Klein-Nishina cross section

Notebook: Óscar Amaro, April 2023 @ GoLP-EPP

Introduction

In this notebook we present some calculations useful when using the KN unpolarized cross section https://en.wikipedia.org/wiki/Klein%E2%80%93Nishina_formula

In[1]:=
```
(* d/dΩ = dσ/(sin θ dθ) → dσ/dθ = sin θ dσ/dΩ *)

Clear[re, λ, λp, λc, dσdΩ, dσdθ, dσdΩ0, ϵ, c, e, ϵeV, dσdΩlow, cdf, getθ]

λp = λ (1 + ϵ (1 - Cos[θ])); (*λp = λ', ϵ = λc/λ = Eγ/mc^2*)
dσdΩ = re^2/2 (λ/λp)^2 (λ/λp + λp/λ - Sin[θ]^2) // Simplify

(* low energy limit and total cross-section *)
dσdΩlow = Limit[dσdΩ, ϵ → 0] // Simplify
2 π Integrate[dσdΩlow Sin[θ], {θ, 0, π}] // Simplify

(* normalized cumulative integral, aka, CDF *)
dσdΩcdf = 2 π Integrate[dσdΩ, θ] // Simplify
(* as expected, the CDF cannot be analytically inverted Solve[dσdΩCDF==v,θ]*)
```

Out[3]=
$$\frac{re^2\left(1+\epsilon-\epsilon\,\mathrm{Cos}[\theta]+\frac{1}{1+\epsilon-\epsilon\,\mathrm{Cos}[\theta]}-\mathrm{Sin}[\theta]^2\right)}{2\,(1+\epsilon-\epsilon\,\mathrm{Cos}[\theta])^2}$$

Out[4]=
$$\frac{1}{4}\,re^2\,(3+\mathrm{Cos}[2\,\theta])$$

Out[5]=
$$\frac{8\,\pi\,re^2}{3}$$

Out[6]=
$$\frac{1}{2\,\epsilon^2}\pi\,re^2\left(2\,\theta+\frac{2\times\left(-2-10\,\epsilon-12\,\epsilon^2+4\,\epsilon^3+11\,\epsilon^4\right)\mathrm{ArcTanh}\left[\sqrt{-1-2\,\epsilon}\;\mathrm{Tan}\left[\frac{\theta}{2}\right]\right]}{(-1-2\,\epsilon)^{5/2}}+\frac{\epsilon^3\,\mathrm{Sin}[\theta]}{(1+2\,\epsilon)\,(1+\epsilon-\epsilon\,\mathrm{Cos}[\theta])^2}-\frac{\epsilon\left(2+8\,\epsilon+11\,\epsilon^2+3\,\epsilon^3\right)\mathrm{Sin}[\theta]}{(1+2\,\epsilon)^2\,(-1-\epsilon+\epsilon\,\mathrm{Cos}[\theta])}\right)$$

In[7]:=
```
dσdΩcdfπ = Limit[dσdΩcdf, θ → π, Direction → "FromBelow"]
```

Out[7]=
$$\frac{\pi^2\,re^2\left(-2-10\,\epsilon-12\,\epsilon^2+4\,\epsilon^3+11\,\epsilon^4+2\,(1+2\,\epsilon)^{5/2}\right)}{2\,\epsilon^2\,(1+2\,\epsilon)^{5/2}}$$

In[8]:=
$$\text{Refine}\left[\left(\frac{\text{d}\sigma\text{d}\Omega}{\text{Limit}[\text{d}\sigma\text{d}\Omega,\ \theta \to 0]} \text{ /. } \{\epsilon\text{eV} \to 510\,998.95\ \text{r}\}\right) \text{ // Simplify},\ \{\theta > 0,\ \text{r} > 0\}\right]$$

Out[8]=
$$\frac{1 + \epsilon - \epsilon \,\text{Cos}[\theta] + \frac{1}{1+\epsilon-\epsilon\,\text{Cos}[\theta]} - \text{Sin}[\theta]^2}{2\,(1 + \epsilon - \epsilon\,\text{Cos}[\theta])^2}$$

In[9]:=
$$\text{Refine}\left[\left(\frac{\text{d}\sigma\text{d}\Omega\text{cdf}}{\text{d}\sigma\text{d}\Omega\text{cdf}\pi} \text{ /. } \{\epsilon\text{eV} \to 510\,998.95\ \text{r}\}\right) \text{ // Simplify},\ \{\theta > 0,\ \text{r} > 0\}\right] \text{ // Simplify}$$

Out[9]=
$$\left((1 + 2\,\epsilon)^{5/2}\left(2\,\theta + \frac{2 \times \left(-2 - 10\,\epsilon - 12\,\epsilon^2 + 4\,\epsilon^3 + 11\,\epsilon^4\right)\text{ArcTanh}\left[\sqrt{-1 - 2\,\epsilon}\ \text{Tan}\left[\frac{\theta}{2}\right]\right]}{(-1 - 2\,\epsilon)^{5/2}} + \frac{\epsilon^3\,\text{Sin}[\theta]}{(1 + 2\,\epsilon)\,(1 + \epsilon - \epsilon\,\text{Cos}[\theta])^2} - \frac{\epsilon\left(2 + 8\,\epsilon + 11\,\epsilon^2 + 3\,\epsilon^3\right)\text{Sin}[\theta]}{(1 + 2\,\epsilon)^2\,(-1 - \epsilon + \epsilon\,\text{Cos}[\theta])}\right)\right) \Big/ \left(\pi\left(-2 - 10\,\epsilon - 12\,\epsilon^2 + 4\,\epsilon^3 + 11\,\epsilon^4 + 2\,(1 + 2\,\epsilon)^{5/2}\right)\right)$$

In[10]:=
```
(* the number of harmonics of KN cross section in the ϵeV→0 limit is very
  small (Sin^2) and the Fourier transform is very simple. however,
in the general case this is not true *)
(*FourierTransform[dσdΩ,θ,kθ]//Simplify*)
```

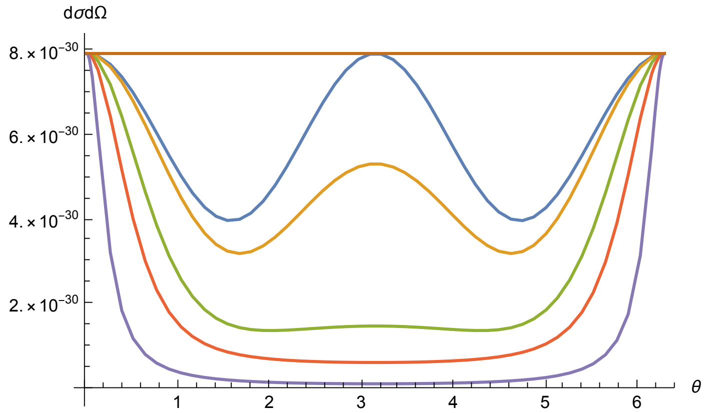
In[11]:=
```
c = 3 × 10^8;
e = 1.6 × 10^-19;
ℏ = 1.05 × 10^-34;
re = 2.82 × 10^-15;
λc = 2.42 × 10^-12;
ϵ = λc / λ;
λ = 2 π c / ω;
ω = e ϵeV / ℏ;
ϵ
(* λ =(2π c)/ω, ℏω/e=ϵeV*)
dσdΩ0 = dσdΩ /. {θ → 0};
Plot[{dσdΩ /. {ϵeV → 2.75}, dσdΩ /. {ϵeV → 60 × 10^3}, dσdΩ /. {ϵeV → 511 × 10^3},
  dσdΩ /. {ϵeV → 1.46 × 10^6}, dσdΩ /. {ϵeV → 10 × 10^6}, dσdΩ0},
 {θ, 0, 2 π}, AxesLabel → {"θ", "dσdΩ"}]

PolarPlot[{dσdΩ /. {ϵeV → 2.75}, dσdΩ /. {ϵeV → 60 × 10^3}, dσdΩ /. {ϵeV → 511 × 10^3},
  dσdΩ /. {ϵeV → 1.46 × 10^6}, dσdΩ /. {ϵeV → 10 × 10^6}, dσdΩ0},
 {θ, 0, 2 π}, AxesLabel → {"θ", "dσdΩ"}]
```

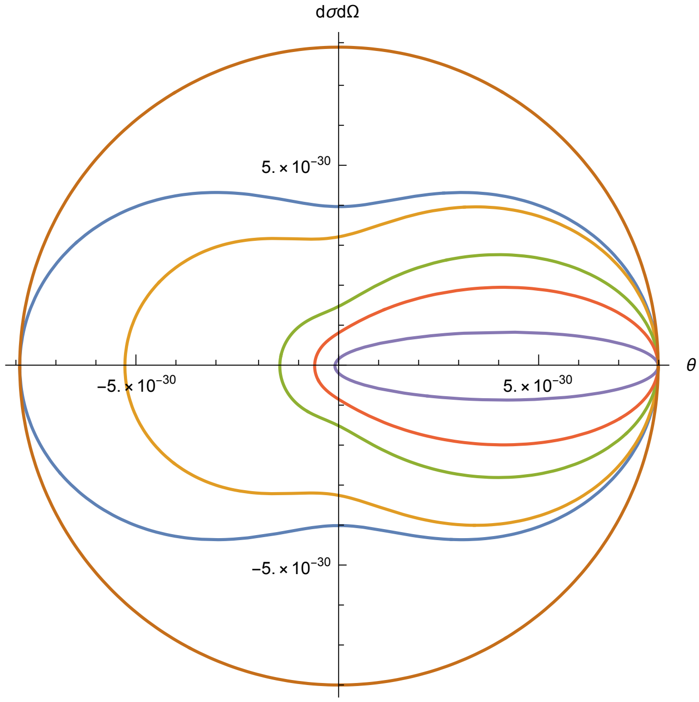
$$\text{Plot}\left[\left\{\frac{\text{d}\sigma\text{d}\Omega\text{cdf}}{\text{d}\sigma\text{d}\Omega\text{cdf}\pi} \text{ /. } \{\epsilon\text{eV} \to 2.75\},\ \frac{\text{d}\sigma\text{d}\Omega\text{cdf}}{\text{d}\sigma\text{d}\Omega\text{cdf}\pi} \text{ /. } \{\epsilon\text{eV} \to 60 \times 10^{3}\},\ \frac{\text{d}\sigma\text{d}\Omega\text{cdf}}{\text{d}\sigma\text{d}\Omega\text{cdf}\pi} \text{ /. } \{\epsilon\text{eV} \to 511 \times 10^{3}\},\ \frac{\text{d}\sigma\text{d}\Omega\text{cdf}}{\text{d}\sigma\text{d}\Omega\text{cdf}\pi} \text{ /. } \{\epsilon\text{eV} \to 1.46 \times 10^{6}\},\ \frac{\text{d}\sigma\text{d}\Omega\text{cdf}}{\text{d}\sigma\text{d}\Omega\text{cdf}\pi} \text{ /. } \{\epsilon\text{eV} \to 10 \times 10^{6}\}\right\},\ \{\theta,\ 0,\ \pi\},\ \text{AxesLabel} \to \{\text{"}\theta\text{"},\ \text{"CDF"}\}\right]$$

Out[19]= $1.95634 \times 10^{-6}\ \epsilon\text{eV}$

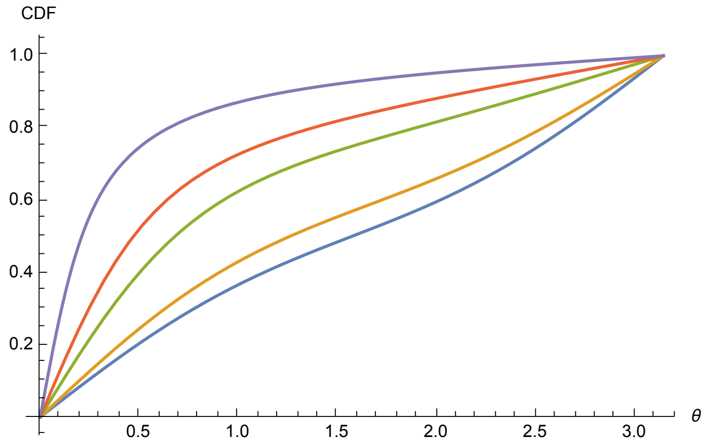
Out[21]=



Out[22]=



Out[23]=



In[24]:=
```
f = dσdΩcdf/(2 dσdΩcdfπ) HeavisideTheta[π - θ] +
   HeavisideTheta[θ - π] (1 - dσdΩcdf/(2 dσdΩcdfπ) /. {θ → 2 π - θ})
Plot[{f /. {ϵeV → 2.75}, f /. {ϵeV → 60 × 10 ^ 3}, f /. {ϵeV → 511 × 10 ^ 3},
   f /. {ϵeV → 1.46 × 10 ^ 6}, f /. {ϵeV → 10 × 10 ^ 6}},
  {θ, 0, 2 π }, AxesLabel → {"θ", "CDF"}]
(*PolarPlot[{f/.{ϵeV→2.75},f/.{ϵeV→60 10^3},f/.{ϵeV→511 10^3},
   f/.{ϵeV→1.46 10^6},f/.{ϵeV→10 10^6}},{θ,0,2π },AxesLabel→{"θ","CDF"}]*)
```

Out[24]=

$$\Bigg(0.159155\left(1+3.91269\times10^{-6}\,\epsilon\text{eV}\right)^{5/2}$$

$$\text{HeavisideTheta}[\pi-\theta]\Bigg(2\,\theta+\frac{1}{\left(-1-3.91269\times10^{-6}\,\epsilon\text{eV}\right)^{5/2}}$$

$$2\times\left(-2-0.0000195634\,\epsilon\text{eV}-4.59273\times10^{-11}\,\epsilon\text{eV}^2+2.99499\times10^{-17}\,\epsilon\text{eV}^3+\right.$$

$$\left.1.61129\times10^{-22}\,\epsilon\text{eV}^4\right)\text{ArcTanh}\left[\sqrt{-1-3.91269\times10^{-6}\,\epsilon\text{eV}}\;\text{Tan}\left[\frac{\theta}{2}\right]\right]+$$

$$\frac{7.48746\times10^{-18}\,\epsilon\text{eV}^3\,\text{Sin}[\theta]}{\left(1+3.91269\times10^{-6}\,\epsilon\text{eV}\right)\left(1+1.95634\times10^{-6}\,\epsilon\text{eV}-1.95634\times10^{-6}\,\epsilon\text{eV}\,\text{Cos}[\theta]\right)^2}-$$

$$\left(1.95634\times10^{-6}\,\epsilon\text{eV}\left(2+0.0000156507\,\epsilon\text{eV}+4.21\times10^{-11}\,\epsilon\text{eV}^2+2.24624\times10^{-17}\,\epsilon\text{eV}^3\right)\right.$$

$$\left.\text{Sin}[\theta]\right)\Big/\left(\left(1+3.91269\times10^{-6}\,\epsilon\text{eV}\right)^2\right.$$

$$\left.\left(-1-1.95634\times10^{-6}\,\epsilon\text{eV}+1.95634\times10^{-6}\,\epsilon\text{eV}\,\text{Cos}[\theta]\right)\right)\Bigg)\Bigg)\Big/$$

$$\left(-2+2\left(1+3.91269\times10^{-6}\,\epsilon\text{eV}\right)^{5/2}-0.0000195634\,\epsilon\text{eV}-4.59273\times10^{-11}\,\epsilon\text{eV}^2+\right.$$

$$\left.2.99499\times10^{-17}\,\epsilon\text{eV}^3+1.61129\times10^{-22}\,\epsilon\text{eV}^4\right)+$$

$$\text{HeavisideTheta}[-\pi+\theta]\Bigg(1-\Bigg(0.159155\left(1+3.91269\times10^{-6}\,\epsilon\text{eV}\right)^{5/2}$$

$$\Bigg(2\times(2\,\pi-\theta)+\frac{1}{\left(-1-3.91269\times10^{-6}\,\epsilon\text{eV}\right)^{5/2}}2\times\left(-2-0.0000195634\,\epsilon\text{eV}-\right.$$

$$\left.4.59273\times10^{-11}\,\epsilon\text{eV}^2+2.99499\times10^{-17}\,\epsilon\text{eV}^3+1.61129\times10^{-22}\,\epsilon\text{eV}^4\right)$$

$$\text{ArcTanh}\left[\sqrt{-1-3.91269\times10^{-6}\,\epsilon\text{eV}}\;\text{Tan}\left[\frac{1}{2}\times(2\,\pi-\theta)\right]\right]-$$

$$\frac{7.48746\times10^{-18}\,\epsilon\text{eV}^3\,\text{Sin}[\theta]}{\left(1+3.91269\times10^{-6}\,\epsilon\text{eV}\right)\left(1+1.95634\times10^{-6}\,\epsilon\text{eV}-1.95634\times10^{-6}\,\epsilon\text{eV}\,\text{Cos}[\theta]\right)^2}+$$

$$\left(1.95634\times10^{-6}\,\epsilon\text{eV}\left(2+0.0000156507\,\epsilon\text{eV}+4.21\times10^{-11}\,\epsilon\text{eV}^2+\right.\right.$$

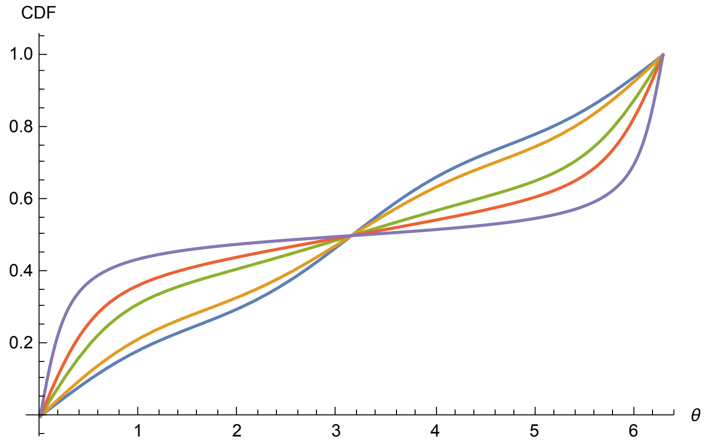
$$\left.\left.2.24624\times10^{-17}\,\epsilon\text{eV}^3\right)\text{Sin}[\theta]\right)\Big/\left(\left(1+3.91269\times10^{-6}\,\epsilon\text{eV}\right)^2\right.$$

$$\left.\left(-1-1.95634\times10^{-6}\,\epsilon\text{eV}+1.95634\times10^{-6}\,\epsilon\text{eV}\,\text{Cos}[\theta]\right)\right)\Bigg)\Bigg)\Big/$$

$$\left(-2+2\left(1+3.91269\times10^{-6}\,\epsilon\text{eV}\right)^{5/2}-0.0000195634\,\epsilon\text{eV}-4.59273\times10^{-11}\,\epsilon\text{eV}^2+\right.$$

$$\left.2.99499\times10^{-17}\,\epsilon\text{eV}^3+1.61129\times10^{-22}\,\epsilon\text{eV}^4\right)\Bigg)$$

Out[25]=



In[26]:=
```
cdf = dσdΩcdf / dσdΩcdfπ // Simplify;
getθ[r_, ϵeVr_] :=
 NSolve[{(cdf /. {ϵeV → ϵeVr}) - r == 0, 0 ≤ θ ≤ π}, θ, Reals][[1, 1, 2]]
getθ[0.001, 10 × 10^6]
```

Out[28]= 0.000333345

In[29]:=
```
Nsmpl = 300;
rdist = RandomReal[{0, 1}, {Nsmpl}];
θdist = ParallelTable[getθ[rdist[[i]], 10 × 10^6], {i, 1, Length[rdist]}];
```

In[32]:=
```
Show[{Histogram[θdist, 40], Plot[{9 × 10^30 dσdΩ /. {ϵeV → 10 × 10^6}},
   {θ, 0, π}, AxesLabel → {"θ", "dσdΩ"}, PlotRange → All]}]
```

Out[32]=