Complex numbers

Textbook: Section 5.1

Motivation

We have seen that the matrix $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ has the characteristic polynomial $c_A(\lambda) = \lambda^2 + 1$ which has no real roots. A root would be a square root $\sqrt{-1}$. Complex numbers solve this problem.

Definition

The set of complex number $\mathbb{C}$ is the set vector space $\mathbb{R}^2$ with an additional multiplication of two complex numbers

$$\begin{pmatrix} a \\ b \end{pmatrix} * \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} ac - bd \\ ad + bc \end{pmatrix}$$

Remark

One often writes these coordinate tuples as vectors in the basis $\{1, i\}$. The coordinate vector $\begin{pmatrix} a \\ b \end{pmatrix}$ corresponds then to $a + ib$.

Discussion

1. Express $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 4 \end{pmatrix}$ in the basis and determine $\begin{pmatrix} 2 \\ 3 \end{pmatrix} * \begin{pmatrix} -1 \\ 4 \end{pmatrix}$
2. Let $W = \operatorname{span}\{1\} \subseteq \mathbb{C}$. Show that W is closed under the multiplication $*$.
3. Can we say that W is isomorphic to $\mathbb{R}$? If so, why?

Definition

Let $z = a + ib \in \mathbb{C}$. We call $Re(z) = a$ the *real part* of z and $Im(z) = b$ the *imaginary part* of z.

Discussion

Let $p(x) = x^2 + 1$ and $q(x) = x^4 + 1$.

1. Find all roots of $p(x)$ in $\mathbb{C}$.
2. Find all solutions to the equation $i \cdot z = 1$. Can you now give meaning to the expression i^{-1}?
3. Find all roots of $q(x)$ in $\mathbb{C}$.
4. What is the inverse of a complex number $a + ib$ in general?

Discussion

Let $w = a + ib$ and consider $\mathbb{C} \xrightarrow{T_w} \mathbb{C}$ to be $T_w(z) = w \cdot z$.

1. Is T_w a linear transformation between real vector spaces? If it is, what is the matrix representing it?
2. Under what condition is T_w invertible?

Discussion

Let $\mathbb{C} \xrightarrow{T} \mathbb{C}$ to be the function $T(a+ib) = a - ib$.

1. Is T_w a linear transformation between real vector spaces? If it is, what is the matrix representing it?
2. What are eigenvalues and eigenspaces of T?
3. What is $T \circ T$? Is T invertible?

Fields

Textbook: Section 5.1

Goal

We would like to replace for vector spaces real numbers $\mathbb{R}$ with complex numbers $\mathbb{C}$ because of their obvious advantage that some characteristic polynomials have roots in $\mathbb{C}$ but not in $\mathbb{R}$. In order to do so, we axiomatize all properties that our prototypes $\mathbb{R}$ and $\mathbb{C}$ satisfy.

Definition (5.1.4)

A *field* is a set F together with two operations called *addition* $(+)$ and *multiplication* $(\cdot)$ if the following axioms are satisfied:

Discussion

Which of the following sets are fields? If they are not field, explain one axiom that does not hold.

$\mathbb{N}$

$\mathbb{Z}$

$\mathbb{Q}$

$\mathbb{R}$

$\mathbb{C}$