Assignment in GSPT
Dynamical Systems 2

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1

We consider the PDE.

$$\phi_t = \epsilon^4 \phi_{x^6} + \epsilon^2 \phi_{x^4} + \phi_{x^2} + f(\phi) \tag{1}$$

Where the subscripts denote partial differentiation. The function f is a polynomial in ϕ.

$$f(\phi) = \phi\left(\phi - a\right)\left(1 - \phi\right)$$

Where a is a parameter satisfying $a \leq \frac{1}{2}$. For this PDE, we are seeking travelling wave solutions. In this case, a travelling front. We thus introduce the variable ξ.

$$\phi(\xi) = \phi(x - ct)$$

There is a correspondence between a travelling front solution to the PDE and a heteroclinic orbit in the ODE system. A travelling front has exponential decay towards two different values, as time goes to $\pm\infty$. This can be said for a heteroclinic as well, as it decays to distinct fixed points. We use this substitution in the PDE in equation 1. We see immediately that the variable does not affect the spatial derivatives, but introduce a factor $-c$ to the temporal derivative. In this new coordinate, we can write the PDE as an ODE. The functional dependence on ξ is suppressed.

$$-c\phi_\xi = \epsilon^4 \phi_{\xi^6} + \epsilon^2 \phi_{\xi^4} + \phi_{\xi^2} + f(\phi)$$

We write the 6th-order ODE as a system of 6 1st-order ODE's. We show how the system is related to the standard form used in the lectures.

$$\dot{y} = \begin{pmatrix} y_2 \\ x_1 \end{pmatrix} = g(x, y, c) \tag{2}$$

With $y = (y_1, y_2)^T$.

$$\epsilon \dot{x} = \begin{pmatrix} x_2 \\ x_3 \\ x_4 \\ -x_3 - x_1 + c y_2 - f(y_1) \end{pmatrix} = f(x, y, c) \tag{3}$$

with $x = (x_1, x_2, x_3, x_4)^T$. This is a fast-slow system, with the slow variables y and the fast variables x. $y \in \mathscr{R}^2, x \in \mathscr{R}^4$. Differentiation is with respect to τ The system is written with respect to the slow time τ. The same system in fast time $t = \epsilon^{-1}\tau$ can be written.

$$y' = \epsilon \begin{pmatrix} y_2 \\ x_1 \end{pmatrix} \tag{4}$$

$$x' = \begin{pmatrix} x_2 \\ x_3 \\ x_4 \\ -x_3 - x_1 + c y_2 - f(y_1) \end{pmatrix} \tag{5}$$

By setting $\epsilon = 0$ in the fast system, we obtain a 4-dimensional system, in which the dynamics on y is trivial.

$$y' = 0 \tag{6}$$

$$x' = \begin{pmatrix} x_2 \\ x_3 \\ x_4 \\ -x_3 - x_1 + c y_2 - f(y_1) \end{pmatrix} \tag{7}$$

By setting$f(x, y, 0) = 0$, we obtain a 2-dimensional manifold of equilibrium points for the layer problem. We call this the critical manifold.

$$\begin{pmatrix} x_2 \\ x_3 \\ x_4 \\ -x_3 - x_1 + c y_2 - f(y_1) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{8}$$

$$\rightarrow \tag{9}$$

$$x1 = c y_2 - f(y_1) \tag{10}$$

$$x_2 = 0 \tag{11}$$

$$x_3 = 0 \tag{12}$$

$$x_4 = 0 \tag{13}$$

$$\tag{14}$$

We take the critical manifold to be $C_0 = \{(x_1, x_2, x_3, x_4, y_1, y_2)|, x1 = c y_2 - f(y_1), x_2 = 0, x_3 = 0, x_4 = 0\}$. The critical manifold, C_0, can be written as a graph over (y_1, y_2) if these belong to some compact domain.

$$h_0(y_1, y_2) = \begin{pmatrix} cy_2 - f(y_1) \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

We can form the linearisation of the critical manifold.

$$D_x f(x_0, 0) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 1 & 0 \end{pmatrix}$$

This matrix has 2 eigenvalues with positive real part and 2 with negative real part. The critical manifold thus has a dimensional stable manifold $W_0^s(C_0)$ and a 2-dimensional unstable manifold $W_0^u(C_0)$. By setting $\epsilon = 0$ in the slow system, we obtain the reduced problem. We use that we can write x by the function h_0.

$$\dot{y} = \begin{pmatrix} y_2 \\ x_1 \end{pmatrix} \tag{15}$$

$$\begin{pmatrix} x_2 \\ x_3 \\ x_4 \\ -x_3 - x_1 + cy_2 - f(y_1) \end{pmatrix} = 0 \tag{16}$$

This shows the dynamics of the slow variable constrained to the critical manifold. The dynamics on the manifold are then in terms of the slow variables.

$$\begin{pmatrix} \dot{y_1} \\ \dot{y_2} \end{pmatrix} = \begin{pmatrix} y_2 \\ cy_2 - f(y_1) \end{pmatrix}$$

This system is known to have a heteroclinic connection from $(y_1, y_2) = (0, 0)$ to $(y_1, y_2) = (1, 0)$, for the value $c_0 = \sqrt{2}\left(\frac{1}{2} - a\right)$. The situation is shown in figure 1.

Two questions are pertinent at this stage, how does the heteroclinic break up for perturbations to c and does it persist when $\epsilon \neq 0$. In order to study this problem, we append an equation for the parameter c to the problem.

$$\begin{aligned} \dot{y_1} &= y_2 \\ \dot{y_2} &= cy_2 - f(y_1) \\ \dot{c} &= 0 \end{aligned}$$

This allows us to graph the problem in 3-dimensions. This is shown in figure 2.

At each point on the c-axis, we can define a distance between the stable and unstable manifolds of the fixed points. We fix a section for constant y_1 and measure the distance there. We name the Melnikov function D. We let it take values of c and ϵ as input.

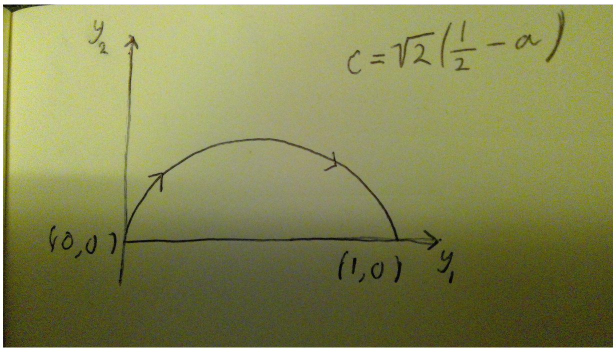

Figure 1: The heteroclinic orbit, when $c = \sqrt{2}\left(\frac{1}{2} - a\right)$ and $\epsilon = 0$.

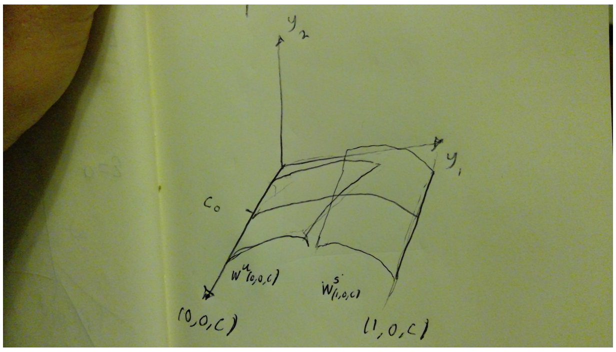

Figure 2: This figure corresponds to the one shown in lecture 4, the functions $p_{i0}(c)$ from there are the c-axes. The Melnikov function defines the transverse distance between W^u and W^s, with $D(c_0)$ defined as where we have the heteroclinic for c_0.

At the known heteroclinic, the distance measure is naturally 0.

$$D(c_0, 0) = 0$$

We now consider the Melnikov integral, which is defined as the derivative of the distance function, with respect to c. We insert the dynamics for the system containing the heteroclinic.

$$\begin{aligned} D'_c(c_0, 0) &= \int_{-\infty}^{\infty} \exp(c) \begin{bmatrix} -cy_2 + f(y_1) & y_2 \end{bmatrix} \begin{bmatrix} 0 \\ y_2 \end{bmatrix} ds \\ &= \int_{-\infty}^{\infty} \exp(-c) y_2^2 ds > 0 \end{aligned}$$

This proves transversality, as both the exponential function and the quadratic are positive. With $D(c_0, 0) = 0$ and $\partial c D(c_0, 0) \neq 0$, we know that $c_*(\epsilon)$ exists, by the Implicit Function Theorem. Thus $c_*(0) = \sqrt{2}\left(\frac{1}{2} - a\right)$, $D(c_*(\epsilon, \epsilon) = 0$ We can expand it in a series in ϵ around c_0.

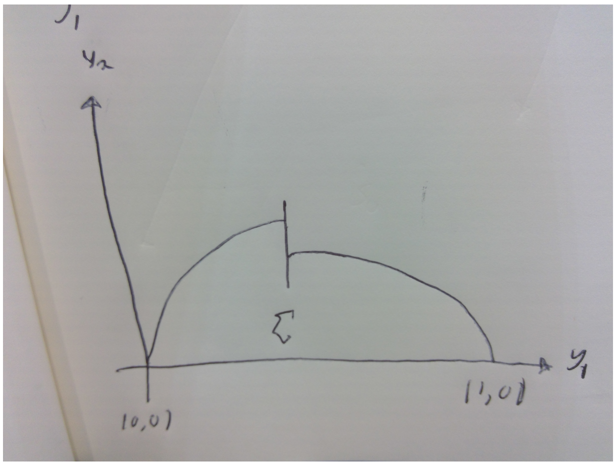

Figure 3: The Heteroclinic system for $c \neq c_0$.The melnikov function quantifies the distance in the plane Σ.

$$
\begin{aligned}
c(\epsilon) &= c(0) + c'(c_0)\epsilon \\
&= c_0 + O(\epsilon)
\end{aligned}
$$

The ODE phase equivalent to the travelling pulse solution of the PDE in eq 1 is a homoclinic. By experimenting in pplane, it is found that such occurs for $c = 0$. We want to prove the existence of this homoclinic for $\epsilon \neq 0$. For $c = 0$, we can write the slow dynamics as a Hamiltonian system.

$$
\begin{aligned}
\dot{y_1} &= y_2 = \partial_{y_2} H(y_1, y_2) \\
\dot{y_2} &= f(y_1) = -\partial_{y_1} H(y_1, y_2)
\end{aligned}
$$

With $H(y_1, y_2) = \frac{1}{2}y^2 + \int f(y_1)dy_1$. The level curves of this system are trajectories. We showed in the exercises that the critical manifold is normal hyperbolic. This means that we can apply Fenichel theory. Looking at the slow flow on the slow manifold.

$$
\begin{aligned}
\dot{y_1} &= y_2 \\
\dot{y_2} &= cy_2 - f(y_1) + O(\epsilon)
\end{aligned}
$$

We know that the homoclinic exists for $c = \epsilon = 0$, so we can use the same approach as before. The geometric situation is illustrated in the lecture notes, week 4.

$$
D(0,0) = 0
$$

We calculate the Melnikov integral again.

$$D_c' = \int_{-\infty}^{\infty} \exp(c)\, y_2^2\, ds > 0$$

Exactly the same as before. So completey analogously, we state that the function $c_*(\epsilon)$ exists and has $c(0) = 0, D(c_*(\epsilon), \epsilon) = 0$. Which shows the existence of a travelling pulse for $0 < \epsilon << 1$. As ϵ goes to 0 from above, the c value of the solutions will go to c_0 for the travelling front and to 0 for the travelling pulse.

2

We want to formulate the problem as a slow-fast system. We introduce $\epsilon\dot{u} = v$, so that the equation can be written as the system.

$$\begin{aligned} \dot{u} &= v \\ \epsilon\dot{v} &= -\alpha(\tau)v - \beta(\tau)u \end{aligned}$$

In order to track the time, we introduce the substitution $\tau = y_2$, to include time explicitly in the system.

$$\begin{aligned} \dot{u} &= v \\ \epsilon\dot{v} &= -\alpha(y_2)v - \beta(y_2)u \\ \dot{y}_2 &= 1 \end{aligned}$$

In this system, we have 2 fast variables and one slow. In order to rectify this, we introduce the eigenbasis for the system, in which $\epsilon = 0$.

$$P = \begin{pmatrix} -1 & 1 \\ \alpha & 0 \end{pmatrix}$$

$$\begin{pmatrix} u \\ v \end{pmatrix} = P \begin{pmatrix} x \\ y_1 \end{pmatrix} = \begin{pmatrix} -x + y_1 \\ \alpha(y_2)x \end{pmatrix}$$

We can now formulate the full system in the new coordinates. To do this, we first have to find the derivatives in terms of the new variables.

$$\begin{aligned} u' &= -x' + y_1' \\ v' &= \epsilon\dot{\alpha}(y_2)x + x'\alpha(y_2) \end{aligned}$$

This is inserted into the DE-system, together with the form of the new variables. The system is now ready for the application of GSPT. Here the slow system.

$$\begin{aligned} \epsilon\dot{x} &= -\left(\alpha - \frac{\beta}{\alpha}\epsilon + \frac{\dot{\alpha}}{\alpha}\epsilon\right)x - \frac{\beta}{\alpha}\epsilon y_1 \\ \dot{y}_1 &= \frac{\beta - \dot{\alpha}}{\alpha}x - \frac{\beta}{\alpha}y_1 \\ \dot{y}_2 &= 1 \end{aligned}$$

And the fast system.

$$
\begin{aligned}
x' &= -\left(\alpha - \frac{\beta}{\alpha}\epsilon + \frac{\dot{\alpha}}{\alpha}\epsilon\right)x - \frac{\beta}{\alpha}\epsilon y_1 \\
y_1' &= \epsilon\left(\frac{\beta - \dot{\alpha}}{\alpha}x - \frac{\beta}{\alpha}y_1\right) \\
y_2' &= \epsilon
\end{aligned}
$$

We now have one fast and two slow variables. The Layer Problem is given once again, by setting $\epsilon = 0$ in the fast system.

$$
\begin{aligned}
x' &= -\alpha x \\
y_1' &= 0 \\
y_2' &= 0
\end{aligned}
$$

We see that the critical manifold for the problem is given by the 2-dimensional manifold $x = 0$. It has stable manifolds foliated by contracting fibers, as shown in **??**.

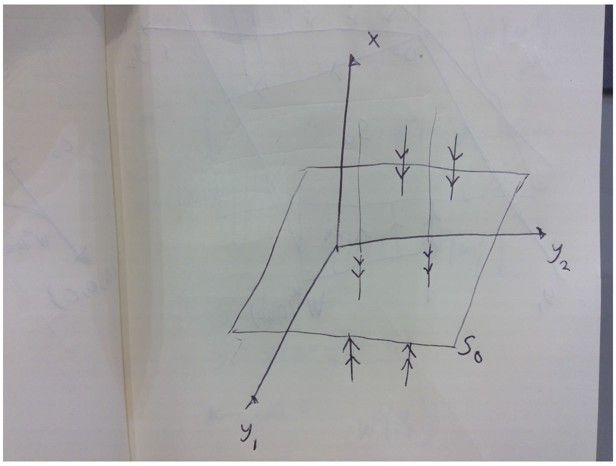

Figure 4: The situation for the fast system, when $\epsilon = 0$. The critical manifold is attracting, as the linearisation has eigenvalue $-\alpha(y_2) < 0$.

The Reduced problem.

$$
\begin{aligned}
0 &= -\alpha x \\
\dot{y}_1 &= \frac{\beta - \dot{\alpha}}{\alpha}x - \frac{\beta}{\alpha}y_1 \\
\dot{y}_2 &= 1
\end{aligned}
$$

As we have $x = 0$, the dynamics constrained on the critical manifold can be given by (y_1, y_2).

$$\dot{y_1} = -\frac{\beta}{\alpha} y_1$$
$$\dot{y_2} = 1$$

We can solve for y_1 as a function of y_2 simply by integrating the dynamical equations. We remember that y_2 is a proxy for time.

$$y_1(y_2) = c \exp\left(-\int_0^{y_2} \frac{\beta(s)}{\alpha(s)}\right) ds$$

Where c is some integration constant and s is the integration variable, not to be confused with the quantities in section 1. The singular solutions can be analyzed together. We remember that the boundary conditions are given as lines in these coordinates. We see that the lines are transversal to the (y_1, y_2)-plane.

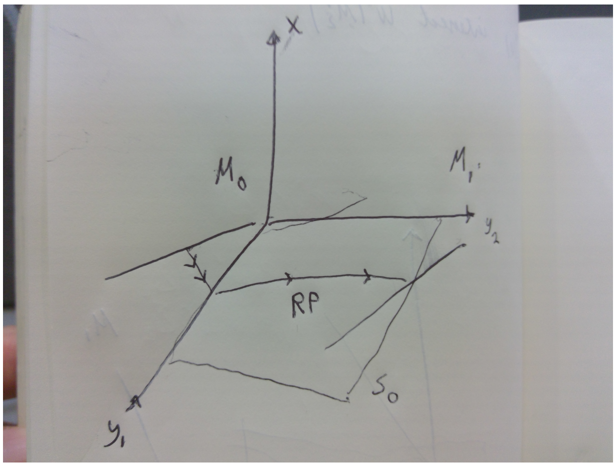

Figure 5: A stitched together version of the Layer and Reduced Problems. M_0 denotes the boundary condition at $y_2 = 0$ and M_1 denotes same at $y_2 = 1$. We see a fast trajectory from the Layer problem

We want to show that we can obtain solutions for $0 < \epsilon << 1$. From Fenichel's Theorem, the critical manifold, being normal hyperbolic, has slow manifold S_ϵ, that is $O(\epsilon)$-close. Orbits are $O(\epsilon)$-close to the unperturbed orbits as well. We want to track a solution curve. We take a small neighbourhood, $O(\epsilon)$-close to a point on the unperturbed IC, M_0. We float this neighbourhood forward in time and call the resulting plane M_0^*. This plane will come exponentially, $O\big(\exp(-\frac{1}{\epsilon})\big)$-close to the slow manifold. This is proved by the Exchange Lemma. We can trade transversality (section 1) for closeness. These tracked solution curves will lie within an order ϵ to the singular solution in the plane $y_2 = 1$. This shows the existence of perturbed solutions for $0 < \epsilon << 1$.

If we instead of a u use a function $f(u)$ in the ODE for the BVP, I don't know what would happen. I tried to do the coordinate transformation with it, but there was no real eureka moment. It might have something to do with the reliance in the Exchange Lemma of all the functions being C-smooth to a given order. That doesn't seem like a satisfying answer, though.

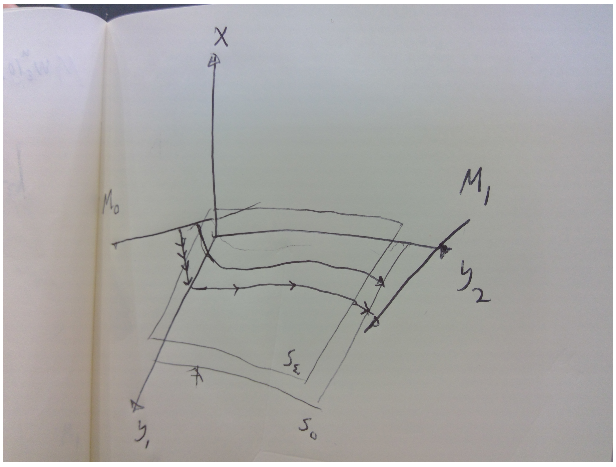

Figure 6: The perturbed solution seen next to the singular from the previous figure. To recap. Shooting manifold will be $O(e^{\frac{1}{\epsilon}}$-close to slow manifold S_ϵ. Slow manifold is $O(\epsilon)$-close to critical manifold S_0. So perturbed solution is $O(\epsilon)$-close to the singular solution at the end point.