Dimension Theorem and its Applications

Textbook: 2.4

Theorem 2.3.17 (Dimension Theorem or Rank-Nullity Theorem)

For any linear transformation $V \xrightarrow{T} W$

$$\dim(V) = \dim(\ker(T)) + \dim(\operatorname{im}(T))$$

Remark

- $\dim(\operatorname{im}(T))$ is the same as the rank of $[T]_\alpha^\beta$ and by abuse of notation also referred to as $\operatorname{rank}(T)$.
- Some books refer to $\dim(\ker(T))$ as the *nullity* of T.

Proof. ∎

Proposition 2.4.2

A linear transformation T is injective if and only if $\ker(T) = \{\vec{0}\}$

Proof. ∎

Example

If a linear transformation $V \xrightarrow{T} W$ is injective, then the image

$$T(\vec{v}_1), \ldots, T(\vec{v}_k)$$

of a linearly independent family in V

$$\vec{v}_1, \ldots, \vec{v}_k$$

under T is linearly independent in W.

True or False

Let $V \xrightarrow{T} W$ be a linear transformation.

- ☐ If $\dim(V) > \dim(W)$, T has to be injective.
- ☐ Can $\mathcal{P}_n(\mathbb{R}) \xrightarrow{\frac{d}{dx}} \mathcal{P}_n(\mathbb{R})$ be injective?
- ☐ T is is surjective if and only if $\dim(\operatorname{im}(T)) = \dim(W)$.
- ☐ If T is an isomorphism, then $\dim(V) = \dim(W)$.

Corollary (2.4.4 & 2.4.5)

A linear transformation might be injective if and only if $\dim(V) \leq \dim(W)$.
Warning: But it doesn't have to be injective!!

Proposition (2.4.7)

A linear transformation $V \xrightarrow{T} W$ is surjective if and only if $\dim(\operatorname{im}(T)) = \dim(W)$

Proof. ■

Discussion

How do the dimensions of V and W obstruct whether or not $V \xrightarrow{T} W$ can be surjective? Explain your answer.
Hint: Think along the lines of Corollary 2.4.4

Proposition (2.4.10)

Let $\dim V = \dim W$. A linear transformation $V \xrightarrow{T} W$ is injective if and only if it is surjective.

Proof. ■

Example (2.4.21)

The Dimension Theorem is really nothing new.

Discussion

Consider the linear map

$$\mathcal{P}_n(\mathbb{R}) \to \mathbb{R}^2$$
$$p(x) \mapsto \begin{pmatrix} p(0) \\ p(5) \end{pmatrix}$$

Argue without computation whether this map is injective or surjective.

Homework

Read Propsition 2.4.11 including the proof and look again at Example 2.4.21 in the book.

Composition

Textbook: Section 2.5

Recall

The *composition* of two transformations $U \xrightarrow{S} V$ and $V \xrightarrow{T} W$ is the transformation

$$U \xrightarrow{T \circ S} W$$

deifned by

$$T \circ S(\vec{u}) = T(S(\vec{u}))$$

Proposition (2.5.1)

In the above setup, $T \circ S$ is linear if T and S are linear.

Proof. ∎

Examples

1.

2. Let $p \in \mathcal{P}_n(\mathbb{R})$ be a polynomial and consider the linear transformations

$$\mathbb{R} \xrightarrow{t_y} \mathbb{R}$$

$$t_y(x) = x - y$$

and

$$\mathcal{P}_n(\mathbb{R}) \xrightarrow{ev_x} \mathbb{R}$$

What is the composition

$$ev_x \circ t_y$$

3. What is the composition $\frac{d}{dx} \circ \frac{d}{dx}$ on $\mathcal{P}_n(\mathbb{R})$?

Discussion 2.5.6

Let $U \xrightarrow{S} V$ and $V \xrightarrow{T} W$ be two composable linear transformations. Can you argue why

1. $\ker(S) \subseteq \ker(T \circ S)$
2. $\operatorname{im}(T \circ S) \subseteq \operatorname{im}(T)$

Observation

Let $U \xrightarrow{S} V$ and $V \xrightarrow{T} W$ be two composable linear transformations. Fix bases α, β and ε of U, V and W respectively. How does the matrix of the composition $T \circ S$ relate to the matrices $[T]_{\beta}^{\varepsilon}$ and $[S]_{\alpha}^{\beta}$?

So the real question is what the composition of matrices should be.

Definition

Let A and B be matrices of size $m \times n$ and $n \times p$. The *product* of the matrices A and B is the matrix AB such that

$$AB\vec{x} = A \cdot (B\vec{x})$$

for all $\vec{x} \in \mathbb{R}^n$.

Proposition (2.5.9)

If A has entries $[a]_{ij}$ and B has entries $[b]_{ij}$, then the (i, j) entry of AB is

$$[AB]_{ij} = \sum_{k=0}^{n} a_{ik} b_{kj}$$

Proof. ■

Intuition

Example Compute the matrix product of ...

Discussion

If A and B are composable matrices, is the rank of A or the rank of AB (possibly) greater? Explain!

Discussion

Recall that the matrix representing $\mathcal{P}_n(\mathbb{R}) \xrightarrow{\frac{d}{dx}} \mathcal{P}_n(\mathbb{R})$ in the basis $\alpha = \{1, x, x^2\}$ is given by

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

Verify that A^2 is the matrix representing $\frac{d^2}{dx^2}$

Remark

Literally everything we know about linear transformations also holds for matrices.

1. $A(BC) = (AB)C$
2. $I \cdot A = AC$
3. $\cdots$

Discussion

Given two composable matrices, is

$$AB = BA$$

true?

The Inverse of a Linear Transformation

Textbook: Section 2.6

Definitions

1. An *inverse* to a linear transformation $V \xrightarrow{T} W$ is another transformation $W \xrightarrow{S} V$ such that

$$S \circ T = \mathrm{id}_V$$

   and

$$T \circ S = \mathrm{id}_W$$

2. A linear transformation that has an inverse is called *invetible*

Proposition 2.6.2 & 2.6.1

1. A linear transformation $V \xrightarrow{T} W$ has an inverse if and only if it is bijective.
2. The inverse of a linear transformation is also linear.

Remark: Remember, we called bjective linear transformations *isomorphisms*, this is now the same as invertible linear transformations.

Proof. ■

Proposition (not in book)

An inverse transformation to $V \xrightarrow{T} W$, if it exists, is unique
We can therefore denote it by *the* inverse and use the notation T^{-1}

Proof. ∎

Examples

1. Rotation $\mathbb{R}^2 \xrightarrow{R_\theta} \mathbb{R}^2$ has the inverse $\mathbb{R}^2 \xrightarrow{R_{-\theta}} \mathbb{R}^2$
2. The evaluation map $\mathcal{P}_n(\mathbb{R}) \xrightarrow{ev_4} \mathbb{R}$ does not have an inverse because it is not surjective for $n \geq 1$.
3. Define for $\vec{x}, \vec{y} \in \mathbb{R}^n$ the dot product $\vec{x} \cdot \vec{y} = x_1y_1 + x_2y_2 + \ldots + x_ny_n$. The projection of a vector $\vec{x}$ onto a vector $\vec{y}$ is not invertible.

Discussion

Define for a vector space V with basis $\alpha = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ the *left shift operator* $V \xrightarrow{L} V$ by

$$T(\alpha_i) = \alpha_{i+1}$$

for $i = 1, 2, 3$ and

$$T(\alpha_4) = \alpha_1$$

What is the matrix representing L in the basis α? Is L invertible?

Proposition 2.6.7

Two finite dimensional vector spaces V and W are isomorphic if and only if $\dim(V) = \dim(W)$.

Proof. ■

Examples

Since the following vector spaces have the same dimension, they have to be isomorphic

1. $$\mathcal{P}_{n-1}(\mathbb{R})$$

   and

   $$\mathbb{R}^n$$

2. $$W = \{\vec{x} \in \mathbb{R}^3 \mid x_1 - x_2 + x_3 = 0\}$$

   and

   $$\mathbb{R}^2$$

Discussion

At how many points do we need to evaluate a polynomial of degree 3 to reconstruct it?

Observation

Let $V \xrightarrow{T} W$ be an invertible linear transformation with inverse T^{-1}. The matrices of T and T^{-1} in some bases α and β satisfy

$$[T]_\alpha^\beta \cdot [T^{-1}]_\beta^\alpha = I_n$$

Remark: All matrices in the remainder of this section are assumed to be square matrices.

This motivates the following.

Definition A matrix A of size $n \times n$ is invertible if there exists another matrix B such that

$$AB = I_n$$

and

$$BA = I_n$$

Remark The results for invertible transformations apply as well to matrices.

1. The inverse of a matrix A is unique if it exists.
2. A matrix A is invertible if and only if it has full rank. This means for an $n \times n$ matrix that its rank, which is the number of leading 1s in $RREF(A)$ is equal to n.

Proposition 2.6.11

For a linear transformation $V \xrightarrow{T} W$ between vector spaces with bases α and β

$$([T]_\alpha^\beta)^{-1} = [T^{-1}]_\beta^\alpha$$

Algorithm (page 119)

How to find the inverse of a matrix A

1. Write A and I_n together in a $n \times 2n$ matrix
$$[A \mid I_n]$$
2. Reduce this matrix to its RREF
$$[A \mid I_n] \rightsquigarrow R$$
3. resulting matrix will contain the inverse of A as the right $n \times n$ block.

$$[A \mid I_n] \rightsquigarrow [I_n \mid A^{-1}]$$

Example (2.6.9)

Discussion

1. Compute the inverse of the left shift operator L as a matrix.
2. Compute the inverse of the matrix

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$