The Determinant

Textbook: Sections 3.1, 3.2 & 3.3

Observation (3.1.6)

When is a 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ invertible?

Definition

We define the determinant funtion $\mathrm{Mat}_2(\mathbb{R}) \xrightarrow{det} \mathbb{R}$ to be $det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$

Remark

1. As we observed above, a 2×2 matrix is invertible if and only if $det(A) \neq 0$.
2. It is helpful to think about $A = [\ \vec{x} \ \ \vec{y}\ ]$ to to consist of column vectors $\vec{x}, \vec{y} \in \mathbb{R}^2$.

Proposition (3.1.1)

The area of the parallelogram spanned by vectors $\vec{x}$ and $\vec{y}$ in $\mathbb{R}^2$ is equal to the determinant of the matrix $A = [\ \vec{x} \ \ \vec{y}\ ]$.

From this it is clear that the area is nonzero
if and only if the vectors spanning the parallelogram
are linearly independent, that is, not parallel.

Observation

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a 2×2 matrix and denote its columns as $\vec{x} = \begin{pmatrix} a \\ c \end{pmatrix}$ and $\vec{y} = \begin{pmatrix} b \\ d \end{pmatrix}$.

We can think of A as the matrix of a linear transformation

$$\mathbb{R}^2 \to \mathbb{R}^2$$

sending

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \mapsto \begin{pmatrix} a \\ c \end{pmatrix}$$
$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} b \\ d \end{pmatrix}$$

Discussion

Compute the determinant, draw the parallelogram and decide if the corresponding transformation is invertible for the following matrices.

1.
$$A = \begin{pmatrix} 1 & 2 \\ 3 & -4 \end{pmatrix}$$

2.
$$A = \begin{pmatrix} 4 & 2 \\ -2 & -1 \end{pmatrix}$$

Remark

1. Let again $A = [\ \vec{x}\ \ \vec{y}]$ be a 2×2 matrix with columns $\vec{x}$ and $\vec{y}$. The determinant function as a funtion of the columns $det(\vec{x}, \vec{y})$ has the following properties
   (a) linear in both arguments

   (b) alternating

   (c) normalized

2. The book calls this the area function $Area(\vec{x}, \vec{y})$ of the vectors spanning the parallelogram.

Proposition (3.1.4)

The determinant funtion is the unique funtion satisfying all properties above.

Proof. In the book ■

Discussion

Can you show that

$$det([\ \vec{x}\ \ \vec{y}\ ]) = 0$$

if $\{\vec{x}, \vec{y}\}$ is linearly dependent by only using the properties above?

Determinant of n x n matrices

Textbook: Sections 3.2

Goal

Generalize the det function to all $n \times n$ matrices as a function of n vectors $\vec{x}_1, \ldots, \vec{x}_n$ with an equivalent set of properties.

1. multilinear

2. alternating

3. normalized

Definition (3.2.4)

Let $A \in \mathrm{Mat}_n(\mathbb{R})$ be a $n \times n$ matrix. Define the *ij-minor* A_{ij} to be the $(n-1) \times (n-1)$ matrix obtained by deleting the i-th row and jth column.

Discussion

Compute for the matrix $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$ the minor A_{23}

Proposition (3.2.5, 3.2.6)

There exists a unique multilinear and alternating function f on the columns of 3×3 matrices that is normalized such that $f(I_n) = 1$.

Definition (3.2.7)

We define the determinant of a 3×3 matrix to be this unique function.

$$det(A) = a_{11}det(A_{11}) - a_{12}det(A_{12}) + a_{13}det(A_{13})$$

Discussion

Compute the determinant of the matrix

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 3 & 1 \\ 0 & 3 & 2 \end{pmatrix}$$

Remarks

1. The formula through which we defined the determinant function is called the *cofactor expansion.*
2. It is also common to call
$$a_{ij}det(A_{ij})$$
the *ij-cofactor* of a matrix A.
3. There is nothing special about the first row, we can expand the $det(A)$ along any row or column.

Example

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 3 & 1 \\ 0 & 3 & 2 \end{pmatrix}$$

Definition We define for an $n \times n$ matrix the determinant function

$$det(A) = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} det(A_{ij})$$

as the cofactor expansion along any row i or likewise along any column.

Theorem 3.2.8

1. There is exactly one alternating multilinear function $\text{Mat}_n(\mathbb{R}) \xrightarrow{f} \mathbb{R}$ such that $f(I_n) = 1$, which is the determinant function defined above.
2. Any other alternating multilinear function f on square $n \times n$ matrices satisfies
$$f(A) = det(A) \cdot f(I_n)$$

Discussion

Write the first line of the cofactor expansion of the determinant for the matrix

$$A = \begin{pmatrix} 1 & -3 & 1 & 4 \\ 8 & -4 & 2 & -2 \\ 3 & -7 & 5 & 3 \\ 0 & 3 & -2 & 4 \end{pmatrix}$$

Exercise: Read Example 3.2.9 in the book and practice computing determinants.

Theorem (3.2.14)

An $n \times n$ matrix A is invertible if and only if $det(A) \neq 0$.

Proof. In the book ■

Remark

Notice that this agrees perfectly with the discussion above in the 2×2 case.

Further properties of determinants

Textbook: Sections 3.3

Proposition (3.3.7)

If A and B are $n \times n$ matrices, then

1. $det(AB) = det(A)det(B)$
2. If A is invertible, then $det(A^{-1}) = \frac{1}{det(A)}$

Proof. ■

This is very useful for transformations, because their matrix representative depends on a choice of basis

Corollary (3.3.8)

For a linear transformation $V \xrightarrow{T} V$ on a vector space of finite dimensional vector space V

$$det([T]_\alpha^\alpha) = det([T_\beta^\beta)$$

for any two bases α and β.

Proof. ■

So we can define independently of the choses basis

Definition (3.3.9)

The determinant $det(T)$ of a linear transformation $V \xrightarrow{T} V$ on a vector space of finite dimension is the determinant of $[T]_\alpha^\alpha$ for any choice of basis α.

Proposition (3.3.11)

A linear transformation $V \xrightarrow{T} V$ on a vector space of finite dimension is invertible if and only if $det(T) \neq 0$.

Proof. ■

Proposition (3.3.12)

For two linear transformations $V \xrightarrow{S} V$ and $V \xrightarrow{S} V$ on a vector space of finite dimension, then

1. $det(ST) = det(S) \cdot det(T)$
2. if T is an isomorphism, then $det(T^{-1}) = \frac{1}{det(T)}$

Proof. ■

Eigenvalues Pt I

Textbook: Section 4.1

Observation

A transformation $T \in \mathcal{L}(V)$ can both stretch and rotate vectors $\vec{v} \in V$. Can we find nonzero vectors which are only stretched? (since $T(\vec{0}) = \vec{0}$ is always true...)

Definition (4.1.2)

Given a linear transformation $V \xrightarrow{T} V$.

1. A nonzero vector $\vec{v} \in V$ is called an *eigenvector* of T if for some scalar $\lambda \in \mathbb{R}$

$$T(\vec{v}) = \lambda \vec{v}$$

   That is, $\vec{v}$ is only stretched by T.

2. If a nonzero vector $\vec{v} \in V$ such that $T(\vec{v}) = \lambda \vec{v}$ exists, we call λ an *eigenvaule* of T corresponding to $\vec{v}$.

One also says: '$\vec{v}$ is an eigenvector of T with eigenvalue λ'

Examples

1. The projection onto the line of slope m in $\mathbb{R}^2$.

2. (4.1.1 in the book)

   The transformation $\mathbb{R}^2 \xrightarrow{T_A} \mathbb{R}^2$ represented by the matrix
   $$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$
   has an eigenvector $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ with eigenvalue $\lambda_1 = 3$ and another eigenvector $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ with eigenvalue $\lambda_2 = 1$.

Question

How do we find eigenvectors and eigenvalues of a transformation $T \in \mathcal{L}(\mathcal{V})$ in general?

This observation proves the following

Proposition (4.1.5)

A nonzero vector $\vec{v} \in V$ is an eigenvector of T with eigenvalue λ if and only if

$$\vec{v} \in \ker(T - \lambda \cdot \mathrm{id}_V)$$

Question

But for which $\lambda \in \mathbb{R}$ can we expect to find eigenvectors?

Proposition (4.1.9)

$\lambda \in \mathbb{R}$ is an eigenvalue of $T \in \mathcal{L}(\mathcal{V})$ if and only if

$$\det(T - \lambda \cdot \mathrm{id}_V) = 0$$

That is, there are eigenvectors $\vec{v} \in V$ with eigenvalue λ.

Proof. ■

Proposition (4.1.6)

For a given eigenvalue λ of $T \in \mathcal{L}(\mathcal{V})$ the set of all eigenvectors

$$E_\lambda = \{\vec{v} \in V \mid T(\vec{v}) = \lambda\vec{v}\}$$

is a subspace of V called the λ-*eigenspace* of T.

Proof. ■

Example (4.1.1)

Let's have another look at example (4.1.1) from the book, Let $\mathbb{R}^2 \xrightarrow{T_A} \mathbb{R}^2$ be the linear transformation represented by the matrix

$$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

Compute its eigenvalues and the corresponding eigenspaces with the theory introduced above.