TP : Saving Private Ryan

SAISON Lou, SEUMEN Janice, DUMONT Quentin,
HOSSEINKHAN Rémy

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Introduction

Context

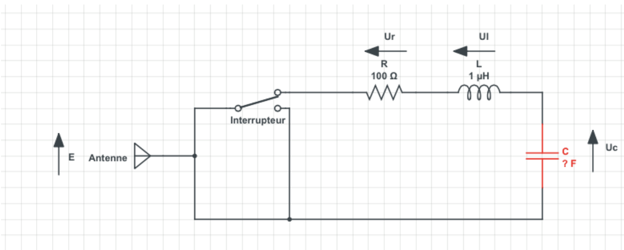
December 1942, France: Dropped with only a few equipment, a soldier named Ryan was charged with a mission of the highest importance, intercept a secret message in order to dismantle a Nazi camp.
Alone, behind enemy lines, Ryan had to use his entire mathematic skills and his precious knowledge on waves and radio devices to achieve his mission.
Most of electronic devices are composed of RLC circuit whose comportment is ruled by a particular differential equation.
Unfortunately, Ryan had a capacity meter and a radio in his bag, but the latest is broken.. He can't read which frequency he is listening to.
Classic military radio RLC circuits are composed of a resistor $R = 100\Omega$*, an inductor* $L = 1\mu H$ *and a capacitor with variable capacity. Most of military secret messages are transmitted at high frequencies (from* $3MHz$ *to* $30MHz$*). Here, Ryan knows the energy offset parameter* E *is equal to* $E = 5\ V$ *and that he has to listen at a frequency of* $29.3MHz$*.*

Problem

Studying the differential equation that caracterized a non-linear RLC circuit help Ryan to carry out his mission.

Step 1: Thanks to CircuitLab construt the RLC circuit.

This is the electric diagram Ryan will use to solve the problem:



RLC Diagram

U_R : voltage across the resistor
U_I : voltage across the inductor
U_C : voltage across the capacitor
E : voltage injected through the antenna in the circuit to offset the energy lost in the resistor (Joule effect)
L : inductor
K : resistor

Step 2: Thanks to physical laws, determine the differential equation that caracterize a RLC circuit.

At $t = 0$ we switch the switchboard on of the RLC circuit represented above:
To study the evolution through time (for $t \geqslant 0$) of the voltage $u = U_C$ across the capacitor, we have to establish a differential equation on u and also determine the initial conditions.
For this purpose, we consider equations which come from physical laws:

Kirchhoff's mesh rule: it shows that the directed sum of the voltage in a mesh is zero, thus:

$$u + U_R + U_L - E = 0 \quad \Leftrightarrow \quad u = E - U_R - U_L \qquad (1)$$

Ohm's Law: $U_R = Ri$ with i be the amperage in the circuit. (2)

Faraday's Law: $U_L = L\frac{di}{dt}$ (3)

Coulomb's Law: $i = \frac{dq}{dt}$ with $q = Cu$ by definition of the electric charge of the capacitor.

Then, $i = \frac{Cdu}{dt}$ (4) and by (4) and (2) we get:

$$U_R = RC\frac{du}{dt} \qquad (5)$$

by (3) and (4) we get:

$$U_L = LC\frac{d^2u}{dt^2} \qquad (6)$$

by (1), (5) and (6) we get:

$$u + RC\frac{du}{dt} + LC\frac{d^2u}{dt^2} = E \qquad (7)$$

Then, we obtain the second order linear, non homogeneous and autonomous equation witch constant coefficients:

$$\frac{d^2u}{dt^2} + \frac{\omega_0}{Q}\frac{du}{dt} + \omega_0^2 u = \frac{E}{LC}$$

where $\omega_0 = \frac{1}{\sqrt{LC}}$ is the angulary frequency and $Q = \frac{1}{R}\sqrt{\frac{L}{C}}$ is the quality factor used to caracterize resonators.
Due to the second derivative, we must determine two constants and therefore two initial conditions.

At $t = 0:\ i(t = 0) \quad \Rightarrow \quad u(0) = R \times 0 = 0$ by (2) and we have $i(t = 0) = C\frac{du}{dt} = 0 \quad \Rightarrow \quad \frac{du}{dt}(0) = 0$ by (5).

Finally, we obtain the differential system :

$$\begin{cases} u''(t) + \frac{\omega_0}{Q}u'(t) + \omega_0^2 u(t) = \frac{E}{LC} & \qquad (NH1) \\ u(0) = 0 \\ u'(0) = 0 \end{cases}$$

where u' and u'' represent respectively the first and second order derivative of the function u.

Step 3: Solve the differential equation giving the time interval of definition of a maximal solution, finding equilibria and stationary solution if they exist. Discuss the existence and uniqueness of the solution.

Let's consider the second order linear, non homogeneous and autonoumous equation with constant coefficients (NH1).
To apply Picard-Lindelöf theorem, we have to transform the equation into a first-order one. (NH1) is equivalent to the linear system of equations:

$$\begin{cases} U'(t) = AU(t) + B & \qquad (NH2) \\ (U_0(0), U_1(0)) = (0,0) \end{cases}$$

with $A = \begin{pmatrix} 0 & 1 \\ -\omega_0^2 & -\frac{\omega_0}{Q} \end{pmatrix}$, $B = \begin{pmatrix} 0 \\ \frac{E}{LC} \end{pmatrix}$, $U(t) = \begin{pmatrix} u_0(t) \\ u_1(t) \end{pmatrix} = \begin{pmatrix} u(t) \\ u'(t) \end{pmatrix}$.
Applying Picard-Lindelöf theorem, there is a unique maximal solution (J, U) of (NH1) such that $U(0) = \begin{pmatrix} u_0(0) \\ u_1(0) \end{pmatrix} = \begin{pmatrix} u(0) \\ u'(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and it is global because solutions of (NH2) are global: $J = \mathbb{R}_+$ (we study the voltage evolution accross the capacitor C reached for $t \geqslant 0$).

Let S_{NH} and S_H denote respectively the sets of solutions of (NH2) and of the associated homogeneous equation:

$$\begin{cases} u''(t) + \frac{\omega_0}{Q}u'(t) + \omega_0^2 u(t) = 0 & \qquad (H1) \\ u(0) = 0 \\ u'(0) = 0 \end{cases}$$

To solve (NH2) we need to find a particular solution U_p .
Since A is an invertible matrix ($det(A) = \frac{\omega_0^3}{Q} > 0$), the autonomous equation (NH2) has a unique equilibrium:
$X^* = -A^{-1}B = \begin{pmatrix} \frac{1}{\omega_0 Q} & \frac{1}{\omega_0^2} \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ \frac{E}{LC} \end{pmatrix} = \begin{pmatrix} E \\ 0 \end{pmatrix}$
Thus a particular solution of (NH2) is the stationary solution $U_p(t) = X^*$, $\forall t \in J$.

Solution of the associated homogeneous equation (H1)
The characteristic polynomial of this equation is $P(U) = U^2 + \frac{\omega_0}{Q}U + \omega_0^2$.
Let Δ denote its discriminant: $\Delta = \frac{\omega_0^2}{Q^2}(1 - 4Q^2)$.
According to the sign of Δ, three case are possible which corresponds to three differents regimes.

1. The pseudo-periodical regime (underdamped response)

It corresponds to the case $\Delta < 0 \Leftrightarrow Q > \frac{1}{2}$.
The polynomial P has two conjugate complex roots of multiplicity one: $u_1 = -\frac{\omega_0}{2Q} + i\frac{\omega_0}{2Q}\sqrt{4Q^2 - 1}$ and $u_2 = -\frac{\omega_0}{2Q} - i\frac{\omega_0}{2Q}\sqrt{4Q^2 - 1}$.
The solutions of (H1) are of the form:

$$u_H(t) = e^{-\frac{\omega_0}{2Q}t}(\lambda\cos(\omega t) + \mu\sin(\omega t)) \quad with \quad (\lambda, \mu) \in \mathbb{R}^2 \quad and \quad \omega = \frac{\omega_0}{2Q}\sqrt{4Q^2 - 1}$$

Then the general solution is:

$$u(t) = E + e^{-\frac{\omega_0}{2Q}t}(\lambda\cos(\omega t) + \mu\sin(\omega t)), \quad \forall t \in J$$

Now we use the two initial conditions to determinate λ and μ :

$$\begin{cases} \text{u(0)} = 0 \\ \text{u'(0)} = 0 \end{cases} \quad \Rightarrow \quad \begin{cases} \lambda = \text{-E} \\ \mu = \text{-}\frac{\omega_0 E}{2Q\omega} = -E\sqrt{4Q^2 - 1} \end{cases}$$

2. Aperiodical regime (overdamped response)

It corresponds to the case $\Delta > 0 \Leftrightarrow Q < \frac{1}{2}$.
P has two reals negatives roots of multiplicity one: $u_1 = -\frac{\omega_0}{2Q}(1 - \sqrt{1-4Q^2})$ and $u_2 = -\frac{\omega_0}{2Q}(1 + \sqrt{1-4Q^2})$.

The solutions of (H1) are of the form:

$$u_H(t) = \lambda e^{-\frac{\omega_0}{2Q}(1-\sqrt{1-4Q^2})t} + \mu e^{-\frac{\omega_0}{2Q}(1+\sqrt{1-4Q^2})t} \quad with \quad (\mu, \lambda) \in \mathbb{R}^2$$

Consequently, the general solution is:

$$u(t) = E + \lambda e^{-\frac{\omega_0}{2Q}(1-\sqrt{1-4Q^2})t} + \mu e^{-\frac{\omega_0}{2Q}(1+\sqrt{1-4Q^2})t}, \quad \forall t \in J \quad with \quad (\mu, \lambda) \in \mathbb{R}^2$$

As usual, we use the initial conditions to find μ and λ :

$$\begin{cases} \text{u(0)} = 0 \\ \text{u'(0)} = 0 \end{cases} \Rightarrow \begin{cases} \lambda = \frac{Eu_2}{u_1 - u_2} \\ \mu = -\frac{Eu_2}{u_1 - u_2} \end{cases}$$

3. Critically damped response

The last case is when $\Delta = 0 \Leftrightarrow Q = \frac{1}{2}$.
P has a unique root of multiplicity two: $u_1 = -\frac{\omega_0}{2Q}$.
The solutions of (H1) are of the form:

$$u_H(t) = \lambda e^{-\frac{\omega_0}{2Q}t} + \mu t e^{-\frac{\omega_0}{2Q}t} \qquad with \quad (\mu, \lambda) \in \mathbb{R}^2$$

Thus, the general solution is:

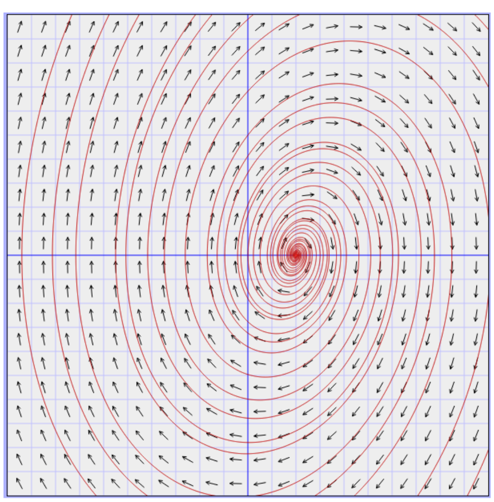
$$u(t) = E + e^{-\frac{\omega_0}{2Q}t}(\mu t + \lambda), \quad \forall t \in J, \qquad with \quad (\mu, \lambda) \in \mathbb{R}^2$$

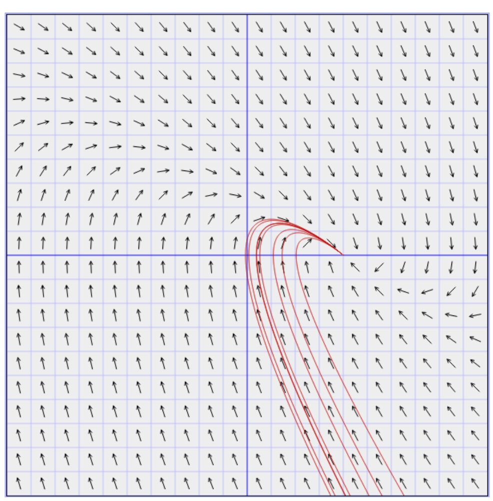
Again, we use the initial conditions to find μ and λ :

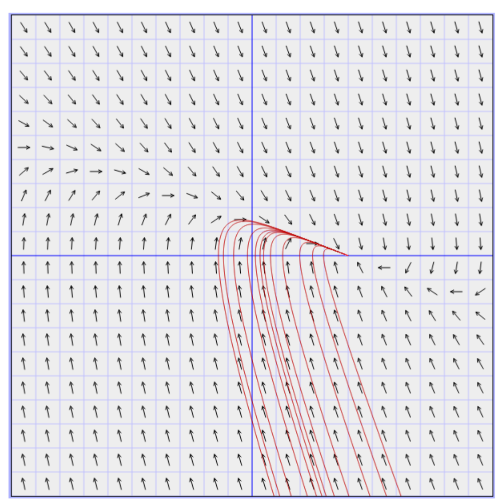
$$\begin{cases} \text{u(0)} = 0 \\ \text{u'(0)} = 0 \end{cases} \Rightarrow \begin{cases} \lambda = -E \\ \mu = -\frac{\omega_0}{2Q}E \end{cases}$$

Step 4: Study the stability of equilibrium, draw the phase portrait and give interpretation.

Let introduce the phase portraits which correspond, from left to right, to the pseudo periodical, the aperiodical and the critical regimes:







We can see on these graphs that in the three cases, the equilibirum X^* is asymptotically stable (attractive and stable).
Let's now prove it formally:

• **Pseudo periodical regime:** We have already found $Re(u_1) = Re(u_2) = -\frac{\omega_0}{2Q} < 0$, then X^* is a hyperbolic equilibrium and since $\max_{\lambda \in SP(A)} Re(\lambda) = -\frac{\omega_0}{2Q} < 0$, X^* is asymptotically stable (stable and attractive equilibrium).

• **Aperiodical regime:** Both u_1 and u_2 are negatives then X^* is hyperbolic and $\max_{\lambda \in SP(A)} Re(\lambda) < 0$ thus X^* is asymptotically stable.

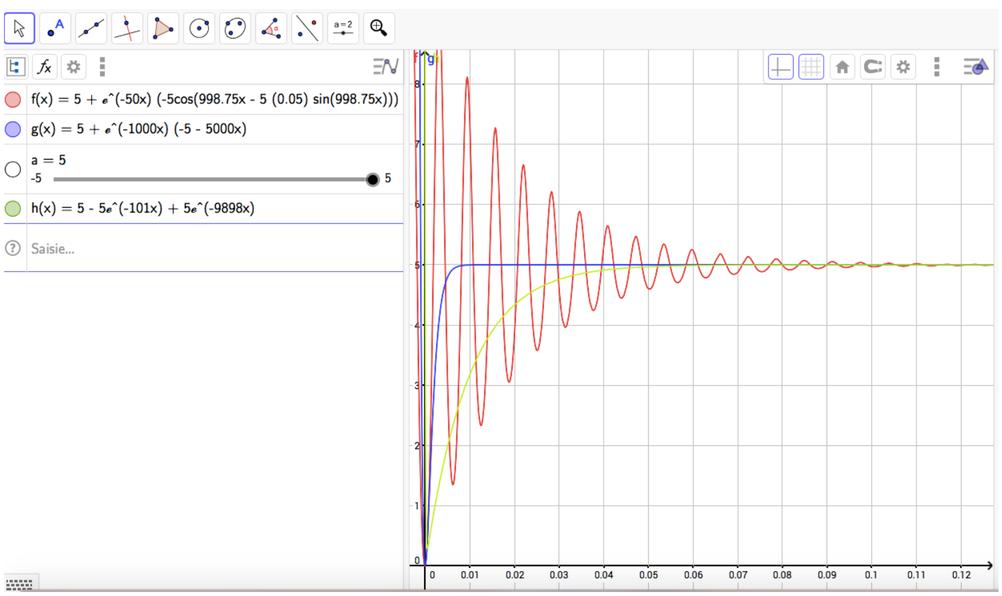
• **Critical regime:** We found a unique root of multiplicity two: $u_1 = \frac{-\omega_0}{2Q} < 0$ then X^* is hyperbolic and $\max_{\lambda \in SP(A)} Re(\lambda) = u_1 < 0$ thus X^* is asymptotically stable.

Interepretation 1 *About case 1, the spiral form of the phase portrait in the case $\Delta < 0 \Leftrightarrow Q > \frac{1}{2}$ shows that the system converges to the equilibrium, that corresponds to the "permanent regime", by making oscillations that match to transitional regime. We deduce that the amplitude of those oscillations are decreasing since the spiral is increasingly "concentric". These damped oscillations caracterize loss of energy dissipated in the resistor: the spiral direction verifies the asymptotical stability of the equilibrium found previously.*

Interepretation 2 *Concerning the critical and the aperiodic regime are different. We don't observe oscillations but an amortization to the stable and attractive equilibrium.*

In all cases, the solution u_H of (H1) corresponds to the "transitional regime" and the particular solution $U_p(t) = E$ to the "permanent regime".

Step 5: Draw the graphical representation of solution, conclude and save Ryan.



The red curve corresponds to the pseudo-periodical regime where $R = 100\ \Omega, L = 1\ Hz, C = 10^{-6}\ F$. The blue curve is the critical regime with $Q = 0.5$ and the green curve represents the aperiodical case with $Q = 0.1$.

To help Ryan, we focus on the pseudo-periodic regime because the others regime are not interesting to detect a particular frequence and create a "tune circuit".
In a pseudo-periodic regime, the solution $u(t)$ has a damped sinusoidal form: its amplitude decrease exponentially ($e^{-\frac{\omega_0}{2Q}t}$).
The graphic (in red for the pseudo-periodical case) gives us the pseudo period $\tau = \frac{2\pi}{\omega}$, ($w$ describe a pseudo-pulsation).

So Ryan deduces from the graph that the period of the signal is $T = \frac{2\pi}{\omega}$. Then, he remembered what Mr Burns, his teacher at military school used to say: *listening a station is to find the capacity of the RLC circuit such that it is going into resonance for the frequency of the station. The RLC circuit is said to be resonating when the frequency of the wave detected is close to the one of the RLC circuit.*
It means that he needs $\omega = \omega_0$ i.e $freq = freq_0 = \frac{\omega_0}{2\pi} = \frac{1}{2\pi\sqrt{LC}}$.
So Ryan has to find C such that $freq = \frac{1}{2\pi\sqrt{LC}}$ which corresponds to $C = (\frac{1}{2\pi freq\sqrt{L}})^2$.

In Ryan's case, the frequence he wants to reach is caracterized by $freq = 29.3\ MHz$ and $L = 1\ \mu H$.
Hence, he deduces $C = 2.951 \times 10^{-11}\ F$ (Faraday). By using the capacity meter, Ryan choose this value of C and is now able to listen to the ennemy station and he gets :

"Allgemein Himmler, der Code Zugriff auf Raketen zu geben ist KP3GT82V.
Wir werden in Kürze treffen Frankreich."