Bases and Dimension

Textbook: Section 1.6

Definition 1.6.1

A family of vectors $\mathcal{B}$ in a vector space V is called *a basis of* V if

1. $\mathcal{B}$ spans V
2. $\mathcal{B}$ is linearly independent

Examples

1. $\{\vec{e}_1, \ldots, \vec{e}_n\}$ is a basis of $\mathbb{R}^n$.
2. We have seen last week that $\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix}\}$ is a basis of $\mathbb{R}^2$
3. Which of families of polynomials we have seen before is a basis of $\mathcal{P}_2(\mathbb{R})$? Can you write down a basis of $\mathcal{P}_2(\mathbb{R})$ that does not contain any monomials?

Theorem 1.6.3

A family of vectors $\mathcal{B}$ is a basis of V if and only if every vector $\vec{v} \in V$ can be written uniquely as a linear combination of vectors in $\mathcal{B}$.

Proof. ■

Remark

1. A vector space does not have just one unique basis as we can easily verify.

Theorem 1.6.6

Let V be a vector space with a finite spanning set. For every linearly independent family S in V, there is a basis $\mathcal{B}$ containing S.

Why do we need a finite spanning set for V? Some vector spaces, such as $\mathcal{P}(\mathbb{R})$ can not be spanned by finitely many polynomials. Can you show why?

Observe that a basis hits a sweet spot of a family that is not too large as that it would contain redundant vectors, but also not too small of a family that it couldn't span the vetcor space.
The *Extend* and *Reduce* theorems from last week give us the following:

A family of vectors that is linearly independent, but not a spanning set can be enlagred to a basis, a family that spans V, but is linearly dependent, can be cut down to form a basis.

Even though a basis is not unique to a vector space, we would like to extract an invariant, a label, something that characterizes the vectors space. This invariant is motivated by the Corollary following below.

Theorem 1.6.10

If V is spanned by a family S with m elements, then no linearly independent family R in V can have more than m elements.

Proof. ■

Corollary 1.6.11

Any two bases $\mathcal{B}$ and $\mathcal{B}'$ of V have the same number of elements

Definitions

1. If a vector space V has a finite basis, we say that V is *finite dimensional*.
2. For a finite dimensional vector space V, the *dimension* of V

$$\dim(V)$$

   is the number of elements of a basis of V.

Discussion

What is the dimension of

$\dim(\mathbb{R}) =$

$\dim(\mathcal{P}_n(\mathbb{R})) =$

$\dim(\mathrm{Mat}_2(\mathbb{R})) =$

$\dim(\mathrm{Mat}_2^{\mathrm{sym}}(\mathbb{R})) =$

$\dim(\mathrm{Mat}_2^{\mathrm{anti}}(\mathbb{R})) =$

Remember that a matrix A is symmetric if $A^T = A$ and antisymmetric if $A^T = -A$.

Discussion

Can you argue that if $U \subseteq V$ is a subspace then $\dim(U) \leq \dim(V)$?
Is the converse true that if $\dim(U) = \dim(V)$ then $U = V$?