Bases and Dimension

Textbook: Section 1.6

Warning: These notes contain probably more content than we can cover in one week. Don't panic, I included some extra material for easy access and reference for you.

Definition 1.6.1

A family of vectors $\mathcal{B}$ in a vector space V is called *a basis of* V if

1. $\mathcal{B}$ spans V
2. $\mathcal{B}$ is linearly independent

Examples

1. $\{\vec{e}_1, \dots, \vec{e}_n\}$ is a basis of $\mathbb{R}^n$.
2. We have seen last week that $\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix}\}$ is a basis of $\mathbb{R}^2$
3. Which families of polynomials of the ones we have seen before are bases of $\mathcal{P}_2(\mathbb{R})$? Can you write down a basis of $\mathcal{P}_2(\mathbb{R})$ that does not contain any monomials?

A vector space does not have just one unique basis as we can easily verify.

Discussion

Given the vector $\begin{pmatrix} 5 \\ 3 \end{pmatrix} \in \mathbb{R}^2$, write down all linear combinations of $\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix}\}$ equal to $\begin{pmatrix} 5 \\ 3 \end{pmatrix} \in \mathbb{R}^2$.

Theorem 1.6.3

A family of vectors $\mathcal{B}$ is a basis of V if and only if every vector $\vec{v} \in V$ can be written uniquely as a linear combination of vectors in $\mathcal{B}$.

Proof. ■

Theorem (Extend & Reduce pt. II)

Let V be a vector space with a finite spanning set.

1. Every linearly independent family S in V may be enlarged to a basis $\mathcal{B}$ containing S. (Theorem 1.6.6)
2. Every family S that spans V may be reduced to a basis $\mathcal{B}$ contained in S.

Proof. This follows from the *Extend* and *Reduce* theorems from last week ■

Remark

1. Why do we need a finite spanning set for V? Some vector spaces, such as $\mathcal{P}(\mathbb{R})$ can not be spanned by finitely many polynomials. Can you show why?
2. Observe that a basis hits a sweet spot as it is not too large to contain redundant vectors, but also not too small to not span the vetcor space.

Corollary

Every finitely spanned vector space has a basis.

Discussion

Consider the vectors $p(x) = 1 + x$ and $q(x) = 1 + x + x^2$ in $\mathcal{P}_2(\mathbb{R})$. Find a third vector $r(x)$ such that the family $\{p, q, r\}$ is a basis for $\mathcal{P}_2(\mathbb{R})$.

Even though a basis is not unique to a vector space, we would like to extract an invariant, a label, something that characterizes the vectors space. This invariant is motivated by the Corollary following below.

Theorem 1.6.10

If V is spanned by a family S with m elements, then no linearly independent family R in V can have more than m elements.

Proof. ■

Corollary 1.6.11

Any two bases $\mathcal{B}$ and $\mathcal{B}'$ of V have the same number of elements

Definitions

1. If a vector space V has a finite basis, we say that V is *finite dimensional.*
2. For a finite dimensional vector space V, the *dimension* of V

$$\dim(V)$$

is the number of elements of a basis of V.

Discussion

What is the dimension of

$\dim(\mathbb{R}) =$

$\dim(\mathcal{P}_n(\mathbb{R})) =$

$\dim(\mathrm{Mat}_2(\mathbb{R})) =$

$\dim(\mathrm{Mat}_2^{\mathrm{sym}}(\mathbb{R})) =$

$\dim(\mathrm{Mat}_2^{\mathrm{anti}}(\mathbb{R})) =$

Remember that a matrix A is symmetric if $A^T = A$ and antisymmetric if $A^T = -A$.

Discussion

- Can you argue that if $U \subseteq V$ is a subspace then $\dim(U) \leq \dim(V)$?
- Is it on the contrary true that for every subspace $U \subseteq V$ $\dim(U) = \dim(V)$ implies $U = V$?

The following result is very useful when we are looking for a basis of a vector space that we already know the dimension of.

Proposition (Corollary of last discussion)

Let V be a vector space of dimension n with a family S containing n vectors, then the following are equivalent.

(a) S is a basis of V

(b) S is linearly independent

(c) S spans V

Proof. ■

Discussion

is the family of polynomials $\{1 + 2x - x^2, 1 + x + x^2\}$ is a basis for the subspace $W = \{a + bx + cx^2 \in \mathcal{P}_2(\mathbb{R}) | -3a + 2b + c = 0\}$?

Since subspaces are vector spaces too, all constructions above apply also to subspaces.

Discussion

Let U and W be subspaces in V, can the sum $\dim(U) + \dim(W)$ be greater than $\dim(V)$?
Argue why it is true or find a counterexample.

Example (continued)

Consider again the subspaces of symmetric and anti symmetric 2×2 matrices, $\mathrm{Mat}_2^{\mathrm{sym}}(\mathbb{R})$ and $\mathrm{Mat}_2^{\mathrm{anti}}(\mathbb{R})$ respectively, in the vector space of all 2×2 matrices $\mathrm{Mat}_2(\mathbb{R})$. Compute the intersection $\mathrm{Mat}_2^{\mathrm{sym}}(\mathbb{R}) \cap \mathrm{Mat}_2^{\mathrm{anti}}(\mathbb{R})$.

This example can be abstractly formalized as

Theorem 1.6.18

Let U and W be finite dimensional subspaces of a vector space V, then

$$\dim(U + W) = \dim(U) + \dim(W) - \dim(U \cap W)$$

Proof. Read the proof in the textbook (voluntarily). ■

Discussion

Let $S = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ be a spanning set of V.

1. What are the possible dimensions of V?
2. Suppose $\{v_1, v_3\}$ is linearly independent, what are the possible dimensions of V?
3. Can you create a basis for V from S?

With the results from this section, we can immediately obtain the following.

Corollary & Discussion

Let V be an n-dimensional vector space and S a linearly independent family with l elements, then

1. $l \leq n$
2. if $l = n$, S is a basis of V.

Discussion

Let $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4$ be vectors in a vector space V and define the subspace $U = \text{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$
Suppose

$$\vec{v}_3 = \vec{v}_1 - \vec{v}_2$$

$$\vec{v}_4 = 2\vec{v}_1 + 3\vec{v}_2 - \vec{v}_3$$

1. What are the possible dimenions of U?
2. Suppose $\dim(U) = 2$. Does this imply that $\{\vec{v}_3, \vec{v}_4\}$ is linealry independent?

Textbook: Section 2.1

Linear Transformatoins

Definition 2.1.1

A function $V \xrightarrow{T} W$ between vector spaces $(V, +_V, \bullet_V)$ and $(W, +_W, \bullet_W)$ is called *linear* if

1. $T(\vec{u} +_V \vec{v}) = T(\vec{v}) +_W T(\vec{v})$ for all $\vec{u}, \vec{v} \in V$
2. $T(\alpha \bullet_V \vec{u}) = \alpha \bullet_W T(\vec{v})$ for all $\vec{u} \in V$ and $\alpha \in \mathbb{R}$.

Remark

- We usually use the expression *linear transformation* or short *transformation* and not linear function.
- Notice that the operations $+$ and $\cdot$ are exactly what distinguishes vector spaces from sets. The requirements for a function to be linear guarantee that it mediates between the operations on the domain and the target.

Examples

1. Consider the function $x \mapsto e^x$, are there vector space structures on $\mathbb{R}$ such that this defines a linear transformation?

Discussion

Given a transformation $V \xrightarrow{T} W$, is it true that

$$T\left(\sum_{i=1}^{k} \alpha_i \vec{v}_i\right) = \sum_{i=1}^{k} \alpha_i T(\vec{v}_i) \qquad ?$$

Discussion

Which of the following functions are linear transformations?

1. $\mathbb{R}^2 \xrightarrow{T} \mathbb{R}$ defined by $T\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \max\{x, y\}$
2. The derivative $\mathcal{P}_n(\mathbb{R}) \xrightarrow{\frac{d}{dx}} \mathcal{P}_n(\mathbb{R})$
$$f(x) \mapsto \frac{df}{dx}(x)$$
3. The transposition of a matrix $A \mapsto A^T$ as a function $\mathrm{Mat}_n(\mathbb{R}) \xrightarrow{T} \mathrm{Mat}_n(\mathbb{R})$
4. The function $\mathrm{Mat}_n(\mathbb{R}) \xrightarrow{RREF} \mathrm{Mat}_n(\mathbb{R})$ that computes the reduced row echolon form of a matrix $A \mapsto \mathrm{RREF}(A)$.
5. The evaluation function $\mathcal{P}_n(\mathbb{R}) \xrightarrow{\mathrm{ev}_7} \mathbb{R}$ evaluating a polynomial at the value 7.
$$\mathrm{ev}_7(p) = p(7)$$

Proposition

For any linear transformation $V \xrightarrow{T} W$ we have

1. $T(\vec{0}_V) = \vec{0}_W$
2. $T(-\vec{v}) = -T(\vec{v})$

Is the converse also true?

Definition

1. A transformation that is bijective is called an *isomorphism*. If there exists and isomorphism between two vector spaces V and W, they are called *isomorphic*.

2. A transformation $V \xrightarrow{T} V$ for which domain and codomain agree is called an *endomorphism*.

3. Given two vector space V and W, the set of all linear transformations from V to W is denoted

$$\mathcal{L}(V, W)$$

or just $\mathcal{L}(V)$ if $V = W$.

Exercise

Show that for any vector spaces V and W the set $\mathcal{L}(V, W)$ is a vector space.
Hint: Compare this to the vector space $\mathcal{F}(\mathbb{R}, \mathbb{R})$ of functions from $\mathbb{R}$ to $\mathbb{R}$.

Recall that a basis in a vector space enbales us to write every vector uniquely as a linear combination. Since a linear combination is determined by its coefficients, all we need to remember for a fixed basis is the n-tuple of coefficients. To formalize this, we define...

Definition

Let V be a vector space with a basis $\mathcal{B} = \{\vec{b}_1, \dots, \vec{b}_n\}$ and for $\vec{v} \in V$

$$\vec{v} = v_1 \cdot \vec{b}_1 + \cdots + v_n \cdot \vec{b}_n$$

the unique representation in this basis. We call $v_1, \dots, v_n$ the *coordinates* of $\vec{v}$ in the basis $\mathcal{B}$ and $[\vec{v}]_{\mathcal{B}} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$ the *coordinate vector* of $\vec{v}$ in the basis $\mathcal{B}$.

Discussion

Consider the basis $\mathcal{B} = \{1, 1+x, 1+x+x^2\}$ of $\mathcal{P}_2(\mathbb{R})$.

1. Determine $[1]_{\mathcal{B}}$, $[x]_{\mathcal{B}}$, $[x^2]_{\mathcal{B}}$ and $[1+x]_{\mathcal{B}}$.
2. Determine $[a+bx+cx^2]_{\mathcal{B}}$ for arbitrary scalars $a, b, c \in \mathbb{R}$ and show that

$$[a+bx+cx^2]_{\mathcal{B}} = a[1]_{\mathcal{B}} + b[x]_{\mathcal{B}} + c[x^2]_{\mathcal{B}}$$

Theorem

Given an n-dimensional vector space V with a basis $\mathcal{B}$. The assignment of its coordinate vector $[\vec{v}]_{\mathcal{B}}$ to every vector $\vec{v} \in V$ is a linear transformation

$$V \xrightarrow{\gamma^{\mathcal{B}}} \mathbb{R}^n$$

Even more is true, $\gamma^{\mathcal{B}}$ is an isomorphism from V to $\mathbb{R}^n$.

Proof. ■

Intuition

Isomorphic vector spaces V and W can be regared as practically the same vector space, because we can use the isomorphism to go back and forth between elements from V to W bijectively. Moreover, an isomorphism identifies the operations on V and W.

The above theorem says that every vector space of dimension n is isomoprhic to $\mathbb{R}^n$ and that the choice of a basis (there are many) tells us how to uniquely identify vectors $\vec{v} \in V$ with n-tuples $(v_1, \ldots, v_n) \in \mathbb{R}^n$.

That is to say $\mathbb{R}^n$ is the prototypical n-dimensional vector space.

Discussion

Suppose $\mathcal{P}_2(\mathbb{R}) \xrightarrow{T} \mathbb{R}$ is given by

$$T(1+x^2) = 5$$
$$T(x-x^2) = 3$$
$$T(1) = 1$$

What is $T(x)$?

Proposition

A transformation $V \xrightarrow{T} W$ is uniquely determined by its values on elements of a basis on V.

Proof. ■