Assignment 6, AI1110

Rajiv Shailesh Chitale (cs21btech11051)

***Abstract*—This document provides a solution to Question 15 from NCERT class 12 Probabilty Ex 13.1**

Question 15: Consider the experiment of throwing a die. If a multiple of 3 comes up, throw the die again and if any other number comes, toss a coin. Find the conditional probability of the event 'the coin shows a tail', given that 'at least one die shows a 3'.

Solution: Some useful random variables are defined below. Y_n describes whether 3 has occurred before n^{th} throw.

Event	Description
$X_n \in \{1,2,3,4,5,6\}$	Number obtained from n^{th} die throw
$Y_n = 0$	$\nexists k < n : X_k = 3$
$Y_n = 1$	$\exists k < n : X_k = 3$
$Z = 0$	Getting heads from a coin toss
$Z = 1$	Getting tails from a coin toss

TABLE I
RANDOM VARIABLES

Let us construct a Markov chain with discrete time t. The states $e_1 \ldots e_5$ describe the outcomes from the latest dice throw or coin toss. States $e_6 \ldots e_{10}$ are similar but they denote that at least one die showed 3 before t.

i	State e_i
1	$X_t = 3 \land Y_t = 0$
2	$X_t = 6 \land Y_t = 0$
3	$X_t = i : i \in \{1,2,4,5\} \land Y_t = 0$
4	$Z = 0 \land Y_t = 0$
5	$Z = 1 \land Y_t = 0$
6	$X_t = 3 \land Y_t = 1$
7	$X_t = 6 \land Y_t = 1$
8	$X_t = i : i \in \{1,2,4,5\} \land Y_t = 1$
9	$Z = 0 \land Y_t = 1$
10	$Z = 1 \land Y_t = 1$

TABLE II
EVENT STATES

The transition probability matrix $\mathbf{P_{ij}} =$

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1/6 & 1/6 & 4/6 & 0 & 0 \\ 1/6 & 1/6 & 4/6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/2 & 1/2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/6 & 1/6 & 4/6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/6 & 1/6 & 4/6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{1}$$

An element in $\mathbf{P_{ij}}$ represents the probability of moving from state e_i to e_j. $\mathbf{Q_t}$ is the state vector at a given t. Initial state vector is provided for $t = 1$

$$\mathbf{Q_1} = \begin{pmatrix} 1/6 & 1/6 & 4/6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{2}$$

$$\mathbf{Q_t} = \mathbf{Q_1} \mathbf{P}^{t-1} \tag{3}$$

The limiting probabilities of states,

$$\lim_{t \to \infty} \mathbf{Q_t} = \begin{pmatrix} 0 & 0 & 0 & 0.4 & 0.4 & 0 & 0 & 0 & 0.1 & 0.1 \end{pmatrix} \tag{4}$$

Required conditional probability is,

$$\lim_{t \to \infty} \Pr\left((Z = 1) \,|\, \Pr\left(Y_t = 1\right)\right) \tag{5}$$

$$= \lim_{t \to \infty} \frac{\Pr\left((Z = 1)\left(Y_t = 1\right)\right)}{\Pr\left(Y_t = 1\right)} \tag{6}$$

$$= \lim_{t \to \infty} \frac{\Pr\left(e_{10}\right)}{\sum_{i=6}^{10} \Pr\left(e_i\right)} \tag{7}$$

$$= \frac{0.1}{0.2} \tag{8}$$

$$= 0.5 \tag{9}$$