

# Quantum state discrimination and Holevo–Helstrom theorem

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# Discrimination of probability distributions

Discrimination of probability distributions:

- Input:  $x \in \mathcal{X}$  drawn from probability distribution  $p_0$  with probability  $\lambda$ , and from  $p_1$  with probability  $1 - \lambda$ .
- Output:  $i \in \{0, 1\}$  that indicates the probability distribution  $p_i$ .

# Maximum probability of success: Discrimination of probability distributions

## Theorem

*The maximum probability of success is equal to*

$$\frac{1 + \|\lambda p_0 - (1 - \lambda)p_1\|_1}{2}$$

where  $\|a\|_1 := \sum_x |a(x)|$ .

## Proof.

Let  $d: \mathcal{X} \rightarrow \{0, 1\}$  be a discriminator. Assume we get 1 point if succeeds, and loose 1 point if fails. Then, expected point is bounded by

$$\begin{aligned} & \lambda \sum_{x \in \mathcal{X}} p_0(x)(-1)^{d(x)} + (1 - \lambda) \sum_{x \in \mathcal{X}} p_1(x)(-1)^{d(x)+1} \\ &= \sum_{x \in \mathcal{X}} (\lambda p_0(x) - (1 - \lambda)p_1(x))(-1)^{d(x)} \leq \|\lambda p_0 - (1 - \lambda)p_1\|_1. \end{aligned}$$

The equality is achieved by  $d(x)$  that is 0 iff  $\lambda p_0(x) \geq (1 - \lambda)p_1(x)$ . On the other hand, the expected point is  $p_{\text{succ}} - p_{\text{fail}} = 2p_{\text{succ}} - 1$ . □

# Discrimination of quantum states

Discrimination of **quantum** states:

- Input: A **quantum** state  $\rho_0$  is given with probability  $\lambda$ , and  $\rho_1$  is given with probability  $1 - \lambda$ .
- Output:  $i \in \{0, 1\}$  that indicates the given state  $\rho_i$ .

## Maximum probability of success: Discrimination of quantum states

### Theorem (Holevo–Helstrom theorem)

*The maximum probability of success is equal to*

$$\frac{1 + \|\lambda\rho_0 - (1 - \lambda)\rho_1\|_1}{2}$$

where  $\|A\|_1 := \text{Tr}(\sqrt{A^\dagger A})$ , which is a sum of the singular values of  $A$ .

### Proof.

Once a measurement  $(P_y)_{y \in \mathcal{Y}}$  is fixed, we get a classical probability distribution  $p_0(y) := \text{Tr}(\rho_0 P_y)$  and  $p_1(y) := \text{Tr}(\rho_1 P_y)$ . In this case, the maximum probability of success is given by

$$\frac{1 + \|\lambda p_0 - (1 - \lambda)p_1\|_1}{2}$$

Hence, it's sufficient to show

$$\max_{(P_y)_{y \in \mathcal{Y}}} \|\lambda p_0 - (1 - \lambda)p_1\|_1 = \|\lambda\rho_0 - (1 - \lambda)\rho_1\|_1.$$

## Holevo–Helstrom theorem

$$\begin{aligned}\max_{(P_y)_{y \in \mathcal{Y}}} \|\lambda \rho_0 - (1 - \lambda) \rho_1\|_1 &= \max_{(P_y)_{y \in \mathcal{Y}}} \sum_{y \in \mathcal{Y}} |\lambda \rho_0(y) - (1 - \lambda) \rho_1(y)| \\&= \max_{(P_y)_{y \in \mathcal{Y}}} \sum_{y \in \mathcal{Y}} |\lambda \text{Tr}(\rho_0 P_y) - (1 - \lambda) \text{Tr}(\rho_1 P_y)| \\&= \max_{(P_y)_{y \in \mathcal{Y}}} \sum_{y \in \mathcal{Y}} |\text{Tr}((\lambda \rho_0 - (1 - \lambda) \rho_1) P_y)|\end{aligned}$$

Let

$$\lambda \rho_0 - (1 - \lambda) \rho_1 = \sum_{x \in \mathcal{X}} \mu_x |\psi_x\rangle \langle \psi_x|$$

be a spectral decomposition. Then,

$$\begin{aligned}\max_{(P_y)_{y \in \mathcal{Y}}} \sum_{y \in \mathcal{Y}} |\text{Tr}((\lambda \rho_0 - (1 - \lambda) \rho_1) P_y)| &= \max_{(P_y)_{y \in \mathcal{Y}}} \sum_{y \in \mathcal{Y}} \left| \sum_{x \in \mathcal{X}} \mu_x \langle \psi_x | P_y | \psi_x \rangle \right| \\&\leq \max_{(P_y)_{y \in \mathcal{Y}}} \sum_{y \in \mathcal{Y}} \sum_{x \in \mathcal{X}} |\mu_x| \langle \psi_x | P_y | \psi_x \rangle = \sum_{x \in \mathcal{X}} |\mu_x| = \|\lambda \rho_0 - (1 - \lambda) \rho_1\|_1.\end{aligned}$$

The maximum is achieved by  $\mathcal{Y} = \mathcal{X}$  and  $P_x = |\psi_x\rangle \langle \psi_x|$ ,  
and  $\mathcal{Y} = \{0, 1\}$  and  $P_0 = \sum_{x: \mu_x \geq 0} |\psi_x\rangle \langle \psi_x|$ ,  $P_1 = \sum_{x: \mu_x < 0} |\psi_x\rangle \langle \psi_x|$ .

## Discrimination of pure states

The maximum probability of success for discrimination of  $|\psi\rangle\langle\psi|$  and  $|\varphi\rangle\langle\varphi|$  is given by

$$\|\lambda |\psi\rangle\langle\psi| - (1 - \lambda) |\varphi\rangle\langle\varphi|\|_1$$

Let  $A := \lambda |\psi\rangle\langle\psi| - (1 - \lambda) |\varphi\rangle\langle\varphi|$ . Then,  $A$  has rank at most **two**. From

$$\mathrm{Tr}(A) = \mu_0 + \mu_1 = \lambda - (1 - \lambda)$$

$$\mathrm{Tr}(A^2) = \mu_0^2 + \mu_1^2 = \lambda^2 + (1 - \lambda)^2 - 2\lambda(1 - \lambda)|\langle\psi|\varphi\rangle|^2,$$

$$2\mu_0\mu_1 = (\mu_0 + \mu_1)^2 - (\mu_0^2 + \mu_1^2) = -2\lambda(1 - \lambda)(1 - |\langle\psi|\varphi\rangle|^2) \leq 0.$$

$$(\mu_0 - \mu_1)^2 = (\mu_0^2 + \mu_1^2) - 2\mu_0\mu_1 = (\lambda + (1 - \lambda))^2 - 4\lambda(1 - \lambda)|\langle\psi|\varphi\rangle|^2$$

$$\|A\|_1 = |\mu_0| + |\mu_1| = |\mu_0 - \mu_1| = \sqrt{1 - 4\lambda(1 - \lambda)|\langle\psi|\varphi\rangle|^2}.$$

# Unitary discrimination

Discrimination of **unitary operators**:

- Input: A **unitary operator**  $U_0$  is given with probability  $\lambda$ , and  $U_1$  is given with probability  $1 - \lambda$  as an oracle  $\mathcal{O}$  that can be used once.
- Output:  $i \in \{0, 1\}$  that indicates the given unitary  $U_i$ .



## Unitary discrimination

If algorithm call the oracle  $\mathcal{O}$  on a state  $|\psi\rangle$ , we get either of  $U_0 |\psi\rangle$  or  $U_1 |\psi\rangle$ .

Then, the maximum probability of success is given by

$$\begin{aligned} & \max_{|\psi\rangle} \|\lambda U_0 |\psi\rangle \langle \psi| U_0^\dagger - (1 - \lambda) U_1 |\psi\rangle \langle \psi| U_1^\dagger\|_1 \\ &= \max_{|\psi\rangle} \sqrt{1 - 4\lambda(1 - \lambda) |\langle \psi| U_0^\dagger U_1 |\psi\rangle|^2} \end{aligned}$$

For  $V := U_0^\dagger U_1$ , let  $V = \sum_x \mu_x |\varphi_x\rangle \langle \varphi_x|$  be a spectral decomposition. Note that  $|\mu_x| = 1$  for all  $x$ . Let  $|\psi\rangle = \sum_x \alpha_x |\varphi_x\rangle$ . Then,  $\langle \psi| U_0^\dagger U_1 |\psi\rangle = \sum_x |\alpha_x|^2 \mu_x$ , which is a convex combination of  $(\mu_x)_x$ . Let  $\theta_{\text{cover}}$  be the smallest angle that covers all eigenvalues  $(\mu_x)_x$ . Then,  $\min_{|\psi\rangle} \langle \psi| U_0^\dagger U_1 |\psi\rangle = \cos(\frac{\theta_{\text{cover}}}{2})$  if  $\theta_{\text{cover}} \leq \pi$ , and 0 otherwise.

## Assignment

- 1 Show the maximum probability of success for discriminating  $|0\rangle$  and  $|+\rangle$  given with the uniform probability.
- 2 Show a binary optimal measurement for the discrimination of  $|0\rangle$  and  $|+\rangle$  given with the uniform probability.
- 3 Show the maximum probability of success for discriminating  $I$  and  $R_Z(\theta)$  given with the uniform probability. Show the input state  $|\psi\rangle$  for the oracle as well.