Announcements for the test

1. Don't quote numbers of Theorems, but their content and why they apply in this case.
2. Famous theorems have names, for example *Extend-, Reduce-* and *Fundamental* Theorem.
3. The test covers everything we did from sections 1.1 - 1.6 and section 2.1, homework problems 1 to 4 and Assignments 1 & 2
4. Please read the *Term-Test-1 information document* about logistics well in advance!
5. *Don't* leave the upload to the last minute! Plan for some technical difficulties, they always occur! :(
6. Read question statements carefully and slowly! Make sure you distinguish *if* vs. *if and only if* in statements. Give a proof or explanation when the question requires one.
7. While we write $\vec{v}$ for vectors, the book and other sources may write $\mathbf{x} \in V$.
8. By abuse of notation, the image of a vector under a transformation $T(\vec{v})$ can sometimes be abbreviated as $T\mathbf{x}$.

Kernel and Image

Textbook: Section 2.3

Definition (2.3.1 & 2.3.10)

For a linear transformation $V \xrightarrow{T} W$, we define

1. the *preimage* $T^{-1}(S)$ of $S \subseteq W$ under T as all $\vec{v} \in V$ that map into S.
2. the *kernel* $\ker(T)$ of T as all $\vec{v} \in V$ that map to $\vec{0}$ under T,
3. the *image* $\operatorname{im}(T)$ of T as all $\vec{w} \in W$ such that $\vec{w} = T(\vec{v})$ for some $\vec{v} \in V$,

Example

- The kernel of $\mathcal{P}_n(\mathbb{R}) \xrightarrow{\frac{d}{dx}} \mathcal{P}_n(\mathbb{R})$ are all constant polynomials, while the image consists of polynomials of degree $n-1$.
- The kernel of the evaluation map $\mathcal{P}_n(\mathbb{R}) \xrightarrow{\mathrm{ev}_2} \mathbb{R}$ are all polynomials that have a root at $x = 2$. What is the image?
- What is the image of the linear transformation defined in example 2.2.2 $\mathbb{R}^2 \xrightarrow{T} \mathbb{R}^2$?

$$T(\vec{e}_1) = \vec{e}_1 + \vec{e}_2$$

$$T(\vec{e}_2) = \vec{2}e_1 - 2\vec{e}_2$$

Proposition 2.3.2 & 2.3.11

For every linear transformation $V \xrightarrow{T} W$

1. $\ker(T)$ is a subspace in V
2. $\operatorname{im}(T)$ is a subspace in W.

Proof. ■

Proposition 2.3.7

Let $V \xrightarrow{T} W$ be a linear transformation between vector spaces V and W with bases α and β respectively.

The subspace $\ker(T)$ is isomorphic to the solution space to the homogeneous system of $[T]_\alpha^\beta$.

That is,

- for every $\vec{v} \in \ker(T)$ the coordinate tuple $[\vec{v}]_\alpha$ solves $[T]_\alpha^\beta \vec{x} = \vec{0}$.
- for every solution $\vec{s} = \begin{pmatrix} s_1 \\ \vdots \\ s_n \end{pmatrix}$ of the system $[T]_\alpha^\beta \vec{x} = \vec{0}$, the *realization* in the basis α, namely

$$(\gamma^\alpha)^{-1}\left(\begin{pmatrix} s_1 \\ \vdots \\ s_n \end{pmatrix}\right)$$

is a vector in $\ker(T)$.

Proof. ■

Example

Find the kernel of the evaluation map $\mathcal{P}_n(\mathbb{R}) \xrightarrow{\mathrm{ev}_2} \mathbb{R}$.

Observation (analogous to 2.3.7)

Let $V \xrightarrow{T} W$ be a linear transformation between vector spaces V and W with bases α and β respectively.

The subspace $\operatorname{im}(T)$ is isomorphic to the space of all $\vec{b} \in \mathbb{R}^m$ such that the system of linear equations $[T]_\alpha^\beta \vec{x} = \vec{b}$ has a solution

That is,

- for every $\vec{w} \in \operatorname{im}(T)$ the system $[T]_\alpha^\beta \vec{x} = [\vec{w}]_\beta$ has a solution.
- for every constant vector $\vec{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$ such that the system $[T]_\alpha^\beta \vec{x} = \vec{b}$ has a solution, the *realization* in the basis β, namely
$$(\gamma^\beta)^{-1}\left(\begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}\right)$$
is a vector in $\operatorname{im}(T)$.

Proof. ■

Discussion

Based on this observation, which $\vec{v} \in V$ will be in the preimage $T^{-1}(\vec{w})$?

Definition

For a matrix $A = [a_1, a_2, \ldots, a_m] \in \mathrm{Mat}_{n,m}(\mathbb{R})$ we denote the span of the columns of A by

$$\mathrm{col}(A) = \mathrm{span}\{a_1, \ldots, a_m\}$$

Lemma

For a matrix $A = [a_1, a_2, \ldots, a_m] \in \mathrm{Mat}_{n,m}(\mathbb{R})$ the system

$$A\vec{x} = \vec{b}$$

has a solution if and only if $\vec{b} \in \mathrm{col}(A)$

Proof. ■

Proposition

Let $V \xrightarrow{T} W$ be a linear transformation between vector spaces V and W with bases α and β respectively.

The image of T is isomorphic to $\mathrm{col}([T]_\alpha^\beta)$ under the coordinate map $W \xrightarrow{\gamma^\beta} \mathbb{R}^m$.

Proof. ■

Example

Find the image of the linear transformation

$$\mathbb{R}^2 \to \mathbb{R}^3$$
$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x+y \\ x \\ y \end{pmatrix}$$

Notice that the columns might not be independent, in which case the columns are a spanning set of the image, but not a basis.

Theorem (Nicholson - Linear algebra with Applications)

Given a linear transformation $V \xrightarrow{T} W$ with matrix $[T]_\alpha^\beta$ for some bases α and β. Let $R = \mathrm{RREF}([T]_\alpha^\beta)$ be the reduced row echolon form of $[T]_\alpha^\beta$.

Then if the leading 1s in are R lie in columns $j_1, j_2, \ldots, j_r$, the columns $j_1, j_2, \ldots, j_r$ of $[T]_\alpha^\beta$ are a basis for $\mathrm{col}([T]_\alpha^\beta)$

Proof. ■

Discussion

Suppose a linear transformation $V \xrightarrow{T} W$ is given in a some bases α and β by

$$[T]_\alpha^\beta = \begin{pmatrix} 1 & 2 & 0 & 1 \\ 1 & 2 & 1 & 0 \end{pmatrix}$$

Find a basis for $\mathrm{im}(T)$ and $\ker(T)$.

A similar question is discussed in Example 2.3.9 in the book.

Discussion

Find the kernel, the image and bases of them for the following transformations in bases of your choice.

1. tbd

Dimension Theorem

Textbook: Section 2.3 & 2.4

Theorem 2.3.17 (Rank-Nullity)

For any linear transformation $V \xrightarrow{T} W$

$$\dim(V) = \dim(\ker(T)) + \dim(\operatorname{im}(T))$$

Remark

- $\dim(\operatorname{im}(T))$ is the same as the rank of $[T]_\alpha^\beta$ and by abuse of notation also referred to as $\operatorname{rank}(T)$.
- Some books refer to $\dim(\ker(T))$ as the *nullity* of T.

Proof. ■

Theorem

A linear transformation T is injective if and only if $\ker(T) = \{\vec{0}\}$

Proof. ■

True or False Let $V \xrightarrow{T} W$ be a linear transformation

- ☐ If T is an isomorphism, then $\dim(V) = \dim(W)$.
- ☐ If $\dim(V) > \dim(W)$, T has to be injective.
- ☐

Composition

Textbook: Section 2.5

Discussion

Without doing a lot of work, can you argue what the matrix representing the composition $F \circ T$ is assuming you know $[F]$ and $[T]$?

Review session

- Vector spaces
- subspaces
- Sum, intersection, direct sum
- Linear independence
- Spanning set
- Basis
- Dimension
- Linear Transformation