Analytical results for a Fokker–Planck equation in the small noise limit

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Asymptotics

In[ ]:=
```
Asymptotic[LambertW[v], v → 0]
Asymptotic[LambertW[v], v → ∞]
```

Out[ ]=

v

Out[ ]=

$\text{Log}[v]$

K

In[ ]:=
```
Clear[K, v]
K = -v / (1 + v^2)
D[K, v] // Simplify
```

Out[ ]=

$$-\frac{v}{1+v^2}$$

Out[ ]=

$$\frac{-1+v^2}{\left(1+v^2\right)^2}$$

Figure 1

In[∘]:=
```
Clear[t, C]
C = 3;
Plot[{Sqrt[LambertW[Exp[-2 t + C]]], Exp[-t + C / 2]},
 {t, 0, 5}, PlotRange → {0, 5}, PlotStyle → {Default, Dashed},
 AxesLabel → {"Time t", "Deterministic trajectory <x(t)>"}]
```

Out[∘]=

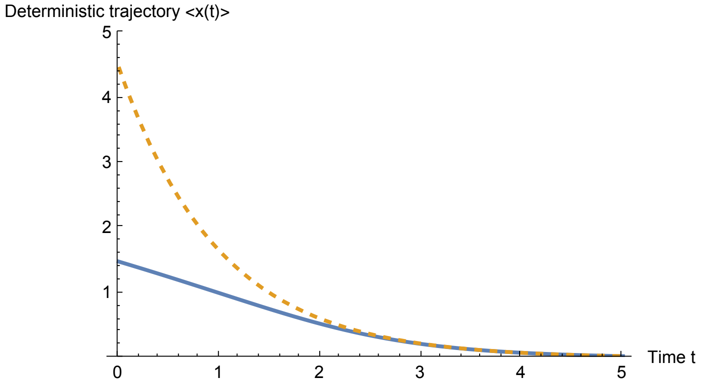

Figure 2 average contribution of noise

In[∘]:=
```
Clear[z0, C, v, t, v0, eq17, K]
K[v_] := -v / (1 + v^2);
v[t_, C_] := Sqrt[LambertW[Exp[-2 t + C]]];
eq17[t_, C_] := K[v[t, C]] / K[Sqrt[Exp[LambertW[C]]]];
eq17[0, 0.8]
Plot[{eq17[t, 8], eq17[t, 0.8], Exp[-t]}, {t, 0, 10},
 PlotRange → {0, 1.5}, PlotStyle → {Default, Dashed, Dotted}]
```

Out[∘]=

```
1.02882
```

Out[∘]=

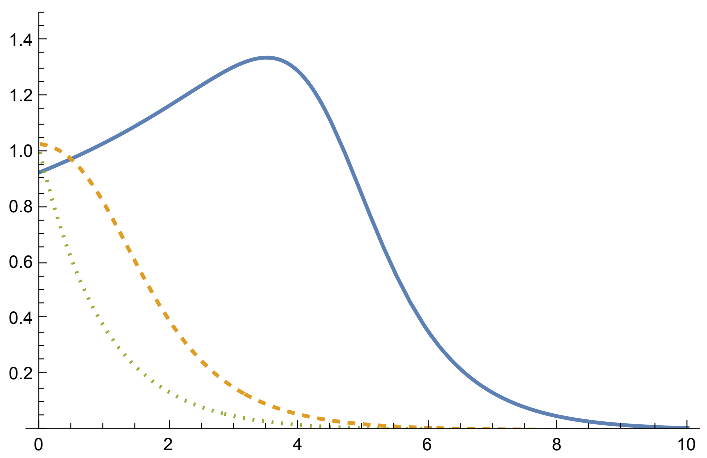

Figure 3 variance of noise contribution

In[•]:=
```
Clear[z0, C, v, t, v0, eq18]
C = 3;
z0 = Exp[LambertW[C]];
v0 = Sqrt[z0];
K = -v / (1 + v^2);
v = Sqrt[LambertW[Exp[-2 t + C]]];
eq18 = (2 v^(-2) + (v0^4 + 12 t - 2 v0^(-2)) - v^4)/2 K^2;
Asymptotic[eq18, t → 0] // N
Plot[{eq18, 1 - Exp[-2 t]}, {t, 0, 10}, PlotRange → {0, 3.2}]
```

Out[•]=

0.374992

Out[•]=

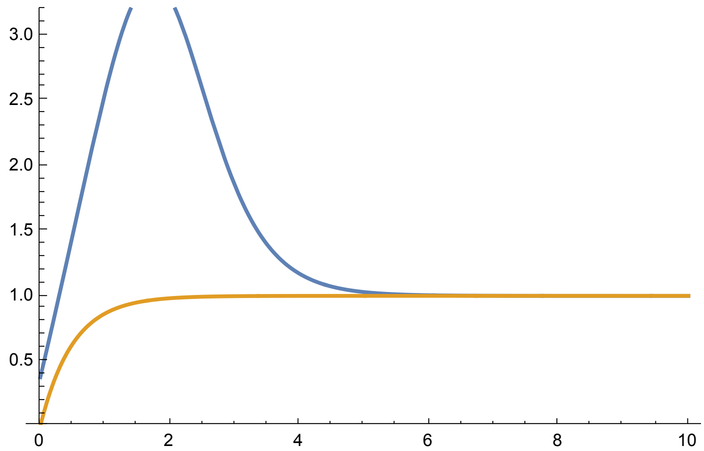

Figure 4 autocorrelation function

In[224]:=

```
Clear[eq24dash, eq24]
Clear[z0, C, v, t, v0, eq18, Kt, Ks, τ, eq18, t, s, v, C]
C = 8;
Kt = -v / (1 + v^2) /. {v → Sqrt[LambertW[Exp[-2 t + C]]]};
Ks = -v / (1 + v^2) /. {v → Sqrt[LambertW[Exp[-2 s + C]]]};
z0 = Exp[LambertW[C]];
v0 = Sqrt[z0];
K = -v / (1 + v^2);
v = Sqrt[LambertW[Exp[-2 t + C]]];
eq18 = (2 v^(-2) + (v0^4 + 12 t - 2 v0^(-2)) - v^4)/2 K^2;
eq24 = eq18 Ks / Kt //. {s → τ + t, t → 1} // N;

Clear[z0, C, v, t, v0, eq18, Kt, Ks, τ, eq18, t, s, v, C]
C = 0.8;
Kt = -v / (1 + v^2) /. {v → Sqrt[LambertW[Exp[-2 t + C]]]};
Ks = -v / (1 + v^2) /. {v → Sqrt[LambertW[Exp[-2 s + C]]]};
z0 = Exp[LambertW[C]];
v0 = Sqrt[z0];
K = -v / (1 + v^2);
v = Sqrt[LambertW[Exp[-2 t + C]]];
eq18 = (2 v^(-2) + (v0^4 + 12 t - 2 v0^(-2)) - v^4)/2 K^2;
eq24dash = eq18 Ks / Kt //. {s → τ + t, t → 1} // N;

Plot[{eq24, eq24dash}, {τ, 0, 8},
 AxesLabel → {"Time difference τ", "Covariance g(t,t+τ)"}]
```

Out[245]=