Eigenvalues and Eigenvectors Pt II

Textbook: Section 4.1

Proposition (4.1.5)

A nonzero vector $\vec{v} \in V$ is an eigenvector of T with eigenvalue λ if and only if

$$\vec{v} \in \ker(T - \lambda \cdot \mathrm{id}_V)$$

Proof. ■

Discussion

For a transformation $T \in \mathcal{L}(\mathcal{V})$ which of these statements are equivalent to λ *is an eigenvalue of* T?

1. $T(\vec{v}) = \lambda\vec{v}$ for some $\vec{v} \in V$
2. $\ker(T - \lambda \mathrm{id}_V) \neq \{0\}$
3. $T - \lambda \mathrm{id}_V$ is not an isomorphism

Question

So for which $\lambda \in \mathbb{R}$ can we expect to find eigenvectors?

Proposition (4.1.9)

$\lambda \in \mathbb{R}$ is an eigenvalue of $T \in \mathcal{L}(\mathcal{V})$ if and only if

$$\det\left(T - \lambda \cdot \mathrm{id}_V\right) = 0$$

That is, there are eigenvectors $\vec{v} \in V$ with eigenvalue λ.

Proof. ■

Proposition (4.1.6)

For a given eigenvalue λ of $T \in \mathcal{L}(\mathcal{V})$ the set of all eigenvectors

$$E_\lambda = \{\vec{v} \in V \mid T(\vec{v}) = \lambda\vec{v}\}$$

together with with the zero vector $\vec{0}$ is a subspace of V called the λ-*eigenspace* of T.

Proof. ■

Example (4.1.1)

Let's have another look at example (4.1.1) from the book. Let $\mathbb{R}^2 \xrightarrow{T_A} \mathbb{R}^2$ be the linear transformation represented by the matrix

$$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

Compute its eigenvalues and the corresponding eigenspaces with the theory introduced above.

Definition

Let A be an $n \times n$ matrix. The polynomial $c_A(\lambda) = \det(A - \lambda I_n)$ is called the *characteristic polynomial* of A.

Following this definition and proposition 4.1.9, we can say that eigenvalues of a matrix A will be the roots of its characteristic polynomial.

Proposition (4.1.12)

Similar matrices have equal characteristic polynomial

Proof. ■

Corollary

The characteristic polynomial for a transformation $T \in (\mathcal{L})$ as

$$c_T(\lambda) = \det\left([T]_\alpha^\alpha - \lambda I_n\right)$$

does not depend on the choice of basis.

True or False

- ☐ $T \in \mathcal{L}(V)$ is an isomorphism if and only if 0 is not an eigenvalue.
- ☐ Every transformation has at least one eigenvalue.
- ☐ There are infinitely many eigenvectors to every eigenvalue of a transformation.
- ☐ There is at least one eigenvector to every eigenvalue of a transformation.
- ☐ An $n \times n$ matrix can have at most n distinct eigenvalues.

Discussion

Suppose the transformation $T \in \mathcal{L}(V)$ is represented by the matrix

$$\begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 2 & 3 & 1 \end{pmatrix}$$

with respect to some basis α.

1. Find the characteristic polynomial $c_T(\lambda)$.
2. Find all eigenvalues of T.
3. Find a basis for each eigenspace of T.

Diagonalizability

Textbook: Section 4.2

Definition (4.2.1)

A linear transformation $T \in \mathcal{L}(V)$ on a finite dimensional vector space V is said to be *diagonalizable* if there exists a basis of V consisting entirely of eigenvectors of T.

Why does this definition makes sense? Try to find the matrix $[T]_\alpha^\alpha$ in a basis of eigenvectors $\alpha = \{\vec{\alpha}_1, \dots, \vec{\alpha}_n\}$

Goal

Find a condition to determine whether a transformation is diagonalizable or not.

Definition

Let λ be an eigenvalue of a linear transformation T on V.

1. The *algebraic multiplicity* m_λ of λ is the degree with which c_T vanishes at λ.
2. The *geometric multiplicity* of λ is the dimension of the eigenspace $E_\lambda(T)$

Proposition (4.2.4)

Let $\vec{v}_1, \dots \vec{v}_k$ be eigenvectors to distinct eigenvalues λ_i of a linear transformation $V \xrightarrow{T} V$, then $\{\vec{v}_1, \dots, \vec{v}_n\}$ is linearly independent in V.

Proof. ■

Corollary (4.2.5)

Let T be a linear transformation on V with distinct eigenvectors $\lambda_1, \ldots, \lambda_k$ and for each eigenvalue λ_j consider a linearly independent family of eigenvectors

$$\{\vec{v}_1^j, \ldots, \vec{v}_{n_j}^j\}$$

in E_{λ_j}. Then the union of all these families of eigenvectors

$$S = \{\vec{v}_1^j, \ldots, \vec{v}_{n_1}^j\} \cup \ldots \cup \{\vec{v}_1^k, \ldots, \vec{v}_{n_k}^k\}$$

is linearly independent in V.

Proof. ■

Discussion

What is the intersection of eigenspaces to distinct eigenvalues?

Proposition (4.2.6)

For every linear transformation T on a finite dimensional vector space V the geometric multiplicity is bounded by 1 and the algebraic multiplicity.

$$1 \leq \dim E_\lambda(T) \leq m_\lambda$$

Proof. ∎

Theorem (4.2.7)

For a linear transformation T on a finite dimensional vector space V with distinct eigenvalues $\lambda_1, \ldots, \lambda_k$. Then T is diagonalizable if and only if

1. $m_{\lambda_1} + \cdots + m_{\lambda_k} = \dim V$
2. for each i, $\dim E_{\lambda_i} = m_{\lambda_i}$

Remark: The first condition can be dropped if we assume that all roots of $c_T(\lambda)$ has are real valued. We will see a discussion later.

Proof. ∎

Example (4.2.3)

Diagonalize, if possible the transformation T on $\mathbb{R}^3$ given by the matrix

$$A = \begin{pmatrix} 2 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 2 \end{pmatrix}$$

Discussion

Can you argue with the transformation given by

$$B = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

why the condition $m_{\lambda_1} + \cdots + m_{\lambda_k} = \dim V$ is important in the above theorem?

Remarks