

Properties of finite amplitude electromagnetic waves propagating in the quantum vacuum

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Notebook: Óscar Amaro, June 2021, November 2022 @ GoLP-EPP

Contact: oscar.amaro@tecnico.ulisboa.pt

Introduction

In this notebook we reproduce some results from the paper.

Prove eq 16+17 is equivalent to eq 13+14 ...

```
(* eq 16 is the easiest to prove *)
Clear[E, B, E0, B0, Ez, By, a, b, x, t, α, β, γ]
```

```
(* eq 15 *)
Ez[x, t] = E0 + a[x, t];
By[x, t] = B0 + b[x, t];
```

```
(* eq 13 *)
D[By[x, t], t] - D[Ez[x, t], x]
```

```
Out[147]= b(0,1)[x, t] - a(1,0)[x, t]
```

```

In[1538]:= (* eq 17 requires a bit more work ... *)
Clear[E, B, E0, B0, Ez, By, a, b, x, t, α, β, γ, eq14, eq17, κ1, κ2]
κ2 =  $\frac{180}{315}$  κ1;

(* eq 15 *)
Ez[x, t] = E0 + a[x, t];
By[x, t] = B0 + b[x, t];

(* eq 14 *)
eq14 = - (1 + 8 κ1 Ez[x, t]^2 + 4 (Ez[x, t]^2 - By[x, t]^2) (κ1 - 3 κ2 Ez[x, t]^2) -
        6 κ2 (Ez[x, t]^2 - By[x, t]^2)^2) D[Ez[x, t], t] +
        (1 - 8 κ1 By[x, t]^2 + 4 (Ez[x, t]^2 - By[x, t]^2) (κ1 + 3 κ2 By[x, t]^2) -
        6 κ2 (Ez[x, t]^2 - By[x, t]^2)^2) D[By[x, t], x] +
        4 × (2 κ1 - 3 κ2 (Ez[x, t]^2 - By[x, t]^2)) Ez[x, t] × By[x, t] ×
        (D[By[x, t], t] + D[Ez[x, t], x]) // Expand // Simplify;

(* eq 18 *)
α = 1 + 8 κ1 (E0 + a[x, t])^2 + 4 ((E0 + a[x, t])^2 - (B0 + b[x, t])^2)
    (κ1 - 3 (E0 + a[x, t])^2 κ2) - 6 κ2 ((E0 + a[x, t])^2 - (B0 + b[x, t])^2)^2;

(* eq 19 *)
β = 4 (E0 + a[x, t]) (B0 + b[x, t]) (2 κ1 - 3 κ2 ((E0 + a[x, t])^2 - (B0 + b[x, t])^2));

(* eq 20 *)
γ = 1 - 8 κ1 (B0 + b[x, t])^2 + 4 ((E0 + a[x, t])^2 - (B0 + b[x, t])^2)
    (κ1 + 3 (B0 + b[x, t])^2 κ2) - 6 κ2 ((E0 + a[x, t])^2 - (B0 + b[x, t])^2)^2;

(* eq 17 *)
eq17 = α D[a[x, t], t] - β (D[a[x, t], x] + D[b[x, t], t]) - γ D[b[x, t], x] // Expand //
Simplify;

(* confirm eq 21 *)
{α, β, γ} /. {a[x, t] → 0, b[x, t] → 0, B0 → E0}

res = eq14 / eq17;
(res /. {b(0,1)[x, t] → a(1,0)[x, t]}) // FullSimplify

(*
E0=RandomReal[];
B0=RandomReal[];
κ1=RandomReal[];
res//FullSimplify
*)
Out[1547]= {1 + 8 E0^2 κ1, 8 E0^2 κ1, 1 - 8 E0^2 κ1}

```

```
Out[1549]= ( (7 - 4 (7 + 6 B0^2 - 6 E0^2) (B0^2 - 3 E0^2) κ1 + 4 κ1 (-72 E0 a[x, t]^3 - 18 a[x, t]^4 -
      b[x, t] (2 B0 + b[x, t]) (7 + 12 B0^2 - 24 E0^2 + 6 b[x, t] (2 B0 + b[x, t])) +
      3 a[x, t]^2 (7 + 8 B0^2 - 36 E0^2 + 8 b[x, t] (2 B0 + b[x, t])) +
      6 E0 a[x, t] (7 + 8 B0^2 - 12 E0^2 + 8 b[x, t] (2 B0 + b[x, t])) ) a^(0,1)[x, t] -
      7 b^(1,0)[x, t] + 4 κ1 (7 + 6 B0^2 - 6 E0^2 - 6 a[x, t] (2 E0 + a[x, t]) +
      6 b[x, t] (2 B0 + b[x, t])) ×
      (-4 (E0 + a[x, t]) (B0 + b[x, t]) a^(1,0)[x, t] -
      (-3 B0^2 + E0^2 + 2 E0 a[x, t] + a[x, t]^2 - 3 b[x, t] (2 B0 + b[x, t])) b^(1,0)[x, t]) ) /
      ( (-7 + 4 (7 + 6 B0^2 - 6 E0^2) (B0^2 - 3 E0^2) κ1 + 4 κ1 (72 E0 a[x, t]^3 + 18 a[x, t]^4 +
      b[x, t] (2 B0 + b[x, t]) (7 + 12 B0^2 - 24 E0^2 + 6 b[x, t] (2 B0 + b[x, t])) -
      3 a[x, t]^2 (7 + 8 B0^2 - 36 E0^2 + 8 b[x, t] (2 B0 + b[x, t])) -
      6 E0 a[x, t] (7 + 8 B0^2 - 12 E0^2 + 8 b[x, t] (2 B0 + b[x, t])) ) )
      a^(0,1)[x, t] + 16 κ1 (E0 + a[x, t]) (B0 + b[x, t])
      (7 + 6 B0^2 - 6 E0^2 - 6 a[x, t] (2 E0 + a[x, t]) + 6 b[x, t] (2 B0 + b[x, t]))
      a^(1,0)[x, t] +
      (7 + 4 (E0^2 - 3 B0^2 (5 + 4 B0^2 - 4 E0^2)) κ1 +
      4 κ1 (-3 b[x, t] (2 B0 + b[x, t]) (5 + 8 B0^2 - 4 E0^2 + 4 b[x, t] (2 B0 + b[x, t])) +
      2 E0 a[x, t] (1 + 12 B0^2 + 12 b[x, t] (2 B0 + b[x, t])) +
      a[x, t]^2 (1 + 12 B0^2 + 12 b[x, t] (2 B0 + b[x, t])) ) ) b^(1,0)[x, t] )
```

```
In[1319]:= (* get eq 23 *)
Clear[eq16, eq17, x, t, a, b, α, β, γ, aa, bb, ν, q, ω, α0, β0, γ0]
(* eq 22 ansatz *)
a[x, t] = aa Exp[-I ω t + I q x];
b[x, t] = bb Exp[-I ω t + I q x];

eq16 = D[b[x, t], t] - D[a[x, t], x] // Simplify;
eq17 =
  α D[a[x, t], t] - β (D[a[x, t], x] + D[b[x, t], t]) - γ D[b[x, t], x] // Expand //
  Simplify;
ω = ν q;
(* *)
eq16
aa + ν bb // Simplify
eq17
ν (bb β - aa α) - (aa β + bb γ) // FullSimplify
```

```
Out[1325]= -i e^(i q (x - t ν)) q
```

```
Out[1326]= i e^(i q (x - t ν)) q
```

```

In[774]:= (* Taylor expand eq 69 ... *)
Clear[κ1, E0, E02]

Series[ $\frac{1 - 8 \kappa_1 E_0^2}{1 + 8 \kappa_1 E_0^2}$ , {E0, 0, 4}]

(1 - 8 κ1 E02) × (1 - 8 κ1 E02) // Expand

Out[775]= 1 - 16 κ1 E02 + 128 κ12 E04 + O[E0]5

Out[776]= 1 - 16 E02 κ1 + 64 E022 κ12

In[1180]:= (* get eq 24 *)
Clear[a, b, v, α0, β0, γ0, sol]
a = -v b;
sol = Solve[v (b β0 - a α0) - (a β0 + b γ0) == 0, v]

(* use relationships by equation 21 *)
{α0, β0, γ0} = {1 + 8 E02 κ1, 8 E02 κ1, 1 - 8 E02 κ1};
(* get eq 25 *)
sol // Simplify

Out[1182]=  $\left\{ \left\{ v \rightarrow \frac{-\beta_0 - \sqrt{\beta_0^2 + \alpha_0 \gamma_0}}{\alpha_0} \right\}, \left\{ v \rightarrow \frac{-\beta_0 + \sqrt{\beta_0^2 + \alpha_0 \gamma_0}}{\alpha_0} \right\} \right\}$ 

Out[1184]=  $\left\{ \{ v \rightarrow -1 \}, \left\{ v \rightarrow \frac{1 - 8 E_0^2 \kappa_1}{1 + 8 E_0^2 \kappa_1} \right\} \right\}$ 

```

```

In[883]:= (* get eq 71 ... *)
(*
Clear[κ1,κ2,E0,v,fp]
v =  $\frac{1-8\kappa_1 E_0^2}{1+8 \kappa_1 E_0^2}$ ;
fp = 4E0  $\left(\kappa_1 \left(-1-3v-3v^2+\frac{3}{v}\right)+3\kappa_2 E_0^2 \left(v^2+3v-3+\frac{1}{v}\right)\right)$  // Simplify;
*)

Clear[κ1, κ2, E0, v, fp]
(* the correct coefficient of E0^4 will not change eq 71 *)
v = 1 - 16 κ1 E0^2 + 128 κ1^2 E0^4;
fp = 4 E0  $\left(\kappa_1 \left(-1-3v-3v^2+\frac{3}{v}\right)+3\kappa_2 E_0^2 \left(v^2+3v-3+\frac{1}{v}\right)\right)$  // Expand;

(* f' will not be a polynome in E0 because of the rational form of v *)

(* this is not equal to eq 71 ...*)
CoefficientList[
  Normal[Series[fp, {E0, 0, 4}]] - 48 E0^3 (12 κ1^2 + κ2) // Expand, E0] [[4]]

(* but 48 E0^3 (16κ1^2+0.5κ2) is *)
Normal[Series[fp, {E0, 0, 4}]] - 48 E0^3 (16 κ1^2 + 0.5 κ2) // Expand

Out[886]= 192 κ1^2 - 24 κ2

Out[887]= 0. - 16 E0 κ1

```

Figure 1: visualize eq 73

```
In[1148]:= Clear[a0, am, fp, x, t]
```

```
a0 = am Sin[x0];
```

```
x = x0 + (1 + fp a0) t
```

```
(* a(x,x0=0) is the first (to the left) point of each curve. for Δt=
0.5 (difference in time is uniform for all curves) then a(x,x0=0) progresses
linearly. the first blue curve for lowest time is for t=0.5 and not t=0 *)
x /. {x0 → 0}
```

```
(* looking at the plot, the amplitude am cannot be 1,
but needs to be around 1.5, which is very close to π/2 *)
```

```
am = π / 2;
```

```
fp = -0.35;
```

```
tab = Table[{x, a0 // N}, {x0, 0, 2 π, 2 π / 100}];
```

```
ListPlot[{(tab /. {t → 0.5}), (tab /. {t → 1}), (tab /. {t → 1.5}), (tab /. {t → 2}),
(tab /. {t → 2.5}), (tab /. {t → 3}), (tab /. {t → 3.5}), (tab /. {t → 0})},
Joined → True, Frame → True, FrameLabel → {"x", "a"},
GridLines → Automatic, AspectRatio → 0.3, ImageSize → Large]
```

```
Out[1150]= x0 + t (1 + am fp Sin[x0])
```

```
Out[1151]= t
```

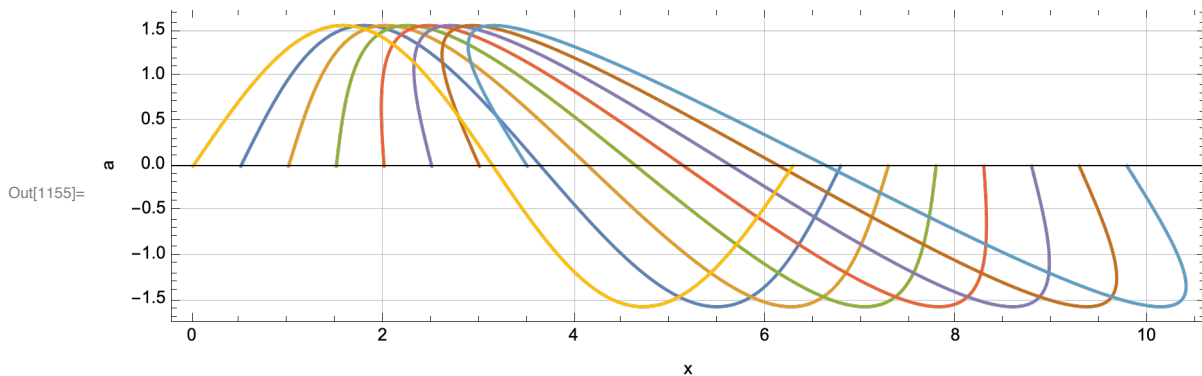


Figure 3

```
In[448]:= (* eq 87 *)
```

```
Clear[a1, a10, t, fp]
```

```
DSolve[{D[a1[t], t] == -fp a1[t]^2, a1[0] == a10}, a1[t], t]
```

```
Out[449]= {{a1[t] →  $\frac{a10}{1 + a10 fp t}$ }}
```

```

In[32]:= Clear[a0, a1, a10, fp, tab]
          a10
a1[t_, a10_, fp_] :=  $\frac{a10}{1 + fp a10 t}$ 
a[x_, t_, a0_, a10_, fp_] := a0 + a1[t, a10, fp] x

(* a *)
tab = Table[a[x, t, -3 t + 1.8, 1, -0.5], {t, 0.2, 1.2, 0.1}];
Plot[tab, {x, 0, 10}, Frame → True, FrameLabel → {"x", "a"},
     PlotLabel → "(a) The rarefaction waves breaks."]

(* b *)
tab = Table[a[x, t, -3 t + 1.8, -1, 0.5], {t, 0.2, 1.2, 0.1}];
Plot[tab, {x, 0, 10}, Frame → True, FrameLabel → {"x", "a"},
     PlotLabel → "(b) The rarefaction waves do not break."]

```

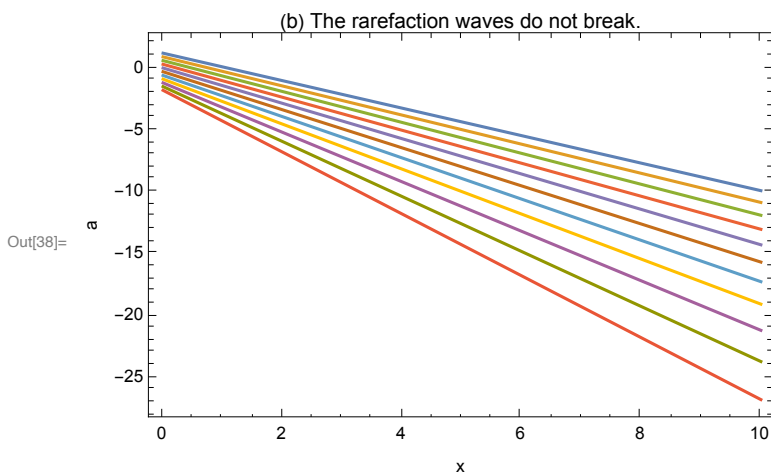
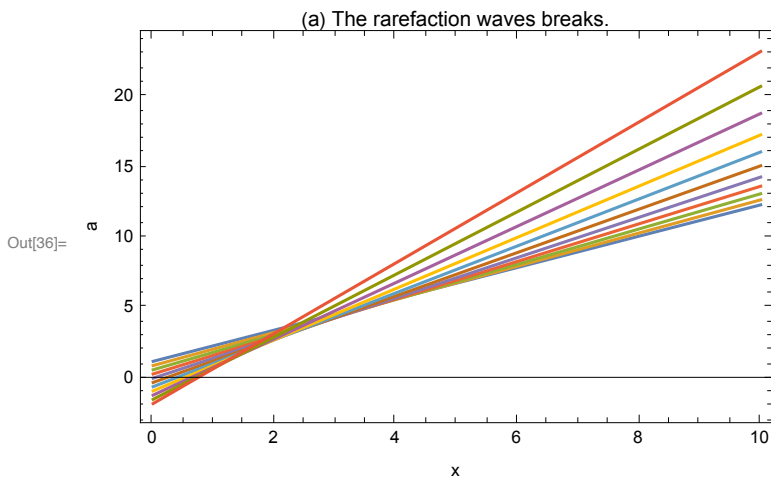


Figure 4

```

In[429]:= (* σγγep in "natural" units, and assuming ω=Ω or ω1=ω2 *)
Clear[σγγep, re, e, me, c, ħ, ω, βe]

βe = Sqrt[1 -  $\frac{me^2 c^4}{\hbar^2 \omega^2}$ ];

σγγep =  $\frac{\pi re^2}{2} (1 - \beta e^2) \times \left( (3 - \beta e^4) \text{Log}\left[\frac{1 + \beta e}{1 - \beta e}\right] - 2 \beta e (2 - \beta e^2) \right)$ ;

re = e^2 / (me c^2);
e = me = c = ħ = 1;
Plot[σγγep, {ω, 0, 10}, PlotRange → {0, 2.5},

Frame → True, FrameLabel → { $\frac{\hbar \omega}{me c^2}$ , "σγγep"}]

Plot[{σγγep,  $\pi re^2 \text{Sqrt}\left[\frac{\hbar^2 \omega^2}{me^2 c^4} - 1\right]$ }, {ω, 0, 2}, PlotRange → {{0, 2}, {0, 5}},

Frame → True, FrameLabel → { $\frac{\hbar \omega}{me c^2}$ , "σγγep"}, ImageSize → Small,

PlotLegends → {"Exact cross section: eq 90", "Near-threshold: eq 89"},
PlotStyle → {Automatic, Red}]

```

