Review Session

Textbook: All of it! [1]

Discussion 1

1. Let $U \subseteq \mathcal{P}_n(\mathbb{C})$ be the subset of polynomials that satisfy $p'(x) = x \cdot p(x)$. Explain why this is a subspace or not.
2. Let W be the line of points in $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{C}^2$ that satisfy the equation

$$5x - 3iy + 2 = 0$$

Explain why this is a subspace or not.

[1] Everything we covered.

Discussion 2

1. Prove that U is a subspace of V if and only if $\text{span}\{U\} = U$.
2. We have seen that $\text{span}\{S\} = \text{span}\{R\}$ for families of vectors S and R in a vector space V if $S \subseteq \text{span}\{R\}$ and $R \subseteq \text{span}\{S\}$. Is the converse true as well?
3. Give an example where $R \neq S$, but $\text{span}\{R\} = \text{span}\{S\}$.

Discussion 3

Why can the vector space $\mathrm{Mat}_2(\mathbb{C})$ of 2×2 matrices be written as a direct sum of symmetric and antisymmetric matrices?

$$\mathrm{Mat}_2(\mathbb{C}) = \mathcal{S} \oplus \mathcal{A}$$

Discussion 4

Suppose $V = U \oplus W$ and S and R are linearly independent families of vectors in U and W respectively. Show that $S \cup R$ is linearly independent in V.

Discussion 5

Complete the family of polynomials $\{1 - x\ ,\ 1 - x + x^2\}$ to find a basis of $\mathcal{P}_3(\mathbb{R})$.

Discussion 6

1. Let T be a transformation from V to W. Is the preimage $T^{-1}(\vec{y})$ for $\vec{y} \in W$ a subspace of V?
2. Fix one vector $\vec{x}_0 \in T^{-1}(\vec{y})$. Show that any other vector $\vec{x} \in T^{-1}(\vec{y})$ can be written as

$$\vec{x} = \vec{x}_0 + \vec{k}$$

for some $\vec{k} \in \ker(T)$.

Discussion 7

Let T be a linear transformation on V.
Explain why the dimension of the kernel $\ker(T)$ is equal to $\dim V - \operatorname{rank}([T]_\alpha^\alpha)$.

Discussion 8

All matrices A in $\mathrm{Mat}_2(\mathbb{C})$ are similar to either $\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$ or $\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$.

Discussion 9

If N is a nilpotent transformation on V, then $\ker(N^j) < \ker(N^{j+1})$ for all $j < \text{index}(N)$.

Discussion 10

Compute the inverse of the left-shift transformation on a vector space V in a given a basis α.

Discussion 11

At how many points do we need to evaluate a polynomial of degree 3 to reconstruct it? Explain your answer in terms of a linear transformation

$$\begin{aligned} \mathcal{P}_3(\mathbb{R}) &\to \mathbb{R}^n \\ p(x) &\mapsto \begin{pmatrix} p(x_1) \\ \vdots \\ p(x_n) \end{pmatrix} \end{aligned}$$

Discussion 12

Two eigenvectors $\vec{v}$ and $\vec{w}$ of T with different eigenvalues λ and σ are linearly independent.

Discussion 13

What are the real and imaginary parts of $\frac{1}{a+ib}$?

Discussion 14

Consider $V = \mathbb{R}$ as a vector space over $\mathbb{Q}$.

1. Is the family of vector $\{1, \sqrt{2}\}$ linearly independent or dependent?
2. What is the dimension of the subspace $\text{span}\{1, \sqrt{2}, \sqrt{3}, \sqrt{4}\}$?
3. What do the above results suggest about $\dim(V)$?