Dynamical Systems 2
Homework 1

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Introduction

Most of the derivations done in this exercise require no more than standard calculus and algebra and so are done without reference. The use of the Jacobi elliptic function in section d) is based on Lecture note 1 of the course.

Exercise 1

In this exercise, we will investigate the Korteweg-de Vries (KdV) equation.

$$u_t - 6uu_x + u_{xxx} = 0 \tag{1}$$

Here $u = u(x,t)$. For the independent variables, we have the following ranges: $x \in]-\infty, \infty[$ and $t \in [0, \infty[$. The subscripts denote partial differentiation.

a)

We use the following ansatz.

$$f(\xi) = u(x,t)$$

The variable in f is defined as $\xi = x - x_0 - vt$. Be begin by evaluating the derivatives that we are going to need in the differential equation. We do this using the Chain Rule.

$$u_t = \frac{\partial f(\xi)}{\partial t} = \frac{\partial f(\xi)}{\partial \xi}\frac{\partial \xi}{\partial t} = -v\frac{\partial f(\xi)}{\partial \xi}$$

$$u_x = \frac{\partial f(\xi)}{\partial x} = \frac{\partial f(\xi)}{\partial \xi}\frac{\partial \xi}{\partial x} = \frac{\partial f(\xi)}{\partial \xi}$$

$$u_{xxx} = \frac{\partial^3 f(\xi)}{\partial \xi^3}$$

The expression for the third derivative follows from the first derivative. Inserting these expression into equation 1, we obtain.

$$-v\frac{\partial f(\xi)}{\partial \xi} - 6f(\xi)\frac{\partial f(\xi)}{\partial \xi} + \frac{\partial^3 f(\xi)}{\partial \xi^3} = 0$$

To ease up on notation, we suppress the functional dependence on the independent variable and denote differentiation with respect to ξ by "$'$". This leads to the following equation.

$$f''' = vf' + 6ff'$$

Integrating with respect to ξ.

$$f'' = vf + 3f^2 + c_1$$

With c_1 being an arbitrary constant of integration. Integrating again, we reach the following equation, using that we can multiply both sides of the equality by 2f' and arrive at.

$$f'^2 = f^2(v+2f) + 2c_1 f + 2c_2$$

Taking the square root of both sides of the equality leaves the desired expression.

$$f' = \pm\sqrt{f^2(v+2f) + 2c_1 f + 2c_2} \tag{2}$$

b)

Equality and linear independence of the polynomial basis functions in f, implies that we can match the coefficients of the monomials on each sides of the equality. This would leave us with 3 equations in 3 unknowns.

$$P(f) = f^2(v+2f) + 2c_1 f + 2c_2 = (f-a_1)(f-a_2)(f-a_3) \tag{3}$$

Collecting the powers of f on both sides of the equality.

$$2f^3 + vf^2 + 2c_1 f + 2c_2 = 2f^3 + (-2a_1 - 2a_2 - 2a_3)f^2 + (2a_1a_2 - 2(-a_1 - a_2)a_3)f - 2a_1a_2a_3$$

We see that f to the third power cancels and so we are left with the following 3 relations, that must be fulfilled to satisfy the equality, due to the linear independence of the powers of f.

$$\begin{aligned} v &= (-2a_1 - 2a_2 - 2a_3) \\ 2c_1 &= (2a_1a_2 - 2(-a_1 - a_2)a_3) \\ 2c_2 &= 2a_1a_2a_3 \end{aligned}$$

In making a sketch of the polynomial, we are told that $a_3 > a_2 > a_1$ and that all three roots are real. According to a simple argument, the function value of the polynomial at $f < a_1$ is negative, as we will have a negative number in each parenthesis. When $a_1 < f < a_2$, we have one positive and two negatives and so, the function value will be positive. When $a_2 < f < a_3$, we have 2 positive and 1 negative contribution, giving a negative function value. Lastly, when $f > a_3$, all contributions are positive and as such, so are the function values. A sketch of the polynomial is provided in figure 1.

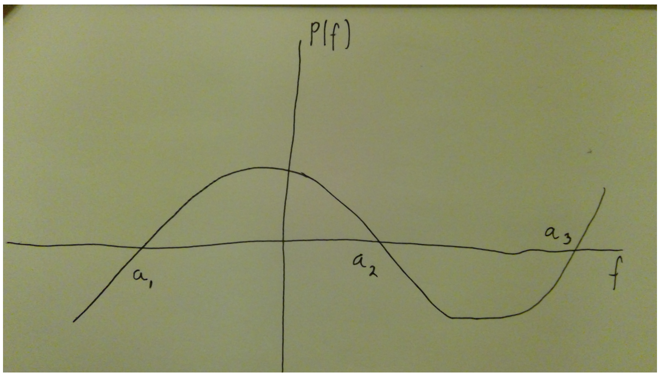

Figure 1: The Polynomial P(f). We see that it is only positive between a_1 and a_2.

c)

We introduce the substitution.

$$f(\theta) = a_1 + (a_2 - a_1)\sin^2(\theta) \tag{4}$$

We insert this into equation 3, cancelling the lone $a_1's$ in the first parenthesis.

$$P(f) = 2(f - a_1)(f - a_2)(f - a_3)$$
$$\rightarrow$$
$$P(f(\theta)) = 2((a_2 - a_1)\sin^2(\theta))(a_1 + (a_2 - a_1)\sin^2(\theta) - a_2)(a_1 + (a_2 - a_1)\sin^2(\theta) - a_3)$$

This is a rather unwieldy expression, I will look at the first two parenthesis in the above and use the trigonometric identity.

$$2((a_2 - a_1)\sin^2(\theta))(a_1 + (a_2 - a_1)\sin^2(\theta) - a_2)$$
$$=$$
$$2((a_2 - a_1)\sin^2(\theta))(-(a_2 - a_1) + (a_2 - a_1)\sin^2(\theta))$$
$$=$$
$$2(-(a_2 - a_1)^2\sin^2(\theta) + (a_2 - a_1)^2\sin^2(\theta)(1 - \cos^2(\theta))$$
$$=$$
$$2(-(a_2 - a_1)^2\sin^2(\theta)\cos^2(\theta))$$

Now let us look at the last parenthesis.

$$\begin{aligned}(a_1 + (a_2 - a_1)\sin^2(\theta) - a_3)\\ =\\ (-(a_3 - a_1) + (a_2 - a_1)\sin^2(\theta))\\ =\\ ((a_3 - a_1)(-1 + k^2\sin^2(\theta))\end{aligned}$$

In the last step, we multiplied by the identity $(a_3 - a_1)/(a_3 - a_1)$ and introduced $k = \sqrt{(a_2 - a_1)/(a_3 - a_1)}$ We now put the found expressions together.

$$\begin{aligned}P(f(\theta)) = 2(-(a_2 - a_1)^2\sin^2(\theta)\cos^2(\theta))((a_3 - a_1)(1 + k^2\sin^2(\theta))\\ =\\ 2(a_2 - a_1)^2(a_3 - a_1)\sin^2(\theta)\cos^2(\theta)(1 - k^2\sin^2(\theta))\end{aligned}$$

Now we can insert this new expression into the ODE in equation 2. Again applying the chain rule to the derivative and rearranging the equation, using that $\frac{df}{d\theta} = 2(a_2 - a_1)\sin(\theta)\cos(\theta))$, from equation 4.

$$\begin{aligned}\frac{df}{d\xi} = \frac{df}{d\theta}\frac{d\theta}{d\xi} = \pm\sqrt{P(\theta)}\\ \rightarrow\\ d\xi = \pm\frac{1}{\sqrt{P(\theta)}}\frac{df}{d\theta}d\theta\\ \rightarrow\\ d\xi = \pm\frac{2(a_2 - a_1)\sin(\theta)\cos(\theta))}{\sqrt{2}(a_2 - a_1)\sqrt{a_3 - a_1}\sin(\theta)\cos(\theta)\sqrt{1 - k^2\sin^2(\theta)}}d\theta = \pm\frac{\sqrt{2}}{\sqrt{a_3 - a_1}\sqrt{1 - k^2\sin^2(\theta)}}d\theta\\ \rightarrow\\ d\xi = \pm\sqrt{\frac{2}{a_3 - a_1}}\frac{d\theta}{\sqrt{1 - k^2\sin^2(\theta)}}\end{aligned}$$

We can integrate these differentials, hoping no confusion arises from the integration variable on the left in the above not being marked. We have that at $\theta = 0$, $f(\theta) = a_1$ and at $\theta = \pi/2$, $f(\theta) = a_2$, so that fixes the lower limit, while the upper limit is a number $\phi \in [0, \pi]$. Thus, we are looking for solutions that oscillate between a_1 and a_2 in the region found to be positive for the polynomial $P(f)$.

$$\xi - \xi_0 = \pm\sqrt{\frac{2}{(a_3 - a_1)}}\int_0^{\phi}\frac{d\theta}{\sqrt{1 - k^2\sin^2(\theta)}} \tag{5}$$

d)

We can rewrite equation 5 on a more familiar form.

$$\pm(\xi-\xi_0)\sqrt{\frac{a_3-a_1)}{2}}=\int_0^{\phi}\frac{d\theta}{\sqrt{1-k^2\sin^2(\theta)}}=F(\phi,k) \tag{6}$$

On the right hand side, we see the incomplete elliptic integral of the first kind, with modulus k. Identifying ϕ as the amplitude function, we can write the following $u=F(\phi,k)$, giving $\phi=F^{-1}(u,k)=\mathrm{am}(u,k)$. So taking the sin of ϕ, is akin to taking the sn of u, as stated in the lecture notes. The ambiguity in sign is eliminated by sn^2 being an even function in its argument.

$$f(\xi)=a_1+(a_2-a_1)\sin^2(\phi)$$
$$\rightarrow$$
$$f(\xi)=a_1+(a_2-a_1)\mathrm{sn}^2\left((\xi-\xi_0)\sqrt{\frac{a_3-a_1)}{2}},k\right)$$
$$\rightarrow$$
$$u(x,t)=a_1+(a_2-a_1)\mathrm{sn}^2\left((x-vt-x_0)\sqrt{\frac{a_3-a_1)}{2}},k\right)$$

Which is the soliton solution we were looking for. In the last step, we replaced our ansatz by the original solution function to the KdV equation. The period of the sn function is given by $4K(k)$, where $K(k)$ is the complete elliptic integral of the first kind.

$$K(k)=\int_0^{\pi/2}\frac{d\theta}{\sqrt{1-k^2\sin^2(\theta)}}$$

To find the period in x of our solution $u(x,t)$, we notice that the periodic function sn appears in the square, we get half the period of the sn. Taking the sn function with our argument. We denote the period of the sn function as $\lambda*$.

$$\mathrm{sn}\left((x-vt-x_0+\lambda*)\sqrt{\frac{a_3-a_1)}{2}},k\right)=\mathrm{sn}\left((x-vt-x_0)\sqrt{\frac{a_3-a_1)}{2}}+4K(k),k\right)$$
$$\rightarrow$$
$$\lambda*=4\sqrt{\frac{2}{(a_3-a_1}}K(k)$$

Half this period and thus the period of the squared sn (in analogy with the geometric argument for the trigonometric functions) is then.

$$\lambda=2\sqrt{\frac{2}{(a_3-a_1)}}K(k)$$

As this is the only periodic function appearing in $u(x,t)$, the period of this function is the λ found. Using maple, we can plot the function $u(x,t)$ for different values

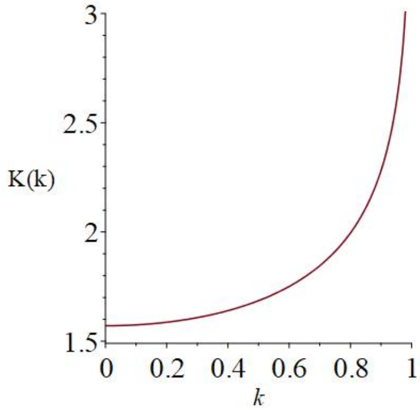

Figure 2: The function K(k). Note that the function increases without bound as k goes toward 1.

of the parameters. The values chosen for a_1, a_2 and a_3, will give different values of k and v. We note that we can get an idea of the dependence of the period on k, by plotting the function $K(k)$ as shown in figure 2.

We can plot $(u(x,t)$ with respect to both x and t, showing that the function is periodic in both variables. I choose $a_1 = 1, a_2 = 2, a_3 = 3$ for simplicity, giving $k = \sqrt{(2-1)/(3-1)} = \sqrt{1/2}$. x_0 is set to be 0.

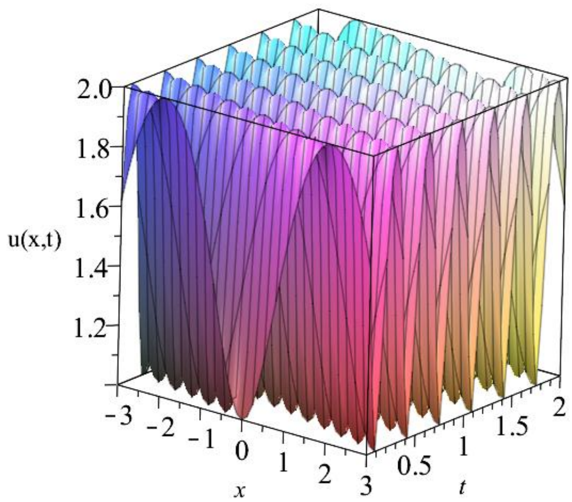

Figure 3: The function $u(x,t)$, showing the periodicity of the function.

I feel that it would be more instructive to plot the functions dependence on the variables one at a time, treating the other variable as a constant. This way, we can also verify the period we found for x. For the constants used, we would expect a spatial period.

$$\lambda = 2\sqrt{\frac{2}{3-1}}\int_0^{\pi/2} \frac{d\theta}{\sqrt{1-1/2\sin^2(\theta)}} \approx 3.71$$

This corresponds very well with what we see in figure 4.

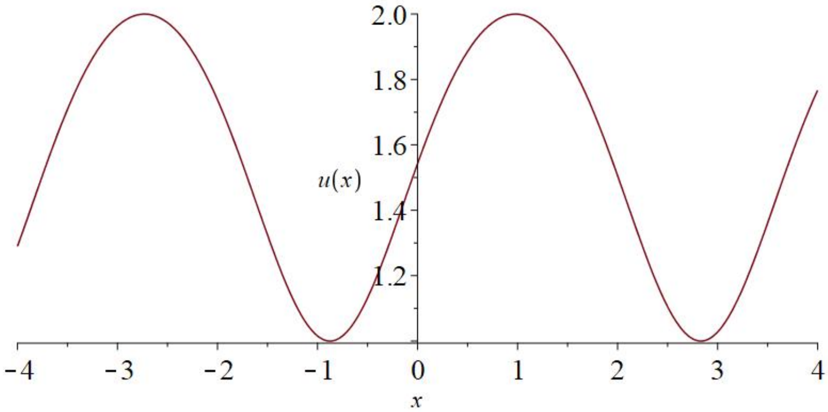

Figure 4: The function $u(x,t)$ for t constant. The function is plotted in the range $-4 \le x \le 4$. The period corresponds well with the analytically found value. We also see that the solution oscillates between the values given to a_1 and a_2.

With fixed x, we see the temporal period being a lot shorter, due to the velocity term. By doing the same analysis as for the spatial period for the temporal one, we see that we get the same expression for the period, but divided by $-v$. With the chosen constants, this corresponds to a period of about 0.3, again in good correspondence with what is seen graphically in figure 5.

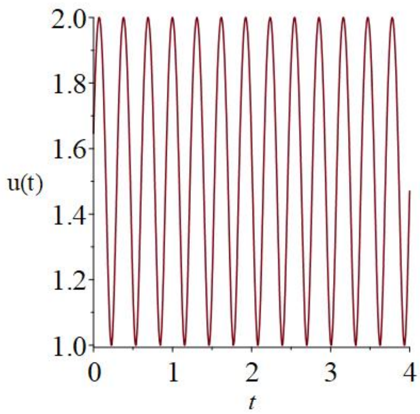

Figure 5: The function $u(x,t)$ for x constant. The function is plotted in the range $0 \le x \le 4$. The period is smaller due to the term v.

We can validate the qualitative considerations made by for example choosing $a_1 = 0.1, a_2 = 0.5$ and $a_3 = 1$, giving $v = -3.2$ and $\lambda = 5.40$ for x. This is shown in figure 6 and 7.

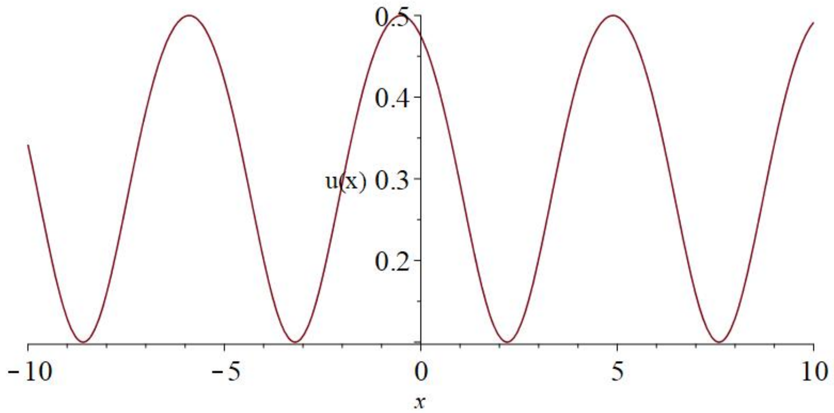

Figure 6: The function $u(x,t)$ for t constant. The function is plotted in the range $-4 \le x \le 4$. The period corresponds well with the analytically found value. We also see that the solution oscillates between the values given to a_1 and a_2.

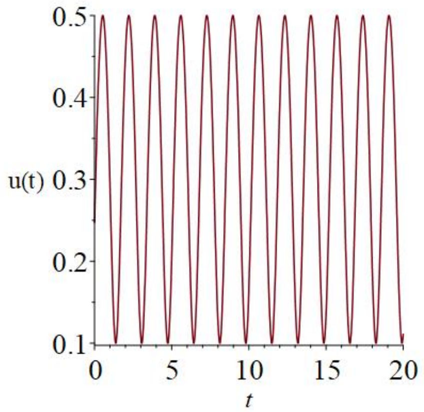

Figure 7: The function $u(x,t)$ for x constant. The function is plotted in the range $0 \le x \le 4$. The period is smaller due to the term v.

For fixed x, we can exemplify some typical temporal trajectories of the system by plotting the u function against its time derivative. We know that.

$$\frac{d\mathrm{sn}(u)}{du} = \mathrm{cn}(u)\mathrm{dn}(u)$$

And so using the chain rule for fixed x, we can plot a dynamical portrait. This is seen in figure 8 for the system with the larger a_1, a_2, a_3 and in figure 9 for the smaller.

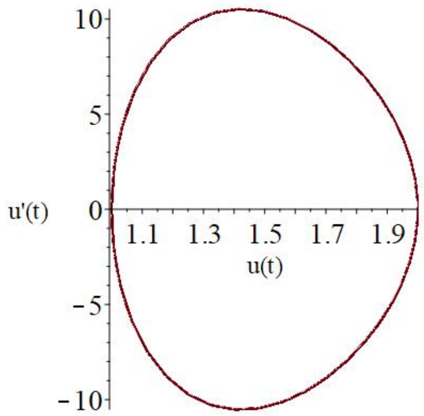

Figure 8: u(t) plotted against u'(t) with $x_0 = 0$ and x=1. $a_1 = 1, a_2 = 2, a_3 = 3$. The derivative is with respect to t

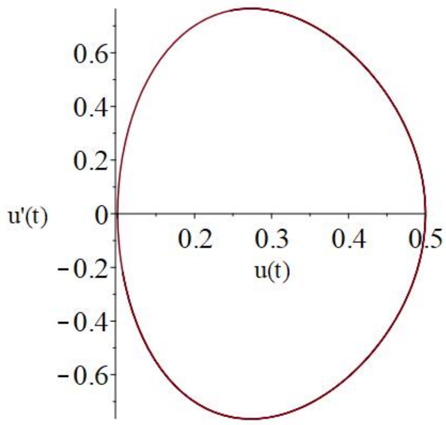

Figure 9: u(t) plotted against u'(t) with $x_0 = 0$ and x=1. $a_1 = 0.1, a_2 = 0.5, a_3 = 1$. The derivative is with respect to t

Appendix

Maple Code

```
with(plots)
```

Plotting the complete elliptic integral of the first kind:

```
plot(EllipticK(k), k = 0 .. 1)
```

Plotting the solution function $u(x,t)$, with fixed $t = 1$.

```
plot(a1+(a2-a1)*JacobiSN(sqrt((a3-a1)*(1/2))*(x-x0-v*1), k)^2, x = -4 .. 4)
```

Plotting the solution function $u(x,t)$, with fixed $x = 1$.

```
plot(a1+(a2-a1)*JacobiSN(sqrt((a3-a1)*(1/2))*(-t*v-x0+1), k)^2, t = 0 .. 4)
```

Plotting the Multivariate function.

```
plot3d(a1+(a2-a1)*JacobiSN(sqrt((a3-a1)*(1/2))*(-t*v+x-x0), k)^2, x = -3 .. 3, t = 0 .. 2)
```