We willen graag nog een poging doen om te laten zien dat geen automaat met hoogstens 5 knopen correspondeerd met een eindige orde machtreeks met order 2^n voor $n \geq 3$. Ik heb een idee dat mogelijk kan werken (de machtreeksen op bepaalde criteria controleren), maar we verwachten dat het te zwak is en dat we nog veel "false positives" zullen vinden. Maar niet geschoten is altijd mis :).

Het idee gebruikt dat de dieptes van de machtreeksen niet zomaar willekeurig zijn.

The depth of $\sigma = \sigma(t) \in t + t\,\mathbf{F}_2[\![t]\!]$ is $d(\sigma) = \mathrm{ord}_t(\sigma(t) - t) - 1$ (and $d(t) = \infty$), so if $\sigma(t) = t + a_k t^k + O(t^{k+1})$ with $a_k \neq 0$, then $d(\sigma) = k - 1$. The lower break sequence of σ with finite order 2^n is defined as the sequence $(b_i)_{i=0}^{n-1}$, where $b_i = d(\sigma^{\circ p^i})$.

The lower break sequence $(b_i)_{i=0}^{n-1}$ corresponds bijectively to the upper break sequence $\langle b^{(i)} \rangle_{i=0}^{n-1}$ which is defined as follows (the "$\langle, \rangle$" are just to distinguish this sequence from the lower break sequence):

$$b^{(0)} = b_0 \quad \text{and} \quad b^{(i)} = b^{(i-1)} + 2^{-i}(b_i - b_{i-1}) \quad \text{for } i > 0. \tag{1}$$

For an element σ of finite order 2^n the elements of the upper break sequence are all integers! Furthermore, the numbers $b^{(i)}$ must also satisfy all of the following 3 conditions:

(1) $\gcd(2, b^{(0)}) = 1$;
(2) For each $i > 0$, $b^{(i)} \geq 2b^{(i-1)}$;
(3) If the above inequality is strict, i.e. $b^{(i)} > 2b^{(i-1)}$, then $\gcd(2, b^{(i)}) = 1$.

So for some $\sigma \in t + t\,\mathbf{F}_2[\![t]\!]$, we can calculate the depths $b_0, \ldots, b_k$ of $\sigma, \ldots, \sigma^{\circ 2^k}$. If the corresponding upper break sequence has non-integral elements or one of the three conditions is not satisfied, then σ does not have finite order.

Example 0.1. For example, there is no finite order series σ with $\sigma = t + t^2 + O(t^3)$ and $\sigma^{\circ 2} = t + t^8 + O(t^9)$, as its lower break sequence $(1, 7)$ corresponds to the upper break sequence $\langle 1, 4 \rangle$ which violates the third condition.

The conditions that $\gcd(2, b^{(0)}) = 1$ is equivalent to σ being of the form $t + t^{2m} + O(t^{2m+1})$ for some integer $m \geq 1$.

For $i > 0$ we must have $b_i \equiv b_{i-1} \mod 2^i$, this is equivalent to upper break sequence consisting out of integers.