Triangular Form

Textbook: Section 6.1
A diagonalizable matrix is in its *normal form* when it is diagonal. And every diagonalizable matrix can be brought to a diagonal form with a change of basis to a *canonical basis.*

Some matrices are not diagonalizable, what are their canonical forms and canonical bases? The remaining sections 6.1-6.4 answer this quetsion step by step.

For this entire lecture, let V be a finite dimensional vector space over a field F.

Definition

A matrix $A \in \mathrm{Mat}_n(F)$ is called *upper triangular* if all entries below the diagonal are 0. For example,

$$\begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix}$$

Definition (6.1.2)

Let $V \xrightarrow{T} V$ be a linear transformation. A subspace $W \subseteq V$ is called *invariant* or *stable* under T if $T(W) \subseteq W$.

Examples

1. Let $F^3 \xrightarrow{P_{xy}} F^3$ be the projection on the xy-plane. Then the xy-plane is an invariant subspace for it.
2. $\{0\}$ and V are always invariant subspaces for any transformation $T \in \mathcal{L}(V)$.

Discussion

Let $V \xrightarrow{T} V$ be a linear transformation such that in a basis $\beta = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$

$$[T]_\beta^\beta = \begin{pmatrix} 2 & 1 & -1 \\ & 3 & 1 \\ & & 3 \end{pmatrix}$$

1. Is $\mathrm{span}\{\vec{v}_1\}$ invariant?
2. Can you find three subspaces $W_1 \subset W_2 \subset W_3$ which are all invariant?

Proposition (6.1.4)

A linear transformation $V \xrightarrow{T} V$ is upper triangular in a basis β if and only if for each i, $1 \leq i \leq \dim(V)$

$$W_i = \text{span}\{\vec{v}_1, \ldots, \vec{v}_i\}$$

is invariant.

Proof. ∎

Definition (6.1.5)

A linear transformation $V \xrightarrow{T} V$ is called triangulizable if there exists a basis β such that $[T]_\beta^\beta$ is upper trianglar.

We skip a technical result from proposition 6.1.6 in the book and directly state

Theorem (6.1.8)

A linear transformation $V \xrightarrow{T} V$ is triangulizable if and only if the characterstic polynomial $c_T(\lambda)$ has $dim(V)$ roots (counted with multiplicity).

Discussion

Let $A \in \text{Mat}_n(\mathbb{C})$. Why is there always an upper trianglar matrix $B \in \text{Mat}_n(\mathbb{C})$ which is similar to A?

Notation

If $p(x) = a_n x^k + \cdots + a_0$ is a polynomial in $\mathcal{P}_k(F)$ and $A \in \mathrm{Mat}_n(F)$, we define

$$p(A) = a_n A^k + \cdots + a_0 I_n$$

Discussion

1. Suppose $A, B \in \mathrm{Mat}_n(F)$ are similar matrices such that $A = QBQ^{-1}$ for an invertible matrix $Q \in \mathrm{Mat}_n(F)$. Show that
$$p(A) = Qp(B)Q^{-1}$$

2. Suppose that
$$A = \begin{pmatrix} \lambda_1 & * & \cdots & * \\ & \lambda_2 & & \vdots \\ & & \ddots & * \\ & & & \lambda_n \end{pmatrix}$$
is an upper triangular matrix. Compute $c_A(\lambda)$ and $c_A(A)$.

Theorem (6.1.12)

Let $V \xrightarrow{T} V$ be a linear transformation and assume that $c_A(\lambda)$ has $dimV$ roots in F. Then $c_T(T) = 0$.

Proof. *(Sketch of proof)* ∎

Nilpotent Normal Forms

Textbook: Section 6.2

The next kind of matrices we want to bring into a normal form are nilpotent matrices. Remember that for the reminder of these notes V is a finite dimensional vector space over a field F.

Definition

A linear transformation $V \xrightarrow{N} V$ is called *nilpotent* if $N^n = 0$ for some $n \geq 1$. The least n such that $N^n = 0$ is called the *index* of the nilpotent transformation.

Discussion

1. Suppose $V \xrightarrow{N} V$ is nilpotent, does N always have an eigenvector?
2. Suppose $V \xrightarrow{N} V$ is nilpotent and λ is an eigenvalue of N. What can λ be?
3. Suppose $V \xrightarrow{T} V$ has only one distinct eigenvalue $\lambda = 0$ of multiplicity $m_\lambda = \dim(V)$. Is T nilpotent?

Observation

For a nilpotent transformation $V \xrightarrow{N} V$ and $\vec{v} \in V$ nonzero

$$N^k(\vec{v}) = 0$$

for $1 \leq k \leq n$, but not necessarily $n = k$!

Deifnition (6.2.1)

Let $V \xrightarrow{N} V$ be a nilpotent transformation on V and $\vec{v} \in V$ nonzero with k as above.

1. The set $\alpha = \{N^{k-1}(\vec{v}), N^{k-2}(\vec{v}), \ldots, \vec{v}\}$ is called the *cycle* generated by $\vec{v}$.
   $\vec{v}$ is called the *inital vector* of this cycle.
2. The subspace generated by this cycle $C(\vec{v}) = \text{span}\{\alpha\}$ is called the *cyclic subspace* generated by $\vec{v}$.
3. We call k the *length* of the cycle.

Discussion

1. Is $\frac{d^2}{dx^2}$ nilpotent on $\mathcal{P}_n(F)$? What is the index?
2. Can you find a polynomial $p \in \mathcal{P}_n(F)$ that generates a cycle of length 3?

Look at Example (6.2.2) in the book for another example.

Proposition (6.2.3)

Let $V \xrightarrow{N} V$ be a nilpotent transformation on V and $\vec{v} \in V$.

1. If $\vec{v}$ generates a cycle of length k, then $N^{k-1}(\vec{v})$ is an eigenvector of N with eigenvalue $\lambda = 0$.
2. $C(\vec{v})$ is an invariant subspace for N.
3. The cycle α generated by $\vec{v}$ is independent and hence a basis for $C(\vec{v})$.

Proof. ∎

Notation (Cycle tableau)

Let $V \xrightarrow{N} V$ be a nilpotent transformation on V.

1. Write for a cycle α of length k a row of k boxes to represent every element in α.

2. If we consider r cycles $\alpha_1, \ldots, \alpha_r$ generated by $\vec{v}_1, \ldots, \vec{v}_r$ each of length $k_1, \ldots, k_r$ write r rows of boxes sorted by length.

   We call this the *cycle tableau* of the cycles $\alpha_1, \ldots \alpha_r$.

Discussion

Consider the following cycle tableau of the cycles $\alpha_1, \alpha_2, \alpha_3$ of a nilpotent transformation $V \xrightarrow{N} V$.

Which of the boxes necessarily correspond to elements in

1. $\ker(N)$
2. $\mathrm{im}(N)$
3. $\mathrm{im}(N^2)$
4. $\ker(N) \cap \mathrm{im}(N^3)$

Discussion

Let $V \xrightarrow{N} V$ be a nilpotent transformation on a vector space V of dimension 6.

1. If $\vec{v}$ generates a cycle α of length 6,
   (a) is α a basis for V?
   (b) What is $[N]_\alpha^\alpha$?
2. If $\alpha_1, \alpha_2, \alpha_3$ are cycles of lengths $1, 2$ and 3 such that $\beta = \alpha_1 \cup \alpha_2 \cup \alpha_3$ is linearly independent,
   (a) is β a basis for V?
   (b) What is $[N]_\beta^\beta$?
3. Does the matrix $[N]_\beta^\beta$ depend on the particular elements in β?

Notation

1. *nilpotent Jordan block* of size $k \times k$

2. direct sum of matrices

3. *nilpotent Jordan matrix*

Goal: For a nilpotent transformation, we want to find a basis consisting of cycles.

How many cycles do we need? Which cycles can we use? How do we find these cycles?

Proposition (6.2.4)

Let $\alpha_1, \ldots, \alpha_r$ be cycles of lengths k_i respectively, generated by $\vec{v}_i$. If the set of eigenvectors

$$\{N^{k_1-1}(\vec{v}_1), \ldots, N^{k_r-1}(\vec{v}_r)\}$$

is linearly independent, then the union

$$\alpha_1 \cup \cdots \cup \alpha_r$$

is linearly independent.

Proof. ■

Definition (6.2.5 & 6.2.7)

Let $V \xrightarrow{N} V$ be a nilpotent transformation.

1. Cycles $\alpha_1, \ldots, \alpha_r$ such that $\alpha_1 \cup \cdots \cup \alpha_r$ is linearly independent are called *non-overlapping* cycles.
2. A basis of V consisting of non-overlapping cycles for N is called a *canonical basis* for N.

Theorem (6.2.8)

Every nilpotent transformation $V \xrightarrow{N} V$, has a canonical basis on V.

Lemma (6.2.9)

The cycle tableau of a canonical basis for N has

$$\dim(\ker(N^j)) - \dim(\ker(N^{j-1}))$$

boxes in column j.

Discussion (6.2.10)

Let $V \xrightarrow{N} V$ be a nilpotent transformation such that

- $\dim(\ker(N)) = 3$
- $\dim(\ker(N^2)) = 5$
- $\dim(\ker(N^3)) = 7$
- $\dim(\ker(N^4)) = 8$

What shape does the cycle tableau of a canonical basis for N must have? What is the canonical form of the transformation N is such a basis?

Corollary (6.2.11)

The canonical form of a nilpotent transformation is unique up to reordering the nilpotent Jordan blocks. (By convention we sort them by size.)

Discussion

Find the nilpotent Jordan form and a canonical basis α of $T(p) = \frac{d^2}{dx^2}p$ on $\mathcal{P}_3(\mathbb{C})$.