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Solution:
Markov Chain

Exercise 1: Consider a MC with a transition matrix given by

$$\mathbb{T} = \begin{bmatrix} 1-a & a \\ b & 1-b \end{bmatrix}, \tag{1}$$

where a and b belongs to $[0,1]$ such that $0 < a+b < 2$ and suppose that the initial system is given by

$$^t\mathbb{P}_{X_0} = {}^t\begin{bmatrix} \mathbb{P}\{X_0 = 0\} & \mathbb{P}\{X_0 = 1\} \end{bmatrix} = {}^t\begin{bmatrix} \alpha & 1-\alpha \end{bmatrix},$$

for some $\alpha \in]0,1[$.

i. Show that

$$\mathbb{T}^n = \frac{1}{a+b}\begin{bmatrix} b & a \\ b & a \end{bmatrix} + \frac{(1-a-b)^n}{a+b}\begin{bmatrix} a & -a \\ -b & b \end{bmatrix}.$$

We need first to compute the characteristic polynomial

$$\begin{aligned} P(\lambda) = det[\mathbb{T} - \lambda] &= \begin{vmatrix} 1-a-\lambda & a \\ b & 1-b-\lambda \end{vmatrix} \\ &= \begin{vmatrix} 1-\lambda & a \\ 1-\lambda & 1-b-\lambda \end{vmatrix} = (1-\lambda)\begin{vmatrix} 1 & a \\ 1 & 1-b-\lambda \end{vmatrix} \\ &= (\lambda-1)(a+b+\lambda-1) \end{aligned}$$

ii. Deduce the

$$\lim_{n\to\infty} \mathbb{T}^n$$

According to i. we have two eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 1-(a+b)$. If E_1 is the eigen-space associated to $\lambda_1 = 1$, then
$(x,y) \in E_1$ if and only if $(\mathbb{T}-1)\begin{vmatrix} x \\ y \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \end{vmatrix}$.

$$\begin{aligned} (\mathbb{T}-1)\begin{vmatrix} x \\ y \end{vmatrix} &= \begin{bmatrix} -a & a \\ b & -b \end{bmatrix}\begin{vmatrix} x \\ y \end{vmatrix} \\ &= \begin{vmatrix} -ax+ay \\ bx-by \end{vmatrix} \end{aligned}$$

since $(a,b) \neq (0,0)$,

we get

$$(\mathbb{T}-1)\left|\begin{array}{c} x \\ y \end{array}\right. = \left|\begin{array}{c} 0 \\ 0 \end{array}\right. \Leftrightarrow x-y=0,$$

and so $E_1 = \mathbb{R}\left|\begin{array}{c} 1 \\ 1 \end{array}\right.$. In the same way we obtain $E_2 = \mathbb{R}\left|\begin{array}{c} -a/b \\ 1 \end{array}\right.$, where we supposed that $b \neq 0$, this is possible because $(a,b) \neq (0,0)$. Finally, if we denote by $P = \left[\begin{array}{cc} -1 & -a/b \\ 1 & -1 \end{array}\right]$, we can write

$$\mathbb{T} = P\left[\begin{array}{cc} 1 & 0 \\ 0 & 1-(a+b) \end{array}\right]P^{-1} = \left[\begin{array}{cc} 1 & -a/b \\ 1 & 1 \end{array}\right]\left[\begin{array}{cc} 1 & 0 \\ 0 & 1-(a+b) \end{array}\right]\frac{1}{a+b}\left[\begin{array}{cc} b & a \\ -b & b \end{array}\right],$$

and so for all $n \geq 1$

$$\mathbb{T}^n = P\left[\begin{array}{cc} 1 & 0 \\ 0 & 1-(a+b) \end{array}\right]^n P^{-1} = \left[\begin{array}{cc} 1 & -a/b \\ 1 & 1 \end{array}\right]\left[\begin{array}{cc} 1 & 0 \\ 0 & (1-(a+b))^n \end{array}\right]\frac{1}{a+b}\left[\begin{array}{cc} b & a \\ -b & b \end{array}\right],$$

which by straightforward computation leads to the desired result.

iii. Deduce then its steady state

$$^t\pi = {}^t\left[\begin{array}{cc} \frac{b}{a+b} & \frac{a}{a+b} \end{array}\right].$$

Since $0 < a+b < 2$ we have $-1 < 1-(a+b) < 1$ and then

$$\lim_{n\to\infty} \mathbb{T}^n = \frac{1}{a+b}\begin{bmatrix} b & a \\ b & a \end{bmatrix}.$$

We deduce that

$$\lim_{n\to\infty} [{}^t\mathbb{T}]^n \left|\begin{array}{c} \alpha \\ 1-\alpha \end{array}\right. = \frac{1}{a+b}\begin{bmatrix} b & b \\ a & a \end{bmatrix}\left|\begin{array}{c} \alpha \\ 1-\alpha \end{array}\right. = \frac{1}{a+b}\left|\begin{array}{c} b \\ a \end{array}\right.$$

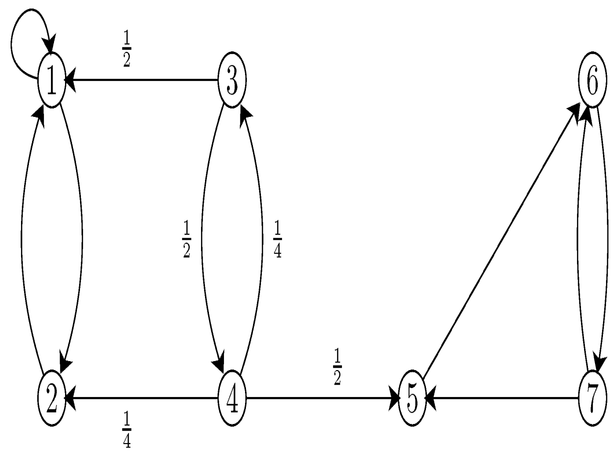
<u>Exercise 2</u>: Consider a MC with three states, its transition matrix is given below

$$\mathbb{T} = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{3} & 0 & \frac{2}{3} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}.$$

i. Draw the state transition diagram for this chain.

ii. If we know that $\mathbb{P}\{X_1 = 1\} = \mathbb{P}\{X_1 = 2\} = \frac{1}{4}$, find $\mathbb{P}\{X_1 = 3, X_2 = 2, X_3 = 1\}$.

<u>Exercise 3</u>: Consider the Markov chain in the figure below



i. Determine its recurrent states and/or classes.
It is clear that apart the states "3" and "4", all other states are recurrent, and we can classify them into two recurrent classes $R_1 = \{1, 2\}$ and $R_2 = \{5, 6, 7\}$.
Let us, for instance, show that the state (1) is recurrent. Following Theorem#3 in the lecture notes, we need to show that

$$\sum_{n\geq 1} \mathbb{P}\{X_n = 1|X_0 = 1\} = \sum_{n\geq 1} [\mathbb{T}^n]_{11} = \infty,$$

according to Exercise 1.i. and for

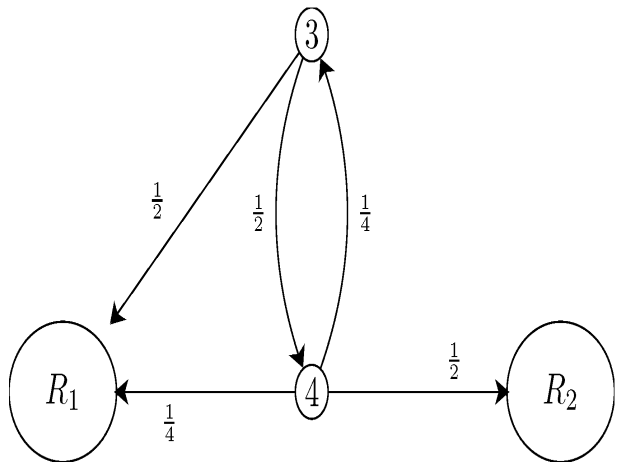
$$\mathbb{T} = \begin{bmatrix} 1-a & a \\ 1 & 0 \end{bmatrix}$$

we have

$$[\mathbb{T}^n]_{11} = \frac{1}{1+a} + \frac{(-a)^n a}{1+a} = \frac{1+a(-a)^n}{1+a},$$

thus $\sum_{n\geq 1} [\mathbb{T}^n]_{11} = \infty$.

ii. Assume that $X_0 = 3$, find the probability that the chain gets absorbed in one of the recurrent state(s).
We can consider that R_1 and R_2 are two states and so reduce our diagram to just 4 states.



Let us denote by

$$a_i^{(1)} = \mathbb{P}\{\text{get absorbed in } R_1 | X_0 = i\},$$

for instance $a_1^{(1)} = 1$ i.e starting from R_1 we get absorbed in R_1 with probability 1. While starting with from R_2 it is impossible to get absorbed in R_1. Now, starting from state 3, we reach R_1, either directly or via state 4 i.e

$$\begin{aligned} a_3^{(1)} &= \frac{1}{2}a_1^{(1)} + \frac{1}{2}a_4^{(1)} \\ &= \frac{1}{2} + \frac{1}{2}a_4^{(1)}, \end{aligned}$$

note that our question is about $a_3^{(1)}$, we need another equation to get it

$$\begin{aligned} a_4^{(1)} &= \frac{1}{4}a_1^{(1)} + \frac{1}{4}a_3^{(1)} + \frac{1}{2}a_2^{(1)} \\ &= \frac{1}{4} + \frac{1}{4}a_3^{(1)}. \end{aligned}$$

We solve the system to get

$$a_3^{(1)} = \frac{5}{7}, \quad a_4^{(1)} = \frac{3}{7}.$$

iii. We would like to find the expected time (average number of steps) until the chain gets absorbed. More precisely, let T be the absorption time, i.e., the first time the chain visits an absorbent state compute

$$\mathbb{E}[T | X_0 = 3].$$

Actually we have two absorbent states R_1 and R_2. Let us denote by $T_i = \mathbb{E}[T|X_0 = i]$, for instance, $T_1 = T_2 = 0$, because once there they stay. While

$$\begin{aligned} T_3 &= 1 + \frac{1}{2}T_1 + \frac{1}{2}T_4 \\ &= 1 + \frac{1}{2}T_4, \end{aligned}$$

and

$$\begin{aligned}T_4 &= 1 + \frac{1}{4}T_1 + \frac{1}{4}T_3 + \frac{1}{2}T_2 \\ &= 1 + \frac{1}{4}T_3.\end{aligned}$$

Solving the above system, we get

$$T_3 = \frac{12}{7}, \quad T_4 = \frac{10}{7}.$$

We answer that starting from state "3" we spend average time $\frac{12}{7}$ to get absorbed.