A novel solution to the Klein–Gordon equation in the presence of a strong rotating electric field

E. Raicher, S. Eliezer, A. Zigler, Physics Letters B 750 (2015) 76–81

Notebook: Óscar Amaro, November 2022 @ GoLP-EPP
Contact: oscar.amaro@tecnico.ulisboa.pt

Introduction

In this notebook we reproduce some results from the paper.

Equation 9: expansion as Fourier series

In[•]:=
```
Clear[y, z, n, c2n, μ, λ, q, eq6]
y[z] = Exp[I μ z] Sum[c2n[n] Exp[2 I z n], n]
eq6 = D[y[z], {z, 2}] + (λ - 2 q (Exp[I 2 z] + Exp[-I 2 z])/2) y[z] // FullSimplify
```

Out[•]= $e^{i z \mu} \sum_n e^{2 i n z} \mathrm{c2n}[n]$

Out[•]= $e^{i z \mu} \left(\left(\lambda - \mu^2 - 2 q \mathrm{Cos}[2 z]\right) \sum_n e^{2 i n z} \mathrm{c2n}[n] + 2 i \mu \sum_n 2 i e^{2 i n z} n\, \mathrm{c2n}[n] + \sum_n -4 e^{2 i n z} n^2 \mathrm{c2n}[n] \right)$

```
Solve[eq6]
```

Equation 19: solving analytically 1st order equation

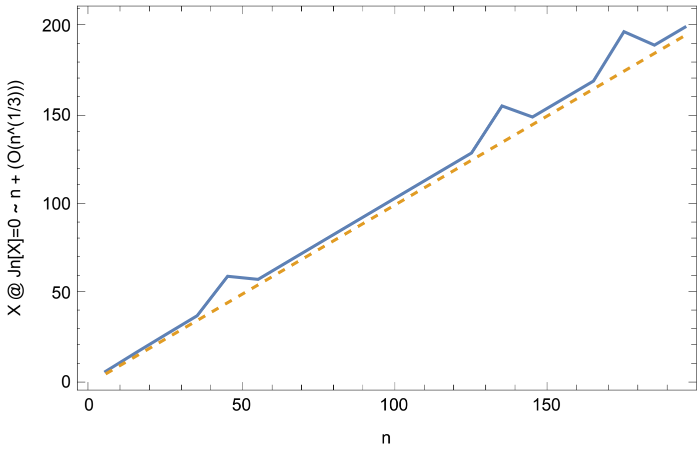
In[•]:=
```
Clear[λ, y, z, q]
DSolve[{y'[z] + I (Sqrt[λ] - q/Sqrt[λ] Cos[2 z]) y[z] == 0}, y[z], z] // Simplify
```

Out[•]= $\left\{\left\{y[z] \to e^{-\frac{i (z \lambda - q \mathrm{Cos}[z] \mathrm{Sin}[z])}{\sqrt{\lambda}}} c_1\right\}\right\}$

Equation 22: Bessel function expansion

In[∘]:=
```
(* first maximum @ X~n *)
Clear[n, X, root]
root[n_] := (FindMaximum[BesselJ[n, X], {X, 0.9 n}] // Quiet)〚2, 1, 2〛
ListPlot[{ParallelTable[{n, root[n]}, {n, 5, 200, 10}],
   ParallelTable[{n, n}, {n, 5, 200, 10}]}, Joined → True,
  Frame → True, FrameLabel → {"n", "X @ Jn[X]=0 ~ n + (O(n^(1/3)))"},
  PlotStyle → {Default, Dashed}]
```

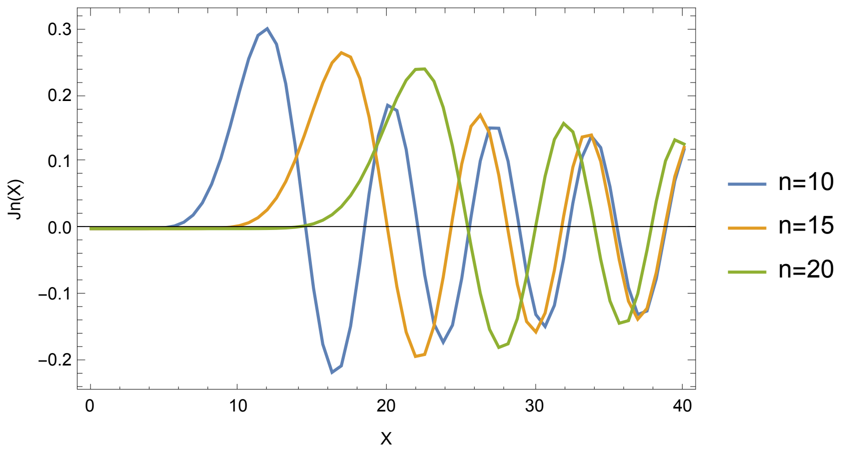
Out[∘]=



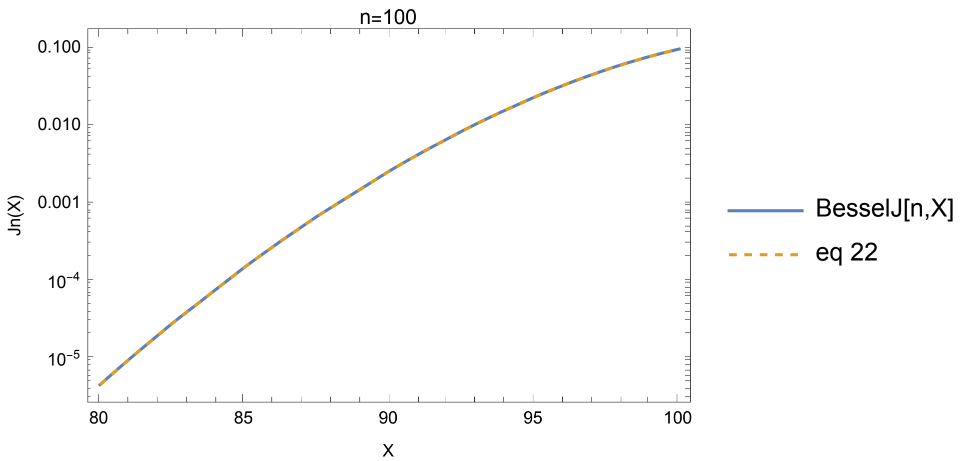
In[31]:=
```
Clear[n, X, tanhα]
Plot[{BesselJ[10, X], BesselJ[15, X], BesselJ[20, X]}, {X, 0, 40}, PlotPoints → 2,
 Frame → True, FrameLabel → {"X", "Jn(X)"}, PlotLegends → {"n=10", "n=15", "n=20"}]

n = 100;
tanhα = Sqrt[1 - (X / n) ^2];
LogPlot[{BesselJ[n, X],
  tanhα/(π √3) BesselK[1 / 3, 1/3 n tanhα^3] Exp[n (tanhα + 1/3 tanhα^3 - ArcTanh[tanhα])]},
 {X, 0.8 n, n}, PlotStyle → {Default, Dashed}, PlotPoints → 2,
 Frame → True, FrameLabel → {"X", "Jn(X)"},
 PlotLegends → {"BesselJ[n,X]", "eq 22"}, PlotLabel → "n=100"]
```

Out[32]=



Out[35]=



Equation 24: simple Taylor series

In[∘]:=
```
Clear[α]
Series[Tanh[α] + 1/3 Tanh[α]^3 - α, {α, 0, 6}]
```

Out[∘]= $-\frac{\alpha^5}{5} + O[\alpha]^7$

Equation 25: simple K1/3 expansion

In[•]:=
```
Clear[u]
Series[BesselK[1 / 3, u], {u, 0, 0}]
```

Out[•]=
$$\frac{\text{Gamma}\left[\frac{1}{3}\right]}{2^{2/3}\,u^{1/3}} + O[u]^{1/3}$$

Figure 1

In[1596]:=
```
(* figure 1: *)
Clear[ea, m, p0, pprp, p, ξ, mph, ν, nstar, pdotϵ, kdotp, nstar, nstarV, ν, νV]
(* "definitions" equation 52 *)
δ = (√2 ea pprp)/(p0^2 + ea^2);
Ω = Sqrt[1 + (ea/p0)^2];
(* in these units *)
m = 1;
ea = 20 m;
(* 3-vector momentum *)
p = {m / 5, m / 5, 0};
(* 0 component in 4-momentum *)
p0 = Refine[Sqrt[1 + Norm[p]^2], {m > 0}];
(* as in main text *)
δ = δ /. {pprp → Norm[p]} // N;
Ω = Ω // N;

ξ = 20;
mph = m / 10;
kdotp = p0 mph;
(* equation 51 *)
pdotϵ = pprp/√2 /. {pprp → Norm[p]} // N;
(* eq 34: new analytical solution *)
nstar = (ea pdotϵ)/(Ω kdotp) // N;
ν = kdotp/mph^2 (Ω - 1) // N;
nstarV = (ea pdotϵ)/kdotp // N;
νV = ea^2/(2 kdotp) // N;

Print["parameters for figure 1:"]
Print["δ=", δ]
Print["Ω=", Ω]
Print["ν=", ν]
Print["νV=", νV]
Plot[{9.5 × 10^-4 × 10^5 Abs[BesselJ[n // Abs, 2 nstarV]],
   2.25 × 10^-4 × 10^5 Abs[BesselJ[n // Abs, 2 nstar]]}, {n, -100, 100},
  PlotRange → {0, 16}, Frame → True, FrameLabel → {"n", "c2n"},
  PlotStyle → {Blue, Red}, PlotLegends → {"Volkov", "Analytical"},
  PlotLabel → "Fig.1 p=(m/5,m/5,0)"]
```

```
parameters for figure 1:
δ=0.0199461
Ω=19.271
ν=189.878
νV=1924.5
```

Out[1618]=

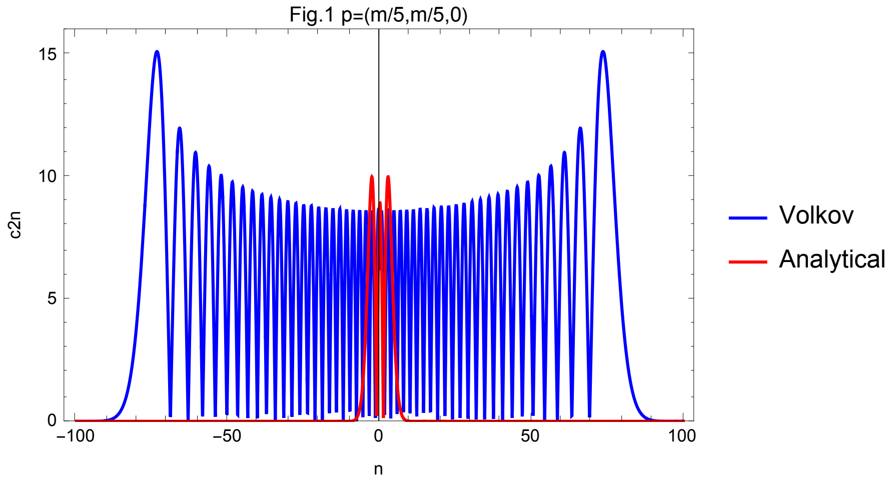


Figure 2

In[1550]:=

```
(* figure 2: *)
Clear[ea, m, p0, pprp, p, ξ, mph, ν,
 nstar, pdotε, kdotp, nstar, nstarV, ν, νV, δ, Ω]
(* "definitions" equation 52 *)
δ = (√2 ea pprp)/(p0^2 + ea^2);
Ω = Sqrt[1 + (ea/p0)^2];
(* in these units *)
m = 1;
ea = 20 m;

p = {5 m, 5 m, 0};
(* 0 component in 4-momentum *)
p0 = Refine[Sqrt[1 + Norm[p]^2], {m > 0}];
(* as in main text *)
δ = δ /. {pprp → Norm[p]} // N;
Ω = Ω // N;

ξ = 20;
mph = m / 10;
kdotp = p0 mph;
(* equation 51 *)
```

```
pdotε = pprp/√2 /. {pprp → Norm[p]} // N;
(* eq 34: new analytical solution *)
nstar = (ea pdotε)/(Ω kdotp) // N;
ν = kdotp/mph^2 (Ω - 1) // N;
nstarV = (ea pdotε)/kdotp // N;
νV = ea^2/(2 kdotp) // N;

Print["parameters for figure 2:"]
Print["δ=", δ]
Print["Ω=", Ω]
Print["ν=", ν]
Print["νV=", νV]
Plot[{3.7 × 10^-4 × 10^5 Abs[BesselJ[n // Abs, 2 nstarV]],
   2.1 × 10^-4 × 10^5 Abs[BesselJ[n // Abs, 2 nstar]]}, {n, -300, 300},
  PlotRange → {0, 4}, Frame → True, FrameLabel → {"n", "c2n"},
  PlotStyle → {Blue, Red}, PlotLegends → {"Volkov", "Analytical"},
  PlotLabel → "Fig.2 p=(5m,5m,0)"]
```

parameters for figure 2:

δ=0.443459

Ω=2.97374

ν=140.953

νV=280.056

Out[1572]=