Dimension Theorem and its Applications

Textbook: 2.4

Theorem 2.3.17 (Dimension Theorem or Rank-Nullity Theorem)

For any linear transformation $V \xrightarrow{T} W$

$$\dim(V) = \dim(\ker(T)) + \dim(\operatorname{im}(T))$$

Remark

- $\dim(\operatorname{im}(T))$ is the same as the rank of $[T]_\alpha^\beta$ and by abuse of notation also referred to as $\operatorname{rank}(T)$.
- Some books refer to $\dim(\ker(T))$ as the *nullity* of T.

Proof. ■

Proposition 2.4.2

A linear transformation T is injective if and only if $\ker(T) = \{\vec{0}\}$

Proof. ■

Example

If a linear transformation $V \xrightarrow{T} W$ is injective, then the image

$$T(\vec{v}_1), \ldots, T(\vec{v}_k)$$

of a linearly independent family in V

$$\vec{v}_1, \ldots, \vec{v}_k$$

under T is linearly independent in W.

True or False

Let $V \xrightarrow{T} W$ be a linear transformation.

- ☐ If $\dim(V) > \dim(W)$, T has to be injective.
- ☐ Can $\mathcal{P}_n(\mathbb{R}) \xrightarrow{\frac{d}{dx}} \mathcal{P}_n(\mathbb{R})$ be injective?
- ☐ T is is surjective if and only if $\dim(\mathrm{im}(T)) = \dim(W)$.
- ☐ If T is an isomorphism, then $\dim(V) = \dim(W)$.

Corollary (2.4.4 & 2.4.5)

A linear transformation might be injective if and only if $\dim(V) \leq \dim(W)$.
Warning: But it doesn't have to be injective!!

Proposition (2.4.7)

A linear transformation $V \xrightarrow{T} W$ is surjective if and only if $\dim(\mathrm{im}(T)) = \dim(W)$

Proof. ■

Discussion

How do the dimensions of V and W obstruct whether or not $V \xrightarrow{T} W$ can be surjective? Explain your answer.
Hint: Think along the lines of Corollary 2.4.4

Proposition (2.4.10)

Let $\dim V = \dim W$. A linear transformation $V \xrightarrow{T} W$ is injective if and only if it is surjective.

Proof. ■

Example (2.4.21)

The Dimension Theorem is really nothing new.

Discussion

Consider the linear map

$$\mathcal{P}_n(\mathbb{R}) \to \mathbb{R}^2$$
$$p(x) \mapsto \begin{pmatrix} p(0) \\ p(5) \end{pmatrix}$$

Argue without computation whether this map is injective or surjective.

Homework

Read Propsition 2.4.11 including the proof and look again at Example 2.4.21 in the book.

Composition

Textbook: Section 2.5

Recall

The *composition* of two transformations $U \xrightarrow{S} V$ and $V \xrightarrow{T} W$ is the transformation

$$U \xrightarrow{T \circ S} W$$

deifned by

$$T \circ S(\vec{u}) = T(S(\vec{u}))$$

Proposition (2.5.1)

In the above setup, $T \circ S$ is linear if T and S are linear.

Proof. ∎

Examples

1.

2. Let $p \in \mathcal{P}_n(\mathbb{R})$ be a polynomial and consider the linear transformations

$$\mathbb{R} \xrightarrow{t_y} \mathbb{R}$$

$$t_y(x) = x - y$$

and

$$\mathcal{P}_n(\mathbb{R}) \xrightarrow{ev_x} \mathbb{R}$$

What is the composition

$$ev_x \circ t_y$$

3. What is the composition $\frac{d}{dx} \circ \frac{d}{dx}$ on $\mathcal{P}_n(\mathbb{R})$?

Discussion 2.5.6

Let $U \xrightarrow{S} V$ and $V \xrightarrow{T} W$ be two composable linear transformations. Can you argue why

1. $\ker(S) \subseteq \ker(T \circ S)$
2. $\operatorname{im}(T \circ S) \subseteq \operatorname{im}(T)$

Observation

Let $U \xrightarrow{S} V$ and $V \xrightarrow{T} W$ be two composable linear transformations. Fix bases α, β and ε of U, V and W respectively. How does the matrix of the composition $T \circ S$ relate to the matrices $[T]_{\beta}^{\varepsilon}$ and $[S]_{\alpha}^{\beta}$?

So the real question is what the composition of matrices should be.

Definition

Let A and B be matrices of size $m \times n$ and $n \times p$. The *product* of the matrices A and B is the matrix AB such that

$$AB\vec{x} = A \cdot (B\vec{x})$$

for all $\vec{x} \in \mathbb{R}^n$.

Proposition (2.5.9)

If A has entries $[a]_{ij}$ and B has entries $[b]_{ij}$, then the (i, j) entry of AB is

$$[AB]_{ij} = \sum_{k=0}^{n} a_{ik} b_{kj}$$

Proof. ■

Intuition

Example Compute the matrix product of ...

Discussion

If A and B are composable matrices, is the rank of A or the rank of AB (possibly) greater? Explain!

Discussion

Recall that the matrix representing $\mathcal{P}_n(\mathbb{R}) \xrightarrow{\frac{d}{dx}} \mathcal{P}_n(\mathbb{R})$ in the basis $\alpha = \{1, x, x^2\}$ is given by

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

Verify that A^2 is the matrix representing $\frac{d^2}{dx^2}$

Remark

Literally everything we know about linear transformations also holds for matrices.

1. $A(BC) = (AB)C$
2. $I \cdot A = AC$
3. $\cdots$

Discussion

Given two composable matrices, is

$$AB = BA$$

true?

The Inverse of a Linear Transformation

Textbook: Section 2.6

tbd