Linear combinations

Textbook: Section 1.3

Definition 1.3.1

1. Let V be a vector space and $\{\vec{v}_1, \ldots, \vec{v}_k\}$ be a family of vectors in V. A *linear combination* of vectors $\vec{v}_1, \ldots, \vec{v}_k$ in S with *coefficients* $\alpha_1, \ldots, \alpha_k \in \mathbb{R}$ is a sum
$$\alpha_1\vec{v}_1 + \alpha_2\vec{v}_2 + \cdots + \alpha_k\vec{v}_k$$
2. A linear combination where all coefficients are zero is called *trivial.*
3. To abbreviate a linear combination as above, we may write
$$\sum_{i=1}^{k} \alpha_i\vec{v}_i$$

Notice that this is the most arbitrary vector we can build with the vectors $\vec{v}_1, \ldots \vec{v}_k$ available in S.

Example

1. Polynomials in $\mathcal{P}_3(\mathbb{R})$ are linear combinations of the *monomials* $\{1, x, x^2, x^3\}$.

2. Let $\left\{ \vec{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \vec{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \vec{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$ be the standard vectors in $\mathbb{R}^3$. Observe that every posible vector in $\mathbb{R}^3$ of the form $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ is a linear combination of these standard vectors.

Definition 1.3.1 (continued)

Let V be a vector space and $S = \{\vec{v}_1, \ldots, \vec{v}_k\}$ be a family of vectors in V. The set of all linear combinations of vectors in S is called the *span of S* and denoted

$$\operatorname{span}\{S\} = \operatorname{span}\{\vec{v}_1, \ldots, \vec{v}_k\}$$

By convention, $\operatorname{span}\{\emptyset\} = \{\vec{0}\}$.

Example

1. $$\mathcal{P}_3(\mathbb{R}) = \operatorname{span}\{1, x, x^2, x^3\}$$

2. $$\operatorname{Mat}_2(\mathbb{R}) = \operatorname{span}\{\qquad ? \qquad\}$$

Discussion

Let R and S be two families of vectors in a vector space V.

1. If R is contained in S, show that $\text{span}\{R\}$ is contained in $\text{span}\{S\}$.
2. Show that we have the following equality of sets.

$$\text{span}\{R \cup S\} = \text{span}\{R\} + \text{span}\{S\}$$

3. Is it true that $R \subseteq \text{span}\{R\}$?

Theorem 1.3.4

For any vector space V and family of vectors $S = \{\vec{v}_1, \ldots, \vec{v}_k\}$ in V

$$\text{span}\{S\} \subseteq V$$

is a subspace of V.

Proof. ■

Corollary

The spans of two families of vectors $R = \{\vec{v}_1, \dots, \vec{v}_k\}$ and $S = \{\vec{w}_1, \dots, \vec{w}_l\}$ are equal

$$\text{span}\{R\} = \text{span}\{S\}$$

if R is contained in span$\{S\}$ and S is contained in span$\{R\}$.

Discussion

Consider in $\mathcal{P}_3(\mathbb{R})$ the family of polynomials

$$S = \{1\ ,\ x - 2x^2\ ,\ 2x^2 + 3x^3\ ,\ 1 + 4x^2\ ,\ 5x^3\}$$

1. Show that $\{1, x, x^2, x^3\} \subseteq \text{span}\{S\} \subseteq \mathcal{P}_3(\mathbb{R})$
2. Use the Corollary above to conclude that $\text{span}\{S\} = \mathcal{P}_3(\mathbb{R})$

Linear Independence

Textbook: Section 1.4

Prelude

We have seen that a vector space can have different spanning sets, for example:

$$\mathbb{R}^2 = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$$
$$\mathbb{R}^2 = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \end{pmatrix} \right\}$$

How do these two spans differ?

Definition 1.4.2

Let $S = \{\vec{v}_1, \dots, \vec{v}_k\}$ be a family of vectors in a vector space V.

1. The family of vectors is called *linearly independent* if only the trivial linear combination of the vectors in S is equal to zero.

   That is,
   $$\alpha_1 \vec{v}_1 + \cdots + \alpha_k \vec{v}_k = \vec{0}$$
   only for the coefficients $\alpha_1 = \alpha_2 = \cdots = \alpha_k = 0$.

2. In he opposite case, when there does exist a nontrivial combination of the vectors in S which is zero, we call the family *linearly dependent*.

Intuition

When does a linear combinatin of vectors equal to zero? It means that the concatenation of 'arrows' representing the vectors results in a loop.

Moreover, such a loop is trivial if it is just a point, i.e. no interior area.

Example

1. The family of polynomials $\{1+x\ ,\ 1-x\ ,\ 1+2x\}$ in $\mathcal{P}_1(\mathbb{R})$ is linearly dependent
2. The family of monomials $\{1\ ,\ x\ ,\ x^2\}$ in $\mathcal{P}_2(\mathbb{R})$ is linearly independet

Discussion

Can yo ufind a linearly dependent family of four vectors in $\mathbb{R}^3$ such that any three of them are linealry independent?

Theorem (Reduce)

If $\{\vec{v}_1,\ldots,\vec{v}_k\}$ is a linearly dependent family of vectors in a vector space V, there exists an index $j \in \{1,\ldots,k\}$ such that

1. $\vec{v}_j \in \operatorname{span}\{\vec{v}_1,\ldots,\vec{v}_{j-1}\}$
2. $\operatorname{span}\{\vec{v}_1,\ldots,\widehat{\vec{v}_j},\ldots,\vec{v}_k\} = \operatorname{span}\{\vec{v}_1,\ldots,\vec{v}_k\}$

Proof. ■

Example

Verify that the above lemma applies to the example in the prelude.

Theorem (Extend)

If $S = \{\vec{v}_1, \ldots, \vec{v}_k\}$ is a linearly independent family of vectors in a vector space V that does not span V, then

1. There exists $\vec{v} \in V$ such that $\vec{v} \notin \text{span}\{S\}$.
2. The family $\{\vec{v}_1, \ldots, \vec{v}_k, \vec{v}\}$ is linearly independent

Proof. ■

Theorem

Let $S = \{\vec{v}_1, \ldots, \vec{v}_k\}$ be a family of vectors in a vector space V. Then S is linearly independent if and only if every vector in $\text{span}\{S\}$ has a unique representation as a linear combination of vectors in S.

That is, if and only if S is linearly independent, we have that

$$\alpha_1\vec{v}_1 + \cdots + \alpha_k\vec{v}_k = \beta_1\vec{v}_1 + \cdots + \beta_k\vec{v}_k$$

is equivalent with $\alpha_i = \beta_i$ for all $i \in \{1, \ldots, k\}$.

Proof. ■

Intuition Can you explain the above theorem graphically?

True or False? (Theorems from above might be helpful)

- ☐ If S is an independent family of vectors and $R \subseteq S$, then R is also independent.
- ☐ A family $\{\vec{v}\}$ of a single vector is always independent.
- ☐ A family $\{\vec{v}, \vec{w}\}$ of two vectors is dependent if and only if $\vec{v}$ and $\vec{w}$ are a multiple of each other.
- ☐ A linearly dependent family $\{\vec{v}_1, \ldots, \vec{v}_k\}$ only contains (sub-) families which are dependent as well.
- ☐ A family $\{\vec{v}_1, \ldots, \vec{v}_k\}$ containing a linearly dependent family $\{\vec{v}_1, \ldots, \vec{v}_k\}$ is also dependent.
- ☐ Is it possible that two families of vectors R and S are not equal, but $\text{span}\{R\} = \text{span}\{S\}$?
- ☐ There is a linearly independent family of fours polynomials in $\mathcal{P}_2(\mathbb{R})$.