Announcements for the test

1. Whatever Sean wrote on quercus.
2. Don't quote numbers of Theorems, but their content and why they apply in this case.
3. Famous theorems have names, for example *Extend-, Reduce-* and *Fundamental* Theorem.
4. The test covers everything we did from sections 1.1 - 1.6 and section 2.1, homework problems 1 to 4 and Assignments 1 & 2
5. Please read the *Term-Test-1 information document* about logistics well in advance!
6. *Don't* leave the upload to the last minute! Plan for some technical difficulties, they always occur! :(

Kernel and Image

Textbook: Section 2.3

Definition (2.3.1 & 2.3.10)

For a linear transformation $V \xrightarrow{T} W$, we define

1. the *preimage* $T^{-1}(S)$ of $S \subseteq W$ under T as all $\vec{v} \in V$ that map into S.
2. the *kernel* $\ker(T)$ of T as all $\vec{v} \in V$ that map to $\vec{0}$ under T,
3. the *image* $\operatorname{im}(T)$ of T as all $\vec{w} \in W$ such that $\vec{w} = T(\vec{v})$ for some $\vec{v} \in V$,

Example

- The kernel of $\mathcal{P}_n(\mathbb{R}) \xrightarrow{\frac{d}{dx}} \mathcal{P}_n(\mathbb{R})$ are all constant polynomials, while the image consists of polynomials of degree $n-1$.
- The kernel of the evaluation map $\mathcal{P}_n(\mathbb{R}) \xrightarrow{\text{ev}_2} \mathcal{P}_n(\mathbb{R})$ are all polynomials that have a root at $x = 2$. What is the image?
- What is the image of the linear transformation defined in example 2.2.2 $\mathbb{R}^2 \xrightarrow{T} \mathbb{R}^2$?

$$T(\vec{e}_1) = \vec{e}_1 + \vec{e}_2$$

$$T(\vec{e}_2) = \vec{2}e_1 - 2\vec{e}_2$$

Proposition 2.3.2 & 2.3.11

For every linear transformation $V \xrightarrow{T} W$

1. $\ker(T)$ is a subspace in V
2. $\mathrm{im}(T)$ is a subspace in W.

Proof. ■

Proposition 2.3.7

The subspace $\ker(T)$ is the solution space to the homogeneous system of $[T]_\alpha^\beta$.

Proof. ■

Example

Example of computation to find $\ker(T)$

Observation

The subspace $\operatorname{im}(T)$ is the space of all $\vec{b} \in \mathbb{R}^n$ such that the system $[T]^\beta_\alpha \vec{x} = \vec{b}$ has a solution.

Proposition 2.3.12

If $\{\vec{v}_1, \ldots, \vec{v}_k\}$ spans V, then $\{T(\vec{v}_1), \ldots, T(\vec{v}_k)\}$ spans $\operatorname{im}(T)$.

Proof. ■

Definition

For a matrix $A = [a_1, a_2, \ldots, a_m] \in \operatorname{Mat}_{n,m}(\mathbb{R})$ we denote the span of the columns of A by

$$\operatorname{col}(A) = \operatorname{span}\{a_1, \ldots, a_m\}$$

Proposition

For every linear transformation $V \xrightarrow{T} W$

$$\operatorname{im}(T) = \operatorname{col}([T]^\beta_\alpha)$$

Proof. ■

Example

Example computation to find $\mathrm{im}(T)$

Notice that the columns might not be independent, in which case the columns are a spanning set of the image, but not a basis.

Theorem

Given a linear transformation $V \xrightarrow{T} W$ with matrix $[T]_\alpha^\beta$ for some bases α and β. Let $R = \mathrm{RREF}([T]_\alpha^\beta)$ be the reduced row echolon form of $[T]_\alpha^\beta$.

Then if the leading 1s in are R lie in columns $j_1, j_2, \ldots, j_r$, the columns $j_1, j_2, \ldots, j_r$ of $[T]_\alpha^\beta$ are a basis for $\mathrm{col}([T]_\alpha^\beta)$

Proof. ■

Discussion

Suppose a linear transformation $V \xrightarrow{T} W$ is given in a some bases α and β by

$$[T]_\alpha^\beta = \begin{pmatrix} 1 & 2 & 0 & 1 \\ 1 & 2 & 1 & 0 \end{pmatrix}$$

Find a basis for $\mathrm{im}(T)$ and $\ker(T)$.

Dimension Theorem

Textbook: Section 2.3 & 2.4

Theorem 2.3.17 (Rank-Nullity)

For any linear transformation $V \xrightarrow{T} W$

$$\dim(V) = \dim(\ker(T)) + \dim(\operatorname{im}(T))$$

Remark

- $\dim(\operatorname{im}(T))$ is the same as the rank of $[T]_\alpha^\beta$ and by abuse of notation also referred to as $\operatorname{rank}(T)$.
- Some books refer to $\dim(\ker(T))$ as the *nullity* of T.

Proof. ■

Theorem

A linear transformation T is injective if and only if $\ker(T) = \{\vec{0}\}$

Proof. ■

True or False Let $V \xrightarrow{T} W$ be a linear transformation

- ☐ If T is an isomorphism, then $\dim(V) = \dim(W)$.
- ☐ If $\dim(V) > \dim(W)$, T has to be injective.
- ☐

Composition

Textbook: Section 2.5

Discussion

Without doing a lot of work, can you argue what the matrix representing the composition $F \circ T$ is assuming you know $[F]$ and $[T]$?

Review session

- Vector spaces
- subspaces
- Sum, intersection, direct sum
- Linear independence
- Spanning set
- Basis
- Dimension
- Linear Transformation