Introduction

Let us start by going over mathematical basics that we will need for this course.

- A *set* is a collection of objects. Examples include the set of integers $\mathbb{Z}$, the set of real numbers $\mathbb{R}$, the set of nonnegative integers $\mathbb{Z}_{\geq 0}$.

  $A \subseteq B$ indicates that every element in A is also an element in B. We say in this case that A is a *subset* of B.

  A subset may be declard by pruning a set by a specified condition. For example the set of even integers or the set of odd integers.

$$\mathbb{E} = \{n \in \mathbb{Z} \mid n \text{ is even } \} \subseteq \mathbb{Z}$$

$$\mathbb{O} = \{n \in \mathbb{Z} \mid n \text{ is odd}\} \subseteq \mathbb{Z}$$

  If $A \subseteq B$ and at least one element of B is not in A, we say that A is properly contained in B, in symbols $A \subsetneq B$.

- A function $A \xrightarrow{f} B$ between sets is an assignment that chooses for every $a \in A$ an element $f(a) = b \in B$. For example

$$\begin{aligned} \mathbb{Z} &\xrightarrow{g} \mathbb{Z} \\ x &\mapsto 2x + 1 \end{aligned}$$

sends every real number x to twice its value plus one. That is, $g(1) = 3$, $g(2) = 5$, and so on ....

domain, codomain, image of x under f

The set of all possible functions between two sets A and B is denoted by $\mathcal{F}(A, B)$.

For any set A, there is the function $A \xrightarrow{\mathrm{id}_A} A$ that sends every element back to itself.

$$\mathrm{id}_A(a) = a \qquad \text{for all } a \in A$$

- A function $A \xrightarrow{f} B$ can have the following properties:

  injective
  No two distinct elements $a \neq b$ in A have the same value $f(a) \neq f(b)$ under f.
  In other words, if f is injective, then $f(a) \neq f(b)$ implies that $a \neq b$.

  surjective
  Every element b in B is the image of some element a in A.
  Equivalently, for all $b \in B$ there is an $a \in A$ such that $b = f(a)$.

  bijective
  The function is both injective and surjective.

  Discussion
  Which of these properties apply to $g(x) = 2x + 1$ defined above as a function from $\mathbb{Z}$ to $\mathbb{Z}$? If not, how can you change the definition to make it bijective?

- Two functions of the form $A \xrightarrow{f} B$ and $B \xrightarrow{g} C$ may be concatinated. This means, whatever the first f functions spits out is fed back into the next function g.

  $$a \mapsto f(a) \mapsto g(f(a))$$

  This is formally called *composition* of f and g and denoted by

  $$(g \circ f)(a) = g(f(a))$$

  Notice, this only makes sense if the domain of g is also the codomain of f.

- An *inverse* of a function $A \xrightarrow{f} B$ is a function in the *opposite* direction $B \xrightarrow{h} A$ with the property that

  $$h(f(a)) = a \qquad \text{for all } a \in A$$

  and

  $$f(h(b)) = b \qquad \text{for all } b \in B$$

  Intuitively, this means h is undoing whatever f was doing. For example, the function $h(y) = \frac{y-1}{2}$ from $\mathbb{O}$ to $\mathbb{Z}$ is an inverse to the function $g(x) = 2x + 1$.

We will wrap up this introduction with the following theorem

Theorem
A function $A \xrightarrow{f} B$ is invertible if and only if it is bijective

Vector spaces

Textbook: Section 1.1

Definition 1.1.1. A (real) vector space $(V, +, \cdot)$ consists of a set V and two operations that we call *addition* $(+)$ and *scalar multiplication*

$$V \times V \xrightarrow{+} V$$

$$\mathbb{R} \times V \xrightarrow{\cdot} V$$

such that the following axioms hold

1. (additive closure) $\vec{x} + \vec{y} \in v$, for all $\vec{x}, \vec{y} \in V$
2. (multiplicative closure) $\alpha \cdot \vec{x} \in V$, for all $\vec{x} \in V$ and scalar $\alpha \in V$
3. (commutativity) $\vec{x} + \vec{y} = \vec{y} + \vec{x}$, for all $\vec{x}, \vec{y} \in V$
4. (additive associativity) $(\vec{x} + \vec{y}) + \vec{z} = \vec{x} + (\vec{y} + \vec{z})$, for all $\vec{x}, \vec{y}, \vec{z} \in V$
5. (additive identity) There exists a vector $\vec{0} \in V$ such that $\vec{x} + \vec{0} = \vec{x}$ for all $\vec{x} \in V$
6. (additive inverse) For each $\vec{x} \in V$, there exists a vector $-\vec{x} \in V$ with the property that $\vec{x} + (-\vec{x}) = \vec{0}$
7. (multiplicative associativity) $(\alpha \cdot \beta) \cdot \vec{x} = \alpha \cdot (\beta \cdot \vec{x})$, for all $\alpha, \beta \in \mathbb{R}$ and $\vec{x} \in V$
8. (distributivity over vector addition) $\alpha \cdot (\vec{x} + \vec{y}) = \alpha\vec{x} + \alpha\vec{y}$, for all $\alpha \in \mathbb{R}$ and $\vec{x}, \vec{y} \in V$
9. (distributivity over scalar addition) $(\alpha + \beta) \cdot \vec{x} = \alpha\vec{x} + \beta\vec{x}$, for all $\alpha, \beta \in \mathbb{R}$ and $\vec{x} \in V$
10. (identity property) $1 \cdot \vec{x} = \vec{x}$, for all $\vec{x} \in V$

Remark

- For elements in a vector space V, we write $\vec{x}, \vec{y}, \ldots \in V$. The textbook writes $\mathbf{x}, \mathbf{y}, \ldots \in V$.
- We often abbreviate $\alpha \cdot \vec{x}$ with $\alpha\vec{x}$.
- Elements in a vetcor space a called vectors. Be aware that anything can be a vector, even functions for example.
- We say that a vector space is *real* if the scalars are real numbers. For now every vector space is real, we will only later allow the scalars to be *complex numbers* and such.

Examples

1. The real numbers $\mathbb{R}$ form a vector space with 'usual' addition $+$ and multiplication $\cdot$.

2. An n-tuple of real numbers can be written as $(v_1, v_2, \ldots, v_n)$ where each $v_i \in \mathbb{R}$. The set of n-tuples is a vector space denoted by $\mathbb{R}^n$.

   What are the operations $+$ and $\cdot$? What is the additive identity element $\vec{0}$?

3. The set $\mathrm{Mat}_{n,m}(\mathbb{R})$ of $n \times m$ matrices with componentwise addition and scalar multiplication.

4. The set
   $$\mathcal{P}_n(\mathbb{R}) = \{a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n | a_0, \ldots, a_n \in \mathbb{R}\}$$
   is a vector space.
   Addition of two polynomials is applied to coefficientwise and the identity element $\vec{0}$ is the polynomial that is constantly zero $p(x) = 0$.

5. The set
   $$\mathcal{F}(\mathbb{R}, \mathbb{R})$$
   of functions from the real numbers to the real numbers is a vector space

Intuition

In many cases vectors may be represented with vectors becasue they too have a direction and a magnitude. But be careful, every analogy has its limitations.

Discussion

(I) Is the set of 2-tuples of integers $\mathbb{Z}^2$ a real vector space?

Hint: To verify that something is a vector space, we need to check *all* axioms in the definition. However, to prove the contrary, it is enough to disprove *one single* axiom!

(II) Is the set $\mathcal{P}_n(\mathbb{R})'$ of polynomials of *exactly* degree n a vector space?

Some Properties of vector spaces

Whenever we introduce a new mathematical *object*, such as a vector space, we may not take anything for granted. In some ways, vectors behave like real numbers (addition, scalar multiplication, zero element, additive inverse ...) but in many ways they do not!
For example, for $3 \in \mathbb{R}$ we can write $\frac{1}{3}$, but for a vector $\vec{v} \in V$ we may not wirte $\frac{1}{\vec{v}}$. To be a good mathematican, it is very helpful to be extremely pedantic!

Theorem (Cancellation)

Let V be a vector space and $\vec{u}, \vec{v}, \vec{w} \in V$. If

$$\vec{u} + \vec{w} = \vec{v} + \vec{w}$$

the

$$\vec{u} = \vec{v}$$

Proof. To practice, write down which property of vector spaces we are using in the following!

$$\begin{aligned}\vec{u} &= \vec{u} + 0 \\ &= \vec{u} + (\vec{w} - \vec{w}) \\ &\vdots\end{aligned}$$

■

Proposition Let V be a vector space and $\vec{v} \in V$, then

$$0\vec{v} = \vec{0}$$

Explain the difference between 0 and $\vec{0}$.

Proof.

$$\begin{aligned}\vec{0} + 0\vec{v} &= 0\vec{v} \\ &= (0+0)\vec{v} \\ &= 0\vec{v} + 0\vec{v}\end{aligned}$$

So by the cancellation theorem we can simplify

$$\vec{0} + 0\vec{v} = 0\vec{v}$$

to

$$\vec{0} = 0\vec{v}$$

■

Proposition Let V be a vector space and $\vec{v} \in V$, then

$$-1 \cdot \vec{v} = -\vec{v}$$

Proof. ■

Notice that the symbol $+$ does *not* necessarily refer to the standard addition, it could be defined in a different way as we can observe in the following. **Discussion** Let $(V, \diamond, \star)$ be a vector space with the following ingredients:

1. The set V of 2-tuples of real numbers $\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$.
2. Addition of 2-tuples
$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \diamond \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} u_1 + v_1 + 1 \\ u_2 + v_2 + 1 \end{pmatrix}$$
3. Scalar multiplication
$$\alpha \star \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} \alpha u_1 + \alpha - 1 \\ \beta u_2 + \beta - 1 \end{pmatrix}$$

Is this a vector vector space? What would the identity element $\vec{0}$ be? What is the inverse of an element $\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$?

Hint Look at the propositions from the previous page.

Subspaces

Textbook: Section 1.2

Definition
A *subspace* U of a vector space $(V, +, \cdot)$ is a subset $U \subseteq V$ that is a vector space in its own right (with the same addition and scalar multiplication of V)