Random Variable Generation, AI1110

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CONTENTS

***Abstract*—This manual provides solutions to simple examples of generation of random numbers**

1 Uniform Random Numbers

Let U be a uniform random variable between 0 and 1.

1.1 Generate 10^6 samples of U using a C program and save into a file called uni.dat .
Solution: Download the following c code. Run it to generate samples of U.

```
wget https://github.com/cs21btech11051Rajiv/
    AI1110_assignments/blob/main/manual1/
    code/q1/1p1.c
```

1.2 Load the uni.dat file into python and plot the empirical CDF of U using the samples in uni.dat. The CDF is defined as

$$F_U(x) = \Pr(U \le x) \tag{1.1}$$

Solution: The empirical CDF of U is plotted in 1.2 using the code below

```
wget https://github.com/cs21btech11051Rajiv/
    AI1110_assignments/blob/main/manual1/
    code/q1/1p2.py
```

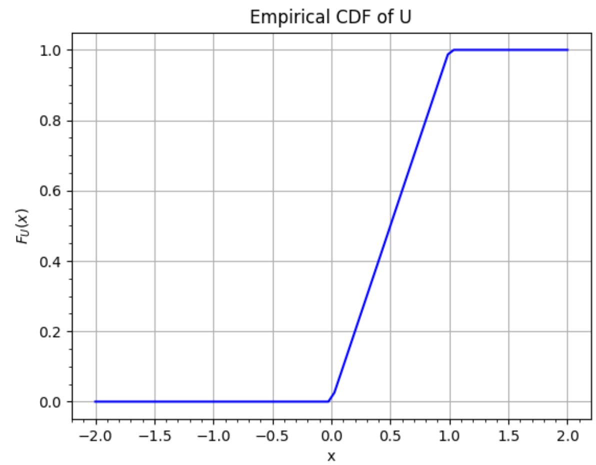

Fig. 1.2. The CDF of U

1.3 Find a theoretical expression for $F_U(x)$.
Solution: The PDF has a uniform distribution,

$$p_U(x) = \begin{cases} \frac{1}{1-0} & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases} \tag{1.2}$$

$$F_U(x) = \int_{\infty}^{x} p_U(x)\, dx \tag{1.3}$$

$$F_U(x) = \begin{cases} 0 & x < 0 \\ \int_0^x 1dx & 0 \le x \le 1 \\ \int_0^1 1dx & 1 < x \end{cases} \tag{1.4}$$

$$F_U(x) = \begin{cases} 0 & x < 0 \\ x & 0 \le x \le 1 \\ 1 & 1 < x \end{cases} \tag{1.5}$$

The theoretical CDF of U is plotted in 1.3 using the code below

```
wget https://github.com/cs21btech11051Rajiv/
    AI1110_assignments/blob/main/manual1/
    code/q1/1p3.py
```

1.4 The mean of U is defined as

$$E[U] = \frac{1}{N}\sum_{i=1}^{N} U_i \tag{1.6}$$

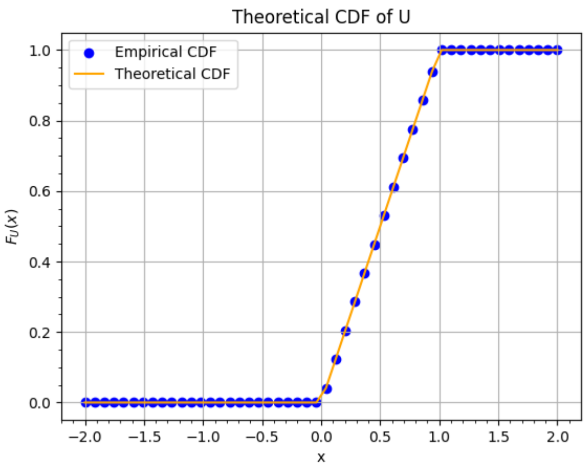

Fig. 1.3. The CDF of U

and its variance as

$$\text{var}\left[U\right] = E\left[\left(U - E\left[U\right]\right)^2\right] \tag{1.7}$$

Write a C program to find the mean and variance of U.

Solution: Run the following C file

```
wget https://github.com/cs21btech11051Rajiv/
    AI1110_assignments/blob/main/manual1/
    code/q1/1p4.c
```

Results were,

$$E\left[U\right] = 0.500007 \tag{1.8}$$

$$\text{var}\left[U\right] = 0.083301 \tag{1.9}$$

1.5 Verify your result theoretically given that

$$E\left[U^k\right] = \int_{-\infty}^{\infty} x^k dF_U\left(x\right) \tag{1.10}$$

Solution: Taking k as 1,

$$E\left[U\right] = \int_{-\infty}^{\infty} x dF_U\left(x\right) \tag{1.11}$$

Using expression (1.5), this simplifies to

$$E\left[U\right] = \int_0^1 x dx \tag{1.12}$$

$$E\left[U\right] = \left.\frac{x^2}{2}\right|_0^1 \tag{1.13}$$

$$E\left[U\right] = 0.5 \tag{1.14}$$

To calculate variance we take k=2

$$\text{var}\left[U\right] = E\left[\left(U - E\left[U\right]\right)^2\right] \tag{1.15}$$

$$\text{var}\left[U\right] = E\left[\left(U - 0.5\right)^2\right] \tag{1.16}$$

$$\text{var}\left[U\right] = E\left[U^2 - U + 0.25\right] \tag{1.17}$$

$$\text{var}\left[U\right] = E\left[U^2\right] - E\left[U\right] + E\left[0.25\right] \tag{1.18}$$

$$\text{var}\left[U\right] = \int_{-\infty}^{\infty} x^2 dF_U\left(x\right) - 0.5 + 0.25 \tag{1.19}$$

$$\text{var}\left[U\right] = \int_0^1 x^2 dx - \frac{1}{4} \tag{1.20}$$

$$\text{var}\left[U\right] = \left.\frac{x^3}{3}\right|_0^1 - \frac{1}{4} \tag{1.21}$$

$$\text{var}\left[U\right] = \frac{1}{12} = 0.0833 \tag{1.22}$$

$$\tag{1.23}$$

2 Central Limit Theorem

2.1 Generate 10^6 samples of the random variable

$$X = \sum_{i=1}^{12} U_i - 6 \tag{2.1}$$

using a C program, where $U_i, i = 1, 2, \dots, 12$ are a set of independent uniform random variables between 0 and 1 and save in a file called gau.dat

Solution: Download the following c code. Run it to generate samples of X.

```
wget https://github.com/cs21btech11051Rajiv/
    AI1110_assignments/blob/main/manual1/
    code/q1/2p1.c
```

2.2 Load gau.dat in python and plot the empirical CDF of X using the samples in gau.dat. What properties does a CDF have?

Solution: The empirical CDF of X is plotted in Fig. 2.2 using the code below

```
wget https://github.com/cs21btech11051Rajiv/
    AI1110_assignments/blob/main/manual1/
    code/q1/2p2.py
```

A CDF is a non-decreasing function. Its value varies from 0 to 1. It is continuous if PDF has finite values.

2.3 Load gau.dat in python and plot the empirical PDF of X using the samples in gau.dat. The PDF of X is defined as

$$p_X\left(x\right) = \frac{d}{dx} F_X\left(x\right) \tag{2.2}$$

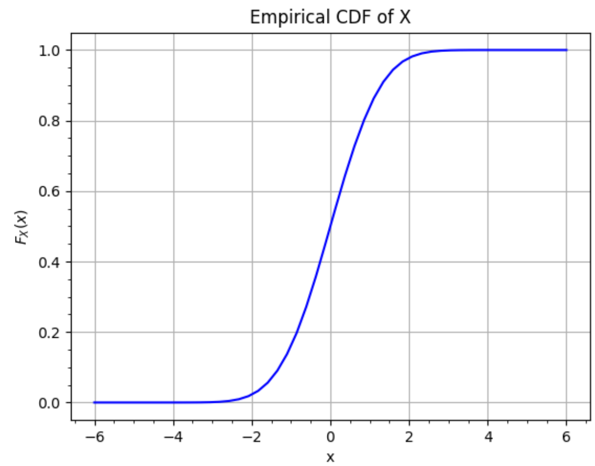

Fig. 2.2. The CDF of X

What properties does the PDF have?
Solution: The empirical PDF of X is plotted in Fig. 2.3 using the code below

```
wget https://github.com/cs21btech11051Rajiv/
    AI1110_assignments/blob/main/manual1/
    code/q2/2p3.c
```

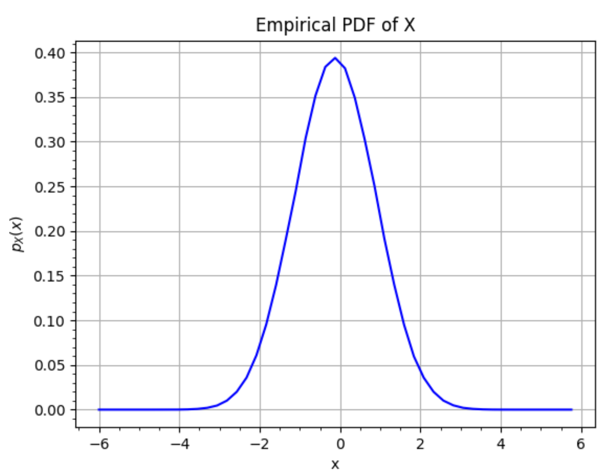

Fig. 2.3. The PDF of X

Values taken by a PDF are non negative. The total area under a PDF is equal to 1.

2.4 Find the mean and variance of X by writing a C program.
Solution: Run the following C file

```
wget https://github.com/cs21btech11051Rajiv/
    AI1110_assignments/blob/main/manual1/
    code/q2/2p4.c
```

Results were,

$$E[X] = 0.000294 \tag{2.3}$$

$$\text{var}[X] = 0.999560 \tag{2.4}$$

2.5 Given that

$$p_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), -\infty < x < \infty, \tag{2.5}$$

repeat the above exercise theoretically.
Solution: The theoretical PDF and CDF of X is plotted in Fig. 2.5 using the code below

```
wget https://github.com/cs21btech11051Rajiv/
    AI1110_assignments/blob/main/manual1/
    code/q2/2p5.c
```

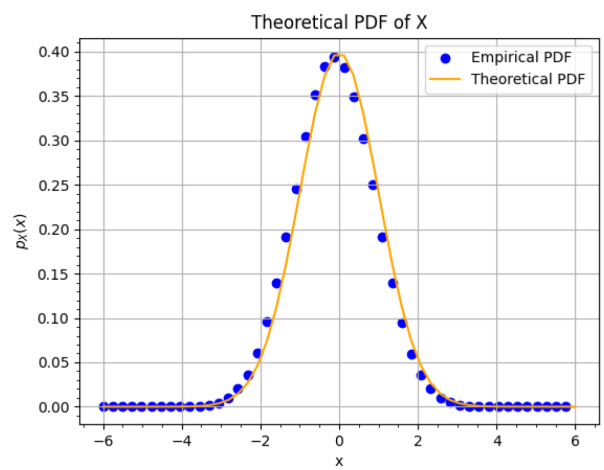

Fig. 2.5. The theoretical PDF of X

Using given equation (1.10),

$$E\left[X^k\right] = \int_{-\infty}^{\infty} x^k dF_X(x) \tag{2.6}$$

$$E[X] = \int_{-\infty}^{\infty} x dF_X(x) \tag{2.7}$$

Using given equation (2.2),

$$E[X] = \int_{-\infty}^{\infty} x p_X(x)\, dx \tag{2.8}$$

$$E[X] = \int_{-\infty}^{\infty} x \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx \tag{2.9}$$

Taking $t = \frac{x^2}{2}$

$$E[X] = \int_{\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{t}{2}\right) dt \tag{2.10}$$

$$E[X] = 0 \tag{2.11}$$

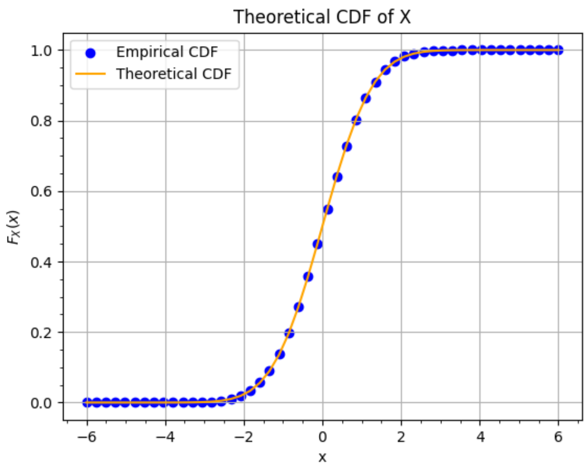

Fig. 2.5. The theoretical CDF of X

To calculate variance we take k=2

$$\text{var}\left[X\right] = E\left[\left(X - E\left[X\right]\right)^2\right] \tag{2.12}$$
$$\text{var}\left[X\right] = E\left[X^2\right] \tag{2.13}$$
$$\text{var}\left[X\right] = \int_{-\infty}^{\infty} x^2 dF_X\left(x\right) \tag{2.14}$$
$$\text{var}\left[X\right] = \int_{-\infty}^{\infty} x^2 p_X\left(x\right) dx \tag{2.15}$$
$$\text{var}\left[X\right] = \int_{-\infty}^{\infty} x^2 \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx \tag{2.16}$$

Integrating by parts, we obtain

$$\text{var}\left[X\right] = -x\frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) \Big|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx \tag{2.17}$$

The second term is the area under the PDF, which is 1

$$\text{var}\left[X\right] = 0 + \int_{-\infty}^{\infty} p_X\left(x\right) dx \tag{2.18}$$
$$\text{var}\left[X\right] = 1 \tag{2.19}$$

The theoretical CDF can be expressed as

$$F_X\left(x\right) = 1 - Q\left(\frac{x - E\left[X\right]}{\text{var}\left[X\right]}\right) \tag{2.20}$$
$$F_X\left(x\right) = 1 - Q\left(x\right) \tag{2.21}$$

3 From Uniform to Other

3.1 Generate samples of

$$V = -2\ln\left(1 - U\right) \tag{3.1}$$

and plot its CDF.
Solution: Run the following files to generate samples and CDF.

```
wget https://github.com/cs21btech11051Rajiv/
    AI1110_assignments/blob/main/manual1/
    code/q3/3p1.c
wget https://github.com/cs21btech11051Rajiv/
    AI1110_assignments/blob/main/manual1/
    code/q3/3p1b.py
```

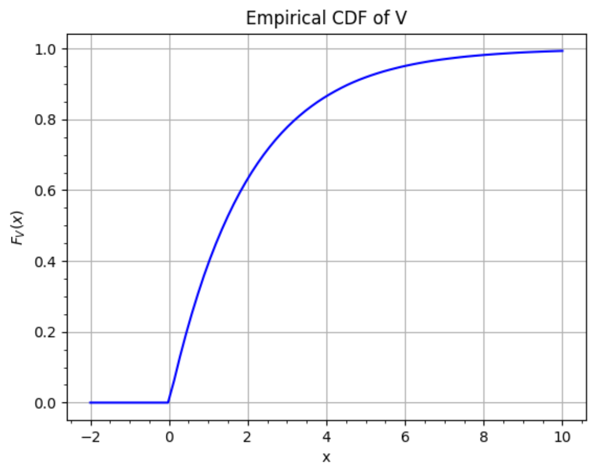

Fig. 3.1. The CDF of V

3.2 Find a theoretical expression for $F_V\left(x\right)$.
Solution:

$$F_V\left(x\right) = Pr\left(V \le x\right) \tag{3.2}$$
$$F_V\left(x\right) = Pr\left(-2\ln\left(1 - U\right) \le x\right) \tag{3.3}$$
$$F_V\left(x\right) = Pr\left(\ln\left(1 - U\right) \ge -\frac{x}{2}\right) \tag{3.4}$$
$$F_V\left(x\right) = Pr\left(1 - U \ge \exp\left(-\frac{x}{2}\right)\right) \tag{3.5}$$
$$F_V\left(x\right) = Pr\left(U \le 1 - \exp\left(-\frac{x}{2}\right)\right) \tag{3.6}$$
$$F_V\left(x\right) = F_U\left(1 - \exp\left(-\frac{x}{2}\right)\right) \tag{3.7}$$

$$F_V\left(x\right) = \begin{cases} 0 & 1 - \exp\left(-\frac{x}{2}\right) < 0 \\ 1 - \exp\left(-\frac{x}{2}\right) & 0 \le 1 - \exp\left(-\frac{x}{2}\right) \le 1 \\ 1 & 1 < 1 - \exp\left(-\frac{x}{2}\right) \end{cases} \tag{3.8}$$

This simplifies to

$$F_V(x) = \begin{cases} 0 & x < 0 \\ 1 - \exp\left(-\frac{x}{2}\right) & x \geq 0 \end{cases} \tag{3.9}$$

The following code plots the theoretical CDF.

```
wget https://github.com/cs21btech11051Rajiv/
    AI1110_assignments/blob/main/manual1/
    code/q3/3p2.py
```

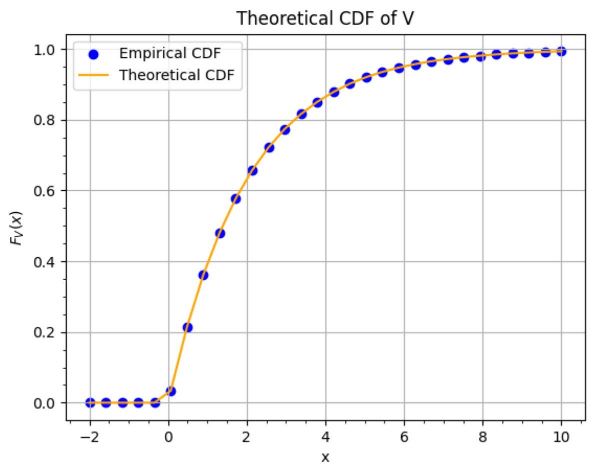

Fig. 3.2. The CDF of V

4 Triangular Distribution

4.1 Generate

$$T = U_1 + U_2 \tag{4.1}$$

Solution: Download the following c code. Run it to generate samples of T.

```
wget https://github.com/cs21btech11051Rajiv/
    AI1110_assignments/blob/main/manual1/
    code/q4/4p1.c
```

4.2 Find the CDF of T. **Solution:** The empirical CDF of T is plotted in 4.2 using the code below

```
wget https://github.com/cs21btech11051Rajiv/
    AI1110_assignments/blob/main/manual1/
    code/q4/4p2.py
```

4.3 Find the PDF of T. **Solution:** The empirical CDF of T is plotted in 4.3 using the code below

```
wget https://github.com/cs21btech11051Rajiv/
    AI1110_assignments/blob/main/manual1/
    code/q4/4p3.py
```

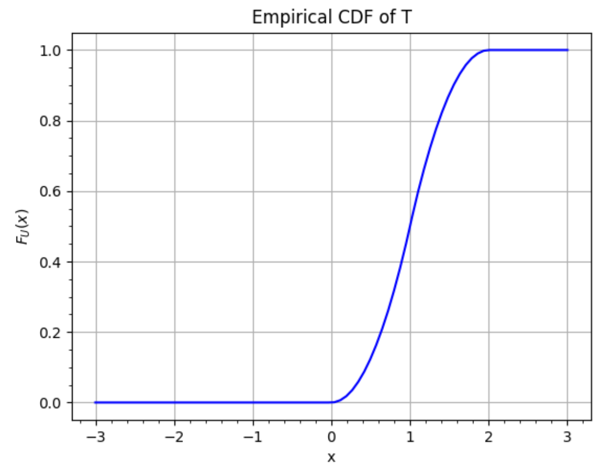

Fig. 4.2. The CDF of T

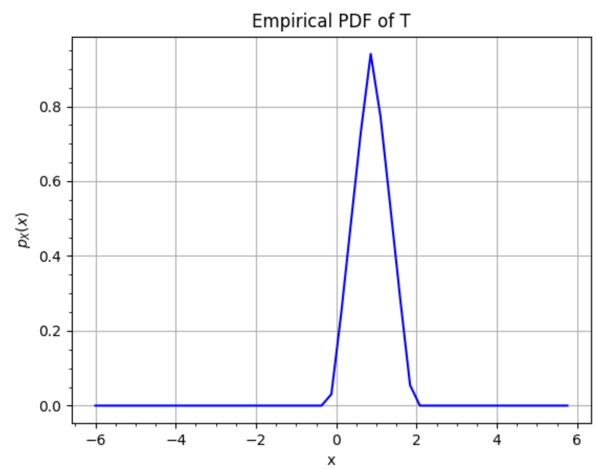

Fig. 4.3. The PDF of T

4.4 Find the theoretical expressions for the PDF and CDF of T.
Solution: The sum of random variables corresponds to the convolution of probability distributions.

$$p_T(x) = p_{U_1+U_2}(x) \tag{4.2}$$

$$p_T(x) = p_{U_1}(x) * p_{U_2}(x) \tag{4.3}$$

$$p_T(x) = \int_{-\infty}^{\infty} p_{U_1}(\tau)\, p_{U_2}(x-\tau)\, d\tau \tag{4.4}$$

$$p_{U_1}(\tau) = \begin{cases} 1 & 0 \leq \tau \leq 1 \\ 0 & \text{otherwise} \end{cases} \tag{4.5}$$

$$p_{U_2}(x-\tau) = \begin{cases} 1 & x-1 \leq \tau \leq x \\ 0 & \text{otherwise} \end{cases} \tag{4.6}$$

$$p_T(x) = \begin{cases} 0 & x < 0 \\ \int_0^x 1 d\tau & 0 < x \leq 1 \\ \int_{x-1}^1 1 d\tau & 1 < x \leq 2 \\ 0 & 2 \leq x \end{cases} \tag{4.7}$$

$$p_T(x) = \begin{cases} 0 & x \leq 0 \\ x & 0 \leq x < 1 \\ 2-x & 1 \leq x < 2 \\ 0 & 2 \leq x \end{cases} \tag{4.8}$$

We integrate PDF to find CDF,

$$F_T(x) = \int_{-\infty}^{x} p_t(z)\, dz \tag{4.9}$$

$$F_T(x) = \begin{cases} \int_{-\infty}^x 0 dz & x \leq 0 \\ F_T(0) + \int_0^x z dz & 0 \leq x < 1 \\ F_T(1) + \int_1^x (2-z)\, dz & 1 \leq x < 2 \\ F_T(2) + \int_2^x 0 dz & 2 \leq x \end{cases} \tag{4.10}$$

$$F_T(x) = \begin{cases} 0 & x \leq 0 \\ \frac{x^2}{2} & 0 \leq x < 1 \\ -\frac{x^2}{2} + 2x - 1 & 1 \leq x < 2 \\ 1 & 2 \leq x \end{cases} \tag{4.11}$$

4.5 Verify your results through a plot.

5 Maximul Likelihood

5.1 Generate

$$Y = AX + N, \tag{5.1}$$

where $A = 5$ dB, $X1\{1, -1\}$, is Bernoulli and $N \sim \mathcal{N}(0, 1)$.

5.2 Plot Y.

5.3 Guess how to estimate X from Y.

5.4 Find

$$P_{e|0} = \Pr\left(\hat{X} = -1 | X = 1\right) \tag{5.2}$$

and

$$P_{e|1} = \Pr\left(\hat{X} = 1 | X = -1\right) \tag{5.3}$$

5.5 Find P_e.

5.6 Verify by plotting the theoretical P_e.

6 Gaussian to Other

6.1 Let $X_1 \sim \mathcal{N}(0, 1)$ and $X_2 \sim \mathcal{N}(0, 1)$. Plot the CDF and PDF of

$$V = X_1^2 + X_2^2 \tag{6.1}$$

6.2 If

$$F_V(x) = \begin{cases} 1 - e^{-\alpha x} & x \geq 0 \\ 0 & x < 0, \end{cases} \tag{6.2}$$

find α.

6.3 Plot the CDF and PDf of

$$A = \sqrt{V} \tag{6.3}$$

7 Conditional Probability

7.1

7.2 Plot

$$P_e = \Pr\left(\hat{X} = -1 | X = 1\right) \tag{7.1}$$

for

$$Y = AX + N, \tag{7.2}$$

where A is Raleigh with $E\left[A^2\right] = \gamma, N \sim \mathcal{N}(0, 1), X \in (-1, 1)$ for $0 \leq \gamma \leq 10$ dB.

7.3 Assuming that N is a constant, find an expression for P_e. Call this $P_e(N)$

7.4 For a function g,

$$E\left[g(X)\right] = \int_{-\infty}^{\infty} g(x) p_X(x)\, dx \tag{7.3}$$

Find $P_e = E\left[P_e(N)\right]$.

7.5 Plot P_e in problems 7.2 and 7.4 on the same graph w.r.t γ. Comment.

8 Two Dimensions

Let

$$\mathbf{y} = A\mathbf{x} + \mathbf{n}, \tag{8.1}$$

where

$$x \in (\mathbf{s}_0, \mathbf{s}_1), \mathbf{s}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathbf{s}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{8.2}$$

$$\mathbf{n} = \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}, n_1, n_2 \sim \mathcal{N}(0, 1). \tag{8.3}$$

8.1 Plot

$$\mathbf{y}|\mathbf{s}_0 \text{ and } \mathbf{y}|\mathbf{s}_1 \tag{8.4}$$

on the same graph using a scatter plot.

8.2 For the above problem, find a decision rule for detecting the symbols $\mathbf{s}_0$ and $\mathbf{s}_1$.

8.3 Plot

$$P_e = \Pr\left(\hat{\mathbf{x}} = \mathbf{s}_1 | \mathbf{x} = \mathbf{s}_0\right) \tag{8.5}$$

with respect to the SNR from 0 to 10 dB.

8.4 Obtain an expression for P_e. Verify this by comparing the theory and simulation plots on the same graph.