

The Determinant

Textbook: Sections 3.1, 3.2 & 3.3

Observation (3.1.6)

When is a 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ invertible?

Definition

We define the determinant function $\text{Mat}_2(\mathbb{R}) \xrightarrow{\det} \mathbb{R}$ to be $\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$

Remark

1. As we observed above, a 2×2 matrix is invertible if and only if $\det(A) \neq 0$.
2. It is helpful to think about $A = [\vec{x} \ \vec{y}]$ to consist of column vectors $\vec{x}, \vec{y} \in \mathbb{R}^2$.

Proposition (3.1.1)

The area of the parallelogram spanned by vectors \vec{x} and \vec{y} in \mathbb{R}^2 is equal to the determinant of the matrix $A = [\vec{x} \ \vec{y}]$.

From this it is clear that the area is nonzero if and only if the vectors spanning the parallelogram are linearly independent, that is, not parallel.

Observation

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a 2×2 matrix and denote its columns as $\vec{x} = \begin{pmatrix} a \\ c \end{pmatrix}$ and $\vec{y} = \begin{pmatrix} b \\ d \end{pmatrix}$.

We can think of A as the matrix of a linear transformation

$$\mathbb{R}^2 \rightarrow \mathbb{R}^2$$

sending

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \mapsto \begin{pmatrix} a \\ c \end{pmatrix}$$
$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} b \\ d \end{pmatrix}$$

Discussion

Compute the determinant, draw the parallelogram and decide if the corresponding transformation is invertible for the following matrices.

1.

$$A = \begin{pmatrix} 1 & 2 \\ 3 & -4 \end{pmatrix}$$

2.

$$A = \begin{pmatrix} 4 & 2 \\ -2 & 1 \end{pmatrix}$$

Remark

1. Let again $A = [\vec{x} \ \vec{y}]$ be a 2×2 matrix with columns \vec{x} and \vec{y} . The determinant function as a function of the columns $\det(\vec{x}, \vec{y})$ has the following properties
 - (a) linear in both arguments
 - (b) alternating
 - (c) normalized
2. The book calls this the area function $Area(\vec{x}, \vec{y})$ of the vectors spanning the parallelogram.

Proposition (3.1.4)

The determinant function is the unique function satisfying all properties above.

Proof. In the book ■

Discussion

Can you show that

$$\det([\vec{x} \ \vec{y}]) = 0$$

if $\{\vec{x}, \vec{y}\}$ is linearly dependent by only using the properties above?

Determinant of $n \times n$ matrices

Textbook: Sections 3.2

Goal

Generalize the \det function to all $n \times n$ matrices as a function of n vectors $\vec{x}_1, \dots, \vec{x}_n$ with an equivalent set of properties.

1. multilinear

2. alternating

3. normalized

Definition

Let $A \in \text{Mat}_n(\mathbb{R})$ be a $n \times n$ matrix. Define the ij -minor A_{ij} to be the $(n-1) \times (n-1)$ matrix obtained by deleting the i -th row and j -th column.

Discussion

Compute for the matrix $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$ the minor A_{23} /

Proposition (3.2.5, 3.2.6)

There exists a unique multilinear and alternating function f on the columns of 3×3 matrices that is normalized such that $f(I_n) = 1$.

Definition (3.2.7)

We define the determinant of a 3×3 matrix to be this unique function.

$$\det(A) = a_{11}\det(A_{11}) - a_{12}\det(A_{12}) + a_{13}\det(A_{13})$$

Discussion

Compute the determinant of the matrix

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 3 & 1 \\ 0 & 3 & 2 \end{pmatrix}$$

Remarks

1. The formula through which we defined the determinant function is called the *cofactor expansion*.
2. It is also common to call

$$a_{ij} \det(A_{ij})$$

the *ij-cofactor* of a matrix A .

3. There is nothing special about the first row, we can expand the $\det(A)$ along any row or column.

Example

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 3 & 1 \\ 0 & 3 & 2 \end{pmatrix}$$

Definition We define for an $n \times n$ matrix the determinant function

$$\det(A) = \sum_{j=1}^n (-1)^{i+j} a_{ij} \det(A_{ij})$$

Theorem 3.2.8

1. There is exactly one alternating multilinear function $\text{Mat}_n(\mathbb{R}) \xrightarrow{f} \mathbb{R}$ such that $f(I_n) = 1$, which is the determinant function defined above.
2. Any other alternating multilinear function f on square $n \times n$ matrices satisfies

$$f(A) = \det(A) \cdot f(I_n)$$

Discussion

Write the first line of the cofactor expansion of the determinant for the matrix

$$A = \begin{pmatrix} 1 & -3 & 1 & 4 \\ 8 & -4 & 2 & -2 \\ 3 & -7 & 5 & 3 \\ 0 & 3 & -2 & 4 \end{pmatrix}$$

Exercise: Read Example 3.2.9 in the book and practice computing determinants.

Theorem (3.2.14)

An $n \times n$ matrix A is invertible if and only if $\det(A) \neq 0$.

Proof. In the book



Remark

Notice that this agrees perfectly with the discussion above in the 2×2 case.

Further properties of determinants

Textbook: Sections 3.3

Proposition (3.3.7)

If A and B are $n \times n$ matrices, then

1. $\det(AB) = \det(A)\det(B)$
2. If A is invertible, then $\det(A^{-1}) = \frac{1}{\det(A)}$

Proof. ■

This is very useful for transformations, because their matrix representative depends on a choice of basis

Corollary (3.3.8)

For a linear transformation $V \xrightarrow{T} V$ on a vector space of finite dimensional vector space V

$$\det([T]_{\alpha}^{\alpha}) = \det([T]_{\beta}^{\beta})$$

for any two bases α and β .

Proof. ■

So we can define independently of the chosen basis

Definition (3.3.9)

The determinant $\det(T)$ of a linear transformation $V \xrightarrow{T} V$ on a vector space of finite dimension is the determinant of $[T]_{\alpha}^{\alpha}$ for any choice of basis α .

Proposition (3.3.11)

A linear transformation $V \xrightarrow{T} V$ on a vector space of finite dimension is invertible if and only if $\det(T) \neq 0$.

Proof. ■

Proposition (3.3.12)

For two linear transformations $V \xrightarrow{S} V$ and $V \xrightarrow{T} V$ on a vector space of finite dimension, then

1. $\det(ST) = \det(S) \cdot \det(T)$
2. if T is an isomorphism, then $\det(T^{-1}) = \frac{1}{\det(T)}$

Proof. ■

Eigenvalues Pt I

Textbook: Section 4.1