# Shor's algorithm

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# Integer factoring and primality test

- $\textsc{PrimalityTest}$:
  Input: $N \in \mathbb{N}$
  Output: YES if $N$ is a prime number, NO if $N$ is a composite number.

- $\textsc{IntegerFactoring}$:
  Input: $N \in \mathbb{N}$
  Output: $a \in \mathbb{N}$ satisfying $a \neq 1, N$ and $a$ divides $N$.

Does there exist an algorithm with time complexity $O((\log N)^c)$ ?

It is known that $\textsc{PrimalityTest} \in \mathrm{P}$ [Agrawal, Kayal, Saxena, 2004].
It is believed that $\textsc{IntegerFactoring} \notin \mathrm{BPP}$.
It is known that $\textsc{IntegerFactoring} \in \mathrm{BQP}$. [Shor 1994]

## Nontrivial square root of 1

$$x^2 = 1 \mod N$$

$$\begin{aligned} x^2 &= 1 \mod N \\ \iff (x-1)(x+1) &= 0 \mod N \end{aligned}$$

If $N$ is a prime number, $x = \pm 1$ are only solution.

If there exists a nontrivial square root of 1, then $N$ must be composite number.

If we find a nontrivial square root $x$ of 1, we can also find a nontrivial factor of $N$ by $\gcd(x \pm 1, N)$.

# Fermat little theorem

## Theorem (Fermat little theorem)

*If $p$ is a prime number, any integer $a$ that is not a multiple of $p$,*

$$a^{p-1} = 1 \mod p.$$

## Proof.

The map

$$x \longmapsto ax \mod p$$

is a bijection on $\{1, \ldots, p-1\}$. Hence,

$$\prod_{x=1}^{p-1} x = \prod_{x=1}^{p-1} (ax) \mod p$$
$$\iff 1 = a^{p-1} \mod p.$$

$\square$

# Fermat test

```
function FERMAT(N)
    loop    k times
        a ← a random integer in [2, N − 2]
        if a^(N−1) ≠ 1 mod N then
            return NO
        end if
    end loop
    return YES
end function
```

**function** $\textsc{Fermat}(N)$
**loop** $k$ times
$a \leftarrow$ a random integer in $[2, N-2]$
**if** $a^{N-1} \neq 1 \mod N$ **then**
**return** NO
**end if**
**end loop**
**return** YES
**end function**

Carmichel numbers ($561 = 3 \cdot 11 \cdot 17$, $1105 = 5 \cdot 13 \cdot 17, \cdots$) passes the Fermat test for all $a$ coprime with $N$.

# Finding the nontirivial square root of 1

We assume that $a^{N-1} = 1 \mod N$ for some integer $a \in [2, N-2]$,

Let $u$ and $d$ be an integer and an odd integer, respectively, satisfying $N - 1 = 2^u d$.

| $a^d$ | $a^{2d}$ | $a^{2^2 d}$ | $\cdots$ | $a^{2^{k-1} d}$ | $a^{2^k d}$ | $\cdots$ | $a^{2^{u-1} d}$ | $a^{2^u d}$ |
|---|---|---|---|---|---|---|---|---|
| $*$ | $*$ | $*$ | $\cdots$ | $z \neq 1$ | 1 | $\cdots$ | 1 | 1 |

$z$ is a square root of 1 modulo $N$.

# Miller–Rabin primality test

**function** $\textsc{Miller–Rabin}(N)$
    Let $u$ and $d$ be an integer and an odd integer, respectively, satisfying $N = 2^u d + 1$
    **loop** $k$ times
        $a \leftarrow$ a random integer in $[2, N-2]$
        $x \leftarrow a^d \bmod N$
        **if** $x = 1$ or $x = N-1$ **then continue**
        **end if**
        **loop** $u-1$ times
            $x \leftarrow x^2 \bmod N$
            **if** $x = N-1$ **then break**
            **end if**
        **end loop**
        **if** $x \neq N-1$ **then return** NO
        **end if**
    **end loop**
    **return** YES
**end function**

# Why Miller–Rabin algorithm doesn't solve IntegerFactoring

In fact, the Miller–Rabin test outputs NO with probability $1 - 1/4^k$ for composite $N$.

The Miller–Rabin algorithm seems to find a nontrivial square root of 1, which means that we can also find a nontrivial factor of $N$, right ?

NO!

| $a^d$ | $a^{2d}$ | $a^{2^2 d}$ | $\cdots$ | $a^{2^u d}$ |
|---|---|---|---|---|
| $*$ | $*$ | $*$ | $\cdots$ | $\neq 1$ |

## Shor's algorithm

**function** $\textsc{Shor}(N\text{: An odd integer})$
  **if** $N = a^b$ for some $a \geq 1$ and $b \geq 2$ **then**
    **return** a
  **end if**
  **loop**
    $a \leftarrow$ a random integer in $[2, N-2]$.
    $b \leftarrow \gcd(a, N)$.
    **if** $b \neq 1$ **then return** $b$
    **end if**
    $r \leftarrow \textsc{OrderFinding}(a, N)$.
    **if** $r$ is odd **then continue**
    **end if**
    **if** $a^{r/2} \neq N - 1$ **then**
      **return** $\gcd(a^{r/2} + 1, N)$
    **end if**
  **end loop**
**end function**

# Eigenvalues of the unitary operator

Let $r$ be the order of $a$ modulo $N$, which is a smallest positive integer satisfying

$$a^r = 1 \mod N.$$

For $a$ that is coprime with $N$, define the unitary operator $U_a$ by

$$U_a |x\rangle = \begin{cases} |ax \bmod N\rangle & \text{if } x < N \\ |x\rangle & \text{Otherwise.} \end{cases}$$

Here, $U_a^r = I$. This means that all eigenvalues of $U_a$ are in the form $e^{2\pi i \frac{s}{r}}$ for $s \in \{0, 1, 2, \ldots, r-1\}$.

By quantum phase estimation for $U_a$, an approximation of $\frac{s}{r}$ can be computed efficiently.

From $0.b_n b_{n-1} \cdots b_1 \approx \frac{s}{r}$, we can extract the denominator $r$ if $s$ is comprime with $r$.

# Eigenvectors of the unitary operator

For $s \in \{0, 1, \dots, r-1\}$,

$$|\psi_s\rangle := \frac{1}{\sqrt{r}} \sum_{j=0}^{r-1} e^{-2\pi i \frac{sj}{r}} \left|a^j \bmod N\right\rangle .$$

$$\begin{aligned}
U_a |\psi_s\rangle &= \frac{1}{\sqrt{r}} \sum_{j=0}^{r-1} e^{-2\pi i \frac{sj}{r}} \left|a^{j+1} \bmod N\right\rangle \\
&= e^{2\pi i \frac{s}{r}} \frac{1}{\sqrt{r}} \sum_{j=0}^{r-1} e^{-2\pi i \frac{s(j+1)}{r}} \left|a^{j+1} \bmod N\right\rangle \\
&= e^{2\pi i \frac{s}{r}} \frac{1}{\sqrt{r}} \sum_{j=0}^{r-1} e^{-2\pi i \frac{sj}{r}} \left|a^{j} \bmod N\right\rangle \\
&= e^{2\pi i \frac{s}{r}} |\psi_s\rangle .
\end{aligned}$$

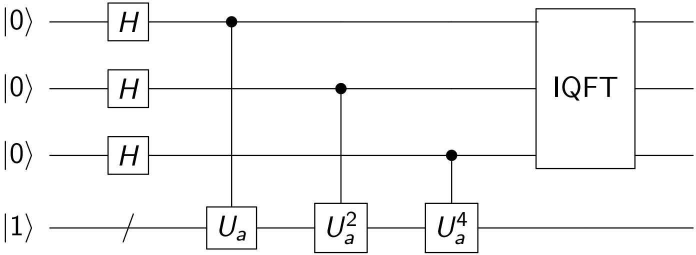
# The uniform sperposition of the eigenvectors

For $s \in \{0, 1, \dots, r-1\}$,

$$|\psi_s\rangle := \frac{1}{\sqrt{r}} \sum_{j=0}^{r-1} e^{-2\pi i \frac{sj}{r}} |a^j \bmod N\rangle .$$

$$\begin{aligned}
\frac{1}{\sqrt{r}} \sum_{s=0}^{r-1} |\psi_s\rangle &= \frac{1}{r} \sum_{s=0}^{r-1} \sum_{j=0}^{r-1} e^{-2\pi i \frac{sj}{r}} |a^j \bmod N\rangle \\
&= \sum_{j=0}^{r-1} \left( \frac{1}{r} \sum_{s=0}^{r-1} e^{-2\pi i \frac{sj}{r}} \right) |a^j \bmod N\rangle \\
&= |1\rangle
\end{aligned}$$

# Quantum phase estimation




For uniformly chosen $s \in \{0, 1, \ldots, r-1\}$, we obtain an approximation of $s/r$.

# Continued fraction

$$\theta = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{a_4 + \cdots}}}}$$

**Theorem**

*Suppose $s/r$ is a rational number satisfying*

$$\left|\frac{s}{r} - \theta\right| \leq \frac{1}{2r^2}.$$

*Then, $s/r$ is a convergent of the continued fraction for $\theta$.*

## The probability that we can calculate $r$ from $s/r$

For uniformly chosen $s \in \{0, 1, \dots, r-1\}$, we obtain an approximation of $s/r$.

From the denominators $d_1$ and $d_2$ of the irreducible fractions of $s_1/r$ and $s_2/r$, we can calculate $r$ by $\mathrm{lcm}(d_1, d_2)$.

$$
\begin{aligned}
\Pr(\mathrm{lcm}(d_1, d_2) \neq r) &= \Pr_{s_1, s_2 \in \{0, \dots, r-1\}}(\gcd(s_1, s_2, r) \neq 1) \\
&= \Pr_{s_1, s_2 \in \{1, \dots, r\}}(\gcd(s_1, s_2, r) \neq 1) \\
&\leq \Pr_{s_1, s_2 \in \{1, \dots, r\}}(\gcd(s_1, s_2) \neq 1) \\
&\leq \sum_{p\text{: prime}} \Pr_{s_1, s_2 \in \{1, \dots, r\}}(p \mid s_1, p \mid s_2) \\
&\leq \sum_{p\text{: prime}} \frac{1}{p^2} \leq 0.4523
\end{aligned}
$$

# Assignments

1. Show all eigenvectors and corresponding eigenvalues of $2^{\lceil \log N \rceil} \times 2^{\lceil \log N \rceil}$ matrix $U_a$.