Universality of quantum circuit

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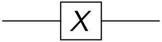
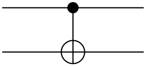
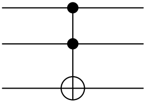
Quantum circuit

- Quantum circuit is a model of computation of Boolean functions which consists of quantum gates.

- Single qubit gate: X gate, Y gate, Z gate, H gate, $S := \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}$ gate $T := \begin{bmatrix} 1 & 0 \\ 0 & \frac{1+i}{\sqrt{2}} \end{bmatrix}$ gate 

- Two qubit gate: CNOT gate 

- Three qubit gate: Toffoli gate 

Spectral norm

For $A \in \mathcal{L}(\mathbb{C}^n, \mathbb{C}^m)$,

$$\|A\| := \max_{|\psi\rangle \in \mathbb{C}^n :\, \langle\psi|\psi\rangle = 1} \sqrt{\langle\psi| A^\dagger A |\psi\rangle}$$

For the singular value decomposition $A = \sum_i \lambda_i |\psi_i\rangle \langle\varphi_i|$,

$$A^\dagger A = \left(\sum_i \lambda_i |\varphi_i\rangle \langle\psi_i|\right) \left(\sum_j \lambda_j |\psi_j\rangle \langle\varphi_j|\right) = \sum_i \lambda_i^2 |\varphi_i\rangle \langle\varphi_i|$$

Hence, $\langle\psi| A^\dagger A |\psi\rangle = \sum_i \lambda_i^2 |\langle\psi|\varphi_i\rangle|^2 \le \max_i \lambda_i^2$

That means $\|A\|$ is the largest singular value of A.

For any unitary matrices $U \in \mathcal{L}(\mathbb{C}^m)$ and $V \in \mathcal{L}(\mathbb{C}^n)$,

$$\|UAV\| = \|A\|.$$

Spectral norm vs diamond norm

The largest probability for discrminating U_0 and U_1 given with probability $1/2$ is $(1+\beta)/2$ where

$$\beta := \frac{1}{2} \max_{|\psi\rangle} \| U_0 |\psi\rangle \langle\psi| U_0^\dagger - U_1 |\psi\rangle \langle\psi| U_1^\dagger \|_1 .$$

Here, $\beta = \sin\left(\frac{\theta_{\text{cover}}}{2}\right)$ if $\theta_{\text{cover}} \leq \pi$, and 1 otherwise.

$$\begin{aligned} \| U_0 - U_1 \| = \| I - U_0^\dagger U_1 \| &= \max_j |1 - e^{i\theta_j}| \\ &\geq \min_\theta \max_j |e^{i\theta} - e^{i\theta_j}| \\ &= 2 \sin\left(\frac{\theta_{\text{cover}}}{4}\right). \end{aligned}$$

If $\theta_{\text{cover}} \leq \pi$, $2\sin(\theta_{\text{cover}}/4) \geq \sin(\theta_{\text{cover}}/2)$. If $\theta_{\text{cover}} > \pi$, $2\sin(\theta_{\text{cover}}/4) \geq 1$. Hence, $\| U_0 - U_1 \| \geq \beta$.

Universality of a quantum circuit

Theorem (Universality of finite gate set)

For any unitary matrix $U \in \mathcal{L}(\mathbb{C}^{2^n})$ and $\epsilon > 0$, there is a quantum circuit with X, Y, Z, H, S, T, CNOT gates computing $\widetilde{U}$ satisfying $\|U - \widetilde{U}\| < \epsilon$.

Proof.

1. Any unitary matrix can be decomposed to a product of two-level unitary matrices.
2. Any two-level unitary matrix can be decomposed to a product of controlled-unitary gates.
3. Any controlled-untary gate can be decomposed to a product of CNOT and arbitrary single-qubit gates.
4. Any single-qubit gate can be approximated by X, Y, Z, H, S and T.

Two-level unitary matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & u_{1,1} & 0 & 0 & 0 & 0 & u_{1,2} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & u_{2,1} & 0 & 0 & 0 & 0 & u_{2,2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

There exist $x \neq y \in \{0,1\}^n$ such that

$$\begin{aligned} |x\rangle &\longmapsto u_{1,1}\,|x\rangle + u_{1,2}\,|y\rangle \\ |y\rangle &\longmapsto u_{2,1}\,|x\rangle + u_{2,2}\,|y\rangle \end{aligned}$$

and for any $z \in \{0,1\}^n \setminus \{x, y\}$, $|z\rangle \longmapsto |z\rangle$.

Two-level unitary matrix

Theorem (Decomposition to two-level unitary matrices)

For any unitary matrix $U \in \mathcal{L}(\mathbb{C}^d)$, there is a sequence $U_1, U_2, \ldots, U_m$ of two-level unitary matrices such that $U = U_1 U_2 \cdots U_m$.

Proof.

We will show that there is a sequence $V_1, V_2, \ldots, V_m$ of two-level unitary matrices such that

$$V_m V_{m-1} \cdots V_1 U = I.$$

Since $U_i := V_i^{-1}$ is two-level unitary, this completes a proof. $\square$

Decomposition to two-level unitary matrix 1/3

$$U = \begin{bmatrix} u_{1,1} & u_{1,2} & u_{1,3} & u_{1,4} \\ u_{2,1} & u_{2,2} & u_{2,3} & u_{2,4} \\ u_{3,1} & u_{3,2} & u_{3,3} & u_{3,4} \\ u_{4,1} & u_{4,2} & u_{4,3} & u_{4,4} \end{bmatrix}$$

If $u_{2,1} = 0$, we skip this step.
If $u_{2,1} \neq 0$, apply the two-level unitary matrix

$$V_1 = \frac{1}{z} \begin{bmatrix} u_{1,1}^* & u_{2,1}^* & 0 & 0 \\ u_{2,1} & -u_{1,1} & 0 & 0 \\ 0 & 0 & z & 0 \\ 0 & 0 & 0 & z \end{bmatrix}$$

for $z := \sqrt{|u_{1,1}|^2 + |u_{2,1}|^2}$.

Decomposition to two-level unitary matrix 2/3

$$V_1 U = \begin{bmatrix} u_{1,1} & u_{1,2} & u_{1,3} & u_{1,4} \\ 0 & u_{2,2} & u_{2,3} & u_{2,4} \\ u_{3,1} & u_{3,2} & u_{3,3} & u_{3,4} \\ u_{4,1} & u_{4,2} & u_{4,3} & u_{4,4} \end{bmatrix}$$

$u_{i,j}$s are not equal to those in the previous slide for $i \in \{1,2\}$.

If $u_{3,1} = 0$, we skip this step.
If $u_{3,1} \neq 0$, apply the two-level unitary matrix

$$V_2 = \frac{1}{z} \begin{bmatrix} u_{1,1}^* & 0 & u_{3,1}^* & 0 \\ 0 & z & 0 & 0 \\ u_{3,1} & 0 & -u_{1,1} & 0 \\ 0 & 0 & 0 & z \end{bmatrix}$$

for $z := \sqrt{|u_{1,1}|^2 + |u_{3,1}|^2}$.

Decomposition to two-level unitary matrix 3/3

$$V_3 V_2 V_1 U = \begin{bmatrix} u_{1,1} & u_{1,2} & u_{1,3} & u_{1,4} \\ 0 & u_{2,2} & u_{2,3} & u_{2,4} \\ 0 & u_{3,2} & u_{3,3} & u_{3,4} \\ 0 & u_{4,2} & u_{4,3} & u_{4,4} \end{bmatrix} = \begin{bmatrix} u_{1,1} & 0 & 0 & 0 \\ 0 & u_{2,2} & u_{2,3} & u_{2,4} \\ 0 & u_{3,2} & u_{3,3} & u_{3,4} \\ 0 & u_{4,2} & u_{4,3} & u_{4,4} \end{bmatrix}$$

$u_{1,1} = 1$ unless $u_{2,1}$, $u_{3,1}$, $u_{4,1}$ are originally 0. In this case, apply one-level unitary for making $u_{1,1} = 1$.

Arbitrary $d \times d$ unitary matrix can be decomposed to a product of at most $d(d-1)/2$ two-level unitary matrices for $d \geq 2$.

Universality of a quantum circuit

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Proof.

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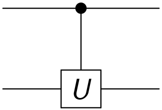
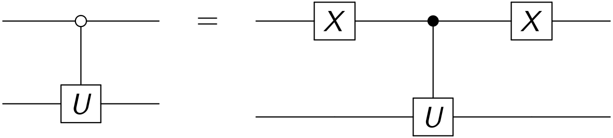
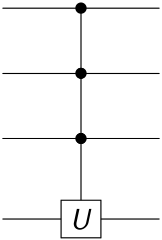
Controlled-unitary

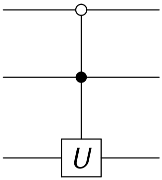






Special cases

$$
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & u_{1,1} & 0 & 0 & 0 & u_{1,2} & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & u_{2,1} & 0 & 0 & 0 & u_{2,2} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
$$



Universality of a quantum circuit

Lemma

Any $2^n \times 2^n$ two-level unitary matrix can be decomposed to a product of controlled-unitary gates.

Proof.

Assume that the two-level unitary matrix acts on a 2-dimentional subspace span$(\{|x\rangle, |y\rangle\})$ for $x \neq y \in \{0,1\}^n$.

Assume that for $i \in \{1, 2, \ldots, n\}$, $x_i = 1$ and $y_i = 0$. Apply at most

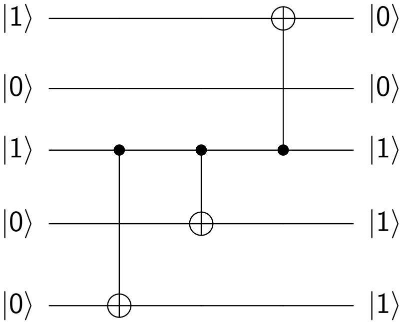
$n - 1$ CNOT gates such that

$$\begin{aligned} & & |x\rangle &\longmapsto |y \oplus e_i\rangle \\ & & |y\rangle &\longmapsto |y\rangle \\ \forall z \neq x, y \qquad & \exists \tilde{z} \neq x, y \qquad & |z\rangle &\longmapsto |\tilde{z}\rangle, \end{aligned}$$

Then, apply "controlled unitary" and reverse the permutation of the basis. □

The first part

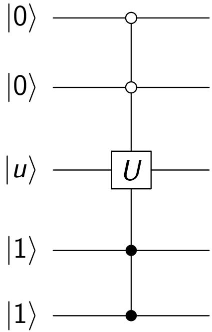
Let $x = \texttt{00101}$, $y = \texttt{11000}$.



$$|00101\rangle \longmapsto |11100\rangle$$
$$|11000\rangle \longmapsto |11000\rangle$$

Controlled-unitary

$$|x\rangle = |00101\rangle \longmapsto |11100\rangle$$
$$|y\rangle = |11000\rangle \longmapsto |11000\rangle$$

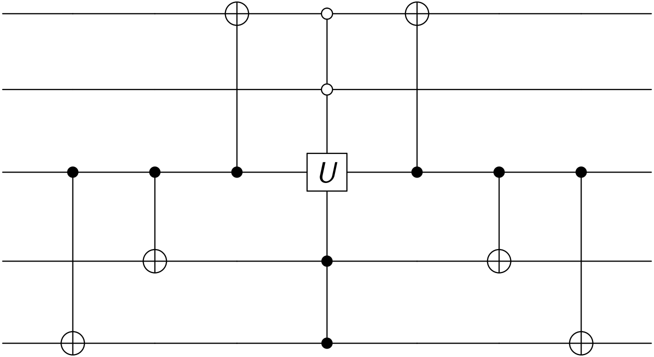


Finally, reverse the basis

$$|11100\rangle \longmapsto |00101\rangle = |x\rangle$$
$$|11000\rangle \longmapsto |11000\rangle = |y\rangle$$

Whole quantum circuit

Let $x = 00101$, $y = 11000$.



$$|00101\rangle \longmapsto |11100\rangle \longmapsto |00101\rangle$$
$$|11000\rangle \longmapsto |11000\rangle \longmapsto |11000\rangle$$

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Assignments

1. Show a decomposition of

$$\frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{bmatrix}$$

into a product of two-level unitary matrices.

2. Show a decomposition of two-level unitary

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a & 0 & 0 & 0 & 0 & c \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & b & 0 & 0 & 0 & 0 & d \end{bmatrix}$$

into a product of controlled-unitary gates.