Random Variable Generation, AI1110

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***Abstract*—This manual provides solutions to simple examples of generation of random numbers**

1 Uniform Random Numbers

Let U be a uniform random variable between 0 and 1.

1.1 Generate 10^6 samples of U using a C program and save into a file called uni.dat .
Solution: Download the following c code. Run it to generate samples of U.

```
wget https://github.com/cs21btech11051Rajiv/
    AI1110_assignments/blob/main/manual1/
    code/q1/1p1.c
```

1.2 Load the uni.dat file into python and plot the empirical CDF of U using the samples in uni.dat. The CDF is defined as

$$F_U(x) = \Pr(U \leq x) \tag{1.1}$$

Solution: The empirical CDF of U is plotted in 1.2 using the code below

```
wget https://github.com/cs21btech11051Rajiv/
    AI1110_assignments/blob/main/manual1/
    code/q1/1p2.py
```

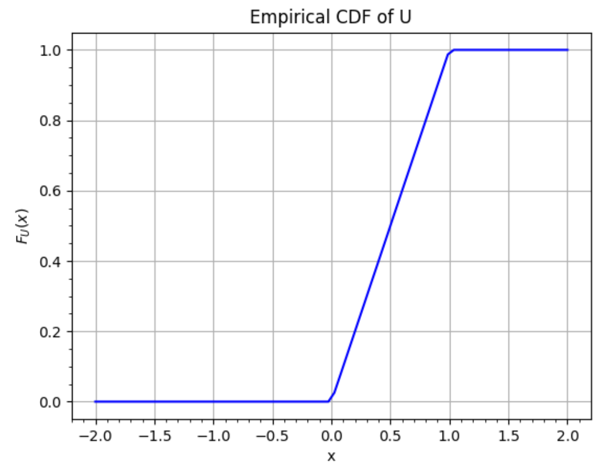

Fig. 1.2. The CDF of U

1.3 Find a theoretical expression for $F_U(x)$.
Solution: The PDF has a uniform distribution,

$$p_U(x) = \begin{cases} \frac{1}{1-0} & 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases} \tag{1.2}$$

$$F_U(x) = \int_{\infty}^{x} p_U(x)\, dx \tag{1.3}$$

$$F_U(x) = \begin{cases} 0 & x < 0 \\ \int_0^x 1dx & 0 \leq x \leq 1 \\ \int_0^1 1dx & 1 < x \end{cases} \tag{1.4}$$

$$F_U(x) = \begin{cases} 0 & x < 0 \\ x & 0 \leq x \leq 1 \\ 1 & 1 < x \end{cases} \tag{1.5}$$

The theoretical CDF of U is plotted in 1.3 using the code below

```
wget https://github.com/cs21btech11051Rajiv/
    AI1110_assignments/blob/main/manual1/
    code/q1/1p3.py
```

1.4 The mean of U is defined as

$$E[U] = \frac{1}{N}\sum_{i=1}^{N} U_i \tag{1.6}$$

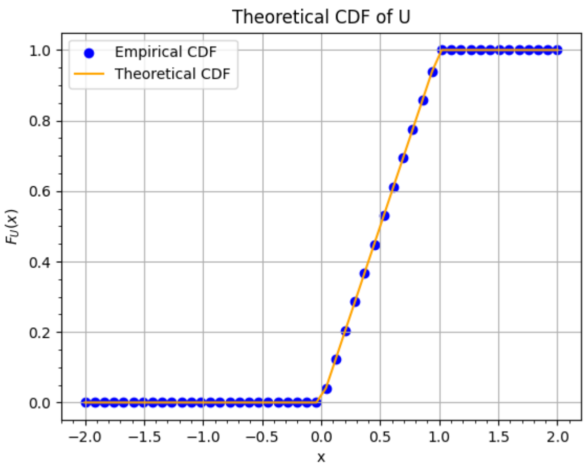

Fig. 1.3. The CDF of U

and its variance as

$$\text{var}[U] = E\left[(U - E[U])^2\right] \tag{1.7}$$

Write a C program to find the mean and variance of U.

Solution: Run the following C file

```
wget https://github.com/cs21btech11051Rajiv/
    AI1110_assignments/blob/main/manual1/
    code/q1/1p4.c
```

Results were,

$$E[U] = 0.500007 \tag{1.8}$$
$$\text{var}[U] = 0.083301 \tag{1.9}$$

1.5 Verify your result theoretically given that

$$E\left[U^k\right] = \int_{-\infty}^{\infty} x^k dF_U(x) \tag{1.10}$$

Solution: Taking k as 1,

$$E[U] = \int_{-\infty}^{\infty} x dF_U(x) \tag{1.11}$$

Using expression (1.5), this simplifies to

$$E[U] = \int_0^1 x dx \tag{1.12}$$
$$E[U] = \left.\frac{x^2}{2}\right|_0^1 \tag{1.13}$$
$$E[U] = 0.5 \tag{1.14}$$

To calculate variance we take k=2

$$\text{var}[U] = E\left[(U - E[U])^2\right] \tag{1.15}$$
$$\text{var}[U] = E\left[(U - 0.5)^2\right] \tag{1.16}$$
$$\text{var}[U] = E\left[U^2 - U + 0.25\right] \tag{1.17}$$
$$\text{var}[U] = E\left[U^2\right] - E[U] + E[0.25] \tag{1.18}$$
$$\text{var}[U] = \int_{-\infty}^{\infty} x^2 dF_U(x) - 0.5 + 0.25 \tag{1.19}$$
$$\text{var}[U] = \int_0^1 x^2 dx - \frac{1}{4} \tag{1.20}$$
$$\text{var}[U] = \left.\frac{x^3}{3}\right|_0^1 - \frac{1}{4} \tag{1.21}$$
$$\text{var}[U] = \frac{1}{12} = 0.0833 \tag{1.22}$$
$$\tag{1.23}$$

2 Central Limit Theorem

2.1 Generate 10^6 samples of the random variable

$$X = \sum_{i=1}^{12} U_i - 6 \tag{2.1}$$

using a C program, where $U_i, i = 1, 2, \dots, 12$ are a set of independent uniform random variables between 0 and 1 and save in a file called gau.dat

Solution: Download the following c code. Run it to generate samples of X.

```
wget https://github.com/cs21btech11051Rajiv/
    AI1110_assignments/blob/main/manual1/
    code/q1/2p1.c
```

2.2 Load gau.dat in python and plot the empirical CDF of X using the samples in gau.dat. What properties does a CDF have?

Solution: The empirical CDF of X is plotted in Fig. 2.2 using the code below

```
wget https://github.com/cs21btech11051Rajiv/
    AI1110_assignments/blob/main/manual1/
    code/q1/2p2.py
```

A CDF is a non-decreasing function. Its value varies from 0 to 1. It is continuous if PDF has finite values.

2.3 Load gau.dat in python and plot the empirical PDF of X using the samples in gau.dat. The PDF of X is defined as

$$p_X(x) = \frac{d}{dx} F_X(x) \tag{2.2}$$

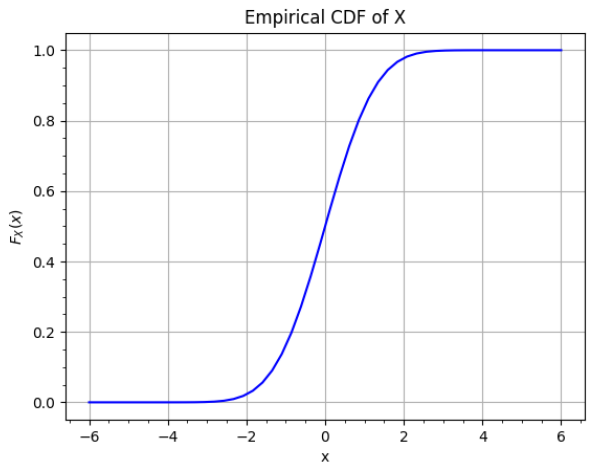

Fig. 2.2. The CDF of X

What properties does the PDF have?
Solution: The empirical PDF of X is plotted in Fig. 2.3 using the code below

```
wget https://github.com/cs21btech11051Rajiv/
    AI1110_assignments/blob/main/manual1/
    code/q2/2p3.c
```

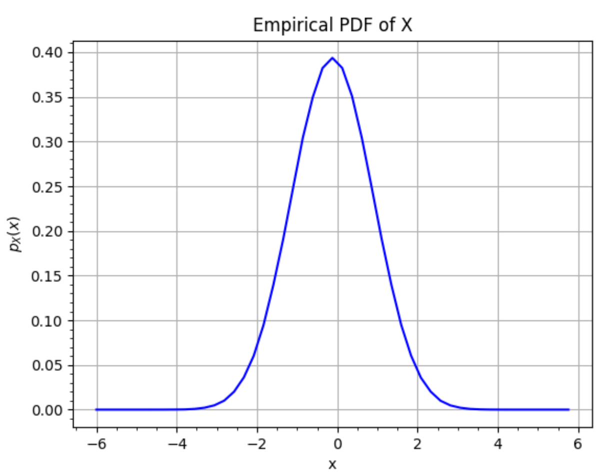

Fig. 2.3. The PDF of X

Values taken by a PDF are non negative. The total area under a PDF is equal to 1.

2.4 Find the mean and variance of X by writing a C program.
Solution: Run the following C file

```
wget https://github.com/cs21btech11051Rajiv/
    AI1110_assignments/blob/main/manual1/
    code/q2/2p4.c
```

Results were,

$$E[X] = 0.000294 \tag{2.3}$$

$$\text{var}[X] = 0.999560 \tag{2.4}$$

2.5 Given that

$$p_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), -\infty < x < \infty, \tag{2.5}$$

repeat the above exercise theoretically.
Solution: The theoretical PDF and CDF of X is plotted in Fig. 2.5 and Fig. 2.5 using the code below

```
wget https://github.com/cs21btech11051Rajiv/
    AI1110_assignments/blob/main/manual1/
    code/q2/2p5.c
```

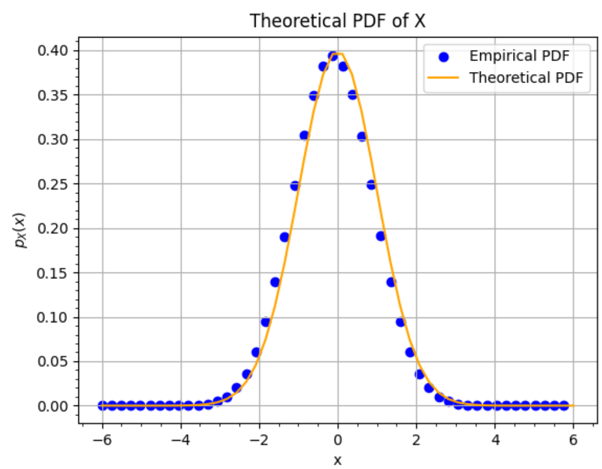

Fig. 2.5. The theoretical PDF of X

Using given equation (1.10),

$$E\left[X^k\right] = \int_{-\infty}^{\infty} x^k dF_X(x) \tag{2.6}$$

$$E[X] = \int_{-\infty}^{\infty} x dF_X(x) \tag{2.7}$$

Using given equation (2.2),

$$E[X] = \int_{-\infty}^{\infty} x p_X(x)\, dx \tag{2.8}$$

$$E[X] = \int_{-\infty}^{\infty} x \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx \tag{2.9}$$

Taking $t = \frac{x^2}{2}$

$$E[X] = \int_{\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{t}{2}\right) dt \tag{2.10}$$

$$E[X] = 0 \tag{2.11}$$

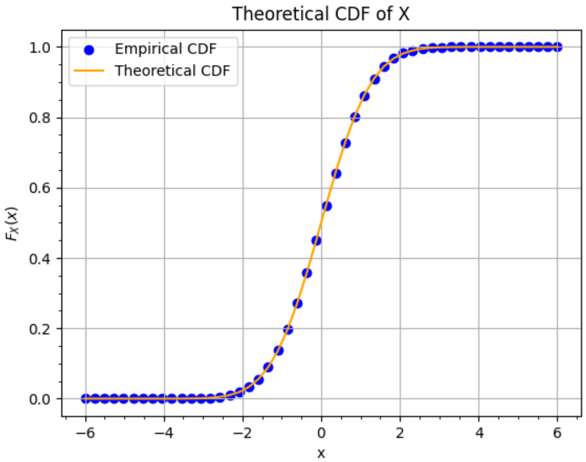

Fig. 2.5. The theoretical CDF of X

To calculate variance we take k=2

$$\text{var}\left[X\right] = E\left[\left(X - E\left[X\right]\right)^2\right] \tag{2.12}$$

$$\text{var}\left[X\right] = E\left[X^2\right] \tag{2.13}$$

$$\text{var}\left[X\right] = \int_{-\infty}^{\infty} x^2 dF_X\left(x\right) \tag{2.14}$$

$$\text{var}\left[X\right] = \int_{-\infty}^{\infty} x^2 p_X\left(x\right) dx \tag{2.15}$$

$$\text{var}\left[X\right] = \int_{-\infty}^{\infty} x^2 \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx \tag{2.16}$$

Integrating by parts, we obtain

$$\text{var}\left[X\right] = -x\frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right)\Big|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx \tag{2.17}$$

The second term is the area under the PDF, which is 1

$$\text{var}\left[X\right] = 0 + \int_{-\infty}^{\infty} p_X\left(x\right) dx \tag{2.18}$$

$$\text{var}\left[X\right] = 1 \tag{2.19}$$

The theoretical CDF can be expressed as

$$F_X\left(x\right) = 1 - Q\left(\frac{x - E\left[X\right]}{\text{var}\left[X\right]}\right) \tag{2.20}$$

$$F_X\left(x\right) = 1 - Q\left(x\right) \tag{2.21}$$

3 From Uniform to Other

3.1 Generate samples of

$$V = -2\ln\left(1 - U\right) \tag{3.1}$$

and plot its CDF.
Solution: Run the following files to generate samples and CDF.

```
wget https://github.com/cs21btech11051Rajiv/
    AI1110_assignments/blob/main/manual1/
    code/q3/3p1.c
wget https://github.com/cs21btech11051Rajiv/
    AI1110_assignments/blob/main/manual1/
    code/q3/3p1b.py
```

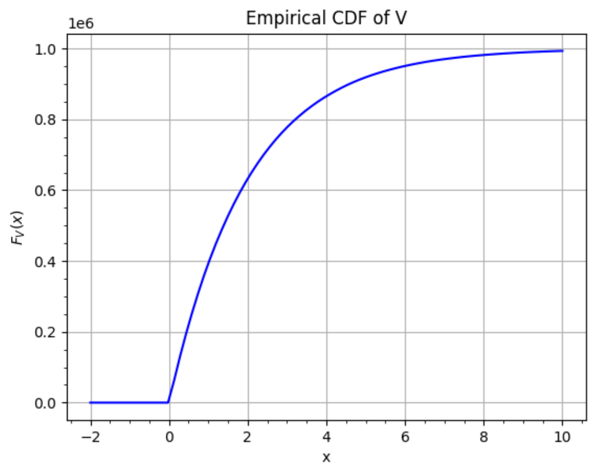

Fig. 3.1. The CDF of V

3.2 Find a theoretical expression for $F_V\left(x\right)$.
Solution:

$$F_V\left(x\right) = Pr\left(V \leq x\right) \tag{3.2}$$

$$F_V\left(x\right) = Pr\left(-2\ln\left(1 - U\right) \leq x\right) \tag{3.3}$$

$$F_V\left(x\right) = Pr\left(\ln\left(1 - U\right) \geq -\frac{x}{2}\right) \tag{3.4}$$

$$F_V\left(x\right) = Pr\left(1 - U \geq \exp\left(-\frac{x}{2}\right)\right) \tag{3.5}$$

$$F_V\left(x\right) = Pr\left(U \leq 1 - \exp\left(-\frac{x}{2}\right)\right) \tag{3.6}$$

$$F_V\left(x\right) = F_U\left(1 - \exp\left(-\frac{x}{2}\right)\right) \tag{3.7}$$

$$F_V\left(x\right) = \begin{cases} 0 & 1 - \exp\left(-\frac{x}{2}\right) < 0 \\ 1 - \exp\left(-\frac{x}{2}\right) & 0 \leq 1 - \exp\left(-\frac{x}{2}\right) \leq 1 \\ 1 & 1 < 1 - \exp\left(-\frac{x}{2}\right) \end{cases} \tag{3.8}$$

This simplifies to

$$F_V(x) = \begin{cases} 0 & x < 0 \\ 1 - \exp\left(-\frac{x}{2}\right) & x \geq 0 \end{cases} \tag{3.9}$$

The following code plots the theoretical CDF.

```
wget https://github.com/cs21btech11051Rajiv/
     AI1110_assignments/blob/main/manual1/
     code/q3/3p2.py
```

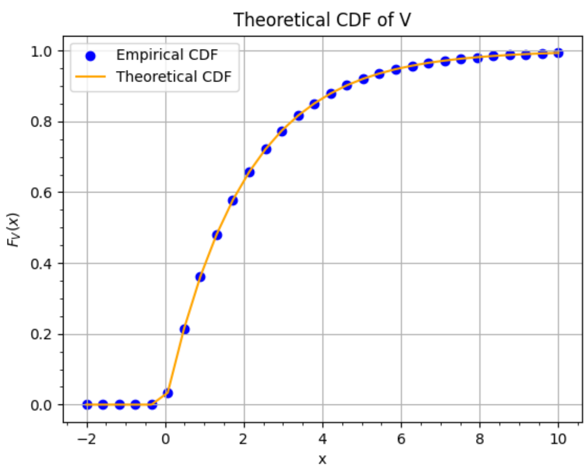

Fig. 3.2. The CDF of V

4 Triangular Distribution

4.1 Generate

$$T = U_1 + U_2 \tag{4.1}$$

Solution: Download the following c code. Run it to generate samples of T.

```
wget https://github.com/cs21btech11051Rajiv/
     AI1110_assignments/blob/main/manual1/
     code/q4/4p1.c
```

4.2 Find the CDF of T.
Solution: The empirical CDF of T is plotted in 4.2 using the code below

```
wget https://github.com/cs21btech11051Rajiv/
     AI1110_assignments/blob/main/manual1/
     code/q4/4p2.py
```

4.3 Find the PDF of T.
Solution: The empirical PDF of T is plotted in 4.3 using the code below

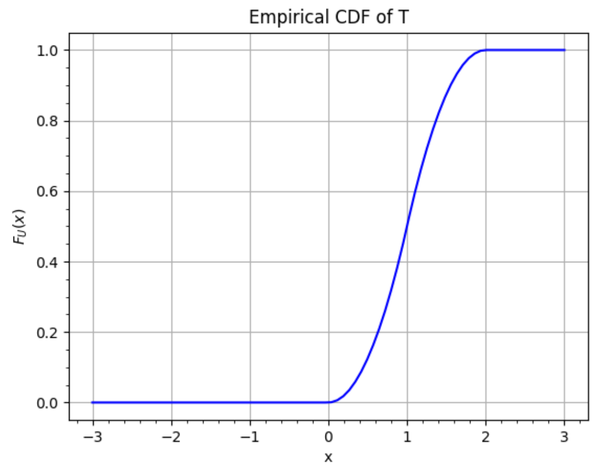

Fig. 4.2. The CDF of T

```
wget https://github.com/cs21btech11051Rajiv/
     AI1110_assignments/blob/main/manual1/
     code/q4/4p3.py
```

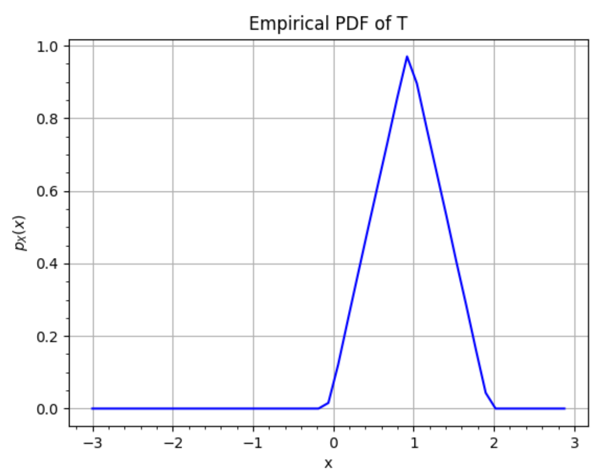

Fig. 4.3. The PDF of T

4.4 Find the theoretical expressions for the PDF and CDF of T.
Solution: The sum of random variables corresponds to the convolution of probability distributions.

$$p_T(x) = p_{U_1+U_2}(x) \tag{4.2}$$

$$p_T(x) = p_{U_1}(x) * p_{U_2}(x) \tag{4.3}$$

$$p_T(x) = \int_{-\infty}^{\infty} p_{U_1}(\tau)\, p_{U_2}(x-\tau)\, d\tau \tag{4.4}$$

$$p_{U_1}(\tau) = \begin{cases} 1 & 0 \le \tau \le 1 \\ 0 & \text{otherwise} \end{cases} \quad (4.5)$$

$$p_{U_2}(x-\tau) = \begin{cases} 1 & x-1 \le \tau \le x \\ 0 & \text{otherwise} \end{cases} \quad (4.6)$$

$$p_T(x) = \begin{cases} 0 & x < 0 \\ \int_0^x 1 d\tau & 0 < x \le 1 \\ \int_{x-1}^1 1 d\tau & 1 < x \le 2 \\ 0 & 2 \le x \end{cases} \quad (4.7)$$

$$p_T(x) = \begin{cases} 0 & x \le 0 \\ x & 0 \le x < 1 \\ 2-x & 1 \le x < 2 \\ 0 & 2 \le x \end{cases} \quad (4.8)$$

We integrate PDF to find CDF,

$$F_T(x) = \int_{-\infty}^{x} p_t(z)\, dz \quad (4.9)$$

$$F_T(x) = \begin{cases} \int_{-\infty}^x 0 dz & x \le 0 \\ F_T(0) + \int_0^x z dz & 0 \le x < 1 \\ F_T(1) + \int_1^x (2-z)\, dz & 1 \le x < 2 \\ F_T(2) + \int_2^x 0 dz & 2 \le x \end{cases} \quad (4.10)$$

$$F_T(x) = \begin{cases} 0 & x \le 0 \\ \frac{x^2}{2} & 0 \le x < 1 \\ -\frac{x^2}{2} + 2x - 1 & 1 \le x < 2 \\ 1 & 2 \le x \end{cases} \quad (4.11)$$

4.5 Verify your results through a plot.
Solution: The theoretical PDF and CDF of X is plotted in Fig. 4.5 and Fig. 4.5 using the code below

```
wget https://github.com/cs21btech11051Rajiv/
    AI1110_assignments/blob/main/manual1/
    code/q4/4p5.c
```

5 Maximum Likelihood

5.1 Generate equiprobable $X \in \{1, -1\}$.
Solution: Download the following c code. Run it to generate samples of X.

```
wget https://github.com/cs21btech11051Rajiv/
    AI1110_assignments/blob/main/manual1/
    code/q5/5p1.c
```

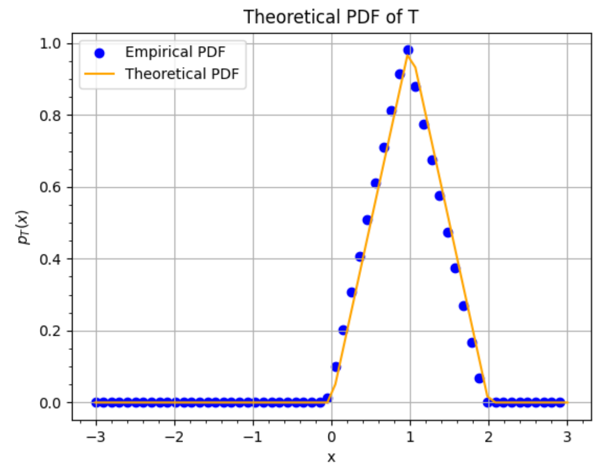

Fig. 4.5. The theoretical PDF of T

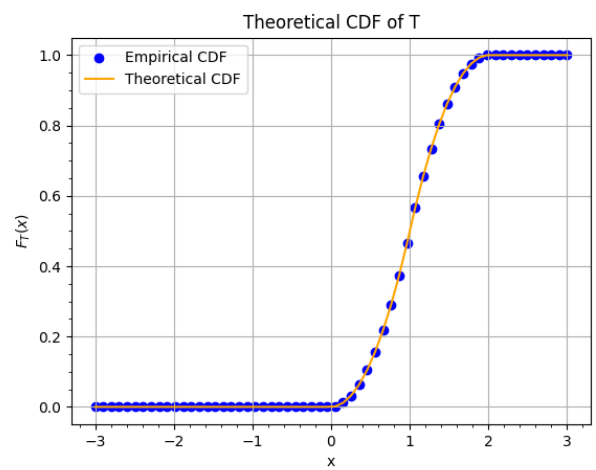

Fig. 4.5. The theoretical CDF of T

5.2 Generate

$$Y = AX + N, \quad (5.1)$$

where $A = 5$ dB, and $N \sim \mathcal{N}(0, 1)$.
Solution: Download the following c code. Run it to generate samples of Y.

```
wget https://github.com/cs21btech11051Rajiv/
    AI1110_assignments/blob/main/manual1/
    code/q5/5p2.c
```

5.3 Plot Y using a scatter plot.
Solution: The scatter plot 5.3 is generated using the code below

```
wget https://github.com/cs21btech11051Rajiv/
    AI1110_assignments/blob/main/manual1/
    code/q1/1p3.py
```

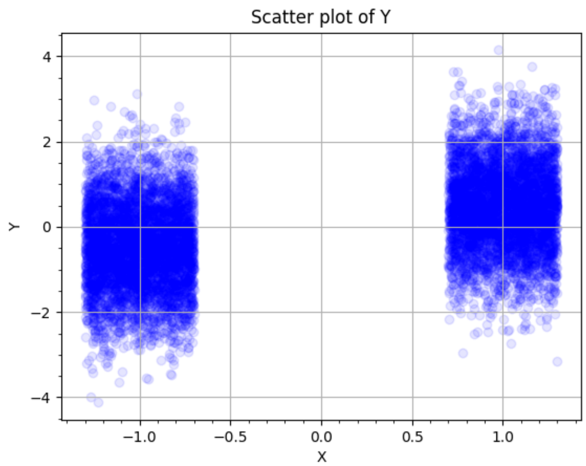

Fig. 5.3. The scatter plot of Y against X

5.4 Guess how to estimate X from Y.
Solution: From the scatter plot we can see that the majority of Y are postiive X is 1, and negative X is -1. We can estimate,

$$\hat{X} = \begin{cases} -1 & Y < 0 \\ 1 & Y \geq 0 \end{cases} \tag{5.2}$$

5.5 Find

$$P_{e|0} = \Pr\left(\hat{X} = -1 | X = 1\right) \tag{5.3}$$

and

$$P_{e|1} = \Pr\left(\hat{X} = 1 | X = -1\right) \tag{5.4}$$

Solution: The following c code finds empirical values for $P_{e|0}$ and $P_{e|1}$.

```
wget https://github.com/cs21btech11051Rajiv/
    AI1110_assignments/blob/main/manual1/
    code/q5/5p5.py
```

The experimental values are

$$P_{e|0} = 0.31068 \tag{5.5}$$
$$P_{e|1} = 0.30915 \tag{5.6}$$

5.6 Find P_e assuming that X has equiprobable symbols.
Solution: The empirical value of P_e is

$$P_e = P_X(1) \times P_{e|0} + P_X(-1) \times P_{e|1} \tag{5.7}$$
$$P_e = 12 \times 0.31068 + 12 \times 0.30915 \tag{5.8}$$
$$P_e = 0.30992 \tag{5.9}$$

The theoretical value of P_e is found below

$$\begin{aligned} P_e = P_X(1) \times \Pr\left(\hat{X} = -1 | X = 1\right) \\ + P_X(-1) \times \Pr\left(\hat{X} = 1 | X = -1\right) \end{aligned} \tag{5.10}$$

$$\begin{aligned} P_e = \frac{1}{2} \times \Pr\left(Y < 0 | X = 1\right) \\ + \frac{1}{2} \times \Pr\left(Y > 0 | X = -1\right) \end{aligned} \tag{5.11}$$

$$\begin{aligned} P_e = \frac{1}{2} \times \Pr\left(A + N < 0\right) \\ + \frac{1}{2} \times \Pr\left(-A + N > 0\right) \end{aligned} \tag{5.12}$$

$$P_e = \frac{1}{2} \times Q(A) + \frac{1}{2} \times Q(A) \tag{5.13}$$

$$P_e = Q(A) \tag{5.14}$$

5.7 Verify by plotting the theoretical P_e with respect to A from 0 to 10 dB.
Solution: The plot of theoretical and empirical P_e is 5.7. The semilog graph of the same is **??**. They are generated using the code below

```
wget https://github.com/cs21btech11051Rajiv/
    AI1110_assignments/blob/main/manual1/
    code/q5/5p7.py
```

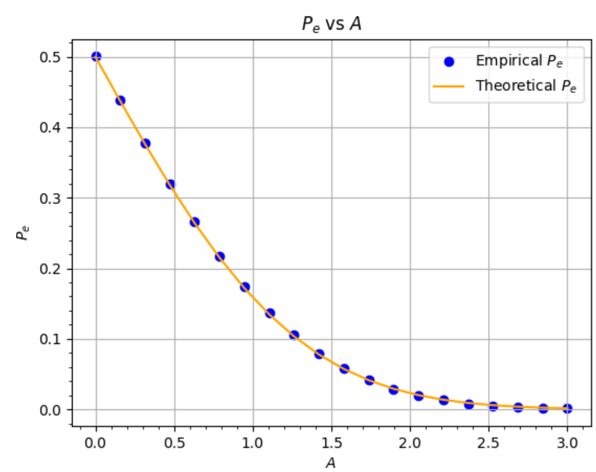

Fig. 5.7. The plot of theoretical and empirical P_e

5.8 Now, consider a threshold δ while estimating X from Y. Find the value of δ that maximizes the theoretical P_e.
Solution: Let us consider some threshold δ

$$\hat{X} = \begin{cases} -1 & Y < \delta \\ 1 & Y > \delta \end{cases} \tag{5.15}$$

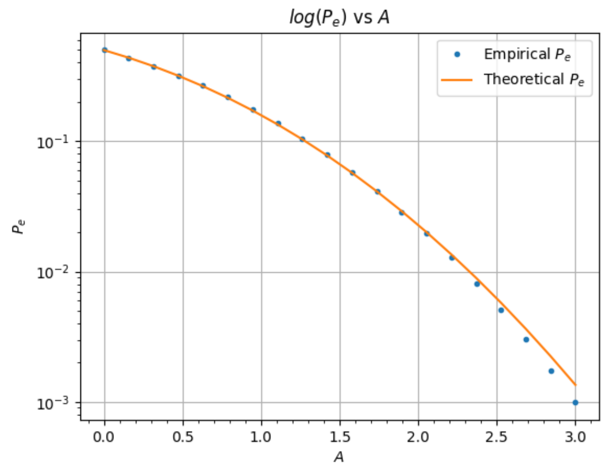

Fig. 5.7. The semilog plot of theoretical and empirical P_e

The theoretical P_e

$$P_e = P_X(1) \times \Pr\left(\hat{X} = -1|X = 1\right) + P_X(-1) \times \Pr\left(\hat{X} = 1|X = -1\right) \tag{5.16}$$

$$P_e = \frac{1}{2} \times \Pr(Y < \delta|X = 1) + \frac{1}{2} \times \Pr(Y > \delta|X = -1) \tag{5.17}$$

$$P_e = \frac{1}{2} \times \Pr(A + N < \delta) + \frac{1}{2} \times \Pr(-A + N > \delta) \tag{5.18}$$

$$P_e = \frac{1}{2}\left[F_N(\delta - A) + 1 - F_N(A + \delta)\right] \tag{5.19}$$

To find maximum P_e, we can differenciate (5.19)

$$0 = \frac{dP_e}{d\delta} \tag{5.20}$$

$$0 = \frac{1}{2}\left[p_N(\delta - A) - p_N(A + \delta)\right] \tag{5.21}$$

$$p_N(\delta - A) = p_N(A + \delta) \tag{5.22}$$

$$\frac{\exp\left(-\frac{(\delta-A)^2}{2}\right)}{\sqrt{2\pi}} = \frac{\exp\left(-\frac{(A+\delta)^2}{2}\right)}{\sqrt{2\pi}} \tag{5.23}$$

$$(\delta - A)^2 = (\delta + A)^2 \tag{5.24}$$

$$A = 0 \text{ or } \delta = 0 \tag{5.25}$$

We discard the first case. Thus $\delta = 0$

5.9 Repeat the above exercise when

$$p_X(1) = p \tag{5.26}$$

Solution: The theoretical P_e

$$P_e = P_X(1) \times \Pr\left(\hat{X} = -1|X = 1\right) + P_X(-1) \times \Pr\left(\hat{X} = 1|X = -1\right) \tag{5.27}$$

$$P_e = (p)\Pr(Y < \delta|X = 1) + (1 - p)\Pr(Y > \delta|X = -1) \tag{5.28}$$

$$P_e = (p)\Pr(A + N < \delta) + (1 - p)\Pr(-A + N > \delta) \tag{5.29}$$

$$P_e = (p)F_N(\delta - A) + (1 - p)\left[1 - F_N(A + \delta)\right] \tag{5.30}$$

To find maximum P_e, we can differenciate (5.30)

$$0 = \frac{dP_e}{d\delta} \tag{5.31}$$

$$0 = (p)p_N(\delta - A) \tag{5.32}$$

$$+ (1 - p)\left[-p_N(A + \delta)\right] \tag{5.33}$$

$$(p)p_N(\delta - A) = (1 - p)p_N(A + \delta) \tag{5.34}$$

$$\frac{(p)\exp\left(-\frac{(\delta-A)^2}{2}\right)}{\sqrt{2\pi}} = \frac{(1-p)\exp\left(-\frac{(A+\delta)^2}{2}\right)}{\sqrt{2\pi}} \tag{5.35}$$

$$\ln(p) - \frac{1}{2}(\delta - A)^2 = \ln(1 - p) - \frac{1}{2}(\delta + A)^2 \tag{5.36}$$

$$\frac{1}{2}(\delta + A)^2 - \frac{1}{2}(\delta - A)^2 = \ln\left(\frac{1 - p}{p}\right) \tag{5.37}$$

The required threshold value is

$$\delta = \frac{1}{2A}\ln\left(\frac{1 - p}{p}\right) \tag{5.38}$$

5.10 Repeat the above exercise using the MAP criterion.

Solution: We need to maximize $p_{X|Y}(x|y)$. We can use Bayes' theorem to represent it as

$$p_{X|Y}(x|y) = \frac{p_{Y|X}(y|x)\, p_X(x)}{p_Y(y)} \tag{5.39}$$

There are two discrete cases,

$$p_{Y|X}(y|x) = \begin{cases} p_{Y|X}(y|1) & X = 1 \\ p_{Y|X}(y|-1) & X = -1 \end{cases} \tag{5.40}$$

$$p_{Y|X}(y|x) = \begin{cases} p_N(y-A) & X = 1 \\ p_N(y+A) & X = -1 \end{cases} \tag{5.41}$$

We can also calculate the denominator,

$$\begin{aligned} p_Y(y) = p_X(1)\, p_{Y|X}(y|1) \\ + p_X(-1)\, p_{Y|X}(y|-1) \end{aligned} \tag{5.42}$$

$$\begin{aligned} p_Y(y) = (p)\, p_N(y-A) \\ + (1-p)\, p_N(y+A) \end{aligned} \tag{5.43}$$

We obtain,

$$\begin{aligned} &p_{X|Y}(x|y) = \\ &\frac{(p)\, p_N(y-A)}{(p)\, p_N(y-A) + (1-p)\, p_N(y+A)}, X = 1 \\ &\frac{(1-p)\, p_N(y+A)}{(p)\, p_N(y-A) + (1-p)\, p_N(y+A)}, X = -1 \end{aligned} \tag{5.44}$$

$$p_{X|Y}(x|y) = \begin{cases} \frac{(p)}{(p)+(1-p)\frac{p_N(y+A)}{p_N(y-A)}} & X = 1 \\ \\ \frac{(1-p)}{(p)\frac{p_N(y-A)}{p_N(y+A)}+(1p)} & X = -1 \end{cases} \tag{5.45}$$

On simplifying

$$p_{X|Y}(x|y) = \begin{cases} \frac{(p)}{(p)+(1-p)e^{-2yA}} & X = 1 \\ \\ \frac{(1-p)}{(p)e^{2yA}+(1-p)} & X = -1 \end{cases} \tag{5.46}$$

For the first value to be maximum,

$$\frac{(p)}{(p) + (1-p)\, e^{-2yA}} > \frac{(1-p)}{(p)\, e^{2yA} + (1-p)} \tag{5.47}$$

$$(p)^2\, e^{2yA} > (1-p)^2\, e^{-2yA} \tag{5.48}$$

$$e^{2yA} > \frac{1-p}{p} \tag{5.49}$$

$$y > \frac{1}{2A} \ln\left(\frac{1-p}{p}\right) \tag{5.50}$$

$$\hat{X}_{MAP} = \begin{cases} \frac{(p)}{(p)+(1-p)e^{-2yA}} & y > \frac{1}{2A}\ln\left(\frac{1-p}{p}\right) \\ \frac{(1-p)}{(p)e^{2yA}+(1-p)} & y \le \frac{1}{2A}\ln\left(\frac{1-p}{p}\right) \end{cases} \tag{5.51}$$

6 Gaussian to Other

6.1 Let $X_1 \sim \mathcal{N}(0,1)$ and $X_2 \sim \mathcal{N}(0,1)$. Plot the CDF and PDF of

$$V = X_1^2 + X_2^2 \tag{6.1}$$

Solution: First we transform X_1 and X_2 to polar form as $R \in [0, \infty)$ and $\Theta \in [0, 2\pi)$.

$$X_1 = R\cos\Theta \tag{6.2}$$

$$X_2 = R\sin\Theta \tag{6.3}$$

The transformation is carried out using the following Jacobian Matrix.

$$\mathbf{J} = \begin{pmatrix} \frac{\partial X_1}{\partial R} & \frac{\partial X_2}{\partial R} \\ \frac{\partial X_1}{\partial \Theta} & \frac{\partial X_2}{\partial \Theta} \end{pmatrix} \tag{6.4}$$

$$\mathbf{J} = \begin{pmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{pmatrix} \tag{6.5}$$

$$|\mathbf{J}| = r \tag{6.6}$$

The equation for transformation is as follows

$$p_{R,\Theta}(r,\theta) = |\mathbf{J}|\, p_{X_1,X_2}(x_1,x_2) \tag{6.7}$$

As X_1 and X_2 are independent,

$$p_{R,\Theta} = |\mathbf{J}|\, p_{X_1}(x_1)\, p_{X_2}(x_2) \tag{6.8}$$

$$p_{R,\Theta} = r\frac{\exp\left(-\frac{x_1^2}{2}\right)}{\sqrt{2\pi}}\frac{\exp\left(-\frac{x_2^2}{2}\right)}{\sqrt{2\pi}} \tag{6.9}$$

$$p_{R,\Theta} = \frac{r}{2\pi}\exp\left(-\frac{x_1^2+x_2^2}{2}\right) \tag{6.10}$$

$$p_{R,\Theta} = \frac{r}{2\pi}\exp\left(-\frac{r^2}{2}\right) \tag{6.11}$$

As R and Θ are independent,

$$p_R(r)\, p_\Theta(\theta) = p_{R,\Theta}(r,\theta) \tag{6.12}$$

$$\therefore p_R(r)\, p_\Theta(\theta) = \frac{r}{2\pi} \exp\left(-\frac{r^2}{2}\right) \tag{6.13}$$

$$\int_0^{2\pi} p_R(r)\, p_\Theta(\theta)\, d\theta = \int_0^{2\pi} \frac{r}{2\pi} \exp\left(-\frac{r^2}{2}\right) d\theta \tag{6.14}$$

We obtain PDF and CDF of R

$$p_R(r) = r \exp\left(-\frac{r^2}{2}\right) \qquad r \geq 0 \tag{6.15}$$

$$F_R(r) = \int_0^r t \exp\left(-\frac{t^2}{2}\right) dt \tag{6.16}$$

$$F_R(r) = -\exp\left(-\frac{r^2}{2}\right) + 1 \tag{6.17}$$

These lead to PDF and CDF of Y

$$Y = R^2 \tag{6.18}$$

$$R = \sqrt{Y} \tag{6.19}$$

$$F_Y(y) = F_R(\sqrt{y}) \tag{6.20}$$

$$\tag{6.21}$$

6.2 If

$$F_V(x) = \begin{cases} 1 - e^{-\alpha x} & x \geq 0 \\ 0 & x < 0, \end{cases} \tag{6.22}$$

find α.

6.3 Plot the CDF and PDf of

$$A = \sqrt{V} \tag{6.23}$$

7 Conditional Probability

7.1 Plot

$$P_e = \Pr\left(\hat{X} = -1 | X = 1\right) \tag{7.1}$$

for

$$Y = AX + N, \tag{7.2}$$

where A is Raleigh with $E\left[A^2\right] = \gamma, N \sim \mathcal{N}(0,1), X \in (-1,1)$ for $0 \leq \gamma \leq 10$ dB.

7.2 Assuming that N is a constant, find an expression for P_e. Call this $P_e(N)$

7.3 For a function g,

$$E\left[g(X)\right] = \int_{-\infty}^{\infty} g(x) p_X(x)\, dx \tag{7.3}$$

Find $P_e = E\left[P_e(N)\right]$.

7.4 Plot P_e in problems 7.1 and 7.3 on the same graph w.r.t γ. Comment.

8 Two Dimensions

Let

$$\mathbf{y} = A\mathbf{x} + \mathbf{n}, \tag{8.1}$$

where

$$x \in (\mathbf{s}_0, \mathbf{s}_1), \mathbf{s}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathbf{s}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{8.2}$$

$$\mathbf{n} = \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}, n_1, n_2 \sim \mathcal{N}(0,1). \tag{8.3}$$

8.1 Plot

$$\mathbf{y}|\mathbf{s}_0 \text{ and } \mathbf{y}|\mathbf{s}_1 \tag{8.4}$$

on the same graph using a scatter plot.

8.2 For the above problem, find a decision rule for detecting the symbols $\mathbf{s}_0$ and $\mathbf{s}_1$.

8.3 Plot

$$P_e = \Pr\left(\hat{\mathbf{x}} = \mathbf{s}_1 | \mathbf{x} = \mathbf{s}_0\right) \tag{8.5}$$

with respect to the SNR from 0 to 10 dB.

8.4 Obtain an expression for P_e. Verify this by comparing the theory and simulation plots on the same graph.