho_0 - (1 - \lambda) \rho_1\|_1.
 \end{aligned}$$

The maximum is achieved by $\mathcal{Y} = \mathcal{X}$ and $P_x = |\psi_x\rangle \langle \psi_x|$,
 and $\mathcal{Y} = \{0, 1\}$ and $P_0 = \sum_{x: \mu_x \geq 0} |\psi_x\rangle \langle \psi_x|$, $P_1 = \sum_{x: \mu_x < 0} |\psi_x\rangle \langle \psi_x|$.

Discrimination of pure states

The maximum probability of success for discrimination of $|\psi\rangle\langle\psi|$ and $|\varphi\rangle\langle\varphi|$ is given by

$$\|\lambda |\psi\rangle\langle\psi| - (1 - \lambda) |\varphi\rangle\langle\varphi|\|_1$$

Let $A := \lambda |\psi\rangle\langle\psi| - (1 - \lambda) |\varphi\rangle\langle\varphi|$. Then, A has rank at most **two**. From

$$\mathrm{Tr}(A) = \mu_0 + \mu_1 = \lambda - (1 - \lambda)$$

$$\mathrm{Tr}(A^2) = \mu_0^2 + \mu_1^2 = \lambda^2 + (1 - \lambda)^2 - 2\lambda(1 - \lambda)|\langle\psi|\varphi\rangle|^2,$$

$$2\mu_0\mu_1 = (\mu_0 + \mu_1)^2 - (\mu_0^2 + \mu_1^2) = -2\lambda(1 - \lambda)(1 - |\langle\psi|\varphi\rangle|^2) \leq 0.$$

$$(\mu_0 - \mu_1)^2 = (\mu_0^2 + \mu_1^2) - 2\mu_0\mu_1 = (\lambda + (1 - \lambda))^2 - 4\lambda(1 - \lambda)|\langle\psi|\varphi\rangle|^2$$

$$\|A\|_1 = |\mu_0| + |\mu_1| = |\mu_0 - \mu_1| = \sqrt{1 - 4\lambda(1 - \lambda)|\langle\psi|\varphi\rangle|^2}.$$

Unitary discrimination

Discrimination of **unitary operators**:

- Input: A **unitary operator** U_0 is given with probability λ , and U_1 is given with probability $1 - \lambda$ as an oracle \mathcal{O} that can be used once.
- Output: $i \in \{0, 1\}$ that indicates the given unitary U_i .

Unitary discrimination

If algorithm call the oracle \mathcal{O} on a state $|\psi\rangle$, we get either of $U_0 |\psi\rangle$ or $U_1 |\psi\rangle$.

Then, the maximum probability of success is given by

$$\begin{aligned} & \max_{|\psi\rangle} \|\lambda U_0 |\psi\rangle \langle \psi| U_0^\dagger - (1 - \lambda) U_1 |\psi\rangle \langle \psi| U_1^\dagger\|_1 \\ &= \max_{|\psi\rangle} \sqrt{1 - 4\lambda(1 - \lambda) |\langle \psi| U_0^\dagger U_1 |\psi\rangle|^2} \end{aligned}$$

For $V := U_0^\dagger U_1$, let $V = \sum_x \mu_x |\varphi_x\rangle \langle \varphi_x|$ be a spectral decomposition. Note that $|\mu_x| = 1$ for all x . Let $|\psi\rangle = \sum_x \alpha_x |\varphi_x\rangle$. Then, $\langle \psi| U_0^\dagger U_1 |\psi\rangle = \sum_x |\alpha_x|^2 \mu_x$, which is a convex combination of $(\mu_x)_x$. Let θ_{cover} be the smallest angle that covers all eigenvalues $(\mu_x)_x$. Then, $\min_{|\psi\rangle} |\langle \psi| U_0^\dagger U_1 |\psi\rangle| = \cos(\frac{\theta_{\text{cover}}}{2})$ if $\theta_{\text{cover}} \leq \pi$, and 0 otherwise.

Assignment

- 1 Show the maximum probability of success for discriminating $|0\rangle$ and $|+\rangle$ given with the uniform probability.
- 2 Show a binary optimal measurement for the discrimination of $|0\rangle$ and $|+\rangle$ given with the uniform probability.
- 3 Show the maximum probability of success for discriminating I and $R_Z(\theta)$ given with the uniform probability. Show the input state $|\psi\rangle$ for the oracle as well.