Dynamical Systems 2
Homework 2

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Exercise 1

In this exercise, we consider the Nonlinear Schrödinger Equation (NLS).

$$iq_z + q_{tt} + 2\left|q\right|^2 q = 0 \tag{1}$$

Where $q(z,t)$ is a complex valued function $q : [0,\infty[\times\mathbb{R}\to\mathbb{C}$. Subscripts denote partial derivatives.

a)

The function q is assumed to follow a general

$$q(z,t) = \rho(z,t)e^{i\phi(z,t)}$$

Here both amplitude function ρ and the phase ϕ are real. We can insert this ansats into the NLS equation. To preserve the overview we evaluate the derivatives that will be needed. The functional dependence is suppressed to ease up on notation.

$$\begin{aligned}
q_z &= \rho_z e^{i\phi} + i\phi_z\rho e^{i\phi} \\
q_t &= \rho_t e^{i\phi} + i\phi_t\rho e^{i\phi} \\
q_{tt} &= \rho_{tt}e^{i\phi} + i\phi_t\rho_t e^{i\phi} + i\phi_{tt}\rho e^{i\phi} + i\phi_t\rho_t e^{i\phi} - \phi_t^2\rho e^{i\phi}
\end{aligned}$$

The derivatives are inserted into the NLS, equation 1.

$$i\left(\rho_z e^{i\phi} + i\phi_z\rho e^{i\phi}\right) + \rho_{tt}e^{i\phi} + i\phi_t\rho_t e^{i\phi} + i\phi_{tt}\rho e^{i\phi} + i\phi_t\rho_t e^{i\phi} - \phi_t^2\rho e^{i\phi} + 2\left(qq^*\right)\rho e^{i\phi} = 0 \tag{2}$$

The absolute value was found by multiplying the complex number q by its complex adjoint.

$$|q|^2 = qq^* = \rho e^{i\phi}\rho e^{-i\phi} = \rho^2$$

For the equation to hold, both the real and complex parts must each equal zero. Gathering terms in equation 2 with respect to whether they are multiplied by an imaginary factor. We first divide by $e^{i\phi}$, as every element in the sum contains this factor.

$$i\left(\rho_z + 2\phi_t\rho_t + \phi_{tt}\rho\right) + \left(-\phi_z\rho + \rho_{tt} - \phi_t^2\rho + 2\rho^3\right) = 0$$

Real and complex parts are set equal to zero individually. If we rearrange the, we are left with the desired expressions.

$$\begin{aligned}-\phi_z\rho + \rho_{tt} - \phi_t^2\rho + 2\rho^3 &= 0\\ \rho_z + 2\phi_t\rho_t + \phi_{tt}\rho &= 0\end{aligned}$$

b)

We proceed by qualified guessing for the form of ρ and ϕ. For ρ we assume $\rho(z,t) = f(\xi)$, where $\xi = z - vt$, so that the amplitude is a travelling wave. We furthermore assume that the wave is localized, meaning that $f^{(p)}(\xi) \to 0$, when $\xi \to \pm\infty$. $p \in \{1,2\}$. For the phase, it is assumed that it is a linear combination of z and t, $\phi(z,t) = az - \kappa t - \sigma_0$. v is the velocity of the wave and we have that parameters v, a, κ, σ, are real numbers. Now the way it is done in the textbook, involves using the assumptions, so we can express the z derivatives in terms of their corresponding t derivatives. This will result in a system of ode's, where we can solve for the unknown phase and obtain an equation in ρ alone. As I see it, this would be quite superfluous in this case, as the first equation of the system decouples without the need for elimination in the second. By direct substitution in the second equation, we see that v and κ must be reciprocal, with a scale factor of 2.

$$\begin{aligned}2vf'\kappa + f' &= 0\\ &\to\\ f'(1+2v\kappa) &= 0\\ &\to\\ \kappa &= -\frac{1}{2v}\end{aligned}$$

We substitute directly into the first equation and use the chain rule to obtain derivatives in the variable ξ.

$$\begin{aligned}\frac{\partial f(\xi)}{\partial t} &= \frac{\partial f(\xi)}{\partial \xi}\frac{\partial \xi}{\partial t} = -vf'(\xi)\\ \frac{\partial f'(\xi)}{\partial t} &= \frac{\partial f'(\xi)}{\partial \xi}\frac{\partial \xi}{\partial t} = v^2 f'(\xi)\end{aligned}$$

$$v^2 f'' + 2f^3 = (a + \kappa^2)f$$

We multiply by an integration factor f' and integrate both sides with respect to ξ.

$$\begin{aligned}
v^2 f'' f' + 2f^3 f' &= (a+\kappa^2) f f' \\
&\rightarrow \\
-v^2 \int f'' f' d\xi + 2\int f^3 f' d\xi &= (a+\kappa^2) \int f f' d\xi \\
&\rightarrow \\
v^2 \frac{(f')^2}{2} + 2\frac{f^4}{4} &= (a+\kappa^2)\frac{f^2}{2}
\end{aligned}$$

Which is the wanted expression.

$$\frac{v^2}{2}(f')^2 + \frac{1}{2}f^4 = \frac{1}{2}(a+\kappa^2)f^2 \tag{3}$$

c)

We now consider the following solution ansatz. The choice is valid, as $\operatorname{sech}^n(x) \rightarrow 0$ for $x \rightarrow \pm\infty$ and $n \in \{1,2\}$. The derivatives all have a factor of sech, effectively forcing them to go to zero in the limit.

$$f(\xi) = \eta \operatorname{sech}(\beta\xi)$$

We will need one derivative.

$$\frac{d}{d\xi}\eta \operatorname{sech}(\beta\xi) = -\eta\beta \tanh(\beta\xi)\operatorname{sech}(\beta\xi)$$

Insertion of the ansatz into equation yields.

$$\begin{aligned}
\frac{v^2}{2}\left(-\eta\beta\tanh(\beta\xi)\operatorname{sech}(\beta\xi)\right)^2 + \frac{1}{2}\left(\eta\operatorname{sech}(\beta\xi)\right)^4 &= \frac{1}{2}\left(a+\kappa^2\right)\left(\eta\operatorname{sech}(\beta\xi)\right)^2 \\
&\rightarrow \\
v^2\eta^2\beta^2\tanh(\beta\xi)^2\operatorname{sech}(\beta\xi)^2 + \eta^4\operatorname{sech}(\beta\xi)^4 &= (a+\kappa^2)\eta^2\operatorname{sech}(\beta\xi)^2
\end{aligned}$$

We have the factor $\eta^2\operatorname{sech}(\beta\xi)^2$ on both sides of the equality, so we divide by it.

$$v^2\beta^2\tanh(\beta\xi)^2 + \eta^2\operatorname{sech}(\beta\xi)^2 = (a+\kappa^2)$$

For this equality to hold for all values of ξ conceivable, the hyperbolic functions have to be eliminated. For this purpose, we use a hyperbolic identity.

$$\tanh(\beta\xi)^2 + \operatorname{sech}(\beta\xi)^2 = 1$$

The coefficient must hence be equal for the two hyperbolic functions, in order to factor the identity. This leaves us with the following condition.

$$v^2\beta^2 = \eta^2 = a + \kappa^2 \tag{4}$$

I would like to take a moment to motivate the choice of the *sech* function as an ansatz. Starting from equation 3, the integration that led to the equation has an arbitrary integration constant, which was set to 0. Not doing that, we can rewrite the equation as follows, with C being the integration constant.

$$f' = \pm\sqrt{\frac{1}{v^2}(a+\kappa^2)f^2 - \frac{1}{v^2}f^4 + C}$$

This looks a lot like the function we used in the last assignment to arrive at a Jacobi elliptic function solution. We could use a substitution in the integral, to put the polynomial in f under the square root on standard form, as we did in assignment 1. With the choice $C = 0$ however, we can work out a solution without the use of elliptic functions.

$$\frac{df}{d\xi} = \pm\frac{f}{v}\sqrt{(a+\kappa^2) - f^2}$$

$$\rightarrow$$

$$\pm\frac{v\,df}{f\sqrt{(a+\kappa^2) - f^2}} = d\xi$$

We recognise the emerging integral on the right hand side as the inverse *sech* function. This way we can isolate a hyperbolic function solution to f, which is what we did with the ansatz. We thus see that changing the value of C, results in qualitatively different solutions to the NLS equation. Taking integration constants 0 and $\xi_0 = f_0 = 0$, we obtain by integrating up to f and ξ.

$$\frac{1}{v}\xi = -\frac{1}{\sqrt{a+\kappa^2}}\operatorname{sech}^{-1}\left(\frac{f}{\sqrt{a+\kappa^2}}\right)$$

Inverting the function and using that sech is even in its argument, we get.

$$f = (a+\kappa^2)\operatorname{sech}\left(\frac{\sqrt{a+\kappa^2}}{v}\xi\right)$$

We see that the constants are equivalent with the ones found by inserting the ansatz.

d)

We now have some knowledge of a possible solution, given our previous assumptions and restrictions on constants.

$$q(z,t) = \rho(z,t)e^{i\phi(z,t)}$$

Using $q(z,t) = f\theta)$, with f being the secant function ansatz, while using the linear combinations ansatz for the phase, together with restrictions on the constants, we can write.

$$q(z,t) = \eta \operatorname{sech}(\beta(z - vt))e^{i(az - \kappa t - \sigma_0)}$$

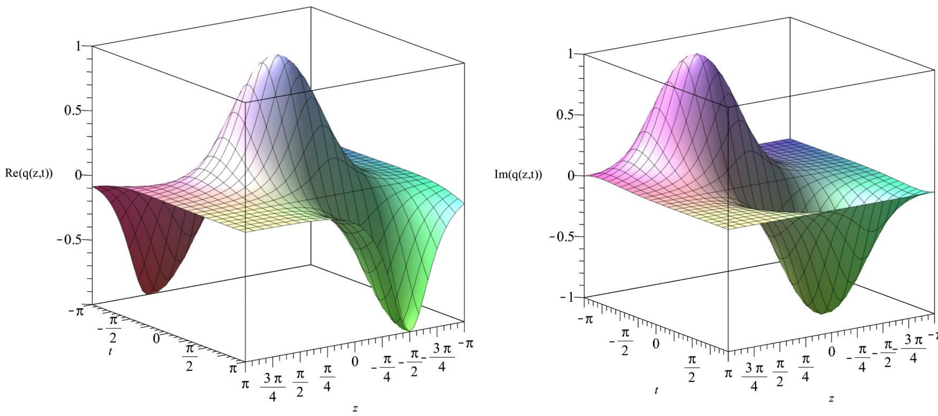
But from equation 4, we have that $a = \eta^2 - \kappa^2$. Also $\beta^2 = \frac{\eta^2}{v^2}$. We also found in that the velocity and κ are inversely related. $\beta = \frac{\eta}{v} = -2\kappa\eta$

$$q(z,t) = \eta \operatorname{sech}(\eta(t + 2\kappa z))e^{i((\eta^2 - \kappa^2)z - \kappa t - \sigma_0)}$$

The complex exponential can be split up in a real and an imaginary part, by using Euler's formula. By using that all the coefficients are real.

$$\Re(q(z,t) = \eta \operatorname{sech}(\eta(t + 2\kappa z)) \cos((\eta^2 - \kappa^2)z - \kappa t - \sigma_0))$$
$$\Im(q(z,t) = \eta \operatorname{sech}(\eta(t + 2\kappa z)) \sin((\eta^2 - \kappa^2)z - \kappa t - \sigma_0))$$

These we can plot as functions of z and t. As the sech function is an even function in its argument, the change of sign in η, changes the sign of the function value. σ_0 amounts to a change in phase and so, it is not qualitatively interesting, hence it has been set to 0 in all the graphs.

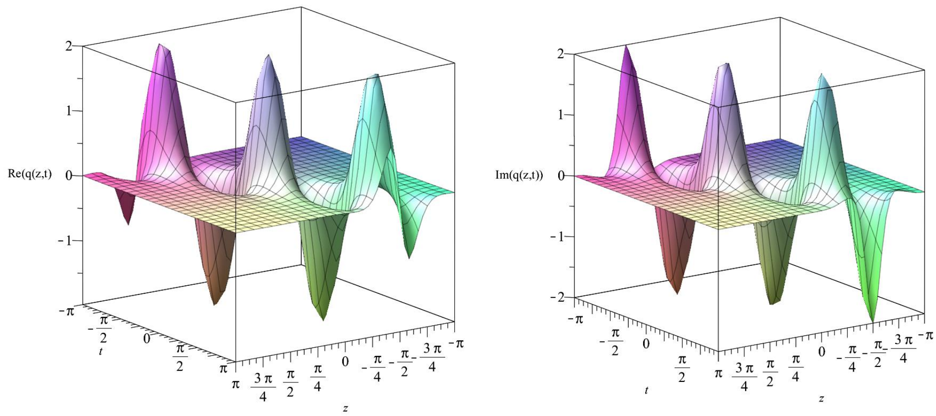


(a) Re(q(z,t))

(b) Im(q(z,t))

Figure 1: The function $q(z,t)$ for $\eta = 1$, $\kappa = 1$, $\sigma_0 = 0$

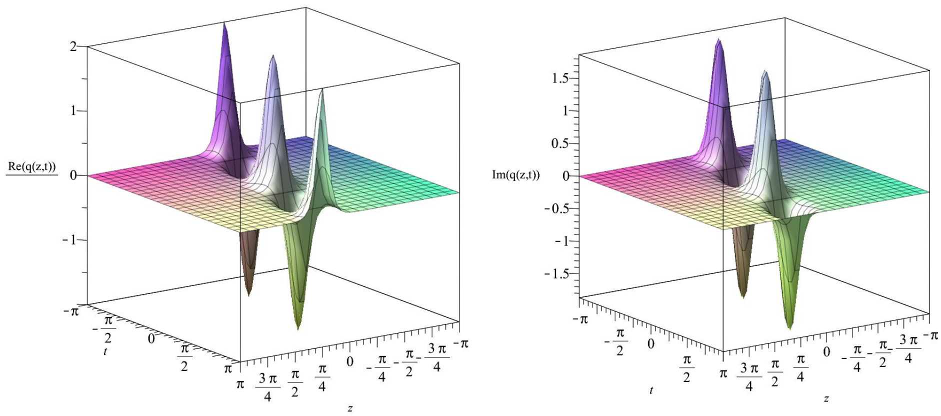
As a final check of consistency, we can insert the found function $q(z,t)$, so see if it satisfies the NLS. Assuming our found solution $q(z,t)$, we can easily take derivatives of the sech and exponential functions. The calculations are quite long although trivial, so I hope the audience will bear with me for having used Maple to check the consistency of the found solution and not reproducing it in this assignment.



(a) Re(q(z,t))

(b) Im(q(z,t))

Figure 2: The function $q(z,t)$ for $\eta = 2$, $\kappa = 1$, $\sigma_0 = 0$



(a) Re(q(z,t))

(b) Im(q(z,t))

Figure 3: The function $q(z,t)$ for $\eta = 2$, $\kappa = -1$, $\sigma_0 = 0$