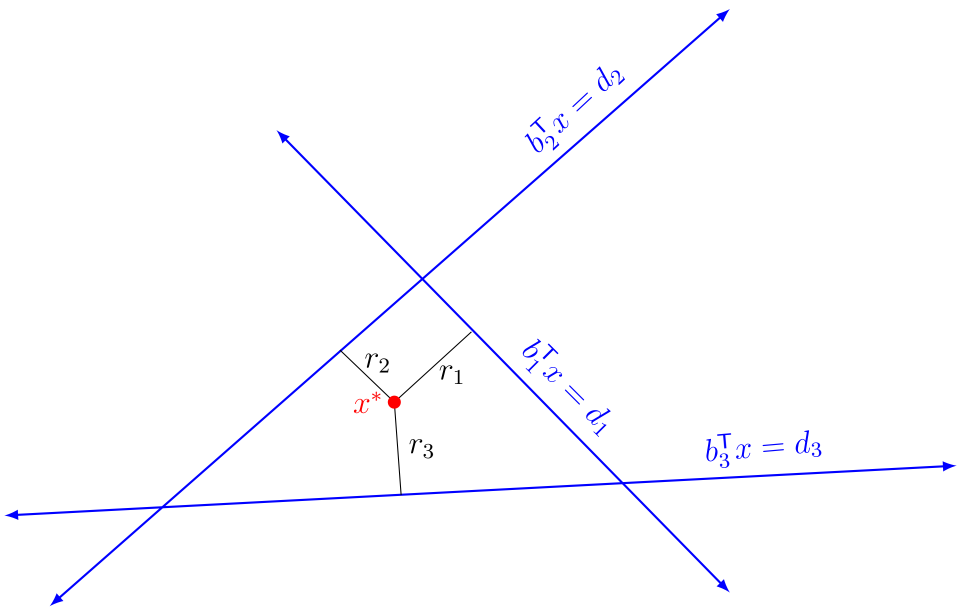
Project #8 : Finding Central Locations

Consider finding an approximate solution to a system of linear equations which may not have a solution. We can think of this problem geometrically as finding a point x^* that is close to each of the equality-defined hyperplanes. This idea is illustrated in $\mathbb{R}^2$ in the figure below.



Three linear equalities are given as $b_k^\intercal x = d_k$ for $k = 1, 2, 3$. This system of equations has no solution. However, we might consider an approximate solution x^* which minimizes the residual distances r_1, r_2, r_3. We will consider three possible scenarios for determining an appropriate x^*. We consider the general case of m linear equalities in n variables.

Method 1: Pseudo-Inverse. In linear algebra, we considered least squares solutions to systems of linear equations. The general concept is described most simply using singular value decomposition (SVD). Given a system $Bx = d$, we use the SVD of B to compute the pseudo-inverse matrix P of B and solution x^* as follows.

$$B = U\Sigma V^\intercal = \widetilde{U}\widetilde{\Sigma}\widetilde{V}^\intercal, \quad P = \widetilde{V}\widetilde{\Sigma}^{-1}\widetilde{U}^\intercal, \quad x^* = Pd.$$

Here, U, Σ, V are the SVD matrices, and the tilde notation indicates submatrices containing row/column entries corresponding to positive singular values. In practice, we do not compute and use P directly, instead allowing software to utilize efficient numerical methods. For example, in Matlab we use

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>> xstar = B\d;
```

and in python we can use

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>>> xstar = scipy.linalg.solve(B,d)
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Method 2: Minimizing Squared Constraint Violation. We can model the problem as a quadratic program that seeks to minimize the sum of the squared residuals. Let $r \in \mathbb{R}^m$ be the vector of residual values and $B \in \mathbb{R}^{m \times n}$ be the matrix with rows b_k^T.

$$\begin{aligned} \min \quad & f(x,r) = \tfrac{1}{2}\|r\|^2 \\ \text{s.t.} \quad & r = Bx - d \\ & x \in \mathbb{R}^n \\ & r \in \mathbb{R}^m \end{aligned}$$

We can write this problem in standard quadratic form as

$$\begin{aligned} \min \quad & \tfrac{1}{2} w^\mathsf{T} G w \\ \text{s.t.} \quad & Aw = b \\ & w \in \mathbb{R}^{n+m} \end{aligned}$$

where

$$w = \begin{bmatrix} x \\ r \end{bmatrix}, \quad G = \begin{bmatrix} 0_{n\times n} & 0_{m\times n} \\ 0_{n\times m} & I_m \end{bmatrix}, \quad A = \begin{bmatrix} B & -I_m \end{bmatrix}, \quad b = d.$$

Method 3: Minimizing Squared Distance Violation. If instead we wish to minimize the actual distances in $\mathbb{R}^n$ instead of the constraint violations, we need to normalize the constraints. In particular, let $\hat{B}$ be the matrix with rows $\hat{b}_k^\mathsf{T} = b_k^\mathsf{T}/\|b_k\|$ and $\hat{d}$ the vector with entries $\hat{d}_k = d_k/\|b_k\|$. Notice that both changes are with respect to the normalizing constants $\|b_k\|$. Now we have a QP identical with that of Method 2, but with the new matrix $\hat{B}$ and new vector $\hat{d}$.

Task 1. Construct optimization code that will solve equality constrained quadratic program (QP) tasks as outlined in class. As usual, keep your code general so that its structure and functionality is independent of the actual problem description. This code should be separate from the unconstrained code which we developed last semester and separate from the derivative-free methods code you have developed earlier this semester. Your code should solve the problem

$$\begin{aligned} \min \quad & f(w) = \tfrac{1}{2} w^\mathsf{T} G w + w^\mathsf{T} c \\ \text{s.t.} \quad & Aw = b \\ & w \in \mathbb{R}^n. \end{aligned}$$

I encourage you to use a similar structure to your previous code. Feel free to consider the structure and functionality of my unconstrained optimization code posted on Canvas.

Task 2. Solve the following problem using all three Methods. Write a short report outlining the task and results. Discuss your results, providing some justification that your results are correct.

Example Problem. Four lookout towers in the national forest have location coordinates $(a) = (10, 12)$, $(b) = (3, 9)$, $(c) = (2, 3)$ and $(d) = (8, 1)$, respectively. The first coordinates (x, y) indicate locate to the east and north, respectively, from a reference ranger station at coordinates $(0, 0)$. The observers at each tower have spotted a fire in the general easterly direction. The observation angles from each tower to the fire are $\theta_a = 95°$, $\theta_b = 84°$, $\theta_c = 75°$ and $\theta_d = 63°$. Angles are reported in degrees east of north, so $0°$ corresponds to north, $45°$ corresponds to north-east, $90°$ corresponds to east, etc. What is the most likely coordinate location of the fire?