Linear Transformations Part II

Textbook: Sections 2.2 & 2.3

The last hour before the lecture next week will be a review session! Please collect your questions and email me if you would like to discuss anything particular. (include MAT224 in subject, thanks)

All vector spaces from now on, unless stated otherwise, will be assumed to be finite dimensional.

What does it take to define a linear transformatio between vector spaces if they have bases?

The matrix of a linear transformation: Remember that a basis is used to encode vectors in n-tuples of coefficient to remember what linear combination of the basis elements we need to recover the vector.

We can use the same idea to encode linear transformations

Example 2.2.2

Let $V = W = \mathbb{R}^2$ with the standard basis $\{\vec{e}_1, \vec{e}_2\}$. Define $V \xrightarrow{T} W$ by

$$T(\vec{e}_1) = \vec{e}_1 + \vec{e}_2$$

$$T(\vec{e}_2) = \vec{2}e_1 - 2\vec{e}_2$$

Algorithm

Given a transformation $V \xrightarrow{T} W$ given as a 'formula', this is how to compute its matrix in two chosen bases α of V and β of W:

1. For each basis element $\vec{b}_i$ in V, compute $T(\vec{b}_i)$.
2. Find the coorindate vector $\gamma^{\mathcal{C}}(T(\vec{b}_i)) = [T(\vec{b}_i)]_{\mathcal{C}}$
3. Assemble these coordinate vectors as columns in a matrix

Discussion

Apply the above algorithm to find the matrix representing the derivative $\frac{d}{dx}$ from $\mathcal{P}_2(\mathbb{R})$ to itself. Choose the basis on $\mathcal{P}_2(\mathbb{R})$ consisting of monomials $\alpha = \{1, x, x^2\}$.

Definition 2.2.6

Let T be a transformation between finite dimensional vector spaces V and W The *matrix of the linear transformation* T with respect to bases α and β is the matrix $[T]_\alpha^\beta$ satisfying

$$[T]_\alpha^\beta \cdot [\vec{v}]_\alpha = [T(\vec{v})]_\beta$$

We now rephrase the algorithm from the previous page in more mathematical terms.

Proposition

In the context of the above definition, the matrix of T can be computed as

$$[T]_\alpha^\beta = \gamma^\beta \circ T \circ (\gamma^\alpha)^{-1}$$

Remark

1. The size of the matrix depends on the dimensions of the vector spaces.

Discussion

On the contrary, given a matrix $A \in \mathrm{Mat}_{n,m}(\mathbb{R})$, does this give us a transformation?

Summary

The upshot of this section is that linear transformations are completely interchangable with matrices!

Discussion (bonus)

Without doing a lot of work, can you argue what the matrix representing the composition $F \circ T$ is assuming you know $[F]$ and $[T]$?

Kernel and Image

Definition Let $V \xrightarrow{T} W$ be a linear transformation.

1. $\ker(T) = \{\vec{v} \in V \mid T(\vec{v}) = 0\}$
2. $\operatorname{im}(T) = \{T(v) \in W \mid \vec{v} \in V\}$

Proposition

For a transformation $V \xrightarrow{T} W$, $\ker(T)$ is a subspace in V and $\operatorname{im}(T)$ a subspace in W.

Proof. ∎

Theorem (Rank Nullity)

For any transformation $V \xrightarrow{T} W$ the dimensions of the kernel and the image add to the dimension of the domain.

$$\dim(V) = \dim(\ker(T)) + \dim(\mathrm{im(T)})$$

It is common to call the dimension of the kernel the *nullity* and the dimension of the image the *rank* of T

$$\dim(V) = \operatorname{null}(T) + \mathrm{rank(T)}$$

Theorem

A transformation $V \xrightarrow{T} W$ is injective if and only if $\ker(T) = \{0\}$.

Proof. ∎