Recall

For two composable linear transformations $U \xrightarrow{S} V$ and $V \xrightarrow{T} W$ and bases α, β and γ respectively, in matrices this means

$$[T \circ S]_\alpha^\gamma = [T]_\beta^\gamma [S]_\alpha^\beta$$

Discussion

Recall that the matrix representing $\mathcal{P}_n(\mathbb{R}) \xrightarrow{\frac{d}{dx}} \mathcal{P}_n(\mathbb{R})$ in the basis $\alpha = \{1, x, x^2\}$ is given by

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

Verify that A^2 is the matrix representing $\frac{d^2}{dx^2}$

Remark

Literally everything we know about linear transformations also holds for matrices.

1. $A(BC) = (AB)C$
2. $I \cdot A = AC$
3. $\text{rank}(A) = \dim(\text{im}(T))$
4. $\cdots$

Discussion

Given two composable matrices, is

$$AB = BA$$

true?

The Inverse of a Linear Transformation

Textbook: Section 2.6

Definitions

1. An *inverse* to a linear transformation $V \xrightarrow{T} W$ is another transformation $W \xrightarrow{S} V$ such that
$$S \circ T = \mathrm{id}_V$$
and
$$T \circ S = \mathrm{id}_W$$
2. A linear transformation that has an inverse is called *invetible.*

Proposition 2.6.2 & 2.6.1

1. A linear transformation $V \xrightarrow{T} W$ has an inverse if and only if it is bijective.
2. The inverse of a linear transformation is also linear.

Remark: Remember, we called bjective linear transformations *isomorphisms*, this is now the same as invertible linear transformations.

Proof. ■

Proposition (not in book)

An inverse transformation to $V \xrightarrow{T} W$, if it exists, is unique
We can therefore denote it by *the* inverse and use the notation T^{-1}

Proof. ■

Examples

1. Rotation $\mathbb{R}^2 \xrightarrow{R_\theta} \mathbb{R}^2$ has the inverse $\mathbb{R}^2 \xrightarrow{R_{-\theta}} \mathbb{R}^2$
2. The evaluation map $\mathcal{P}_n(\mathbb{R}) \xrightarrow{ev_4} \mathbb{R}$ does not have an inverse because it is not injective for $n \geq 1$.
3. Define for $\vec{x}, \vec{y} \in \mathbb{R}^n$ the
   (a) *dot product*
   $$\vec{x} \cdot \vec{y} = x_1 y_1 + x_2 y_2 + \ldots + x_n y_n$$
   (b) *norm*
   $$\| \vec{x} \|^2 = \vec{x} \cdot \vec{x} = x_1^2 + x_2^2 + \ldots + x_n^2$$
   The projection of a vector $\vec{x}$ onto a vector $\vec{y}$ is not invertible.
   $$\text{proj}_{\vec{x}}(\vec{y}) = \frac{\vec{x} \cdot \vec{y}}{\| \vec{x} \|^2} \cdot \vec{x}$$

Exercise Fix a vector $\begin{pmatrix} a \\ b \end{pmatrix}$ in $\mathbb{R}^2$ and compute the matrix representing the linear transformation

$$\vec{y} \mapsto \text{proj}_{\begin{pmatrix} a \\ b \end{pmatrix}}(\vec{y})$$

Discussion

Define for a vector space V with basis $\alpha = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ the *left shift operator* $V \xrightarrow{L} V$ by

$$T(\alpha_i) = \alpha_{i+1}$$

for $i = 1, 2, 3$ and

$$T(\alpha_4) = \alpha_1$$

What is the matrix representing L in the basis α? Is L invertible?

Proposition 2.6.7

Two finite dimensional vector spaces V and W are isomorphic if and only if $\dim(V) = \dim(W)$.

Proof. ∎

Examples

Since the following vector spaces have the same dimension, they have to be isomorphic

1. $$\mathcal{P}_{n-1}(\mathbb{R})$$
   and
   $$\mathbb{R}^n$$
2. $$W = \{\vec{x} \in \mathbb{R}^3 \mid x_1 - x_2 + x_3 = 0\}$$
   and
   $$\mathbb{R}^2$$

Discussion

At how many points do we need to evaluate a polynomial of degree 3 to reconstruct it? Explain your answer in terms of a linear transformation

$$\begin{aligned} \mathcal{P}_3(\mathbb{R}) &\to \mathbb{R}^n \\ p(x) &\mapsto \begin{pmatrix} p(x_1) \\ \vdots \\ p(x_n) \end{pmatrix} \end{aligned}$$

Observation

Let $V \xrightarrow{T} W$ be an invertible linear transformation with inverse T^{-1}. The matrices of T and T^{-1} in some bases α and β satisfy

$$[T]_{\alpha}^{\beta} \cdot [T^{-1}]_{\beta}^{\alpha} = I_n$$

Note: All matrices in the remainder of this section are assumed to be square matrices.

This motivates the following.

Definition A matrix A of size $n \times n$ is invertible if there exists another matrix B such that

$$AB = I_n$$

and

$$BA = I_n$$

The results for invertible transformations apply as well to matrices.

Proposition

1. The inverse of a matrix A is unique if it exists.
2. A matrix A is invertible if and only if it has full rank. This means for an $n \times n$ matrix that its rank, which is the number of leading 1s in $RREF(A)$, is equal to n.

Proposition 2.6.11

For a linear transformation $V \xrightarrow{T} W$ between vector spaces with bases α and β

$$([T]_\alpha^\beta)^{-1} = [T^{-1}]_\beta^\alpha$$

Proof. ■

Algorithm (page 119)

How to find the inverse of a matrix A

1. Write A and I_n together in a $n \times 2n$ matrix
$$[A \mid I_n]$$
2. Reduce this matrix to its RREF
$$[A \mid I_n] \rightsquigarrow R$$
3. resulting matrix will contain the inverse of A as the right $n \times n$ block.

$$[A \mid I_n] \rightsquigarrow [I_n \mid A^{-1}]$$

Example (2.6.9)

Discussion

1. Compute the inverse of the left shift operator L as a matrix.
2. Compute the inverse of the matrix

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

Change of Basis

Textbook: Section 2.6

The choice of the right basis makes some computations a lot easier. Take for example the projection onto the line of slope m in $\mathbb{R}^2$.

Example

Question

Given a vector space V with two bases α and β, a vector $\vec{v} \in V$ has two different coordinate representations $[v]_\alpha$ and $[v]_\beta$.

How can we change from the basis α to the basis β?

Example (2.7.1)

Proposition (2.7.3)

Given a vector space V with two bases α and β. The coordinate tuples of a vector $\vec{v} \in V$ are related by the matrix representing the identity transformation

$$V \xrightarrow{\mathrm{id}_V} V$$

from the basis α to the basis β.

$$[I]_\alpha^\beta \cdot [v]_\alpha = [v]_\beta$$

We therefore call $[I]_\alpha^\beta$ the *change of basis* matrix from α to β.

Proof. ∎

Discussion

1. Are change of basis matrices invertible? If so, what are there inverses?
2. Compute the change of basis matrix going in $\mathcal{P}_2(\mathbb{R})$ from $\alpha = \{1 + x + x^2, 1, x\}$ to $\beta = \{1, x, x^2\}$.

Proposition

Let $V \xrightarrow{T} W$ be a linear transformation between vector spaces with bases α and β that is represented by

$$[T]_\alpha^\beta$$

When we introduce new bases α' and β' on V and W respectively with change of basis matrices $[I_V]_\alpha^{\alpha'}$ and $[I_W]_\beta^{\beta'}$, the matrix representing T in these bases is given by

$$[T]_{\alpha'}^{\beta'} = [I_W]_\beta^{\beta'}[T]_\alpha^\beta[I_V]_{\alpha'}^\alpha$$

Definition

If we forget that these matrices come from linear transformations, we may define for two $n \times n$ matrices A and B related by

$$A = PBP^{-1}$$

for some invertible $n \times n$ matrix P to be *similar* matrices.

Proposition

Similar matrices have the same rank.

Example

Change the basis for the transformation of $\mathrm{proj}_{\vec{x}}(\vec{y})$ from the standard basis to a more convenient basis.

The Determinant

Textbook: Section 3.1

tbd