Random Variable Generation, AI1110

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CONTENTS

Abstract—**This manual provides solutions to simple examples of generation of random numbers**

1 UNIFORM RANDOM NUMBERS

Let U be a uniform random variable between 0 and 1.

1.1 Generate 10^6 samples of U using a C program and save into a file called uni.dat .
Solution: Download the following c code. Run it to generate samples of U.

```
wget https://github.com/cs21btech11051Rajiv/
    AI1110_assignments/blob/main/manual1/
    code/q1/1p1.c
```

1.2 Load the uni.dat file into python and plot the empirical CDF of U using the samples in uni.dat. The CDF is defined as

$$F_U(x) = \Pr(U \le x) \tag{1.1}$$

Solution: The empirical CDF of U is plotted in 1.2 using the code below

```
wget https://github.com/cs21btech11051Rajiv/
    AI1110_assignments/blob/main/manual1/
    code/q1/1p2.py
```
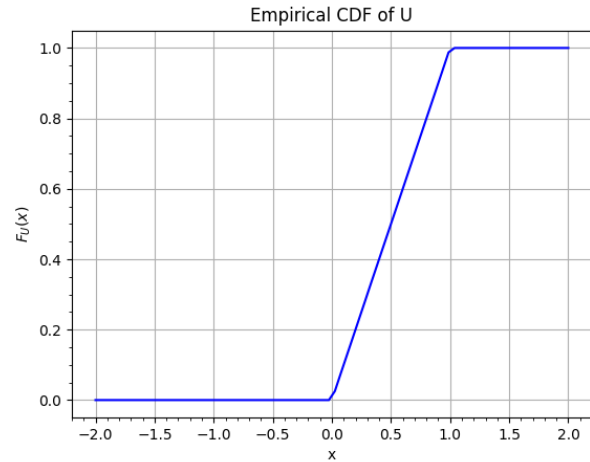

Fig. 1.2. The CDF of U

1.3 Find a theoretical expression for $F_U(x)$.
Solution: The PDF has a uniform distribution,

$$p_U(x) = \begin{cases} \frac{1}{1-0} & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases} \tag{1.2}$$

$$F_U(x) = \int_{\infty}^{x} p_U(x)\, dx \tag{1.3}$$

$$F_U(x) = \begin{cases} 0 & x < 0 \\ \int_0^x 1 dx & 0 \le x \le 1 \\ \int_0^1 1 dx & 1 < x \end{cases} \tag{1.4}$$

$$F_U(x) = \begin{cases} 0 & x < 0 \\ x & 0 \le x \le 1 \\ 1 & 1 < x \end{cases} \tag{1.5}$$

The theoretical CDF of U is plotted in 1.3 using the code below

```
wget https://github.com/cs21btech11051Rajiv/
    AI1110_assignments/blob/main/manual1/
    code/q1/1p3.py
```

1.4 The mean of U is defined as

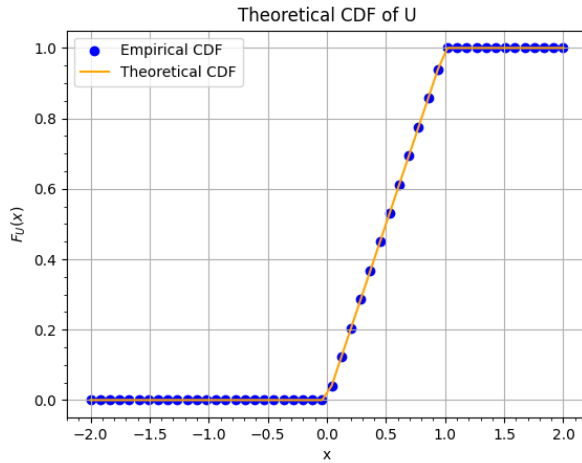$$E[U] = \frac{1}{N}\sum_{i=1}^{N} U_i \tag{1.6}$$

Fig. 1.3. The CDF of U

and its variance as

$$\text{var}\,[U] = E\left[(U - E\,[U])^2\right] \tag{1.7}$$

Write a C program to find the mean and variance of U.
Solution: Run the following C file

wget https://github.com/cs21btech11051Rajiv/
    AI1110_assignments/blob/main/manual1/
    code/q1/1p4.c

Results were,

$$E\,[U] = 0.500007 \tag{1.8}$$
$$\text{var}\,[U] = 0.083301 \tag{1.9}$$

1.5 Verify your result theoretically given that

$$E\left[U^k\right] = \int_{-\infty}^{\infty} x^k dF_U\,(x) \tag{1.10}$$

Solution: Taking k as 1,

$$E\,[U] = \int_{-\infty}^{\infty} x dF_U\,(x) \tag{1.11}$$

Using expression (1.5), this simplifies to

$$E\,[U] = \int_{0}^{1} x dx \tag{1.12}$$

$$E\,[U] = \left.\frac{x^2}{2}\right|_0^1 \tag{1.13}$$

$$E\,[U] = 0.5 \tag{1.14}$$

To calculate variance we take k=2

$$\text{var}\,[U] = E\left[(U - E\,[U])^2\right] \tag{1.15}$$
$$\text{var}\,[U] = E\left[(U - 0.5)^2\right] \tag{1.16}$$
$$\text{var}\,[U] = E\left[U^2 - U + 0.25\right] \tag{1.17}$$
$$\text{var}\,[U] = E\left[U^2\right] - E\,[U] + E\,[0.25] \tag{1.18}$$
$$\text{var}\,[U] = \int_{-\infty}^{\infty} x^2 dF_U\,(x) - 0.5 + 0.25 \tag{1.19}$$
$$\text{var}\,[U] = \int_{0}^{1} x^2 dx - \frac{1}{4} \tag{1.20}$$
$$\text{var}\,[U] = \left.\frac{x^3}{3}\right|_0^1 - \frac{1}{4} \tag{1.21}$$
$$\text{var}\,[U] = \frac{1}{12} = 0.0833 \tag{1.22}$$

$$\tag{1.23}$$

2 CENTRAL LIMIT THEOREM

2.1 Generate 10^6 samples of the random variable

$$X = \sum_{i=1}^{12} U_i - 6 \tag{2.1}$$

using a C program, where $U_i, i = 1, 2, \ldots, 12$ are a set of independent uniform random variables between 0 and 1 and save in a file called gau.dat
Solution: Download the following c code. Run it to generate samples of X.

wget https://github.com/cs21btech11051Rajiv/
    AI1110_assignments/blob/main/manual1/
    code/q1/2p1.c

2.2 Load gau.dat in python and plot the empirical CDF of X using the samples in gau.dat. What properties does a CDF have?
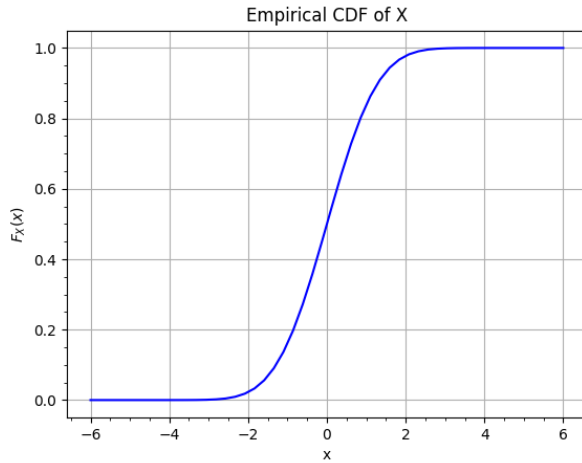**Solution:** The empirical CDF of X is plotted in Fig. 2.2 using the code below

wget https://github.com/cs21btech11051Rajiv/
    AI1110_assignments/blob/main/manual1/
    code/q1/2p2.py

A CDF is a non-decreasing function. Its value varies from 0 to 1. It is continuous if PDF has finite values.

2.3 Load gau.dat in python and plot the empirical PDF of X using the samples in gau.dat. The PDF of X is defined as
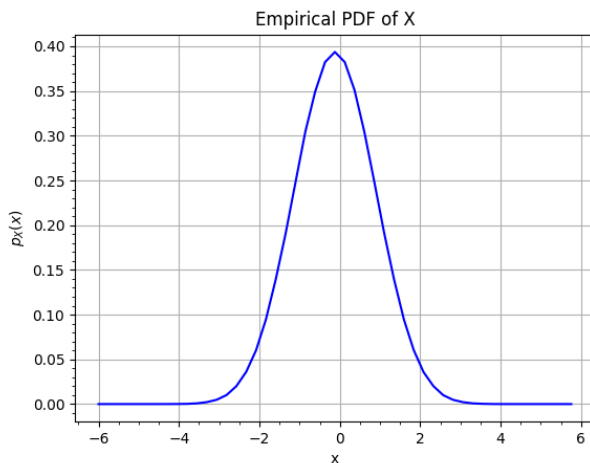
$$p_X\,(x) = \frac{d}{dx} F_X\,(x) \tag{2.2}$$

Empirical CDF of X

Fig. 2.2. The CDF of X

What properties does the PDF have?
Solution: The empirical PDF of X is plotted in Fig. 2.3 using the code below

wget https://github.com/cs21btech11051Rajiv/
AI1110_assignments/blob/main/manual1/
code/q2/2p3.c

Fig. 2.3. The PDF of X

Values taken by a PDF are non negative. The total area under a PDF is equal to 1.

2.4 Find the mean and variance of X by writing a C program.
Solution: Run the following C file

wget https://github.com/cs21btech11051Rajiv/
AI1110_assignments/blob/main/manual1/
code/q2/2p4.c

Results were,

$$E[X] = 0.000294 \qquad (2.3)$$
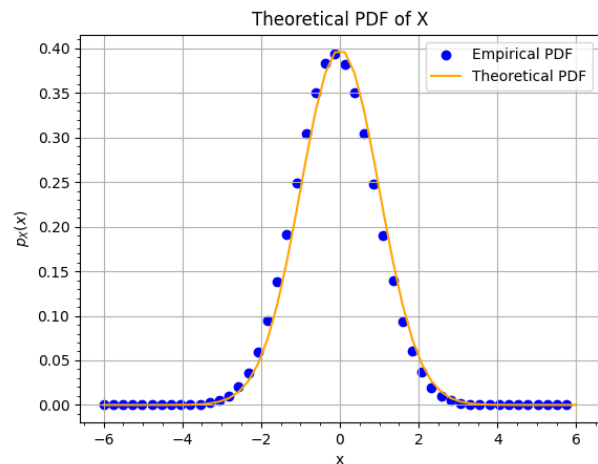$$\text{var}[X] = 0.999560 \qquad (2.4)$$

2.5 Given that
$$p_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}, -\infty < x < \infty, \quad (2.5)$$

repeat the above exercise theoretically.
Solution: The theoretical PDF and CDF of X is plotted in Fig. 2.5 and Fig. 2.5 using the code below

wget https://github.com/cs21btech11051Rajiv/
AI1110_assignments/blob/main/manual1/
code/q2/2p5.c

Fig. 2.5. The theoretical PDF of X

Using given equation (1.10),

$$E[X^k] = \int_{-\infty}^{\infty} x^k dF_X(x) \qquad (2.6)$$

$$E[X] = \int_{-\infty}^{\infty} x dF_X(x) \qquad (2.7)$$

Using given equation (2.2),

$$E[X] = \int_{-\infty}^{\infty} x p_X(x) \, dx \qquad (2.8)$$

$$E[X] = \int_{-\infty}^{\infty} x \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \, dx \qquad (2.9)$$

Taking $t = \frac{x^2}{2}$

$$E[X] = \int_{\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{t}{2}} dt \qquad (2.10)$$
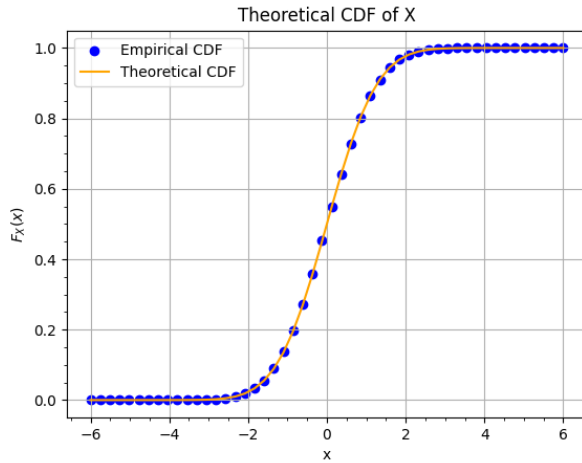
$$E[X] = 0 \qquad (2.11)$$

Fig. 2.5. The theoretical CDF of X

To calculate variance we take k=2

$$\text{var}\left[X\right] = E\left[(X - E\left[X\right])^2\right] \quad (2.12)$$

$$\text{var}\left[X\right] = E\left[X^2\right] \quad (2.13)$$

$$\text{var}\left[X\right] = \int_{-\infty}^{\infty} x^2 dF_X\left(x\right) \quad (2.14)$$

$$\text{var}\left[X\right] = \int_{-\infty}^{\infty} x^2 p_X\left(x\right) dx \quad (2.15)$$

$$\text{var}\left[X\right] = \int_{-\infty}^{\infty} x^2 \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \quad (2.16)$$

Integrating by parts, we obtain

$$\text{var}\left[X\right] = -x\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \Big|_{-\infty}^{\infty}$$
$$+ \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \quad (2.17)$$

The second term is the area under the PDF, which is 1

$$\text{var}\left[X\right] = 0 + \int_{-\infty}^{\infty} p_X\left(x\right) dx \quad (2.18)$$

$$\text{var}\left[X\right] = 1 \quad (2.19)$$

The theoretical CDF can be expressed as

$$F_X\left(x\right) = 1 - Q\left(\frac{x - E\left[X\right]}{\text{var}\left[X\right]}\right) \quad (2.20)$$

$$F_X\left(x\right) = 1 - Q\left(x\right) \quad (2.21)$$

3 FROM UNIFORM TO OTHER

3.1 Generate samples of

$$V = -2\ln\left(1 - U\right) \quad (3.1)$$

and plot its CDF.

Solution: Run the following files to generate samples and CDF.

```
wget https://github.com/cs21btech11051Rajiv/
    AI1110_assignments/blob/main/manual1/
    code/q3/3p1.c
wget https://github.com/cs21btech11051Rajiv/
    AI1110_assignments/blob/main/manual1/
    code/q3/3p1b.py
```
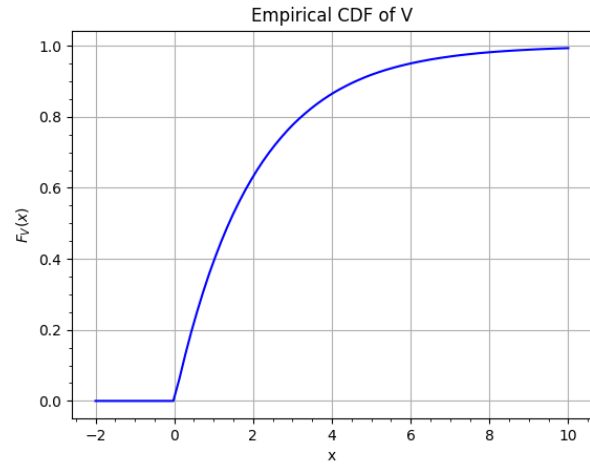

Fig. 3.1. The CDF of V

3.2 Find a theoretical expression for $F_V\left(x\right)$.
Solution:

$$F_V\left(x\right) = Pr\left(V \leq x\right) \quad (3.2)$$

$$F_V\left(x\right) = Pr\left(-2\ln\left(1 - U\right) \leq x\right) \quad (3.3)$$

$$F_V\left(x\right) = Pr\left(\ln\left(1 - U\right) \geq -\frac{x}{2}\right) \quad (3.4)$$

$$F_V\left(x\right) = Pr\left(1 - U \geq e^{-\frac{x}{2}}\right) \quad (3.5)$$

$$F_V\left(x\right) = Pr\left(U \leq 1 - e^{-\frac{x}{2}}\right) \quad (3.6)$$

$$F_V\left(x\right) = F_U\left(1 - e^{-\frac{x}{2}}\right) \quad (3.7)$$

$$F_V\left(x\right) = \begin{cases} 0 & 1 - e^{-\frac{x}{2}} < 0 \\ 1 - e^{-\frac{x}{2}} & 0 \leq 1 - e^{\left(-\frac{x}{2}\right)} \leq 1 \\ 1 & 1 < 1 - e^{-\frac{x}{2}} \end{cases} \quad (3.8)$$

This simplifies to

$$F_V\left(x\right) = \begin{cases} 0 & x < 0 \\ 1 - e^{-\frac{x}{2}} & x \geq 0 \end{cases} \quad (3.9)$$

The following code plots the theoretical CDF.
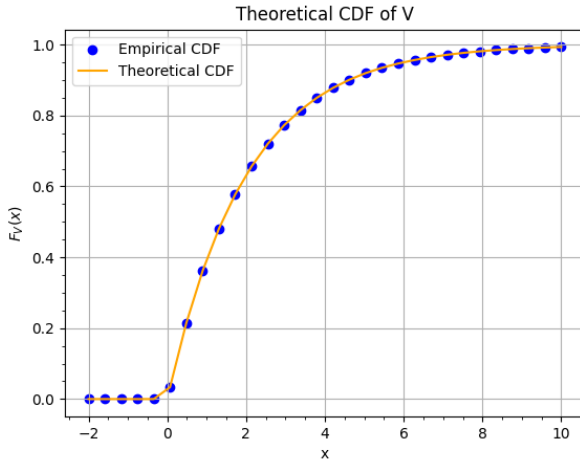
Fig. 4.2. The CDF of T

Fig. 3.2. The CDF of V

4 TRIANGULAR DISTRIBUTION

4.1 Generate

$$T = U_1 + U_2 \tag{4.1}$$

Solution: Download the following c code. Run it to generate samples of T.

4.2 Find the CDF of T.

Solution: The empirical CDF of T is plotted in 4.2 using the code below

4.3 Find the PDF of T.

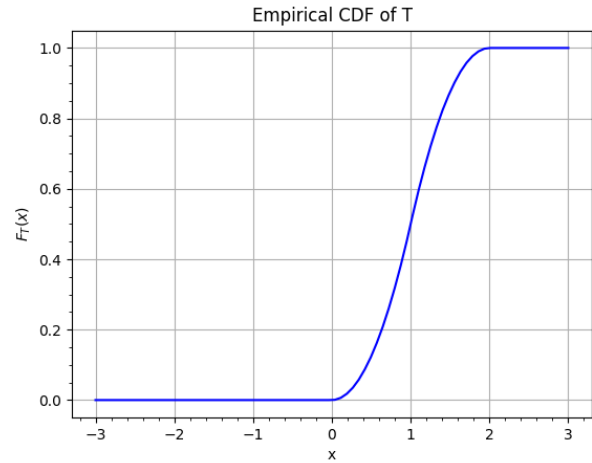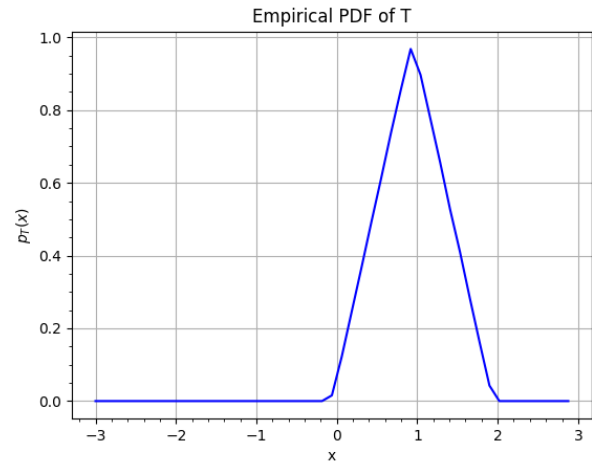**Solution:** The empirical PDF of T is plotted in 4.3 using the code below

Fig. 4.3. The PDF of T

4.4 Find the theoretical expressions for the PDF and CDF of T.

Solution: The sum of random variables corresponds to the convolution of probability distributions.

$$p_T(x) = p_{U_1 + U_2}(x) \tag{4.2}$$

$$p_T(x) = p_{U_1}(x) * p_{U_2}(x) \tag{4.3}$$

$$p_T(x) = \int_{-\infty}^{\infty} p_{U_1}(\tau)\, p_{U_2}(x - \tau)\, d\tau \tag{4.4}$$

$$p_{U_1}(\tau) = \begin{cases} 1 & 0 \leq \tau \leq 1 \\ 0 & \text{otherwise} \end{cases} \tag{4.5}$$

$$p_{U_2}(x - \tau) = \begin{cases} 1 & x - 1 \leq \tau \leq x \\ 0 & \text{otherwise} \end{cases} \tag{4.6}$$

$$p_T(x) = \begin{cases} 0 & x < 0 \\ \int_0^x 1\,d\tau & 0 < x \le 1 \\ \int_{x-1}^1 1\,d\tau & 1 < x \le 2 \\ 0 & 2 \le x \end{cases} \quad (4.7)$$

$$p_T(x) = \begin{cases} 0 & x \le 0 \\ x & 0 \le x < 1 \\ 2 - x & 1 \le x < 2 \\ 0 & 2 \le x \end{cases} \quad (4.8)$$

We integrate PDF to find CDF,

$$F_T(x) = \int_{-\infty}^{x} p_t(z)\,dz \quad (4.9)$$

$$F_T(x) = \begin{cases} \int_{-\infty}^x 0\,dz & x \le 0 \\ F_T(0) + \int_0^x z\,dz & 0 \le x < 1 \\ F_T(1) + \int_1^x (2 - z)\,dz & 1 \le x < 2 \\ F_T(2) + \int_2^x 0\,dz & 2 \le x \end{cases} \quad (4.10)$$

$$F_T(x) = \begin{cases} 0 & x \le 0 \\ \frac{x^2}{2} & 0 \le x < 1 \\ -\frac{x^2}{2} + 2x - 1 & 1 \le x < 2 \\ 1 & 2 \le x \end{cases} \quad (4.11)$$

4.5 Verify your results through a plot.
Solution: The theoretical PDF and CDF of X is plotted in Fig. 4.5 and Fig. 4.5 using the code below

```
wget https://github.com/cs21btech11051Rajiv/
    AI1110_assignments/blob/main/manual1/
    code/q4/4p5.c
```

5 MAXIMUM LIKELIHOOD

5.1 Generate equiprobable $X \in \{1, -1\}$.
Solution: Download the following c code. Run it to generate samples of X.

```
wget https://github.com/cs21btech11051Rajiv/
    AI1110_assignments/blob/main/manual1/
    code/q5/5p1.c
```

5.2 Generate
$$Y = AX + N, \quad (5.1)$$

where $A = 5$ dB, and $N \sim \mathcal{N}(0, 1)$.
Solution: Download the following c code. Run it to generate samples of Y.
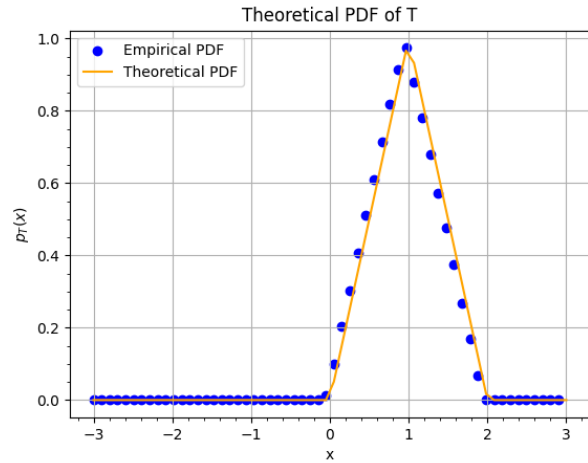

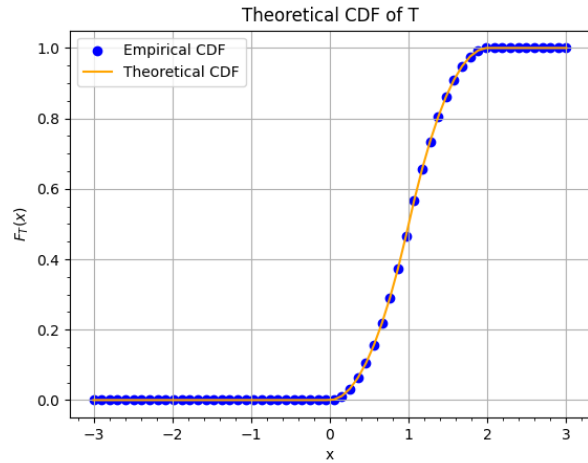Fig. 4.5. The theoretical PDF of T

Fig. 4.5. The theoretical CDF of T

```
wget https://github.com/cs21btech11051Rajiv/
    AI1110_assignments/blob/main/manual1/
    code/q5/5p2.c
```

5.3 Plot Y using a scatter plot.
Solution: The scatter plot 5.3 is generated using the code below

```
wget https://github.com/cs21btech11051Rajiv/
    AI1110_assignments/blob/main/manual1/
    code/q1/1p3.py
```

5.4 Guess how to estimate X from Y.
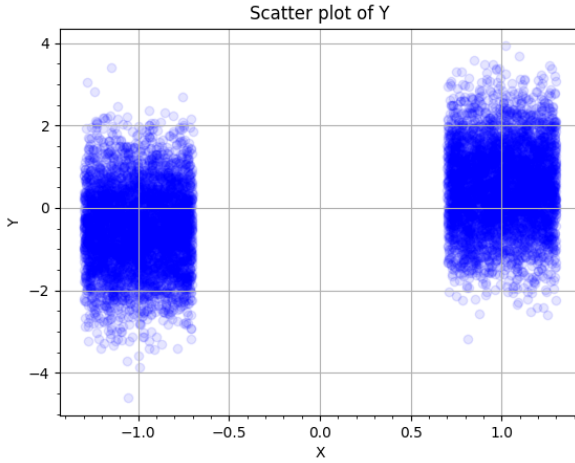**Solution:** From the scatter plot we can see that the majority of Y are postiive X is 1, and

Fig. 5.3. The scatter plot of Y against X

negative X is -1. We can estimate,

$$\hat{X} = \begin{cases} -1 & Y < 0 \\ 1 & Y \geq 0 \end{cases} \tag{5.2}$$

5.5 Find

$$P_{e|0} = \Pr\left(\hat{X} = -1 | X = 1\right) \tag{5.3}$$

and

$$P_{e|1} = \Pr\left(\hat{X} = 1 | X = -1\right) \tag{5.4}$$

Solution: The following c code finds empirical values for $P_{e|0}$ and $P_{e|1}$.

wget https://github.com/cs21btech11051Rajiv/
AI1110_assignments/blob/main/manual1/
code/q5/5p5.py

The experimental values are

$$P_{e|0} = 0.31068 \tag{5.5}$$

$$P_{e|1} = 0.30915 \tag{5.6}$$

5.6 Find P_e assuming that X has equiprobable symbols.
Solution: The empirical value of P_e is

$$P_e = P_X(1) \times P_{e|0} + P_X(-1) \times P_{e|1} \tag{5.7}$$

$$P_e = \tfrac{1}{2} \times 0.31068 + \tfrac{1}{2} \times 0.30915 \tag{5.8}$$

$$P_e = 0.30992 \tag{5.9}$$

The theoretical value of P_e is found below

$$P_e = P_X(1) \times \Pr\left(\hat{X} = -1 | X = 1\right)$$
$$+ P_X(-1) \times \Pr\left(\hat{X} = 1 | X = -1\right) \tag{5.10}$$

$$P_e = \frac{1}{2} \times \Pr\left(Y < 0 | X = 1\right)$$
$$+ \frac{1}{2} \times \Pr\left(Y > 0 | X = -1\right) \tag{5.11}$$

$$P_e = \frac{1}{2} \times \Pr\left(A + N < 0\right)$$
$$+ \frac{1}{2} \times \Pr\left(-A + N > 0\right) \tag{5.12}$$

$$P_e = \frac{1}{2} \times Q(A) + \frac{1}{2} \times Q(A) \tag{5.13}$$

$$P_e = Q(A) \tag{5.14}$$

5.7 Verify by plotting the theoretical P_e with respect to A from 0 to 10 dB.
Solution: The plot of theoretical and empirical P_e is 5.7. The semilog graph of the same is 5.7. They are generated using the code below

wget https://github.com/cs21btech11051Rajiv/
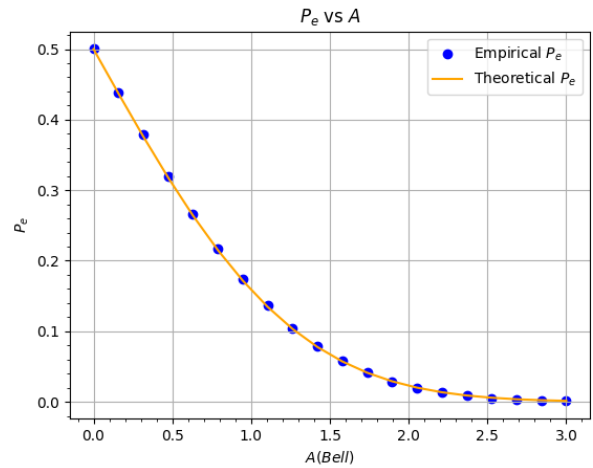AI1110_assignments/blob/main/manual1/
code/q5/5p7.py

Fig. 5.7. The plot of theoretical and empirical P_e

5.8 Now, consider a threshold δ while estimating X from Y. Find the value of δ that maximizes the theoretical P_e.
Solution: Let us consider some threshold δ

$$\hat{X} = \begin{cases} -1 & Y < \delta \\ 1 & Y > \delta \end{cases} \tag{5.15}$$

The theoretical P_e

$$P_e = P_X(1) \times \Pr\left(\hat{X} = -1 | X = 1\right)$$
$$+ P_X(-1) \times \Pr\left(\hat{X} = 1 | X = -1\right) \tag{5.16}$$
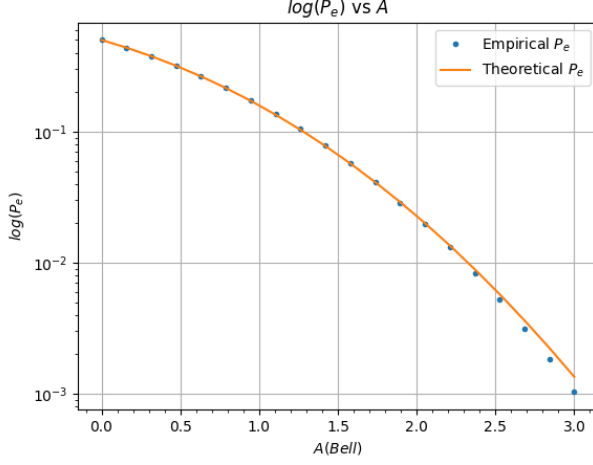
Fig. 5.7. The semilog plot of theoretical and empirical P_e

$$P_e = \frac{1}{2} \times \Pr(Y < \delta | X = 1)$$
$$+ \frac{1}{2} \times \Pr(Y > \delta | X = -1) \tag{5.17}$$

$$P_e = \frac{1}{2} \times \Pr(A + N < \delta)$$
$$+ \frac{1}{2} \times \Pr(-A + N > \delta) \tag{5.18}$$

$$P_e = \frac{1}{2}\left[F_N(\delta - A) + 1 - F_N(A + \delta)\right] \tag{5.19}$$

To find maximum P_e, we can differenciate (5.19)

$$0 = \frac{dP_e}{d\delta} \tag{5.20}$$

$$0 = \frac{1}{2}\left[p_N(\delta - A) - p_N(A + \delta)\right] \tag{5.21}$$

$$p_N(\delta - A) = p_N(A + \delta) \tag{5.22}$$

$$\frac{e^{-\frac{(\delta-A)^2}{2}}}{\sqrt{2\pi}} = \frac{e^{-\frac{(A+\delta)^2}{2}}}{\sqrt{2\pi}} \tag{5.23}$$

$$(\delta - A)^2 = (\delta + A)^2 \tag{5.24}$$

$$A = 0 \text{ or } \delta = 0 \tag{5.25}$$

We discard the first case. Thus $\delta = 0$

5.9 Repeat the above exercise when

$$p_X(1) = p \tag{5.26}$$

Solution: The theoretical P_e

$$P_e = P_X(1) \times \Pr\left(\hat{X} = -1 | X = 1\right)$$
$$+ P_X(-1) \times \Pr\left(\hat{X} = 1 | X = -1\right) \tag{5.27}$$

$$P_e = (p)\Pr(Y < \delta | X = 1)$$
$$+ (1 - p)\Pr(Y > \delta | X = -1) \tag{5.28}$$

$$P_e = (p)\Pr(A + N < \delta)$$
$$+ (1 - p)\Pr(-A + N > \delta) \tag{5.29}$$

$$P_e = (p)F_N(\delta - A)$$
$$+ (1 - p)\left[1 - F_N(A + \delta)\right] \tag{5.30}$$

To find maximum P_e, we can differenciate (5.30)

$$0 = \frac{dP_e}{d\delta} \tag{5.31}$$

$$0 = (p)p_N(\delta - A) \tag{5.32}$$

$$+ (1 - p)\left[-p_N(A + \delta)\right] \tag{5.33}$$

$$(p)p_N(\delta - A) = (1 - p)p_N(A + \delta) \tag{5.34}$$

$$\frac{(p)e^{-\frac{(\delta-A)^2}{2}}}{\sqrt{2\pi}} = \frac{(1 - p)e^{-\frac{(A+\delta)^2}{2}}}{\sqrt{2\pi}} \tag{5.35}$$

$$\ln(p) - \frac{1}{2}(\delta - A)^2 = \ln(1 - p) - \frac{1}{2}(\delta + A)^2 \tag{5.36}$$

$$\frac{1}{2}(\delta + A)^2 - \frac{1}{2}(\delta - A)^2 = \ln\left(\frac{1 - p}{p}\right) \tag{5.37}$$

The required threshold value is

$$\delta = \frac{1}{2A}\ln\left(\frac{1 - p}{p}\right) \tag{5.38}$$

5.10 Repeat the above exercise using the MAP criterion.

Solution: We need to maximize $p_{X|Y}(x|y)$. We can use Bayes' theorem to represent it as

$$p_{X|Y}(x|y) = \frac{p_{Y|X}(y|x)p_X(x)}{p_Y(y)} \tag{5.39}$$

There are two discrete cases,

$$p_{Y|X}(y|x) = \begin{cases} p_{Y|X}(y|1) & X = 1 \\ p_{Y|X}(y|-1) & X = -1 \end{cases} \tag{5.40}$$

$$p_{Y|X}(y|x) = \begin{cases} p_N(y - A) & X = 1 \\ p_N(y + A) & X = -1 \end{cases} \tag{5.41}$$

We can also calculate the denominator,

$$p_Y(y) = p_X(1)\, p_{Y|X}(y|1) \\ + p_X(-1)\, p_{Y|X}(y|-1) \tag{5.42}$$

$$p_Y(y) = (p)\, p_N(y - A) \\ + (1 - p)\, p_N(y + A) \tag{5.43}$$

We obtain,

$$p_{X|Y}(x|y) =$$

$$\frac{(p)\, p_N(y - A)}{(p)\, p_N(y - A) + (1 - p)\, p_N(y + A)}, X = 1$$

$$\frac{(1 - p)\, p_N(y + A)}{(p)\, p_N(y - A) + (1 - p)\, p_N(y + A)}, X = -1 \tag{5.44}$$

$$p_{X|Y}(x|y) = \begin{cases} \dfrac{(p)}{(p) + (1-p)\frac{p_N(y+A)}{p_N(y-A)}} & X = 1 \\[4mm] \dfrac{(1-p)}{(p)\frac{p_N(y-A)}{p_N(y+A)} + (1p)} & X = -1 \end{cases} \tag{5.45}$$

On simplifying

$$p_{X|Y}(x|y) = \begin{cases} \dfrac{(p)}{(p) + (1-p)e^{-2yA}} & X = 1 \\[4mm] \dfrac{(1-p)}{(p)e^{2yA} + (1-p)} & X = -1 \end{cases} \tag{5.46}$$

For the first value to be maximum,

$$\frac{(p)}{(p) + (1 - p)\, e^{-2yA}} > \frac{(1 - p)}{(p)\, e^{2yA} + (1 - p)} \tag{5.47}$$

$$(p)^2\, e^{2yA} > (1 - p)^2\, e^{-2yA} \tag{5.48}$$

$$e^{2yA} > \frac{1 - p}{p} \tag{5.49}$$

$$y > \frac{1}{2A} \ln\left(\frac{1 - p}{p}\right) \tag{5.50}$$

$$\hat{X}_{MAP} = \begin{cases} \dfrac{(p)}{(p)+(1-p)e^{-2yA}} & y > \frac{1}{2A}\ln\left(\frac{1-p}{p}\right) \\[4mm] \dfrac{(1-p)}{(p)e^{2yA}+(1-p)} & y \le \frac{1}{2A}\ln\left(\frac{1-p}{p}\right) \end{cases} \tag{5.51}$$

6 GAUSSIAN TO OTHER

6.1 Let $X_1 \sim \mathcal{N}(0,1)$ and $X_2 \sim \mathcal{N}(0,1)$. Plot the CDF and PDF of

$$V = X_1^2 + X_2^2 \tag{6.1}$$

Solution: First we transform X_1 and X_2 to polar form as $R \in [0, \infty)$ and $\Theta \in [0, 2\pi)$.

$$X_1 = R\cos\Theta \tag{6.2}$$
$$X_2 = R\sin\Theta \tag{6.3}$$

The transformation is carried out using the following Jacobian Matrix.

$$\mathbf{J} = \begin{pmatrix} \frac{\partial X_1}{\partial R} & \frac{\partial X_2}{\partial R} \\ \frac{\partial X_1}{\partial \Theta} & \frac{\partial X_2}{\partial \Theta} \end{pmatrix} \tag{6.4}$$

$$\mathbf{J} = \begin{pmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{pmatrix} \tag{6.5}$$

$$|\mathbf{J}| = r \tag{6.6}$$

The equation for transformation is as follows

$$p_{R,\Theta}(r,\theta) = |\mathbf{J}|\, p_{X_1,X_2}(x_1,x_2) \tag{6.7}$$

As X_1 and X_2 are independent,

$$p_{R,\Theta} = |\mathbf{J}|\, p_{X_1}(x_1)\, p_{X_2}(x_2) \tag{6.8}$$

$$p_{R,\Theta} = r\frac{e^{-\frac{x_1^2}{2}}}{\sqrt{2\pi}}\frac{e^{-\frac{x_2^2}{2}}}{\sqrt{2\pi}} \tag{6.9}$$

$$p_{R,\Theta} = \frac{r}{2\pi}e^{-\frac{x_1^2+x_2^2}{2}} \tag{6.10}$$

$$p_{R,\Theta} = \frac{r}{2\pi}e^{-\frac{r^2}{2}} \tag{6.11}$$

As R and Θ are independent,

$$p_R(r)\, p_\Theta(\theta) = p_{R,\Theta}(r,\theta) \tag{6.12}$$

$$\therefore p_R(r)\, p_\Theta(\theta) = \frac{r}{2\pi}e^{-\frac{r^2}{2}} \tag{6.13}$$

$$\int_0^{2\pi} p_R(r)\, p_\Theta(\theta)\, d\theta = \int_0^{2\pi} \frac{r}{2\pi}e^{-\frac{r^2}{2}}\, d\theta \tag{6.14}$$

We obtain PDF and CDF of R.

$$p_R(r) = \begin{cases} re^{-\frac{r^2}{2}} & r \geq 0 \\ 0 & r < 0 \end{cases} \qquad (6.15)$$

$$F_R(r) = \int_0^r te^{-\frac{t^2}{2}} dt \qquad r \geq 0 \qquad (6.16)$$

$$F_R(r) = \begin{cases} -e^{-\frac{r^2}{2}} + 1 & r \geq 0 \\ 0 & r < 0 \end{cases} \qquad (6.17)$$

These lead to PDF and CDF of V.

$$V = R^2 \qquad (6.18)$$
$$F_V(v) = Pr(V < v) \qquad (6.19)$$
$$F_V(v) = Pr(R^2 < v) \qquad (6.20)$$
$$F_V(v) = Pr(R < \sqrt{v}) \qquad (6.21)$$
$$F_V(v) = F_R(\sqrt{v}) \qquad (6.22)$$

$$F_V(v) = \begin{cases} 1 - e^{-\frac{v}{2}} & v \geq 0 \\ 0 & v < 0 \end{cases} \qquad (6.23)$$

$$p_V(v) = \begin{cases} \frac{1}{2}e^{-\frac{v}{2}} & v \geq 0 \\ 0 & v < 0 \end{cases} \qquad (6.24)$$

The plot of theoretical and empirical CDF of V is 6.1. The plot of theoretical and empirical PDF of V is 6.1. These are generated using the code below

```
wget https://github.com/cs21btech11051Rajiv/
    AI1110_assignments/blob/main/manual1/
    code/q6/6p1cdf.py
```

```
wget https://github.com/cs21btech11051Rajiv/
    AI1110_assignments/blob/main/manual1/
    code/q6/6p1pdf.py
```

6.2 If

$$F_V(x) = \begin{cases} 1 - e^{-\alpha x} & x \geq 0 \\ 0 & x < 0, \end{cases} \qquad (6.25)$$

find α.
Solution: From (6.23) we obtain,

$$\alpha = \frac{1}{2} \qquad (6.26)$$

6.3 Plot the CDF and PDf of
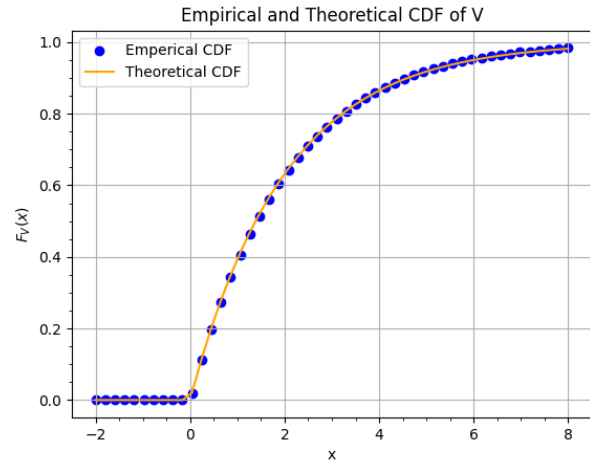
$$A = \sqrt{V} \qquad (6.27)$$
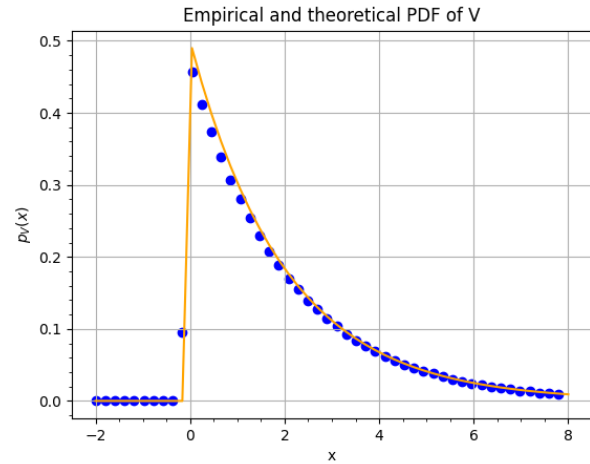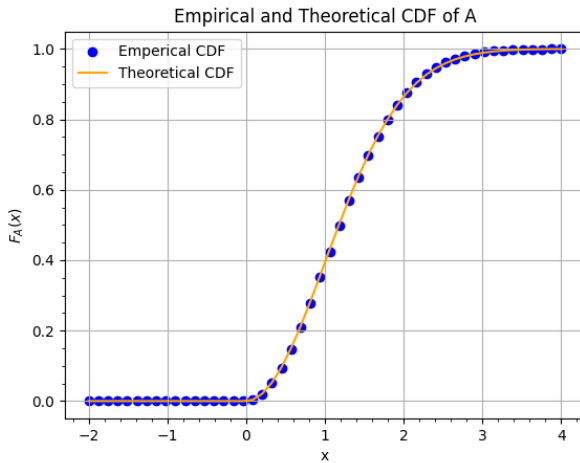

Fig. 6.1. The plot of theoretical and empirical F_V

Fig. 6.1. The plot of theoretical and empirical p_V

Solution:

$$V = A^2 \qquad (6.28)$$

A is identical to R and its theoretical distrubtions are given in (6.15) and (6.17). The plot of theoretical and empirical CDF of A is 6.3. The plot of theoretical and empirical PDF of A is 6.3. These are generated using the code below

```
wget https://github.com/cs21btech11051Rajiv/
    AI1110_assignments/blob/main/manual1/
    code/q6/6p3cdf.py
```
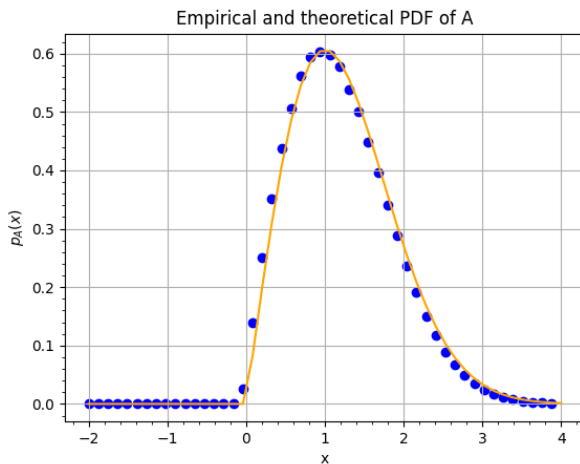
Fig. 6.3. The plot of theoretical and empirical F_A

Fig. 6.3. The plot of theoretical and empirical $p_V A$

7 CONDITIONAL PROBABILITY

7.1 Plot

$$P_e = \Pr\left(\hat{X} = -1|X = 1\right) \qquad (7.1)$$

for

$$Y = AX + N, \qquad (7.2)$$

where A is Raleigh with $E\left[A^2\right] = \gamma, N \sim \mathcal{N}(0,1), X \in (-1,1)$ for $0 \leq \gamma \leq 10$ dB.

7.2 Assuming that N is a constant, find an expression for P_e. Call this $P_e(N)$

7.3 For a function g,

$$E\left[g(X)\right] = \int_{-\infty}^{\infty} g(x)p_X(x)\, dx \qquad (7.3)$$

Find $P_e = E\left[P_e(N)\right]$.

7.4 Plot P_e in problems 7.1 and 7.3 on the same graph w.r.t γ. Comment.

8 TWO DIMENSIONS

Let

$$\mathbf{y} = A\mathbf{x} + \mathbf{n}, \qquad (8.1)$$

where

$$x \in (\mathbf{s}_0, \mathbf{s}_1), \mathbf{s}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathbf{s}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad (8.2)$$

$$\mathbf{n} = \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}, n_1, n_2 \sim \mathcal{N}(0,1). \qquad (8.3)$$

8.1 Plot

$$\mathbf{y}|\mathbf{s}_0 \text{ and } \mathbf{y}|\mathbf{s}_1 \qquad (8.4)$$

on the same graph using a scatter plot.

8.2 For the above problem, find a decision rule for detecting the symbols $\mathbf{s}_0$ and $\mathbf{s}_1$.

8.3 Plot

$$P_e = \Pr\left(\hat{\mathbf{x}} = \mathbf{s}_1|\mathbf{x} = \mathbf{s}_0\right) \qquad (8.5)$$

with respect to the SNR from 0 to 10 dB.

8.4 Obtain an expression for P_e. Verify this by comparing the theory and simulation plots on the same graph.