# Linear Transformations Part II

**Textbook:** Section 2.2

### Announcements

- The last hour before the lecture next week will be a review session! Please collect your questions and email me if you would like to discuss anything particular. (include MAT224 in subject, thanks)
- All vector spaces from now on, unless stated otherwise, will be assumed to be finite dimensional.

Remember that a basis encodes a vector $\vec{v} \in V$ as an $n$-tuple. We can use the same idea to encode linear transformations

### Example 2.2.2

Let $V = W = \mathbb{R}^2$ with the standard basis $\{\vec{e}_1, \vec{e}_2\}$. Define $V \xrightarrow{T} W$ by

$$T(\vec{e}_1) = \vec{e}_1 + \vec{e}_2$$

$$T(\vec{e}_2) = \vec{2}e_1 - 2\vec{e}_2$$

**Algorithm**

Given a linear transformation $V \xrightarrow{T} W$ given as a 'formula', this is how to compute its matrix in two chosen bases

$$\alpha = \{\vec{\alpha}_1, \dots \vec{\alpha}_m\}$$

of $V$ and

$$\beta = \{\vec{\beta}_1, \dots \vec{\beta}_n\}$$

of $W$

1. For each basis element $\vec{\alpha}_i$ in $V$, compute $T(\vec{\alpha}_i)$.
2. Find the coorindate vector $\gamma^\beta(T(\vec{\alpha}_i)) = [T(\vec{\alpha}_i)]_\beta$.
3. Assemble these coordinate vectors as columns in a matrix

**Discussion**

Apply the above algorithm to find the matrix representing the derivative $\frac{d}{dx}$ from $\mathcal{P}_2(\mathbb{R})$ to itself. Choose the basis on $\mathcal{P}_2(\mathbb{R})$ consisting of monomials $\alpha = \{1, x, x^2\}$.

**Definition 2.2.6**

Let $T$ be a linear transformation between finite dimensional vector spaces $V$ and $W$ with bases $\alpha$ and $\beta$ respectively. The *matrix of the linear transformation* $T$ with respect to bases $\alpha$ and $\beta$ is the matrix $[T]_\alpha^\beta$ satisfying

$$[T]_\alpha^\beta \cdot [\vec{v}]_\alpha = [T(\vec{v})]_\beta$$

**Discussion**

1. What does he 'size' of the matrix $[T]_\alpha^\beta$ depend on?
2. What is the matrix of the identity transformation $V \xrightarrow{\mathrm{id}_V} V$?

**Exmaple**

We compute the matrix of the linear transformation $\mathcal{P}_3(\mathbb{R}) \xrightarrow{\mathrm{ev}_2} \mathbb{R}$

**Discussion**

On the contrary, given a matrix $A \in \mathrm{Mat}_{n,m}(\mathbb{R})$, does this give us a linear transformation? What are the domain and codomain?

**Summary**

1. The upshot of this section is that linear transformations are completely interchangable with matrices! The identification depends on the choice of bases for the domain and codomain.

2. The operations

   - matrix of a linear transformation $[T]_{\alpha}^{\beta}$
   - linear transformation of a matrix $T_A$

   are inverse to each other.

We now rephrase the *algorithm* from before in more mathematical terms. (Remember, abstraction is a powerful tool!)

### Proposition

In the context of the above definition, the matrix of $T$ can be computed as

$$[T]_\alpha^\beta = \gamma^\beta \circ T \circ (\gamma^\alpha)^{-1}$$

*Proof.* ■

### Discussion

Without doing a lot of work, can you argue what the matrix representing the composition $F \circ T$ is assuming you know $[F]$ and $[T]$?

**Definition (2.3.1 & 2.3.10)**

For a linear transformation $V \xrightarrow{T} W$, we define

1. the *preimage* $T^{-1}(S)$ of $S \subseteq W$ under $T$ as all $\vec{v} \in V$ that map into $S$.
2. the *kernel* $\ker(T)$ of $T$ as all $\vec{v} \in V$ that map to $\vec{0}$ under $T$,
3. the *image* $\operatorname{im}(T)$ of $T$ as all $\vec{w} \in W$ such that $\vec{w} = T(\vec{v})$ for some $\vec{v} \in V$,

**Example**

- The kernel of $\mathcal{P}_n(\mathbb{R}) \xrightarrow{\frac{d}{dx}} \mathcal{P}_n(\mathbb{R})$ are all constant polynomials, while the image consists of polynomials of degree $n-1$.
- The kernel of the evaluation map $\mathcal{P}_n(\mathbb{R}) \xrightarrow{\mathrm{ev}_2} \mathcal{P}_n(\mathbb{R})$ are all polynomials that have a root at $x = 2$. What is the image?
- What is the image of the linear transformation defined in example 2.2.2 $\mathbb{R}^2 \xrightarrow{T} \mathbb{R}^2$?

$$T(\vec{e}_1) = \vec{e}_1 + \vec{e}_2$$

$$T(\vec{e}_2) = \vec{2}e_1 - 2\vec{e}_2$$

**Proposition 2.3.2 & 2.3.11**

For every linear transformation $V \xrightarrow{T} W$

1. $\ker(T)$ is a subspace in $V$
2. $\operatorname{im}(T)$ is a subspace in $W$.

*Proof.* ■

**Proposition 2.3.7**

The subspace $\ker(T)$ is the solution space to the homogeneous system of $[T]_\alpha^\beta$.

*Proof.* ■

**Example**

Example of computation to find $\ker(T)$

**Observation**

The subspace $\text{im}(T)$ is the space of all $\vec{b} \in \mathbb{R}^n$ such that the system $[T]_\alpha^\beta \vec{x} = \vec{b}$ has a solution.

**Proposition 2.3.12**

If $\{\vec{v}_1, \ldots, \vec{v}_k\}$ spans $V$, then $\{T(\vec{v}_1), \ldots, T(\vec{v}_k)\}$ spans $\text{im}(T)$.

*Proof.* ■

**Definition**

For a matrix $A = [a_1, a_2, \ldots, a_m] \in \text{Mat}_{n,m}(\mathbb{R})$ we denote the span of the columns of $A$ by

$$\text{col}(A) = \text{span}\{a_1, \ldots, a_m\}$$

**Proposition**

For every linear transformation $V \xrightarrow{T} W$

$$\text{im}(T) = \text{col}([T]_\alpha^\beta)$$

*Proof.* ■

**Example**

Example computation to find $\operatorname{im}(T)$

Notice that the columns might not be independent, in which case the columns are a spanning set of the image, but not a basis.

**Theorem**

Given a linear transformation $V \xrightarrow{T} W$ with matrix $[T]_\alpha^\beta$ for some bases $\alpha$ and $\beta$. Let $R = \mathrm{RREF}([T]_\alpha^\beta)$ be the reduced row echolon form of $[T]_\alpha^\beta$.

Then if the leading 1s in are $R$ lie in columns $j_1, j_2, \ldots, j_r$, the columns $j_1, j_2, \ldots, j_r$ of $[T]_\alpha^\beta$ are a basis for $\operatorname{col}([T]_\alpha^\beta)$

*Proof.* ∎

**Discussion**

Suppose a linear transformation $V \xrightarrow{T} W$ is given in a some bases $\alpha$ and $\beta$ by

$$[T]_\alpha^\beta = \begin{pmatrix} 1 & 2 & 0 & 1 \\ 1 & 2 & 1 & 0 \end{pmatrix}$$

Find a basis for $\operatorname{im}(T)$ and $\ker(T)$.

**Theorem 2.3.17 (Rank-Nullity)**

For any linear transformation $V \xrightarrow{T} W$

$$\dim(V) = \dim(\ker(T)) + \dim(\mathrm{im}(T))$$

**Remark**

- $\dim(\mathrm{im}(T))$ is the same as the rank of $[T]_\alpha^\beta$ and by abuse of notation also referred to as $\mathrm{rank}(T)$.
- Some books refer to $\dim(\ker(T))$ as the *nullity* of $T$.

*Proof.* ■

**Theorem**

A linear transformation $T$ is injective if and only if $\ker(T) = \{\vec{0}\}$

*Proof.* ■

**True or False** Let $V \xrightarrow{T} W$ be a linear transformation

- ☐ If $T$ is an isomorphism, then $\dim(V) = \dim(W)$.
- ☐ If $\dim(V) > \dim(W)$, $T$ has to be injective.
- ☐