Dispersion Analysis of the Continuous and Discontinuous Galerkin Formulations.

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Abstract. The dispersion relation of the semi-discrete continuous and discontinuous Galerkin formulations are analysed for the linear advection equation. In the context of an spectral/hp element discretisation on an equispaced mesh the problem can be reduced to a $P \times P$ eigenvalue problem where P is the polynomial order. The analytical dispersion relationships for polynomial order up to $P = 3$ and the numerical values for $P = 10$ are presented demonstrating similar dispersion properties but show that the discontinuous scheme is more diffusive.

1 Introduction

In this paper we derive the phase properties of the discontinuous and continuous hp element Galerkin formulation [1,2] of the linear advection equation. To gain a better insight into the phase properties of these schemes we analytically construct an $P \times P$ eigenvalue problem which completely describes the phase properties of both the continuous and discontinuous Galerkin schemes on an equi-spaced mesh. This analysis shows that the discontinuous Galerkin formulation has a comparable dispersion relationship as the continuous version although the discontinuous formulation show significant damping at higher frequencies.

2 Continuous and Discontinuous Galerkin Formulation

Considering the linear advection equation:

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 0, \tag{1}$$

we assume an equispaced discretisation within which the solution is a approximated by $u(x,t) \simeq u^e(x,t) = \sum_{p=0}^{p=P} \phi_p(x)\hat{u}_p^e(t)$ within the e^{th} elemental region $[x_l^e, x_u^e]$. Taking the inner product with respect to the expansion basis $\phi_q(x)$ we obtain the elemental Galerkin approximation:

$$\left(\phi_q(x), \frac{\partial u^e}{\partial t}(x,t)\right)_e + \left(\phi_q(x), \frac{\partial u^e}{\partial x}(x,t)\right)_e \qquad \forall\ q. \tag{2}$$

In the standard continuous Galerkin formulation $\phi_p(x)$ is typically defined in terms of an interior and boundary type decomposition so that a globally C^0 continuous can be constructed. Introducing the elemental matrices

$\boldsymbol{M}[q,p] = (\phi_q, \phi_p), \boldsymbol{D}[q,p] = (\phi_q, \frac{\partial \phi_p}{\partial x})$ the global form of equation (2) can be represented in matrix form as:

$$\boldsymbol{Z}\underline{\boldsymbol{M}}\boldsymbol{Z}^{\boldsymbol{T}}\hat{\boldsymbol{u}}_t + \boldsymbol{Z}\underline{\boldsymbol{D}}\boldsymbol{Z}^{\boldsymbol{T}}\hat{\boldsymbol{u}} = 0, \tag{3}$$

where $\boldsymbol{Z}$ is the matrix operation of direct stiffness assembly. In equation (3) the use of the underlined matrix $\underline{\boldsymbol{M}}$ represents the extension of the local matrices $\boldsymbol{M}$ to a global system of block diagonal matrices. A similar extension has also been assumed for $\hat{\boldsymbol{u}} = \boldsymbol{Z}\underline{\hat{\boldsymbol{u}}}^e$ where $\hat{\boldsymbol{u}}^e$ is a local vector of expansion coefficients, i.e. $\underline{\boldsymbol{u}}^e[p] = \hat{u}^e_p$. See [3] for more details.

For the discontinuous Galerkin formulation we integrate the second term in equation (2) by parts to obtain:

$$\left(\phi_q, \frac{\partial u^e}{\partial t}\right)_e - \left(\frac{\partial \phi_q(x)}{\partial x}, u^e\right)_e + [\phi_q(x) u^e]^{x^e_u}_{x^e_l}. \tag{4}$$

To allow information to propagate from one elemental region to another the boundary flux is upwinded which is denoted as $u^e|_x = \tilde{u}^e|_x$. For the linear advection equation $\tilde{u}^e|_x$ is defined as:

$$\tilde{u}^e|_x = u^e_{x^e_u} \;\text{ if } x = x^e_u, \qquad \tilde{u}^e|_x = u^{e-1}_{x^{e-1}_u} \;\text{ if } x = x^e_l. \tag{5}$$

A numerically more convenient form is obtained by integrating the second term in equation (4) by parts again and substituting in the definition of $u^e(x,t)$ to arrive at the equation:

$$\sum_e \left[\left(\phi_q, \textstyle\sum_p \phi_p\right)_e \frac{\partial \hat{u}_p}{\partial t} + \left(\phi_q, \textstyle\sum_p \frac{\partial \phi_p}{\partial x}\right)_e \hat{u}_p + [\phi_q(\tilde{u}^e - u^e)]^{x^e_u}_{x^e_l} \right] = 0.$$

This equation can be represented in an elemental matrix form as:

$$\boldsymbol{M}\frac{\partial \boldsymbol{u^e}}{\partial t} + \boldsymbol{D}\boldsymbol{u^e} + \boldsymbol{F}\boldsymbol{u^e} + \boldsymbol{G}\boldsymbol{u^{e-1}} = 0 \tag{6}$$

where $\boldsymbol{M}, \boldsymbol{D}$ have the same definition as before and applying the upwinding condition (5) $\boldsymbol{F}[q,p] = \phi_q(x_l)\phi_p(x_l)$, $\boldsymbol{G}[q,p] = -\phi_q(x_l)\phi_p(x_u)$.

3 Phase Analysis

Considering an equispaced mesh of N_{el} elements within a periodic region $[x_a, x_b]$ the element matrices for both formulations become:

$$\begin{array}{ll} \boldsymbol{M}[q,p] = \frac{h}{2}(\phi_q(\zeta), \phi_p(\zeta)), & \boldsymbol{D}[q,p] = (\phi_q(\zeta), \phi'_p(\zeta)), \\ \boldsymbol{F}[q,p] = \phi_q(-1)\phi_p(-1), & \boldsymbol{G}[q,p] = -\phi_q(-1)\phi_p(1) \end{array}$$

where $h/2$ is the Jacobian of the mapping from the region $[x^e_l, x^e_u]$ to $[-1,1]$ and $h = (x_b - x_a)/N_{el}$.

The global matrix system for the semi-discrete advection equation using the discontinuous formulations can be written as:

$$\begin{bmatrix} \boldsymbol{M} & 0 & \cdots & 0 \\ 0 & \boldsymbol{M} & \cdots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & \cdots & \boldsymbol{M} \end{bmatrix} \frac{\partial \underline{\hat{\mathbf{u}}}^e}{\partial t} + \begin{bmatrix} (\boldsymbol{D}+\boldsymbol{F}) & 0 & \cdots & \boldsymbol{G} \\ \boldsymbol{G} & (\boldsymbol{D}+\boldsymbol{F}) & \cdots & 0 \\ 0 & \ddots & \ddots & 0 \\ 0 & 0 & \boldsymbol{G} & (\boldsymbol{D}+\boldsymbol{F}) \end{bmatrix} \underline{\hat{\mathbf{u}}}^e = 0 \qquad (7)$$

In our formulation we assume an orthonormal basis of Legendre polynomials $L_p(\zeta)$ and so $\boldsymbol{M} = \boldsymbol{I}h/2$. We now seek a solution to the semi-discrete problem of the form

$$\underline{\hat{\mathbf{u}}}^e = e^{-i\omega t}[\boldsymbol{\alpha} e^{0i\theta}, \boldsymbol{\alpha} e^{i\theta}, \boldsymbol{\alpha} e^{i2\theta}, \cdots, \boldsymbol{\alpha} e^{i(N_{el}-1)\theta}]^T$$

where $i = \sqrt{-1}$ and $\boldsymbol{\alpha}$ is a vector of $P+1$ constants. For periodicity $e^{i\theta N_{el}} = 1$ which implies that $\theta_n = 2\pi n/N_{el}$ for $0 \leq n < N_{el}$. Substituting this expression into equation (7) we obtain the eigenvalue problem

$$\begin{gathered} [-(i\omega h)\boldsymbol{I} + \boldsymbol{A}]\,\boldsymbol{\alpha} = 0 \quad \text{where} \qquad (8) \\ \boldsymbol{A}[q,p] = 2\left\{\boldsymbol{D}[q,p] + \boldsymbol{F}[q,p] \,+\, \boldsymbol{G}[q,p]e^{-i\theta}\right\}, \qquad \boldsymbol{\alpha}[p] = \alpha_p \end{gathered}$$

For $P = 0$, which is the Godunov scheme, $\phi_0(\zeta) = 1/\sqrt{2}$ and so $\boldsymbol{D} = 0, \boldsymbol{F} = 1/2, \boldsymbol{G} = -1/2$ and the eigenvalue problem reduces to $[-i\omega h + 1 - e^{-i\theta}]\alpha_0 = 0$. Therefore $ih\omega = (1 - e^{-i\theta})$ which is identical to the upwinded first order finite difference scheme. When $P = 1$, $\phi_0(\zeta) = 1/\sqrt{2}, \phi_1(\zeta) = \sqrt{3/2}\zeta$ and

$$\boldsymbol{A} = \begin{bmatrix} 1 - e^{-i\theta} & \sqrt{3}(1 - e^{-i\theta}) \\ \sqrt{3}(e^{-i\theta} - 1) & 3(1 + e^{-i\theta}) \end{bmatrix}$$

which leads to the eigenvalues $ih\omega_{1,2} = 2 + e^{-i\theta} \pm \sqrt{e^{-2i\theta} + 10e^{-i\theta} - 2}$.

The analytic eigenvalues for the next two polynomial orders were obtained using Mathematica and found to be, for $P = 2$:

$$ih\omega_1 = 3 - e^{-i\theta} - \beta^{-1/3}(14 - e^{-i\theta} + 3e^{i\theta}) + \beta^{1/3}e^{-i\theta} \qquad (9)$$

$$\begin{aligned} ih\omega_{2,3} &= 3 - e^{-i\theta} + \beta^{-1/3}(14 - e^{-i\theta} + 3e^{i\theta})(1 \pm i\sqrt{3})/2 \\ &\quad -\beta^{1/3}e^{-i\theta}(1 \mp i\sqrt{3})/2 \end{aligned} \qquad (10)$$

where

$$\alpha = 3 - 166e^{i\theta} + 1872e^{2i\theta} - 18e^{3i\theta} + 9e^{4i\theta}$$

$$\beta = -1 + 21e^{i\theta} - 75e^{2i\theta} + 3e^{3i\theta} + 2\sqrt{\alpha}e^{i\theta}$$

and for $P=3$ are

$$ih\omega_{1,2} = 4 + e^{-i\theta} - \frac{\sqrt{\gamma}}{2} \pm \frac{\sqrt{\delta-}}{2}, \qquad ih\omega_{3,4} = 4 + e^{-i\theta} + \frac{\sqrt{\gamma}}{2} \pm \frac{\sqrt{\delta+}}{2} \quad (11)$$

where

$$\begin{aligned}
\alpha &= 24 + 2796e^{i\theta} + 342225e^{2i\theta} + 6271880e^{3i\theta} + 326640e^{4i\theta} + 1056e^{5i\theta} - 96e^{6i\theta} \\
\beta &= -8 + 924e^{-i\theta} + 291e^{-2i\theta} - 2e^{-3i\theta} + \sqrt{\alpha}e^{-3i\theta} \\
\gamma &= -16 + 72e^{-i\theta} + 4e^{-2\theta} + 4(50/\beta)^{1/3}(-e^{-2i\theta} - 66e^{-i\theta} + 2) + 2(20\beta)^{1/3} \\
\delta^{\pm} &= 144e^{-i\theta} - 32 + 8e^{-2i\theta} - 4(50/\beta)^{1/3}(-e^{-2i\theta} - 66e^{-i\theta} + 2) - 2(20\beta)^{1/3} \\
&\quad \pm (\gamma)^{-1/2}(16e^{-3i\theta} + 432e^{-2i\theta} + 1968e^{-i\theta} + 64).
\end{aligned}$$

In the above expressions the n^{th} root of a complex number is considered to be $z^{1/n} = |z|^{1/n}e^{\angle z/n}$.

For the continuous Galerkin method we also seek a solution of the form

$$\underline{\hat{\boldsymbol{u}}}^e = e^{-i\omega t}[\boldsymbol{\alpha}e^{0i\theta}, \boldsymbol{\alpha}e^{i\theta}, \boldsymbol{\alpha}e^{i2\theta}, \cdots, \boldsymbol{\alpha}e^{i(N_{el}-1)\theta}]^T$$

however this time $\boldsymbol{\alpha}$ is a vector of P constants due to the assembly operation involved with the continuous scheme. Assuming that the vertex degrees of freedom are defined for $p=0$ and P in the definition of $\phi_p(\zeta)$ then substituting this expression into globally assembled matrix system (3) we obtain the eigenvalue problem

$$\begin{aligned}
&[-i\omega\boldsymbol{B} + \boldsymbol{A}]\,\boldsymbol{\alpha} = 0 \qquad \text{where} \qquad (12) \\
\boldsymbol{B}[q,p] &= \boldsymbol{M}[q,p] + \boldsymbol{M}[q,P]e^{i\theta}\delta_{p0} + \boldsymbol{M}[P,p]e^{-i\theta}\delta_{q0} + \boldsymbol{M}[P,P]\delta_{p0}\delta_{q0} \\
\boldsymbol{A}[q,p] &= \boldsymbol{D}[q,p] + \boldsymbol{D}[q,P]e^{i\theta}\delta_{p0} + \boldsymbol{D}[P,p]e^{-i\theta}\delta_{q0} + \boldsymbol{D}[P,P]\delta_{p0}\delta_{q0} \\
&0 \le p,q < P
\end{aligned}$$

Using a continuous basis of the form: $\phi_0(\zeta) = \frac{1-\zeta}{2}, \phi_p(\zeta) = \frac{1-\zeta}{2}\frac{1+\zeta}{2}P_p^{1,1}(\zeta)(0 < p < P)$ and $\phi_P(\zeta) = \frac{1+\zeta}{2}$, where $P_p^{1,1}(\zeta)$ is the Jacobi polynomial, for $P=1$ we find that

$$\boldsymbol{M} = \frac{h}{2}\begin{bmatrix} 2/3, \; 1/3 \\ 1/3, \; 2/3 \end{bmatrix}, \boldsymbol{D} = \begin{bmatrix} -1/2, \; 1/2 \\ -1/2, \; 1/2 \end{bmatrix}$$

and so $\boldsymbol{A} = \frac{e^{i\theta} - e^{-i\theta}}{2} = i\sin\theta$ and $\boldsymbol{B} = \frac{h}{3}(2 + \frac{e^{i\theta}+e^{-i\theta}}{2}) = \frac{h}{3}(2+\cos\theta)$ which gives us the dispersion relations

$$i\omega\frac{h}{3}(2+\cos\theta) + i\sin\theta = 0 \qquad \Rightarrow \qquad h\omega = \frac{-3\sin(\theta)}{2+\cos\theta}.$$

which is identical to the fourth order compact finite difference scheme using a three-point stencil. For $P=2$ we obtain:

$$h\omega_{1,2} = \frac{4\sin\theta \pm 2\sqrt{40\sin^4(\theta/2) + 9\sin^2\theta}}{\cos\theta - 3}$$

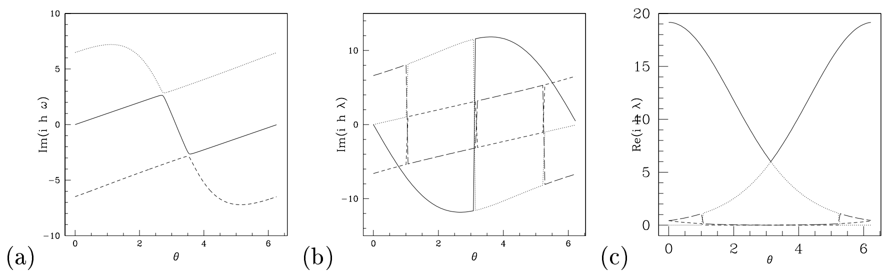

Fig. 1. Dispersion relations for (a) continuous scheme at $P = 3$ and (b,c) discontinuous scheme at $P = 3$.

and finally for $P = 3$ the eigenvalues are:

$$\begin{aligned}
\alpha &= 90\cos 2\theta + 5760\cos\theta + 6750 \\
\beta &= 540\sin 3\theta + 153900\sin\theta - 77760\sin 2\theta \\
\gamma &= \sqrt{4\alpha^3 + \beta^2} - i\beta \\
ih\omega_1 &= \frac{15e^{2\theta} - 15 - \alpha(2/\gamma)^{1/3}e^{i\theta} + e^{i\theta}(\gamma/2)^{1/3}}{3(1 + 8e^{i\theta} + e^{2i\theta})} \qquad (13) \\
ih\omega_{2,3} &= \frac{15e^{2\theta} - 15 + \alpha/(4\gamma)^{1/3}(1 \pm \sqrt{3}i)e^{i\theta} - (\gamma/16)^{1/3}(1 \mp \sqrt{3}i)e^{i\theta}}{3(1 + 8e^{i\theta} + e^{2i\theta})} \qquad (14)
\end{aligned}$$

4 Discussion

Figure 1 illustrates the dispersion relation for the continuous and discontinuous formulations at a polynomial order of $P = 3$. The continuous formulation provides a purely imaginary phase solution (see eqn.(13)). Figures 1(b) and (c) show the imaginary and real components of the analytic phase relation for the discontinuous scheme given by equation (11). These plots completely define the phase relationship for any number of elements. In both imaginary component of the solution we see a linear growth with θ which represents the analytic dispersion relation for the linear advection equation. At very high frequencies the curves decay back to zero. The very dispersive modes of the discontinuous scheme are associated with a very fast damping as indicated by figure 1(c). Two notable features of the imaginary components shown in figure 1 are the branch jumping of the solutions and the multiple roots for each value of θ. The cause of the branch jumping is due to the restriction in argument of the trigonometric functions in the computer program between $-\pi < \theta < \pi$. When we evaluate the square root of a complex number of argument π a small perturbation, ϵ to the argument causes a jump from $\pi - \epsilon$ to $-\pi + \epsilon$ which leads to a change in sign of the complex component of the

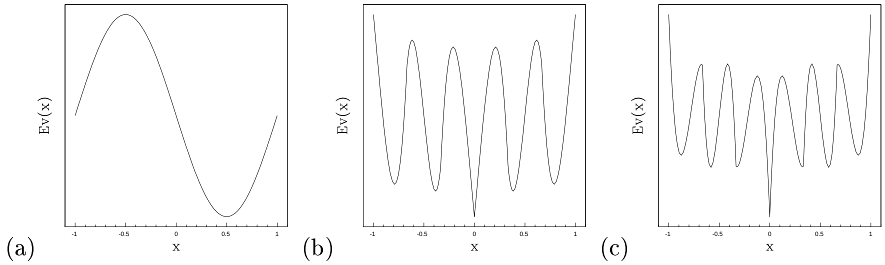

Fig. 2. Eigenvectors of discrete continuous problem $N_{el} = 6$, $P = 3$. (a)$\omega = 3.142 \approx \pi$ (b) $\omega = -16.08 \approx -5\pi$ and (c) $\omega = 21.60 \approx 7\pi$.

root. The multiple roots for a given θ are simply due to the ability of a high order expansion to resolve more than one eigenfunction within an element. This point is illustrated in figure 2 where we consider the eigenfunctions of the discrete problem for the continuous formulation when $N_{el} = 6, P = 3$. In this problem the first non-zero value of θ_n is $2\pi/6 \approx 1$ which produces three discrete values of $i\omega h$ (see figure 1(a)). For the analytic solution the three eigenvalues corresponding to the wavenumbers $\omega = \pi, -5\pi, 7\pi$ and should have eigenvalues of the form $e^{\omega x}$. As shown in figure 2 all branches produce a discernible approximation to these eigenfunctions.

In figure 3 we show a similar series of plot as shown in figure 1 for a polynomial order of $P = 10$. These plots were numerically evaluating from the systems given by equation (8) and (12). All results have been validated against a numerical eigenvalue evaluation of the complete semi-discrete system similar to equation (7) using LAPACK.

The ultimate aim of this investigation is to compare the relative advantages and disadvantages of the continuous and discontinuous formulations. Certainly from the implementation point of view the local elemental characteristic of the discontinuous scheme is very efficient. Both formulations also have comparable phase properties. However in the discontinuous formula-

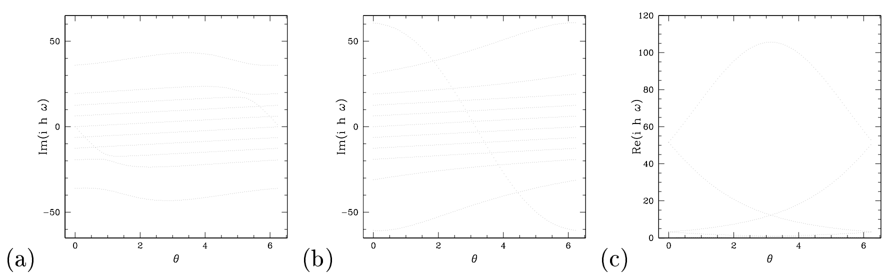

Fig. 3. Dispersion relations for (a) continuous scheme at $P = 10$ and (b,c) discontinuous scheme at $P = 10$.

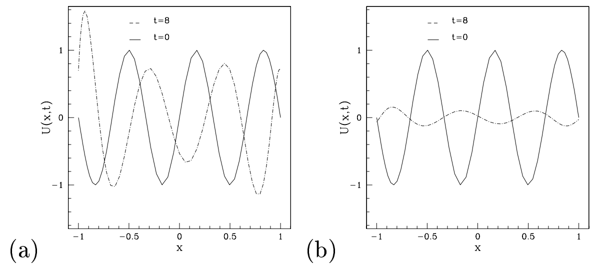

Fig. 4. Solution of the linear advection equation at $t = 0, 4$ for initial conditions of $\sin(3\pi x)$ with $N_{el} = 1, P = 10$. (a) continuous and (b) discontinuous.

tions the range of the dispersion relation is larger due to the greater number of degrees of freedom for a given polynomial order. Nevertheless at higher frequencies there is a significant damping which could be eliminated using a centred flux as investigated in [4]. Although the non-diffusive nature of the continuous scheme appears mathematically attractive it can lead to equally erroneous solution due to the poor phase properties of the higher frequencies. To illustrate this point in figure 4 we compare the two methods for $u(x, 0) = \sin(3\pi x)$ in $-1 < x < 1$ with $N_{el} = 1, P = 10$. The solid line shows the initial condition and the dotted line gives the solution at $t = 8$. From these figures we see the diffusive nature of the discontinuous scheme however we also note the the continuous scheme gives rise to a solution of magnitude greater than 1. Figure 2 demonstrates that the discrete eigenfunctions are not pure sinusoidal waves and so we would expect the projection of the initial conditions to involve more than one discrete eigenfunction. Although all of these frequencies are non-diffusive the poor phase property of the high frequency components can force the solution to become erroneous leading to the observed peak as the discrete eigenfunctions move in and out of phase.

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References

B. Cockburn and C.W. Shu: TVB Runge-Kutta projection discontinuous Galerkin finite element methods for conservation laws II, General framework Math. Comp. **52** (1989) 411-435

I. Lomtev, C.W. Quillen and G. Karniadakis: Spectral/hp Methods for Viscous Compressible Flows on Unstructured 2D Meshes. J. Comp. Phys. Submitted (1997)

G.Em. Karniadakis and S.J. Sherwin: Spectral/*hp* Element Methods for CFD. Oxford University Press (1999)

F.Q.Hu, M.Y.Hussaini and P. Rasetarinera: An analysis of the discontinuous Galerkin method for wave propagation problems. Submitted to JCP (1999)