Linear combinations

Textbook: Section 1.3

Definition 1.3.1

1. Let V be a vector space and $\{\vec{v}_1, \dots, \vec{v}_k\}$ be a collection of vectors in V. A *linear combination* of vectors $\vec{v}_1, \dots, \vec{v}_k$ in S with *coefficients* $\alpha_1, \dots, \alpha_k \in \mathbb{R}$ is a sum
$$\alpha_1\vec{v}_1 + \alpha_2\vec{v}_2 + \dots + \alpha_k\vec{v}_k$$
2. A linear combination where all coefficients are zero is called *trivial.*
3. To abbreviate a linear combination as above, we may write
$$\sum_{i=1}^{k} \alpha_i\vec{v}_i$$

Notice that this is the most arbitrary vector we can build with the vectors $\vec{v}_1, \dots \vec{v}_k$ available in S.

Example

1. Polynomials in $\mathcal{P}_3(\mathbb{R})$ are linear combinations of the *monomials* $\{1, x, x^2\}$.

2. Let $\vec{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ $\vec{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $\vec{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ be the standard vectors in $\mathbb{R}^3$. Observe that every posible vector in $\mathbb{R}^3$ of the form $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ is a linear combination of these standard vectors.

Definition 1.3.1 (continued)

Let V be a vector space and $S = \{\vec{v}_1, \ldots, \vec{v}_k\}$ be a collection of vectors in V. The set of all linear combinations of vectors in S is called the *span of* S and denoted

$$\operatorname{span}\{S\} = \operatorname{span}\{\vec{v}_1, \ldots, \vec{v}_k\}$$

By convention, $\operatorname{span}\{\emptyset\} = \{\vec{0}\}$.

Example

1. $$\mathcal{P}_2(\mathbb{R}) = \operatorname{span}\{1, x, x^2\}$$

2. $$\operatorname{Mat}_2(\mathbb{R}) = \operatorname{span}\{ \qquad ? \qquad \}$$

Discussion

Let $R = \{\vec{v}_1, \ldots, \vec{v}_k\}$ and $S = \{\vec{v}_1, \ldots, \vec{v}_k, \vec{v}_{k+1}, \ldots, \vec{v}_{k+m}\}$ be two collections of vectors in a vector space V such that R is contained in S. Show that $\text{span}\{R\}$ is contained in $\text{span}\{S\}$.

Theorem 1.3.4

Show that for any vector space V and collection of vectors $S = \{\vec{v}_1, \ldots, \vec{v}_k\}$ in V

$$\text{span}\{S\} \subseteq V$$

is a subspace of V.

Proof. ■

Discussion

Does the set of polynomials $\{1 - 2x^2\ ,\ x^2 + x\ ,\ x^3 - 3x^2\ ,\ 1\}$ span $\mathcal{P}_3(\mathbb{R})$?

Linear Independence

Textbook: Section 1.4

Prelude

We have seen that a vector space can have different spanning sets, for example:

$$\mathbb{R}^2 = \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$$
$$\mathbb{R}^2 = \text{span}\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \end{pmatrix} \right\}$$

How do these two spans differ?

Definition 1.4.2