Complex numbers

Textbook: Section 5.1

Motivation

We have seen that the matrix $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ has the characteristic polynomial $c_A(\lambda) = \lambda^2 + 1$ which has no real roots. A root would be a square root $\sqrt{-1}$. Complex numbers solve this problem.

Definition

The set of complex number $\mathbb{C}$ is the set vector space $\mathbb{R}^2$ with an additional multiplication of two complex numbers

$$\begin{pmatrix} a \\ b \end{pmatrix} * \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} ac - bd \\ ad + bc \end{pmatrix}$$

Remark

One often writes these coordinate tuples as vectors in the basis $\{1, i\}$. The coordinate vector $\begin{pmatrix} a \\ b \end{pmatrix}$ corresponds then to $a + ib$.

Discussion

1. Express $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 4 \end{pmatrix}$ in the basis and determine $\begin{pmatrix} 2 \\ 3 \end{pmatrix} * \begin{pmatrix} -1 \\ 4 \end{pmatrix}$
2. Let $W = \operatorname{span}\{1\} \subseteq \mathbb{C}$. Show that W is closed under the multiplication $*$.
3. Can we say that W is isomorphic to $\mathbb{R}$? If so, why?

Definition

Let $z = a + ib \in \mathbb{C}$. We call $Re(z) = a$ the *real part* of z and $Im(z) = b$ the *imaginary part* of z.

Discussion

Let $p(x) = x^2 + 1$ and $q(x) = x^4 + 1$.

1. Find all roots of $p(x)$ in $\mathbb{C}$.
2. Find all solutions to the equation $i \cdot z = 1$. Can you now give meaning to the expression i^{-1}?
3. Find all roots of $q(x)$ in $\mathbb{C}$.
4. What is the inverse of a complex number $a + ib$ in general?

Discussion

Let $w = a + ib$ and consider $\mathbb{C} \xrightarrow{T_w} \mathbb{C}$ to be $T_w(z) = w \cdot z$.

1. Is T_w a linear transformation between real vector spaces? If it is, what is the matrix representing it?
2. Under what condition is T_w invertible?

Discussion

Let $\mathbb{C} \xrightarrow{T} \mathbb{C}$ to be the function $T(a+ib) = a - ib$.

1. Is T_w a linear transformation between real vector spaces? If it is, what is the matrix representing it?
2. What are eigenvalues and eigenspaces of T?
3. What is $T \circ T$? Is T invertible?

Fields

Textbook: Section 5.1

Goal

We would like to replace for vector spaces real numbers $\mathbb{R}$ with complex numbers $\mathbb{C}$ because of their obvious advantage that some characteristic polynomials have roots in $\mathbb{C}$ but not in $\mathbb{R}$. In order to do so, we axiomatize all properties that our prototypes $\mathbb{R}$ and $\mathbb{C}$ satisfy.

Definition (5.1.4)

A *field* is a set F together with two operations called *addition* $(+)$ and *multiplication* $(\cdot)$ if the following axioms are satisfied:

Discussion

Which of the following sets are fields? If they are not field, explain one axiom that does not hold.

$\mathbb{N}$

$\mathbb{Z}$

$\mathbb{Q}$

$\mathbb{R}$

$\mathbb{R}^+ = \{x \in \mathbb{R} \mid x \geq 0\}$

$\mathbb{C}$

Example

The finite field $\mathbb{F}_p$ with p elements where p is a prime.

Exercise

1. Write out the addition and multiplication tables for the field $\mathbb{F}_3$.
2. How can you tell that addition and multiplication are commutative?
3. How can you conclude that 0 is the additive identity?
4. How can you conclude that every element has an additive inverse?
5. How can you conclude that 1 is the multiplicative identity?
6. How can you conclude that every non-zero element has a multiplicative inverse?

Discussion

1. What are the roots of $p(x) = x^2 - 1$ in $\mathbb{F}_2$? How about $\mathbb{F}_3$?
2. For each of the fields $\mathbb{R}, \mathbb{Q}, \mathbb{C}$ and $\mathbb{F}_2$, if possible give an example of
   (a) a polynomial that has a root in the field
   (b) a polynomial that does not have a root in the field.

Definition (5.1.11)

A field F is called algebraically closed if every non-constant polynomial with coefficients in F has a root in F.

Discussion

Decide which of the fields $\mathbb{R}, \mathbb{Q}, \mathbb{C}$ and $\mathbb{F}_2$ is algebraically closed.

Theorem (5.1.12 Fundamental Theorem of Algebra)

$\mathbb{C}$ is algebraically closed, that is, every plynomial of degree n

$$p(x) = a_n x^n + \cdots + a_0$$

has n roots counted with multiplicity.

Vector Spaces over a Field

Textbook: Section 5.2

Definition 5.2.1
A vector space $(V, +, \cdot)$ over a field F consists of a set V and two operations that we call *addition* $(+)$ and *scalar multiplication* $\cdot$

$$V \times V \xrightarrow{+} V$$
$$F \times V \xrightarrow{\cdot} V$$

such that the following axioms hold

1. (additive closure) $\vec{x} + \vec{y} \in V$, for all $\vec{x}, \vec{y} \in V$
2. (multiplicative closure) $\alpha \cdot \vec{x} \in V$, for all $\vec{x} \in V$ and scalars $\alpha \in F$
3. (commutativity) $\vec{x} + \vec{y} = \vec{y} + \vec{x}$, for all $\vec{x}, \vec{y} \in V$
4. (additive associativity) $(\vec{x} + \vec{y}) + \vec{z} = \vec{x} + (\vec{y} + \vec{z})$, for all $\vec{x}, \vec{y}, \vec{z} \in V$
5. (additive identity) There exists a vector $\vec{0} \in V$ such that $\vec{x} + \vec{0} = \vec{x}$ for all $\vec{x} \in V$
6. (additive inverse) For each $\vec{x} \in V$, there exists a vector $-\vec{x} \in V$ with the property that $\vec{x} + (-\vec{x}) = \vec{0}$
7. (multiplicative associativity) $(\alpha \cdot \beta) \cdot \vec{x} = \alpha \cdot (\beta \cdot \vec{x})$, for all $\alpha, \beta \in F$ and $\vec{x} \in V$
8. (distributivity over vector addition) $\alpha \cdot (\vec{x} + \vec{y}) = \alpha\vec{x} + \alpha\vec{y}$, for all $\alpha \in F$ and $\vec{x}, \vec{y} \in V$
9. (distributivity over scalar addition) $(\alpha + \beta) \cdot \vec{x} = \alpha\vec{x} + \beta\vec{x}$, for all $\alpha, \beta \in F$ and $\vec{x} \in V$
10. (identity property) $1 \cdot \vec{x} = \vec{x}$, for all $\vec{x} \in V$, $1 \in F$

Examples

1. F^n, the set of n-tuples in a field F
2. $\mathrm{Mat}_n(F)$, the set of $n \times n$ matrices with entries from F.
3. $\mathcal{P}_n(F)$, the set of polynomials of degree n with coefficients in F.
4. $\mathbb{C}^n$ may be thought of as a complex vector space or a real vector space.

Remark

Specifying the field for a vector space is important! The vectors $\begin{pmatrix} i \\ 2 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 2i \end{pmatrix}$ in are dependent in $\mathcal{C}^2$ as a complex vector space, but independent in $\mathcal{C}^2$ as a real vector space.

Discussion

Consider $V = \mathbb{R}$ as a vector space over $\mathbb{Q}$.

1. Is the family of vector $\{1, \sqrt{2}\}$ linearly independent or dependent?
2. What is the dimension of the subspace $\text{span}\{1, \sqrt{2}, \sqrt{3}, \sqrt{4}\}$?
3. What do the above results suggest about $\dim(V)$?

Remark

Everything we did about vector spaces, matrices, inverse matrices, determinants, eigenvalues, ... can be applied to a vector space over a field. See Example (5.2.8-10) in the book for examples.