

Quantum teleportation

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Time evolution of a system

Time evolution of a system is represented by a map from a state to a state.

T : The set of states \rightarrow the set of states.

$$pT(\rho_1) + (1 - p)T(\rho_2) = T(p\rho_1 + (1 - p)\rho_2)$$

for any density matrices ρ_1, ρ_2 and $p \in [0, 1]$.

$T : \mathcal{H}(V) \rightarrow \mathcal{H}(W)$ must be **linear** (a proof is needed).

Schrödinger picture and Heisenberg picture

T^\dagger : The set of binary measurements \rightarrow the set of binary measurements.

$$\langle T(\rho), P \rangle = \langle \rho, T^\dagger(P) \rangle$$

for any $\rho \in \mathcal{H}(V)$ and $P \in \mathcal{H}(W)$. T^\dagger is an **adjoint** map of T .

$$\begin{aligned} \langle T_3(T_2(T_1(\rho))), P \rangle &= \langle T_2(T_1(\rho)), T_3^\dagger(P) \rangle \\ &= \langle T_1(\rho), T_2^\dagger(T_3^\dagger(P)) \rangle = \langle \rho, T_1^\dagger(T_2^\dagger(T_3^\dagger(P))) \rangle \end{aligned}$$

No-cloning theorem

$$\begin{aligned}|0\rangle\langle 0| &\longmapsto |0\rangle\langle 0| \otimes |0\rangle\langle 0| \\ |1\rangle\langle 1| &\longmapsto |1\rangle\langle 1| \otimes |1\rangle\langle 1|\end{aligned}$$

From the linearity,

$$\frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|) \longmapsto \frac{1}{2}(|0\rangle\langle 0| \otimes |0\rangle\langle 0| + |1\rangle\langle 1| \otimes |1\rangle\langle 1|)$$

This is not equal to

$$\frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|) \otimes \frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|).$$

Axioms for quantum channel

$$T: \mathcal{H}(V) \rightarrow \mathcal{H}(W).$$

- ① Trace-preserving: $\text{Tr}(T(\rho)) = \text{Tr}(\rho)$.
- ② Positive : $T(\rho) \succeq 0$ for any $\rho \succeq 0$.
- ③ Completely positive: $\text{id} \otimes T$ is positive.

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Positive but not completely positive 1/2

$T \in \mathcal{L}(\mathcal{H}(\mathbb{C}^2))$: **transposition** according to $\{|0\rangle, |1\rangle\}$.

The transposition is obviously trace-preserving.

The transposition is **positive**.

Proof.

For any $A \succeq 0$ and $|\psi\rangle \in \mathbb{C}^2$,

$$\langle \psi | T(A) | \psi \rangle = \langle \psi | A^T | \psi \rangle = \langle \psi | A^* | \psi \rangle = (\langle \psi |^* A | \psi \rangle^*)^* \geq 0$$

□

Positive but not completely positive 2/2

But, the transposition is **not** completely positive.

Proof.

For $|\Phi\rangle := \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$,

$$\begin{aligned} & (\text{id}_{\mathcal{H}(\mathbb{C}^2)} \otimes T)(|\Phi\rangle\langle\Phi|) \\ &= (\text{id}_{\mathcal{H}(\mathbb{C}^2)} \otimes T) \left(\frac{1}{2} \left(|0\rangle\langle 0| \otimes |0\rangle\langle 0| + |0\rangle\langle 1| \otimes |0\rangle\langle 1| \right. \right. \\ &\quad \left. \left. + |1\rangle\langle 0| \otimes |1\rangle\langle 0| + |1\rangle\langle 1| \otimes |1\rangle\langle 1| \right) \right) \\ &= \frac{1}{2} \left(|0\rangle\langle 0| \otimes |0\rangle\langle 0| + |0\rangle\langle 1| \otimes |1\rangle\langle 0| \right. \\ &\quad \left. + |1\rangle\langle 0| \otimes |0\rangle\langle 1| + |1\rangle\langle 1| \otimes |1\rangle\langle 1| \right) \end{aligned}$$

$$|00\rangle \mapsto |00\rangle \quad |01\rangle \mapsto |10\rangle \quad |10\rangle \mapsto |01\rangle \quad |11\rangle \mapsto |11\rangle$$

Hence, $|01\rangle - |10\rangle \mapsto |10\rangle - |01\rangle$. $(\text{id}_{\mathcal{H}(\mathbb{C}^2)} \otimes T)(|\Phi\rangle\langle\Phi|)$ is not positive semidefinite.



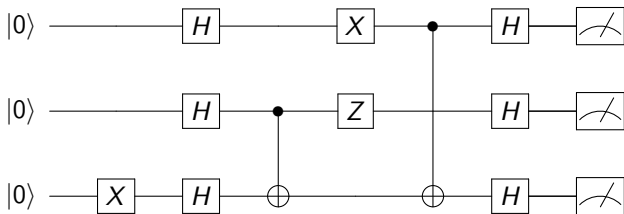
Unitary operations

$$\rho \longmapsto U\rho U^\dagger.$$

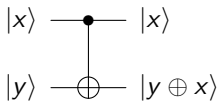
- ① Trace-preserving: $\text{Tr}(U\rho U^\dagger) = \text{Tr}(\rho)$.
- ② Completely positive: $(\text{id} \otimes T)(\rho) = (I \otimes U)\rho(I \otimes U^\dagger)$.

In the most of quantum computing, only **pure** states and **unitary** operations are used.

Quantum circuit



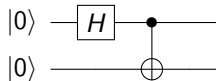
Controlled not



$$\text{CNOT } |x\rangle |y\rangle \mapsto |x\rangle |y \oplus x\rangle$$

$$\text{CNOT} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

Bell states and quantum circuit



$$\begin{aligned} |0\rangle |0\rangle &\longmapsto \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) |0\rangle = \frac{1}{\sqrt{2}}(|0\rangle |0\rangle + |1\rangle |0\rangle) \\ &\longmapsto \frac{1}{\sqrt{2}}(|0\rangle |0\rangle + |1\rangle |1\rangle) \end{aligned}$$

$$\begin{aligned} |x\rangle |y\rangle &\longmapsto \frac{1}{\sqrt{2}}(|0\rangle + (-1)^x |1\rangle) |y\rangle = \frac{1}{\sqrt{2}}(|0\rangle |y\rangle + (-1)^x |1\rangle |y\rangle) \\ &\longmapsto \frac{1}{\sqrt{2}}(|0\rangle |y\rangle + (-1)^x |1\rangle |\bar{y}\rangle). \end{aligned}$$

Conditional density operator

A probability of outcome of local measurement in a joint system is

$$P(a, b) = \text{Tr}(\rho_{V \otimes W}(P_a \otimes Q_b)).$$

$$P(a | b) = \frac{1}{P(b)} \text{Tr}(\rho_{V \otimes W}(P_a \otimes Q_b)) = \frac{1}{P(b)} \text{Tr}(\text{Tr}_W(\rho_{V \otimes W}(I_V \otimes Q_b))P_a).$$

$$\rho_{V|Q_b} := \frac{1}{P(b)} \text{Tr}_W(\rho_{V \otimes W}(I_V \otimes Q_b)).$$

For $\rho_{V \otimes W} = |\varphi\rangle_{V \otimes W} \langle \varphi|_{V \otimes W}$ and $Q_b = |\psi_b\rangle_W \langle \psi_b|_W$,

$$\text{Tr}_W(\rho_{V \otimes W}(I \otimes Q_b)) = \text{Tr}_W(|\varphi\rangle_{V \otimes W} \langle \varphi|_{V \otimes W} (I_V \otimes |\psi_b\rangle_W \langle \psi_b|_W))$$

Conditional density operator for pure state

For $\rho = |\varphi\rangle_{V \otimes W} \langle \varphi|_{V \otimes W}$ and $Q_b = |\psi_b\rangle_W \langle \psi_b|_W$,

$$\text{Tr}_W(\rho_{V \otimes W} (I_V \otimes Q_b)) = \text{Tr}_W(|\varphi\rangle_{V \otimes W} \langle \varphi|_{V \otimes W} (I \otimes |\psi_b\rangle_W \langle \psi_b|_W))$$

From an expression $|\varphi\rangle_{V \otimes W} = \sum_{i,j} \varphi_{i,j} |i\rangle_V |\psi_j\rangle_W$,

$$\begin{aligned} & \text{Tr}_W(|\varphi\rangle_{V \otimes W} \langle \varphi|_{V \otimes W} (I_V \otimes |\psi_b\rangle_W \langle \psi_b|_W)) \\ &= \text{Tr}_W \left(\sum_{i,j,k,l} \varphi_{i,j} \varphi_{k,l}^* |i\rangle_V |\psi_j\rangle_W \langle k|_V \langle \psi_l|_W (I_V \otimes |\psi_b\rangle_W \langle \psi_b|_W) \right) \\ &= \text{Tr}_W \left(\sum_{i,j,k,l} \varphi_{i,j} \varphi_{k,l}^* |i\rangle_V \langle k|_V \otimes |\psi_j\rangle_W \langle \psi_l|_W (I_V \otimes |\psi_b\rangle_W \langle \psi_b|_W) \right) \\ &= \sum_{i,j,k,l} \varphi_{i,j} \varphi_{k,l}^* |i\rangle_V \langle k|_V \text{Tr}(|\psi_j\rangle_W \langle \psi_l|_W |\psi_b\rangle_W \langle \psi_b|_W) \\ &= \sum_{i,k} \varphi_{i,b} \varphi_{k,b}^* |i\rangle_V \langle k|_V = \left(\sum_i \varphi_{i,b} |i\rangle_V \right) \left(\sum_k \varphi_{k,b}^* \langle k|_V \right) \end{aligned}$$

$$|\varphi\rangle_{V \otimes W} = \sum_{i,j} \varphi_{i,j} |i\rangle_V |\psi_j\rangle_W \mapsto \frac{1}{\sqrt{P(b)}} \sum_i \varphi_{i,b} |i\rangle_V$$

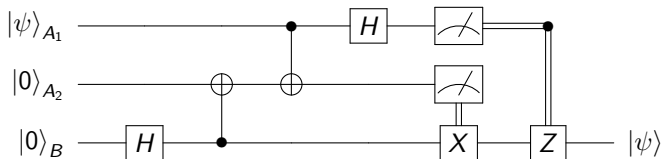
Examples of conditional density operator

For $|\psi\rangle_{V\otimes W} := \sum_{i,j=0}^1 \alpha_{i,j} |i\rangle_V |j\rangle_W$, we measure the system W by $(|0\rangle\langle 0|, |1\rangle\langle 1|)$.

if the outcome is 0, the state $\frac{1}{\sqrt{|\alpha_{0,0}|^2 + |\alpha_{1,0}|^2}} \sum_{i=0}^1 \alpha_{i,0} |i\rangle_V$.

if the outcome is 1, the state $\frac{1}{\sqrt{|\alpha_{0,1}|^2 + |\alpha_{1,1}|^2}} \sum_{i=0}^1 \alpha_{i,1} |i\rangle_V$.

Quantum teleportation



Quantum teleportation

$$|\Phi\rangle_{A_2 \otimes B} = \frac{1}{\sqrt{2}} \left(|0\rangle_{A_2} |0\rangle_B + |1\rangle_{A_2} |1\rangle_B \right)$$

$$(H_{A_1} \otimes I_{A_2} \otimes I_B)(\text{CNOT}_{A_1 A_2} \otimes I_B) |\psi\rangle_{A_1} |\Phi\rangle_{A_2 \otimes B}$$

For $|\psi\rangle_{A_1} = \alpha |0\rangle_{A_1} + \beta |1\rangle_{A_1}$

$$|\psi\rangle_{A_1} |\Phi\rangle_{A_2 \otimes B} = \left(\alpha |0\rangle_{A_1} + \beta |1\rangle_{A_1} \right) \frac{1}{\sqrt{2}} \left(|0\rangle_{A_2} |0\rangle_B + |1\rangle_{A_2} |1\rangle_B \right)$$

$$\xrightarrow{\text{CNOT}_{A_1 A_2}} \frac{\alpha}{\sqrt{2}} \left(|0\rangle_{A_1} |0\rangle_{A_2} |0\rangle_B + |0\rangle_{A_1} |1\rangle_{A_2} |1\rangle_B \right)$$

$$+ \frac{\beta}{\sqrt{2}} \left(|1\rangle_{A_1} |1\rangle_{A_2} |0\rangle_B + |1\rangle_{A_1} |0\rangle_{A_2} |1\rangle_B \right)$$

$$\xrightarrow{H_{A_1}} \frac{\alpha}{2} \left(|000\rangle_{A_1 A_2 B} + |100\rangle_{A_1 A_2 B} + |011\rangle_{A_1 A_2 B} + |111\rangle_{A_1 A_2 B} \right)$$

$$+ \frac{\beta}{2} \left(|010\rangle_{A_1 A_2 B} - |110\rangle_{A_1 A_2 B} + |001\rangle_{A_1 A_2 B} - |101\rangle_{A_1 A_2 B} \right)$$

Quantum teleportation

$$\begin{aligned} & \frac{\alpha}{2} (|000\rangle_{A_1 A_2 B} + |100\rangle_{A_1 A_2 B} + |011\rangle_{A_1 A_2 B} + |111\rangle_{A_1 A_2 B}) \\ & + \frac{\beta}{2} (|010\rangle_{A_1 A_2 B} - |110\rangle_{A_1 A_2 B} + |001\rangle_{A_1 A_2 B} - |101\rangle_{A_1 A_2 B}) \\ & = \frac{1}{2} \left[|00\rangle_{A_1 A_2} (\alpha |0\rangle_B + \beta |1\rangle_B) + |01\rangle_{A_1 A_2} (\alpha |1\rangle_B + \beta |0\rangle_B) \right. \\ & \quad \left. + |10\rangle_{A_1 A_2} (\alpha |0\rangle_B - \beta |1\rangle_B) + |11\rangle_{A_1 A_2} (\alpha |1\rangle_B - \beta |0\rangle_B) \right] \end{aligned}$$

According to the measurement outcome of $A_1 A_2$

$00 \Rightarrow I$

$01 \Rightarrow X$

$10 \Rightarrow Z$

$11 \Rightarrow ZX$

Assignments

- ① Show the state vector $|\psi\rangle \in \mathbb{C}^2$ of the Bell state

$$\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \in \mathbb{C}^2 \otimes \mathbb{C}^2$$

when $|0\rangle\langle 0|$ is measured at the second system.

- ② Show the state vector $|\psi\rangle \in \mathbb{C}^2$ of the Bell state when $|+\rangle\langle +|$ is measured at the second system.
- ③ Show the state vector $|\psi\rangle \in \mathbb{C}^2$ of the Bell state when $|\varphi\rangle\langle \varphi|$ is measured at the second system where $|\varphi\rangle := \alpha|0\rangle + \beta|1\rangle$.