Bases and Dimension

Textbook: Section 1.6

Definition 1.6.1

A family of vectors $\mathcal{B}$ in a vector space V is called *a basis of* V if

1. $\mathcal{B}$ spans V
2. $\mathcal{B}$ is linearly independent

Examples

1. $\{\vec{e}_1, \ldots, \vec{e}_n\}$ is a basis of $\mathbb{R}^n$.
2. We have seen last week that $\{\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}-1\\1\end{pmatrix}\}$ is a basis of $\mathbb{R}^2$
3. Which of families of polynomials we have seen before is a basis of $\mathcal{P}_2(\mathbb{R})$? Can you write down a basis of $\mathcal{P}_2(\mathbb{R})$ that does not contain any monomials?

Theorem 1.6.3

A family of vectors $\mathcal{B}$ is a basis of V if and only if every vector $\vec{v} \in V$ can be written uniquely as a linear combination of vectors in $\mathcal{B}$.

Proof. ■

Remark

1. A vector space does not have just one unique basis as we can easily verify.

Theorem 1.6.6

Let V be a vector space with a finite spanning set. For every linearly independent family S in V, there is a basis $\mathcal{B}$ containing S.

Why do we need a finite spanning set for V? Some vector spaces, such as $\mathcal{P}(\mathbb{R})$ can not be spanned by finitely many polynomials. Can you show why?

Observe that a basis hits a sweet spot of a family that is not too large as that it would contain redundant vectors, but also not too small of a family that it couldn't span the vetcor space.
The *Extend* and *Reduce* theorems from last week give us the following:

A family of vectors that is linearly independent, but not a spanning set can be enlagred to a basis, a family that spans V, but is linearly dependent, can be cut down to form a basis.

Even though a basis is not unique to a vector space, we would like to extract an invariant, a label, something that characterizes the vectors space. This invariant is motivated by the Corollary following below.

Theorem 1.6.10

If V is spanned by a family S with m elements, then no linearly independent family R in V can have more than m elements.

Proof. ■

Corollary 1.6.11

Any two bases $\mathcal{B}$ and $\mathcal{B}'$ of V have the same number of elements

Definitions

1. If a vector space V has a finite basis, we say that V is *finite dimensional.*
2. For a finite dimensional vector space V, the *dimension* of V

$$\dim(V)$$

is the number of elements of a basis of V.

Discussion

What is the dimension of

$\dim(\mathbb{R}) =$

$\dim(\mathcal{P}_n(\mathbb{R})) =$

$\dim(\mathrm{Mat}_2(\mathbb{R})) =$

$\dim(\mathrm{Mat}_2^{\mathrm{sym}}(\mathbb{R})) =$

$\dim(\mathrm{Mat}_2^{\mathrm{anti}}(\mathbb{R})) =$

Remember that a matrix A is symmetric if $A^T = A$ and antisymmetric if $A^T = -A$.

Discussion

- Can you argue that if $U \subseteq V$ is a subspace then $\dim(U) \leq \dim(V)$?
- Is it on the contrary true that for every subspace $U \subseteq V$ $\dim(U) = \dim(V)$ implies $U = V$?

The following result is very useful when we are looking for a basis of space that we already know the dimension of.

Proposition

Let V be a vector space of dimension n with a family S containing n vectors, then the following are equivalent.

(a) S is a basis of V

(b) S is linearly independent

(c) S spans V

Since subspaces are vector spaces too, all constructions above apply, to subspaces.

Discussion

Let U and W be subspaces in V, can the sum $\dim(U) + \dim(W)$ be greater than $\dim(V)$?
Argue why it might be true or find a counter example.

Example (continued)

Consider again the subspaces of symmetric and anti symmetric 2×2 matrices, $\mathrm{Mat}_2^{\mathrm{sym}}(\mathbb{R})$ and $\mathrm{Mat}_2^{\mathrm{anti}}(\mathbb{R})$ respectively, in the vector space of all 2×2 matrices $\mathrm{Mat}_2(\mathbb{R})$. Compute the intersection $\mathrm{Mat}_2^{\mathrm{sym}}(\mathbb{R}) \cap \mathrm{Mat}_2^{\mathrm{anti}}(\mathbb{R})$.

As a final result, we will state without proof

Theorem 1.6.18

Let U and W be finite dimensional subspaces of a vector space V, then

$$\dim(U + W) = \dim(U) + \dim(W) - \dim(U \cap W)$$

Proof. Read the proof in the textbook (voluntarily). ■

Corollary

Every finite dimensional vector space has basis.

Textbook: Section 2.1

Linear Transformatoins

Definition 2.1.1

A function $V \xrightarrow{T} W$ between vector spaces $(V, +_V, \bullet_V)$ and $(W, +_W, \bullet_W)$ is called *linear* if

1. $T(\vec{u} +_V \vec{v}) = T(\vec{v}) +_W T(\vec{v})$ for all $\vec{u}, \vec{v} \in V$
2. $T(\alpha \bullet_V \vec{u}) = \alpha \bullet_W T(\vec{v})$ for all $\vec{u} \in V$ and $\alpha \in \mathbb{R}$.

Remark

- We usually use the expression *linear transformation* or short *transformation* and not linear function.
- Notice that the operations $+$ and $\cdot$ are exactly what distinguishes vector spaces from sets. The requirements for a function to be linear guarantee that it mediates between the operations on the domain and the target.

Examples

1. ...tbd

Discussion

Given a transformation $V \xrightarrow{T} W$, is it true that

$$T\left(\sum_{i=1}^{k} \alpha_i \vec{v}_i\right) = \sum_{i=1}^{k} \alpha_i T(\vec{v}_i) \qquad ?$$

Discussion

Which of the following functions are linear transformations?

1. The derivative of a polynomial $\mathcal{P}_n(\mathbb{R}) \to \mathcal{P}_n(\mathbb{R})$ $p(x) \mapsto \frac{d}{dx}p(x)$

Proposition

For any linear transformation $V \xrightarrow{T} W$, $T(\vec{0}_V) = \vec{0}_W$.

Proposition

A transformation $V \xrightarrow{T} W$ is uniquely specified by its values on elements of a basis on V.

Recall that a basis in a vector space enbales us to write every vector uniquely as a linear combination. Since a linear combination is determined by its coefficients, all we need to remember for a fixed basis is the n-tuple of coefficients. To formalize this, we define...

Definition

Let V be a vector space with a basis $\mathcal{B} = \{\vec{b}_1, \ldots, \vec{b}_n\}$ and for $\vec{v} \in V$

$$\vec{v} = v_1 \cdot \vec{b}_1 + \cdots + v_n \cdot \vec{b}_n$$

the unique representation in this basis. We call $[\vec{v}]_{\mathcal{B}} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$ the *coordinate vector* of $\vec{v}$ in the basis $\mathcal{B}$.

Proposition

Given an n-dimensional vector space V with a basis $\mathcal{B}$. The assignment of its coordinate vector $[\vec{v}]_{\mathcal{B}}$ to every vector $\vec{v} \in V$ is a linear transformation

$$V \xrightarrow{\gamma^{\mathcal{B}}} \mathbb{R}^n$$