Jordan Canonical Form

Textbook: Section 6.3 & 6.4

Goal: We Would would like to improve the upper triangular form of triangulizable matrices.

Discussion

Suppose we have the linear transformation $T(p) = \frac{d^2}{dx^2}p + p$ on $\mathcal{P}_3(\mathbb{C})$.

1. What are the eigenvalues of T?
2. Why is $N = T - I$ nilpotent?
3. What is the nilpotent Jordan form of N?
4. Find a canonical basis α of N, what is $[T]_\alpha^\alpha$ in this basis?

Notation

1. *Jordan block* of size $k \times k$ with eigenvalue λ, $J_m(\lambda)$

2. *Jordan matrix* $J = J_{m_1}(\lambda_1) \oplus \cdots \oplus J_{m_k}(\lambda_k)$

Discussion

Let T be a linear transformation on a complex vector space of dimension 4 with only a single eigenvalue λ_1.

1. What is the characteristic polynomial $c_T(\lambda)$?
2. Is $T - \lambda_1 I$ nilpotent?
3. List all possible Jordan forms of T.

Hint: Corollary (6.1.11) says that a triangulizable matrix can in particular be written in a canonical form where the diagonal only contains eigenvalues.

Problem: What if T on V has more than one distinct eigenvalue?

Notation

1. Let T be a linear transformation on V with an invariant subspace $W \subseteq V$. We denote by $T|_W$ the restriction of T to W.
2. Suppose $V = W_1 \oplus W_2$ is the direct sum of two invariant subspaces of T. Then T is fully determined by its restrictions to W_1 and W_2. Moreover, if α and β are bases of W_1 and W_2 respectively

$$[T]_\gamma^\gamma = [T]_\alpha^\alpha \oplus [T]_\beta^\beta$$

in the basis $\gamma = \alpha \cup \beta$ as a direct sum of matrices.

Goal

Why does this help us? If we can find invariant subspaces $W_1, \ldots, W_k$ such that $T|_{W_i}$ has only one distinct eigenvalue λ_i, we can find the Jordan canonical form of $T|_{W_i}$ and take the direct sum for each invarinat subspace.

Definition (6.3.2)

Let T be a linear transformation on a finite-dimensional vector space V with an eigenvalue λ of multiplicity m

1. The λ-*generalized eigenspace* K_λ is the kernel of the transformation $(T - \lambda I)^m$ on V.
2. The nonzero elements of K_λ are called *generalized eigenvectors* of T with eigenvalue λ.

Discussion

Let V be a vector space with basis $\alpha = \{\vec{v}_1, \ldots, \vec{v}_5\}$ and T a linear transformation on V such that

$$[T]_\alpha^\alpha = J_2(2) \oplus J_2(3) \oplus J_1(3)$$

1. Find all eigenvalues of T with their multiplicity.
2. Find all the eigenspaces of T.
3. Find all of the generalized eigenspaces of T.
4. Is T diagonalizable?

Discussion

Let V be a vector space with basis $\alpha = \{\vec{v}_1, \ldots, \vec{v}_5\}$ and T a linear transformation on V such that

$$[T]_\alpha^\alpha = J_2(2) \oplus J_2(3) \oplus J_1(3)$$

1. Are the generalized eigenspaces invariant?
2. Can you write V as a direct sum of subspaces such that T restricted to each of them has exactly one eigenvalue?

Discussion

Let T be a linear transformation on a V be a finite dimensional vector space with eigenvalue λ of multiplicity m.

1. Show that $T|_{K_\lambda}$ has only the eigenvalue λ
2. Is $(T - \lambda I)|_{K_\lambda}$ nilpotent?
3. Show that K_λ is an invariant subspace in V.

Proposition (6.3.4)

Let T be a linear transformation on a V be a finite dimensional vector space such that $c_T(\lambda)$ has $\dim(V)$ roots.

1. For each eigenvalue λ of T, K_λ is an invariant subspace of V.
2. If $\lambda_1, \dots, \lambda_k$ are all the distinct eigenvalues of T, then $V = K_{\lambda_1} \oplus \cdots \oplus K_{\lambda_k}$.
3. If λ is an eigenvalue of multiplicity m, then $\dim(K_\lambda) = m$.

Proof. ■

Theorem (6.3.6) (Jordan Canonical Form)

Let T be a linear transformation on a finite dimensinal vector space V whose characteristic polynomial has $\dim(V)$ roots in the field F over which V is defined. Then

- V has a *canonical basis* γ in which $[T]_\gamma^\gamma$ is a Jordan matrix.
- Moreover, this decomposition of $[T]_\gamma^\gamma$ into Jordan blocks is unique up to reordering of the Jordan blocks.

We call this the *Jordan canonical form* of a linear transformation.

Proof. ■

Discussion

Let T be a linear transformation on a vector space V such that

1. $c_T(\lambda) = (\lambda - 2)^3(\lambda + 1)^4(\lambda - 5)$
2. The dimension of the eigenspaces E_2, E_{-1} and E_5 are 1, 2 and 1, respectively.
3. $(T + I)^2|_{K_{-1}} = 0$

Find the Jordan canonical form of T with the eigenvalues listed in the order 2, -1, 5.

Discussion

All matrices A in $\mathrm{Mat}_2(\mathbb{C})$ are similar to either $\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$ or $\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$.

In general, we can prove

Corollary

Two matrices are similar if and only if they have the same Jordan canonical form up to reordering.

Exercise 2 in section 6.4