

Random Variable Generation, AI1110

Rajiv Shailesh Chitale (cs21btech11051)

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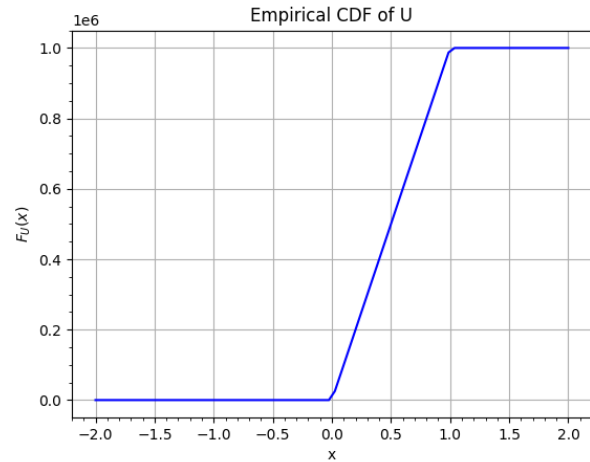


Fig. 1.2. The CDF of U

Abstract—This manual provides solutions to simple examples of generation of random numbers

1 UNIFORM RANDOM NUMBERS

Let U be a uniform random variable between 0 and 1.

1.1 Generate 10^6 samples of U using a C program and save into a file called uni.dat .

Solution: Download the following c code. Run it to generate samples of U .

```
wget https://github.com/cs21btech11051Rajiv/AI1110_assignments/blob/main/manual1/code/q1/1p1.c
```

1.2 Load the uni.dat file into python and plot the empirical CDF of U using the samples in uni.dat. The CDF is defined as

$$F_U(x) = \Pr(U \leq x) \quad (1.1)$$

Solution: The empirical CDF of U is plotted in 1.2 using the code below

```
wget https://github.com/cs21btech11051Rajiv/AI1110_assignments/blob/main/manual1/code/q1/1p2.py
```

1.3 Find a theoretical expression for $F_U(x)$.

Solution: The PDF has a uniform distribution,

$$p_U(x) = \begin{cases} \frac{1}{1-0} & 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases} \quad (1.2)$$

$$F_U(x) = \int_{-\infty}^x p_U(x) dx \quad (1.3)$$

$$F_U(x) = \begin{cases} 0 & x < 0 \\ \int_0^x 1 dx & 0 \leq x \leq 1 \\ \int_0^1 1 dx & 1 < x \end{cases} \quad (1.4)$$

$$F_U(x) = \begin{cases} 0 & x < 0 \\ x & 0 \leq x \leq 1 \\ 1 & 1 < x \end{cases} \quad (1.5)$$

The theoretical CDF of U is plotted in 1.3 using the code below

```
wget https://github.com/cs21btech11051Rajiv/AI1110_assignments/blob/main/manual1/code/q1/1p3.py
```

1.4 The mean of U is defined as

$$E[U] = \frac{1}{N} \sum_{i=1}^N U_i \quad (1.6)$$

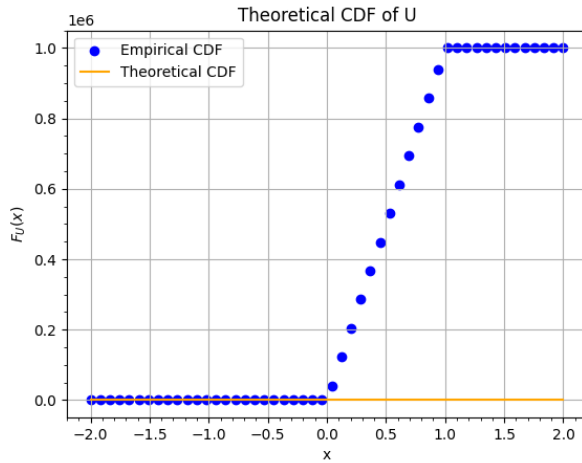


Fig. 1.3. The CDF of U

and its variance as

$$\text{var}[U] = E[(U - E[U])^2] \quad (1.7)$$

Write a C program to find the mean and variance of U .

Solution: Run the following C file

```
wget https://github.com/cs21btech11051Rajiv/
AI1110_assignments/blob/main/manual1/
code/q1/1p4.c
```

Results were,

$$E[U] = 0.500007 \quad (1.8)$$

$$\text{var}[U] = 0.083301 \quad (1.9)$$

1.5 Verify your result theoretically given that

$$E[U^k] = \int_{-\infty}^{\infty} x^k dF_U(x) \quad (1.10)$$

Solution: Taking k as 1,

$$E[U] = \int_{-\infty}^{\infty} x dF_U(x) \quad (1.11)$$

Using expression (1.5), this simplifies to

$$E[U] = \int_0^1 x dx \quad (1.12)$$

$$E[U] = \frac{x^2}{2} \Big|_0^1 \quad (1.13)$$

$$E[U] = 0.5 \quad (1.14)$$

To calculate variance we take $k=2$

$$\text{var}[U] = E[(U - E[U])^2] \quad (1.15)$$

$$\text{var}[U] = E[(U - 0.5)^2] \quad (1.16)$$

$$\text{var}[U] = E[U^2 - U + 0.25] \quad (1.17)$$

$$\text{var}[U] = E[U^2] - E[U] + E[0.25] \quad (1.18)$$

$$\text{var}[U] = \int_{-\infty}^{\infty} x^2 dF_U(x) - 0.5 + 0.25 \quad (1.19)$$

$$\text{var}[U] = \int_0^1 x^2 dx - \frac{1}{4} \quad (1.20)$$

$$\text{var}[U] = \frac{x^3}{3} \Big|_0^1 - \frac{1}{4} \quad (1.21)$$

$$\text{var}[U] = \frac{1}{12} = 0.0833 \quad (1.22)$$

$$(1.23)$$

2 CENTRAL LIMIT THEOREM

2.1 Generate 10^6 samples of the random variable

$$X = \sum_{i=1}^{12} U_i - 6 \quad (2.1)$$

using a C program, where $U_i, i = 1, 2, \dots, 12$ are a set of independent uniform random variables between 0 and 1 and save in a file called gau.dat

Solution: Download the following c code. Run it to generate samples of X .

```
wget https://github.com/cs21btech11051Rajiv/
AI1110_assignments/blob/main/manual1/
code/q1/2p1.c
```

2.2 Load gau.dat in python and plot the empirical CDF of X using the samples in gau.dat. What properties does a CDF have?

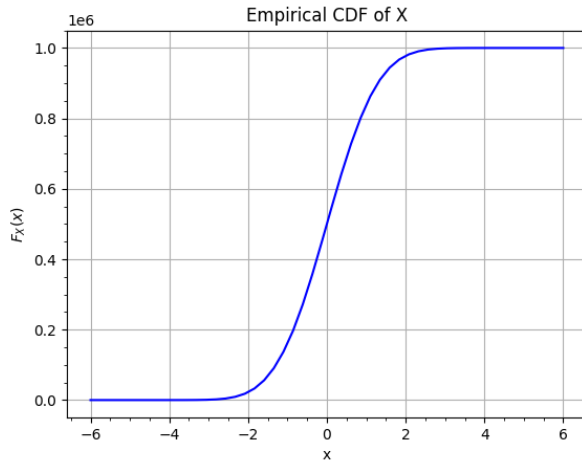
Solution: The empirical CDF of X is plotted in Fig. 2.2 using the code below

```
wget https://github.com/cs21btech11051Rajiv/
AI1110_assignments/blob/main/manual1/
code/q1/2p2.py
```

A CDF is a non-decreasing function. Its value varies from 0 to 1. It is continuous if PDF has finite values.

2.3 Load gau.dat in python and plot the empirical PDF of X using the samples in gau.dat. The PDF of X is defined as

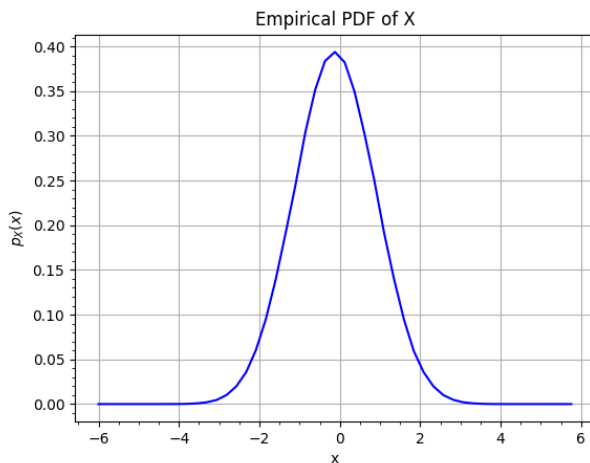
$$p_X(x) = \frac{d}{dx} F_X(x) \quad (2.2)$$

Fig. 2.2. The CDF of X

What properties does the PDF have?

Solution: The empirical PDF of X is plotted in Fig. 2.3 using the code below

```
wget https://github.com/cs21btech11051Rajiv/
  AI1110_assignments/blob/main/manual1/
  code/q2/2p3.c
```

Fig. 2.3. The PDF of X

Values taken by a PDF are non negative. The total area under a PDF is equal to 1.

2.4 Find the mean and variance of X by writing a C program.

Solution: Run the following C file

```
wget https://github.com/cs21btech11051Rajiv/
  AI1110_assignments/blob/main/manual1/
  code/q2/2p4.c
```

Results were,

$$E[X] = 0.000294 \quad (2.3)$$

$$\text{var}[X] = 0.999560 \quad (2.4)$$

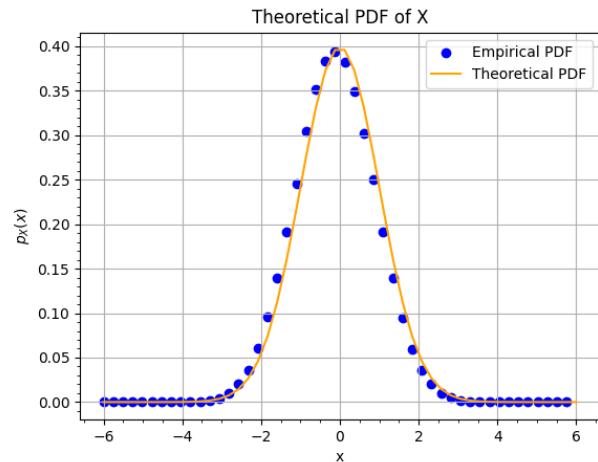
2.5 Given that

$$p_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), -\infty < x < \infty, \quad (2.5)$$

repeat the above exercise theoretically.

Solution: The theoretical PDF and CDF of X is plotted in Fig. 2.5 and Fig. 2.5 using the code below

```
wget https://github.com/cs21btech11051Rajiv/
  AI1110_assignments/blob/main/manual1/
  code/q2/2p5.c
```

Fig. 2.5. The theoretical PDF of X

Using given equation (1.10),

$$E[X^k] = \int_{-\infty}^{\infty} x^k dF_X(x) \quad (2.6)$$

$$E[X] = \int_{-\infty}^{\infty} x dF_X(x) \quad (2.7)$$

Using given equation (2.2),

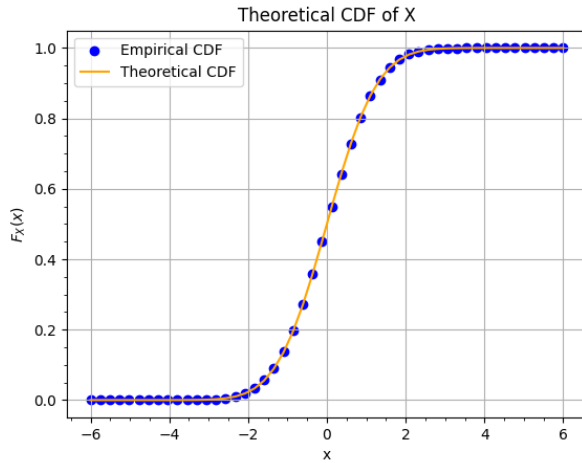
$$E[X] = \int_{-\infty}^{\infty} x p_X(x) dx \quad (2.8)$$

$$E[X] = \int_{-\infty}^{\infty} x \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx \quad (2.9)$$

Taking $t = \frac{x^2}{2}$

$$E[X] = \int_{\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{t}{2}\right) dt \quad (2.10)$$

$$E[X] = 0 \quad (2.11)$$

Fig. 2.5. The theoretical CDF of X

To calculate variance we take $k=2$

$$\text{var}[X] = E[(X - E[X])^2] \quad (2.12)$$

$$\text{var}[X] = E[X^2] \quad (2.13)$$

$$\text{var}[X] = \int_{-\infty}^{\infty} x^2 dF_X(x) \quad (2.14)$$

$$\text{var}[X] = \int_{-\infty}^{\infty} x^2 p_X(x) dx \quad (2.15)$$

$$\text{var}[X] = \int_{-\infty}^{\infty} x^2 \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx \quad (2.16)$$

Integrating by parts, we obtain

$$\begin{aligned} \text{var}[X] &= -x \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) \Big|_{-\infty}^{\infty} \\ &\quad + \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx \end{aligned} \quad (2.17)$$

The second term is the area under the PDF, which is 1

$$\text{var}[X] = 0 + \int_{-\infty}^{\infty} p_X(x) dx \quad (2.18)$$

$$\text{var}[X] = 1 \quad (2.19)$$

The theoretical CDF can be expressed as

$$F_X(x) = 1 - Q\left(\frac{x - E[X]}{\sqrt{\text{var}[X]}}\right) \quad (2.20)$$

$$F_X(x) = 1 - Q(x) \quad (2.21)$$

3 FROM UNIFORM TO OTHER

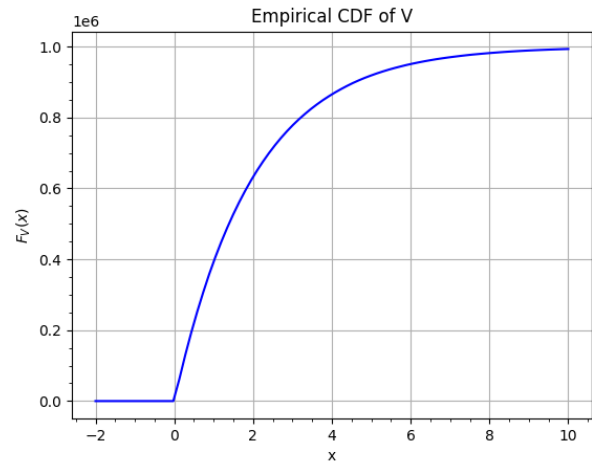
3.1 Generate samples of

$$V = -2 \ln(1 - U) \quad (3.1)$$

and plot its CDF.

Solution: Run the following files to generate samples and CDF.

```
wget https://github.com/cs21btech11051Rajiv/
  AI1110_assignments/blob/main/manual1/
  code/q3/3p1.c
wget https://github.com/cs21btech11051Rajiv/
  AI1110_assignments/blob/main/manual1/
  code/q3/3p1b.py
```

Fig. 3.1. The CDF of V

3.2 Find a theoretical expression for $F_V(x)$.

Solution:

$$F_V(x) = \Pr(V \leq x) \quad (3.2)$$

$$F_V(x) = \Pr(-2 \ln(1 - U) \leq x) \quad (3.3)$$

$$F_V(x) = \Pr\left(\ln(1 - U) \geq -\frac{x}{2}\right) \quad (3.4)$$

$$F_V(x) = \Pr\left(1 - U \geq \exp\left(-\frac{x}{2}\right)\right) \quad (3.5)$$

$$F_V(x) = \Pr\left(U \leq 1 - \exp\left(-\frac{x}{2}\right)\right) \quad (3.6)$$

$$F_V(x) = F_U\left(1 - \exp\left(-\frac{x}{2}\right)\right) \quad (3.7)$$

$$F_V(x) = \begin{cases} 0 & 1 - \exp\left(-\frac{x}{2}\right) < 0 \\ 1 - \exp\left(-\frac{x}{2}\right) & 0 \leq 1 - \exp\left(-\frac{x}{2}\right) \leq 1 \\ 1 & 1 < 1 - \exp\left(-\frac{x}{2}\right) \end{cases} \quad (3.8)$$

This simplifies to

$$F_V(x) = \begin{cases} 0 & x < 0 \\ 1 - \exp\left(-\frac{x}{2}\right) & x \geq 0 \end{cases} \quad (3.9)$$

The following code plots the theoretical CDF.

```
wget https://github.com/cs21btech11051Rajiv/
  AI1110_assignments/blob/main/manual1/
  code/q3/3p2.py
```

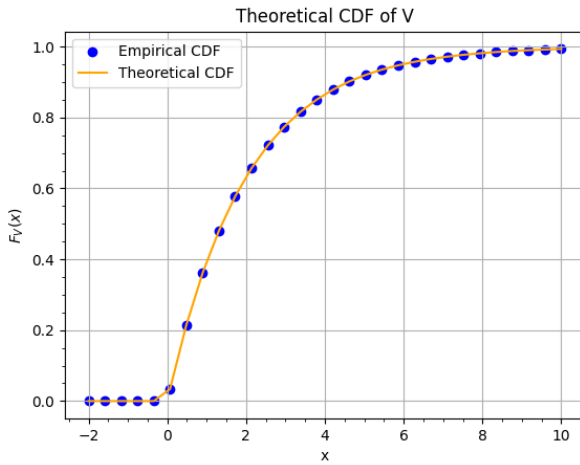


Fig. 3.2. The CDF of V

4 TRIANGULAR DISTRIBUTION

4.1 Generate

$$T = U_1 + U_2 \quad (4.1)$$

Solution: Download the following c code. Run it to generate samples of T .

```
wget https://github.com/cs21btech11051Rajiv/
  AI1110_assignments/blob/main/manual1/
  code/q4/4p1.c
```

4.2 Find the CDF of T .

Solution: The empirical CDF of T is plotted in 4.2 using the code below

```
wget https://github.com/cs21btech11051Rajiv/
  AI1110_assignments/blob/main/manual1/
  code/q4/4p2.py
```

4.3 Find the PDF of T . **Solution:** The empirical PDF of T is plotted in 4.3 using the code below

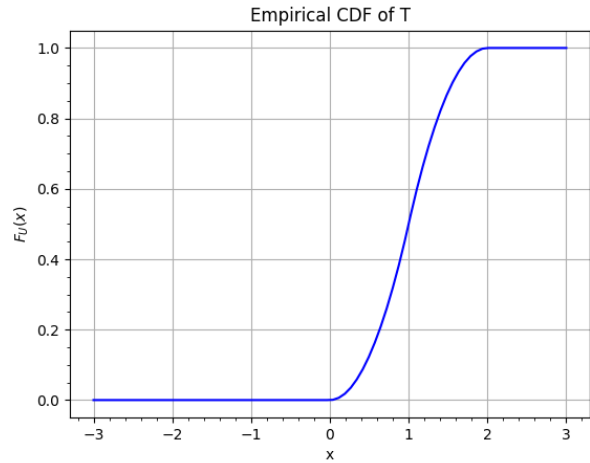


Fig. 4.2. The CDF of T

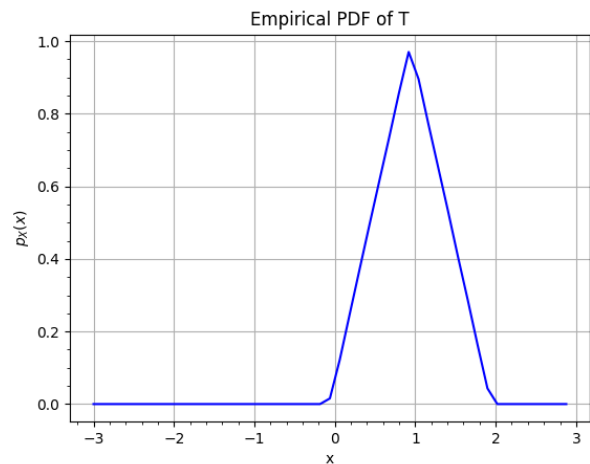


Fig. 4.3. The PDF of T

```
wget https://github.com/cs21btech11051Rajiv/
  AI1110_assignments/blob/main/manual1/
  code/q4/4p3.py
```

4.4 Find the theoretical expressions for the PDF and CDF of T .

Solution: The sum of random variables corresponds to the convolution of probability distributions.

$$p_T(x) = p_{U_1+U_2}(x) \quad (4.2)$$

$$p_T(x) = p_{U_1}(x) * p_{U_2}(x) \quad (4.3)$$

$$p_T(x) = \int_{-\infty}^{\infty} p_{U_1}(\tau) p_{U_2}(x - \tau) d\tau \quad (4.4)$$

$$p_{U_1}(\tau) = \begin{cases} 1 & 0 \leq \tau \leq 1 \\ 0 & \text{otherwise} \end{cases} \quad (4.5)$$

$$p_{U_2}(x - \tau) = \begin{cases} 1 & x - 1 \leq \tau \leq x \\ 0 & \text{otherwise} \end{cases} \quad (4.6)$$

$$p_T(x) = \begin{cases} 0 & x < 0 \\ \int_0^x 1 d\tau & 0 < x \leq 1 \\ \int_{x-1}^1 1 d\tau & 1 < x \leq 2 \\ 0 & 2 \leq x \end{cases} \quad (4.7)$$

$$p_T(x) = \begin{cases} 0 & x \leq 0 \\ x & 0 \leq x < 1 \\ 2 - x & 1 \leq x < 2 \\ 0 & 2 \leq x \end{cases} \quad (4.8)$$

We integrate PDF to find CDF,

$$F_T(x) = \int_{-\infty}^x p_t(z) dz \quad (4.9)$$

$$F_T(x) = \begin{cases} \int_{-\infty}^x 0 dz & x \leq 0 \\ F_T(0) + \int_0^x z dz & 0 \leq x < 1 \\ F_T(1) + \int_1^x (2 - z) dz & 1 \leq x < 2 \\ F_T(2) + \int_2^x 0 dz & 2 \leq x \end{cases} \quad (4.10)$$

$$F_T(x) = \begin{cases} 0 & x \leq 0 \\ \frac{x^2}{2} & 0 \leq x < 1 \\ -\frac{x^2}{2} + 2x - 1 & 1 \leq x < 2 \\ 1 & 2 \leq x \end{cases} \quad (4.11)$$

4.5 Verify your results through a plot.

Solution: The theoretical PDF and CDF of X is plotted in Fig. 4.5 and Fig. 4.5 using the code below

```
wget https://github.com/cs21btech11051Rajiv/
AI1110_assignments/blob/main/manual1/
code/q4/4p5.c
```

5 MAXIMUL LIKELIHOOD

5.1 Generate

$$Y = AX + N, \quad (5.1)$$

where $A = 5$ dB, $X_1 \{1, -1\}$, is Bernoulli and $N \sim \mathcal{N}(0, 1)$.

5.2 Plot Y .

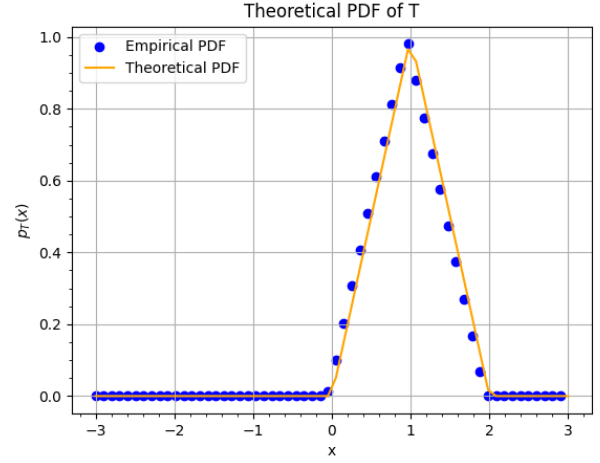


Fig. 4.5. The theoretical PDF of T

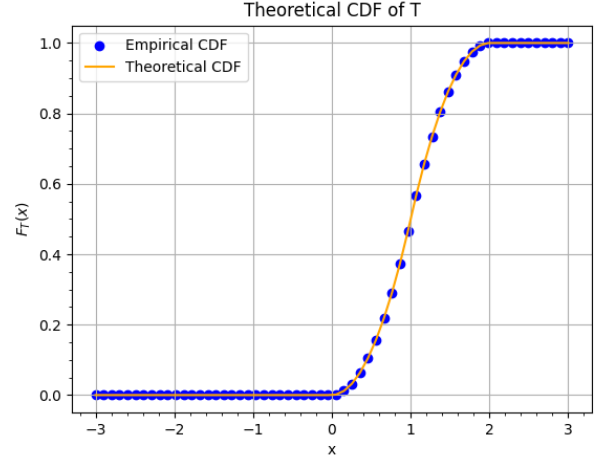


Fig. 4.5. The theoretical CDF of T

5.3 Guess how to estimate X from Y .

5.4 Find

$$P_{e|0} = \Pr(\hat{X} = -1 | X = 1) \quad (5.2)$$

and

$$P_{e|1} = \Pr(\hat{X} = 1 | X = -1) \quad (5.3)$$

5.5 Find P_e .

5.6 Verify by plotting the theoretical P_e .

6 GAUSSIAN TO OTHER

6.1 Let $X_1 \sim \mathcal{N}(0, 1)$ and $X_2 \sim \mathcal{N}(0, 1)$. Plot the CDF and PDF of

$$V = X_1^2 + X_2^2 \quad (6.1)$$

6.2 If

$$F_V(x) = \begin{cases} 1 - e^{-\alpha x} & x \geq 0 \\ 0 & x < 0, \end{cases} \quad (6.2)$$

find α .

6.3 Plot the CDF and PDF of

$$A = \sqrt{V} \quad (6.3)$$

7 CONDITIONAL PROBABILITY

7.1

7.2 Plot

$$P_e = \Pr(\hat{X} = -1 | X = 1) \quad (7.1)$$

for

$$Y = AX + N, \quad (7.2)$$

where A is Rayleigh with $E[A^2] = \gamma$, $N \sim \mathcal{N}(0, 1)$, $X \in (-1, 1)$ for $0 \leq \gamma \leq 10$ dB.

7.3 Assuming that N is a constant, find an expression for P_e . Call this $P_e(N)$

7.4 For a function g ,

$$E[g(X)] = \int_{-\infty}^{\infty} g(x)p_X(x) dx \quad (7.3)$$

Find $P_e = E[P_e(N)]$.

7.5 Plot P_e in problems 7.2 and 7.4 on the same graph w.r.t γ . Comment.

8 TWO DIMENSIONS

Let

$$\mathbf{y} = A\mathbf{x} + \mathbf{n}, \quad (8.1)$$

where

$$x \in (\mathbf{s}_0, \mathbf{s}_1), \mathbf{s}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathbf{s}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (8.2)$$

$$\mathbf{n} = \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}, n_1, n_2 \sim \mathcal{N}(0, 1). \quad (8.3)$$

8.1 Plot

$$\mathbf{y}|\mathbf{s}_0 \text{ and } \mathbf{y}|\mathbf{s}_1 \quad (8.4)$$

on the same graph using a scatter plot.

8.2 For the above problem, find a decision rule for detecting the symbols \mathbf{s}_0 and \mathbf{s}_1 .

8.3 Plot

$$P_e = \Pr(\hat{\mathbf{x}} = \mathbf{s}_1 | \mathbf{x} = \mathbf{s}_0) \quad (8.5)$$

with respect to the SNR from 0 to 10 dB.

8.4 Obtain an expression for P_e . Verify this by comparing the theory and simulation plots on the same graph.