Linear combinations

Textbook: Section 1.3

Definition 1.3.1

1. Let V be a vector space and $\{\vec{v}_1, \dots, \vec{v}_k\}$ be a collection of vectors in V. A *linear combination* of vectors $\vec{v}_1, \dots, \vec{v}_k$ in S with *coefficients* $\alpha_1, \dots, \alpha_k \in \mathbb{R}$ is a sum
$$\alpha_1\vec{v}_1 + \alpha_2\vec{v}_2 + \cdots + \alpha_k\vec{v}_k$$
2. A linear combination where all coefficients are zero is called *trivial.*
3. To abbreviate a linear combination as above, we may write
$$\sum_{i=1}^{k} \alpha_i\vec{v}_i$$

Notice that this is the most arbitrary vector we can build with the vectors $\vec{v}_1, \dots \vec{v}_k$ available in S.

Example

1. Polynomials in $\mathcal{P}_3(\mathbb{R})$ are linear combinations of the *monomials* $\{1, x, x^2\}$.

2. Let $\vec{e}_1 = \begin{pmatrix}1\\0\\0\end{pmatrix}$ $\vec{e}_2 = \begin{pmatrix}0\\1\\0\end{pmatrix}, \vec{e}_3 = \begin{pmatrix}0\\0\\1\end{pmatrix}$ be the standard vectors in $\mathbb{R}^3$. Observe that every posible vector in $\mathbb{R}^3$ of the form $\begin{pmatrix}x\\y\\z\end{pmatrix}$ is a linear combination of these standard vectors.

Definition 1.3.1 (continued)

Let V be a vector space and $S = \{\vec{v}_1, \ldots, \vec{v}_k\}$ be a collection of vectors in V. The set of all linear combinations of vectors in S is called the *span of S* and denoted

$$\operatorname{span}\{S\} = \operatorname{span}\{\vec{v}_1, \ldots, \vec{v}_k\}$$

By convention, $\operatorname{span}\{\emptyset\} = \{\vec{0}\}$.

Example

1. $$\mathcal{P}_2(\mathbb{R}) = \operatorname{span}\{1, x, x^2\}$$

2. $$\operatorname{Mat}_2(\mathbb{R}) = \operatorname{span}\{\quad ? \quad\}$$

Discussion

Let $R = \{\vec{v}_1, \ldots, \vec{v}_k\}$ and $S = \{\vec{v}_1, \ldots, \vec{v}_k, \vec{v}_{k+1}, \ldots, \vec{v}_{k+m}\}$ be two collections of vectors in a vector space V such that R is contained in S. Show that $\text{span}\{R\}$ is contained in $\text{span}\{S\}$.

Theorem 1.3.4

Show that for any vector space V and collection of vectors $S = \{\vec{v}_1, \ldots, \vec{v}_k\}$ in V

$$\text{span}\{S\} \subseteq V$$

is a subspace of V.

Proof. ■

Discussion

Does the set of polynomials $\{1 - 2x^2\ ,\ x^2 + x\ ,\ x^3 - 3x^2\ ,\ 1\}$ span $\mathcal{P}_3(\mathbb{R})$?

Linear Independence

Textbook: Section 1.4

Prelude

We have seen that a vector space can have different spanning sets, for example:

$$\mathbb{R}^2 = \operatorname{span}\left\{ \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix} \right\}$$
$$\mathbb{R}^2 = \operatorname{span}\left\{ \begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}-1\\1\end{pmatrix}, \begin{pmatrix}2\\0\end{pmatrix} \right\}$$

How do these two spans differ?

Definition 1.4.2

Let $S = \{\vec{v}_1, \ldots, \vec{v}_k\}$ be a collection of vectors in a vector space V.

1. The collection of vectors is called *linearly independent* if only the trivial linear combination of the vectors in S is equal to zero.

   That is,
   $$\alpha_1\vec{v}_1 + \cdots + \alpha_k\vec{v}_k = \vec{0}$$
   only for the coefficients $\alpha_1 = \alpha_2 = \cdots = \alpha_k = 0$.

2. In he opposite case, when there does exist a nontrivial combination of the vectors in S which is zero, we call the collection *linearly dependent.*

Intuition

When does a linear combinatin of vectors equal to zero? It means that the concatenation of ‘arrows‘ representing the vectors results in a loop.

Moreover, such a loop is trivial if it is just a point, i.e. no interior area.

Example

1. The collection of polynomials $\{1+x\ ,\ 1-x\ ,\ 1+2x\}$ in $\mathcal{P}_1(\mathbb{R})$ is linearly dependent
2. The collection of monomials $\{1\ ,\ x\ ,\ x^2\}$ in $\mathcal{P}_2(\mathbb{R})$ is linearly independet

Discussion

Decide if the following collections of vectors is linearly independent or dependent.

1. todo
2. todo
3. todo

Lemma

If $\{\vec{v}_1, \ldots, \vec{v}_k\}$ is a linearly dependent collection of vectors in a vector space V, there exists an index $j \in \{1, \ldots, k\}$ such that

1. $\vec{v}_j \in \operatorname{span}\{\vec{v}_1, \ldots, \vec{v}_{j-1}\}$
2. $\operatorname{span}\{\vec{v}_1, \ldots, \widehat{\vec{v}_j}, \ldots, \vec{v}_k\} = \operatorname{span}\{\vec{v}_1, \ldots, \vec{v}_k\}$

Proof. ∎

Example

More examples?!

Theorem

Let $S = \{\vec{v}_1, \dots, \vec{v}_k\}$ be a collection of vectors in a vector space V.
Then S is linearly independent if and only if every vector in span$\{S\}$ has a unique representation as a linear combination.

That is, if and only if S is linearly independent, we have that

$$\alpha_1\vec{v}_1 + \cdots + \alpha_k\vec{v}_k = \beta_1\vec{v}_1 + \cdots + \beta_k\vec{v}_k$$

is equivalent with $\alpha_i = \beta_i$ for all $i \in \{1, \dots, k\}$.

Proof. ■

Theorem (Extend & Reduce)

Given ...

True or False? (Theorems from above might be helpful)

☐ If S is an independent collection of vectors and $R \subseteq S$, then R is also independent.

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