See chapter 9 in Regression and Other Stories.

```
html"""
<style>
    main {
        margin: 0 auto;
        max-width: 2000px;
        padding-left: max(160px, 10%);
        padding-right: max(160px, 10%);
    }
</style>
"""

```

```
using Pkg ✓ , DrWatson ✓
```

```
begin
    using GLM ✓

    # Specific to ROSStanPluto
    using StanSample ✓

    # Graphics related
    using GLMakie ✓

    # Common data files and functions
    using RegressionAndOtherStories ✓
end
```

```
Replacing docs for `RegressionAndOtherStories.
rames.DataFrame, AbstractString}` in module `R
`
```

9.1 Propagating uncertainty in inference using posterior simulations.

hibbs =

	year	growth	vote	inc_party_candidate
1	1952	2.4	44.6	"Stevenson"
2	1956	2.89	57.76	"Eisenhower"
3	1960	0.85	49.91	"Nixon"
4	1964	4.21	61.34	"Johnson"
5	1968	3.02	49.6	"Humphrey"
6	1972	3.62	61.79	"Nixon"
7	1976	1.08	48.95	"Ford"
8	1980	-0.39	44.7	"Carter"
9	1984	3.86	59.17	"Reagan"
10	1988	2.27	53.94	"Bush, Sr."
⋮ more				
16	2012	0.95	52.0	"Obama"

```
hibbs =
CSV.read(ros_datadir("ElectionsEconomy",
"hibbs.csv"), DataFrame)
```

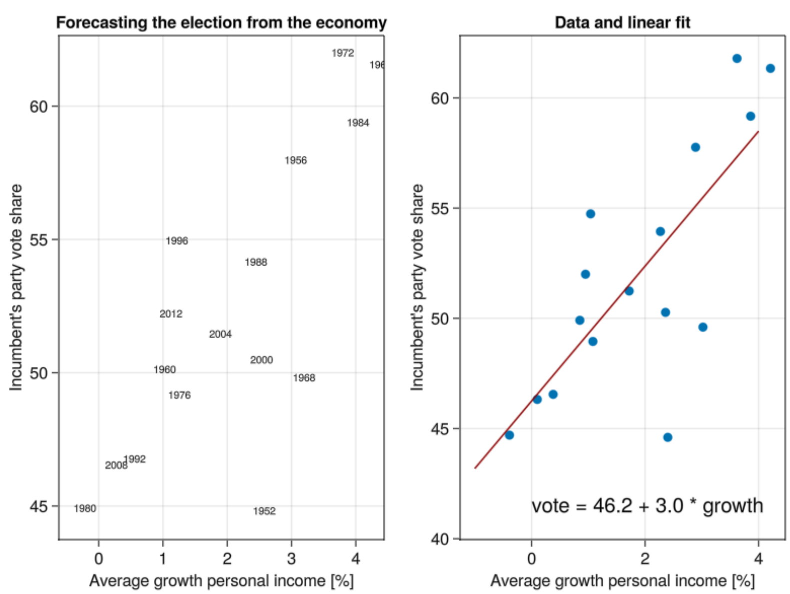
```
hibbs_lm =
StatsModels.TableRegressionModel{LinearModel{Gl

vote ~ 1 + growth

Coefficients:
```

	Coef.	Std. Error	t	Pr(>|
(Intercept)	46.2476	1.62193	28.51	<1e
growth	3.06053	0.696274	4.40	0.0

```
hibbs_lm = lm(@formula(vote ~ growth),
hibbs)
```



```julia
let
    fig = Figure()
    hibbs.label = string.(hibbs.year)
    xlabel = "Average growth personal income [%]"
    ylabel = "Incumbent's party vote share"
    let
        title = "Forecasting the election from the economy"
        ax = Axis(fig[1, 1]; title, xlabel, ylabel)
        for (ind, yr) in enumerate(hibbs.year)
            annotations!("$(yr)"; position=(hibbs.growth[ind], hibbs.vote[ind]), textsize=10)
        end
    end
    let
        x = LinRange(-1, 4, 100)
        title = "Data and linear fit"
        ax = Axis(fig[1, 2]; title, xlabel, ylabel)
        scatter!(hibbs.growth, hibbs.vote)
        lines!(x, coef(hibbs_lm)[1] .+ coef(hibbs_lm)[2] .* x; color=:darkred)
        annotations!("vote = 46.2 + 3.0 * growth"; position=(0, 41))
    end
    fig
end
```

```
stan7_1 = "
data {
    int<lower=1> N;      // total number
    of observations
    vector[N] growth;    // Independent
    variable: growth
    vector[N] vote;      // Dependent
    variable: votes
}
parameters {
    real b;              // Coefficient
    independent variable
    real a;              // Intercept
    real<lower=0> sigma; // dispersion
    parameter
}
model {
    vector[N] mu;

    // priors including constants
    a ~ normal(50, 20);
    b ~ normal(2, 10);
    sigma ~ exponential(1);

    mu = a + b * growth;

    // likelihood including constants
    vote ~ normal(mu, sigma);
}";
```

	parameters	mean	mcse	std	
1	"b"	3.04897	0.0162602	0.654979	1
2	"a"	46.2878	0.0365967	1.5194	4
3	"sigma"	3.57757	0.0159071	0.613961	2

```
let
    data = (N=nrow(hibbs), vote=hibbs.vote,
    growth=hibbs.growth)
    global m7_1s = SampleModel("hibbs",
    stan7_1)
    global rc7_1s = stan_sample(m7_1s;
    data)
    success(rc7_1s) && describe(m7_1s)
end
```

```
/var/folders/l7/pr04h0650q5dvqttnvs8s2c00000gr
updated.
```

1489.69

```
let
    ss7_1s = describe(m7_1s)
    ss7_1s[:sigma, :ess]
end
```

1489.69

```
let
    ss7_1s = describe(m7_1s; showall=true)
    ss7_1s[:sigma, :ess]
end
```

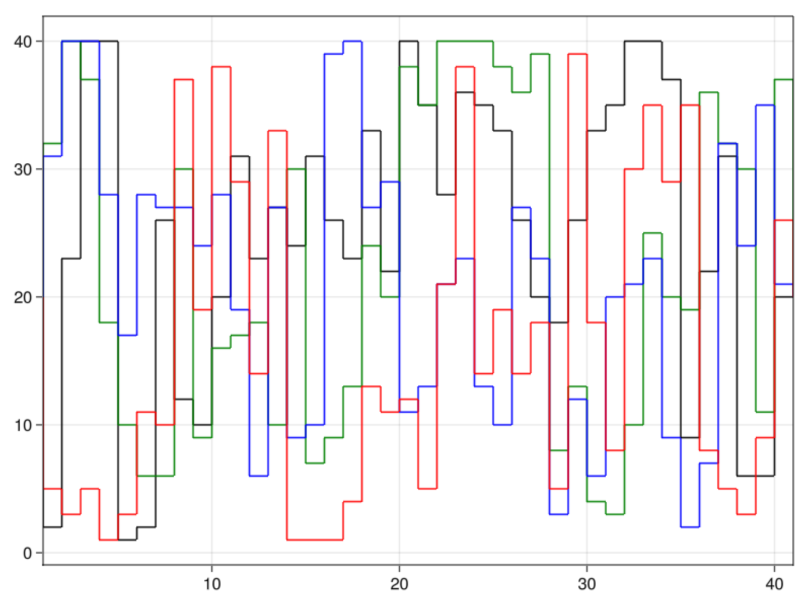
	parameters	mean	mcse	
1	"lp__"	-31.9128	0.0383475	1.3
2	"accept_stat__"	0.917887	0.00977819	0.1
3	"stepsize__"	0.404689	0.0258293	0.0
4	"treedepth__"	2.666	0.0150726	0.6
5	"n_leapfrog__"	7.88	0.205667	4.1
6	"divergent__"	0.0	NaN	0.0
7	"energy__"	33.3892	0.0489152	1.7
8	"b"	3.04897	0.0162602	0.6
9	"a"	46.2878	0.0365967	1.5
10	"sigma"	3.57757	0.0159071	0.6

```
describe(m7_1s; showall=true)
```

post7_1s =

	b	a	sigma
1	1.90114	48.6989	2.77909
2	3.10169	47.0837	3.10472
3	4.40427	44.2424	4.01044
4	4.60825	45.0937	4.07456
5	1.63412	47.5276	3.7154
6	1.74813	47.3943	3.79968
7	3.18921	45.6586	3.21523
8	2.71001	47.9793	4.27731
9	2.62185	47.9963	4.25759
10	2.98121	46.9598	4.32174
⋮ more			
4000	2.04827	49.2821	3.21009

```
post7_1s = success(rc7_1s) &&
read_samples(m7_1s, :dataframe)
```




```
trankplot(post7_1s, "b")
```

ms7_1s =

	parameters	median	mad_sd	mean	st
1	"a"	46.303	1.504	46.288	1.51
2	"b"	3.036	0.622	3.049	0.65
3	"sigma"	3.508	0.577	3.578	0.61

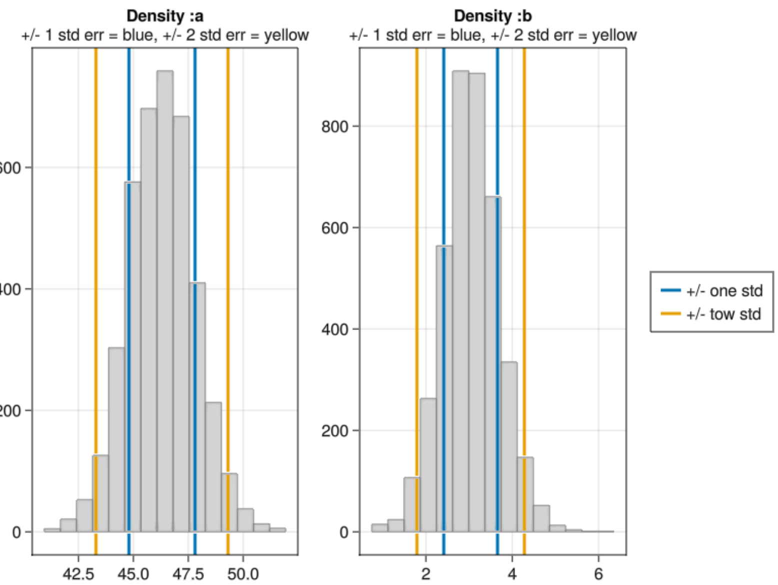
```
ms7_1s = model_summary(post7_1s, [:a, :b,
:sigma])
```

```
sims = 4000×3 Matrix{Float64}:
        1.90114  48.6989  2.77909
        3.10169  47.0837  3.10472
        4.40427  44.2424  4.01044
        4.60825  45.0937  4.07456
        1.63412  47.5276  3.7154
        1.74813  47.3943  3.79968
        3.18921  45.6586  3.21523
        ⋮
        3.0609   46.2324  3.51815
        3.5505   45.7074  3.81199
        2.95468  45.8916  3.8862
        2.86051  46.8871  5.21757
        1.58464  48.6092  3.16146
        2.04827  49.2821  3.21009
```

```
sims = Array(post7_1s)
```

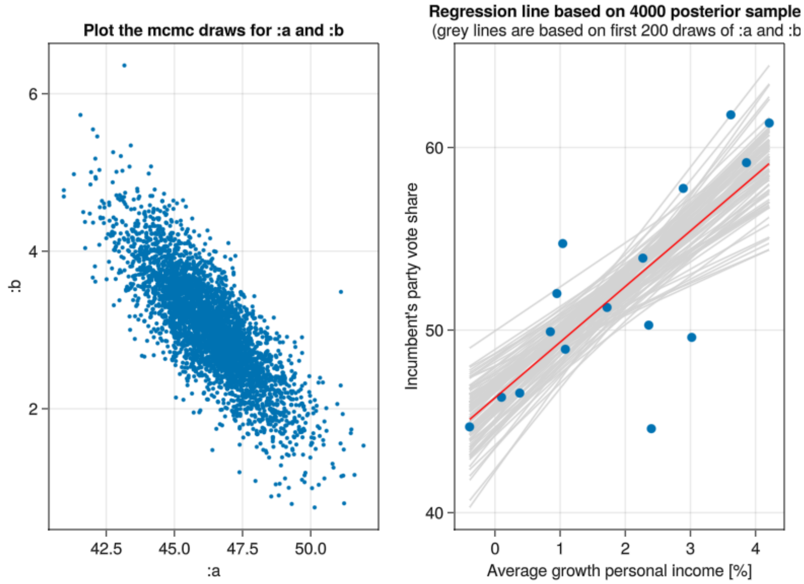
```
1×3 Matrix{Float64}:
 3.03601  46.3033  3.50782
```

```
median(sims; dims=1)
```


Density :a
+/- 1 std err = blue, +/- 2 std err = yellow
Density :b
+/- 1 std err = blue, +/- 2 std err = yellow
+/- one std
+/- tow std
0
200
400
600
800
42.5
45.0
47.5
50.0
2
4
6

```
let
    f = Figure()
    ax = Axis(f[1, 1]; title="Density :a",
    subtitle="+/- 1 std err = blue, +/- 2
    std err = yellow")
    hist!(post7_1s.a; bins=15, color =
    :lightgrey, strokewidth = 1,
    strokecolor = :grey)
    one = vlines!([ms7_1s[:a, :median] -
    ms7_1s[:a, :mad_sd], ms7_1s[:a,
    :median] + ms7_1s[:a, :mad_sd]];
    linewidth=3)
    two = vlines!([ms7_1s[:a, :median] -
    2ms7_1s[:a, :mad_sd], ms7_1s[:a,
    :median] + 2ms7_1s[:a, :mad_sd]];
    linewidth=3)

    ax = Axis(f[1, 2]; title="Density :b",
    subtitle="+/- 1 std err = blue, +/- 2
    std err = yellow")
    hist!(post7_1s.b; bins=15, color =
    :lightgrey, strokewidth = 1,
    strokecolor = :grey)
    vlines!([ms7_1s[:b, :median] -
    ms7_1s[:b, :mad_sd], ms7_1s[:b,
    :median] + ms7_1s[:b, :mad_sd]];
    linewidth=3)
    vlines!([ms7_1s[:b, :median] -
    2ms7_1s[:b, :mad_sd], ms7_1s[:b,
    :median] + 2ms7_1s[:b, :mad_sd]];
    linewidth=3)
    Legend(f[1, 3], [one, two], ["+/- one
    std", "+/- tow std"])
    f
end
```


Plot the mcmc draws for :a and :b
:b
:a
6
4
2
42.5
45.0
47.5
50.0
Regression line based on 4000 posterior sample
(grey lines are based on first 200 draws of :a and :b
Incumbent's party vote share
Average growth personal income [%]
60
50
40
0
1
2
3
4

```
let
    growth_range =
    LinRange(minimum(hibbs.growth),
    maximum(hibbs.growth), 200)
    votes = mean.(link(post7_1s, (r,x) ->
    r.a + x * r.b, growth_range))

    xlabel = "Average growth personal
    income [%]"
    ylabel = "Incumbent's party vote share"

    fig = Figure()

    ax = Axis(fig[1, 1]; title="Plot the
    mcmc draws for :a and :b",
    xlabel=":a", ylabel=":b")
    scatter!(post7_1s.a, post7_1s.b;
    markersize=4)

    xlabel = "Average growth personal
    income [%]"
    ylabel="Incumbent's party vote share"
    ax = Axis(fig[1, 2]; title="Regression
    line based on 4000 posterior samples",
        subtitle = "(grey lines are based
        on first 200 draws of :a and :b)",
        xlabel, ylabel)
    for i in 1:100
        lines!(growth_range, post7_1s.a[i]
        .+ post7_1s.b[i] .* growth_range,
        color = :lightgrey)
    end
    scatter!(hibbs.growth, hibbs.vote)
    lines!(growth_range, votes, color =
    :red)
    fig
end
```

9.2 Prediction and uncertainty.

	x	y
1	-2.0	50
2	-1.0	44
3	0.0	50
4	1.0	47
5	2.0	56

```
let
    x = LinRange(-2, 2, 5)
    y = [50, 44, 50, 47, 56]
    global sexratio = DataFrame(x = x, y =
    y)
end
```

```
stan9_1 = "
data {
    int<lower=1> N; // total number of
    observations
    vector[N] x;    // Independent
    variable: growth
    vector[N] y;    // Dependent variable:
    votes
}
parameters {
    real b;              // Coefficient
    independent variable
    real a;              // Intercept
    real<lower=0> sigma; // dispersion
    parameter
}
model {
    vector[N] mu;

    // priors including constants
    a ~ normal(50, 5);
    b ~ normal(0, 5);
    sigma ~ uniform(0, 10);

    mu = a + b * x;

    // likelihood including constants
    y ~ normal(mu, sigma);
}";
```

	parameters	mean	mcse	std	
1	"b"	1.25238	0.0676678	1.74543	-1
2	"a"	49.4801	0.0416165	2.41213	45
3	"sigma"	5.72269	0.0849459	1.88623	3.

```
let
    data = (N = nrow(sexratio), x =
    sexratio.x, y = sexratio.y)
    global m9_1s = SampleModel("m9_1s",
    stan9_1)
    global rc9_1s = stan_sample(m9_1s;
    data)
    success(rc9_1s) && describe(m9_1s)
end
```

```
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updated.
```
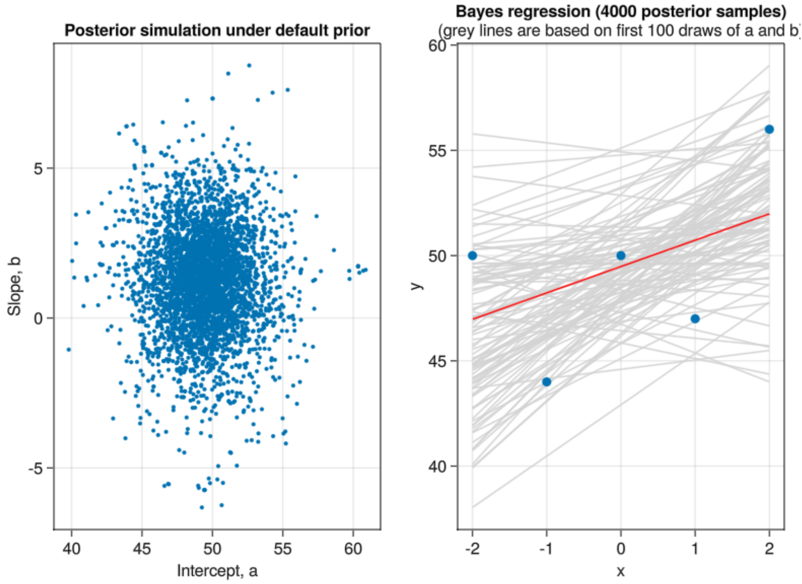

	parameters	median	mad_sd	mean	std
1	"a"	49.474	2.147	49.48	2.41
2	"b"	1.304	1.572	1.252	1.74
3	"sigma"	5.514	2.058	5.723	1.88

```
if success(rc9_1s)
    post9_1s = read_samples(m9_1s,
    :dataframe)
    sm9_1s = model_summary(post9_1s, [:a,
    :b, :sigma])
end
```


Posterior simulation under default prior
Slope, b
Intercept, a
5
0
-5
40
45
50
55
60
Bayes regression (4000 posterior samples)
(grey lines are based on first 100 draws of a and b)
y
x
60
55
50
45
40
-2
-1
0
1
2

```
let
    x_range = LinRange(minimum(sexratio.x),
    maximum(sexratio.x), 200)
    y = mean.(link(post9_1s, (r,x) -> r.a
    + x * r.b, x_range))

    xlabel = "x"
    ylabel = "y"

    fig = Figure()

    ax = Axis(fig[1, 1]; title="Posterior
    simulation under default prior",
    xlabel="Intercept, a", ylabel="Slope,
    b")
    scatter!(post9_1s.a, post9_1s.b;
    markersize=4)

    ax = Axis(fig[1, 2]; title="Bayes
    regression (4000 posterior samples)",
        subtitle = "(grey lines are based
        on first 100 draws of a and b)",
        xlabel, ylabel)
    for i in 1:100
        lines!(x_range, post9_1s.a[i] .+
        post9_1s.b[i] .* x_range, color =
        :lightgrey)
    end
    scatter!(sexratio.x, sexratio.y)
    lines!(x_range, y, color = :red)
    fig
end
```

```
stan9_2 = "
data {
    int<lower=1> N; // total number of
    observations
    vector[N] x;    // Independent
    variable: growth
    vector[N] y;    // Dependent variable:
    votes
}
parameters {
    real b;               // Coefficient
    independent variable
    real a;               // Intercept
    real<lower=0> sigma; // dispersion
    parameter
}
model {
    vector[N] mu;

    // priors including constants
    a ~ normal(48.8, 0.2);
    b ~ normal(0, 0.2);
    sigma ~ uniform(0, 10);

    mu = a + b * x;

    // likelihood including constants
    y ~ normal(mu, sigma);
}";
```

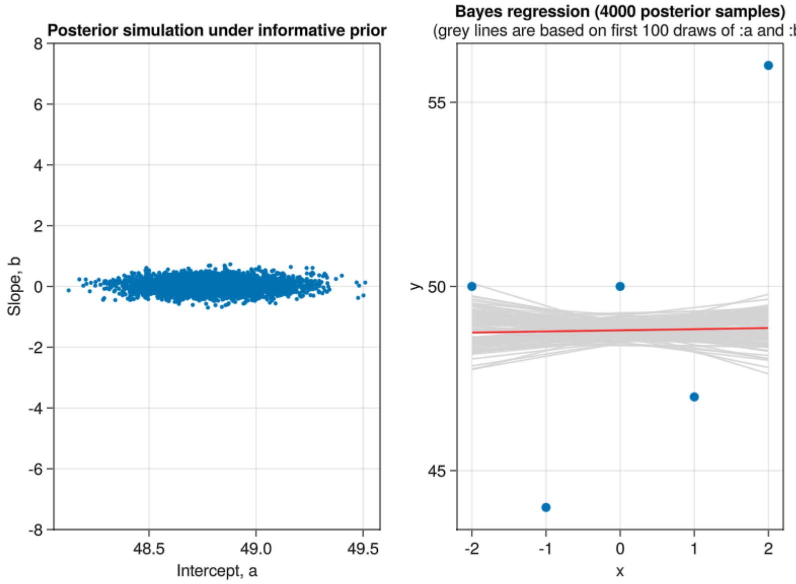
	parameters	mean	mcse	std
1	"b"	0.0308248	0.0034984	0.19705
2	"a"	48.8087	0.00329835	0.19532
3	"sigma"	5.06993	0.0303117	1.66184

```
let
    data = (N = nrow(sexratio), x =
    sexratio.x, y = sexratio.y)
    global m9_2s = SampleModel("m9_2s",
    stan9_2)
    global rc9_2s = stan_sample(m9_2s;
    data)
    success(rc9_2s) && describe(m9_2s)
end
```

```
/var/folders/l7/pr04h0650q5dvqttnvs8s2c00000gr
updated.
```

	parameters	median	mad_sd	mean	st
1	"a"	48.809	0.195	48.809	0.19
2	"b"	0.03	0.194	0.031	0.19
3	"sigma"	4.758	1.573	5.07	1.66

```
if success(rc9_2s)
    post9_2s = read_samples(m9_2s,
    :dataframe)
    sm9_2s = model_summary(post9_2s, [:a,
    :b, :sigma])
end
```


Posterior simulation under informative prior
8
6
4
2
0
-2
-4
-6
-8
Slope, b
48.5
49.0
49.5
Intercept, a
Bayes regression (4000 posterior samples)
(grey lines are based on first 100 draws of :a and :b
55
50
45
y
-2
-1
0
1
2
x

```
let
    x_range = LinRange(minimum(sexratio.x),
    maximum(sexratio.x), 200)
    y = mean.(link(post9_2s, (r,x) -> r.a
    + x * r.b, x_range))

    xlabel = "x"
    ylabel = "y"

    fig = Figure()

    ax = Axis(fig[1, 1]; title="Posterior
    simulation under informative prior",
    xlabel="Intercept, a", ylabel="Slope,
    b")
    ylims!(ax, -8, 8)
    scatter!(post9_2s.a, post9_2s.b;
    markersize=4)

    ax = Axis(fig[1, 2]; title="Bayes
    regression (4000 posterior samples)",
        subtitle = "(grey lines are based
        on first 100 draws of :a and :b)",
        xlabel, ylabel)
    for i in 1:100
        lines!(x_range, post9_2s.a[i] .+
        post9_2s.b[i] .* x_range, color =
        :lightgrey)
    end
    scatter!(sexratio.x, sexratio.y)
    lines!(x_range, y, color = :red)
    fig
end
```

9.3 Prior information and Bayesian synthesis.

Prior based on a previously-fitted model using economic and political condition.

```
begin
    theta_hat_prior = 0.524
    se_prior = 0.041
end;
```

Survey of 400 people, of whom 190 say they will vote for the Democratic candidate.

```
begin
    n = 400
    y = 190
end;
```

Data estimate.

```
theta_hat_data = 0.475
```

```
theta_hat_data = y/n
```

```
se_data = 0.02496873044429772
```

```
se_data = √((y/n)*(1-y/n)/n)
```

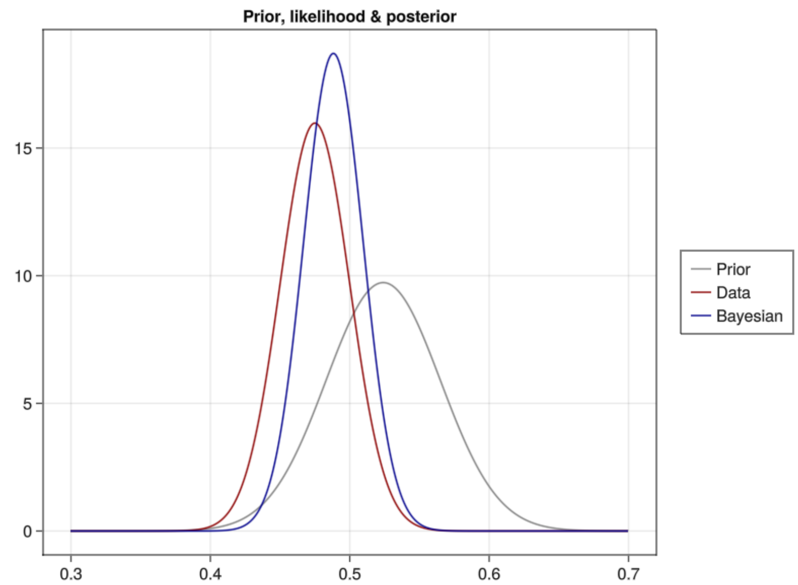
Bayes estimate.

```
theta_hat_bayes = 0.48825635323153693
```

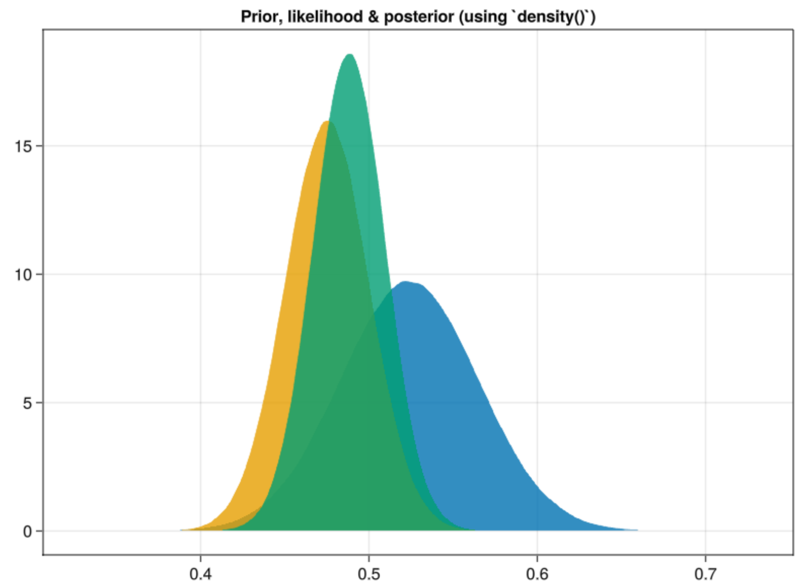
```
theta_hat_bayes =
(theta_hat_prior/se_prior^2 +
    theta_hat_data/se_data^2)
    /(1/se_prior^2 + 1/se_data^2)
```

```
se_bayes = 0.02132543263776925
```

```
se_bayes = sqrt(1/(1/se_prior^2 +
1/se_data^2))
```




```
let
    x = 0.3:0.001:0.7
    f = Figure()
    ax = Axis(f[1, 1], title="Prior, likelihood & posterior")
    prior = lines!(f[1, 1], x, pdf.(Normal(theta_hat_prior, se_prior), x), color=:gray)
    data = lines!(x, pdf.(Normal(theta_hat_data, se_data), x),color=:darkred)
    bayes = lines!(x, pdf.(Normal(theta_hat_bayes, se_bayes), x), color=:darkblue)
    Legend(f[1, 2], [prior, data, bayes], ["Prior", "Data", "Bayesian"])

    current_figure()
end
```




```julia
let
    f = Figure()
    ax = Axis(f[1, 1], title="Prior, likelihood & posterior (using `density()`)")
    density!(rand(Normal(theta_hat_prior, se_prior), Int(1e6)), lab="prior")
    density!(rand(Normal(theta_hat_data, se_data), Int(1e6)), lab="likelihood")
    density!(rand(Normal(theta_hat_bayes, se_bayes), Int(1e6)), lab="bayes")
    current_figure()
end
```

9.4 Example of Bayesian inference: beauty and sex ratio.