Project #10 : Propeller Performance Optimization

This project was inspired by one of my students from a couple of years ago.]. A hovercraft rotor system can a adjust three performance parameters while in flight. These parameters are:

- Rotor diameter D (meters), with capability $1 \leq D \leq 4$.
- Angle of attack α (degrees), with capability $0 \leq \alpha \leq 10$.
- Rotor speed n (10^3 RPS), with capability $1 \leq n \leq 7$.

These capabilities affect the efficiency characteristics of flight. In particular, we can monitor thrust (T) and torque (Q) as a function of these variables. The power use efficiency is then measured as $F = T^{3/2}/nDQ$. Our goal is to understand the optimal adjustment of the parameters which achieves an optimal efficiency at a particular value of thrust. That is, we have a set of optimization problems (one for each chosen value of thrust T_0):

$$\begin{aligned}
\max_{D,\alpha,n} \quad & F = \frac{T^{2/3}}{nDQ} \\
\text{s.t.} \quad & T = T_0 \\
& 1 \leq D \leq 4 \\
& 0 \leq \alpha \leq 10 \\
& 1 \leq n \leq 7 \\
& \text{where } T = T(D,\alpha,n) \text{ and } Q = Q(D,\alpha,n)
\end{aligned}$$

However, the thrust and torque functions are not known analytically. Instead we have experimental data in which, for various configurations, the thrust and torque have been measured. But before we tackle that issue, we can make a simple change in the optimization problem by noting that, for constant $T = T_0$, we have the equivalent problem:

$$\begin{aligned}
\min_{D,\alpha,n} \quad & z = nDQ(D,\alpha,n) \\
\text{s.t.} \quad & T(D,\alpha,n) = T_0 \\
& 1 \leq D \leq 4 \\
& 0 \leq \alpha \leq 10 \\
& 1 \leq n \leq 7
\end{aligned}$$

Creating Thrust and Torque Functions

Now, suppose we have the experimental data set $\{(D_k, \alpha_k, n_k, T_k, Q_k)\}$, $k = 1, 2, \ldots, m$. For each triplet of parameters (D_k, α_k, n_k), the thrust and torque (T_k, Q_k) were measured. We can use this data to approximate T and Q as smooth functions. Our thrust function should satisfy $T(D_k, \alpha_k, n_k) \approx T_k$ for all data m data vectors.

We consider a thrust function of the form

$$T(D,\alpha,n) = \sum_{j=1}^{p} a_j \phi_j(D,\alpha,n)$$

where the ϕ_j are fixed nonparametric functions and $p < m$ to help avoid overfitting. The a_j are determined by the unconstrained optimization problem

$$\min_{a} \quad f(a) = \sum_{k=1}^{m} \left(\sum_{j=1}^{p} a_j \phi_j(D_k,\alpha_k,n_k) - T_k \right)^2$$

This convex quadratic problem has the stationary point solution (a^*) given by

$$\begin{bmatrix} \Phi_{11}(x_k) & \Phi_{12}(x_k) & \cdots & \Phi_{1p}(x_k) \\ \Phi_{21}(x_k) & \Phi_{22}(x_k) & \cdots & \Phi_{2p}(x_k) \\ \vdots & \vdots & \ddots & \vdots \\ \Phi_{p1}(x_k) & \Phi_{p2}(x_k) & \cdots & \Phi_{pp}(x_k) \end{bmatrix} \begin{bmatrix} a_1^* \\ a_2^* \\ \vdots \\ a_p^* \end{bmatrix} = \begin{bmatrix} \Theta_1(x_k) \\ \Theta_2(x_k) \\ \vdots \\ \Theta_p(x_k) \end{bmatrix}.$$

Here

$$\Phi_{ij}(x_k) := \sum_{k=1}^{m} \phi_i(D_k,\alpha_k,n_k)\phi_j(D_k,\alpha_k,n_k),$$

$$\Theta_j(x_k) := \sum_{k=1}^{m} T_k \phi_j(D_k,\alpha_k,n_k).$$

And, finally we have

$$T(D,\alpha,n) = \sum_{j=1}^{p} a_j^* \phi_j(D,\alpha,n).$$

The functions ϕ_j are arbitrary, though we need to consider nonparametric twice-continuously differentiable functions. For example, a general affine fit to the data is given by $T(D,\alpha,n) = a_1 + a_2 D + a_3\alpha + a_4 n$, and we can make the definitions $\phi_1 = 1$, $\phi_2 = D$, $\phi_3 = \alpha$, $\phi_4 = n$.

A similar formulation leads to a fit function for the torque Q, possibly with different functions $\hat{\phi}_j$:

$$Q(D,\alpha,n) = \sum_{j=1}^{r} b_j^* \hat{\phi}_j(D,\alpha,n).$$

The Optimization Problem

We now have a complete problem formulation with explicit constraint functions:

$$
\begin{aligned}
\min_{x,y} \quad & f(x) = x_3 x_1 Q(x_1, x_2, x_3) \\
\text{s.t.} \quad & c_1(x,y) = x_1 + y_1^2 - 4 = 0 \\
& c_2(x,y) = x_2 + y_2^2 - 10 = 0 \\
& c_3(x,y) = x_3 + y_3^2 - 7 = 0 \\
& c_4(x,y) = -x_1 + y_4^2 + 1 = 0 \\
& c_5(x,y) = -x_2 + y_5^2 = 0 \\
& c_6(x,y) = -x_3 + y_6^2 + 1 = 0 \\
& c_7(x,y) = T(x_1, x_2, x_3) - T_0 = 0 \\
& x = (D, \alpha, n) \in \mathbb{R}^3 \\
& y \in \mathbb{R}^6
\end{aligned}
$$

The Augmented Lagrangian Formulation

Using the Augmented Lagrangian approach, we introduce quadratic penalty terms weighted by $mu > 0$ and Lagrangian terms. We then have the unconstrained subproblem formulation:

$$
\begin{aligned}
\min_{x,y,\lambda} \quad & L(x,y,\lambda) = f(x) - c(x,y)\lambda + \tfrac{\mu}{2} c(x,y)^{\mathsf{T}} c(x,y) \\
\text{s.t.} \quad & x \in \mathbb{R}^3, \quad y \in \mathbb{R}^6
\end{aligned}
$$

where $c(x,y) \in \mathbb{R}^7$ is the vector of constraint evaluations. We use unconstrained gradient based methods to provide updates in (x,y) and the fixed update $\lambda^{k+1} = \lambda^k - \mu c(x,y)$ to provide effective Lagrange multiplier improvement.

The Task

Solve the propeller optimization problem using the Augmented Lagrangian Method. Determine the figure of merit $F(T_0)$ as well as the corresponding optimal parameters $D^*(T_0)$, $\alpha^*(T_0)$, $n^*(T_0)$ for a range of thrust values T_0. The propeller data is provided on Canvas.