Linear Transformations Part II

Textbook: Section 2.2

Announcements

- The last hour before the lecture next week will be a review session! Please collect your questions and email me if you would like to discuss anything particular. (include MAT224 in subject, thanks)
- All vector spaces from now on, unless stated otherwise, will be assumed to be finite dimensional.

Remember that a basis encodes a vector $\vec{v} \in V$ as an n-tuple. We can use the same idea to encode linear transformations

Example 2.2.2

Let $V = W = \mathbb{R}^2$ with the standard basis $\{\vec{e}_1, \vec{e}_2\}$. Define $V \xrightarrow{T} W$ by

$$T(\vec{e}_1) = \vec{e}_1 + \vec{e}_2$$

$$T(\vec{e}_2) = \vec{2}e_1 - 2\vec{e}_2$$

Algorithm

Given a transformation $V \xrightarrow{T} W$ given as a 'formula', this is how to compute its matrix in two chosen bases

$$\alpha = \{\vec{\alpha}_1, \dots \vec{\alpha}_m\}$$

of V and

$$\beta = \{\vec{\beta}_1, \dots \vec{\beta}_n\}$$

of W

1. For each basis element $\vec{\alpha}_i$ in V, compute $T(\vec{\alpha}_i)$.
2. Find the coorindate vector $\gamma^\beta(T(\vec{\alpha}_i)) = [T(\vec{\alpha}_i)]_\beta$.
3. Assemble these coordinate vectors as columns in a matrix

Discussion

Apply the above algorithm to find the matrix representing the derivative $\frac{d}{dx}$ from $\mathcal{P}_2(\mathbb{R})$ to itself. Choose the basis on $\mathcal{P}_2(\mathbb{R})$ consisting of monomials $\alpha = \{1, x, x^2\}$.

Definition 2.2.6

Let T be a transformation between finite dimensional vector spaces V and W with bases α and β respectively. The *matrix of the linear transformation* T with respect to bases α and β is the matrix $[T]_\alpha^\beta$ satisfying

$$[T]_\alpha^\beta \cdot [\vec{v}]_\alpha = [T(\vec{v})]_\beta$$

Discussion

1. What does he 'size' of the matrix $[T]_\alpha^\beta$ depend on?
2. What is the matrix of the identity transformation $V \xrightarrow{\mathrm{id}_V} V$?

Exmaple

We compute the matrix of the linear transformation $\mathcal{P}_3(\mathbb{R}) \xrightarrow{\mathrm{ev}_2} \mathbb{R}$

Discussion

On the contrary, given a matrix $A \in \mathrm{Mat}_{n,m}(\mathbb{R})$, does this give us a transformation? What are the domain and codomain?

Summary

1. The upshot of this section is that linear transformations are completely interchangable with matrices! The identification depends on the choice of bases for the domain and codomain.
2. The operations
   - matrix of a transformation $[T]_{\alpha}^{\beta}$
   - transformation of a matrix T_A

   are inverse to each other.

We now rephrase the *algorithm* from before in more mathematical terms. (Remember, abstraction is a powerful tool!)

Proposition

In the context of the above definition, the matrix of T can be computed as

$$[T]_\alpha^\beta = \gamma^\beta \circ T \circ (\gamma^\alpha)^{-1}$$

Proof. ■

Discussion

Without doing a lot of work, can you argue what the matrix representing the composition $F \circ T$ is assuming you know $[F]$ and $[T]$?

Definition

Let $V \xrightarrow{T} W$ be a linear transformation and pick $\vec{w} \in W$. The *preimage of* $\vec{w}$ *under* T is the subset of all $\vec{v} \in V$ that map to $\vec{w}$ under T.

$$T^{-}1(w)\{\vec{v} \in V \mid T(\vec{v}) = \vec{w}\}$$

We do *not* say that T is invertible here and T^{-1} is *not* the inverse of T