COMP30026, Models of Computation

Assignment 1 - Written Component

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Tutorial: Friday 9:00am, Mr Colton Carner

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1. **Consider the four first-order predicate formulas:**

$$\begin{aligned}
F &= \forall x\ (\neg P(x,x)) \\
G &= \forall x\ \forall y\ \forall z\ (P(x,y) \land P(y,z) \Rightarrow P(x,z)) \\
G' &= \forall x\ \forall y\ \forall z\ (P(x,y) \land P(y,z) \Rightarrow \neg P(x,z)) \\
H &= \forall x\ \forall y\ (P(x,y) \Rightarrow \neg P(y,x))
\end{aligned}$$

(a) **Show that $F \lor G \lor G' \lor H$ is not valid.**

Answer. To prove $F \lor G \lor G' \lor H$ is not valid, one simply needs to provide an interpretation (counterexample) $\mathbb{I}$ under which it is $\mathbf{f}$.

Consider the interpretation $\mathbb{I}$ where the domain $D = \mathbb{N}$ (the natural numbers starting from 1) and defining the predicate

$$P(x,y) = \begin{cases} \mathbf{t} & x+y \text{ is odd} \\ \mathbf{t} & x = y = 1 \\ \mathbf{f} & \text{otherwise} \end{cases}$$

Under this interpretation:

- Consider F. As $P(1,1) \equiv \mathbf{t}$ then $\neg P(1,1) \equiv \mathbf{f}$, hence $F = \forall x(\neg P(x,x)) \equiv \mathbf{f}$ under $\mathbb{I}$.
- Consider G. Now $P(2,3) \land P(3,4) \equiv \mathbf{t}$ but $P(2,4) \equiv \mathbf{f}$. Hence a counterexample exists for G where $x = 2, y = 3, z = 4$ and so $G \equiv \mathbf{f}$ under $\mathbb{I}$.
- Consider G'. Now $P(1,1) \land P(1,1) \equiv \mathbf{t}$ but $\neg P(1,1) \equiv \mathbf{f}$. Hence a counterexample exists for G' where $x = y = z = 1$ and so $G' \equiv \mathbf{f}$ under $\mathbb{I}$.
- Consider H. $P(1,1) \Rightarrow \neg P(1,1) \equiv \mathbf{t} \Rightarrow \mathbf{f} \equiv \mathbf{f}$ and so a counterexample exists for H where $x = y = 1$; so $H \equiv \mathbf{f}$ under $\mathbb{I}$.

Hence, $\mathbb{I} \nvDash F \lor G \lor G' \lor H$ and so it is not valid by definition.

(b) **Show that $F \land G' \land H$ is satisfiable.**

Answer. To prove $F \land G' \land H$ is satisfiable, one simply needs to provide an interpretation (example) $\mathbb{I}$ under which it is $\mathbf{t}$.

Consider the interpretation $\mathbb{I}$ where the domain D is the set of all nodes in a linked list where each node can have at most one child (succeeding node) and each node is unique. Also define the predicate

$$P(x,y) = \begin{cases} \mathbf{t} & x \text{ is the parent node of } y \\ \mathbf{f} & \text{otherwise.} \end{cases}$$

Under this interpretation:

- Consider F. $P(x,x)$ is impossible for any node (it cannot precede itself) and so $\forall x \neg P(x,x) \equiv \mathbf{t}$. Hence $F \equiv \mathbf{t}$ under $\mathbb{I}$.
- Consider G'. As nodes are unique and have exactly one child, for any node x, y, z then if $P(x,y) \land P(y,z)$ then x is immediately followed by y and y immediately followed by z, so $P(x,z)$ must be $\mathbf{f}$. Hence $G' \equiv \mathbf{t}$ under $\mathbb{I}$ by definition.
- Consider H. For any x, y if $P(x,y)$ then x is immediately followed by y, thus it is impossible that $P(y,x)$. Thus $\forall x \forall y\ P(x,y) \Rightarrow \neg P(y,x)$ and so $H \equiv \mathbf{t}$ under $\mathbb{I}$.

Overall $\mathbb{I} \vDash F \land G' \land H$ and so is satisfiable.

(c) **Show that $(F \wedge G) \Rightarrow H$ is valid.**

Answer.

This shall be done by showing instead that $\neg\{(F \wedge G) \Rightarrow H\}$ or $F \wedge G \wedge \neg H$ is unsatisfiable under any interpretation.

First, to find the clausal form for each.

For F:

$$\begin{aligned} F &= \forall x(\neg P(x,x)) \\ &\neg P(x,x) \quad \text{[Dropping universals]} \end{aligned}$$

yielding the clause $\{\neg P(x,x)\}$.

For G:

$$\begin{aligned} G &= \forall x\ \forall y\ \forall z\ (P(x,y) \wedge P(y,z) \Rightarrow P(x,z)) \\ &\forall x\ \forall y\ \forall z\ (\neg P(x,y) \vee \neg P(y,z) \vee P(x,z)) \quad \text{[Simplifying } \Rightarrow\text{]} \\ &\neg P(x,y) \vee \neg P(y,z) \vee P(x,z) \quad \text{[Dropping universals]} \end{aligned}$$

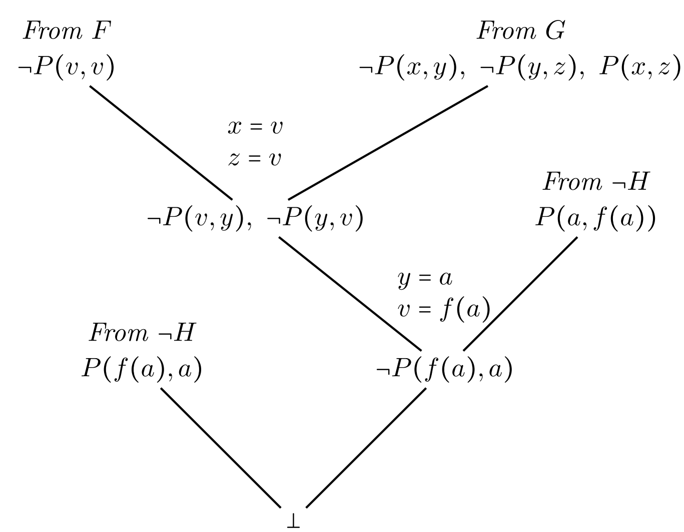
Yielding the clause $\{\neg P(x,y),\ \neg P(y,z),\ P(x,z)\}$.

For $\neg H$:

$$\begin{aligned} \neg H &= \neg\forall x\ \forall y\ (P(x,y) \Rightarrow \neg P(y,x)) \\ &\exists x\ \exists y\ (P(x,y) \wedge P(y,x)) \quad \text{[Dragging } \neg \text{ inwards]} \\ &\exists x\ P(x,f(x)) \wedge P(f(x),x) \quad \text{[Skolemizing]} \\ &P(a,f(a)) \wedge P(f(a),a) \quad \text{[Further skolemizing]} \end{aligned}$$

yielding the clauses $\{P(a,f(a))\}$ and $\{P(f(a),a)\}$.

Drawing a resolution tree, it is clear that the statement $F \wedge G \wedge \neg H$ is unsatisfiable and thus that $F \wedge G \Rightarrow H$ is valid:



2. **For this challenge use the following predicates:**

- $F(x)$**, which stands for "the force is with** x**"**
- $J(x)$**, which stands for "**x **is a Jedi master"**
- $E(x, y)$**, which stands for "**x **exceeds** y**"**
- $P(x, y)$**, which stands for "**x **is a pupil of** y**"**
- $V(x)$**, which stands for "**x **is venerated"**

and use the constant 'a**' to denote Yoda.**

(a) **For each of the statements** $S_1, \dots S_4$**, express it *as a formula in first-order predicate logic*:**

S_1: **"The force is with every Jedi master."**
S_2: **"Yoda exceeds some Jedi master."**
S_3: **"A Jedi master is venerated if (1) the force is with him/her *and* (2) all his/her pupils exceed him/her."**
S_4: **"Every Jedi master without pupils is venerated."**

Answer.

$$S_1 = \forall x(J(x) \Rightarrow F(x))$$

$$S_2 = \exists x(J(x) \wedge E(a, x))$$

$$S_3 = \forall x(J(x) \wedge F(x) \wedge \forall y(P(y, x) \Rightarrow E(y, x)) \Rightarrow V(x))$$

$$S_4 = \forall x(J(x) \wedge \forall y(\neg P(y, x)) \Rightarrow V(x))$$

(b) **Translate** S_1**–**S_3 **to clausal form.**

Answer. S_1:

$$\begin{aligned} S_1 &= \forall x(J(x) \Rightarrow F(x)) \\ &\forall x(\neg J(x) \vee F(x)) \quad \text{[Simplifying } \Rightarrow\text{]} \\ &\neg J(x) \vee F(x) \quad \text{[Removing universals, as no skolemizing necessary]} \end{aligned}$$

Which yields the clausal form $\{\neg J(x), F(x)\}$.
Secondly

$$\begin{aligned} S_2 &= \exists x(J(x) \wedge E(a, x)) \\ &J(b) \wedge E(a, b) \quad \text{[Use skolem constant to remove existential]} \end{aligned}$$

Which yields the clauses $\{J(b)\}, \{E(a, b)\}$.
Thirdly

$$\begin{aligned} S_3 &= \forall x(J(x) \wedge F(x) \wedge \forall y(P(y, x) \Rightarrow E(y, x)) \Rightarrow V(x)) \\ &\forall x(\neg\{J(x) \wedge F(x) \wedge \forall y(\neg P(y, x) \vee E(y, x))\} \vee V(x)) \quad \text{[Simplify } \Rightarrow\text{]} \\ &\forall x(\neg J(x) \vee \neg F(x) \vee \neg\forall y(\neg P(y, x) \vee E(y, x)) \vee V(x)) \quad \text{[Dragging } \neg \text{ inward]} \\ &\forall x(\neg J(x) \vee \neg F(x) \vee V(x) \vee \exists y(P(y, x) \wedge \neg E(y, x))) \quad \text{[Dragging } \neg \text{ inward]} \\ &\forall x(\neg J(x) \vee \neg F(x) \vee V(x) \vee (P(f(x), x) \wedge \neg E(f(x), x))) \quad \text{[Skolem function to remove } \exists y\text{]} \\ &\neg J(x) \vee \neg F(x) \vee V(x) \vee (P(f(x), x) \wedge \neg E(f(x), x)) \quad \text{[Removing universal quantifier]} \\ &\{\neg J(x) \vee \neg F(x) \vee V(x) \vee P(f(x), x)\} \wedge \{\neg J(x) \vee \neg F(x) \vee V(x) \vee \neg E(f(x), x)\} \quad \text{[Distributive law]} \end{aligned}$$

Which yields the final clauses

$$\{\neg J(x), \neg F(x), V(x), P(f(x), x)\} \text{ and } \{\neg J(x), \neg F(x), V(x), \neg E(f(x), x)\}$$

(c) **Translate *the negation of* S_4 to clausal form.**

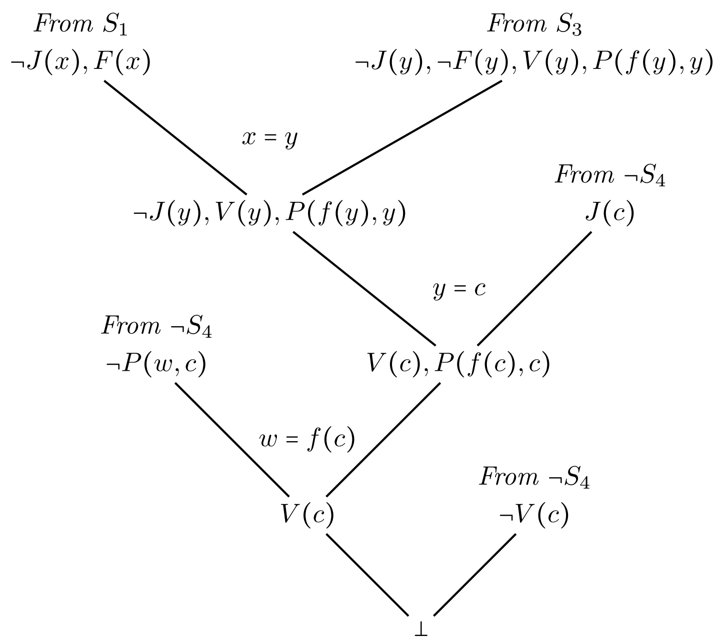
Answer. Simplfying $\neg S_4$:

$$\begin{aligned}
\neg S_4 = {}& \neg\forall x(J(x) \wedge \forall y(\neg P(y,x)) \Rightarrow V(x)) \\
& \neg\forall x(\neg\{J(x) \wedge \forall y(\neg P(y,x))\} \vee V(x)) \quad \text{[Simplfiying } \Rightarrow\text{]} \\
& \exists x(\neg\neg\{J(x) \wedge \forall y(\neg P(y,x))\} \wedge \neg V(x)) \quad \text{[Dragging } \neg \text{ inward]} \\
& \exists x(J(x) \wedge \forall y(\neg P(y,x)) \wedge \neg V(x)) \quad \text{[Negations cancel]} \\
& J(c) \wedge \forall y \neg P(y,c) \wedge \neg V(c) \quad \text{[Using skolem constant to remove } \exists x\text{]} \\
& J(c) \wedge \neg P(y,c) \wedge \neg V(c) \quad \text{[Free to remove universal quantifiers]}
\end{aligned}$$

Which yields the separate clauses $\{J(c)\}$, $\{\neg P(y,c)\}$, $\{\neg V(c)\}$.

(d) **Give a proof by resolution to show that S_4 follows from the other statements.**

Answer. To prove that $S_1 \wedge S_2 \wedge S_3 \models S_4$, attempt to resolve $S_1 \wedge S_2 \wedge S_3 \wedge \neg S_4$ to a contradiction. After reparametrising to distinguish variables, the following proves that the clauses resolve to a contradiction (and hence that $S_1 \wedge S_2 \wedge S_3 \Rightarrow S_4$):



3. **(Note: description omitted.)**

(a) **Consider the Boolean vector function $\boldsymbol{f_a}(b_1, b_2) = (\neg b_1, b_1 \oplus b_2)$. Show that this function is reversible.**

Answer. For a function to be reversible, there needs to be a clear one-to-one relation between the input space and output vector. In other words, the complete truth table needs each output vector to be unique.

Investigating the truth table for $\boldsymbol{f_a}$:

b_1	b_2	$\boldsymbol{f_a}(\neg b_1, b_1 \oplus b_2)$
f	**f**	$(\mathbf{t}, \mathbf{f})$
f	**t**	$(\mathbf{t}, \mathbf{t})$
t	**f**	$(\mathbf{f}, \mathbf{t})$
t	**t**	$(\mathbf{f}, \mathbf{f})$

Clearly each output vector has a unique input combination, and so $\boldsymbol{f_a}$ is reversible.

(b) **Consider the Boolean vector function $\boldsymbol{f_b}(b_1, b_2, b_3) = (b_1 \oplus b_2, b_2 \oplus b_3, b_3 \oplus b_1)$. Show that this function is *not* reversible.**

Answer. It is readily deducible that

$$\boldsymbol{f_b}(\mathbf{t}, \mathbf{f}, \mathbf{t}) = (\mathbf{t} \oplus \mathbf{f}, \mathbf{f} \oplus \mathbf{t}, \mathbf{t} \oplus \mathbf{t}) = (\mathbf{t}, \mathbf{t}, \mathbf{f})$$

Furthermore,

$$\boldsymbol{f_b}(\mathbf{f}, \mathbf{t}, \mathbf{f}) = (\mathbf{f} \oplus \mathbf{t}, \mathbf{t} \oplus \mathbf{f}, \mathbf{f} \oplus \mathbf{f}) = (\mathbf{t}, \mathbf{t}, \mathbf{f})$$

So two different input combinations yield the same output; this violates the definition of a reversible function and so $\boldsymbol{f_b}$ is not reversible.

(c) **Is the Boolean vector function $\boldsymbol{f_c}(b_1, b_2, b_3) = (b_1 \oplus b_3, \neg b_3, b_1 \oplus b_2)$ reversible? Justify your answer.**

Answer. By similar technique to (a):

b_1	b_2	b_3	$(b_1 \oplus b_3, \neg b_3, b_1 \oplus b_2)$
$\mathbf{f}$	$\mathbf{f}$	$\mathbf{f}$	$(\mathbf{f}, \mathbf{t}, \mathbf{f})$
$\mathbf{f}$	$\mathbf{f}$	$\mathbf{t}$	$(\mathbf{t}, \mathbf{f}, \mathbf{f})$
$\mathbf{f}$	$\mathbf{t}$	$\mathbf{f}$	$(\mathbf{f}, \mathbf{t}, \mathbf{t})$
$\mathbf{f}$	$\mathbf{t}$	$\mathbf{t}$	$(\mathbf{t}, \mathbf{f}, \mathbf{t})$
$\mathbf{t}$	$\mathbf{f}$	$\mathbf{f}$	$(\mathbf{t}, \mathbf{t}, \mathbf{t})$
$\mathbf{t}$	$\mathbf{f}$	$\mathbf{t}$	$(\mathbf{f}, \mathbf{f}, \mathbf{t})$
$\mathbf{t}$	$\mathbf{t}$	$\mathbf{f}$	$(\mathbf{t}, \mathbf{t}, \mathbf{f})$
$\mathbf{t}$	$\mathbf{t}$	$\mathbf{t}$	$(\mathbf{f}, \mathbf{f}, \mathbf{f})$

All outputs are distinctly unique and so all map to a unique input combination. thus, $\boldsymbol{f_c}$ is reversible.

(d) **Give a formula, in terms of n, for the fraction of distinct n-dimensional Boolean vector functions that are reversible.**

Answer. The formula is

$$\frac{(2^n)!}{(2^n)^{2^n}}$$

This is because of the following:

- The number of n-dimensional binary numbers is 2^n
- For a generic n-dimensional Boolean vector function $(f(n))$, any input combination can have any n-dimensional vector output, i.e. 2^n possible outputs.
- As there are also 2^n possible inputs for any $f(n)$, in total there are $(2^n)^{2^n}$ total mappings of outputs to inputs.
- For reversible $f(n)$, there must be unique outputs for the 2^n possible inputs; thus, there are 2^n unique outputs.
- The number of ways of arranging these are $(2^n)!$ (total reversible functions); so overall, the ratio is $(2^n)!/(2^n)^{2^n}$.