Definition. The depth of $\sigma = \sigma(t) \in t + t\,\mathbf{F}_2[\![t]\!]$ is defined by $d(\sigma) = \mathrm{ord}_t(\sigma(t) - t) - 1$ (and $d(t) = \infty$), so if $\sigma(t) = t + a_k t^k + O(t^{k+1})$ with $a_k \neq 0$, then $d(\sigma) = k - 1$. The lower break sequence of σ with finite order 2^n is defined as the sequence $(b_i)_{i=0}^{n-1}$, where $b_i = d(\sigma^{\circ p^i})$.

The lower break sequence $(b_i)_{i=0}^{n-1}$ corresponds bijectively to the upper break sequence $\langle b^{(i)} \rangle_{i=0}^{n-1}$ which is defined as follows (the brackets "$\langle , \rangle$" are there to distinguish both sequences):

$$b^{(0)} = b_0 \quad \text{and} \quad b^{(i)} = b^{(i-1)} + 2^{-i}(b_i - b_{i-1}) \quad \text{for } i > 0. \tag{1}$$

For an element σ of finite order 2^n the elements of the upper break sequence are all integers. Furthermore, the numbers $b^{(i)}$ must also satisfy all of the following 3 conditions:

(1) $\gcd(2, b^{(0)}) = 1$;
(2) For each $i > 0$, $b^{(i)} \geq 2b^{(i-1)}$;
(3) If the above inequality is strict, i.e. $b^{(i)} > 2b^{(i-1)}$, then $\gcd(2, b^{(i)}) = 1$.

Starting with $\sigma \in t + t\,\mathbf{F}_2[\![t]\!]$, one can calculate the depths $b_0, \ldots, b_k$ of $\sigma, \ldots, \sigma^{\circ 2^k}$. If the corresponding upper break sequence has non-integral elements or if one of the three conditions is not satisfied, then σ does not have finite order.

Example. There is no finite order series σ satisfying $\sigma = t + t^2 + O(t^3)$ and $\sigma^{\circ 2} = t + t^8 + O(t^9)$. The first two terms of its lower break sequence are $(1, 7)$ which corresponds to the upper break sequence $\langle 1, 4 \rangle$. This latter sequence violates the third condition.

The conditions that $\gcd(2, b^{(0)}) = 1$ is equivalent to σ being of the form $t + t^{2m} + O(t^{2m+1})$ for some integer $m \geq 1$, here $b^{(0)} = 2m - 1$.

For $i > 0$ we have $b_i \equiv b_{i-1} \mod 2^i$. This is equivalent to the upper break sequence consisting solely out of integers.