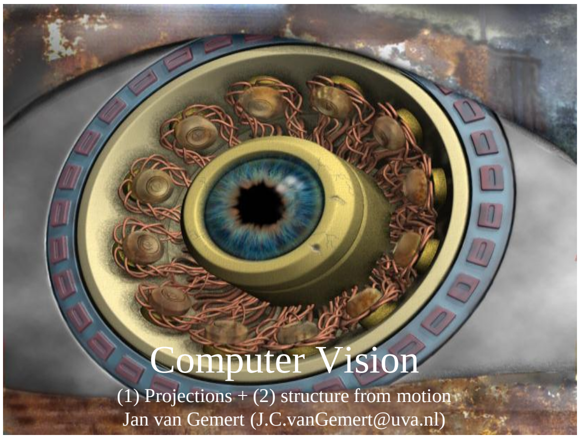


Computer Vision

(1) Projections + (2) structure from motion
Jan van Gemert (J.C.vanGemert@uva.nl)

2

Projection

– Szeliski chapter 2

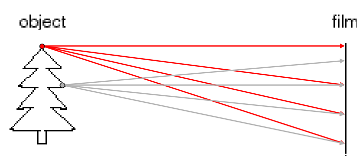
3

Projection

– Szeliski chapter 2

4

Let's design a camera

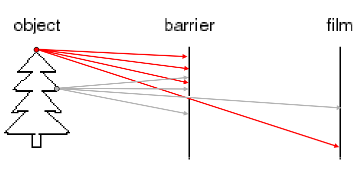


- Idea 1: put a piece of film in front of an object
- Do we get a reasonable image?

Slide by Steve Seitz

5

Pinhole camera

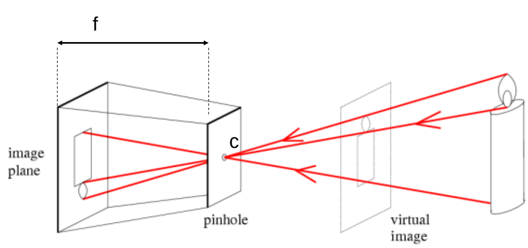


- Add a barrier to block off most of the rays
  - This reduces blurring
  - The opening is known as the **aperture**

Slide by Steve Seitz

6

Pinhole camera



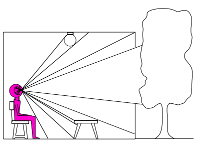
f = focal length
c = center of the camera

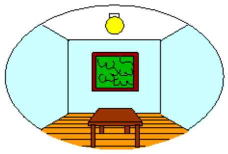
Figure from Forsyth

7

Dimensionality Reduction Machine (3D to 2D)

3D world *2D image*





Point of observation

What have we lost?

- Angles
- Distances (lengths)

Slide by A. Efros
Figures © Stephen E. Palmer, 2002

8

Projection properties

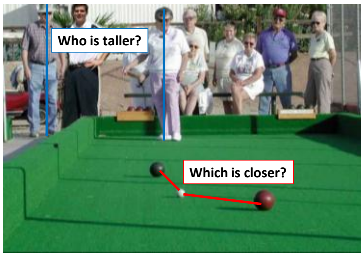
- Many-to-one: any points along same *visual ray* map to same point in image
- Points → points
  - But projection of points on focal plane is undefined
- Lines → lines (collinearity is preserved)
  - But line through focal point (visual ray) projects to a point
- Planes → planes (or half-planes)
  - But plane through focal point projects to line

9

Projective Geometry

What is lost?

- Length



10

Length is not preserved

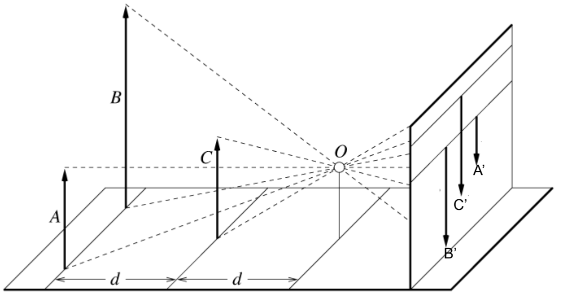


Figure by David Forsyth

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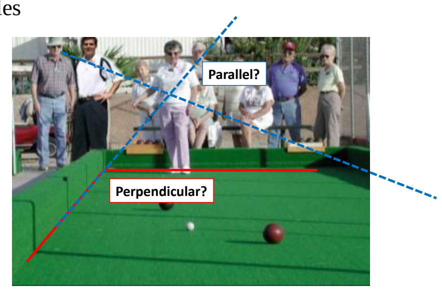
Projective Geometry

What is lost?

- Length
- Angles



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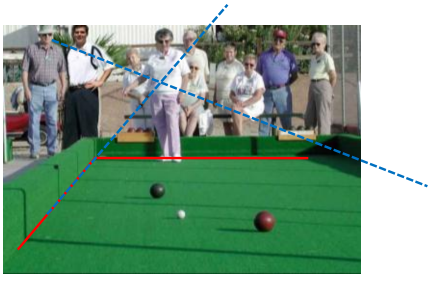
Projective Geometry

What is preserved?

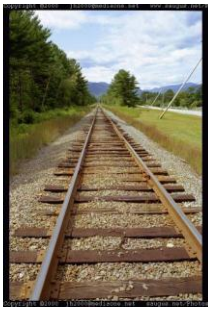
- Straight lines are still straight



13

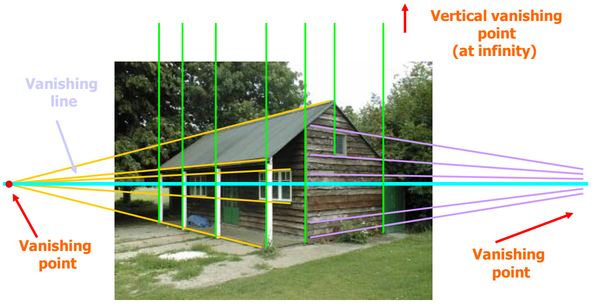
Vanishing points and lines

Parallel lines in the world intersect in the image at a "vanishing point"



14

Vanishing points and lines



Slide from Efros, Photo from Criminisi

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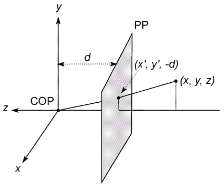
Modeling projection



- The coordinate system
  - We will use the pin-hole model as an approximation
  - Put the optical center (**C**enter **O**f **P**rojection) at the origin
  - Put the image plane (**P**rojection **P**lane) *in front* of the COP
    - Why?
  - The camera looks down the *negative* z axis

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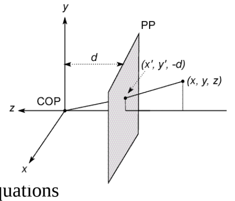
Modeling projection



- Projection equations
  - Compute intersection with PP of ray from (x,y,z) to COP
  - Derived using similar triangles

$$(x, y, z) \rightarrow (-d\frac{x}{z},\ -d\frac{y}{z},\ -d)$$

- We get the projection by throwing out the z coordinate:

$$(x, y, z) \rightarrow (-d\frac{x}{z},\ -d\frac{y}{z})$$

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Homogeneous coordinates

- Is this a linear transformation?

no—division by z is nonlinear

Trick: add one more coordinate:

$$(x, y) \Rightarrow \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \qquad (x, y, z) \Rightarrow \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

homogeneous image coordinates

homogeneous scene coordinates

Converting *from* homogeneous coordinates

$$\begin{bmatrix} x \\ y \\ w \end{bmatrix} \Rightarrow (x/w, y/w) \qquad \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \Rightarrow (x/w, y/w, z/w)$$

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Perspective Projection Matrix

- Projection is a matrix multiplication using homogeneous coordinates:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1/f & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z/f \end{bmatrix} \Rightarrow (f\frac{x}{z}, f\frac{y}{z})$$

divide by the third coordinate

19

Perspective Projection Matrix

- Projection is a matrix multiplication using homogeneous coordinates:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1/f & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z/f \end{bmatrix} \Rightarrow (f\frac{x}{z}, f\frac{y}{z})$$

divide by the third coordinate

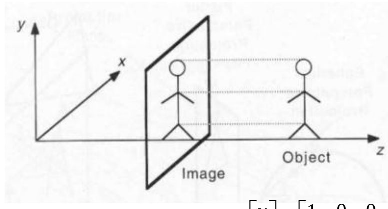
In practice: split into lots of different coordinate transformations…

$$\begin{pmatrix} \text{2D} \\ \text{point} \\ \text{(3x1)} \end{pmatrix} = \begin{pmatrix} \text{Camera to} \\ \text{pixel coord.} \\ \text{trans. matrix} \\ \text{(3x3)} \end{pmatrix} \begin{pmatrix} \text{Perspective} \\ \text{projection matrix} \\ \text{(3x4)} \end{pmatrix} \begin{pmatrix} \text{World to} \\ \text{camera coord.} \\ \text{trans. matrix} \\ \text{(4x4)} \end{pmatrix} \begin{pmatrix} \text{3D} \\ \text{point} \\ \text{(4x1)} \end{pmatrix}$$

20

Orthographic Projection

- Special case of perspective projection
  - Distance from the COP to the image plane is ‘infinite’



  - Also called “parallel projection”
  - What’s the projection matrix?

$$\begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

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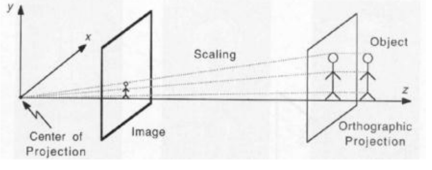
Scaled Orthographic Projection

- Special case of perspective projection
  - Object dimensions are small compared to distance to camera



  - Also called “weak perspective”
  - What’s the projection matrix?

$$\begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \begin{bmatrix} f & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & 0 & s \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

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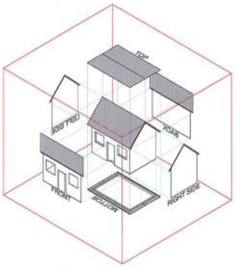
Example of a scaled orthographic projection



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Questions?

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Camera Obscura

Gemma Frisius, 1558

- Basic principle known to Mozi (470-390 BCE), Aristotle (384-322 BCE)
- Drawing aid for artists: described by Leonardo da Vinci (1452-1519)

Source: A. Efros

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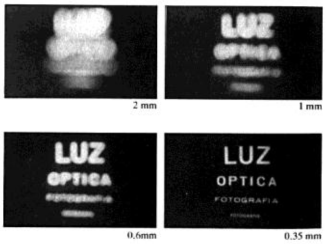
Shrinking the aperture

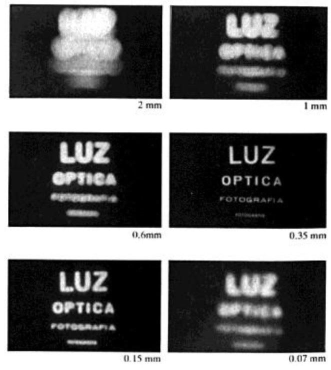


- Why not make the aperture as small as possible?
  - Less light gets through
  - Diffraction effects…

Slide by Steve Seitz

26

Shrinking the aperture



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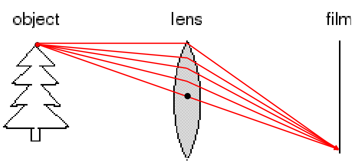
Adding a lens

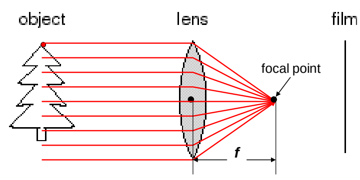


- A lens focuses light onto the film
  - Rays passing through the center are not deviated

Slide by Steve Seitz

28

Adding a lens

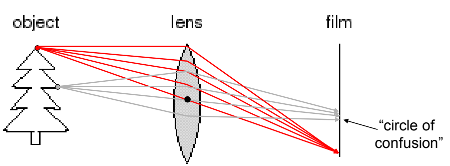


- A lens focuses light onto the film
  - Rays passing through the center are not deviated
  - All parallel rays converge to one point on a plane located at the *focal length f*

Slide by Steve Seitz

29

Adding a lens



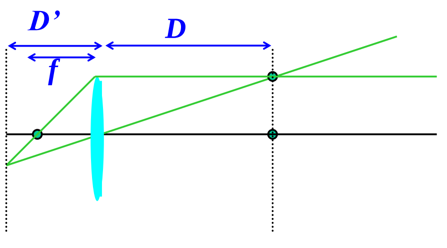
- A lens focuses light onto the film
  - There is a specific distance at which objects are "in focus"
    - other points project to a "circle of confusion" in the image

Slide by Steve Seitz

30

Thin lens formula

Relation between the focal length (f), the distance of the object from the camera (D), and the distance at which the object will be in focus (D')



Frédo Durand's slide

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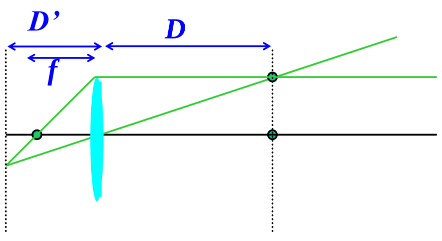
Thin lens formula

Similar triangles everywhere!



Frédo Durand's slide

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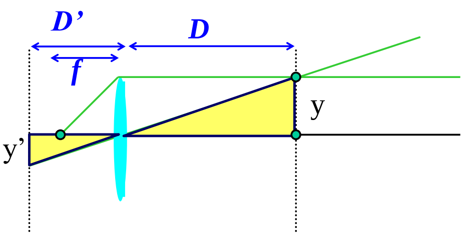
Thin lens formula

$$y'/y = D'/D$$



Frédo Durand's slide

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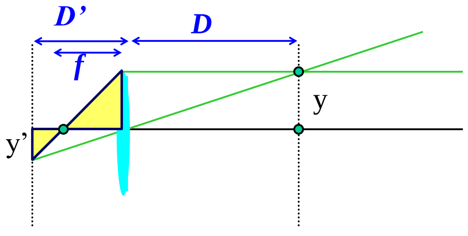
Thin lens formula

$$y'/y = D'/D$$

$$y'/y = (D'-f)/f$$



Frédo Durand's slide

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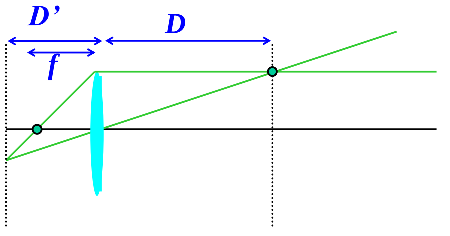
Thin lens formula

$$\boxed{\frac{1}{D'} + \frac{1}{D} = \frac{1}{f}}$$

Any point satisfying the thin lens equation is in focus.



Frédo Durand's slide

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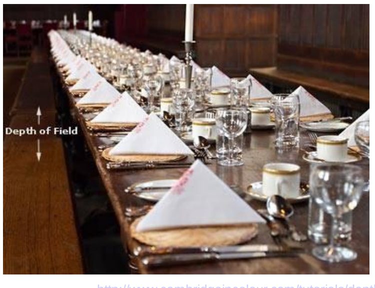
Depth of Field



http://www.cambridgeincolour.com/tutorials/depth-of-field.htm

Slide by A. Efros

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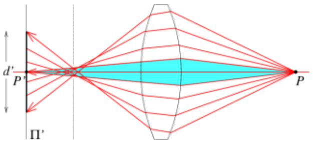
Lens flaws: Spherical aberration

- Spherical lenses don't focus light perfectly
- Rays farther from the optical axis focus closer



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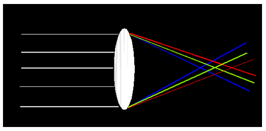
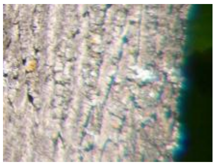
Lens Flaws: Chromatic Aberration

- Lens has different refractive indices for different wavelengths: causes color fringing



Near Lens Center

Near Lens Outer Edge



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Questions?

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Basic 2D Transformations

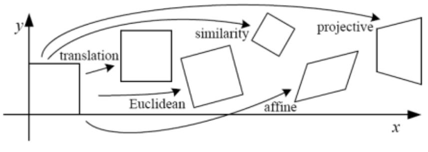

Figure 2.4: *Basic set of 2D planar transformations*

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Basic 2D Transformations

Name	Matrix	# D.O.F.	Preserves:	Icon
translation	$\begin{bmatrix} I \mid t \end{bmatrix}_{2\times 3}$	2	orientation $+\cdots$	
rigid (Euclidean)	$\begin{bmatrix} R \mid t \end{bmatrix}_{2\times 3}$	3	lengths $+\cdots$	
similarity	$\begin{bmatrix} sR \mid t \end{bmatrix}_{2\times 3}$	4	angles $+\cdots$	
affine	$\begin{bmatrix} A \end{bmatrix}_{2\times 3}$	6	parallelism $+\cdots$	
projective	$\begin{bmatrix} \tilde{H} \end{bmatrix}_{3\times 3}$	8	straight lines	

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2D Affine Transformations

Basic 2D transformations as 3x3 matrices

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Translate

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Scale

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\Theta & -\sin\Theta & 0 \\ \sin\Theta & \cos\Theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Rotate

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & \boldsymbol{sh_x} & 0 \\ \boldsymbol{sh_y} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Shear

Source: Alyosha Efros

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2D Affine Transformations

$$\begin{bmatrix} x' \\ y' \\ w \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ w \end{bmatrix}$$

Affine transformations are combinations of …

- Linear transformations, and
- Translations

Parallel lines remain parallel

Grauman

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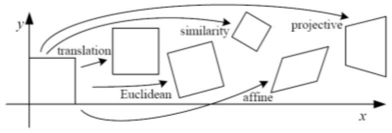
Projective Transformations

$$\begin{bmatrix} x' \\ y' \\ w' \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} x \\ y \\ w \end{bmatrix}$$

Projective transformations:
- Affine transformations, and
- Projective warps

Parallel lines do not necessarily remain parallel



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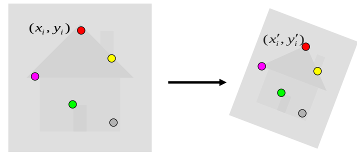
Fitting an affine transformation

- Assuming we know the correspondences, how do we get the transformation?



$$\begin{bmatrix} x_i' \\ y_i' \end{bmatrix} = \begin{bmatrix} m_1 & m_2 \\ m_3 & m_4 \end{bmatrix} \begin{bmatrix} x_i \\ y_i \end{bmatrix} + \begin{bmatrix} t_1 \\ t_2 \end{bmatrix}$$

Grauman

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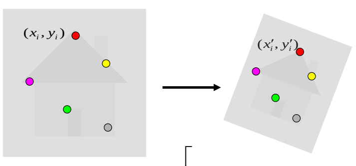
Fitting an affine transformation

- Assuming we know the correspondences, how do we get the transformation?



$$\begin{bmatrix} x_i' \\ y_i' \end{bmatrix} = \begin{bmatrix} m_1 & m_2 \\ m_3 & m_4 \end{bmatrix} \begin{bmatrix} x_i \\ y_i \end{bmatrix} + \begin{bmatrix} t_1 \\ t_2 \end{bmatrix} \qquad \begin{bmatrix} \quad \\ \\ \\ \end{bmatrix} \begin{bmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \\ t_1 \\ t_2 \end{bmatrix} = \begin{bmatrix} \quad \\ \\ \\ \end{bmatrix}$$

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Fitting an affine transformation

$$\begin{bmatrix} & & \cdots & & & \\ x_i & y_i & 0 & 0 & 1 & 0 \\ 0 & 0 & x_i & y_i & 0 & 1 \\ & & \cdots & & & \end{bmatrix} \begin{bmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \\ t_1 \\ t_2 \end{bmatrix} = \begin{bmatrix} \cdots \\ x_i' \\ y_i' \\ \cdots \end{bmatrix}$$

- How many matches (correspondence pairs) do we need to solve for the transformation parameters?
- Linear system with six unknowns. Each match gives us 2 linearly independent equations: need at least 3 to solve

Grauman

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Panoramas

Obtain a wider angle view by combining multiple images.

Grauman

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How to stitch together a panorama?

- Basic Procedure
  - Take a sequence of images from the same position
    - Rotate the camera about its optical center
  - Compute transformation between second image and first
  - Transform the second image to overlap with the first
  - Blend the two together to create a mosaic
  - (If there are more images, repeat)
- …but **wait**, why should this work at all?
  - What about the 3D geometry of the scene?

Source: Steve Seitz

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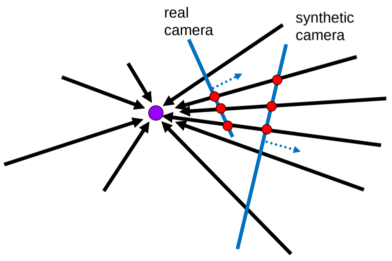
Panoramas: generating synthetic views



Can generate any synthetic camera view
as long as it has **the same center of projection**!

Source: Alyosha Efros

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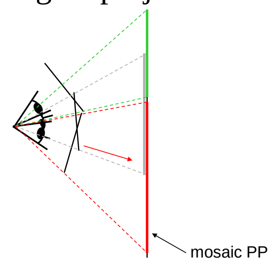
Image reprojection



The mosaic has a natural interpretation in 3D
- The images are reprojected onto a common plane
- The mosaic is formed on this plane
- Mosaic is a *synthetic wide-angle camera*

Source: Steve Seitz

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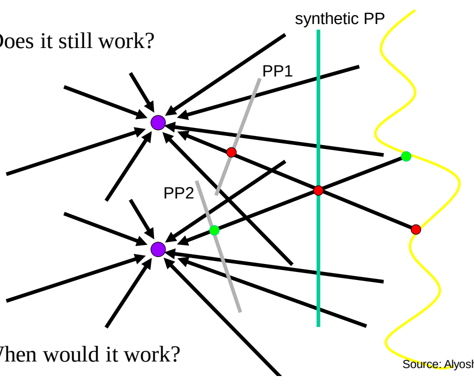
changing camera center

Does it still work?



When would it work?

Source: Alyosha Efros

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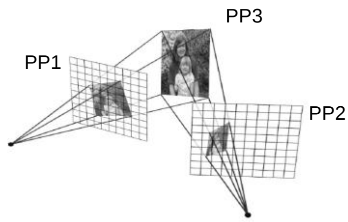
Planar scene (or far away)



PP3 is a projection plane of both centers of projection, so we are OK!

This is how big aerial photographs are made

Source: Alyosha Efros

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Grauman

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Homography (projective transform)

How to relate two images from the same camera center?

- how to map a pixel from PP1 to PP2?

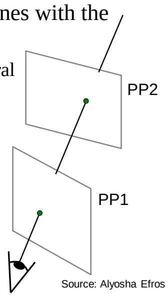
A 2D **image warp** from one image to another.

A mapping between any two Projective Planes with the same center of projection

- rectangle should map to arbitrary quadrilateral
- parallel lines aren't preserved
- but must preserve straight lines

called **Homography**

$$\underbrace{\begin{bmatrix} wx' \\ wy' \\ w \end{bmatrix}}_{\mathbf{p'}} = \underbrace{\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \end{bmatrix}}_{\mathbf{H}} \underbrace{\begin{bmatrix} x \\ y \\ 1 \end{bmatrix}}_{\mathbf{p}}$$



Source: Alyosha Efros

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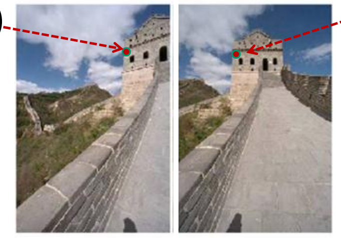
Homography

(x, y)



$\left(\frac{wx'}{w}, \frac{wy'}{w}\right) = (x', y')$

To **apply** a given homography **H**

- Compute **p'** = **Hp** (regular matrix multiply)
- Convert **p'** from homogeneous to image coordinates

$$\underbrace{\begin{bmatrix} wx' \\ wy' \\ w \end{bmatrix}}_{\mathbf{p'}} = \underbrace{\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \end{bmatrix}}_{\mathbf{H}} \underbrace{\begin{bmatrix} x \\ y \\ 1 \end{bmatrix}}_{\mathbf{p}}$$

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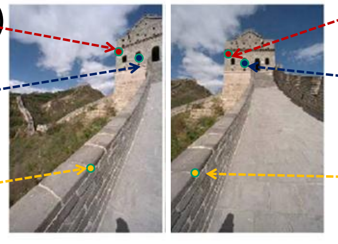
Homography

(x_1, y_1) (x_2, y_2) $\vdots$ (x_n, y_n)



(x'_1, y'_1) (x'_2, y'_2) $\vdots$ (x'_n, y'_n)

To **compute** the homography given pairs of corresponding points in the images, we need to set up an equation where the parameters of **H** are the unknowns…

Grauman

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Solving for homographies

p' = Hp

$$\begin{bmatrix} wx' \\ wy' \\ w \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Can set scale factor i=1. So, there are 8 unknowns.

Set up a system of linear equations:

Ah = b

where vector of unknowns h = $[a,b,c,d,e,f,g,h]^T$

Need at least 8 eqs, but the more the better…

Solve for h. If overconstrained, solve using least-squares:

$$\min \|Ah - b\|^2$$

Grauman

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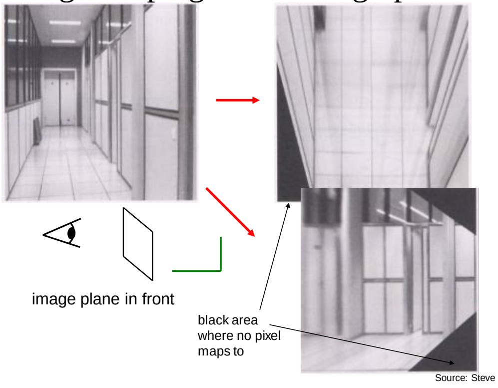
Image warping with homographies



Source: Steve Seitz

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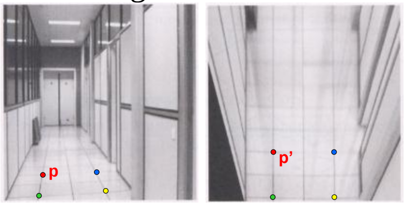
Image rectification



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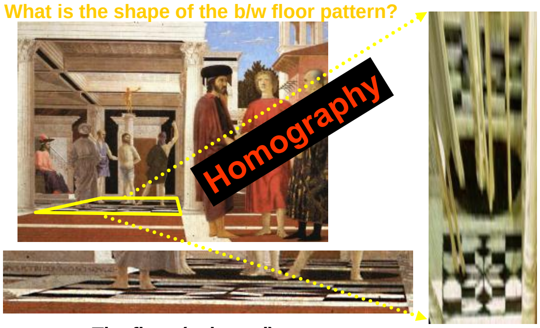
Analysing patterns and shapes

What is the shape of the b/w floor pattern?



The floor (enlarged)

Automatically rectified floor

Slide: Criminisi

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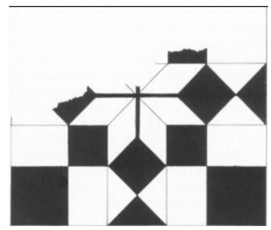
Analysing patterns and shapes

Automatic rectification



From Martin Kemp *The Science of Art*
(manual reconstruction)

Slide: Criminisi

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Analysing patterns and shapes

St. Lucy Altarpiece, **D. Veneziano**

What is the (complicated) shape of the floor pattern?

Automatically rectified floor

Slide: Criminisi

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Analysing patterns and shapes

Automatic rectification

From Martin Kemp, *The Science of Art*
(manual reconstruction)

Slide from Criminisi

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Questions?

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Recap: How to stitch together a panorama?

- Basic Procedure
  - Take a sequence of images from the same position
    - Rotate the camera about its optical center
  - Compute transformation between second image and first
  - Transform the second image to overlap with the first
  - Blend the two together to create a mosaic
  - (If there are more images, repeat)

Source: Steve Seitz

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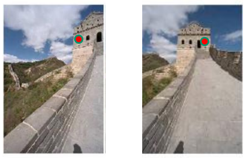
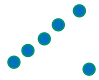
Outliers

- **Outliers** can hurt the quality of our parameter estimates, e.g.,
  - an erroneous pair of matching points from two images
  - an edge point that is noise, or doesn't belong to the line we are fitting.





Grauman

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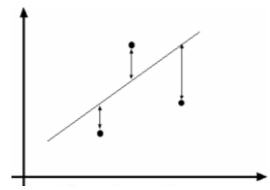
Example: least squares line fitting

- Assuming all the points that belong to a particular line are known



Grauman

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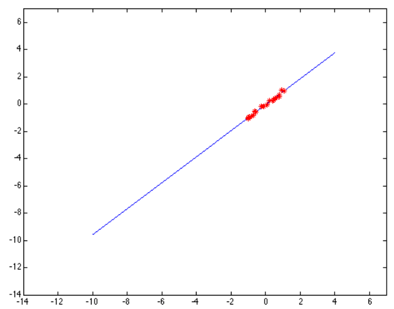
Outliers affect least squares fit



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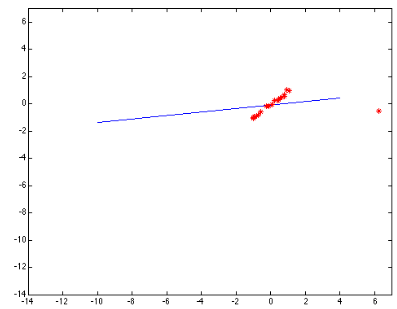
Outliers affect least squares fit



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RANSAC

- RANdom Sample Consensus

- Approach: we want to avoid the impact of outliers, so let's look for "inliers", and use those only.

- Intuition: if an outlier is chosen to compute the current fit, then the resulting line won't have much support from rest of the points.

Fischler & Bolles in '81.

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RANSAC

- RANSAC loop:

1. Randomly select a *seed group* of points on which to base transformation estimate (e.g., a group of matches)
2. Compute transformation from seed group
3. Find *inliers* to this transformation
4. If the number of inliers is sufficiently large, re-compute least-squares estimate of transformation on all of the inliers

- Keep the transformation with the largest number of inliers

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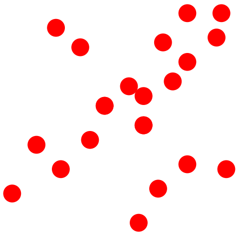
RANSAC

(RANdom SAmple Consensus) :

Fischler & Bolles in '81.



Algorithm:

1. **Sample** (randomly) the number of points required to fit the model
2. **Solve** for model parameters using samples
3. **Score** by the fraction of inliers within a preset threshold of the model

Repeat 1-3 until the best model is found with high confidence

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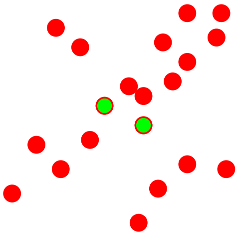
RANSAC

Line fitting example



Algorithm:

1. **Sample** (randomly) the number of points required to fit the model (#=2)
2. **Solve** for model parameters using samples
3. **Score** by the fraction of inliers within a preset threshold of the model

Repeat 1-3 until the best model is found with high confidence

Illustration by Savarese

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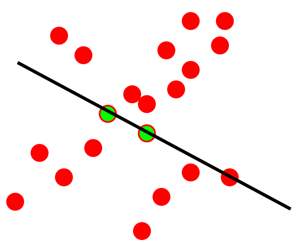
RANSAC

Line fitting example



Algorithm:

1. **Sample** (randomly) the number of points required to fit the model (#=2)
2. **Solve** for model parameters using samples
3. **Score** by the fraction of inliers within a preset threshold of the model

Repeat 1-3 until the best model is found with high confidence

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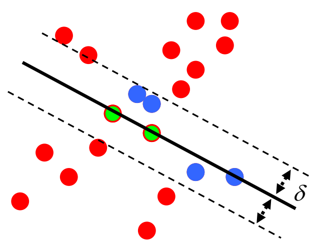
RANSAC

Line fitting example

$N_I = 6$



Algorithm:

1. **Sample** (randomly) the number of points required to fit the model (#=2)
2. **Solve** for model parameters using samples
3. **Score** by the fraction of inliers within a preset threshold of the model

Repeat 1-3 until the best model is found with high confidence

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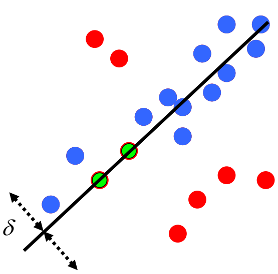
RANSAC



$N_I = 14$

Algorithm:

1. **Sample** (randomly) the number of points required to fit the model (#=2)
2. **Solve** for model parameters using samples
3. **Score** by the fraction of inliers within a preset threshold of the model

Repeat 1-3 until the best model is found with high confidence

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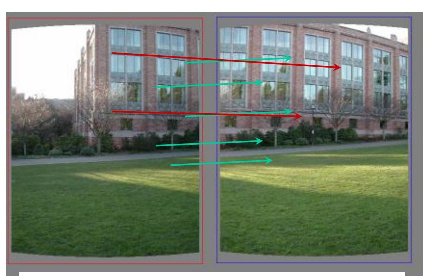
RANSAC example: Translation



Putative matches

Source: Rick Szeliski

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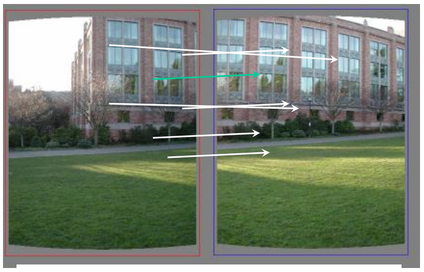
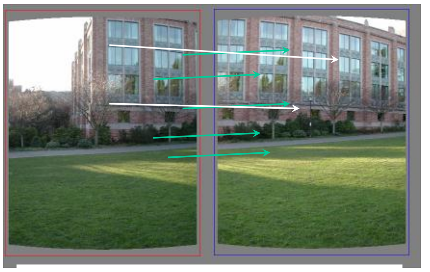
RANSAC example: Translation



Select *one* match, count *inliers*

81

RANSAC example: Translation



Select *one* match, count *inliers*

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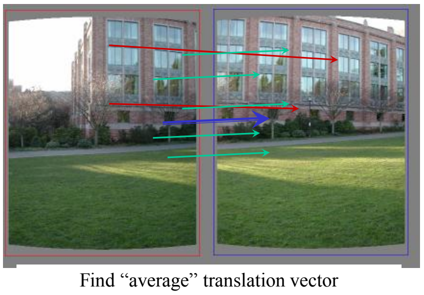
RANSAC example: Translation



Find “average” translation vector

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Feature-based alignment outline

Source: L. Lazebnik

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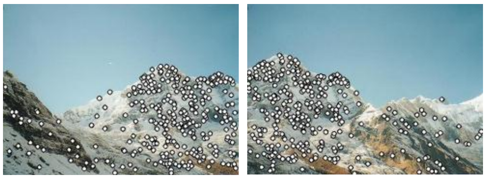
Feature-based alignment outline



- Extract features

Source: L. Lazebnik

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Feature-based alignment outline

- Extract features
- Compute *putative matches*

Source: L. Lazebnik

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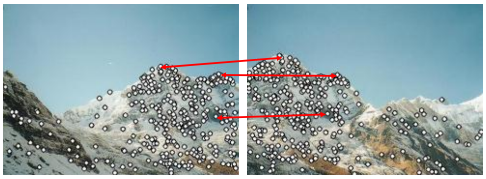
Feature-based alignment outline



- Extract features
- Compute *putative matches*
- Loop:
  - *Hypothesize* transformation *T* (small group of putative matches that are related by *T*)

Source: L. Lazebnik

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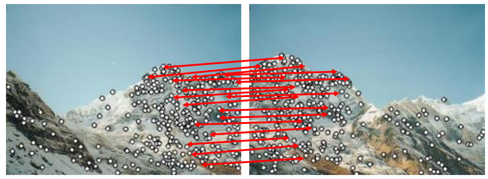
Feature-based alignment outline



- Extract features
- Compute *putative matches*
- Loop:
  - *Hypothesize* transformation *T* (small group of putative matches that are related by *T*)
  - *Verify* transformation (search for other matches consistent with *T*)

Source: L. Lazebnik

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Feature-based alignment outline

- Extract features
- Compute *putative matches*
- Loop:
  - *Hypothesize* transformation *T* (small group of putative matches that are related by *T*)
  - *Verify* transformation (search for other matches consistent with *T*)

Source: L. Lazebnik

How well does this work?

Test on 100s of examples…

How well does this work?

Test on 100s of examples…

…still too many failures (5-10%)
for consumer application

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Matching Mistakes: False Positive

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Matching Mistakes: False Positive

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Matching Mistake: False Negative

- Moving objects: large areas of disagreement (ghosting)

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Matching Mistakes

- Accidental alignment
  - repeated / similar regions
- Failed alignments
  - moving objects / parallax
  - low overlap
  - “feature-less” regions (more variety?)
- No 100% reliable algorithm?

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How can we fix these?

- Tune the feature detector
- Tune the feature matcher (cost metric)
- Tune the RANSAC stage (motion model)
- Tune the verification stage
- Use “higher-level” knowledge
  - e.g., typical camera motions

- → Sounds like a big “learning” problem
  - Need a large training/test data set (panoramas)

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Questions?

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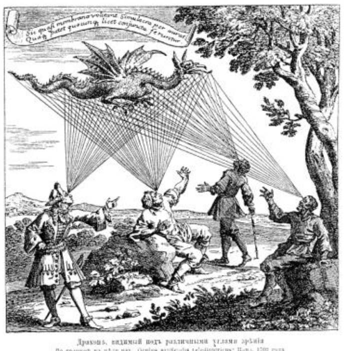
Structure from motion



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Multiple-view geometry questions

- **Scene geometry (structure):** Given 2D point matches in two or more images, where are the corresponding points in 3D?
- **Correspondence (stereo matching):** Given a point in just one image, how does it constrain the position of the corresponding point in another image?
- **Camera geometry (motion):** Given a set of corresponding points in two or more images, what are the camera matrices for these views?

Slide: S. Lazebnik

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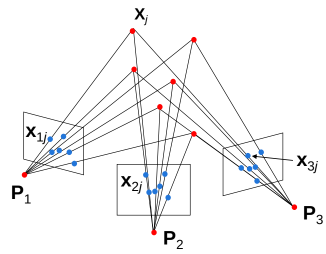
Structure from motion

- Given: m images of n fixed 3D points
  - $\mathbf{x}_{ij} = \mathbf{P}_i \mathbf{X}_j,\ i = 1, \dots, m, \quad j = 1, \dots, n$
- Problem: estimate m projection matrices $\mathbf{P}_i$ and n 3D points $\mathbf{X}_j$ from the mn correspondences $\mathbf{x}_{ij}$



Slide: S. Lazebnik

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Structure from motion ambiguity

- If we scale the entire scene by some factor k and, at the same time, scale the camera matrices by the factor of $1/k$, the projections of the scene points in the image remain exactly the same:

$$\mathbf{x} = \mathbf{PX} = \left(\frac{1}{k}\mathbf{P}\right)(k\mathbf{X})$$

It is impossible to recover the absolute scale of the scene!

Slide: S. Lazebnik

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Structure from motion ambiguity

- If we scale the entire scene by some factor k and, at the same time, scale the camera matrices by the factor of $1/k$, the projections of the scene points in the image remain exactly the same
- More generally: if we transform the scene using a transformation $\mathbf{Q}$ and apply the inverse transformation to the camera matrices, then the images do not change

$$\mathbf{x} = \mathbf{PX} = \left(\mathbf{PQ}^{-1}\right)\left(\mathbf{QX}\right)$$

Slide: S. Lazebnik

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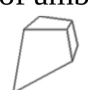
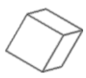
Types of ambiguity

Projective 15dof	$\begin{bmatrix} A & t \\ v^T & v \end{bmatrix}$		Preserves intersection and tangency
Affine 12dof	$\begin{bmatrix} A & t \\ 0^T & 1 \end{bmatrix}$		Preserves parallellism, volume ratios
Similarity 7dof	$\begin{bmatrix} sR & t \\ 0^T & 1 \end{bmatrix}$		Preserves angles, ratios of length
Euclidean 6dof	$\begin{bmatrix} R & t \\ 0^T & 1 \end{bmatrix}$		Preserves angles, lengths

- With no constraints on the camera matrix or on the scene, we get a *projective* reconstruction
- Need additional information to *upgrade* the reconstruction to affine, similarity, or Euclidean

Slide: S. Lazebnik

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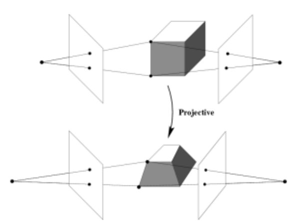
Projective ambiguity



$$\mathbf{x} = \mathbf{PX} = \left(\mathbf{PQ}_\mathbf{P}^{-1}\right)\left(\mathbf{Q}_\mathbf{P}\mathbf{X}\right)$$

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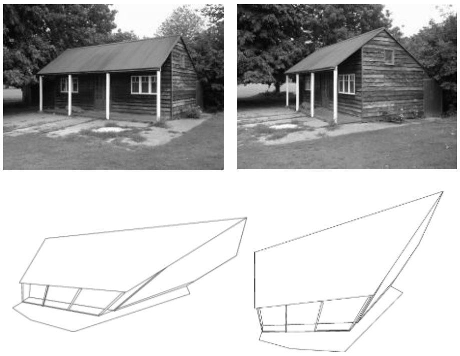
Projective ambiguity



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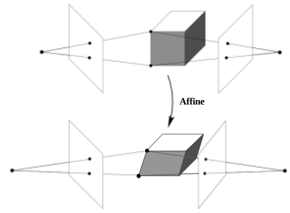
Affine ambiguity



$$\mathbf{x} = \mathbf{P}\mathbf{X} = \left(\mathbf{P}\mathbf{Q}_{\mathbf{A}}^{-1}\right)\left(\mathbf{Q}_{\mathbf{A}}\mathbf{X}\right)$$

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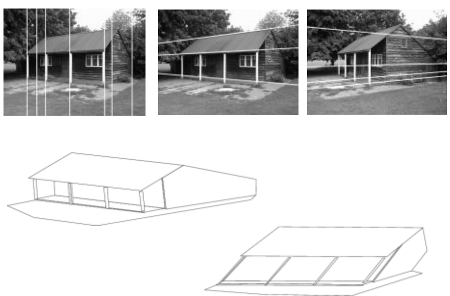
Affine ambiguity



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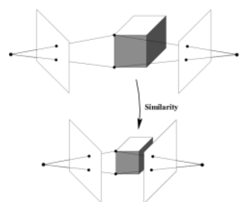
Similarity ambiguity



$$\mathbf{x} = \mathbf{P}\mathbf{X} = \left(\mathbf{P}\mathbf{Q}_{\mathbf{S}}^{-1}\right)\left(\mathbf{Q}_{\mathbf{S}}\mathbf{X}\right)$$

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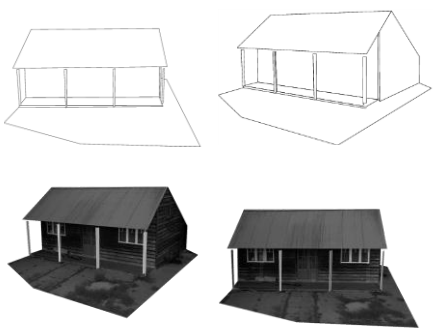
Similarity ambiguity



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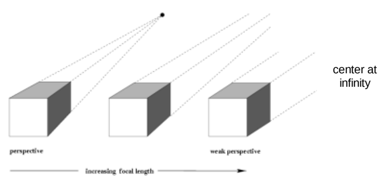
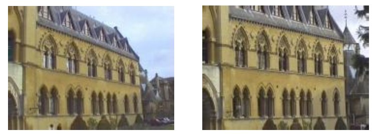
Affine Structure from motion





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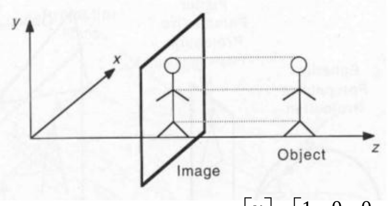
Recall: Orthographic Projection

- Special case of perspective projection
  - Distance from the COP to the image plane is 'infinite'



  - Also called "parallel projection"
  - Projection matrix:

$$\begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

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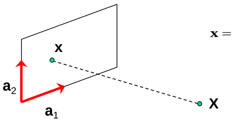
Affine cameras

- A general affine camera combines the effects of an affine transformation of the 3D space, orthographic projection, and an affine transformation of the image:

$$\mathbf{P} = [3\times3\,\text{affine}] \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} [4\times4\,\text{affine}] = \begin{bmatrix} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{b} \\ \mathbf{0} & \mathbf{1} \end{bmatrix}$$

Affine projection is a linear mapping + translation in inhomogeneous coordinates



$$\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \mathbf{AX} + \mathbf{b}$$

Projection of world origin

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Affine structure from motion

- Given: m images of n fixed 3D points:
  - $\mathbf{x}_{ij} = \mathbf{A}_i \mathbf{X}_j + \mathbf{b}_i$, $i = 1, \dots, m, \; j = 1, \dots, n$
- Problem: use the mn correspondences $\mathbf{x}_{ij}$ to estimate m projection matrices $\mathbf{A}_i$ and translation vectors $\mathbf{b}_i$, and n points $\mathbf{X}_j$
- The reconstruction is defined up to an arbitrary *affine* transformation $\mathbf{Q}$ (12 degrees of freedom):

$$\begin{bmatrix} \mathbf{A} & \mathbf{b} \\ \mathbf{0} & \mathbf{1} \end{bmatrix} \rightarrow \begin{bmatrix} \mathbf{A} & \mathbf{b} \\ \mathbf{0} & \mathbf{1} \end{bmatrix} \mathbf{Q}^{-1}, \qquad \begin{pmatrix} \mathbf{X} \\ \mathbf{1} \end{pmatrix} \rightarrow \mathbf{Q} \begin{pmatrix} \mathbf{X} \\ \mathbf{1} \end{pmatrix}$$

- We have $2mn$ knowns and $8m + 3n$ unknowns (minus 12 dof for affine ambiguity)
- Thus, we must have $2mn >= 8m + 3n - 12$
- For two views, we need four point correspondences

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Affine structure from motion

- Centering: subtract the centroid of the image points (removes translation)

$$\hat{\mathbf{x}}_{ij} = \mathbf{x}_{ij} - \frac{1}{n}\sum_{k=1}^{n} \mathbf{x}_{ik} = \mathbf{A}_i \mathbf{X}_j + \mathbf{b}_i - \frac{1}{n}\sum_{k=1}^{n} (\mathbf{A}_i \mathbf{X}_k + \mathbf{b}_i)$$
$$= \mathbf{A}_i \left(\mathbf{X}_j - \frac{1}{n}\sum_{k=1}^{n} \mathbf{X}_k \right) = \mathbf{A}_i \hat{\mathbf{X}}_j$$

- For simplicity, assume that the origin of the world coordinate system is at the centroid of the 3D points
- After centering, each normalized point $\mathbf{x}_{ij}$ is related to the 3D point $\mathbf{X}_i$ by $\hat{\mathbf{x}}_{ij} = \mathbf{A}_i \mathbf{X}_j$

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Affine structure from motion

- Let's create a $2m \times n$ data (measurement) matrix:

$$\mathbf{D} = \begin{bmatrix} \hat{\mathbf{x}}_{11} & \hat{\mathbf{x}}_{12} & \cdots & \hat{\mathbf{x}}_{1n} \\ \hat{\mathbf{x}}_{21} & \hat{\mathbf{x}}_{22} & \cdots & \hat{\mathbf{x}}_{2n} \\ & & \ddots & \\ \hat{\mathbf{x}}_{m1} & \hat{\mathbf{x}}_{m2} & \cdots & \hat{\mathbf{x}}_{mn} \end{bmatrix}$$

cameras ($2m$)

points (n)

C. Tomasi and T. Kanade. Shape and motion from image streams under orthography: A factorization method. *IJCV*, 9(2):137-154, November 1992.

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Affine structure from motion

- Let's create a $2m \times n$ data (measurement) matrix:

$$\mathbf{D} = \begin{bmatrix} \hat{\mathbf{x}}_{11} & \hat{\mathbf{x}}_{12} & \cdots & \hat{\mathbf{x}}_{1n} \\ \hat{\mathbf{x}}_{21} & \hat{\mathbf{x}}_{22} & \cdots & \hat{\mathbf{x}}_{2n} \\ & & \ddots & \\ \hat{\mathbf{x}}_{m1} & \hat{\mathbf{x}}_{m2} & \cdots & \hat{\mathbf{x}}_{mn} \end{bmatrix} = \begin{bmatrix} \mathbf{A}_1 \\ \mathbf{A}_2 \\ \vdots \\ \mathbf{A}_m \end{bmatrix} \begin{bmatrix} \mathbf{X}_1 & \mathbf{X}_2 & \cdots & \mathbf{X}_n \end{bmatrix}$$

points ($3 \times n$)

cameras ($2m \times 3$)

The measurement matrix $\mathbf{D} = \mathbf{MS}$ must have rank 3!

C. Tomasi and T. Kanade. Shape and motion from image streams under orthography: A factorization method. *IJCV*, 9(2):137-154, November 1992.

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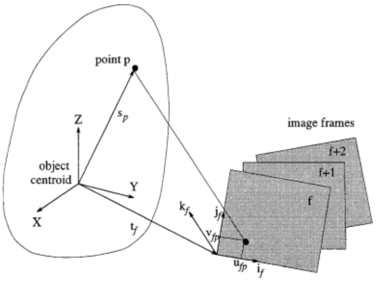
Affine structure from motion

Main idea: You only need M camera orientations and N points



Because Affine transformation is linear, use a linear factorization

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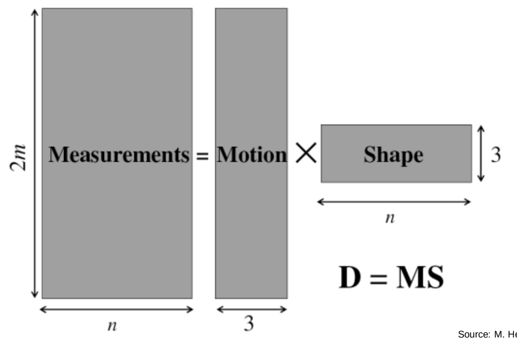
Factorizing the measurement matrix



Source: M. Hebert

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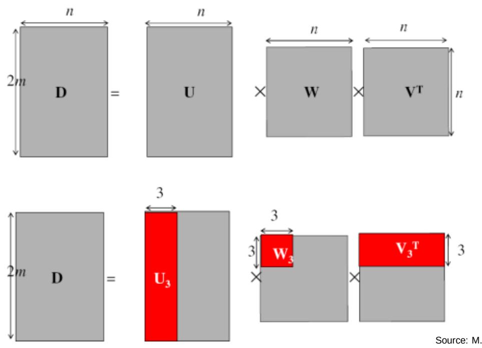
Factorizing the measurement matrix

- Singular value decomposition of D:



Source: M. Hebert

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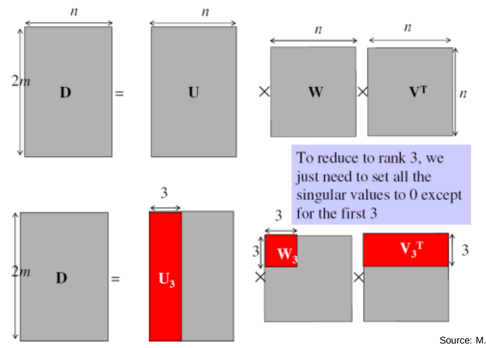
Factorizing the measurement matrix

- Singular value decomposition of D:



Source: M. Hebert

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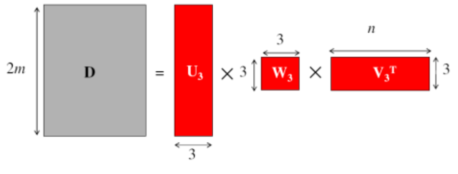
Factorizing the measurement matrix

- Obtaining a factorization from SVD:



Source: M. Hebert

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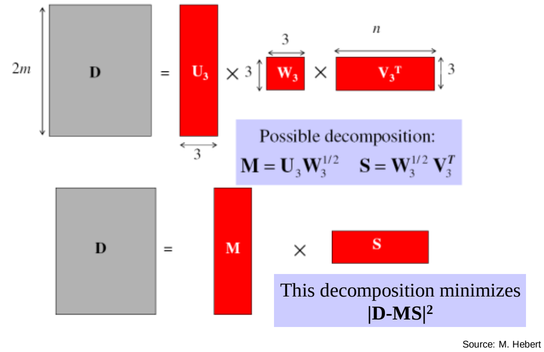
Factorizing the measurement matrix

- Obtaining a factorization from SVD:



Source: M. Hebert

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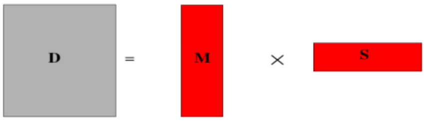
Affine ambiguity



- The decomposition is not unique. We get the same **D** by using any 3×3 matrix **C** and applying the transformations **M** → **MC, S** →**C**$^{-1}$**S**
  (The true M and S are a linear transformation of M, S)
- That is because we have only an affine transformation and we have not enforced any Euclidean constraints (like forcing the image axes to be perpendicular, for example)

Source: M. Hebert

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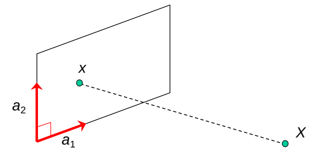
Eliminating the affine ambiguity

- Orthographic: image axes are perpendicular and scale is 1



$\mathbf{a}_1 \cdot \mathbf{a}_2 = 0$

$|\mathbf{a}_1|^2 = |\mathbf{a}_2|^2 = 1$

$\tilde{\mathbf{a}}_{i1}^T \mathbf{C}\mathbf{C}^T \tilde{\mathbf{a}}_{i1}^T = 1$

$\tilde{\mathbf{a}}_{i2}^T \mathbf{C}\mathbf{C}^T \tilde{\mathbf{a}}_{i2}^T = 1$

$\tilde{\mathbf{a}}_{i1}^T \mathbf{C}\mathbf{C}^T \tilde{\mathbf{a}}_{i2}^T = 0$

- This translates into 3m equations in $\mathbf{L} = \mathbf{CC}^T$:

  $\mathbf{A}_i \mathbf{L} \mathbf{A}_i^T = \mathbf{Id}$, $\quad i = 1, \ldots, m$

  - Solve for **L**
  - Recover **C** from **L** by Cholesky decomposition: $\mathbf{L} = \mathbf{CC}^T$
  - Update **M** and **S**: $\mathbf{M} = \mathbf{MC}$, $\mathbf{S} = \mathbf{C}^{-1}\mathbf{S}$

Lazebnik

Source: M. Hebert

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Algorithm summary

- Given: m images and n features $\mathbf{x}_{ij}$
- For each image i, center the feature coordinates
- Construct a 2m × n measurement matrix **D**:
  - Column j contains the projection of point j in all views
  - Row i contains one coordinate of the projections of all the n points in image i
- Factorize **D**:
  - Compute SVD: $\mathbf{D} = \mathbf{U}\,\mathbf{W}\,\mathbf{V}^T$
  - Create $\mathbf{U}_3$ by taking the first 3 columns of **U**
  - Create $\mathbf{V}_3$ by taking the first 3 columns of **V**
  - Create $\mathbf{W}_3$ by taking the upper left 3 × 3 block of **W**
- Create the motion and shape matrices:
  - $\mathbf{M} = \mathbf{U}_3\mathbf{W}_3^{1/2}$ and $\mathbf{S} = \mathbf{W}_3^{1/2}\,\mathbf{V}_3^T$ (**or** $\mathbf{M} = \mathbf{U}_3$ and $\mathbf{S} = \mathbf{W}_3\mathbf{V}_3^T$)
- Eliminate affine ambiguity

Source: M. Hebert

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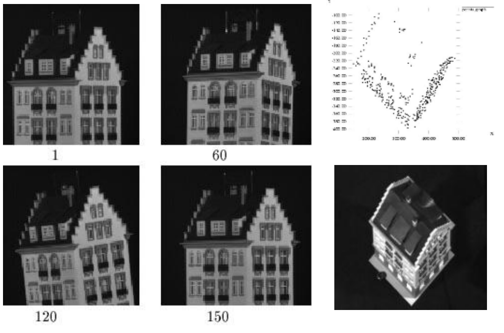
Reconstruction results



C. Tomasi and T. Kanade. Shape and motion from image streams under orthography: A factorization method. *IJCV*, 9(2):137-154, November 1992.

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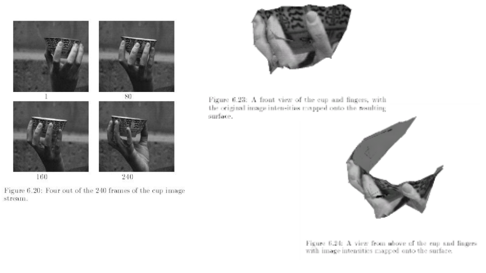
Results



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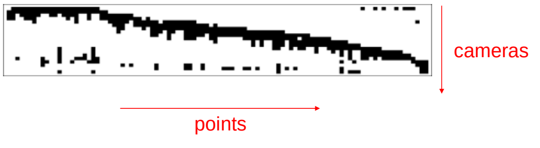
Dealing with missing data

- So far, we have assumed that all points are visible in all views
- In reality, the measurement matrix typically looks something like this:



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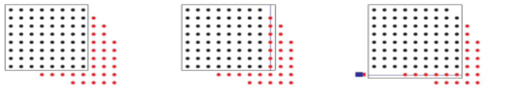
Dealing with missing data

- Possible solution: decompose matrix into dense sub-blocks, factorize each sub-block, and fuse the results



(1) Perform factorization on a dense sub-block

(2) Solve for a new 3D point visible by at least two known cameras (linear least squares)

(3) Solve for a new camera that sees at least three known 3D points (linear least squares)

F. Rothganger, S. Lazebnik, C. Schmid, and J. Ponce. Segmenting, Modeling, and Matching Video Clips Containing Multiple Moving Objects. PAMI 2007.

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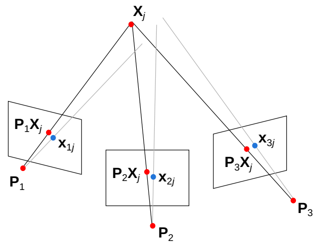
Bundle adjustment

- Non-linear method for refining structure and motion
- Minimizing reprojection error

$$E(\mathbf{P},\mathbf{X}) = \sum_{i=1}^{m}\sum_{j=1}^{n} D\left(\mathbf{x}_{ij}, \mathbf{P}_i\mathbf{X}_j\right)^2$$



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Further Factorization work

Factorization with uncertainty
(Irani & Anandan, IJCV'02)

Factorization for indep. moving objects
(Costeira and Kanade '94)

Factorization for articulated objects
(Yan and Pollefeys '05)

Factorization for dynamic objects
(Bregler et al. 2000, Brand 2001)

Perspective factorization
(Sturm & Triggs 1996, …)

Factorization with outliers and missing pts.
(Jacobs '97 (affine), Martinek & Pajdla'01 Aanaes'02 (perspective))

Pollefeys

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Questions?

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Summary

- Projections
- Pin-hole camera
- Transformation
- Homography
- Ransac
- Structure from Motion