Project in Dynamical Systems 2

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Solitary and Cnoidal waves

We will be working with the modified Boussinesq equation.

$$u_{xx} - u_{tt} + u_x u_{xx} + u_{xxtt} = 0 \tag{1}$$

Here $u = u(x,t)$ and the independent variable have domains $x \in]-\infty,\infty[$ and $t \in [0,t]$. Subscripts denote partial differentiation with respect to the variable written. We begin by inserting the solution ansatz $u(x,t) = f(\xi)$, where $\xi = x - x_0 - vt$. "$'$" denotes differentiation with respect to ξ.

a)

By using the Chain Rule, we can evaluate the relevant derivatives.

$$\begin{aligned} u_x = \frac{\partial f(\xi)}{\partial x} = \frac{\partial f(\xi)}{\partial \xi}\frac{\partial \xi}{\partial x} &= f' \\ u_{xx} &= f'' \\ u_{tt} &= v^2 f'' \\ u_{xxtt} &= v^2 f'''' \end{aligned}$$

So the equation for f can be written.

$$f'' - v^2 f'' + f' f'' + v^2 f'''' = 0$$

Introducing $f'(\xi) = g(\xi)$, this can be rewritten.

$$g' - v^2 g' + g g' + v^2 g''' = 0$$

We attempt to solve this 3rd order Differential Equation by integration. Integrating once, using the boundary conditions at $\pm\infty$.

$$v^2 g'' = -\frac{1}{2} g^2 - g + v^2 g + c_1$$

We multiply by $2g'$ and integrate once again.

$$v^2\left(g'\right)^2 = -\frac{1}{3}g^3 - g^2 + v^2 g^2 + 2c_1 g + c_2 \tag{2}$$

Using the boundary conditions, we arrive at a first order differential equation for g.

$$g' = \pm\sqrt{-\frac{1}{3v^2}g^3 - \left(\frac{1}{v^2}-1\right)g^2} = \pm g\sqrt{-\frac{1}{3v^2}g - \left(\frac{1}{v^2}-1\right)}$$

We can separate the variables g and ξ.

$$\frac{dg}{d\xi} = \pm g\sqrt{-\frac{1}{3v^2}g - \left(\frac{1}{v^2}-1\right)}$$

$$\rightarrow$$

$$d\xi = \pm\frac{1}{g\sqrt{-\frac{1}{3v^2}g - \left(\frac{1}{v^2}-1\right)}}dg$$

We integrate and use [5] to look up the integral.

$$\int_{\xi_0}^{\xi} d\xi = \pm\int_0^g \frac{1}{g\sqrt{-\frac{1}{3v^2}g - \left(\frac{1}{v^2}-1\right)}}dg$$

$$\rightarrow$$

$$\xi - \xi_0 = \frac{2v\tanh^{-1}\left(\frac{\sqrt{-\frac{3g}{v^2}+\frac{9(v^2-1)}{v^2}}\,v}{3\sqrt{v^2-1}}\right)}{\sqrt{v^2-1}}$$

We invert the above expression to obtain g.

$$g = \frac{3v^2-3}{\cosh\left(\frac{(\xi-\xi_0)\sqrt{v^2-1}}{2v}\right)^2} = \left(3v^2-3\right)\mathrm{sech}\left(\frac{(\xi-\xi_0)\sqrt{v^2-1}}{2v}\right)^2$$

The result of plotting this function in time and space is seen in figure 1. The corresponding for f can be seen in figure 2. f is found by integrating and setting the arbitrary integration constant equal to 0. The antiderivative of the squared sech function is the tanh function.

$$f == \left(3v^2-3\right)\tanh\left(\frac{(\xi-\xi_0)\sqrt{v^2-1}}{2v}\right) + C_0$$

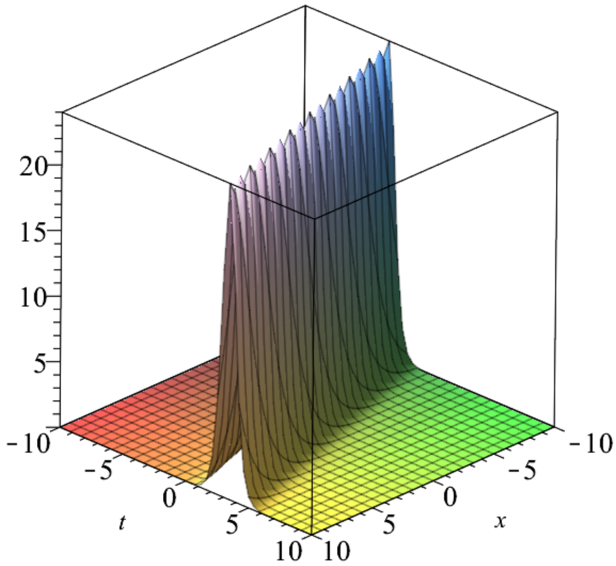


Figure 1: The solution $g(\xi)$, using $v = 3$. We see what looks like a solitary wave, propagating in time, as the shape stays constant.

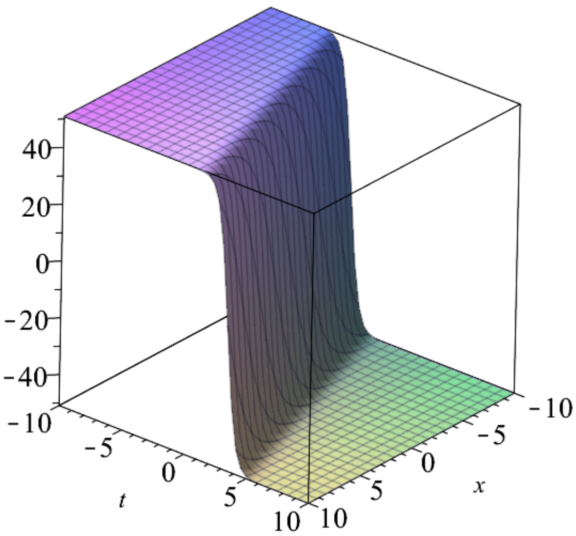


Figure 2: f as a function of time and space. We see a wavefront instead of the solitary wave. The constant of integration is set equal to 0.

b)

In the more general case, the derivation develops as above, until we reach equation 2. In this case, we do not assume the restrictive boundary conditions and instead we are left with the following as a differential equation for g.

$$g' = \pm\sqrt{-\frac{1}{3v^2}g^3 - \left(\frac{1}{v^2}-1\right)g^2 + \frac{2c_1}{v^2}g + \frac{c_2}{v^2}} \tag{3}$$

This does not factor as simply. We introduce the function P to describe the third order polynomial in g under the root.

$$P(g) = -\frac{1}{3v^2}g^3 - \left(\frac{1}{v^2}-1\right)g^2 + \frac{2c_1}{v^2}g + \frac{c_2}{v^2} \tag{4}$$

Defining the constants v, c_1, c_2, so that $a_{1,2,3}$ satisfy $a_1 < a_2 < a_3$, we simplify the polynomial by factorizing it.

$$p(g) = \left(-\frac{1}{3v^2}\right)(g-a_1)(g-a_2)(g-a_3) \tag{5}$$

The factor is chosen for the third order term to vanish, so that we can fix the coefficients. We introduce a substitution inspired by its use in [4].

$$g(\theta) = a_3 - (a_3-a_2)\sin^2(\theta) \tag{6}$$

We insert this function into equation 5.

$$\begin{aligned} P(\theta) &= \left(-\frac{1}{3v^2}\right)\left(a_3-(a_3-a_2)\sin^2(\theta)-a_1\right)\left(a_3-(a_3-a_2)\sin^2(\theta)-a_2\right)\left(a_3-(a_3-a_2)\sin^2(\theta)-a_3\right) \\ &= \left(-\frac{1}{3v^2}\right)\left(a_3-(a_3-a_2)\sin^2(\theta)-a_1\right)\left(a_3-(a_3-a_2)\sin^2(\theta)-a_2\right)\left((a_3-a_2)\sin^2(\theta)\right) \end{aligned}$$

In the same way as we did in assignment 1, this is simplified, using trigonometric identities. The second two parenthesis, can be easily simplified.

$$\begin{aligned} &\left(a_3-(a_3-a_2)\sin^2(\theta)-a_2\right)\left((a_3-a_2)\sin^2(\theta)\right) \\ &= (-a_3+a_2)^2\sin^2(\theta)\cos^2(\theta) \end{aligned}$$

The first parenthesis as well, introducing $k = \sqrt{\frac{a_3-a_2}{a_3-a_1}}$.

$$\begin{aligned} &\left(a_3-(a_3-a_2)\sin^2(\theta)-a_1\right) \\ &= (a_3-a_1)(-1+k^2\sin^2(\theta)) \end{aligned}$$

So we state the polynomial P in the new variable θ.

$$P(\theta) = \left(-\frac{1}{3v^2}\right)(-a_3+a_2)\sin^2(\theta)\cos^2(\theta)(a_3-a_1)(-1+k^2\sin^2(\theta))$$

We use this to solve the original ODE, equation 3. Applying the Chain Rule to rewrite the differential in the new quantities.

$$\frac{dg}{d\xi} = \frac{dg}{d\theta}\frac{d\theta}{d\xi} = \pm\sqrt{P(\theta)}$$

$$\rightarrow$$

$$d\xi = \pm\frac{1}{\sqrt{P(\theta}}\frac{dg}{d\theta}d\theta$$

We insert the previously defined quantities. $\frac{dg}{d\theta}$ is readily found from the definition of g in equation 6.

$$d\xi = \pm\frac{-2(a_3-a_2)\sin(\theta)\cos(\theta)}{\sqrt{\left(-\frac{1}{3v^2}\right)-(a_3-a_2)^2\sin^2(\theta)\cos^2(\theta)(a_3-a_1)(-1+k^2\sin^2(\theta))}}d\theta$$

$$=$$

$$\pm\sqrt{\frac{3}{-\frac{1}{v^2}(a_3-a_1)}}\frac{d\theta}{\sqrt{1-k^2\sin^2(\theta)}}$$

Integration then gives us an opportunity to introduce the elliptic function we seek. Using ϕ as upper limit on the integral (see assignment 1), we arrive at.

$$(\xi-\xi_0)\sqrt{\frac{-\frac{1}{v^2}(a_3-a_1)}{3}} = \int_0^{\phi}\frac{d\theta}{\sqrt{1-k^2\sin^2(\theta)}} = F(\phi,k)$$

As stated in Appendix C of [4], we can invert this and use the relation $\phi = F^{-1}(u,k) = \mathrm{am}(u,k)$, so $\sin(\phi)$ is equivalent to taking sn of u. We go back to the anzats for g, equation 6.

$$g(\xi) = a_3-(a_3-a_2)\mathrm{sn}^2\left((\xi-\xi_0)\sqrt{\frac{-\frac{1}{v^2}(a_3-a_1)}{3}},k\right)$$

Reductive Perturbation Theory

We now consider a different variable substitution, in which we define, $u(x,t) = u(\xi,\tau)$, where $\xi = \varepsilon^p(x-vt)$ and $\tau = \varepsilon^q t$.

a)

We formulate equation 1 in terms of these quantities. First we have to evaluate some derivatives.

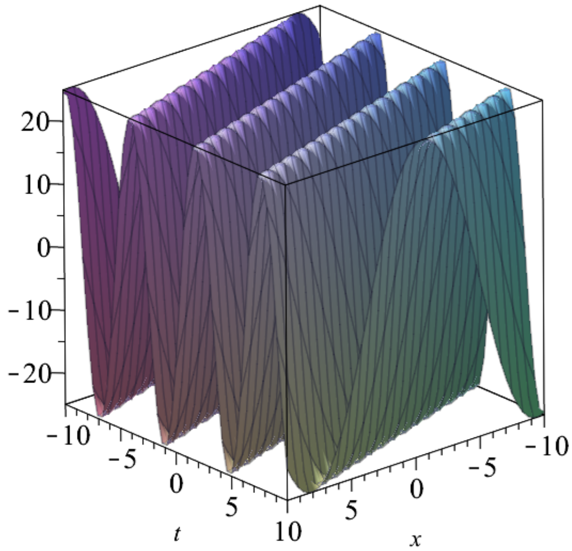


Figure 3: $g(\xi)$ using $a_3 > a_2 > a_1$, with $v = 3$, showing the temporal and spatial aspects of the sn function. In structure, it is basically the same as the solution we found in assignment 1 and [1] to the KdV equation.

$$
\begin{aligned}
u_x = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \tau}\frac{\partial \tau}{\partial x} &= u_\xi \varepsilon^p \\
u_{xx} &= u_{\xi\xi}\varepsilon^{2p} \\
u_t = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial t} + \frac{\partial u}{\partial \tau}\frac{\partial \tau}{\partial t} &= -v\varepsilon^p u_\xi + \varepsilon^q u_\tau \\
u_{tt} &= v^2\varepsilon^{2p}u_{\xi\xi} - 2v\varepsilon^{p+q}u_{\xi\tau} + \varepsilon^{2q}u_{\tau\tau} \\
u_{xxtt} &= v^2\varepsilon^{4p}u_{\xi\xi\xi\xi} - 2v\varepsilon^{3p}\varepsilon^q u_{\xi\xi\xi\tau} + \varepsilon^{2p}\varepsilon^{2q}u_{\xi\xi\tau\tau}
\end{aligned}
$$

A simple test for the last equation above is to do the derivatives the other way around and compare. We replace the derivatives in equation 1 with their (ξ, τ) counterparts.

$$
\begin{aligned}
\varepsilon^{2p}u_{\xi\xi} - \left(v^2\varepsilon^{2p}u_{\xi\xi} - 2v\varepsilon^{p+q}u_{\xi\tau} + \varepsilon^{2q}u_{\tau\tau}\right) + \varepsilon^{3p}u_\xi u_{\xi\xi} + v^2\varepsilon^{4p}u_{\xi\xi\xi\xi} - 2v\varepsilon^{3p+q}u_{\xi\xi\xi\tau} + \varepsilon^{2p+2q}u_{\xi\xi\tau\tau} &= 0 \\
&\rightarrow \\
\varepsilon^{2p}\left(1 - v^2\right)u_{\xi\xi} + 2v\varepsilon^{p+q}u_{\xi\tau} - \varepsilon^{2q}u_{\tau\tau} + \varepsilon^{3p}u_\xi u_{\xi\xi} + v^2\varepsilon^{4p}u_{\xi\xi\xi\xi} - 2v\varepsilon^{3p+q}u_{\xi\xi\xi\tau} + \varepsilon^{2p+2q}u_{\xi\xi\tau\tau} &= 0
\end{aligned}
$$

We now consider an asymptotic expansion of $u(\xi, \tau)$ into powers of ε.

$$
u(\xi, \tau) = \varepsilon u_1(\xi, \tau) + \varepsilon^2 u_2(\xi, \tau)
$$

Inserting this whole expression, we could define a range of different equations, depending on choice of p, q, in that the factor of each distinct power of ε should be equal to 0. Instead of writing the entire expansion, we are told that we want to investigate the function $w(\xi,\tau) = u_{1\xi}(\xi\tau)$. We see that the kdV-equation appears in the function w in the expansion. The coefficients of ε in front of these parts should be equal, to reduce to the KdV-equation for that order. This gives the conditions. k denoting the order desired.

$$\begin{aligned} 3p+2 &= k \\ p+q+1 &= k \\ 4p+1 &= k \end{aligned}$$

Where the plus one is from u_1 and plus 2 is from the product of two of those functions, when we take a product of derivatives. The system has the solution $p = 1$, $q = 3$, which gives the order $k = 5$. There are no other terms of this order, so with the chosen values.

$$\varepsilon^5 \left(2 v w_\tau + w w_\xi + v^2 w_{\xi\xi\xi}\right) = 0$$

Now this makes me doubt the approach, as there is no way to fix v in order to make this equation satisfy the KdV-equation, if we are interested in the constants. We way have a mistake in the variable change or some more fundamental problem, but I have not been able to find it.

Energy Method

We can consider introducing a perturbation to equation 1.

$$u_{xx} - u_{tt} + u_x u_{xx} + u_{xxtt} = \varepsilon R(u, u_t, u_x) \tag{7}$$

a)

The Lagrangian and Hamiltonian for the unperturbed Boussinesq equation can be found by considering the Lagrangian density giving rise to equation 1.[2]

$$\mathscr{L}(u_t, u_x, u_{xt}) = \frac{1}{2}u_t^2 - \frac{1}{2}u_x^2 - \frac{1}{6}u_x^3 + \frac{1}{2}u_{xt}^2 \tag{8}$$

We want to minimize the action J, defined by the temporal integral of the Lagrangian on domain $t \in [t_1, t_2]$. The Lagrangian is the spatial integral of the Lagrangian density on some domain $x \in [0, L]$.

$$J = \int_{t_1}^{t_2} \int_0^L \mathscr{L}(u_t, u_x, u_{xt}) dx dt$$

With the boundary conditions.

$$u(0,t) = 0$$
$$u(L,t) = 0$$

We let $\bar{u}(x,t)$ denote the solution to the minimization problem. We then define an auxiliary function η, denoting a perturbation from the optimal path. [3]

$$\bar{u}(x,t) = u(x,t) + \alpha\eta(x,t)$$

Where α is a parameter and η vanishes on the boundary of the region considered.

$$\eta(0,t) = 0$$
$$\eta(L,t) = 0$$
$$\eta(x,t_1) = 0$$
$$\eta(x,t_2) = 0$$

We want to minimise the functional with respect to α. We can take the differentiation inside the integral. Then we apply the Chain Rule.

$$\frac{\partial J}{\partial \alpha} = \int_{t_1}^{t_2}\int_0^L \frac{\partial \mathscr{L}(u_t, u_x, u_{xt})}{\partial \alpha} dxdt$$
$$= \int_{t_1}^{t_2}\int_0^L \frac{\partial \mathscr{L}}{\partial u_t}\frac{\partial u_t}{\partial \alpha} + \frac{\partial \mathscr{L}}{\partial u_x}\frac{\partial u_x}{\partial \alpha} + \frac{\partial \mathscr{L}}{\partial u_{xt}}\frac{\partial u_{xt}}{\partial \alpha} dxdt$$
$$=$$

To minimize clutter, we exploit the linearity of the integration operator and evaluate each term in the integral separately.

$$\int_{t_1}^{t_2}\int_0^L \frac{\partial \mathscr{L}}{\partial u_t}\eta_t dxdt = \int_0^L \left[\frac{\partial \mathscr{L}}{\partial u_t}\eta\right]_{t_1}^{t_2} dx - \int_{t_1}^{t_2}\int_0^L \frac{\partial}{\partial t}\left\{\frac{\partial \mathscr{L}}{\partial u_t}\right\}\eta dxdt$$

$$\int_{t_1}^{t_2}\int_0^L \frac{\partial \mathscr{L}}{\partial u_x}\eta_x dxdt = \int_{t_1}^{t_2} \left[\frac{\partial \mathscr{L}}{\partial u_x}\eta\right]_0^L dt - \int_{t_1}^{t_2}\int_0^L \frac{\partial}{\partial x}\left\{\frac{\partial \mathscr{L}}{\partial u_x}\right\}\eta dxdt$$

For the second derivative, we will need to integrate partially twice.

$$\int_{t_1}^{t_2}\int_0^L \frac{\partial \mathscr{L}}{\partial u_{xt}}\frac{\partial u_{xt}}{\partial \alpha} dxdt = \int_{t_1}^{t_2}\int_0^L \frac{\partial \mathscr{L}}{\partial u_{xt}}\eta_{xt} dxdt$$
$$=$$
$$\int_{t_1}^{t_2} \left[\frac{\partial \mathscr{L}}{\partial u_{xt}}\eta_t\right]_0^L dt - \int_{t_1}^{t_2}\int_0^L \frac{\partial}{\partial x}\left\{\frac{\mathscr{L}}{u_{xt}}\right\}\eta_t dxdt$$
$$=$$
$$\int_{t_1}^{t_2} \left[\frac{\partial \mathscr{L}}{\partial u_{xt}}\eta_t\right]_0^L dt - \int_0^L \left[\frac{\partial \mathscr{L}}{\partial u_{xt}}\eta_x\right]_{t_1}^{t_2} dx + \int_{t_1}^{t_2}\int_0^L \frac{\partial^2}{\partial xt}\left\{\frac{\mathscr{L}}{u_{xt}}\right\}\eta dxdt$$

Now we can use the boundary conditions, but we see that we need to impose conditions on the first order derivatives of η, that are analogous to the conditions we placed on η itself.

$$\begin{aligned}\eta_x(0,t) &= 0\\ \eta_x(L,t) &= 0\\ \eta_t(x,t_1) &= 0\\ \eta_t(x,t_2) &= 0\end{aligned}$$

The Euler-Lagrange equation for the Lagrangian density in equation 8, can then be formulated.

$$-\frac{\partial}{\partial t}\left(\frac{\mathscr{L}}{\partial u_t}\right)-\frac{\partial}{\partial x}\left(\frac{\mathscr{L}}{\partial u_x}\right)+\frac{\partial^2}{\partial x\partial t}\left(\frac{\partial\mathscr{L}}{\partial u_{xt}}\right)=0$$

We evaluate this by inserting the Lagrangian density into the Lagrange equation, obtaining the PDE from equation 1. A Hamiltonian density for the system can be formulated directly from the Lagrangian density, by defining the generalized momentum density.

$$\mathscr{P}=\frac{\partial\mathscr{L}}{\partial u_t}=u_t$$

This gives the Hamiltonian density.

$$\mathscr{H}=\mathscr{P}u_t-\mathscr{L}=\frac{1}{2}u_t^2+\frac{1}{2}u_x^2+\frac{1}{6}u_x^3-\frac{1}{2}u_{xt}^2$$

The full Hamiltonian is then the integral of the density over the spatial domain.

$$H=\int_0^L \mathscr{H}dx \tag{9}$$

A key assumption is that for the unperturbed equation, the time derivative of the Hamiltonian can be set equal to zero. This implies a conserved quantity, identified here as the energy. Introducing the dissipation from equation 7, would in all probability result in a reduction in energy.

b)

Taking the time derivative of the Hamiltonian, we simply apply the Chain rule on each term as the integral is over x.

$$\frac{dH}{dt}=\int_0^L\left(u_tu_{tt}+u_xu_{xt}+\frac{1}{2}u_x^2u_{xt}-u_{xt}u_{xtt}\right)dx \tag{10}$$

We want to simplify the integral. We use that the spatial derivative u_x goes to 0 as the boundary of the domain is reached. We use partial integration. By the linearity property of integration, we can split up the terms and do them one by one.

$$\int_0^L u_x u_{xt} dx = [u_x u_t]_0^L - \int_0^L u_{xx} u_t dx$$

$$\int_0^L \frac{1}{2} u_x^2 u_{xt} dx = \left[\frac{1}{2} u_x^2 u_t\right]_0^L - \int_0^L u_x u_{xx} u_t dx$$

$$\int_0^L u_{xt} u_{xtt} dx = [u_{xtt} u_t]_0^L - \int_0^L u_{xxtt} u_t dx$$

We see that all of the terms in the integral in equation 10 have a common factor u_t.

$$\frac{dH}{dt} = \int_0^L u_{tt} u_t - u_{xx} u_t - u_x u_{xx} u_t + u_{xxtt} u_t dx \tag{11}$$

If we define the perturbation $R(u, u_x, u_t) = u_t$, we can write.

$$\int_0^L (u_{tt} - u_{xx} - u_x u_{xx} + u_{xxtt})\, u_t dx = -\varepsilon \int_0^L u_t^2 dx \tag{12}$$

Considering a single solitary wave, we want a localized function, that has a permanent form. Calling this function f, the amplitude A and the independent variable $z = x - vt$, we can insert into the Hamiltonian, equation 9, where we suppress functional dependence on z in the notation and $v = 1$ for convenience.

$$H = \int_0^L \left(A^2 \frac{1}{2} f'^2 + A^2 \frac{1}{2} f'^2 + A^3 \frac{1}{6} f'^3 + A^2 \frac{1}{2} f''^2\right) dx \tag{13}$$

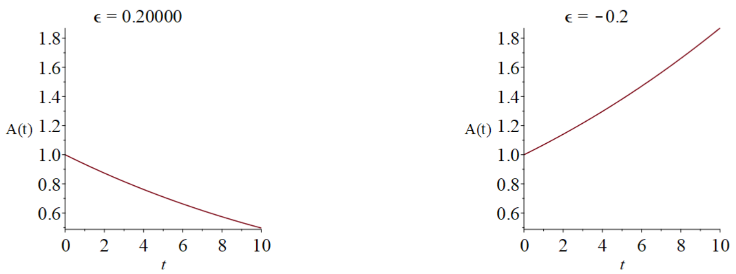
The right hand side of the time variation in H from equation 12 can be evaluated similarly.

$$-\varepsilon \int_0^L u_t^2 dx = -\varepsilon \int_0^L A^2 f'^2 dx$$

Now if we assume that the amplitude is a slowly varying function of time, we can evaluate the time derivative of equation 13. Integrals can be handled symbolically and A can be taken outside the integral as it has no spatial dependence.

$$\frac{dH}{dt} = 2AA'K + \frac{1}{2} A^2 A' K + AA'Q = -\varepsilon A^2 K \tag{14}$$

Where $K = \int_0^L f'^2 dx$ and $Q = \int_0^L f''^2 dx$. In the case we treated in the lecture, this ODE had a simple, exponential solution, as the K's cancelled and we had no Q. In this case, the solution will depend on the value of the integrals and the solution to the ODE will in general be quite complicated. Fixing K and Q can allow us to find closed form solutions to 14, or to use numerical methods. This can give us a qualitative impression of how the amplitude behaves. Fixing $K = Q = 1$, the analytical solution can be found the exponential of a Lambert W function, which I have not encountered before. The solution was found by a symbolical mathematics tool, so I see no reason to reproduce it here. Instead, it is plotted in figure 4, for small positive and small negative ε. For $\varepsilon = 0$, the amplitude remains constant. It makes sense that our perturbation introduces a dampening of the amplitude in time, as we chose a dissipative term.



(a) Positive ε

(b) Negative ε

Figure 4: The amplitude of the perturbed Boussinesq equation for different values of the perturbation ε

A natural next step from here would be to solve the original PDE, equation 7 numerically, in order to asses the qualitative assertions made.

References

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