Stochastic Differential Equations
Assignment 1

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1 Preliminary Remarks

The Matlab code for all the simulations is enclosed in an .m file called $MAIN\_assignment1.m$. When references are given to definitions without source, the course notes are meant.

Question 1. A stochastic experiment

A stochastic experiment consists of selecting a point in the following set.

$$\Omega = \{(x, y) : x > 0, x^2 + y^2 < 1\}$$

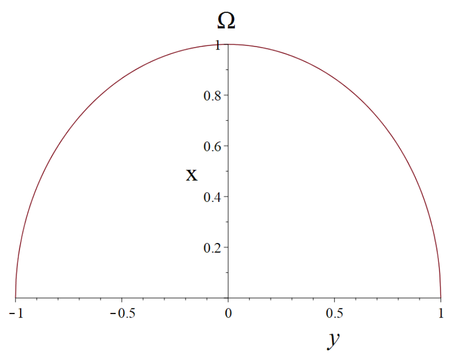
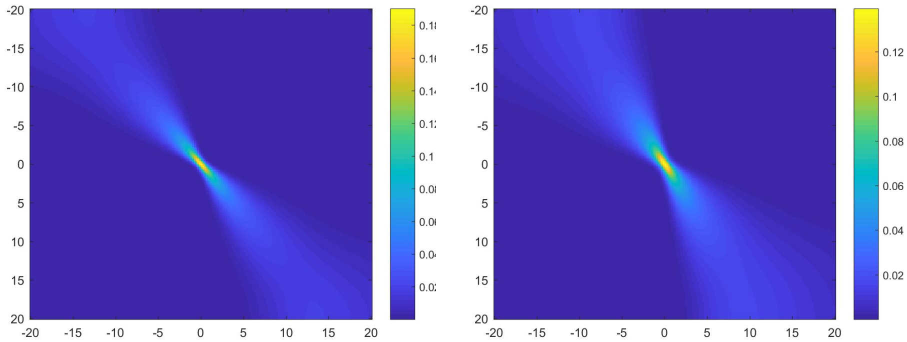


Figure 1: The set Ω is defined as the area bounded by the semicircle and the y-axis, excluding the boundary.

The set forms a semi-circle, as illustrated in figure 1 The σ-algebra, we call $\mathscr{F}$ and it is composed of the smallest subset so that for any $a, b \in \mathscr{R}$, the following sets are events: $\{(x, y) \in \Omega : a \le x \le b\}$ and $\{x, y) \in \Omega : a \le y \le b\}$. On Ω, we define the random variables $X(x, y) = x$, $Y(x, y) = y$.

The sample space Ω is sketched in figure 1. We next sketch some events belonging to $X \le a$ for $0 < a < 1$ and $Y \le b$ for $-1 < b < 1$.
The cumulative distribution functions concern the probabilities of events. As we are told that the probability is proportional to the area of the sample space and that from



(a) (b)

Figure 2: Sketches of events on the probability space Ω. (a) Two events. The chequered event denotes $\omega \in \omega : X(\omega) \leq 1/4$. The lined event that includes the previous, denotes $\omega \in \omega : X(\omega) \leq 1/2$. (b) Two events. The chequered event denotes $\omega \in \omega : -1/4 \leq Y(\omega) \leq 1/4$. The lined event that includes the previous, denotes $\omega \in \omega : -1/2 \leq Y(\omega) \leq 1/2$.

the axioms of probability $P(\Omega) = 1$, we can associate the probability of an event with the proportion of the total area enclosed in the event. This is equivalent to saying that the joint distribution $f(x, y)$ is uniform on the half disk.

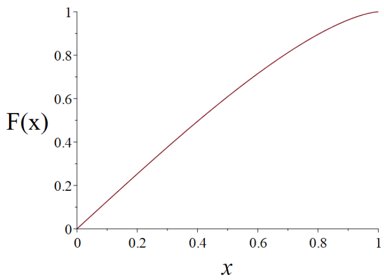
$$f_{X,Y}(x, y) = \frac{2}{\pi} \qquad \text{for} \qquad x > 0, x^2 + y^2 < 1$$

The marginal densities can then be found by considering the "graph" of the half circle, shown in figure 1. We see that for X the proportion of the half disk enclosed is given by the area under the function $2\sqrt{1+x^2}$ (height over x of the circle, times 2 for the symmetry over the x-axis), divided by the area of the half circle, $\pi/2$. The consideration for Y is similar, we just flip the figure, take the same function with respect to y, but do not multiply by 2. We could have fo The cumulative density functions are then found by taking the integrals.

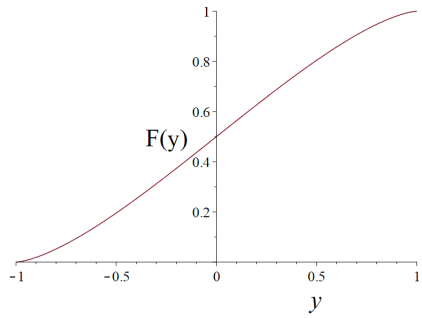
$$F_X(x) = P(\omega : X(\omega) < x) = P(X^{-1}(0, x]) = \int_0^1 \frac{4\sqrt{1-x^2}}{\pi} dx$$
$$F_Y(y) = P\big(\omega : Y(\omega) < y\big) = P(Y^{-1}(-1, y]) = \int_{-1}^1 \frac{2\sqrt{1-y^2}}{\pi} dy$$

From this, the conditional distribution of X with respect to Y can be found from the corresponding densities. We naturally assume that it lies on the midpoint between a point on the y-axis and the height of the circle from this point, for all y in the domain considered. We can use this heuristic as a test of the calculation.

$$f_{X|Y} = \frac{\frac{2}{\pi}}{\frac{2\sqrt{1-y^2}}{\pi}} = \frac{1}{\sqrt{1-y^2}}$$



(a)



(b)

Figure 3: The cumulative distribution functions for x (a) and y (b). They line up with our notion of the probability as the proportional area.

We then take the expectation of X given this conditional distribution.

$$Z = \mathbb{E}(X|Y) = \int_0^{\sqrt{1-y^2}} \frac{x}{\sqrt{1-y^2}} dx = \frac{\sqrt{1-y^2}}{2}$$

Which is what we expected. Now we wish to check whether the defining property of the conditional expectation is fulfilled. We consider a Y-measurable event H. We take as this event $H = \{\omega \in \Omega : -0.5 \le Y(\omega) \le 0.5\}$.

$$\mathbb{E}(Z 1_H) = \mathbb{E}(X 1_H)$$

$$\int_{-0.5}^{0.5} \int_0^{\sqrt{1-y^2}} x dx dy = \int_{-0.5}^{0.5} \int_0^{\sqrt{1-y^2}} \frac{\sqrt{1-y^2}}{2} dx dy = 0.458$$

Where the number itself is truncated after 3 digits. The equality holds for any area attempted. To be even more sure, the integrals where taken with variable limits and their behaviour plotted. They are identical over the range of the semicircle.

Question 2. An Ito process

We define the process $\{I_t t \ge 0\}$.

$$I_t = \int_0^t \sigma \sqrt{1 + B_s^2} dB_s$$

Here σ is a constant and $\{B_t, t \ge 0\}$ is brownian motion.

a

To show that I_t is well defined, we want to make sure that the process being integrated is $\mathscr{L}_2$ integrable in the sense of definition 6.3.2. We will use the probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}, P)$, where Ω is an underlying probability space, $\mathscr{F}$ is a σ-algebra on this space, P a probability measure on these sets and $\{\mathscr{F}_t\}$ is the natural filtration of the process we are considering. Calling the process $G_t = \sigma\sqrt{1+B_t^2}$, it is adapted to its own filtration by definition. That it is measurable follows from the function G_t being well behaved. To show that the process has locally integrable variance, we use, in sequence, the nonnegativity of the integrand, the stochastic version of Fubini's theorem and linearity of expectation.

$$\mathbb{E}\int_0^t \left|\sigma\sqrt{1+B_s^2}\right|^2 ds = \int_0^t \mathbb{E}\left(\sigma^2\left(1+B_s^2\right)\right) ds = \int_0^t \sigma^2 + \sigma^2 s ds = \sigma^2\left(t+\frac{t^2}{2}\right)$$

Now we determine the mean and the variance of the process. We are helped by the mean of a stochastic integral being 0. This can be understood intuitively by the discrete formulation of the Ito integral. We are summing over discrete increments of brownian motion and these have a mean of 0.

$$\mathbb{E}(I_t) = 0$$

Having shown the integral is well-defined, we can find the variance by using Ito's isometry.

$$\mathbb{V}(I_t) = \mathbb{E}\left(I_t^2\right) - \mathbb{E}(I_t)^2 = \mathbb{E}\left(\left(\int_0^t \sqrt{1+B_s^2} dB_s\right)^2\right) = \int_0^t \mathbb{E}\left(\left|\sqrt{1+B_s^2}\right|^2\right) ds$$

Basically the Ito isometry reduces the calculation to the one we did to ascertain whether the process was well defined.

b

Using the results from the last section, we utilize the explicit Euler-Maruyama method to provide a numerical solution for the process I_t. In general in this assignment, this method is used, as the step sizes are small enough and the problems are not stiff enough to warrant stability concerns.
Next we simulate 1000 realizations and construct a histogram over the distribution of I_{10}, the value of I_t at time $T = 10$. We use the same values for constants and step size as the previous realizations. The result is shown in figure 5.
We see that some realizations deviate a lot from the mean but that most remain inside the single standard deviation.

c

We define $X_t = I_t^2$. Using Ito's lemma, we can find the differential of the new process. Naming $h(x,t) = x^2$, we find.

$$\frac{\partial h(x,t)}{\partial t} = 0, \frac{\partial h(x,t)}{\partial x} = 2x, \frac{\partial^2 h(x,t)}{\partial x^2} = 2$$

Just inserting these quantities yields the SDE for the process X_t.

$$dX_t = \left(1+B_t^2\right) dt + 2X_t\sqrt{1+B_t^2} dB_t$$

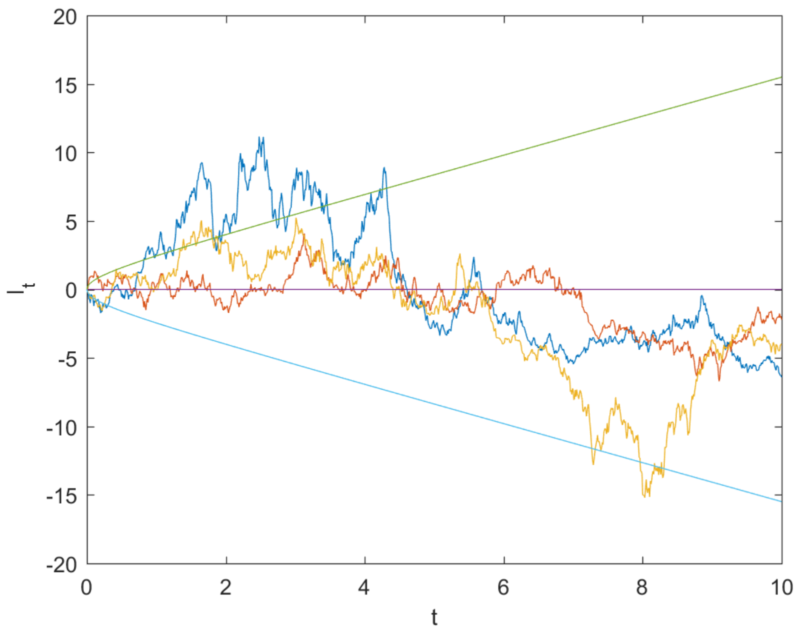


Figure 4: 3 realizations of the process I_t, using $\sigma = 2$. A step size of $h = 0.01$ was used. The mean of the process and $\pm$ one standard variation is are graphed.

Question 3. An Ito or a Stratonovich equation

We consider the two coupled Ito SDE equations, describing stochastic processes X_t and Y_t.

$$\begin{aligned} dX_t &= -\lambda X_t dt + \sigma dB_t, \\ dY_t &= -\sin Y_t dt + sX_t \cos Y_t dt \end{aligned}$$

Here λ, σ and s are constants. B_t is a brownian motion process.

a

We write the equations as a system.

$$\begin{pmatrix} dX_t \\ dY_t \end{pmatrix} = \begin{pmatrix} -\lambda X_t \\ -\sin Y_t + sX_t \cos Y_t \end{pmatrix} dt + \begin{pmatrix} \sigma \\ 0 \end{pmatrix} dB_t$$

We first consider uniqueness of a the solution. This is guaranteed if the system satisfies a Lipschitz condition. We are working in $\mathbb{R}^2$, a finite-dimensional, normed vector space. This means that all norms are equivalent. Taking the L_1 norm on $\mathbb{R}^2$, saying $\mathbf{X} = (X_t, Y_t)^T$.

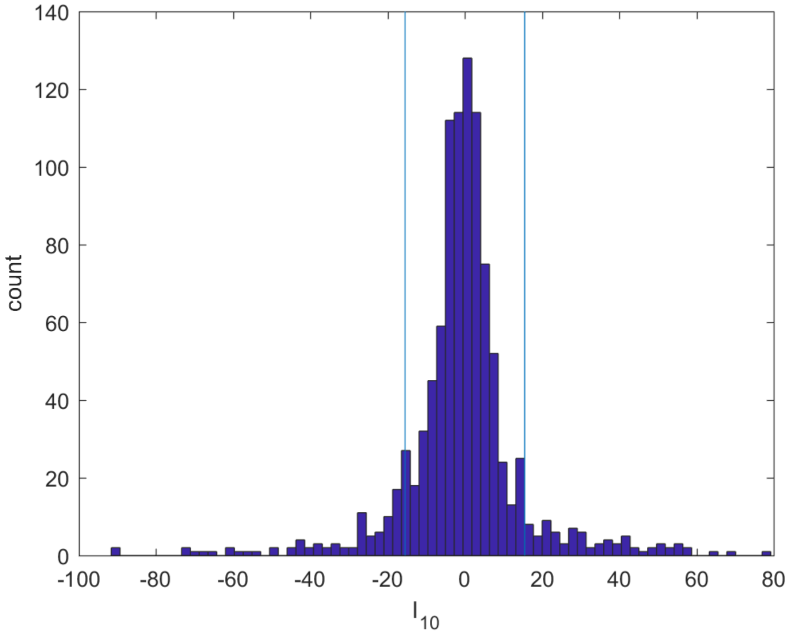


Figure 5: Histogram over the distribution of I_{10}. Vertical lines denote $\pm$ one standard deviation at $T = 10$.

$$
\begin{aligned}
\left|\left|f(\mathbf{X_1}) - f(\mathbf{X_2})\right|\right| &= |-\lambda X_1 + \lambda X_2| + |-\sin Y_1 + sX_1 \cos Y_1 + \sin Y_2 - sX_2 \cos Y_2| \\
&\leq |\lambda (X_2 - X_1)| + |sX_1 \cos Y_1 - sX_2 \cos Y_2| + |\sin Y_2 - \sin Y_1| \\
&\leq \lambda |X_1 - X_2| + s|X_1 - X_2| + 2 = K||\mathbf{X_1} - \mathbf{X_2}|| + 2
\end{aligned}
$$

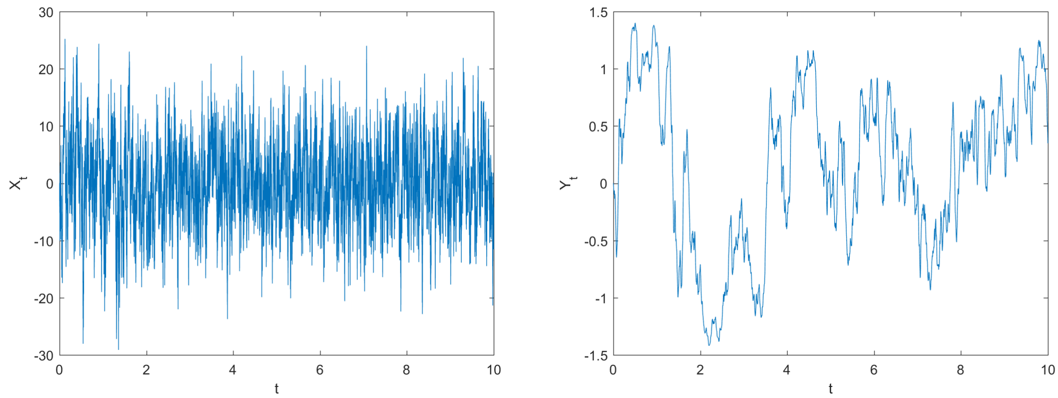
Where $K = \lambda + s$. The same could be seen by taking and bounding the Jacobian of the functions f and g. This is an equivalent condition, but we have elected to show the one in the notes. The diffusion intensity g obviously satisfies a Lipschitz constant. Now for the existence of a solution. We state the condition from the notes. With $x = (X_t, Y_t)$.

$$
x^T f(x,t) = -\lambda X_t^2 - Y_t \sin Y_t + sY_t X_t \cos Y_t \leq C \cdot (|X_t| + |Y_t|)^2 \leq C \cdot \left(1 + ||x||^2\right)
$$

FOr $C = \max(-\lambda, s)$. Which rules out explosions in the drift term. Again the diffusion intensity is trivially bounded as it is a constant. So the theorems do show existence and uniqueness for the system considered.

b

The simulation of the system is shown in figure 6. The standard Euler-Maruyama method is used.



(a) (b)

Figure 6: Solution for the SDE system. (a) X_t and (b) Y_t. We see that the driving force is very large in magnitude, due to the constants used and work on a much shorter time scale. The pendulum exhibits periodic motion.

c

We first find an explicit expression for the process $\{X_t\}$. Inspired by exercise 7.73, we begin considering the process obtained by multiplying X_t by $e^{\lambda t}$. We call this process Q_t. The resulting SDE for the new process can be found by applying Ito's lemma. For $h(x,t) = e^{\lambda t}x$.

$$\frac{\partial h(x,t)}{\partial t} = \lambda e^{\lambda t}x, \frac{\partial h(x,t)}{\partial x} = e^{\lambda t}, \frac{\partial^2 h(x,t)}{\partial x^2} = 0$$

$$dQ_t = \lambda e^{\lambda t} X_t dt + e^{\lambda t} dX_t \tag{1}$$

By multiplying $e^{\lambda t}$ on the original SDE, we get the following.

$$e^{\lambda t} dX_t = -e^{\lambda t}\lambda X_t dt + e^{\lambda t}\sigma dB_t \tag{2}$$

Combining equations 1, 2, we get the following.

$$dQ_t = \sigma e^{\lambda t} dB_t$$

Integrating this equation and rearranging the resulting expression, we obtain a solution formula for X_t.

$$Q_t = X_t e^{\lambda t} = X_0 + \int_0^t \sigma e^{\lambda s} dB_s$$

$$\rightarrow$$

$$X_t = X_0 e^{-\lambda t} + \int_0^t \sigma e^{\lambda(s-t)} dB_s$$

To make things simple, we take $X_0 = 0$ and use that the expectation of a stochastic integral is 0. The autocovariance function can then be stated.

$$\mathrm{cov}(X_t, X_{t+h}) = \mathbb{E}(X_t X_{t+h})$$

We find the function and use that the t-exponentials can be taken outside the expectation with different signs. Furthermore we use a tacit assumption that the interval of integration is the same in the integrals.

$$\begin{aligned}\mathbb{E}(X_t X_{t+h}) &= \mathbb{E}\left(\int_0^t \sigma e^{\lambda(u-t)} dB_u \int_0^s \sigma e^{\lambda(v-t-h))} dB_v\right) \\ &= \sigma^2 e^{-\lambda|h|} \mathbb{E}\left(\int_0^t e^{-\lambda u} dB_u \int_0^s e^{-\lambda v} dB_v\right)\end{aligned}$$

Now we can use the Ito isometry and that the expectation of a deterministic function is an identity.

$$\rho_X(h) = \mathbb{E}(X_t X_{t+h}) = \frac{\sigma^2}{2\lambda} e^{-\lambda|h|}$$

To find the variance spectrum, we take the Fourier transform.

$$\begin{aligned}S_X(\omega) = \int_{-\infty}^{\infty} \rho_X(h) e^{-i\omega h} dh = \frac{\sigma^2}{2\lambda}\left(\int_{-\infty}^{0} e^{\lambda h} e^{-i\omega h} dh + \int_0^{\infty} e^{-\lambda h} e^{-i\omega h} dh\right) \\ = \frac{\sigma^2}{2\lambda}\left(\frac{1}{\lambda - i\omega} + \frac{1}{\lambda + i\omega}\right) = \frac{1}{1+\omega^2}\end{aligned}$$

We see that at low frequencies, the spectrum goes to unity, namely, towards white noise. To verify, we plot the B_t used in the previous sections of the question and $\int_0^t X_s ds$ as a function of time. The integral is evaluated numerically. The result can be found in figure 7.

As we know that Brownian motion is integrated white noise, it would stand to reason that the process X_t approximates white noise for the parameters chosen, based on our numerical investigations.

d

Instead of modelling the noisy pendulum as an SDE system, we can instead incorporate the noise into the equation for the position of the pendulum itself. We have been presented for 2 different ways of doing this. The difference comes down to how the stochastic integral is discretized. First we try an Ito integral for the noise. The result is displayed in figure 8.

e

The Stratonovich integral is then used for the same process as above. THe result is displayed in figure 9.

An immediate result is that the noise is much better approximated by the Stratonovich noise. This makes sense, as we are using a midpoint rule in evaluating the integral.

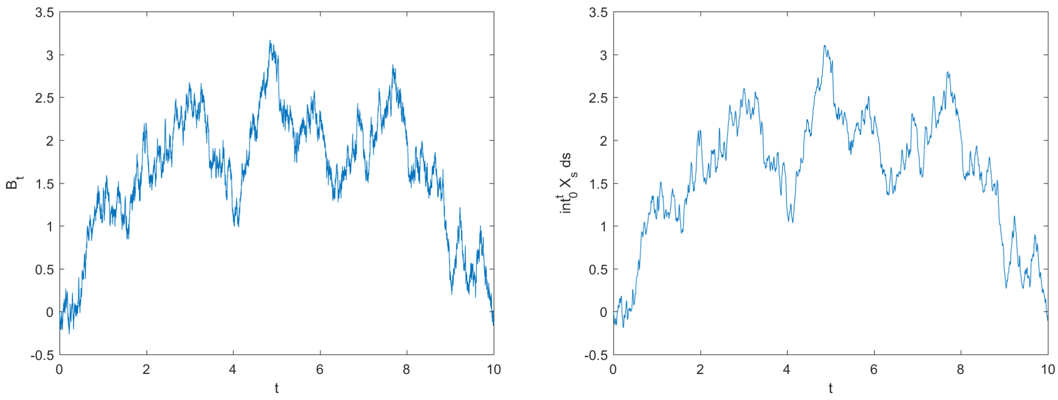


Figure 7: (a) The brownian motion B_t on the interval. (b) The integral $\int_0^t X_s ds$ as a function of time, over the interval. We see that the two correspond to a large degree.

While the Ito integral seems to overcompensate for trends in the solution, the error in the Stratonovich integral seems more evenly distributed. Again, this makes sense from the definitions of these integrals.

f

We define the horizontal position, $Z_t = \sin Y_t^S$. As the process is given in terms of a Stratonovich integral, we can use the ordinary rules of calculus to find the corresponding differential. For $h(x,t) = \sin(x)$, we then have.

$$\frac{\partial h(x,t)}{\partial t} = 0, \frac{\partial h(x,t)}{\partial x} = \cos(x)$$

The resulting Stratonovich equation is then found by transforming coordinates.

$$dZ_t = -\cos(Y_t)\sin(Y_t)dt + s\cos^2(Y_t) \circ dB_t$$
$$\rightarrow$$
$$dZ_t = -Z_t\sqrt{1-Z_t^2}dt + s\left(1-Z_t^2\right) \circ dB_t$$

To find the corresponding Ito equation, we observe that only the drift term is different. Using the drift $f_S(x) = x\sqrt{1-x^2}$ and the noise intensity $g(x) = s\left(1-x^2\right)$. We need $g'(x) = -2sx$.

$$f_I(x) = f_S(x) + \frac{1}{2}\frac{\partial g(x)}{\partial x}g(x) = -x\sqrt{1-x^2} - s^2\left(x - x^3\right)$$

Which gives the Ito equation.

$$dZ_t = -Z_t\sqrt{1-Z_t^2} - s^2\left(Z_t - Z_t^3\right) + s\left(1-Z_t^2\right)dB_t$$

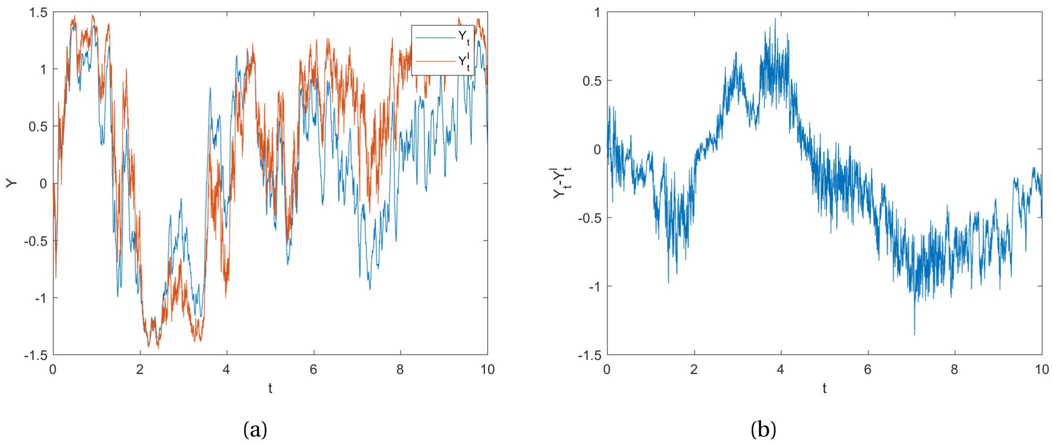
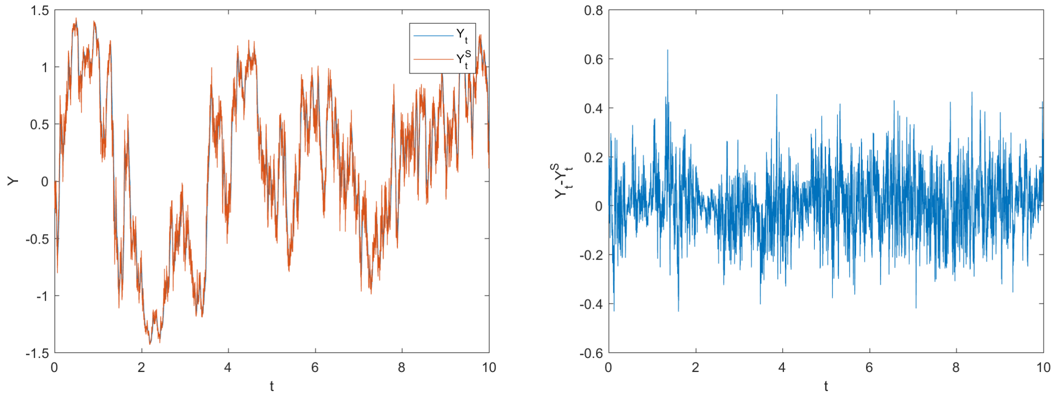


Figure 8: (a) Comparison between the solution to Y_t in the SDE system and the approximation using Ito noise. (b) The difference between the approximations as a function of time

Now this is all well and good, but to verify the claim we simulate the solution of the resulting Stratonovich equation with the stochastic Heun method and the solution to the Ito equation with the explicit Euler-Maruyama method. The result is displayed in figure 10.



(a) (b)

Figure 9: (a) Comparison between the solution to Y_t in the system system and the approximation using Stratonovich noise. (b) The difference between the approximations as a function of time.

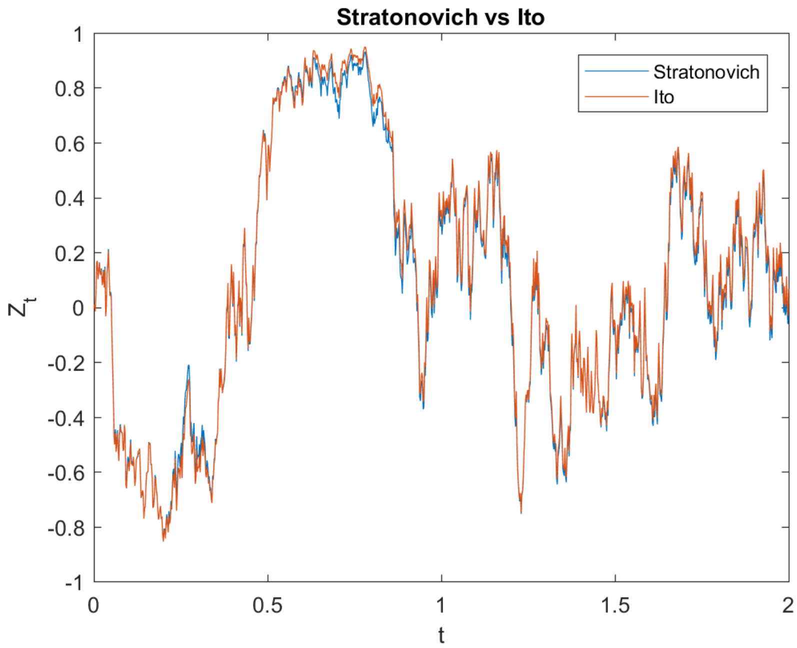


Figure 10: Comparison between numerical solutions of the Stratonovich equation found from the rules of Stratonovich calculus, versus the solution to the Ito equation found by transforming the drift term in the Stratonovich equation. They are seen to overlap to a great extent. Also interesting that they stay within the range of the sine function.