Linear Algebra II
MAT224

Lennart Döppenschmitt

January 13, 2021

Welcome,

my name is Lennart, I will be your instructor for Linear Algebra II this term. I am a graduate student in the maths department and with my advisor I work on differential geometry and the moduli space of Kähler metrics. Apart from math (surprisingly similar though), I enjoy rock climbing.

You can reach me with questions about the course via **email:** *lennart at math dot toronto dot edu*
For math questions this course uses *Piazza.*

My **office hours** take place on Wednesdays at 2 pm (EST) on Zoom.

A few things about the class:

- Details can be found in the syllabus on Quercus `q.utoronto.ca`.
- The content will build on what you have learned in MAT223 with a particular focus on abstraction. Thinking abstractly is an important skill and makes arguments more powerful as it applies to a wider range of applications.
- A weekly class schedule, which we will follow closely, can be found at the end of the syllabus.
- All lectures will be recorded and made availble on YouTube for *one* week, link will follow. Office hours will *not* be recorded.

About the lectures:

- Our lectures will be a mix of me explaining new math to you, time for you to try out what you just learned in short exercises and discussions and then a discussion round where we discuss the answer together.
- I will try to give Definitions, Theorems, etc. the same numbers as in the textbook.
- These notes will be available on Quercus. I encourage you to have a copy of them ready for the lecture to fill in the blanks and scribble down ideas for the discussions.
- Please let me know when you find mistakes in the notes, there will be plenty.

About Zoom:

- We will use the same Zoom meeting ID for lectures and office hours. Please do **NOT** share the ID or password publically.
- To join you need a Zoom account with an official UofT email.
- Please mute yourself in Zoom.
- For questions, you can *virtually* raise your hand and unmute yourself when I ask you to.
- Please only use the chat function for lecture relevant messages and please don't spoil answers to give everyone time to think about the questions.

Thank you to Professor Sean Uppal and Jeffrey Im for providing me with their old slides from a previous iteration of this course as a reference!

Introduction

Let us start by going over mathematical basics that we will need for this course.

- A *set* is a collection of objects. Examples include the set of integers $\mathbb{Z}$, the set of real numbers $\mathbb{R}$, the set of nonnegative integers $\mathbb{Z}_{\geq 0}$.

  $A \subseteq B$ indicates that every element in A is also an element in B. We say in this case that A is a *subset* of B.

  A subset may be declared by pruning a set with a specified condition. For example, the set of even integers and the set of odd integers are

  $$\mathbb{E} = \{n \in \mathbb{Z} \mid n \text{ is even } \} \subseteq \mathbb{Z}$$

  $$\mathbb{O} = \{n \in \mathbb{Z} \mid n \text{ is odd}\} \subseteq \mathbb{Z}$$

  If $A \subseteq B$ and at least one element of B is not in A, we say that A is properly contained in B, in symbols $A \subsetneq B$.

- A function $A \xrightarrow{f} B$ between sets is an assignment that chooses for every $a \in A$ an element $f(a) = b \in B$. For example,

$$\begin{aligned} \mathbb{Z} &\xrightarrow{g} \mathbb{Z} \\ x &\mapsto 2x + 1 \end{aligned}$$

  sends every real number x to twice its value plus one. That is, $g(1) = 3$, $g(2) = 5$, and so on ....

  The notation $A \xrightarrow{f} B$ moreover describes that we are allowed to apply f to elements $a \in A$, we therefore call A the *domain* of f.

  The set B is called the *codomain* or *target* of f, this is where f takes values in.

  for $a \in A$ we call $f(a)$ the *image of a under f*.

  The set of all possible functions between two sets A and B is denoted by $\mathcal{F}(A, B)$.

  For any set A, there is the function $A \xrightarrow{\mathrm{id}_A} A$ that sends every element back to itself.

$$\mathrm{id}_A(a) = a \qquad \text{for all } a \in A$$

- A function $A \xrightarrow{f} B$ can have the following properties:

  injective
  No two distinct elements $a \neq b$ in A have the same value $f(a) \neq f(b)$ under f.
  In other words, if f is injective, then $f(a) = f(b)$ implies that $a = b$.

  surjective
  Every element b in B is the image of some element a in A.
  Equivalently, for all $b \in B$ there is an $a \in A$ such that $b = f(a)$.

  bijective
  The function is both injective and surjective.

  Which of these properties apply to $g(x) = 2x + 1$ defined above as a function from $\mathbb{Z}$ to $\mathbb{Z}$? If not, how can you change the definition to make it bijective?

- Two functions of the form $A \xrightarrow{f} B$ and $B \xrightarrow{g} C$ may be concatinated. This means, whatever the first f functions spits out is fed back into the next function g.

  $$a \mapsto f(a) \mapsto g(f(a))$$

  This is formally called *composition* of f and g and denoted by

  $$(g \circ f)(a) = g(f(a))$$

  Notice, this only makes sense if the domain of g is also the codomain of f.

- An *inverse* of a function $A \xrightarrow{f} B$ is a function in the *opposite* direction $B \xrightarrow{h} A$ with the property that

  $$h(f(a)) = a \qquad \text{for all } a \in A$$

  and

  $$f(h(b)) = b \qquad \text{for all } b \in B$$

  Intuitively, this means h is undoing whatever f was doing. For example, the function $h(y) = \frac{y-1}{2}$ from $\mathbb{O}$ to $\mathbb{Z}$ is an inverse to the function $g(x) = 2x + 1$.

We will wrap up this introduction with the following theorem

Theorem

A function $A \xrightarrow{f} B$ is invertible if and only if it is bijective

Vector spaces

Textbook: Section 1.1

Definition 1.1.1
A (real) vector space $(V, +, \cdot)$ consists of a set V and two operations that we call *addition* $(+)$ and *scalar multiplication* $\cdot$

$$V \times V \xrightarrow{+} V$$

$$\mathbb{R} \times V \xrightarrow{\cdot} V$$

such that the following axioms hold

1. (additive closure) $\vec{x} + \vec{y} \in V$, for all $\vec{x}, \vec{y} \in V$
2. (multiplicative closure) $\alpha \cdot \vec{x} \in V$, for all $\vec{x} \in V$ and scalars $\alpha \in \mathbb{R}$
3. (commutativity) $\vec{x} + \vec{y} = \vec{y} + \vec{x}$, for all $\vec{x}, \vec{y} \in V$
4. (additive associativity) $(\vec{x} + \vec{y}) + \vec{z} = \vec{x} + (\vec{y} + \vec{z})$, for all $\vec{x}, \vec{y}, \vec{z} \in V$
5. (additive identity) There exists a vector $\vec{0} \in V$ such that $\vec{x} + \vec{0} = \vec{x}$ for all $\vec{x} \in V$
6. (additive inverse) For each $\vec{x} \in V$, there exists a vector $-\vec{x} \in V$ with the property that $\vec{x} + (-\vec{x}) = \vec{0}$
7. (multiplicative associativity) $(\alpha \cdot \beta) \cdot \vec{x} = \alpha \cdot (\beta \cdot \vec{x})$, for all $\alpha, \beta \in \mathbb{R}$ and $\vec{x} \in V$
8. (distributivity over vector addition) $\alpha \cdot (\vec{x} + \vec{y}) = \alpha\vec{x} + \alpha\vec{y}$, for all $\alpha \in \mathbb{R}$ and $\vec{x}, \vec{y} \in V$
9. (distributivity over scalar addition) $(\alpha + \beta) \cdot \vec{x} = \alpha\vec{x} + \beta\vec{x}$, for all $\alpha, \beta \in \mathbb{R}$ and $\vec{x} \in V$
10. (identity property) $1 \cdot \vec{x} = \vec{x}$, for all $\vec{x} \in V$, $1 \in \mathbb{R}$

Remark

- Think of a vector space as set, only that you are able to do more with its elements (you can add two of them together and multiply them by a number). Plus, there are 10 ‘rules’ you know your vector space obeys to.
- For elements in a vector space V, we write $\vec{x}, \vec{y}, \ldots \in V$. The textbook writes $\mathbf{x}, \mathbf{y}, \ldots \in V$.
- We often abbreviate $\alpha \cdot \vec{x}$ with $\alpha\vec{x}$.
- Elements in a vetcor space a called vectors. Be aware that anything can be a vector, even functions for example.
- We say that a vector space is *real* if the scalars are real numbers. For now every vector space is real, we will only later allow the scalars to be *complex numbers* and such.

Examples

1. The real numbers $\mathbb{R}$ form a vector space with 'usual' addition $+$ and multiplication $\cdot$.

2. An n-tuple of real numbers can be written as $\vec{v} = (v_1, v_2, \ldots, v_n)$ where each $v_i \in \mathbb{R}$. The set of n-tuples is a vector space denoted by $\mathbb{R}^n$.

   The operations addition and componentwise are performed componentwise. The additive identity is the zero tuple $\vec{0} = (0, \ldots, 0)$.

3. The set $\mathrm{Mat}_{n,m}(\mathbb{R})$ of $n \times m$ matrices with componentwise addition and scalar multiplication.

4. The set of polynomials of degree at most n

   $$\mathcal{P}_n(\mathbb{R}) = \{a_0 + a_1x + a_2x^2 + \cdots + a_nx^n | a_0, \ldots, a_n \in \mathbb{R}\}$$

   is a vector space.
   Addition of two polynomials is applied to coefficientwise and the identity element $\vec{0}$ is the polynomial that is constantly zero $p(x) = 0$.

   Because every term in a polynomial looks like a_ix^i for some value of i between 0 and n, we can abbreviate

   $$\sum_{i=0}^{n} a_ix^i = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$$

5. The set

   $$\mathcal{F}(\mathbb{R}, \mathbb{R})$$

   of functions from the real numbers to the real numbers is a vector space.

   What might be the operations $+$ and $\cdot$?

Intuition

In many cases vectors may be represented with arrows becasue they, too, have a direction and a magnitude. But be careful, every analogy has its limitations.

Discussion

(I) Is the set of 2-tuples of integers $\mathbb{Z}^2$ a real vector space?

Hint: To verify that something is a vector space, we need to check *all* axioms in the definition. However, to prove the contrary, it is enough to disprove *one single* axiom!

(II) Is the set $\mathcal{P}_n(\mathbb{R})'$ of polynomials of *exactly* degree n a vector space?

Some Properties of vector spaces

Whenever we introduce a new mathematical *object*, such as a vector space, we may not take anything for granted. In some ways, vectors behave like real numbers (addition, scalar multiplication, zero element, additive inverse ...) but in many ways they do not!
For example, for $3 \in \mathbb{R}$ we can write $\frac{1}{3}$, but for a vector $\vec{v} \in V$ we may not wirte $\frac{1}{\vec{v}}$. To be a good mathematican, it is very helpful to be extremely pedantic!

Theorem (Cancellation)

Let V be a vector space and $\vec{u}, \vec{v}, \vec{w} \in V$. If

$$\vec{u} + \vec{w} = \vec{v} + \vec{w}$$

the

$$\vec{u} = \vec{v}$$

Proof.

$$\begin{aligned} \vec{u} &= \vec{u} + \vec{0} \\ &= \vec{u} + (\vec{w} - \vec{w}) \\ &= (\vec{u} + \vec{w}) - \vec{w} \\ &= (\vec{v} + \vec{w}) - \vec{w} \\ &= \vec{v} + (\vec{w} - \vec{w}) \\ &= \vec{v} + \vec{0} \\ &= \vec{v} \end{aligned}$$

■

Proposition Let V be a vector space and $\vec{v} \in V$, then

$$0\vec{v} = \vec{0}$$

Explain the difference between 0 and $\vec{0}$.

Proof.

$$\begin{aligned} \vec{0} + 0\vec{v} &= 0\vec{v} \\ &= (0 + 0)\vec{v} \\ &= 0\vec{v} + 0\vec{v} \end{aligned}$$

So by the cancellation theorem we can simplify

$$\vec{0} + 0\vec{v} = 0\vec{v}$$

to

$$\vec{0} = 0\vec{v}$$

■

Proposition Let V be a vector space and $\vec{v} \in V$, then

$$-1 \cdot \vec{v} = -\vec{v}$$

Proof.

$$\begin{aligned} -\vec{v} &= -\vec{v} + \vec{0} \\ &= -\vec{v} + 0\vec{v} \\ &= -\vec{v} + (1 + (-1))\vec{v} \\ &= -\vec{v} + (1 + (-1))\vec{v} \\ &= -\vec{v} + 1\vec{v} + (-1) \cdot \vec{v} \\ &= -\vec{v} + \vec{v} + (-1) \cdot \vec{v} \\ &= \vec{0} + (-1) \cdot \vec{v} \\ &= -1 \cdot \vec{v} \end{aligned}$$

■

Notice that the symbol $+$ does *not* necessarily refer to the standard addition, it could be defined in a different way as we can observe in the following.

Discussion

Let V be the set of 2-tuples of real numbers $\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$.

Define addition of 2-tuples as

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \diamond \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} u_1 + v_1 + 1 \\ u_2 + v_2 + 1 \end{pmatrix}$$

Let scalar multiplication be given by

$$\alpha \star \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} \alpha u_1 + \alpha - 1 \\ \beta u_2 + \beta - 1 \end{pmatrix}$$

Is there an additive identity? Are there inverses? Is $(V, \diamond, \star)$ a vector space?

Hint Look at the propositions from the previous page.

Subspaces

Textbook: Section 1.2

Definition
A *subspace* U of a vector space $(V, +, \cdot)$ is a subset $U \subseteq V$ that is a vector space in its own right (with the same addition and scalar multiplication of V).

This is the same idea as for subsets, only that the sets have been 'upgraded' to vector spaces.

Exmaples

1. As a first example, consider the vector spaces of polynomials $\mathcal{P}_n(\mathbb{R})$ and of functions $\mathcal{F}(\mathbb{R}, \mathbb{R})$ we have already encountered. Now, every polynomial with real coefficients is automatically a function $\mathbb{R} \to \mathbb{R}$.

   Is this enough to be a subspace?

   We also need to check that the operations of addition and scalar multiplication of polynomials are the same when we consider polynomials to be functions.

2. Sine we can view a vector in $\mathbb{R}^2$, that is a tuple with 2 entries $\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$, also as a vector in $\mathbb{R}^3$, namely a tuple with 3 entries, by adding the value 0 at the last entry,

$$\begin{pmatrix} v_1 \\ v_2 \\ 0 \end{pmatrix}$$

   it is clear that $\mathbb{R}^2 \subseteq \mathbb{R}^3$ as sets.

   As before, we need to check addition and scalar multiplication.

Discussion

Let V be a real vector space.

1. Can $U \subseteq V$ be a subspace if U is empty (i.e. has no elements)?
2. Suppose U is a subset of V, which vector space axioms does U automatically inherit from V?
3. Following the previous question, which axioms are left to check for U to be a vector space?

As we observed, we don't need to start from scratch to check all 10 axioms for subspaces.

Theorem 1.2.8

Let V be a vector space with a subset U. Then $U \subseteq V$ is a subspace if and only if

1. U is nonempty
2. $\alpha\vec{u} + \vec{v} \in U$ for all $\vec{u}, \vec{v} \in U$ and $\alpha \in \mathbb{R}$.

Proof. ■

We are now equipped with a very powerful tool to quickly test if a subset is a subspace.

Examples / Discussion

Decide which of the following subsets in vector spaces are subspaces.

1. Continuous functions $C^1(\mathbb{R},\mathbb{R})$ contained in all functions $\mathcal{F}(\mathbb{R},\mathbb{R})$
2. Invertible $n \times n$ matrices $\{A \in \mathrm{Mat}_n(\mathbb{R}) | A \text{ is invertible}\}$ contained in all matrices $\mathrm{Mat}_n(\mathbb{R})$
3. The set $\left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \mid x + y = 1 \right\}$ in $\mathbb{R}^2$
4. Nonnegative real numbers $\mathbb{R}_{\geq 0} = \{r \in \mathbb{R} \mid r \geq 0\}$ in $\mathbb{R}$
5. The plane of vectors $\left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; y = 0 \right\}$ in $\mathbb{R}^3$
6. Only the zero element $\{\vec{0}\}$ in $\mathbb{R}^2$
7. The space of *even* and *odd* functions respectively
$$\mathcal{F}(\mathbb{R},\mathbb{R})^{\text{even}} = \{f \in \mathcal{F}(\mathbb{R},\mathbb{R}) \mid f(x) = f(-x)\}$$
$$\mathcal{F}(\mathbb{R},\mathbb{R})^{\text{odd}} = \{f \in \mathcal{F}(\mathbb{R},\mathbb{R}) \mid -f(x) = f(-x)\}$$
in all functions $\mathcal{F}(\mathbb{R},\mathbb{R})$

Hint : Remember that the easiest way to disprove a *general* statement is to give a counterexample.

Theorem

Let $U, W \subseteq V$ be subspaces in a vector space V, then

1. $U \cap W$ is a subspace in V (Theorem 1.2.13)
2. $U + W$ is a subspace in V

Proof. ■