Quantum circuit

Ryuhei Mori

Tokyo Institute of Technology

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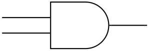
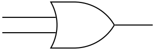
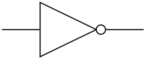
Boolean circuit

- Boolean function $f \colon \{0,1\}^n \to \{0,1\}$.

- Boolean circuit is a model of computation of a Boolean functions which consists of logic gates.

- AND gate: 

- OR gate: 

- NOT gate: 

Universality of $\{\text{AND}, \text{OR}, \text{NOT}\}$

Theorem

For any Boolean function $f \colon \{0,1\}^n \to \{0,1\}$, there is a Boolean circuit with AND, OR and NOT gates computing f.

Proof.

The proof is an induction on n. Theorem is trivial for $n = 1$.
Assume that Theorem holds for Boolean functions $n \leq k - 1$.

$$f(x_1, \dots, x_k) = (x_k \wedge f(x_1, \dots, x_{k-1}, 1)) \vee (\overline{x_k} \wedge f(x_1, \dots, x_{k-1}, 0)).$$

$\square$

Size of Boolean circuits

Size of Boolean circuit := # of AND/OR gates in the Boolean circuit.
Let $C(f)$ be a smallest size of Boolean circuit computing $f\colon \{0,1\}^n \to \{0,1\}$.
Let $s(n) := \max_{f\colon \{0,1\}^n \to \{0,1\}} C(f)$.

$$f(x_1,\dots,x_n) = (x_n \wedge f(x_1,\dots,x_{n-1},1)) \vee (\overline{x_n} \wedge f(x_1,\dots,x_{n-1},0))$$

$$\begin{aligned}
s(n) &\le c + 2s(n-1) \\
\frac{s(n)}{2^n} &\le \frac{c}{2^n} + \frac{s(n-1)}{2^{n-1}} \\
&\le c\left(\frac{1}{2^n} + \frac{1}{2^{n-1}} + \cdots \frac{1}{2}\right) + s(0) \le c + s(0)
\end{aligned}$$

$$s(n) = O(2^n).$$

Lower bound of size of Boolean circuits

The number of Boolean functions with n variables is 2^{2^n}.

The number of Boolean circuits of size s is at most

$$(8(n+s)^2)^s.$$

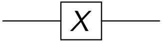
This means

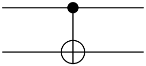
$$\begin{aligned} & (8(n+s(n))^2)^{s(n)} \geq 2^{2^n} \\ \iff & s(n)\log(8(n+s(n))^2) \geq 2^n \\ \implies & s(n) \geq \frac{2^n}{3n} \quad \text{for sufficiently large } n. \end{aligned}$$

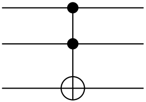
In fact, $s(n) = \frac{2^n}{n}(1+o(1))$.

Quantum circuit

- Quantum circuit is a model of computation of Boolean functions which consists of quantum gates.

- Single qubit gate: X gate, Y gate, Z gate, H gate, $S := \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}$ gate 

- Two qubit gate: CNOT gate 

- Three qubit gate: Toffoli gate 

“Classical computation” by a quantum circuit

Lemma

For any function $f\colon \{0,1\}^n \to \{0,1\}$, *there is a quantum circuit on* $n+1+w$ *qubits which consists of* X, $CNOT$ *and* *Toffoli* *gates for* U *satisfying*

$$U|x\rangle|y\rangle|0\rangle^{\otimes w} = |x\rangle|y \oplus f(x)\rangle|0\rangle^{\otimes w}$$

for all $x \in \{0,1\}^n$, $y \in \{0,1\}$. *Here, the number* w *of working qubits (ancilla) is at most* $C(f)+1$ *and the number* g *of quantum gates is at most* $O(C(f))$.

A sketch of a proof.

Translate a Boolean circuit to a quantum circuit. □

Universality of a quantum circuit

Theorem (Universality of finite gate set)

For any unitary matrix $U \in \mathcal{L}(\mathbb{C}^{2^n})$ and $\epsilon > 0$, there is a quantum circuit with X, Y, Z, H, S, CNOT, Toffoli gates computing $\widetilde{U}$ satisfying $\|U - \widetilde{U}\| < \epsilon$.

Proof.

In the next lecture. $\square$

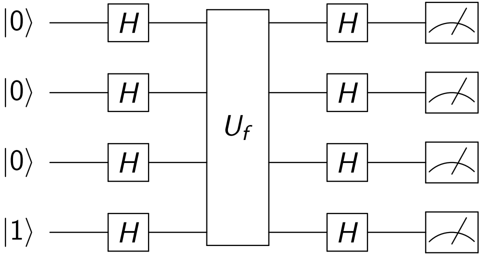
Oracle model

- Input is given by an oracle.
- Classical oracle: oracle gate $i \mapsto x_i$.
- Quantum oracle: quantum oracle gate $U|i\rangle|y\rangle = |i\rangle|y \oplus x_i\rangle$.
- Query complexity: the number of oracle calls.
- Circuit size: the number of total quantum gates.

Deutsch–Jozsa problem

- There is a hidden Boolean function $f\colon \{0,1\}^n \to \{0,1\}$ that is a constant or balanced.
- Quantum oracle $U_f \left|x\right\rangle \left|y\right\rangle = \left|x\right\rangle \left|y \oplus f(x)\right\rangle$.
- Goal is to determine whether f is constant or balanced.
- Classical deterministic algorithm needs $2^{n-1} + 1$ oracle calls.
- Deutsch–Jozsa algorithm solves this problem by single oracle call (and $O(n)$ gates).

Deutsch–Jozsa algoritnm



$$|0\rangle^{\otimes n}|1\rangle \overset{H^{\otimes(n+1)}}{\longmapsto} |+\rangle^{\otimes n}|-\rangle = \frac{1}{2^{n/2}} \sum_{x\in\{0,1\}^n} |x\rangle|-\rangle$$

$$\overset{U_f}{\longmapsto} \frac{1}{2^{n/2}} \sum_{x\in\{0,1\}^n} (-1)^{f(x)}|x\rangle|-\rangle$$

$$\overset{H^{\otimes(n+1)}}{\longmapsto} \frac{1}{2^n} \sum_{x\in\{0,1\}^n} \sum_{z\in\{0,1\}^n} (-1)^{f(x)}(-1)^{\langle x,z\rangle}|z\rangle|1\rangle$$

The probability of outcome

$$
\begin{aligned}
&\frac{1}{2^n} \sum_{x\in\{0,1\}^n} \sum_{z\in\{0,1\}^n} (-1)^{f(x)}(-1)^{\langle x,z\rangle} |z\rangle |1\rangle \\
&= \sum_{z\in\{0,1\}^n} \left(\frac{1}{2^n} \sum_{x\in\{0,1\}^n} (-1)^{f(x)}(-1)^{\langle x,z\rangle} \right) |z\rangle |1\rangle
\end{aligned}
$$

The coefficient of $|0\rangle^{\otimes n} |1\rangle$ is $S := \frac{1}{2^n} \sum_x (-1)^{f(x)}$. Here, $S^2 = 1$ if $f(x)$ is constant, and $S^2 = 0$ if $f(x)$ is balanced.

In the Deutsch–Jozsa algorithm, the first n qubits are measured and output "constant" if all-zero is measured, and output "balanced", otherwise.

Assignments

1. Describe quantum circuits computing the following Boolean functions, i.e., quantum circuit U satisfies

$$U |x\rangle |y\rangle |0\rangle^{\otimes w} = |x\rangle |y \oplus f(x)\rangle |0\rangle^{\otimes w}$$

for some w.

   A. $f(x_1, x_2, x_3, x_4) := x_1 \oplus x_2 \oplus x_3 \oplus x_4$.
   B. $f(x_1, x_2, x_3, x_4) := x_1 \wedge x_2 \wedge x_3 \wedge x_4$.
   C. $f(x_1, x_2, x_3, x_4) := (x_1 \vee x_2) \wedge (x_3 \vee x_4)$.
   D. $f(x_1, x_2, x_3) :=$ Majority of x_1, x_2 and x_3.

2. (Advanced) For fixed $S \subseteq \{1, 2, \dots, n\}$, a Boolean function f is either $g_a(x) = a + \sum_{i \in S} x_i \mod 2$ for $a \in \{0, 1\}$ or $h(x)$ satisfying $\sum_x (-1)^{g_0(x)+h(x)} = 0$. Show quantum algorithm that distinguishes the two cases $f(x) = g_0(x)$ or $g_1(x)$ and $f(x) = h(x)$ for $h(x)$ satsifying the above condition.