Exact Solutions of a Fokker-Planck Equation

MJ Englefield, J Stat Phys, Vol. 52 (1988)
Notebook: Óscar Amaro, January 2023 @ GoLP-EPP

Introduction

In this notebook we reproduce some results from the paper.

5. DISCUSSION AND CONCLUSION:
“This paper has given exact, explicit solutions involving only elementary functions for a Fokker-Planck equation with a potential of the form u = 2 h^2 log(ϕ(x))... In the potential there are four parameters that can be chosen to approximate any other required form, while the solutions contain parameters that can be chosen to simulate a required initial configuration”

If ψ solves eq 2.4 then W solves eq 2.1

In[∘]:=
```
Clear[V, W, ψ, u, x, t, h, eq21]
W = ψ[x, t] Exp[-u[x]/(2 h^2)];
(* eq 2.5 *)
V = 1/(4 h^2) D[u[x], x]^2 - 0.5 D[u[x], {x, 2}];
(* eq 2.1 and replace dψdt by eq 2.4 *)
eq21 = (h^2 D[W, {x, 2}] + D[D[u[x], x] W, x] - D[W, t]) // Simplify
eq21 /. {ψ^(0,1)[x, t] → h^2 ψ^(2,0)[x, t] - V ψ[x, t]} // FullSimplify
```

Out[∘]=
$$\frac{e^{-\frac{u[x]}{2h^2}}\left(-\psi[x,t]\left(u'[x]^2-2h^2u''[x]\right)-4h^2\psi^{(0,1)}[x,t]+4h^4\psi^{(2,0)}[x,t]\right)}{4h^2}$$

Out[∘]= 0.

eq 2.3 solves eq 2.1

In[∘]:=
```
Clear[x, t, w, u, h, LHS]
(* eq 2.3 time-independent function, h is constant, not a function of x *)
w = Exp[-u[x]/h^2];
(* eq 2.1 *)
LHS = h^2 D[w, {x, 2}] + D[D[u[x], x] w, x]
LHS // Simplify
```

Out[∘]= $-\frac{e^{-\frac{u[x]}{h^2}} u'[x]^2}{h^2} + e^{-\frac{u[x]}{h^2}} u''[x] + h^2 \left(\frac{e^{-\frac{u[x]}{h^2}} u'[x]^2}{h^4} - \frac{e^{-\frac{u[x]}{h^2}} u''[x]}{h^2}\right)$

Out[∘]= 0

eq 2.8 is solution to eq 2.4 if V=0 (heat equation)

In[∘]:=
```
(* if V(x)=0, a solution to 2.4 is *)
Clear[ψ, x, t, h, a, β, V]
(* eq 2.8 *)
ψ = (1 + 2 β h^2 t)^(-1/2) Exp[-β (x - a)^2/(2 × (1 + 2 β h^2 t))];
V = 0;
(* eq 2.4 *)
h^2 D[ψ, {x, 2}] - V ψ - D[ψ, t] // Simplify
```

Out[∘]= 0

In[∘]:=
```
(* cosh αx satisfies 2.6 with λ=-h²α² → replace Exp[-u/2h²] with Cosh[α x]*)
Clear[t, x, a, β, h, ψ, W, u, eq21, f]
f = Cosh[α x];
λ = -h^2 α^2;
-h^2 D[f, {x, 2}] - λ f
```

Out[∘]= 0

eq 2.9 is solution to eq 2.1

```
Clear[t, x, a, β, h, ψ, W, u, eq21, α, ρ, k, ν]
(* definition before eq 2.15 *)
ρ = 1/(1 + 2 β h^2 t);
(* eq 2.15 *)
W = -k Sqrt[ρ] Sech[ν x] (ν Tanh[ν x] + ρ β (x - a))
     Exp[-ν^2 h^2 t - β ρ (x - a)^2 / 2] + 0.5 ν Sech[ν x]^2;
(* potential defined after eq 2.15*)
u = h^2 Log[Cosh[ν x]^2];
eq21 = h^2 D[W, {x, 2}] + D[D[u, x] W, x] - D[W, t] // FullSimplify
```

Out[∘]= 0

eq 2.15 is solution to eq 2.1

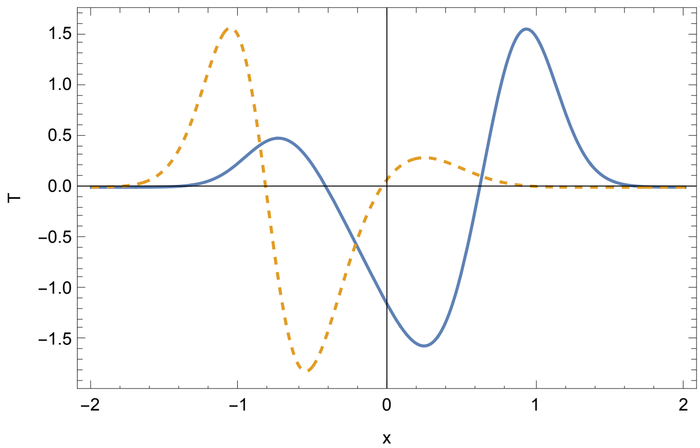
```
In[∘]:= Clear[t, x, a, β, h, ψ, W, u, eq21, α]
(* eq 2.8 *)
ψ = (1 + 2 β h^2 t)^(-1/2) Exp[-β (x - a)^2/(2 × (1 + 2 β h^2 t))];
(* eq 2.9 *)
W = ψ Cosh[α x] Exp[-h^2 α^2 t] // Simplify;
(* inverted potential defined after eq 2.9*)
u = h^2 Log[Sech[α x]^2];
eq21 = h^2 D[W, {x, 2}] + D[D[u, x] W, x] - D[W, t] // FullSimplify // Expand
```

Out[∘]= 0

Figure 1 transient solutions

In[29]:=
```
Clear[T, β, a, x, t, ν, γ, ω, θ, ρ, ϕ]
(* definition before eq 2.15 *)
ρ = 1/(1 + 2 β h^2 t);
(* eq 3.5 *)
T = (-β ρ - ν^2 + β^2 ρ^2 (x - a)^2 + ((γ^2 - ν^2) (β ρ (x - a) + ν Tanh[ν x + ω]))/(γ Coth[γ x + θ] - ν Tanh[ν x + ω]))
    Sqrt[ρ]/ϕ Exp[-γ^2 h^2 t - (β ρ (x - a)^2)/2];
ϕ = h γ Cosh[γ x + θ] - h ν Tanh[ν x + ω] Sinh[γ x + θ];
(* parameters *)
h = 0.35355; ν = 2.59; γ = 2.67; ω = 0.285; θ = -0.63;

Plot[{h^2 T /. {β → 6.4, a → 0.106, t → 0}, h^2 T /. {β → 8, a → -0.53, t → 0}},
 {x, -2, +2}, Frame → True, FrameLabel → {"x", "T"}, PlotStyle → {Default, Dashed}]

(* integral = 0 on (-∞,+∞) *)
NIntegrate[T /. {β → 6.4, a → 0.106, t → 0}, {x, -∞, +∞}] // Quiet
NIntegrate[T /. {β → 8, a → -0.53, t → 0}, {x, -∞, +∞}] // Quiet
```

Out[34]=



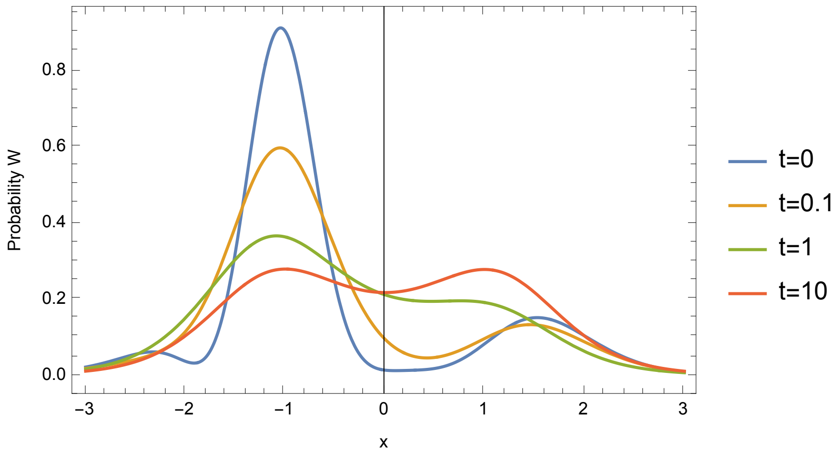
Out[35]= 1.80411×10^{-16}

Out[36]= -1.05471×10^{-15}

Figure 2

In[37]:=
```
Clear[T, β, a, x, t, ν, γ, ω, θ, ρ, k1, k2, a1, a2, f, ϕ, c, β1, β2, W, μ, h, g]
(* definition before eq 2.15 *)
ρ = 1/(1 + 2 β h^2 t);
μ = -h^2 γ^2;

(* eq 3.1 *)
ϕ = h γ Cosh[γ x + θ] - h ν Tanh[ν x + ω] Sinh[γ x + θ];

(* eq 3.3 *)
g = (h^2 γ (γ^2 - ν^2))/(2 ϕ^2);

(* eq 3.4 ??? *)
f = Sinh[γ x + θ] (Exp[(ν^2 - γ^2) h^2 t])/(ϕ^2 Cosh[ν x + ω]);

(* eq 3.5 *)
T = (-β ρ - ν^2 + β^2 ρ^2 (x - a)^2 + ((γ^2 - ν^2) (β ρ (x - a) + ν Tanh[ν x + ω]))/(γ Coth[γ x + θ] - ν Tanh[ν x + ω]))
   Sqrt[ρ]/ϕ Exp[-γ^2 h^2 t - (β ρ (x - a)^2)/2];

(* eq 3.6 *)
W = g + c f + k1 (T /. {β → β1, a → a1}) + k2 ( T /. {β → β2, a → a2});

c = -0.177;
β1 = 4.7;
a1 = -1.1;
k1 = -0.095;
β2 = 1;
a2 = 0.5;
k2 = 0.09;

h = 1;
γ = 1.293;
ν = 1.054;
θ = ω = 0;

Plot[{W /. {t → 0}, W /. {t → 0.1}, W /. {t → 1}, W /. {t → 10}},
 {x, -3, +3}, Frame → True, FrameLabel → {"x", "Probability W"},
 PlotRange → All, PlotLegends → {"t=0", "t=0.1", "t=1", "t=10"}]

Plot[ϕ, {x, -2, +2}, Frame → True, FrameLabel → {"x", "Potential ϕ"}]
```

Out[56]=


Probability W
0.0
0.2
0.4
0.6
0.8
−3
−2
−1
0
1
2
3
x
t=0
t=0.1
t=1
t=10

Out[57]=

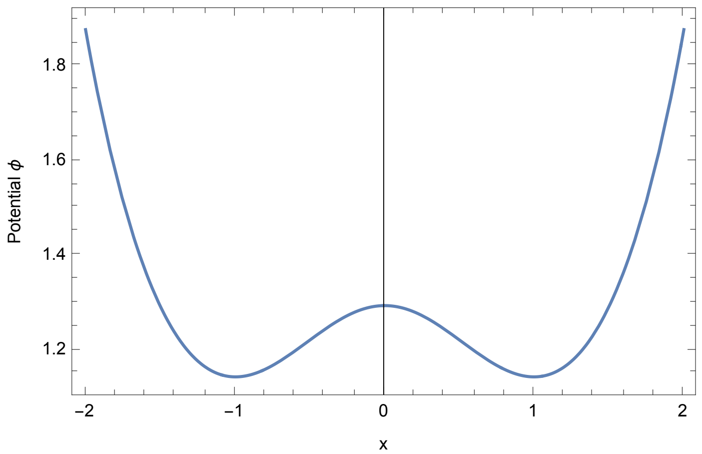
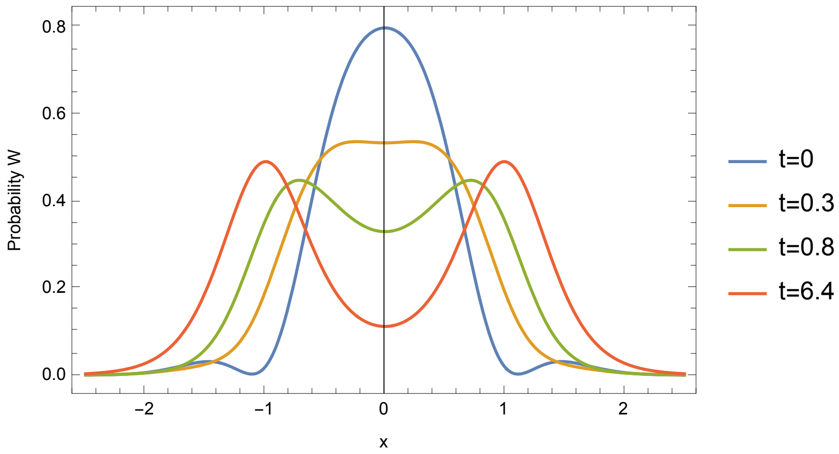

Potential ϕ
1.2
1.4
1.6
1.8
−2
−1
0
1
2
x

Figure 3

```
In[58]:= Clear[T, β, a, x, t, ν, γ, ω, θ, ρ, k1, k2, a1, a2, f, ϕ, c, β1, β2, W, μ, h, g]
(* definition before eq 2.15 *)
ρ = 1/(1 + 2 β h^2 t);
μ = -h^2 γ^2;

(* eq 3.1 *)
ϕ = h γ Cosh[γ x + θ] - h ν Tanh[ν x + ω] Sinh[γ x + θ];

(* eq 3.3 *)
g = (h^2 γ (γ^2 - ν^2))/(2 ϕ^2);

(* eq 3.4 ??? *)
f = Sinh[γ x + θ] (Exp[(ν^2 - γ^2) h^2 t])/(ϕ^2 Cosh[ν x + ω]);

(* eq 3.5 *)
T = (-β ρ - ν^2 + β^2 ρ^2 (x - a)^2 + ((γ^2 - ν^2) (β ρ (x - a) + ν Tanh[ν x + ω]))/(γ Coth[γ x + θ] - ν Tanh[ν x + ω]))
   (√ρ/ϕ) Exp[-γ^2 h^2 t - (β ρ (x - a)^2)/2];

(* eq 3.6 *)
W = g + c f + k1 (T /. {β → β1, a → a1}) + k2 ( T /. {β → β2, a → a2});

c = 0;
β1 = 3.6;
a1 = 0;
k1 = -0.06833;
β2 = 0.9;
a2 = 0;
k2 = -0.014833;

h = 0.4082;
γ = 2.152;
ν = 2.038;
θ = ω = 0;

Plot[{W /. {t → 0}, W /. {t → 0.3}, W /. {t → 0.8}, W /. {t → 6.4}},
  {x, -2.5, +2.5}, Frame → True, FrameLabel → {"x", "Probability W"},
  PlotRange → All, PlotLegends → {"t=0", "t=0.3", "t=0.8", "t=6.4"}]

Plot[ϕ, {x, -2, +2}, Frame → True, FrameLabel → {"x", "Potential ϕ"}]
```

Out[77]=



Out[78]=

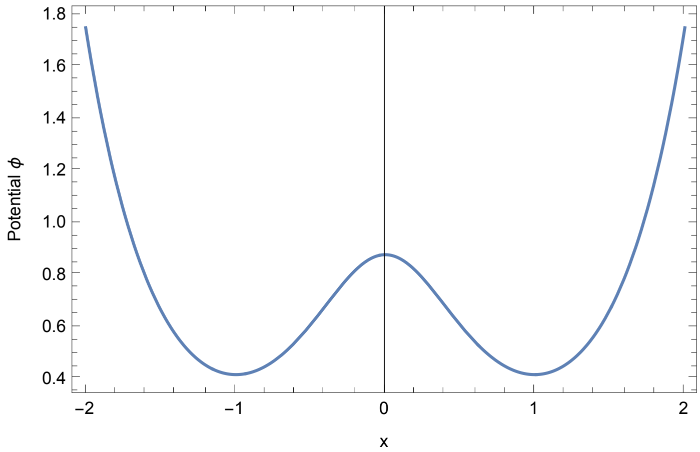


Figure 4

In[∘]:=

```
Clear[T, β, a, x, t, ν, γ, ω, θ, ρ, k1, k2, a1, a2, f, ϕ, c, β1, β2, W, μ, h, g]
(* definition before eq 2.15 *)
```

$\rho = \dfrac{1}{1 + 2\,\beta\,h^2\,t};$

$\mu = -h^2\,\gamma^2;$

```
(* eq 3.1 *)
ϕ = h γ Cosh[γ x + θ] - h ν Tanh[ν x + ω] Sinh[γ x + θ];

(* eq 3.3 *)
```

$g = \dfrac{h^2\,\gamma\,(\gamma^2 - \nu^2)}{2\,\phi^2};$

```
(* eq 3.4 ??? *)
```

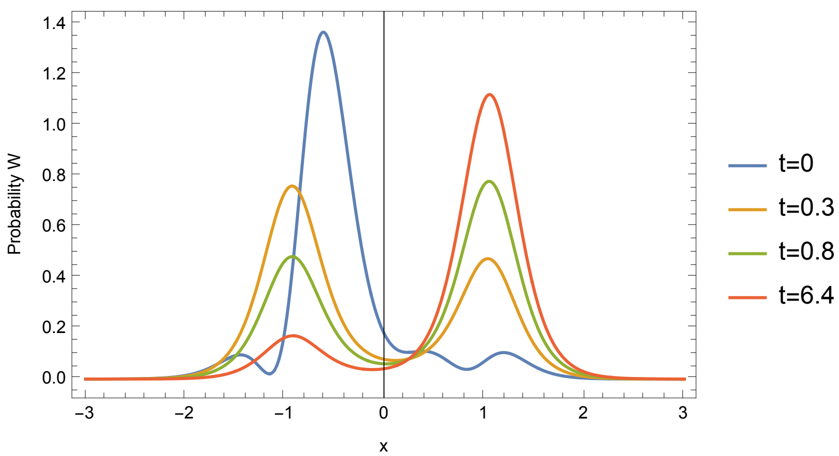
$f = \mathrm{Sinh}[\gamma\,x + \theta]\,\dfrac{(\mathrm{Exp}[(\nu^2 - \gamma^2)\,h^2\,t])}{\phi^2\,\mathrm{Cosh}[\nu\,x + \omega]};$

```
(* eq 3.5 *)
```

$$T = \left(-\beta\,\rho - \nu^2 + \beta^2\,\rho^2\,(x-a)^2 + \frac{\left(\gamma^2-\nu^2\right)\,(\beta\,\rho\,(x-a) + \nu\,\mathrm{Tanh}[\nu\,x+\omega])}{\gamma\,\mathrm{Coth}[\gamma\,x+\theta] - \nu\,\mathrm{Tanh}[\nu\,x+\omega]}\right)$$
$$\frac{\sqrt{\rho}}{\phi}\,\mathrm{Exp}\left[-\gamma^2\,h^2\,t - \frac{\beta\,\rho\,(x-a)^2}{2}\right];$$

```
(* eq 3.6 *)
W = g + c f + k1 (T /. {β → β1, a → a1}) + k2 ( T /. {β → β2, a → a2});

c = -0.13;
β1 = 6.4;
a1 = 0.106;
k1 = -0.01375;
β2 = 8.0;
a2 = -0.53;
k2 = -0.0625;

h = 0.35355;
γ = 2.67;
ν = 2.59;
θ = -0.63;
ω = 0.285;

Plot[{W /. {t → 0}, W /. {t → 6}, W /. {t → 18}, W /. {t → 100}},
 {x, -3, +3}, Frame → True, FrameLabel → {"x", "Probability W"},
 PlotRange → All, PlotLegends → {"t=0", "t=0.3", "t=0.8", "t=6.4"}]

Plot[ϕ, {x, -2, +2}, Frame → True, FrameLabel → {"x", "Potential ϕ"}]
```

Out[∘]=



Out[ ]=