Solutions to David A.Cox "Galois Theory"

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13 Chapter 13 : LAGRANGE, COMPUTING GALOIS GROUPS

13.1 QUARTIC POLYNOMIALS

Ex. 13.1.1 *Let $f \in F[x]$ be separable of degree n, and let $\alpha_1, \ldots, \alpha_n$ be the roots of f in a splitting field $F \subset L$ of f. In Section 6.3 we used the action of the Galois group on the roots to construct a one-to-one group homomorphism $\phi_1 : \mathrm{Gal}(L/F) \to S_n$. Now let $\beta_1, \ldots, \beta_n$ be the same roots, possibly written in a different order. This gives $\phi_2 : Gal(L/F) \to S_n$. To relate ϕ_1 and ϕ_2, note that there is $\gamma \in S_n$ such that $\beta_i = \alpha_{\gamma(i)}$ for $1 \leq i \leq n$. Now define the conjugation map $\hat{\gamma} : S_n \to S_n$ by $\hat{\gamma}(\tau) = \gamma^{-1}\tau\gamma$.*

(a) Prove that $\phi_2 = \hat{\gamma} \circ \phi_1$.

(b) Let $G \subset S_n$ be the image of ϕ_1. Explain why part (a) justifies the assertion made in the text that "if we change the labels, then G gets replaced with a conjugate subgroup".

Proof. (a) By definition of the isomorphism $\phi_1 : \mathrm{Gal}(L/F) \to S_n$ in Section 6.3, if $\tau_1 = \phi_1(\sigma)$, then

$$\sigma(\alpha_i) = \alpha_{\tau_1(i)}, \qquad i = 1, \ldots, n. \tag{1}$$

As $\beta_1, \ldots, \beta_n$ are the same roots in a different order, there exist a permutation $\gamma \in S_n$ such that

$$\beta_i = \alpha_{\gamma(i)}, \qquad i = 1, \ldots, n. \tag{2}$$

This numbering of the roots is associate to the isomorphism ϕ_2. If $\tau_2 = \phi_2(\sigma)$, then

$$\sigma(\beta_i) = \beta_{\tau_2(i)}, \qquad i = 1, \ldots, n. \tag{3}$$

Therefore, for all $i = 1, \ldots, n$, using (2), (3), and (2) again,

$$\sigma(\alpha_{\gamma(i)}) = \sigma(\beta_i) = \beta_{\tau_2(i)} = \alpha_{\gamma(\tau_2(i))}. \tag{4}$$

Now, with the substitution $i \to \gamma(i)$ in (1), we get

$$\sigma(\alpha_{\gamma(i)}) = \alpha_{\tau_1(\gamma(i))}. \tag{5}$$

Thus , by (4),(5), $\alpha_{\gamma(\tau_2(i))} = \alpha_{\tau_1(\gamma(i))}$ for all i. Since $i \mapsto \alpha_i$ is one-to-one,

$$\gamma(\tau_2(i)) = \tau_1(\gamma(i)), \qquad i = 1, \ldots, n,$$

so

$$\gamma\tau_2 = \tau_1\gamma.$$

Therefore $\tau_2 = \gamma^{-1}\tau_1\gamma$, so $\phi_2(\sigma) = \hat{\gamma}(\phi_1(\sigma))$, for all $\sigma \in \mathrm{Gal}(L/F)$:

$$\phi_2 = \hat{\gamma} \circ \phi_1.$$

(b) Let G the image of ϕ_1 in S_n: $G = \{\phi_1(\sigma) \mid \sigma \in \mathrm{Gal}(L/F)\} \subset S_n$.
Similarly the image of ϕ_2 is $G' = \{\phi_2(\sigma) \mid \sigma \in \mathrm{Gal}(L/F)\} \subset S_n$.
Since $\phi_2(\sigma) = \gamma^{-1}\phi_1(\sigma)\gamma$ for all $\sigma \in \mathrm{Gal}(L/F)$ by part (a),

$$G' = \gamma^{-1}G\gamma.$$

So, if we change the labels, then G gets replaces with a conjugate subgroup.

□

Ex. 13.1.2 *Prove that A_4 is the only subgroup of S_4 with 12 elements.*

Proof. Let H a subgroup of S_n such that $[S_n : H] = 2$. Then H is normal in S_n (by Exercise 12.1.20). Thus $S_n/H \simeq \{1, -1\}$. So there exists a group homomorphism

$$\varphi : S_n \to \{1, -1\}, \qquad \text{with } \ker(\varphi) = H.$$

Any two transpositions $\tau_1 = (a\,b), \tau_2 = (c\,d)$ of S_n are conjugate: if $\gamma = (a\,c)(b\,d)$, then $\tau_2 = \gamma\tau_1\gamma^{-1}$ (even if $b = c$).

Since $\{1, -1\} \simeq \mathbb{Z}/2\mathbb{Z}$ is abelian,

$$\begin{aligned}\varphi(\tau_2) &= \varphi(\gamma)\varphi(\tau_1)\varphi(\gamma)^{-1} \\ &= \varphi(\gamma)\varphi(\gamma)^{-1}\varphi(\tau_1) \\ &= \varphi(\tau_1)\end{aligned}$$

So $\tau_1, \tau_2 \in H$, or $\tau_1, \tau_2 \in S_n \setminus H$.

If τ_1, τ_2 are in $S_n \setminus H$, then $\varphi(\tau_1\tau_2) = \varphi(\tau_1)\varphi(\tau_2) = (-1) \times (-1) = 1$, so $\tau_1\tau_2 \in H$. In both cases $\tau_1\tau_2 \in H$.

Since every permutation σ of A_n is the product of an even number of transpositions, $\sigma \in H$, so $A_n \subset H$. As $|A_n| = |H| = n!/2$, $H = A_n$.

A_n is the only subgroup of S_n with $n!/2$ elements. □

Ex. 13.1.3 *Explain carefully why (13.6) follows from Exercise 9 of section 2.4.*

Proof. By definition,

$$y_1 = x_1x_2 + x_3x_4, \qquad y_2 = x_1x_3 + x_2x_4, \qquad y_3 = x_1x_4 + x_2x_3.$$

By Exercise 2.4.9, we know that

$$\Delta(\theta) = (y_1{-}y_2)^2(y_1{-}y_3)^2(y_2{-}y_3)^2 = [(x_1{-}x_4)(x_2{-}x_3)(x_1{-}x_3)(x_2{-}x_4)(x_1{-}x_2)(x_3{-}x_4)]^2 = \Delta$$

As the evaluation is a ring homomorphism, if we applied the evaluation defined by $x_1 \mapsto \alpha_1, \ldots, x_4 \mapsto \alpha_4$ to this equality in $F[x_1, x_2, x_3, x_4]$, we obtain that the roots

$$\beta_1 = \alpha_1\alpha_2 + \alpha_3\alpha_4, \qquad \beta_2 = \alpha_1\alpha_3 + \alpha_2\alpha_4, \qquad \beta_3 = \alpha_1\alpha_4 + \alpha_2\alpha_3,$$

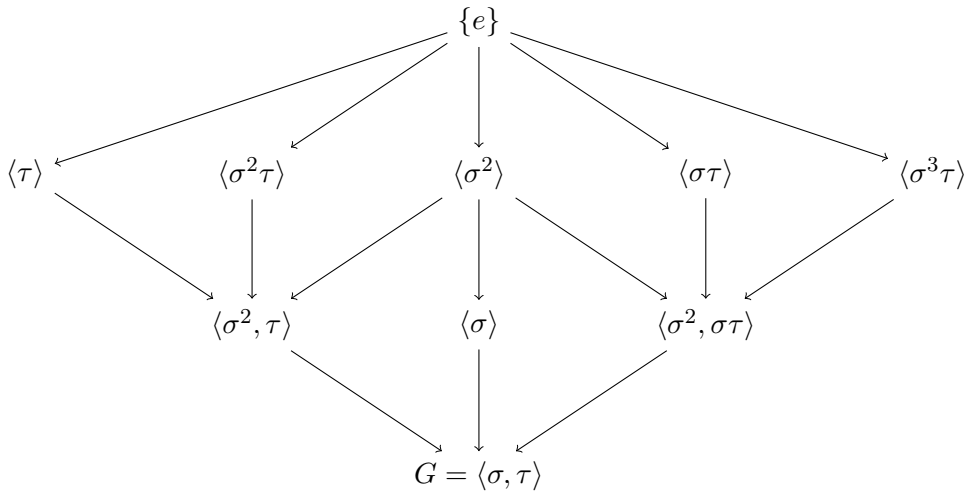
are the images of y_1, y_2, y_3 and satisfy

$$\begin{aligned}\Delta(\theta_f) &= (\beta_1-\beta_2)^2(\beta_1-\beta_3)^2(\beta_2-\beta_3)^2\\ &= [(\alpha_1-\alpha_4)(\alpha_2-\alpha_3)(\alpha_1-\alpha_3)(\alpha_2-\alpha_4)(\alpha_1-\alpha_2)(\alpha_3-\alpha_4)]^2\\ &= \Delta(f)\end{aligned}$$

□

Ex. 13.1.4 *Use Example 7.3.4 from Chapter 7 to show that (13.8) gives all subgroups of* $\langle (1\,3\,2\,4), (1\,2)\rangle$ *of order 4 or 8.*

Proof. We obtain all subgroups of $D_8 \simeq \langle \sigma, \tau\rangle$, where $\sigma = (1\,3\,2\,4), \tau = (1\,2)$, in Exercise 7.3.3



If G is a subgroup of order 4 or 8, then G is one of the four groups

$$\langle \sigma^2, \tau\rangle, \quad \langle \sigma\rangle, \quad \langle \sigma^2, \sigma\tau\rangle, \quad \langle \sigma, \tau\rangle,$$

Moreover $\sigma^2 = (1\,2)(3\,4)$ and $\sigma\tau = (1\,4)(2\,3)$, so

$$\langle \sigma^2, \tau\rangle = \langle (1\,2)(3\,4), (1\,2)\rangle = \langle (3\,4), (1\,2)\rangle,$$

and

$$\langle \sigma^2, \sigma\tau\rangle = \langle (1\,2)(3\,4), (1\,4)(2\,3)\rangle = \langle (1\,2)(3\,4), (1\,3)(2\,4)\rangle$$

is the group of double transpositions $\{(), (1\,2)(3\,4), (1\,4)(2\,3), (1\,3)(2\,4)\}$.

Therefore G is one of the four groups given in the text

$$\langle (1\,2), (3\,4)\rangle, \qquad \langle (1\,2)(3\,4), (1\,3)(2\,4)\rangle, \qquad \langle (1\,3\,2\,4)\rangle, \qquad \langle (1\,3\,2\,4), (1\,2)\rangle.$$

□

Ex. 13.1.5 *Let F be a field of characteristic $\neq 2$, and let $g \in F[x]$ be a monic cubic polynomial that has a root in F. Prove that g splits completely over F if and only if $\Delta(g) \in F^2$.*

Proof. Let $g = (x-\alpha_1)(x-\alpha_2)(x-\alpha_3)$, where $\alpha_1, \alpha_2, \alpha_3$ lie in some splitting field of F, and $\alpha_1 \in F$.

- If g splits completely over F, then $\alpha_1, \alpha_2, \alpha_3$ lie in F, therefore
  $\delta = (\alpha_1-\alpha_2)(\alpha_1-\alpha_3)(\alpha_2-\alpha_3) \in F$, so $\Delta(g) = \delta^2 \in F^2$.
- Conversely, suppose that $\Delta(g) \in F^2$. Then $\Delta(g) = a^2$, $a \in F$, so $\delta = \pm a \in F$. Since $\alpha_1 \in F$, the Euclidean division of $g(x)$ by $x - \alpha_1 \in F[x]$ gives

  $$g(x) = (x-\alpha_1)(x^2+px+q), \qquad p, q \in F.$$

  Then $x^2+px+q = (x-\alpha_2)(x-\alpha_3)$, hence $\alpha_2+\alpha_3 = -p \in F, \alpha_2\alpha_3 = q \in F$, and

  $$(\alpha_1-\alpha_2)(\alpha_1-\alpha_3) = \alpha_1^2 + p\alpha_1 + q \in F.$$

  If $\alpha_1 = \alpha_2$, then $\alpha_3 = -p-\alpha_2 = -p-\alpha_1 \in F$, so g splits completely over F, and similarly the same conclusion is true if $\alpha_1 = \alpha_3$.

  In the remaining case, $(\alpha_1-\alpha_2)(\alpha_1-\alpha_3) \neq 0$, so

  $$\alpha_2 - \alpha_3 = \delta[(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)]^{-1} \in F.$$

  Since $\alpha_2+\alpha_3 \in F$, and $\alpha_2-\alpha_3 \in F$, and since the characteristic of F is not 2,

  $$\alpha_2 = \frac{1}{2}[(\alpha_2+\alpha_3)+(\alpha_2-\alpha_3)] \in F, \alpha_3 = \frac{1}{2}[(\alpha_2+\alpha_3)-(\alpha_2-\alpha_3)] \in F.$$

  Therefore $g = (x-\alpha_1)(x-\alpha_2)(x-\alpha_3)$ splits completely over F.

□

Ex. 13.1.6 *This exercise is concerned with the proof of part (c) of Theorem 13.1.1. Let $f(x) = x^4 - c_1x^3 + c_2x^2 - c_3x + c_4$ as in the theorem.*

(a) Suppose that f has roots $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ such that $\alpha_1+\alpha_2-\alpha_3-\alpha_4 = \alpha_1\alpha_2-\alpha_3\alpha_4 = 0$. Prove that f is not separable.

(b) Let β be a root of the resolvent $\theta_f(y)$. Use part (a) to prove that $4\beta + c_1^2 - 4c_2$ and $\beta^2 - 4c_4$ can't both vanish when f is separable.

(c) Suppose that $4\beta + c_1^2 - 4c_2 = 0$ in part (c) of Theorem 13.1.1. Prove carefully that G is conjugate to $\langle (1\,3\,2\,4), (1\,2)\rangle$ if and only if $\Delta(f)(\beta^2-4c_4) \notin (F^*)^2$.

Proof. (a) If $\alpha_1+\alpha_2-\alpha_3-\alpha_4 = \alpha_1\alpha_2-\alpha_3\alpha_4 = 0$, then

$$s := \alpha_1+\alpha_2 = \alpha_3+\alpha_4$$
$$p := \alpha_1\alpha_2 = \alpha_3\alpha_4$$

Thus $x^2 - sx + p = (x-\alpha_1)(x-\alpha_2) = (x-\alpha_3)(x-\alpha_4)$, therefore

$$\{\alpha_1, \alpha_2\} = \{\alpha_3, \alpha_4\}.$$

Since $\alpha_3 = \alpha_1$ or $\alpha_3 = \alpha_2$, f is not separable.

(b) If β is a root of the resolvent θ_f, we can relabel the roots of f so that $\beta = \alpha_1\alpha_2 + \alpha_3\alpha_4$ and

$$4\beta + c_1^2 - 4c_2 = (\alpha_1 + \alpha_2 - \alpha_3 - \alpha_4)^2.$$

Since $\beta^2 - 4c_4 = (\alpha_1\alpha_2 - \alpha_3\alpha_4)^2$, if $4\beta + c_1^2 - 4c_2$ and $\beta^2 - 4c_4$ both vanish, then $\alpha_1 + \alpha_2 - \alpha_2 - \alpha_4 = 0$ and $\alpha_1\alpha_2 - \alpha_3\alpha_4 = 0$. Then by part (a) f is not separable.

Therefore $4\beta + c_1^2 - 4c_2$ and $\beta^2 - 4c_4$ can't both vanish when f is separable.

(c) Suppose that $4\beta + c_1^2 - 4c_2 = 0$ in part (c) of Theorem 13.1.1, where $\theta_f(y)$ has a unique root β in F. Therefore $\theta_f(y) = (y-\beta)(y-\beta')(y-\beta'')$, where $\beta' \notin F, \beta'' \notin F$.

If θ_f was not separable, then $\beta' = \beta''$, and $\theta_f(t) = (y-\beta)(y-\beta')^2 \in F[y], \beta \in F$, thus $(y-\beta')^2 = y^2 - 2\beta' y + \beta'^2 \in F[y]$, which implies that $2\beta' \in F$.

Since the characteristic of F is not 2, $\beta' \in F$. This is a contradiction, so θ_f is separable. Since the discriminant of θ_f and f are equal, f is separable.

Then by part (b), $\beta^2 - 4c_4 \neq 0$, and since f is separable, $\Delta(f) \neq 0$, so

$$\Delta(f)(\beta^2 - 4c_4) \neq 0.$$

We know that $G = \langle (1\,3\,2\,4) \rangle$ or $G = \langle (1\,3\,2\,4), (1,2) \rangle$.

- Suppose that $G = \langle (1\,3\,2\,4) \rangle$. Then $\mathrm{Gal}(L/F) = \langle \sigma \rangle$, where σ corresponds to $(1\,3\,2\,4)$. We choose

$$\sqrt{\Delta(f)(\beta^2 - 4c_4)} = \sqrt{\Delta(f)}(\alpha_1\alpha_2 - \alpha_3\alpha_4).$$

Since $(1\,3\,2\,4) = (1\,3)(3\,2)(2\,4) \notin A_4$, $\sigma(\sqrt{\Delta(f)}) = -\sqrt{\Delta(f)}$, and

$$\sigma(\alpha_1\alpha_2 - \alpha_3\alpha_4) = \alpha_3\alpha_4 - \alpha_2\alpha_1 = -(\alpha_1\alpha_2 - \alpha_3\alpha_4).$$

Therefore σ fixes $\sqrt{\Delta(f)(\beta^2 - 4c_4)}$, so $\sqrt{\Delta(f)(\beta^2 - 4c_4)} \in F^*$, and

$$\Delta(f)(\beta^2 - 4c_4) \in (F^*)^2.$$

- Suppose that $G = \langle (1\,3\,2\,4), (1,2) \rangle$. Then $\mathrm{Gal}(L/F) = \langle \sigma, \tau \rangle$, where τ corresponds to $(1\,2)$. $\tau(\sqrt{\Delta(f)}) = -\sqrt{\Delta(f)}$ and $\tau(\alpha_1\alpha_2 - \alpha_3\alpha_4) = \alpha_2\alpha_1 - \alpha_3\alpha_4 = \alpha_1\alpha_2 - \alpha_3\alpha_4$, so $\tau(\sqrt{\Delta(f)(\beta^2 - 4c_4)}) = -\sqrt{\Delta(f)(\beta^2 - 4c_4)}$. Since the characteristic is not 2, and $\Delta(f)(\beta^2 - 4c_4) \neq 0, \sqrt{\Delta(f)(\beta^2 - 4c_4)} \notin F$, so

$$\Delta(f)(\beta^2 - 4c_4) \notin (F^*)^2.$$

Therefore G is conjugate to $\langle (1\,3\,2\,4), (1\,2) \rangle$ if and only if $\Delta(f)(\beta^2 - 4c_4) \notin (F^*)^2$.

□

Ex. 13.1.7 *In Exercise 18 of section 12.1 you found the roots of $f = x^4 + 2x^2 - 4x + 2 \in \mathbb{Q}[x]$ using the formula developed in that section. At the end of the exercise, we said that "this quartic is especially simple". Justify this assertion using Theorem 13.1.1*

Proof. By Exercise 12.1.18,

$$\theta_f(y) = y^3 - 2y^2 - 8y = y(y-4)(y+2).$$

Since $\theta_f(y)$ splits completely over F, by Theorem 13.1.1,

$$G = \langle (1\,2)(3\,4), (1\,3)(2\,4) \rangle \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}.$$

(This result was already proved in Exercise 12.1.18, since the splitting field of f is $\mathbb{Q}(i, \sqrt{2})$.) □

Ex. 13.1.8 *In Example 10.3.10, we showed that the roots of* $f = 7m^4 - 16m^3 - 21m^2 + 8m + 4 \in \mathbb{Q}[m]$ *can be constructed using origami. Show that the splitting field of* f *is an extension of* $\mathbb{Q}$ *of degree 24. By the results of Section 10.1, it follows that the roots of* f *are not constructible with straightedge and compass, since* 24 *is not a power of* 2.

Proof. The discriminant of $g = \frac{1}{7}f$ is

$$\Delta(g) = \frac{174446784}{117649} = 2^6 \cdot 3^6 \cdot 3739 \cdot 7^{-6},$$

so $\Delta(g)$ is not a square in $\mathbb{Q}$.

The Ferrari resolvent is

$$\theta_f(y) = y^3 + 3y^2 - \frac{240}{49}y - \frac{3824}{343}.$$

and

$$7^3\theta_f(y) = 343y^3 + 1029y^2 - 1680y - 3824$$

has no root in $\mathbb{Q}$, so is irreducible over $\mathbb{Q}$.

By theorem 13.1.1, $G = S_4$. Therefore the splitting field L of f has degree

$$[L : \mathbb{Q}] = |G| = 24.$$

Sage instructions :

```
var('m')
R.<m> = QQ[m]
f= 7*m^4-16*m^3-21*m^2+8*m+4
g=f/7
d=g.discriminant()
d.factor()
```

$$2^6 \cdot 3^6 \cdot 7^{-6} \cdot 3739$$

```
R.<y> = QQ[]
l = f.coefficients(sparse=False);
c1 = -l[3]/l[4]; c2 = l[2]/l[4];c3 = -l[1]/l[4]; c4 = l[0]/l[4];
theta_f = y^3 -c2*y^2 +(c1*c3-4*c4)*y - c3^2-c1^2*c4 + 4*c2*c4;
```

$$y^3 + 3y^2 - \frac{240}{49}y - \frac{3824}{343}$$

```
theta_f.is_irreducible()
```

True

□

Ex. 13.1.9 *As in Example 13.1.3, let $f = x^4 + ax^3 + bx^2 + ax + 1 \in F[x]$, and let α be a root of f in some splitting field of f over F. Show that α^{-1} is also a root of f, and then use (13.5) to conclude that 2 is a root of the resolvent $\theta_f(y)$.*

Proof. If α is a root of f in some splitting field L of F, then $\alpha^4 + a\alpha^3 + b\alpha^2 + a\alpha + 1 = 0$. If we divide by α^4, we obtain $1 + a\alpha^{-1} + b\alpha^{-2} + a\alpha^{-3} + \alpha^{-4}$, so $f(\alpha^{-1}) = 0$. Note that

$$\begin{aligned} x^4 + ax^3 + bx^2 + ax + 1 &= x^2 \left[\left(x^2 + \frac{1}{x^2} \right) + a \left(x + \frac{1}{x} \right) + b \right] \\ &= x^2 \left[\left(x + \frac{1}{x} \right)^2 + a \left(x + \frac{1}{x} \right) + b - 2 \right] \end{aligned}$$

As 0 is not a root of f, the roots of f are the roots of $z = x + \frac{1}{x}$, where z is a root of $z^2 + az + b - 2$, so the roots of f are the roots of the two polynomials

$$x^2 - z_1 x + 1, \qquad x^2 - z_2 x + 1,$$

where z_1, z_2 are the roots in L of

$$z^2 + az + b - 2.$$

If we relabel the roots so that α_1, α_2 are the roots of $x^2 - z_1 x + 1$, and α_3, α_4 the roots of $x^2 - z_2 x + 1$, then $\alpha_1\alpha_2 = 1, \alpha_3\alpha_4 = 1$, therefore $\beta_1 = \alpha_1\alpha_2 + \alpha_3\alpha_4 = 2$ is a root of the Ferrari resolvent $\theta_f(y)$. □

Ex. 13.1.10 *As in Example 13.1.4, let $f = x^4 + bx^2 + d \in F[x]$, where $d \notin F^2$. Compute $\Delta(f)$ and $\theta_f(y)$.*

Proof. The discriminant of f is

$$\Delta(f) = 16b^4 d - 128b^2 d^2 + 256d^3 = 16\, d\, (b^2 - 4d)^2.$$

The Ferrari resolvent is

$$\theta_f(y) = y^3 - by^2 - 4dy + 4bd = (y - b)(y^2 - 4d).$$

Sage instructions:

```
R.<x,b,d> = QQ[]
f=x^4+b*x^2+d
c1 = 0; c2 = b; c3 = 0; c4 = d;
theta_f = x^3 - c2*x^2 + (c1*c3-4*c4)*x - c3^2-c1^2*c4 + 4*c2*c4;
factor(theta_f)
```

$$(-x + b) \cdot (-x^2 + 4d)$$

```
Delta = theta_f.discriminant(x)
factor(Delta)
```

$$(16) \cdot d \cdot (-b^2 + 4d)^2$$

Thus $\theta_f(y) = (y - b)(y - 2\sqrt{d})(y + 2\sqrt{d})$ has a unique root in F if $d \notin F^2$, and the discriminant is not a square in F^2. □

Ex. 13.1.11 *In Example 13.1.7 we showed that if $f = x^4 + ax^3 + bx^2 + ax + 1 \in \mathbb{Z}[x]$ is irreducible over $\mathbb{Q}$, then its Galois group is $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ if and only if there is $c \in \mathbb{Q}$ such that $4a^2 + c^2 = (b+2)^2$.*

(a) Show that $c \in \mathbb{Z}$, and use the irreducibility of f to prove that $c \neq 0$. Hence we may assume that $c > 0$, so that $(2a, c, b+2)$ is a Pythagorean triple.

(b) Show that $3^2 + 4^2 = 5^2, 5^2 + 12^2 = 13^2, 7^2 + 24^2 = 25^2$, and $8^2 + 15^2 = 17^2$ give two examples of polynomials with $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ as Galois group (two of the triples give reducible polynomials).

Proof. (a) $c \in \mathbb{Q}$ is such that $c^2 = n \in \mathbb{Z}$. Write $c = a/b, b > 0, a \wedge b = 1$. Then $a^2 = nb^2$. If $b \neq 1$, there is a prime p such that $p \mid b$. But then $p \mid a^2$, thus $p \mid a$, in contradiction with $a \wedge b = 1$. So $c \in \mathbb{Z}$.

If $c = 0$, then $(b+2)^2 = 4a^2$, so $b + 2 = 2\varepsilon a$, $b = -2 + 2\varepsilon a$, where $\varepsilon = \pm 1$.

In Exercise 9, we saw that

$$f = x^4 + ax^3 + bx^2 + ax + 1 = (x^2 - z_1 x + 1)(x^2 - z_2 x + 1),$$

where z_1, z_2 are the roots of $z^2 + az + b - 2$. Here $b = -2 + 2\varepsilon a$, so z_1, z_2 are the roots of

$$z^2 + az - 4 + 2\varepsilon a = (z + a - 2\varepsilon)(z + 2\varepsilon),$$

so

$$z_1 = -a + 2\varepsilon \in \mathbb{Z}, \qquad z_2 = -2\varepsilon \in \mathbb{Z},$$

so f is not irreducible over $\mathbb{Q}$, in contradiction with the hypothesis. We have proved that $c \neq 0$ if f is irreducible, and so $(2a, c, b+2)$ is a Pythagorean triple.

(b) $3^2 + 4^2 = 5^2$ gives $a = 2, b = 3$, and $f = x^4 + 2x^3 + 3x^2 + 2x + 1 = (x^2 + x + 1)^2$ is not irreducible.

$5^2 + 12^2 = 13^2$ gives $a = 6, b = 11$, and $f = x^4 + 6x^3 + 11x^2 + 6x + 1 = (x^2 + 3x + 1)^2$ is not irreducible.

$7^2 + 24^2 = 25^2$ gives $a = 12, b = 23$, and $f = x^4 + 12x^3 + 23x^2 + 12x + 1$ which is irreducible. So the Galois group of

$$f = x^4 + 12x^3 + 23x^2 + 12x + 1$$

is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

Verification with Sage:

```
R.<x> = QQ[]
f= x^4 + 12*x^3 + 23*x^2 + 12*x + 1
f.is_irreducible()
```

True

```
G = f.galois_group()
G.gens()
```

$$[(1,2)(3,4), (1,4)(2,3)]$$

```
G.structure_description()
```

$$C2 \times C2$$

□

$8^2 + 15^2 = 17^2$ gives $a = 4, b = 15$, and $f = x^4 + 4x^3 + 15x^2 + 4x + 1$, which is irreducible. The Galois group of

$$f = x^4 + 4x^3 + 15x^2 + 4x + 1$$

is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

Note: the polynomial associate to $7^2 + 24^2 = 25^2$ is

$$\begin{aligned} f &= x^4 + 12x^3 + 23x^2 + 12x + 1 \\ &= (x^2 + 6x + 1)^2 - 15x^2 \\ &= (x^2 + (6 + \sqrt{15})x + 1)(x^2 + (6 - \sqrt{15})x + 1) \end{aligned}$$

The discriminant of the first factor is $\Delta_1 = 47 + 12\sqrt{15}$ and the discriminant of the second is $\Delta_2 = 47 - 12\sqrt{15}$. Since

$$\left(\sqrt{47 + 12\sqrt{15}}\right)\left(\sqrt{47 - 12\sqrt{15}}\right) = \sqrt{47^2 - 144 \times 15} = \sqrt{49} = 7 \in \mathbb{Q}^*),$$

the splitting field of f over $\mathbb{Q}$ is $\mathbb{Q}\left(\sqrt{47 + 12\sqrt{15}}\right)$, which is a quadratic extension of a quadratic extension. The minimal polynomial of $a = \sqrt{47 + 12\sqrt{15}}$ is $x^4 - 94x^2 + 49$, whose Galois group is also $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ (here $d = 49$ is a square).

Ex. 13.1.12 *This exercise is concerned with the proof of Proposition 13.1.5.*

(a) Prove (13.12).

(b) Prove that the two polynomials h_1 and h_2 defined in the proof of the proposition factor as $h_1 = (y - (\alpha_1 + \alpha_2))(y - (\alpha_3 + \alpha_4))$ and $h_2 = (y - \alpha_1\alpha_2)(y - \alpha_3\alpha_4)$.

Proof. (a) Let $g = y^2 + Ay + B \in F[y]$ and let $F \subset F(\sqrt{a})$, $a \in F$, be a quadratic extension.

If $\Delta(g) = 0$ then $a\Delta(g) = 0 \in F^2$. Suppose now that g is irreducible over F.

- Suppose that g splits completely over $F(\sqrt{a})$, so

$$g = (y - y_1)(y - y_2), \qquad y_1, y_2 \in F(\sqrt{a}).$$

Then $\Delta(g) = (y_1 - y_2)^2 = A^2 - 4B \in F$. We choose $\sqrt{\Delta(g)} = y_2 - y_1 \in F(\sqrt{a})$. Here $\deg(g) = 2$, and g is irreducible over F, therefore the roots of g

$$\begin{aligned} y_1 &= \frac{1}{2}((y_1 + y_2) + (y_1 - y_2)) = \frac{1}{2}\left(-A - \sqrt{\Delta(g)}\right), \\ y_2 &= \frac{1}{2}((y_1 + y_2) - (y_1 - y_2)) = \frac{1}{2}\left(-A + \sqrt{\Delta(g)}\right), \end{aligned}$$

are not in F, and this is equivalent to

$$\sqrt{\Delta(g)} \notin F.$$

Since $\sqrt{\Delta(g)} \in F(\sqrt{a})$, and $\sqrt{\Delta(g)} \notin F$,

$$\sqrt{\Delta(g)} = u + v\sqrt{a}, \qquad u, v \in F, \qquad v \neq 0.$$

Therefore

$$\begin{aligned} u^2 &= \left(\sqrt{\Delta(g)} - v\sqrt{a}\right)^2 \\ &= \Delta(g) + av^2 - 2v\sqrt{a}\sqrt{\Delta(g)} \end{aligned}$$

Since $v \neq 0$, and $\text{char}(F) \neq 2$,

$$\sqrt{a}\sqrt{\Delta(g)} = \frac{\Delta(g) + av^2 - u^2}{2v} \in F,$$

so

$$a\Delta(g) \in F^2.$$

- Conversely, suppose that $a\Delta(g) \in F^2$. Here $a \neq 0$ since $F(\sqrt{a})$ is a quadratic extension of F. There exists $w \in F$ such that $a\Delta(g) = w^2$.
  We choose $\sqrt{\Delta(g)}$ such that

  $$\sqrt{\Delta(g)} = \frac{w}{\sqrt{a}} = \frac{w}{a}\sqrt{a} \in F(\sqrt{a}).$$

  Then

  $$\begin{aligned} y_1 &= \frac{1}{2}((y_1 + y_2) + (y_1 - y_2)) = \frac{1}{2}\left(-A - \sqrt{\Delta(g)}\right), \\ y_2 &= \frac{1}{2}((y_1 + y_2) - (y_1 - y_2)) = \frac{1}{2}\left(-A + \sqrt{\Delta(g)}\right), \end{aligned}$$

  are in $F(\sqrt{a})$, so $g = (y - y_1)(y - y_2)$ splits completely over $F(\sqrt{a})$.
  Finally, if $\Delta(g) = 0$, $g = (y - y_0)^2$, where $y_0 = -A/2 \in F$, splits completely over F, a fortiori over $F(\sqrt{a})$.

Conclusion:

Let $g = y^2 + Ay + B$ and $F(\sqrt{a})$ a quadratic extension of F, with $\text{char}(F) \neq 2$. If $\Delta(g) = 0$, or if g is irreducible over F, then

$$g \text{ splits completely over } F(\sqrt{a}) \iff a\Delta(g) \in F^2.$$

(b)

$$\begin{aligned} &(y - (\alpha_1 + \alpha_2))(y - (\alpha_3 + \alpha_4)) \\ &= y^2 - (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4)y + (\alpha_1\alpha_3 + \alpha_1\alpha_4 + \alpha_2\alpha_3 + \alpha_2\alpha_4) \\ &= y^2 - c_1 y + (\alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_1\alpha_4 + \alpha_2\alpha_3 + \alpha_2\alpha_4 + \alpha_3\alpha_4) - (\alpha_1\alpha_2 + \alpha_3\alpha_4) \\ &= y^2 - c_1 y + c_2 - \beta \end{aligned}$$

so

$$h_1 = y^2 - c_1 y + c_2 - \beta = (y - (\alpha_1 + \alpha_2))(y - (\alpha_3 + \alpha_4)).$$

Similarly

$$\begin{aligned}&(y-\alpha_1\alpha_2)(y-\alpha_3\alpha_4)\\&=y^2-(\alpha_1\alpha_2+\alpha_3\alpha_4)y+\alpha_1\alpha_2\alpha_3\alpha_4\\&=y^2-\beta y+c_4\end{aligned}$$

so

$$h_2=y^2-\beta y+c_4=(y-\alpha_1\alpha_2)(y-\alpha_3\alpha_4).$$

We have proved that h_1, h_2 split completely over L. Since $\deg(h_1)=2$, h_1 splits over a quadratic extension $F\subset M$, with $M\subset L$. But the unique such quadratic extension is $F(\sqrt{\Delta(f)})$ (since $\mathrm{Gal}(L/F)\simeq\mathbb{Z}/4\mathbb{Z}$ has a unique subgroup of index 2). Therefore $M=F(\sqrt{\Delta(f)})$, and h_1 splits completely over $F(\sqrt{\Delta(f)})$, and also h_2. □

Ex. 13.1.13 *Suppose that $f\in F[x]$ satisfies the hypothesis of part (c) of Theorem 13.1.1, and let α be a root of f. Prove that $G\simeq\mathbb{Z}/4\mathbb{Z}$ if f splits completely over $F(\alpha)$, and $G\simeq D_8$ otherwise. This gives a version of part (c) that doesn't use resolvents. Since we can factor over extension fields by Section 4.2, this method is useful in practice.*

Proof. With the hypothesis of part (c), $\Delta(f)\notin F^2$, so $\Delta(f)\neq 0$ and f is separable.

- If $G\simeq\mathbb{Z}/4\mathbb{Z}$, then $G=\langle\sigma\rangle\subset S_4$, where σ corresponds to $\tilde{\sigma}\in\mathrm{Gal}(L/F)$. Write $G_\alpha=\mathrm{Stab}_G(\alpha)$. Since f is irreducible, $\mathcal{O}_\alpha=\{\alpha_1,\alpha_2,\alpha_3,\alpha_4\}$ is the set of the four roots of f, therefore $4=|\mathcal{O}_\alpha|=(G:G_\alpha)$, so $G_\alpha=\{e\}$. Hence $\tilde{\sigma}^i(\alpha)\neq\tilde{\sigma}^j(\alpha)$ if $1\leq i<j\leq 4$. We choose the numbering of the roots such that $\alpha_1=\alpha$, and $\tilde{\sigma}(\alpha_1)=\alpha_3, \tilde{\sigma}(\alpha_3)=\alpha_2, \tilde{\sigma}(\alpha_2)=\alpha_4$ are the four distinct roots of f, so $\sigma=(1\,3\,2\,4)$.

  $$f=(x-\alpha_1)(x-\alpha_3)(x-\alpha_2)(x-\alpha_4)=(x-\alpha)(x-\tilde{\sigma}(\alpha))(x-\tilde{\sigma}^2(\alpha))(x-\tilde{\sigma}^3(\alpha)).$$

  As $\Delta(f)\notin F^2$, $F(\sqrt{\Delta(f)})$ is a quadratic extension of F.

  Since the only subgroup of G are $\{e\}\subset H=\langle\sigma^2\rangle\subset G=\langle\sigma\rangle$, by the Galois correspondence, the only intermediate fields of $F\subset L$ are $F\subset F(\sqrt{\Delta(f)})\subset L$, and the fixed field of $H=\langle\sigma^2\rangle$ is $L_H=F(\sqrt{\Delta(f)})$.

  If $F(\alpha)\subset F(\sqrt{\Delta(f)})$, then $\alpha\in F(\sqrt{\Delta(f)})=L_H$, therefore $\sigma^2(\alpha)=\alpha$, and so $\alpha_2=\alpha_1$, in contradiction with the separability of f. Hence $F(\alpha)\not\subset F(\sqrt{\Delta(f)})$, so

  $$F(\alpha)=L=F(\alpha_1,\alpha_2,\alpha_3,\alpha_4).$$

  Then f splits completely over $F(\alpha)$.

- If $G\not\simeq\mathbb{Z}/4\mathbb{Z}$, then by Theorem 13.1.1, $G\simeq D_8$. Therefore $[L:F]=|G|=8$, and $[F(\alpha):F]=\deg(f)=4$, which implies $F(\alpha)\neq L=F(\alpha_1,\alpha_2,\alpha_3,\alpha_4)$. Therefore one of the roots α_i is not in $F(\alpha)$, and so $f=(x-\alpha_1)(x-\alpha_2)(x-\alpha_3)(x-\alpha_4)$ doesn't splits completely over $F(\alpha)$.

Conclusion. Let f be a quadratic polynomial, and let α be a root of f.
If $\Delta(f)\notin F^2$ and $\theta_f(y)$ is reducible over F, then

$$\begin{aligned}f\text{ splits completely over }F(\alpha)&\iff\mathrm{Gal}_F(f)\simeq\mathbb{Z}/4\mathbb{Z},\\f\text{ doesn't split completely over }F(\alpha)&\iff\mathrm{Gal}_F(f)\simeq D_8.\end{aligned}$$

□

Example 1: $f = x^4 - 12x^2 + 18$ over $\mathbb{Q}$.

```
R.<x> = QQ[]
f = x^4-12*x^2 + 18
print(f.is_irreducible())
factor(f.discriminant()), f.discriminant().is_square()
```

True

$$(2^{11} \cdot 3^6, \text{False}).$$

```
l = f.coefficients(sparse=False);
c1 = -l[3]/l[4]; c2 = l[2]/l[4];c3 = -l[1]/l[4]; c4 = l[0]/l[4];
S.<y> = QQ[]
theta_f = y^3 -c2*y^2 +(c1*c3-4*c4)*y - c3^2-c1^2*c4 + 4*c2*c4;
factor(theta_f)
```

$$(y + 12) \cdot (y^2 - 72)$$

```
K.<a>= NumberField(f)
S.<x> = K[]
f = x^4-12*x^2 + 18
factor(f)
```

$$(x - a) \cdot (x + a) \cdot (x - \frac{1}{3}a^3 + 3a) \cdot (x + \frac{1}{3}a^3 - 3a)$$

These results prove that the Galois group of $f = x^4 - 12x^2 + 18$ over $\mathbb{Q}$ is isomorphic to $\mathbb{Z}/4\mathbb{Z}$. Verification with Sage:

```
R.<x> = QQ[]
f = x^4-12*x^2 + 18
f.galois_group().gens()
```

$$[(1, 2, 3, 4)]$$

```
f.galois_group().structure_description()
```

$$C4$$

Example 2: $f = x^4 - 2$ over $\mathbb{Q}$.

```
R.<x> = QQ[]
f = x^4-2
print(f.is_irreducible())
factor(f.discriminant()), f.discriminant().is_square()
```

True

$$(-1 \cdot 2^{11}, \text{False})$$

```
l = f.coefficients(sparse=False);
c1 = -l[3]/l[4]; c2 = l[2]/l[4];c3 = -l[1]/l[4]; c4 = l[0]/l[4];
S.<y> = QQ[]
theta_f = y^3 -c2*y^2 +(c1*c3-4*c4)*y - c3^2-c1^2*c4 + 4*c2*c4;
factor(theta_f)
```

$$y \cdot (y^2 + 8)$$

```
K.<a>= NumberField(f)
S.<x> = K[]
f = x^4-2
factor(f)
```

$$(x - a) \cdot (x + a) \cdot (x^2 + a^2)$$

Thus the Galois group of $x^4 - 2$ over $\mathbb{Q}$ is D_8. Verification with Sage:

```
R.<x> = QQ[]
f = x^4-2
f.galois_group().gens()
```

$$[(1, 2, 3, 4), (1, 3)]$$

```
f.galois_group().structure_description()
```

$$D4$$

Example 3: $f = x^4 - 18x^2 + 9$ over $\mathbb{Q}$.

```
R.<x> = QQ[]
f = x^4-18*x^2 + 9
print(f.is_irreducible())
factor(f.discriminant()), f.discriminant().is_square()
```

$$\text{True}$$

$$(2^{14} \cdot 3^6, \text{True})$$

```
l = f.coefficients(sparse=False);
c1 = -l[3]/l[4]; c2 = l[2]/l[4];c3 = -l[1]/l[4]; c4 = l[0]/l[4];
S.<y> = QQ[]
theta_f = y^3 -c2*y^2 +(c1*c3-4*c4)*y - c3^2-c1^2*c4 + 4*c2*c4;
factor(theta_f)
```

$$(y - 6) \cdot (y + 6) \cdot (y + 18)$$

The Galois group of $f = x^4 - 18x^2 + 9$ over $\mathbb{Q}$ is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Verification with Sage:

```
R.<x> = QQ[]
f = x^4-18*x^2 + 9
f.galois_group().gens()
```

$$[(1, 2)(3, 4), (1, 4)(2, 3)]$$

```
f.galois_group().structure_description()
```

$$C2 \times C2$$

Ex. 13.1.14 *Use Theorem 13.1.1 to compute the Galois groups of the following polynomials in* $\mathbb{Q}[x]$*:*

(a) $x^4 + 4x + 2$.

(b) $x^4 + 8x + 12$.

(c) $x^4 + 1$.

(d) $x^4 + x^3 + x^2 + x + 1$.

(e) $x^4 - 2$.

Proof. (a) $f = x^4 + 4x + 2$.

$\Delta(f) = -2^8 \cdot 19$ is not a square in $\mathbb{Q}$, and $\theta_f(y) = y^3 - 8y - 16$ is irreducible over $\mathbb{Q}$, so $\mathrm{Gal}_{\mathbb{Q}}(f) \simeq S_4$ (part (a) of Theorem 13.1.11).

(b) $f = x^4 + 8x + 12$.

$\Delta(f) = 2^{12} \cdot 3^4$ is a square in $\mathbb{Q}$, and $\theta_f(y) = y^3 - 48y - 64$ is irreducible over $\mathbb{Q}$, so $\mathrm{Gal}_{\mathbb{Q}}(f) \simeq A_4$ (part (a) of Theorem 13.1.11).

(c) $f = x^4 + 1$.

$\Delta(f) = 2^8$ is a square in $\mathbb{Q}$ and $\theta_f(y) = y(y-2)(y+2)$ splits completely over $\mathbb{Q}$, so $\mathrm{Gal}_{\mathbb{Q}}(f) \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ (part (b) of Theorem 13.1.11).

(d) $f = x^4 + x^3 + x^2 + x + 1$.

$\Delta(f) = 5^3$ is not a square, and $\theta_f(y) = (y-2)(y^2+y+1)$ has a unique root in $\mathbb{Q}$, so part (c) of Theorem 13.1.1 applies. Let ζ a root of f. Then

$$f = (x-\zeta)(x-\zeta^2)(x-\zeta^3)(x-\zeta^4)$$

splits completely over $\mathbb{Q}(\zeta)$. By Exercise 13,

$$G \simeq \mathbb{Z}/4\mathbb{Z}.$$

(we know already this result, since $f = \Phi_5$.)

(e) $f = x^4 - 2$.

By Exercise 13, Example 2, $\Delta(f) = -2^{11}$ is not a square, and $\theta_f(y) = y(y^2+8)$ has a unique root in $\mathbb{Q}$. Moreover if $a = \sqrt[4]{2}$,

$$f = (x-a)(x+a)(x^2+a^2)$$

doesn't splits completely over $\mathbb{Q}$, so

$$G \simeq D_8.$$

□

Ex. 13.1.15 *In the situation of Theorem 13.1.1, assume that $\theta_f(y)$ has a root in F. In the proof of the theorem, we used (13.5) and (13.7) to show that G is conjugate to a subgroup of D_8. Show that the weaker assertion that $|G| = 4$ or 8 can be proved directly from (12.17).*

Proof. By (12.17), the roots of the quartic $f = x^4 - c_1x^3 + c_2x^2 - c_3x + c_4$ are

$$\alpha = \frac{1}{4}\left(c_1 + \varepsilon_1\sqrt{4y_1 + c_1^2 - 4c_2} + \varepsilon_2\sqrt{4y_2 + c_1^2 - 4c_2} + \varepsilon_3\sqrt{4y_3 + c_1^2 - 4c_2}\right),$$

where y_1, y_2, y_3 are the roots of the Ferrari resolvent

$$\theta_f(y) = y^3 - c_2y^2 + (c_1c_3 - 4c_4)y - c_3^2 - c_1^2c_4 + 4c_2c_4,$$

and the $\varepsilon_i = \pm 1$ are chosen so that the product of the radicals $t_i = +\varepsilon_i\sqrt{4y_i + c_1^2 - 4c_2}$ is

$$t_1t_2t_3 = c_1^3 - 4c_1c_2 + 8c_3.$$

Let $L = F(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ the splitting field of F.

Here $\theta_f(y)$ has a root in F, say y_1. Thus

$$\theta_f(y) = (y - y_1)g(y),$$

where $g(y) = y^2 + ay + b \in F[y]$. Therefore the roots y_2, y_3 of g are in $F(\sqrt{\delta})$, where $\delta = a^2 - 4b \in F$ is the discriminant of g. Moreover $t_1 = \alpha_1 + \alpha_2 - \alpha_3 - \alpha_4 = \sqrt{4y_1 + c_1^2 - 4c_2} \in L$, and similarly $t_2, t_3 \in L$, so $F(t_1, t_2, t_3) \subset L$, and by (12.17), $L \subset F(t_1, t_2, t_3)$, therefore

$$L = F(t_1, t_2, t_3) = F\left(\sqrt{4y_1 + c_1^2 - 4c_2}, \sqrt{4y_2 + c_1^2 - 4c_2}, \sqrt{4y_3 + c_1^2 - 4c_2}\right).$$

Since $\Delta(\theta_f) = \Delta(f) \neq 0$, there is at most one t_i equal to 0. So we can choose the numbering such that $t_1t_2 \neq 0$ (perhaps $t_3 = 0$). Since $t_1t_2t_3 = c_1^3 - 4c_1c_2 + 8c_3 \in F$, $t_3 \in F(t_1, t_2)$, so

$$L = F(t_1, t_2, t_3) = F(t_1, t_2) = F\left(\sqrt{4y_1 + c_1^2 - 4c_2}, \sqrt{4y_2 + c_1^2 - 4c_2}\right).$$

Note that $t_i^2 = 4y_i + c_1^2 - 4c_2 \in L$, so $y_i \in L$, $i = 1, 2, 3$, so $\sqrt{\delta} = y_2 - y_3 \in L$, therefore $L(\sqrt{\delta}) = L$. Consider the chain of inclusions

$$F \subset F\left(\sqrt{4y_1 + c_1^2 - 4c_2}\right) \subset F\left(\sqrt{4y_1 + c_1^2 - 4c_2}, \sqrt{\delta}\right)$$
$$\subset F\left(\sqrt{4y_1 + c_1^2 - 4c_2}, \sqrt{\delta}, \sqrt{4y_2 + c_1^2 - 4c_2}\right) = L.$$

Since $4y_1 + c_1^2 - 4c_2 \in F, \delta \in F$ and $4y_2 + c_1^2 - 4c_2 \in F(\sqrt{\delta})$, the degree of each extension is 1 or 2, so

$$[L : F] \mid 8.$$

Moreover $L \supset F(\alpha_1)$, and the minimal polynomial of α_1 is f, so

$$[L : F] \geq [F(\alpha_1) : F] = \deg(f) = 4.$$

Since $|G| = [L : F]$,

$$|G| = 4 \text{ or } |G| = 8.$$

□

Ex. 13.1.16 *Consider the subgroups* $\langle (1\,2), (3\,4)\rangle$ *and* $\langle (1\,2)(3\,4), (1\,3)(2\,4)\rangle$ *of* S_4.

(a) Prove that these subgroups are isomorphic but not conjugate. This shows that when classifying subgroups of a given group, it can happen that nonconjugate subgroups can be isomorphic as abstract groups.

(b) Explain why the subgroup $\langle (1\,2), (3\,4)\rangle$ *isn't mentioned in Theorems 13.1.1 and 13.1.6.*

Proof. (a)

$$H_1 = \langle (1\,2), (3\,4)\rangle = \{(), (1\,2), (3\,4), (1\,2)(3\,4)\},$$
$$H_2 = \langle (1\,2)(3\,4), (1\,3)(2\,4)\rangle = \{(), (1\,2)(3\,4), (1\,3)(2\,4), (1\,4)(2\,3)\}$$

are both isomorphic to the Klein's group $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

Every conjugate of $(1\,2) \in H_1$ by $\sigma \in S_4$ is $(\sigma(1)\,\sigma(2))$, which is not in H_2. The subgroups H_1, H_2 are not conjugate.

(b) $H_1 = \langle (1\,2), (3\,4)\rangle$ is not a transitive subgroup of S_4 (the orbit of 1 is $\{1, 2\}$), so isn't mentioned in Theorems 13.1.1 and 13.1.6.

□

13.2 QUINTIC POLYNOMIALS

Ex. 13.2.1 *As explained in the text, we can regard* $\mathrm{AGL}(1, \mathbb{F}_5)$ *as a subgroup of* S_5.

(a) Prove that $\mathrm{AGL}(1, \mathbb{F}_5)$ *is generated by* $(1\,2\,3\,4\,5)$ *and* $(1\,2\,4\,3)$.

(b) Prove that $\mathrm{AGL}(1, \mathbb{F}_5) \cap A_5$ *is generated by* $(1\,2\,3\,4\,5)$ *and* $(1\,4)(2\,3)$.

(c) Prove that the group of part (b) is isomorphic to the dihedral group D_{10} *of order* 10.

(d) Prove that $\langle (1\,2\,3\,4\,5)\rangle, \mathrm{AGL}(1, \mathbb{F}_5) \cap A_5$, *and* $\mathrm{AGL}(1, \mathbb{F}_5)$ *are the only subgroups of* $\mathrm{AGL}(1, \mathbb{F}_5)$ *containing* $\langle (1\,2\,3\,4\,5)\rangle$.

Proof. (a) Let $r : \mathbb{F}_5 \to \mathbb{F}_5, x \mapsto x + 1$ and $s : \mathbb{F}_5 \to \mathbb{F}_5, x \mapsto 2x$, corresponding to the permutations $\rho = (1\,2\,3\,4\,5)$ and $\sigma = (1\,2\,4\,3)$.

Since 2 is a generator of $\mathbb{F}_5^*$ ($2^2 \equiv -1 \mod 5$), every $a \in \mathbb{F}_p^*$ is of the form $a = 2^k,\ k \in \mathbb{N}$, so every $f \in \mathrm{AGL}(1, \mathbb{F}_5)$, defined by $x \mapsto ax + b,\ a = 2^k \in \mathbb{F}_p^*, b \in \mathbb{F}_p$ is equal to $f = r^b \circ s^k$. Therefore $\mathrm{AGL}(1, \mathbb{F}_5) = \langle r, s\rangle$, and the corresponding subgroup G of S_5, isomorphic to $\mathrm{AGL}(1, \mathbb{F}_5)$, is generated by $\rho = (1\,2\,3\,4\,5)$ and $\sigma = (1\,2\,4\,3)$.

(b) By part (a), every permutation χ of $\mathrm{AGL}(1, \mathbb{F}_5)$ is of the form $\chi = \rho^b\sigma^k,\ 0 \le b \le 4, 0 \le k \le 3$. Since $\rho \in A_5$ and $\sigma \in S_5 \setminus A_5$, $\chi \in A_5$ if and only if k is even. Moreover, since $\sigma^4 = e$, for each integer l, $\sigma^{2l} = e$ or $\sigma^{2l} = \sigma^2$, so

$$\begin{aligned}\mathrm{AGL}(1, \mathbb{F}_5) \cap A_5 &= \{\rho^k \mid 0 \le k \le 4\} \cup \{\rho^k\sigma^2 \mid 0 \le k \le 4\} \\ &= \{e, \rho, \rho^2, \rho^3, \rho^4, \sigma^2, \rho\sigma^2, \rho^2\sigma^2, \rho^3\sigma^2, \rho^4\sigma^2\}.\end{aligned}$$

Thus

$$\mathrm{AGL}(1, \mathbb{F}_5) \cap A_5 = \langle \rho, \sigma^2\rangle = \langle (1\,2\,3\,4\,5), (1\,4)(2\,3)\rangle.$$

(c) For every $x \in \mathbb{F}_p$, $(s^2 \circ r)(x) = 4(x+1)$, and $(r^{-1} \circ s^2)(x) = 4x-1 = 4x+4 = 4(x+1)$, so $s^2 \circ r = r^{-1} \circ s^2$ and $\sigma^2\rho = \rho^{-1}\sigma^2$.

Write $\sigma' = \sigma^2$. Since $\mathrm{AGL}(1, \mathbb{F}_5) \cap A_5 = \langle \rho, \sigma' \rangle$, the relations

$$\rho^5 = e, \qquad \sigma'^2 = e, \qquad \sigma'\rho = \rho^{-1}\sigma'$$

characterizes the dihedral group D_{10}.

(d) Let $H \supsetneq \langle (1\,2\,3\,4\,5) \rangle$ a subgroup of $\mathrm{AGL}(1, \mathbb{F}_5)$. By part (a), H contains an element $\rho^b\sigma^k$, with $k \in \{1, 2, 3\}$.

Since $\rho \in H$, $\sigma^k = \rho^{-b}(\rho^b\sigma^k) \in H$.

If $k = 1$, then $\sigma \in H$, and if $k = 3$, then $\sigma^3 = \sigma^{-1} \in H$. In both cases, $\sigma \in H$. Since $\mathrm{AGL}(1, \mathbb{F}_5)$ is generated by $\rho = (1\,2\,3\,4\,5)$ and $\sigma = (1\,2\,4\,3)$, then $H = \mathrm{AGL}(1, \mathbb{F}_5)$.

It remains the case where H contains σ^2 and doesn't contain σ. Then $H \supset \langle \rho, \sigma^2 \rangle$. No element of the form $\rho^b\sigma^{2k+1}$ is in H, otherwise $\sigma \in H$, so

$$H = \langle \rho, \sigma^2 \rangle = \mathrm{AGL}(1, \mathbb{F}_5) \cap A_5.$$

Thus the only subgroups of $\mathrm{AGL}(1, \mathbb{F}_5)$ containing $\langle (1\,2\,3\,4\,5) \rangle$ are

$$\langle (1\,2\,3\,4\,5) \rangle, \qquad \mathrm{AGL}(1, \mathbb{F}_5) \cap A_5, \qquad \mathrm{AGL}(1, \mathbb{F}_5).$$

□

Ex. 13.2.2 *This exercise will consider some simple properties of S_5.*

(a) Prove that $\langle (1\,2\,3\,4\,5) \rangle$ is a 5-Sylow subgroup of S_5 and more generally is a 5-Sylow subgroup of any subgroup $G \subset S_5$ containing $\langle (1\,2\,3\,4\,5) \rangle$.

(b) Prove that S_5 has twenty-four 5-cycles.

Proof. (a) As $|S_5| = 5! = 5 \cdot 24$, where $\gcd(5, 24) = 1$, any subgroup of S_5 with order 5 is a 5-Sylow of S_5, so $\langle (1\,2\,3\,4\,5) \rangle$ is a 5-Sylow of S_5.

Let G be a subgroup of S_5 containing $\langle (1\,2\,3\,4\,5) \rangle$. Then 5 divides $|G|$ and $|G|$ divides $5! = 5 \cdot 24$, so $|G| = 5d$, where $d \mid 24$, thus $\gcd(5, d) = 1$. Therefore $\langle (1\,2\,3\,4\,5) \rangle$ is a 5-Sylow of G.

(b) There are 5! arrangements $(a_1, a_2, a_3, a_4, a_5)$, with distinct a_i in $\{1, 2, 3, 4, 5\}$. The 5 arrangements $(a_1, a_2, a_3, a_4, a_5), (a_2, a_3, a_4, a_5, a_1), \ldots$ correspond to the same permutation $(a_1\,a_2\,a_3\,a_4\,a_5)$, so there are $5!/5 = 24$ 5-cycles in S_5.

□

Ex. 13.2.3 *Let $G \subset S_5$ be transitive, and let N be the number of subgroups of G of order 5. In this exercise, you will use an argument from [Postnikov] to prove that $N = 1$ or 6 without using the Sylow Theorems. Let $C = \{\tau \in S_5 \setminus G \mid \tau \text{ is a 5-cycle}\}$.*

(a) Prove that $\sigma \cdot \tau = \sigma\tau\sigma^{-1}$ defines an action of G on C.

(b) Let $\tau \in S_5$ be a 5-cycle. Prove that $\sigma \in S_5$ satisfies $\sigma\tau\sigma^{-1} = \tau$ if and only if $\sigma \in \langle \tau \rangle$.

(c) Use parts (a) and (b) to prove that $|G|$ divides $|C|$.

(d) Prove that $4N + |C| = 24$.

(e) Use parts (c) and (d) to prove that $N = 1$ *or* 6.

Proof. (a) Let $\sigma \in G$ and $\tau \in S_5 \setminus G$. If $\sigma\tau\sigma^{-1} \in G$, then $\tau \in G$, in contradiction with the hypothesis. So, if $\sigma \in G$,

$$\tau \in C \Rightarrow \sigma \cdot \tau \in C.$$

Moreover, if $\sigma, \sigma' \in G$, and $\tau \in C$, then $e \cdot \tau = e\tau e^{-1} = \tau$, and

$$\sigma \cdot (\sigma' \cdot \tau) = \sigma \cdot (\sigma'\tau\sigma'^{-1}) = \sigma\sigma'\tau\sigma'^{-1}\sigma^{-1} = (\sigma\sigma') \cdot \tau.$$

Therefore $\sigma \cdot \tau = \sigma\tau\sigma^{-1}$ defines an action of G on C.

(b) Let $\tau = (a_1\, a_2\, a_3\, a_4\, a_5) \in S_5$ be a 5-cycle.

- Suppose that $\sigma \in S_5$ satisfies $\sigma\tau\sigma^{-1} = \tau$. Then $(\sigma(a_1)\, \sigma(a_2)\, \sigma(a_3)\, \sigma(a_4)\, \sigma(a_5)) = (a_1\, a_2\, a_3\, a_4\, a_5)$.
  Therefore $\sigma(a_1) \in \{a_1, a_2, a_3, a_4, a_5\}$, so $\sigma(a_1) = a_i$ for some $i \in [\![1, 5]\!]$.
  Then, for all $j \in [\![1, 5]\!]$, since $\sigma\tau = \tau\sigma$,

$$\begin{aligned}\sigma(a_j) &= (\sigma\tau^{j-1})(a_1)\\ &= (\tau^{j-1}\sigma)(a_1)\\ &= \tau^{j-1}(a_i)\\ &= \tau^{j-1}(\tau^{i-1}(a_1))\\ &= \tau^{i-1+j-1}(a_1)\\ &= \tau^{i-1}(a_j),\end{aligned}$$

  Since $\{a_1, a_2, a_3, a_4, a_5\} = \{1, 2, 3, 4, 5\}$, $\sigma = \tau^{i-1} \in \langle\tau\rangle$.
- Conversely, suppose that $\sigma \in \langle\tau\rangle$. Since $\langle\tau\rangle$ is cyclic, it is an Abelian subgroup, therefore $\sigma\tau = \tau\sigma$, so $\sigma\tau\sigma^{-1} = \tau$.

Conclusion: If $\tau \in S_5$ is a 5-cycle,

$$\forall \sigma \in S_5, \quad \sigma\tau\sigma^{-1} = \tau \iff \sigma \in \langle\tau\rangle.$$

(c) By part (b), the stabilizer of $\tau \in C$ in G is

$$\mathrm{Stab}_G(\tau) = \langle\tau\rangle \cap G.$$

Since $\tau \in C$, $\tau \notin G$. If $\tau^k \in G$ for some $k \in \{2, 3, 4\}$, since $k \wedge 5 = 1$, $uk + 5v = 1$ for some integers u, v, so $\tau = \tau^{uk}\tau^{5v} = (\tau^k)^u \in G$, which is a contradiction, so $\tau, \tau^2, \tau^3, \tau^4$ are not in G, so $\langle\tau\rangle \cap G = \{e\}$. Therefore

$$G_\tau = \mathrm{Stab}_G(\tau) = \{e\}.$$

If $\mathcal{O}_\tau$ is the orbit of τ for the action defined in part (a),

$$|\mathcal{O}_\tau| = (G : G_\tau) = |G|.$$

As $C = \coprod_{\tau \in S} \mathcal{O}_\tau$, where S is a complete system of the representatives of the orbits, if $m = |S|$ is the number of orbits, $|C| = m|G|$, so

$$|G| \text{ divides } |C|.$$

(d) By Exercise 2, S_5 has 24 5-cycles.

$$\begin{aligned}\{\tau \in S_5 \mid \tau \text{ is a 5-cycle}\} &= \{\tau \in G \mid \tau \text{ is a 5-cycle}\} \cup \{\tau \in S_5 \setminus G \mid \tau \text{ is a 5-cycle}\} \\ &= \{\tau \in G \mid \tau \text{ is a 5-cycle}\} \cup C,\end{aligned}$$

where the union is a disjoint union.

Moreover, G has N subgroups of order 5, and the intersection of two such subgroups is $\{e\}$, so G has $N \times 4$ 5-cycles. Therefore

$$24 = 4N + |C|.$$

(e) Since $G \subset S_5$ is a transitive subgroup, by Lemma 13.2.1, $5 \mid |G|$, and by part (c), $|G| \mid |C|$, so part (d) implies

$$5 \mid 24 - 4N.$$

Therefore $4N \equiv 24 \equiv 4 \pmod 5$, so $N \equiv 1 \pmod 5$, and since $4N \leq 24$, $N \leq 6$, so

$$N = 1 \text{ or } N = 6.$$

□

Ex. 13.2.4 *Prove that (13.19) gives coset representatives of* $\mathrm{AGL}(1, \mathbb{F}_5)$ *in* S_5.

Proof. As the index $(S_5 : \mathrm{AGL}(1, \mathbb{F}_5)) = 120/20 = 6$, it is sufficient to verify that the 6 permutations

$$S = \{e, (1\,2\,3), (2\,3\,4), (3\,4\,5), (1\,4\,5), (1\,2\,5)\}$$

are in distinct coset, by verifying that the 15 permutations $uv^{-1} \notin \mathrm{AGL}(1, \mathbb{F}_5)$, where

$$u, v \in S, \qquad u \neq v.$$

$$\begin{aligned}\mathrm{AGL}(1, \mathbb{F}_5) = \{&e, (1,2,4,3), (1,2,3,4,5), (1,3,5,2,4), (1,4,5,2), (1,3,2,5), \\ &(1,4)(2,3), (1,4,2,5,3), (2,5)(3,4), (1,5,3,4), (2,3,5,4), (1,3)(4,5), \\ &(1,3,4,2), (1,5)(2,4), (1,2)(3,5), (1,5,4,3,2), (2,4,5,3), (1,4,3,5), \\ &(1,2,5,4), (1,5,2,3)\}\end{aligned}$$

$$\begin{aligned}&\{uv^{-1} \mid u \in S, v \in S, u \neq v\} = \\ &\{(1,3,2), (2,4,3), (3,5,4), (1,5,4), (1,5,2), (1,2,3,5,4), (2,3,5), \\ &(1,3,4), (1,5,4,2,3), (1,5,2,3,4), (1,3,4,5,2), (2,4,5), (1,5,3), \\ &(1,5,3,4,2), (1,2,4)\}\end{aligned}$$

Sage instructions

```
S5 = SymmetricGroup(5)
a = S5([(1,2,3,4,5)])
b = S5([(1,2,4,3)])
G = PermutationGroup([a,b])
l =  [S5([]),S5([(1,2,3)]),S5([(2,3,4)]),S5([(3,4,5)]),S5([(1,4,5)]),S5([(1,2,5)])]
[u*v^(-1) in G for u in l for v in l if u< v]
```

[False, False, ..., False]

So S is a set of coset representatives of $\mathrm{AGL}(1, \mathbb{F}_5)$ in S_5. □

Ex. 13.2.5 *Complete the proof of part (b) of Theorem 13.2.6. Then prove part (c).*

Proof. • In the context of the proof of part (b) of Theorem 13.2.6, $A_5 \subset G$. Let

$$\tau_1 = e, \quad \tau_2 = (1\,2\,3), \quad \tau_3 = (2\,3\,4), \quad \tau_4 = (3\,4\,5), \quad \tau_5 = (1\,4\,5), \quad \tau_6 = (1\,2\,5),$$

and $h_i = \tau_i \cdot h,\ i = 1, \ldots, 6$, where $h = u^2$ and

$$\begin{aligned} u = {} & x_1x_2 + x_2x_3 + x_3x_4 + x_4x_5 + x_5x_1 \\ & - x_1x_3 - x_3x_5 - x_5x_2 - x_2x_4 - x_4x_1. \end{aligned}$$

By definition,

$$\beta_i = h_i(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5).$$

Since $A_5 \subset G$ and $\tau_i \in A_5$, there exists $\sigma_i \in \mathrm{Gal}(L/F)$ such that σ_i maps to τ_i for every $i \in [\![1, 6]\!]$, so

$$\sigma_i(\alpha_j) = \alpha_{\tau_i(j)}, \qquad i \in [\![1, 6]\!], \qquad j \in [\![1, 5]\!].$$

Then, for all $i \in [\![1, 6]\!]$,

$$\begin{aligned} \sigma_i(\beta_1) &= \sigma_i(h(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5)) \\ &= h(\sigma_i(\alpha_1), \sigma_i(\alpha_2), \sigma_i(\alpha_3), \sigma_i(\alpha_4), \sigma_i(\alpha_5)) \\ &= h(\alpha_{\tau_i(1)}, \alpha_{\tau_i(2)}, \alpha_{\tau_i(3)}, \alpha_{\tau_i(4)}, \alpha_{\tau_i(5)}) \\ &= (\tau_i \cdot h)(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5) \\ &= h_i(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5) \\ &= \beta_i \end{aligned}$$

So

$$\sigma_i(\beta_1) = \beta_i, \qquad i = 1, \ldots, 6.$$

By assumption, some of the root β_j is in F, therefore $\beta_1 = \sigma_j^{-1}(\beta_j) = \beta_j \in F$, and $\beta_i = \sigma_i(\beta_1) = \beta_1$ for all $i \in [\![1, 6]\!]$. We obtain the identity

$$(y - \beta_1)^6 = (y^3 + b_2y^2 + b_4y + b_6)^2 - 2^{10}\Delta(f)y.$$

Multiplying this out, we obtain

$$\begin{aligned} & y^6 - 6\beta_1y^5 + 15\beta_1^2y^4 - 20\beta_1^3y^3 + 15\beta_1^4y^2 - 6\beta_1^5y + \beta_1^6 = \\ & y^6 + 2b_2y^5 + (b_2^2 + 2b_4)y^4 + (2b_6 + 2b_2b_4)y^3 + (b_4^2 + 2b_2b_6)y^2 + (2b_4b_6 - 2^{10}\Delta(f))y + b_6^2, \end{aligned}$$

so

$$\begin{aligned} -6\beta_1 &= 2b_2, \\ 15\beta_1^2 &= b_2^2 + 2b_4, \\ -20\beta_1^3 &= 2b_2b_4 + 2b_6. \end{aligned}$$

Therefore, since F has characteristic $\neq 2$,

$$\begin{aligned} b_2 &= -3\beta_1, \\ b_4 &= \frac{1}{2}(15\beta_1^2 - 9\beta_1^2) \\ &= 3\beta_1^2, \\ b_6 &= \frac{1}{2}(-20\beta_1^3 + 18\beta_1^3) \\ &= -\beta_1^3, \end{aligned}$$

so

$$b_2 = -3\beta_1, \qquad b_4 = 3\beta_1^2, \qquad b_6 = -\beta_1^3.$$

The precedent identity becomes

$$\begin{aligned} (y-\beta_1)^6 &= (y^3 + b_2y^2 + b_4y + b_6)^2 - 2^{10}\Delta(f)y \\ &= (y^3 - 3\beta_1y^2 + 3\beta_1^2y - \beta_1^3)^2 - 2^{10}\Delta(f)y \\ &= (y-\beta_1)^6 - 2^{10}\Delta(f)y. \end{aligned}$$

Hence $2^{10}\Delta(f) = 0$. Yet F has characteristic $\neq 2$, and $\Delta(f) \neq 0$, since f is separable. This contradiction completes the proof of the theorem.

- We prove part (c) of Theorem 13.2.6.

  Suppose that G is conjugate to $\langle(1\,2\,3\,4\,5)\rangle$. Let $L = F(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5)$ be the splitting field of f. Then $\mathrm{Gal}(L/F) = \langle\sigma\rangle$, where σ corresponds to $(1\,2\,3\,4\,5)$.

  $[L : F] = |\mathrm{Gal}(L/F)| = 5$, so the tower Theorem implies that $[F(\alpha) : F]$ divides 5.

  Since f is irreducible, $\alpha \notin F$, so $[F(\alpha) : F] \neq 1$, $[F(\alpha) : F] = 5$ and $L = F(\alpha)$. Therefore $\alpha_i \in F(\alpha)$, $i = 1, \ldots, 5$ and so $f = (x-\alpha_1)(x-\alpha_2)(x-\alpha_3)(x-\alpha_4)(x-\alpha_5)$ splits completely over $F(\alpha)$.

  Conversely suppose that f splits completely over $F(\alpha)$. Then $\alpha_i \in F(\alpha)$, $i = 1, \ldots, 5$, so the splitting field of f is $L = F(\alpha)$. Therefore $|\mathrm{Gal}(L/F)| = [L : F] = 5$ is prime, so $\mathrm{Gal}(L/F)$ is cyclic. $G \simeq \mathrm{Gal}(L/F)$ is cyclic of order 5, so $G = \langle\tau\rangle$, where $\tau \in S_5$ is a permutation of order 5. τ is a product of disjoint cycles whose order is the least common multiple of the length of the cycles, so τ is a 5-cycle.

  $G = \langle(a_1\,a_2a_3a_4a_5)\rangle = \sigma\langle(1\,2\,3\,4\,5)\rangle\sigma^{-1}$, where σ is defined by $\sigma(i) = a_i$, $i = 1, \ldots, 5$. G is conjugate to $\langle(1\,2\,3\,4\,5)\rangle$.

□

Ex. 13.2.6 *In this exercise, you will use Maple or Mathematica (or Sage!), to prove (13.23) and (13.24).*

(a) The first step is to enter (13.17) and call it, for example $u1$*. Then use substitution commands and (13.19) to create* $u2, \ldots u6$*. For example,* $u2$ *is obtained by applying* $(1\,2\,3)$ *to* $u1$*. In Maple, this is done via the command*

```
u2 := subs({x1=x2,x2=x3,x3=x1},u1);
```

whereas in Mathematica one uses

```
u2 := u1 /. {x1->x2,x2->x3,x3->x1}
```

(b) Now multiply out $\Theta(y) = (y-u1)\cdots(y-u6)$ *and use the methods of section 2.3 to express the coefficients of* $\Theta(y)$ *in terms of the elementary symmetric polynomials.*

(c) Show that your results imply (13.23) and (13.24).

Proof. Sage instructions :

```
R.<y,x,x1,x2,x3,x4,x5,y1,y2,y3,y4,y5> = PolynomialRing(QQ, order = 'degrevlex')
elt = SymmetricFunctions(QQ).e()
e = [elt([i]).expand(5).subs(x0=x1, x1=x2, x2=x3, x3 = x4, x4 = x5)
               for i in range(6)]
J = R.ideal(e[1]-y1, e[2]-y2, e[3]-y3,e[4]-y4,e[5]-y5)
G = J.groebner_basis()
u1 =x1*x2 + x2*x3 + x3*x4 + x4*x5 + x5*x1 - x1*x3 - x3*x5 -x5*x2 - x2*x4 -x4*x1
u2 = u1.subs(x1 = x2, x2 = x3, x3 = x1)
u3 = u1.subs(x2 = x3, x3 = x4, x4 = x2)
u4 = u1.subs(x3 = x4, x4 = x5, x5 = x3)
u5 = u1.subs(x1 = x4, x4 = x5, x5 = x1)
u6 = u1.subs(x1 = x2, x2 = x5, x5 = x1)
f1 = (y-u1) * (y-u2) * (y-u3) * (y-u4) * (y-u5) * (y-u6)
var('sigma_1,sigma_2,sigma_3,sigma_4,sigma_5')
g = f1.reduce(G).subs(y1=sigma_1, y2=sigma_2, y3=sigma_3, y4=sigma_4, y5= sigma_5)
h = g.collect(y);
```

Now we can verify (13.23) and (13.24):

```
h.coefficient(y^5),h.coefficient(y^3)
```

$$(0, 0)$$

```
B2 = h.coefficient(y^4); B2
```

$$-3\,\sigma_2^2 + 8\,\sigma_1\sigma_3 - 20\,\sigma_4$$

```
B4 = h.coefficient(y^2); B4
```

$$\begin{aligned} &3\,\sigma_2^4 - 16\,\sigma_1\sigma_2^2\sigma_3 + 16\,\sigma_1^2\sigma_3^2 + 16\,\sigma_1^2\sigma_2\sigma_4 - 64\,\sigma_1^3\sigma_5 + 16\,\sigma_2\sigma_3^2 - 8\,\sigma_2^2\sigma_4 \\ &- 112\,\sigma_1\sigma_3\sigma_4 + 240\,\sigma_1\sigma_2\sigma_5 + 240\,\sigma_4^2 - 400\,\sigma_3\sigma_5 \end{aligned}$$

```
B6 = h.subs(y = 0);B6
```

$$\begin{aligned} &-\sigma_2^6 + 8\,\sigma_1\sigma_2^4\sigma_3 - 16\,\sigma_1^2\sigma_2^2\sigma_3^2 - 16\,\sigma_1^2\sigma_2^3\sigma_4 + 64\,\sigma_1^3\sigma_2\sigma_3\sigma_4 \\ &- 64\,\sigma_1^4\sigma_4^2 - 16\,\sigma_2^3\sigma_3^2 + 64\,\sigma_1\sigma_2\sigma_3^3 + 28\,\sigma_2^4\sigma_4 - 112\,\sigma_1\sigma_2^2\sigma_3\sigma_4 - 128\,\sigma_1^2\sigma_3^2\sigma_4 + 224\,\sigma_1^2\sigma_2\sigma_4^2 \\ &+ 48\,\sigma_1\sigma_2^3\sigma_5 - 192\,\sigma_1^2\sigma_2\sigma_3\sigma_5 + 384\,\sigma_1^3\sigma_4\sigma_5 - 64\,\sigma_3^4 + 224\,\sigma_2\sigma_3^2\sigma_4 - 176\,\sigma_2^2\sigma_4^2 - 64\,\sigma_1\sigma_3\sigma_4^2 \\ &- 80\,\sigma_2^2\sigma_3\sigma_5 + 640\,\sigma_1\sigma_3^2\sigma_5 - 640\,\sigma_1\sigma_2\sigma_4\sigma_5 - 1600\,\sigma_1^2\sigma_5^2 + 320\,\sigma_4^3 - 1600\,\sigma_3\sigma_4\sigma_5 + 4000\,\sigma_2\sigma_5^2 \end{aligned}$$

The coefficient c_1 of y in $h = \Theta(y)$ is not symmetric in $x_1, \ldots, x_5$, but we verify that $c_1 = 2^5\sqrt{\Delta}\,y$, computing first $\sqrt{\Delta}$:

```
c1  = f1.coefficient(y)
x = [1,x1,x2,x3,x4,x5]
sqrtDelta = 1
for i in range(1,6):
    for j in range(i+1,6):
        sqrtDelta *= (x[i] -x[j])
sqrtDelta
c1 + 2^5 * sqrtDelta
```

$$0$$

So (13.23) and (13.24) are verified. □

Ex. 13.2.7 *Consider* $\mathrm{AGL}(1,\mathbb{F}_5) \cap A_5 \subset S_5$*, and let* u *be defined as in (13.17).*

(a) Prove that the symmetry group of u *is* $\mathrm{AGL}(1,\mathbb{F}_5) \cap A_5$*.*

(b) Prove that (13.19) gives coset representatives of $\mathrm{AGL}(1,\mathbb{F}_5) \cap A_5$ *in* A_5*.*

Proof. (a) Let G be the symmetry group of u.

- If $\sigma \in G$, then $\sigma \cdot u = u$, therefore $\sigma \cdot h = \sigma \cdot u^2 = u^2 = h$. By Lemma 13.2.4, $\sigma \in \mathrm{AGL}(1,\mathbb{F}_5)$, so $G \subset \mathrm{AGL}(1,\mathbb{F}_5)$. $G \neq \mathrm{AGL}(1,\mathbb{F}_5)$, otherwise $(1\,2\,4\,3) \in G$, but $(1\,2\,4\,3) \cdot u = -u \neq u$ (see (13.2.B)). Therefore $G \subsetneq \mathrm{AGL}(1,\mathbb{F}_5)$.
  Moreover $(1\,2\,3\,4\,5) \cdot u = u$, so $\langle (1\,2\,3\,4\,5) \rangle \subset G$ and G is transitive. By Theorem 13.2.2,
  $$G \subset \mathrm{AGL}(1,\mathbb{F}_5) \cap A_5.$$
- If $\chi \in \mathrm{AGL}(1,\mathbb{F}_5) \cap A_5$, by Exercise 1 part (b),
  $$\chi = (1\,2\,3\,4\,5)^k[(1\,4)(2\,3)]^l, \qquad k,l \in \mathbb{N}.$$
  $(1\,2\,3\,4\,5) \cdot u = u$ and $(1\,2\,4\,3) \cdot u = -u$, therefore $(1\,4)(2\,3) \cdot u = (1\,2\,4\,3)^2 \cdot u = u$. Thus $\chi \in G$.
  $$G = \mathrm{AGL}(1,\mathbb{F}_5) \cap A_5.$$

(b) In Exercise 4, we verified that for $u, v \in S, u \neq v$, with
$$S = \{e, (1\,2\,3), (2\,3\,4), (3\,4\,5), (1\,4\,5), (1\,2\,5)\} \subset A_5,$$
then $uv^{-1} \notin \mathrm{AGL}(1,\mathbb{F}_5)$, a fortiori $uv^{-1} \notin \mathrm{AGL}(1,\mathbb{F}_5) \cap A_5$.
Moreover the index $(A_5 : \mathrm{AGL}(1,\mathbb{F}_5) \cap A_5) = 60/10 = 6$, so S is a complete system of coset representatives of $\mathrm{AGL}(1,\mathbb{F}_5) \cap A_5$ in A_5.

□

Ex. 13.2.8 *Let* $u_1, \ldots, u_6$ *be as in the proof of Proposition 13.2.5, and let* $\tau \in S_5$ *be a transposition.*

(a) For each i*, prove that* $\tau \cdot u_i = -u_j$ *for some* j*.*

(b) Let $\Theta(y) = \prod_{i=1}^{6}(y - u_i)$ *and write this polynomial as*
$$\Theta(y) = y^6 + B_1y^5 + B_2y^4 + B_3y^3 + B_4y^2 + B_5y + B_6.$$
Use part (a) to show that $\tau \cdot B_i = (-1)^i B_i$ *for* $i = 1, \ldots, 6$*.*

(c) *Explain how part (b) and the results of Chapter 2 imply that the coefficients B_2, B_4, B_6 are polynomials in $\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5$. This explains why the formulas (13.24) exist.*

(d) *Use Exercise 3 of Section 7.4 to show that the coefficients B_1, B_3, B_5 must be of the form $B\sqrt{\Delta}$, where B is a polynomial in $\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5$.*

(e) *Note that $\sqrt{\Delta}$ has degree 10 as a polynomial in x_1, x_2, x_3, x_4, x_5. By considering the degrees of B_1, B_3, B_5 as polynomials in x_1, x_2, x_4, x_4, x_5, show that part (d) implies that $B_1 = B_3 = 0$ and that B_5 is a constant multiple of $\sqrt{\Delta}$. This explains (13.23).*

Proof. (a) Let $\tau \in S_5 \setminus A_5$ be a transposition, and write $\sigma = (1\,2\,4\,3) \in S_5 \setminus A_5$. We know that $\sigma \cdot u = -u$.

Since $\sigma \in S_5 \setminus A_5$, S_5 is the disjoint union $S_5 = A_5 \cup A_5\sigma$, so $S_5 \setminus A_5 = A_5\sigma$. Since $\tau\tau_i \in S_5 \setminus A_5$, then $\tau\tau_i \in A_5\sigma$, so

$$\tau\tau_i = \psi\sigma, \qquad \psi \in A_5.$$

By Exercise 7, $\{\tau_1, \tau_2, \cdots, \tau_6\} = \{e, (1\,2\,3), (2\,3\,4), (3\,4\,5), (1\,4\,5), (1\,2\,5)\}$ is a complete system of coset representatives of $\mathrm{AGL}(1, \mathbb{F}_5) \cap A_5$ in A_5. Therefore

$$\psi = \tau_j\varphi, \qquad j \in [\![1, 6]\!], \qquad \varphi \in \mathrm{AGL}(1, \mathbb{F}_5) \cap A_5.$$

Since $\varphi \in \mathrm{AGL}(1, \mathbb{F}_5) \cap A_5$, by Exercise 7 part (a), $\varphi \cdot u = u$. Therefore

$$\begin{aligned} \tau \cdot u_i &= (\tau\tau_i) \cdot u \\ &= (\tau_j\varphi\sigma) \cdot u \\ &= -(\tau_j\varphi) \cdot u = -\tau_j u = -u_j \end{aligned}$$

For each $i \in [\![1, 6]\!]$, there exists $j \in [\![1, 6]\!]$ such that $\tau \cdot u_i = -u_j$.

(b) Let

$$\Theta(y) = \prod_{i=1}^{6}(y - u_i) = y^6 + B_1y^5 + B_2y^4 + B_3y^3 + B_4y^2 + B_5y + B_6.$$

Note that if $\tau \cdot u_i = \tau \cdot u_j$, $i, j \in [\![1, 6]\!]$, then $\tau^2 \cdot u_i = \tau^2 \cdot u_j$, so $u_i = u_j$ and $i = j$. Therefore τ maps the set $\{u_1, \ldots, u_6\}$ on $\{-u_1, \ldots, -u_6\}$. Consequently

$$\begin{aligned} \tau \cdot \Theta(y) &= \prod_{i=1}^{6}(y - \tau \cdot u_i) \\ &= \prod_{j=1}^{6}(y + u_j) \\ &= y^6 - B_1y^5 + B_2y^4 - B_3y^3 + B_4y^2 - B_5y + B_6 \end{aligned}$$

Since

$$\tau \cdot \Theta(y) = y^6 + \tau \cdot B_1y^5 + \tau \cdot B_2y^4 + \tau \cdot B_3y^3 + \tau \cdot B_4y^2 + \tau \cdot B_5y + \tau \cdot B_6,$$

we conclude

$$\tau \cdot B_i = (-1)^i B_i, \qquad i = 1, \ldots, 6.$$

(c) For $i = 2, 4, 6$, $\tau \cdot B_i = B_i$ for every transposition τ. Since every $\sigma \in S_5$ is a product of transpositions, for all $\sigma \in S_5$, $\sigma \cdot B_i = B_i$, where $B_i \in F[x_1, \ldots, x_5]$, therefore $B_i \in F[\sigma_1, \ldots, \sigma_5]$.

B_2, B_4, B_6 are polynomials in $\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5$.

(d) For $i = 1, 3, 5$, $\tau \cdot B_i = -B_i$, thus B_i is invariant under A_5. Since the characteristic of F is not 2, by Exercise 7.4.3, $B_i = C_i + D_i\sqrt{\Delta}$, where C_i, D_i are polynomials in the σ_i. Then $-C_i - D_i\sqrt{\Delta} = -B_i = \tau \cdot B_i = C_i - D_i\sqrt{\Delta}$, so $C_i = 0$.

$$B_i = D_i\sqrt{\Delta}, \qquad D_i \in F[\sigma_1, \ldots, \sigma_5] \qquad \text{for } i = 1, 3, 5.$$

(e) Since $\sqrt{\Delta} = \prod_{1 \le i < j \le 5}(x_i - x_j)$, $\sqrt{\Delta}$ has degree $1 + 2 + 3 + 4 = 10$ as a polynomial in x_1, x_2, x_3, x_4, x_5. $B_1 = u_1 + u_2 + u_3 + u_4 + u_5$, with $\deg(u_i) = 2$, thus $\deg(D_1\sqrt{\Delta}) \le 2$. Therefore $D_1 = 0$.

$B_3 = u_1u_2u_3 + \cdots$, so $\deg(B_3) = \deg(D_3\sqrt{\Delta}) \le 6$. Therefore $D_3 = 0$, and $B_3 = 0$.

$B_5 = u_1u_2u_3u_4u_5 + \cdots$, so $\deg(B_5) \le 10$. Therefore $\deg(D_5) \le 0$, so $D_5 = c \in F$ is a constant.

$$\Theta(y) = y^6 + B_2y^4 + B_4y^2 + B_6 + c\sqrt{\Delta}y, \qquad B_2, B_4, B_6 \in F[\sigma_1, \ldots, \sigma_5], \qquad c \in F.$$

By Exercise 7, $c = -2^5$. □

Ex. 13.2.9 *This exercise will prove the first equivalence of Proposition 13.2.7.*

(a) First suppose that $\theta_f(y)$ is irreducible. Prove that $|G|$ is divisible by 6, and explain why this implies that $A_5 \subset G$.

(b) Now suppose that $A_5 \subset G$. Prove that $\mathrm{Gal}(L/F)$ acts transitively on $\beta_1, \ldots, \beta_6$. However, we don't know that $\beta_1, \ldots, \beta_6$ are distinct.

(c) Let $p(y)$ be the minimal polynomial of β_1 over F. By part (b), it is also the minimal polynomial of $\beta_2, \ldots, \beta_6$. Prove that $\theta_f(y) = p(y)^m$, where $m = 1, 2, 3,$ or 6. The proof of Theorem 13.2.6 shows that $m = 6$ cannot occur, and $m = 1$ implies that $\theta_f(y)$ is irreducible over F. It remains to consider what happens when $m = 2$ or 3.

(d) Show that $(y^3 + ay^2 + by + c)^2 = \theta_f(y)$ implies that $\Delta(f) = 0$. Hence this case can't occur.

(e) Show that $(y^2 + ay + b)^3 = \theta_f(y)$ implies that $4b = a^2$, and then use this to show that $\Delta(f) = 0$.

Proof. (a) Suppose that $\theta_f(y) = \prod_{i=1}^{6}(y - \beta_i)$ is irreducible over F. Then $\theta_f(y)$ is the minimal polynomial of $\beta_1 = h(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5)$ over F. Therefore

$$[F(\beta_1) : F] = \deg \theta_f(y) = 6.$$

Since $\beta_1 = h(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5) \in F(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5) = L$,

$$F \subset F(\beta_1) \subset F(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5) = L.$$

By the Tower Theorem,

$$[F(\beta_1) : F] \mid [L : F],$$

therefore

$$6 \mid [L : F] = |\mathrm{Gal}(L/F)| = |G|.$$

Since $6 \nmid |\mathrm{AGL}(1, \mathbb{F}_5)| = 20$, G is not a subgroup of $\mathrm{AGL}(1, \mathbb{F}_5)$. By Theorem 13.2.2, since G is a transitive subgroup of S_5, $G = A_5$ or $G = S_5$:

$$A_5 \subset G.$$

(b) Now suppose that $A_5 \subset G$. Then

$$G \supset \{\tau_1, \ldots, \tau_6\} = \{e, (1\,2\,3), (2\,3\,4), (3\,4\,5), (1\,4\,5), (1\,2\,5)\},$$

so $\tau_i \in G$ and the corresponding σ_i are in $\mathrm{Gal}(L/F)$. By Exercise 13.2.5, $\sigma_i(\beta_1) = \beta_i$, thus the orbit of β_1 under the action of $\mathrm{Gal}(L/F)$ is $\mathcal{O}_{\beta_1} = \{\beta_1, \ldots, \beta_6\}$. This is sufficient to prove that $\mathrm{Gal}(L/F)$ acts transitively on $\beta_1, \ldots, \beta_6$.

(c) Let $p(y)$ be the minimal polynomial of β_1 over F. There exists $\sigma_i \in \mathrm{Gal}(L/F)$ such that $\sigma_i(\beta_1) = \beta_i$, and since $p(y) \in F[y]$, $0 = \sigma_i(p(\beta_1)) = p(\sigma_i(\beta_1)) = p(\beta_i)$, so β_i is a root of p, where p is irreducible. Therefore p is the minimal polynomial over F of $\beta_1, \ldots, \beta_6$.

Under the hypothesis of Theorem 13.2.7 (and 13.2.6), $F \subset L$ is a separable extension, so β_1 is separable, therefore

$$p(x) = (x - \gamma_1) \cdots (x - \gamma_r),$$

where $\gamma_1, \ldots, \gamma_r$ are distinct. Since p is the minimal polynomial over F of $\beta_1, \ldots, \beta_6$, each β_j is a γ_i for some $i, 1 \leq i \leq r$, and since $p(y)$ divides $\theta_f(y)$, each γ_i is a β_j, so $\{\gamma_1, \ldots, \gamma_r\} = \{\beta_1, \ldots, \beta_6\}$, and $\gamma_1, \ldots, \gamma_r$ are the distinct roots of $\theta_f(y)$.

Let k_i the order of multiplicity of β_i in $\theta_f(y)$, so $\theta_f(y) = (y - \beta_i)^{k_i} q_i(y)$, $q_i(y) \in L[y]$. Let $\sigma \in \mathrm{Gal}(L/F)$ such that $\sigma(\beta_i) = \beta_j$. Applying σ to $\theta_f(y)$, we obtain $\theta_f(y) = (y - \beta_j)^{k_i} (\sigma \cdot q_i)(y)$, so $k_j \geq k_i$, and similarly $k_i \geq k_j$, so the distinct γ_i have the same order of multiplicity m in $\theta_f(y)$. Therefore

$$\theta_f(y) = (x - \gamma_1)^m \cdots (x - \gamma_r)^m = p(y)^m.$$

Since $6 = \deg(\theta_f(y)) = m \deg(p(y))$, $m \mid 6$, so $m = 1, 2, 3$ or 6.

$m = 6$ gives $\theta_f(y) = (x - \beta_1)^6$. Since the characteristic of f is not 2, this is impossible by the proof of Theorem 13.2.6. It remains to prove the impossibility of $m = 2$ or $m = 3$.

(d) If $m = 2$,

$$\theta_f(y) = (y^3 + ay^2 + by + c)^2, \qquad a, b, c \in F.$$

By Proposition 13.2.5, this gives

$$(y^3 + b_2y^2 + b_4y + b_6)^2 - 2^{10}\Delta(f)y = (y^3 + ay^2 + by + c)^2,$$

so

$$\begin{aligned} 2^{10}\Delta(f)y &= (y^3 + b_2y^2 + b_4y + b_6)^2 - (y^3 + ay^2 + by + c)^2 \\ &= [(b_2 - a)y^2 + (b_4 - b)y + (b_6 - c)][2y^3 + (b_2 + a)y^2 + (b_4 + b)y + (b_6 + c)] \end{aligned}$$

Therefore the coefficient in y^5 is $2(b_2 - a) = 0$. Since the characteristic is not 2,

$$b_2 = a.$$

Using $b_2 = a$, the coefficient in y^4 is $2(b_4 - b) = 0$, so

$$b_4 = b,$$

and then the coefficient in y^3 is $2(b_6 - c)$, so

$$b_6 = c.$$

Therefore $2^{10}\Delta(f)y = 0$. Since the characteristic is not 2, $\Delta(f) = 0$, in contradiction with the assumed separability of f.

(e) If $m = 3$, there exist coefficients $a, b \in F$ such that

$$\theta_f(y) = (y^2 + ay + b)^3 = (y^3 + b_2y^2 + b_4y + b_6)^2 - 2^{10}\Delta(f)y.$$

$$\begin{aligned} 0 = \theta_f(y) - (y^2 + ax + b)^3 = &-(3\,a - 2\,b_2)y^5 - \left(3\,a^2 - b_2^2 + 3\,b - 2\,b_4\right)y^4 \\ &- \left(a^3 + 6\,ab - 2\,b_2b_4 - 2\,b_6\right)y^3 - \left(3\,a^2b + 3\,b^2 - b_4^2 - 2\,b_2b_6\right)y^2 \\ &- \left(3\,ab^2 - 2\,b_4b_6 + 1024\,\Delta(f)\right)y - b^3 + b_6^2. \end{aligned}$$

We obtain b_2, b_4, b_6 with the equations corresponding to the coefficients of y^5, y^4, y^3:

$$\begin{cases} 0 &= -3\,a + 2\,b_2, \\ 0 &= -3\,a^2 + b_2^2 - 3\,b + 2\,b_4, \\ 0 &= -a^3 - 6\,ab + 2\,b_2b_4 + 2\,b_6, \end{cases}$$

which gives

$$b_2 = \frac{3}{2}\,a, \qquad b_4 = \frac{3}{8}\,a^2 + \frac{3}{2}\,b, \qquad b_6 = -\frac{1}{16}\,a^3 + \frac{3}{4}\,ab.$$

If we substitute these values in the coefficient of y^3, we obtain

$$\begin{aligned} a^3 + 6\,ab - 2\,b_2b_4 - 2\,b_6 &= a^3 + 6\,ab - 2\left(\frac{3}{2}\,a\right)\left(\frac{3}{8}\,a^2 + \frac{3}{2}\,b\right) + 2\left(\frac{1}{16}\,a^3 + \frac{3}{4}\,ab\right) \\ &= 0. \end{aligned}$$

The coefficient of y^2 gives $-\frac{3}{64}\left(a^4 - 8\,a^2b + 16\,b^2\right) = -\frac{3}{64}(a^2 - 4b)^2 = 0$.

If we suppose that the characteristic is not 3, then

$$a^2 = 4b.$$

The coefficient of y gives

$$\begin{aligned} 0 &= -\frac{3}{64}\,a^5 + \frac{3}{8}\,a^3b - \frac{3}{4}\,ab^2 - 2^{10}\,\Delta(f) \\ &= -\frac{3}{64}\,a\left(a^2 - 4b\right)^2 - 2^{10}\,\Delta(f) \\ &= -2^{10}\,\Delta(f) \end{aligned}$$

Therefore

$$\Delta(f) = 0.$$

Since f is separable, this is a contradiction, so $\theta_f(y)$ is irreducible.

It remains the case where the characteristic is 3. Then the equation

$$\theta_f(y) = (y^2 + ay + b)^3 = (y^3 + b_2y^2 + b_4y + b_6)^2 - 2^{10}\Delta(f)y$$

gives the system

$$\begin{cases} 0 &= 2\,b_2, \\ 0 &= b_2^2 + 2\,b_4, \\ 0 &= -a^3 + 2\,b_2b_4 + 2\,b_6, \end{cases}$$

Therefore $b_2 = b_4 = 0$, so the initial equation gives

$$y^6 + a^3y^3 + b^3 = y^6 + 2b_6y^3 - 2^{10}\Delta(f)y + b_6^2,$$

and we have the same contradiction $\Delta(f) = 0$, and the same conclusion:

$$\theta_f(y) \text{ is irreducible over } F.$$

We give here the corresponding Sage instructions:

```
y,b2,b4,b6,Delta,a,b,c = var('y,b2,b4,b6,Delta,a,b,c')
u = (y^3+b2*y^2+b4*y+b6)^2 - 2^10*Delta*y - (y^2+a*y+b)^3
u = u.expand().collect(y); u
```

$$-(3\,a - 2\,b_2)y^5 - \left(3\,a^2 - b_2^2 + 3\,b - 2\,b_4\right)y^4 - \left(a^3 + 6\,ab - 2\,b_2b_4 - 2\,b_6\right)y^3 - b^3$$
$$-\left(3\,a^2b + 3\,b^2 - b_4^2 - 2\,b_2b_6\right)y^2 + b_6^2 - \left(3\,ab^2 - 2\,b_4b_6 + 1024\,\Delta\right)y$$

```
eq = [u.coefficient(y^i) for i in range(3,6)]
solve(eq,b2,b4,b6)
```

$$\left[\left[b_2 = \frac{3}{2}\,a, b_4 = \frac{3}{8}\,a^2 + \frac{3}{2}\,b, b_6 = -\frac{1}{16}\,a^3 + \frac{3}{4}\,ab\right]\right]$$

```
v = u.coefficient(y^3)
w = v.subs(b2 == 3/2*a, b4 == 3/8*a^2 + 3/2*b, b6 == -1/16*a^3 + 3/4*a*b)
w.expand()
```

$$0$$

```
s = u.coefficient(y^2)
t = s.subs(b2 == 3/2*a, b4 == 3/8*a^2 + 3/2*b, b6 == -1/16*a^3 + 3/4*a*b)
t.expand().factor()
```

$$-\frac{3}{64}\left(a^2 - 4\,b\right)^2$$

```
p = u.coefficient(y)
q = p.subs(b2 == 3/2*a, b4 == 3/8*a^2 + 3/2*b, b6 == -1/16*a^3 + 3/4*a*b)
q.expand()
```

$$-\frac{3}{64}\,a^5 + \frac{3}{8}\,a^3b - \frac{3}{4}\,ab^2 - 1024\,\Delta$$

```
q.expand().subs(b = a^2/4)
```

$$-1024\Delta$$

We obtained $\Delta(f) = 0$. □

Ex. 13.2.10 *This exercise will prove the second equivalence of Proposition 13.2.7. Note that one direction follows trivially from Theorem 13.2.6. So we can assume that* $G \subset \mathrm{AGL}(1, \mathbb{F}_5)$ *and that* $\theta_f(y) = (y - \beta_1)g(y)$ *where* $\beta_1 \in F$.

(a) Use $(1\,2\,3\,4\,5) \in G$ *to prove that* $\mathrm{Gal}(L/F)$ *acts transitively on* $\beta_2, \ldots, \beta_6$. *As in the previous exercise, we don't know if* $\beta_2, \ldots, \beta_6$ *are distinct.*

(b) Let $p(y)$ *be the minimal polynomial of* β_2 *over* F. *By part (a), it is also the minimal polynomial of* $\beta_2, \ldots, \beta_6$. *Prove that* $\theta_f(y) = (y - \beta_1)p(y)^m$, *where* $m = 1$ *or* 5. *If* $m = 1$, *then we are done. So we need to rule out* $m = 5$.

(c) Show that $(y-\beta_1)(y-\beta_2)^5 = \theta_f(y)$ *implies that* $\beta_1 = \beta_2$, *and then use this to show that* $\Delta(f) = 0$.

Proof. If $\theta_f(y)$ has a root $\beta \in F$, then by Theorem 13.2.6(b), G is conjugate to a subgroup of $\mathrm{AGL}(1, \mathbb{F}_5)$.

Conversely, assume that G is conjugate to a subgroup of $\mathrm{AGL}(1, \mathbb{F}_5)$. Relabeling the roots, we may assume that $\langle(1\,2\,3\,4\,5)\rangle \subset G \subset \mathrm{AGL}(1, \mathbb{F}_5)$, and by Theorem 13.2.6(b), $\theta_f(y)$ has a root $\beta_1 \in F$, so $\theta_f(y) = (y - \beta_1)g(y)$.

(a) Write $\rho = (1\,2\,3\,4\,5)$ and $\tilde{\rho} \in \mathrm{Gal}(L/F)$ the corresponding automorphism. Then

$$\tilde{\rho}(\alpha_1) = \alpha_2, \ \ldots, \quad \tilde{\rho}(\alpha_4) = \alpha_5, \quad \tilde{\rho}(\alpha_5) = \alpha_1,$$

and $\sigma_i \in \mathrm{Gal}(L/F)$ corresponds to τ_i.

We name the left coset representatives of $\mathrm{AGL}(1, \mathbb{F}_5)$ given in S_5:

$$\tau_1 = e, \ \tau_2 = (1\,2\,3), \ \tau_3 = (2\,3\,4), \ \tau_4 = (3\,4\,5), \ \tau_5 = (1\,4\,5), \ \tau_6 = (1\,2\,5).$$

Note that these cosets representatives verify $\rho\tau_1\rho^{-1} = \tau_1 = e$, and

$$\rho\tau_2\rho^{-1} = \tau_3, \quad \cdots, \quad \rho\tau_5\rho^{-1} = \tau_6, \quad \rho\tau_6\rho^{-1} = \tau_2.$$

By definition, $h_i = \tau_i \cdot h, \ i = 1, \ldots, 6$, where $h = u^2$ and u is given in (13.17), and

$$\sigma_i(\beta_1) = (\tau_i \cdot h)(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5) = \beta_i$$

(see Exercise 13.2.5). Since $\rho \in \mathrm{AGL}(1, 5)$, $\rho \cdot h = h$, therefore, for $2 \le i \le 5$

$$(\rho\tau_i) \cdot h = (\rho\tau_i\rho^{-1}) \cdot (\rho \cdot h) = \tau_{i+1} \cdot h,$$

and $(\rho\tau_6) \cdot h = \tau_2 \cdot h$.

If $\tilde{\varphi} \in \mathrm{Gal}(L/F)$ corresponds to some $\varphi \in S_5$, then $\tilde{\varphi}(\alpha_i) = \alpha_{\varphi(i)}, \ i = 1, \ldots, 5$, so

$$\tilde{\varphi}(h(\alpha_1, \ldots, \alpha_5)) = h(\alpha_{\varphi(1)}, \ldots, \alpha_{\varphi(5)}) = (\varphi \cdot h)(\alpha_1, \ldots, \alpha_5).$$

Since $\tilde{\rho} \circ \sigma_i \in \mathrm{Gal}(L/F)$ corresponds to $\rho\tau_i$,

$$\begin{aligned}\tilde{\rho}(\beta_i) &= \tilde{\rho}(\sigma_i(\beta_1)) \\ &= (\tilde{\rho} \circ \sigma_i)(h(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5)) \\ &= [(\rho\,\tau_i) \cdot h](\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5) \\ &= [(\rho\tau_i\rho^{-1}) \cdot (\rho \cdot h)](\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5) \\ &= (\tau_{i+1}h)(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5) \\ &= \beta_{i+1}, \qquad i = 2, 3, 4, 5,\end{aligned}$$

and similarly $\tilde{\rho}(\beta_6) = \beta_2$. So the images of β_i by the automorphism $\tilde{\rho}$ corresponding to $\rho = (1\,2\,3\,4\,5)$ are given by

$$\beta_2 \mapsto \beta_3 \mapsto \beta_4 \mapsto \beta_5 \mapsto \beta_6 \mapsto \beta_2, \qquad \beta_1 \mapsto \beta_1,$$

therefore $\mathrm{Gal}(L/F)$ acts transitively on $\beta_2, \ldots, \beta_6$.

(b) Let $p(y) \in F[y]$ be the minimal polynomial of β_2 over F. Since $\tilde{\rho} \in \mathrm{Gal}(L, F)$, $p(\beta_3) = p(\tilde{\rho}(\beta_2)) = \tilde{\rho}(p(\beta_2)) = 0$, so β_3, and similarly $\beta_4, \beta_5, \beta_6$ are roots of p, so p is the minimal polynomial of $\beta_2, \ldots, \beta_6$.

$\theta_f(y) = (y - \beta_1)g(y)$, therefore $g(y) \in F[y]$.

$F \subset L$ is a separable extension, so β_2 is separable, therefore

$$p(y) = (y - \gamma_1) \cdots (y - \gamma_r),$$

where $\gamma_1, \ldots, \gamma_r$ are distinct. As p is the minimal polynomial of β_2 over F, and $g(\beta_2) = 0$, with $g \in F[y]$, $p(y)$ divides $\theta_f(y)$, so each γ_i is a β_j, and each β_j, $2 \leq j \leq 6$, is a root of p so is a γ_i. Therefore $\{\gamma_1, \ldots, \gamma_r\} = \{\beta_2, \ldots, \beta_6\}$, and $\gamma_1, \ldots, \gamma_r$ are the distinct roots of g.

Since $\mathrm{Gal}(L/F)$ acts transitively on $\beta_2, \ldots, \beta_6$, then the distinct roots $\gamma_1, \cdots, \gamma_r$ have the same order of multiplicity m (as in Exercise 9). Therefore $g(y) = p(y)^m$, so

$$\theta_f(y) = (y - \beta_1)p(y)^m.$$

Since $5 = \deg(g) = m\deg(p)$, $m \mid 5$, so $m = 1$ or $m = 5$. We need to rule out $m = 5$.

(c) If $m = 5$,

$$\theta_f(y) = (y - \beta_1)(y - \beta_2)^5.$$

Then, with some formal computations,

$$\begin{aligned}0 &= (y^3 + b_2y^2 + b_4y + b_6)^2 - 2^{10}\Delta(f)y - (y - \beta_1)(y - \beta_2)^5 \\ &= (2\,b_2 + \beta_1 + 5\,\beta_2)y^5 + \big(b_2^2 - 5\,\beta_1\beta_2 - 10\,\beta_2^2 + 2\,b_4\big)y^4 + 2\,\big(5\,\beta_1\beta_2^2 + 5\,\beta_2^3 + b_2b_4 + b_6\big)y^3 \\ &\quad - \big(10\,\beta_1\beta_2^3 + 5\,\beta_2^4 - b_4^2 - 2\,b_2b_6\big)y^2 + \big(5\,\beta_1\beta_2^4 + \beta_2^5 + 2\,b_4b_6 - 1024\,\Delta(f)\big)y - \beta_1\beta_2^5 + b_6^2.\end{aligned}$$

The coefficients of y^5, y^4, y^3 give

$$\begin{aligned}0 &= 2\,b_2 + \beta_1 + 5\,\beta_2, \\ 0 &= b_2^2 - 5\,\beta_1\beta_2 - 10\,\beta_2^2 + 2\,b_4, \\ 0 &= 10\,\beta_1\beta_2^2 + 10\,\beta_2^3 + 2\,b_2b_4 + 2\,b_6,\end{aligned}$$

so

$$\begin{aligned}
b_2 &= -\frac{1}{2}\,\beta_1 - \frac{5}{2}\,\beta_2,\\
b_4 &= -\frac{1}{8}\,\beta_1^2 + \frac{5}{4}\,\beta_1\beta_2 + \frac{15}{8}\,\beta_2^2,\\
b_6 &= -\frac{1}{16}\,\beta_1^3 + \frac{5}{16}\,\beta_1^2\beta_2 - \frac{15}{16}\,\beta_1\beta_2^2 - \frac{5}{16}\,\beta_2^3.
\end{aligned}$$

Substituting these values in the equation, we obtain

$$\begin{aligned}
0 &= \frac{5}{64}\left(\beta_1^4 - 4\,\beta_1^3\beta_2 + 6\,\beta_1^2\beta_2^2 - 4\,\beta_1\beta_2^3 + \beta_2^4\right)y^2 + by + c,\\
&= \frac{5}{64}\,(\beta_1 - \beta_2)^4 y^2 + by + c,
\end{aligned}$$

where $b, c \in F(\beta_1, \beta_2)$ are constant.

If the characteristic is not 5, then $\beta_1 = \beta_2$, so $\theta_f(y) = (y - \beta_1)^6$. But the proof of Theorem 13.2.6 shows that this implies that $\Delta(f) = 0$, and this is a contradiction. Thus $m = 1$ and g is irreducible over F. This proves Proposition 13.2.7. in characteristic $\neq 5$.

It remains the case where the characteristic is 5. Then the coefficients of $\theta_f(y) - (y - \beta_1)(y - \beta_2)^5$ give the system of equations

$$\begin{aligned}
0 &= 2b_2 + \beta_1\\
0 &= b_2^2 + 2b_4\\
0 &= b_2 b_4 + b_6\\
0 &= b_4^2 + 2b_2 b_6\\
0 &= \beta_2^5 + 2b_4 b_6 - 2^{10}\Delta(f)\\
0 &= -\beta_1\beta_2^5 + b_6^2
\end{aligned}$$

(where the fourth equation is useless, since in characteristic 5, the coefficient of y^2 is always 0).

We obtain

$$\begin{aligned}
b_4 &= -\frac{1}{2}b_2^2\\
b_6 &= +\frac{1}{2}b_2^3
\end{aligned}$$

The first equation gives $\beta_1 = -2b_2$, and the last gives

$$-2b_2\beta_2^5 = \frac{1}{4}b_2^6.$$

If $b_2 \neq 0$, $\beta_2^5 = -\frac{1}{8}b_2^5$, so

$$2^{10}\Delta(f) = \beta_2^5 + 2b_4 b_6 = -\frac{1}{8}b_2^5 - \frac{1}{2}b_2^5 = -\frac{5}{8}b_2^5 = 0.$$

Therefore $\Delta(f) = 0$, and this is a contradiction.

If $b_2 = 0$, then $b_2 = b_4 = b_6 = 0$, so $\beta_1 = 0$, and the system reduces to a unique equation

$$0 = \beta_2^5 - 2^{10}\Delta(f).$$

In this case

$$\theta_f(y) = y(y - \beta_2)^5 = y(y^5 - \beta_2^5) = y(y^5 - 2^{10}\Delta(f)).$$

I don't see an immediate contradiction ...

The second part of Theorem 13.2.7 is proved here only if the characteristic is not 5.

Sage instructions for part (e):

```
x,y,beta1,beta2,Delta,b2,b4,b6 = var('x,y,beta1,beta2,Delta,b2,b4,b6')
p = (y^3 + b2*y^2 + b4*y + b6)^2 - 2^10*Delta*y - (y-beta1)*(y-beta2)^5
p = p.expand().collect(y)
R.<y>=QQ[]
l = [p.coefficient(y,i) for i in range(5,-1,-1)]
eq = l[:3]
solve(eq,b2,b4,b6)
q=p.subs(b2 == -1/2*beta1 - 5/2*beta2,
      b4 == -1/8*beta1^2 + 5/4*beta1*beta2 + 15/8*beta2^2,
      b6 == -1/16*beta1^3 + 5/16*beta1^2*beta2 -15/16*beta1*beta2^2 - 5/16*beta2
q=q.expand().collect(y)
q.coefficient(y,2).factor()
```

$$\frac{5}{64}\,(\beta_1 - \beta_2)^4$$

□

Ex. 13.2.11 *Show that the table preceding Example 13.2.8 follows from the diagram (13.16) and Theorem 13.2.6.*

Proof.

- Suppose that $\theta_f(y)$ has no root in F (lines 1 and 2 of the table). By Theorem 13.2.6 (b), G is not conjugate to a subgroup of $\mathrm{AGL}(1, \mathbb{F}_5)$. Therefore by diagram (13.6) and Theorem 13.2.2, $G = A_5$ or $G = S_5$ (no conjugacy here). By Theorem 13.2.6 (a), $G = A_5$ if $\Delta(f) \in F^2$, and $G = S_5$ otherwise.

- Suppose now that $\theta_f(y)$ has a root in F (lines 3,4,5 of the table). Then, by Theorem 13.2.6 (b) (and Theorem 13.2.2), G is conjugate to a subgroup of $\mathrm{AGL}(1, \mathbb{F}_5)$ containing $\langle (1\,2\,3\,4\,5) \rangle$.

  So, by diagram (13.16), G is conjugate to $\mathrm{AGL}(1, \mathbb{F}_5)$, $\mathrm{AGL}(1, \mathbb{F}_5) \cap A_5$, or $\langle (1\,2\,3\,4\,5) \rangle$.

  If $\Delta(f) \notin F^2$, $G \not\subset A_5$, therefore $G = \mathrm{AGL}(1, \mathbb{F}_5)$. This is the third line of the table.

  If $\Delta(f) \in F^2$, $G \subset A_5$, therefore G is conjugate to $\mathrm{AGL}(1, \mathbb{F}_5) \cap A_5$, or $\langle (1\,2\,3\,4\,5) \rangle$.

  By theorem 13.2.6 (c), G is conjugate to $\langle (1\,2\,3\,4\,5) \rangle$ if and only if f splits completely over $F(\alpha)$, and this gives the two last lines of the table.

□

Ex. 13.2.12 *Let $f = x^5 - 6x + 3 \in \mathbb{Q}[x]$. Compute $\Delta(f)$ and $\theta_f(y)$ and show that $\theta_f(y)$ is irreducible over $\mathbb{Q}$.*

Proof. By the Schönemann-Eisenstein Criterion for $p = 3$, we know that f is irreducible over $\mathbb{Q}$.

The discriminant $f = x^5 + ax + b, a, b \in \mathbb{Q}$ is given by

$$\Delta(f) = 256a^5 + 3125b^4,$$

so

$$\Delta(f) = -256 \cdot 6^5 + 3125 \cdot 3^4 = -1737531.$$

If we apply on the resolvent $\theta_f(y)$ the evaluation $\sigma_1 \mapsto 0, \sigma_2 \mapsto 0, \sigma_3 \mapsto 0, \sigma_4 \mapsto a, \sigma_5 \mapsto -b$, we obtain

$$\theta_f(y) = \left(y^3 - 20\,ay^2 + 240\,a^2y + 320\,a^3\right)^2 - 2^{10}(256\,a^5 + 3125\,b^4)y$$

With $a = -6, b = 3$, we obtain

$$\begin{aligned}\theta_f(y) &= \left(y^3 + 120\,y^2 + 8640\,y - 69120\right)^2 + 2^{10}\,1737531\,y \\ &= y^6 + 240\,y^5 + 31680\,y^4 + 1935360\,y^3 + 58060800\,y^2 + 584838144\,y + 4777574400\end{aligned}$$

The Schönemann-Eisenstein Criterion doesn't apply.

With Sage, we obtain

```
R.<y> = QQ[]
p=y^6 + 240*y^5 + 31680*y^4 + 1935360*y^3 + 58060800*y^2 + 584838144*y + 4777574400
p.is_irreducible()
```

True

$\theta_f(y)$ is irreducible over $\mathbb{Q}$. A fortiori, $\theta_f(y)$ has no root in $\mathbb{Q}$.
Since $\Delta(f) < 0$ is not a square in $\mathbb{Q}$, the Galois group of f is S_5. □

Ex. 13.2.13 *Let $f = x^5 - 2 \in \mathbb{Q}(\sqrt{5})[x]$ be as in Example 13.2.9.*

(a) Compute $\Delta(f)$ and $\theta_f(y)$.

(b) In Section 6.4 we showed that the Galois group of f over $\mathbb{Q}$ is isomorphic to $\mathrm{AGL}(1, \mathbb{F}_5)$. Use this and the Galois correspondence to show that the Galois group over $\mathbb{Q}(\sqrt{5})$ is isomorphic to $\mathrm{AGL}(1, \mathbb{F}_5) \cap A_5$.

Proof.

(a) We use the formulas of Exercise 12 for $f = x^5 + ax + b$:

$$\begin{aligned}\Delta(f) &= 256a^5 + 3125b^4 \\ \theta_f(y) &= \left(y^3 - 20\,ay^2 + 240\,a^2y + 320\,a^3\right)^2 - 2^{10}\Delta(f)y\end{aligned}$$

With $a = 0, b = -2$, we obtain

$$\begin{aligned}\Delta(f) &= 50000 = 2^4\,5^5 \\ \theta_f(y) &= y^6 - 2^{14}\,5^5\,y\end{aligned}$$

Let L the splitting field of $x^5 - 2$ over $\mathbb{Q}$. $\Delta(f)$ is not a square in $\mathbb{Q}$, and $\theta_f(y)$ has a root 0 in $\mathbb{Q}$. So, by Theorem 13.2.6 and Exercise 11, $\mathrm{Gal}(L/\mathbb{Q})$ is isomorphic to $\mathrm{AGL}(1, \mathbb{F}_5)$. This result is already proved in Section 6.4.

(b) We know that $\zeta_5 = (\zeta_5 \sqrt[5]{2})/\sqrt[5]{2} \in L$, and also $\sqrt{5} = \zeta_5 + \zeta_5^{-1} - \zeta_5^2 - \zeta_5^{-2} \in L$ (see the quadratic Gauss sum page 249).

Since $\mathbb{Q} \subset \mathbb{Q}(\sqrt{5})$ is a quadratic extension, by the Galois correspondence, $\mathrm{Gal}(L/\mathbb{Q}(\sqrt{5}))$ is a subgroup of index 2 in $\mathrm{Gal}(L/\mathbb{Q})$ and the subgroup $H \subset S_5$ corresponding to $\mathrm{Gal}(L/\mathbb{Q}(\sqrt{5}))$ has index 2 in $G \simeq \mathrm{AGL}(1, \mathbb{F}_5)$. Thus $|H| = 10$, and since $5 \mid |H|$, H contains a 5-cycle and is a transitive subgroup of S_5. By Theorem 13.2.2, H is conjugate to $\langle (1\,2\,3\,4\,5) \rangle$ or to $\mathrm{AGL}(1, \mathbb{F}_5) \cap A_5$. Since $(G : H) = 2$, H is conjugate to $\mathrm{AGL}(1, \mathbb{F}_5) \cap A_5$, so

$$\mathrm{Gal}(L/\mathbb{Q}(\sqrt{5})) \simeq H \simeq \mathrm{AGL}(1, \mathbb{F}_5) \cap A_5.$$

□

Ex. 13.2.14 *Let $f = x^5 + px^3 + \frac{1}{5}p^2x + q \in \mathbb{Q}[x]$ be as in Example 13.2.10, and assume that f is irreducible over $\mathbb{Q}$.*

(a) Compute $\Delta(f)$ and $\theta_f(y)$.

(b) Factor $\theta_f(y) \in \mathbb{Q}[x]$, and conclude that $5p^2 \in \mathbb{Q}$ is a root of $\theta_f(y)$.

(c) Show that the substitution $x = z - \frac{p}{5z}$ transforms f into $z^5 - \frac{p^5}{5^5 z^5} + q$.

(d) Use part (c) to give an elementary proof that f is solvable by radicals over $\mathbb{Q}$.

Proof. (a) We obtain the discriminant with Sage:

```
S.<p,q,x> = QQ[]
f = x^5 + p*x^3 +(1/5)*p^2*x + q
Delta = f.discriminant(x);Delta.factor()
```

$$\left(\frac{1}{3125}\right) \cdot (4p^5 + 3125q^2)^2$$

So

$$\Delta(f) = \frac{1}{5^5} \cdot (4p^5 + 3125q^2)^2.$$

We use the following procedure to compute a sextic resolvent with the same method as in Exercise 6:

```
def resolvent(f):
    l = f.coefficients(sparse = False)
    R.<Delta,x1,x2,x3,x4,x5,y1,y2,y3,y4,y5,y,P,Q,e> = PolynomialRing(QQ, order =
    elt = SymmetricFunctions(QQ).e()
    e = [elt([i]).expand(5).subs(x0=x1, x1=x2, x2=x3, x3 = x4, x4 = x5)
                    for i in range(6)]
    J = R.ideal(e[1]-y1, e[2]-y2, e[3]-y3,e[4]-y4,e[5]-y5)
    G = J.groebner_basis()
    u1 =x1*x2 + x2*x3 + x3*x4 + x4*x5 + x5*x1 - x1*x3 - x3*x5 -x5*x2 - x2*x4 -x4
    u2 = u1.subs(x1 = x2, x2 = x3, x3 = x1)
    u3 = u1.subs(x2 = x3, x3 = x4, x4 = x2)
    u4 = u1.subs(x3 = x4, x4 = x5, x5 = x3)
```

```
u5 = u1.subs(x1 = x4, x4 = x5, x5 = x1)
u6 = u1.subs(x1 = x2, x2 = x5, x5 = x1)
f1 = (y-u1) * (y-u2) * (y-u3) * (y-u4) * (y-u5) * (y-u6)
var('sigma_1,sigma_2,sigma_3,sigma_4,sigma_5')
g = f1.reduce(G).subs(y1=sigma_1, y2=sigma_2, y3=sigma_3, y4=sigma_4, y5= si
h = g.collect(y);
B2 = h.coefficient(y,4)
B4 = h.coefficient(y,2)
B6 = h.coefficient(y,0)
b2 = B2.subs(sigma_1 = -l[4], sigma_2= l[3],sigma_3 = -l[2], sigma_4 = l[1],
b4 = B4.subs(sigma_1 = -l[4], sigma_2= l[3],sigma_3 = -l[2], sigma_4 = l[1],
b6 = B6.subs(sigma_1 = -l[4], sigma_2= l[3],sigma_3 = -l[2],
            sigma_4 = l[1], sigma_5 = -l[0])
theta_f = [(y^3+b2*y^2+b4*y+b6)^2 - 2^10*Delta*y,b2,b4,b6]
return theta_f
```

Then we obtain b_2, b_4, b_6 and $\theta_f(y)$:

```
K.<p,q> = QQ[]
S.<x> =PolynomialRing(K, order = 'degrevlex')
f = x^5 + p*x^3 +(1/5)*p^2*x + q
resolvent(f)[1:4]
```

$$\left(-7\,p^2, 11\,p^4, \frac{3}{25}\,p^6 + 4000\,pq^2\right)$$

```
theta_f=resolvent(f)[0];theta_f
```

$$\theta_f(y) = \frac{1}{625}\left(3p^6 + 275p^4y - 175p^2y^2 + 100000pq^2 + 25y^3\right)^2 - 1024\Delta(f)y$$
$$= \left(y^3 - 7\,p^2y^2 + 11p^4y + \frac{3}{25}\,p^6 + 4000\,pq^2\right)^2 - 2^{10}\Delta(f)y$$

We obtained the results given in the text.

(b) To find the rational root of f we write

```
theta_f.subs(Delta = Delta).factor()
```

$$\frac{1}{3125}\,(5\,p^2 - y)\,(9\,p^{10} - 1625\,p^8y + 74250\,p^6y^2 + 600000\,p^5q^2 - 81250\,p^4y^3$$
$$+ 50000000\,p^3q^2y + 28125\,p^2y^4 - 25000000\,pq^2y^2 - 3125\,y^5 + 10000000000\,q^4)$$

Thus

$$\theta_f(5p^2) = 0.$$

By Corollary 13.2.11, f is solvable by radicals over $\mathbb{Q}$.

(c) The substitution $x = z - \frac{p}{5z}$ is obtained by

```
z = var('z')
g = f.subs(x = z - p/(5*z))
g.expand()
```

$$z^5 + q - \frac{p^5}{3125\, z^5}$$

Thus

$$g(z) = f\left(z - \frac{p}{5z}\right) = z^5 - \frac{p^5}{5^5\, z^5} + q.$$

(d) Let $\beta \in \mathbb{C}$.

$$\begin{aligned} g(\beta) = 0 &\iff \beta^{10} + q\,\beta^5 - \left(\frac{p}{5}\right)^5 = 0 \\ &\iff \left(\beta^5 + \frac{q}{2}\right)^2 - \left[\left(\frac{q}{2}\right)^2 + \left(\frac{p}{5}\right)^5\right] = 0 \\ &\iff \left[\beta^5 + \frac{q}{2} - \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{5}\right)^5}\right]\left[\beta^5 + \frac{q}{2} - \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{5}\right)^5}\right] \end{aligned}$$

So the 10 roots of g are

$$\beta_{k,\varepsilon} = \zeta^k \sqrt[5]{-\frac{q}{2} + \varepsilon\sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{5}\right)^5}}, \qquad \varepsilon = \pm 1,\ k = 0, 1, 2, 3, 4.$$

where $\zeta = \zeta_5 = e^{2i\pi/5}$.

Let α be a root of f in $\mathbb{C}$. There exists $\beta \in \mathbb{C}$ such that $\alpha = \beta - \frac{p}{5\beta}$, so

$$0 = f(\alpha) = f\left(\beta - \frac{p}{5\beta}\right) = g(\beta).$$

Since $g(\beta) = 0$, $\beta = \beta_{k,\varepsilon}$ for some $k \in \{0, \ldots, 4\}, \varepsilon \in \{-1, 1\}$. If L is the splitting field of f in $\mathbb{C}$, then

$$L \subset \mathbb{Q}(\beta_{0,1}, \ldots, \beta_{4,1}, \beta_{0,-1}, \ldots, \beta_{4,-1}).$$

Write $\delta = \left(\frac{q}{2}\right)^2 + \left(\frac{p}{5}\right)^5 \in \mathbb{Q}$. Since $\beta_{k,\varepsilon} \in \mathbb{Q}\left(\sqrt{\delta}, \zeta, \sqrt[5]{-\frac{q}{2} + \varepsilon\sqrt{\delta}}\right)$,

$$L \subset \mathbb{Q}\left(\zeta_5, \sqrt{\delta}, \sqrt[5]{-\frac{q}{2} + \sqrt{\delta}}, \sqrt[5]{-\frac{q}{2} - \sqrt{\delta}}\right),$$

where $\delta, q \in \mathbb{Q}$, so L is included in some radical extension of $\mathbb{Q}$.

Therefore f is solvable by radicals over $\mathbb{Q}$.

Note: We can choose $\sqrt[5]{-\frac{q}{2} - \sqrt{\delta}}$ so that $\sqrt[5]{-\frac{q}{2} + \sqrt{\delta}}\sqrt[5]{-\frac{q}{2} - \sqrt{\delta}} = -\frac{p}{5} \in \mathbb{Q}$. Therefore

$$L \subset \mathbb{Q}\left(\zeta_5, \sqrt{\delta}, \sqrt[5]{-\frac{q}{2} + \sqrt{\delta}}\right),$$

where the chain of inclusions

$$\mathbb{Q} \subset \mathbb{Q}(\zeta_5) \subset \mathbb{Q}\left(\zeta_5, \sqrt{\delta}\right) \subset \mathbb{Q}\left(\zeta_5, \sqrt{\delta}, \sqrt[5]{-\frac{q}{2} + \sqrt{\delta}}\right)$$

proves that this last field is a radical extension.

□

Ex. 13.2.15 *As in Theorem 13.2.12, let $f = x^5 + ax + b$. Compute $\Delta(f)$ and $\theta_f(y)$.*

Proof. With the same Sage procedure as in Exercise 14, we obtain:

```
S.<a,b,x> = QQ[]
f = x^5 + a*x + b
Delta = f.discriminant(x);Delta.factor()
```

$$\Delta(f) = 256a^5 + 3125b^4,$$

```
K.<a,b> = QQ[]
S.<x> =PolynomialRing(K, order = 'degrevlex')
f = x^5 + a*x + b
theta_f=resolvent(f)[0];theta_f.subs(Delta = f.discriminant())
```

$$\theta_f(y) = \left(y^3 - 20\,ay^2 + 240\,a^2y + 320\,a^3\right)^2 - 2^{10}(256\,a^5 + 3125\,b^4)y.$$

□

Ex. 13.2.16 *Let $f = x^5 + ax + b \in F[x]$, where f is separable and irreducible and F has characteristic 5. The goal of this exercise is to prove the observation of [28] that the Galois group of f over F is solvable.*

(a) Prove that $a \neq 0$.

(b) Use Exercise 5 from Section 6.2 to show that the Galois group of f over F is cyclic when $a = -1$.

(c) Show that there is a Galois extension $F \subset L$ with solvable Galois group such that f is equivalent (as defined in the Mathematical Notes) to a polynomial of the form $x^5 - x + b'$ for some $b' \in L$.

(d) Conclude that the Galois group of f over F is solvable.

(e) Show that there is a field F of characteristic 5 and a monic, separable, irreducible quintic $g \in F[x]$ that cannot be transformed to one in Bring-Jerrard form defined over any Galois extension $F \subset L$ with solvable Galois group.

In [28] Ruppert explores the geometric reasons why things go wrong in characteristic 5.

Proof. (a) If $a = 0$, then $f = x^5 + b$, so $f = x^5 - \alpha^5$, where α is a root of f in some extension of F. Since the characteristic of F is 5, $f = x^5 - \alpha^5 = (x - \alpha)^5$ is not separable, in contradiction with the hypothesis, so $a \neq 0$.

(b) If $a = -1$, by Exercise 5.3.16 and 6.2.5, we know that

$$x^5 - x + b = (x - \alpha)(x - \alpha - 1)(x - \alpha - 2)(x - \alpha - 3)(x - \alpha - 4),$$

where α is a root of f in some extension. Then $K = F(\alpha)$ is the splitting field of f over F. By part (c) of exercise 6.2.5, we know also that

$$\varphi \begin{cases} \mathrm{Gal}(L/F) & \to & \mathbb{Z}/5\mathbb{Z} \\ \sigma & \mapsto & \sigma(\alpha) - \alpha \end{cases}$$

is a group isomorphism, so $\mathrm{Gal}(L/F)$ is cyclic of order 5.

(c) We search λ such that

$$x^5 - x + b' = \lambda^{-5}((\lambda x)^5 + a(\lambda x) + b)$$

for some b'.

This is equivalent to

$$\lambda^5 x^5 - \lambda^5 x + \lambda^5 b' = \lambda^5 x^5 + a\lambda x + b,$$

so $a\lambda = -\lambda^5, \lambda^4 = -a$. Let L a splitting field of $x^4 + a$, and choose

$$\lambda = \sqrt[4]{-a}$$

a fixed root of $x^4 + a$ in L.

Since L is a splitting field of $x^4 + a \in F[x]$ over F, $F \subset L$ is a Galois extension. The characteristic is 5, so $2^2 = -1$, and

$$x^4 + a = x^4 - \lambda^4 = (x^2 + \lambda^2)(x^2 - \lambda^2) = (x + 2\lambda)(x - 2\lambda)(x - \lambda)(x + \lambda)$$

splits completely over F. Therefore $L = F(\lambda)$ is the splitting field of $x^4 + a$ over F.

So there exists λ in some solvable Galois extension L of F such that $x^5 - x + b' = \lambda^{-5}((\lambda x)^5 + a(\lambda x) + b)$ with $b' = (\sqrt[4]{-a})^{-5}b$, where $\lambda, b \in L$.

(d) If $\beta \in L$, β is a root of f if and only if $\lambda^{-1}\beta$ is a root of $x^5 - x + b'$. If α is a fixed root of $x^5 - x + b'$, then by part (b) the roots of $x^5 - x + b'$ are $\alpha, \alpha+1, \alpha+2, \alpha+3, \alpha+4$, so the roots of f are

$$\beta_0 = \lambda\alpha, \beta_1 = \lambda(\alpha + 1), \beta_2 = \lambda(\alpha + 2), \beta_3 = \lambda(\alpha + 3), \beta_4 = \lambda(\alpha + 4).$$

A splitting field of f over F is

$$K = F(\beta_0, \ldots, \beta_4) = F(\lambda\alpha, \lambda(\alpha + 1), \ldots, \lambda(\alpha + 4)).$$

Since $\lambda = \lambda(\alpha + 1) - \lambda\alpha = \beta_1 - \beta_0 \in K$, and $\alpha = (\lambda\alpha)/\lambda = \beta_0/(\beta_1 - \beta_0) \in K$, $F(\lambda, \alpha) \subset K$, and $\lambda\alpha, \ldots, \lambda(\alpha + 4) \in F(\lambda, \alpha)$, so $K \subset F(\lambda, \alpha)$:

$$K = F(\lambda, \alpha)$$

is the splitting field of $f = x^5 + ax + b$ over F.

Since $F(\lambda) \subset L \subset K$, $K = L(\alpha)$.

Since f is irreducible over F, $5 = \deg(f) = [F(\beta_0) : F] \mid [K : F]$.

$\mathrm{Gal}(L/F)$ is isomorphic to a subgroup of S_4, so $5 \nmid [L : F] = |\mathrm{Gal}(L/F)|$.

Since $[K : F] = [K : L][L : F]$, $5 \mid [K : L] = [L[\alpha] : L]$, where $\alpha^5 - \alpha + b' = 0$. Therefore $x^5 - x + b'$ is irreducible over L and by part (b),

$$\mathrm{Gal}(K/L) \simeq \mathbb{Z}/5\mathbb{Z} \text{ is cyclic.}$$

Since $F \subset L$ is a Galois extension,

$$\mathrm{Gal}(K/F)/\mathrm{Gal}(K/L) \simeq \mathrm{Gal}(L/F).$$

$\mathrm{Gal}(L/F)$ is isomorphic to a subgroup of S_4, so is solvable, and $\mathrm{Gal}(K/L)$ is cyclic, a fortiori solvable. Therefore $\mathrm{Gal}(K/F)$ is solvable:

The Galois group of f over F is solvable.

(e) Let $F = \mathbb{F}_5(\sigma_1, \ldots, \sigma_5) \subset \mathbb{F}_5(x_1, \ldots, x_5)$. The Galois group of $f = x^5 - \sigma_1 x^4 + \sigma_2 x^3 - \sigma_3 x^2 + \sigma_4 x - \sigma_5$ is S_5, and S_5 is not solvable. Therefore f cannot be equivalent to a polynomial $x^5 + ax + b$ whose Galois group over F is solvable.

□

Ex. 13.2.17 *Following Example 13.2.14, consider the equations $x^3 + 3x + 1 = 0$, and $y = a + bx + x^2$.*

(a) Use Maple or Mathematica and Section 2.3 to eliminate x and obtain (13.26).

(b) Show that coefficients of y^2 and y in (13.26) both vanish if and only if $a = 2$ and $b^2 + b - 1 = 0$.

(c) The equation for y becomes trivial to solve when $a = 2$ and $b = (\sqrt{5} - 1)/2$. We could then solve for x using $y = a + bx + x^2$, but there is a better way to proceed. Note that

$$x^3 = -bx^2 - ax + yx$$

follows from $y = a + bx + x^2$. Furthermore, we can use $y = a + bx + x^2$ to eliminate the x^2 in the above equation. Then use $x^3 + 3x + 1 = 0$ to obtain an equation in which x appears only to the first power. Solving this gives a formula for x in terms of y. The general version of this argument can be found in [Lagrange, p.223].

Proof. (a) We eliminate x between the two polynomials

$$f = x^3 + 3x + 1,$$
$$g = x^2 + bx + (a - y),$$

with the resultant $\mathrm{Res}_x(f, g) = \det(S)$, where

$$S = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & b & 1 & 0 \\ 3 & 0 & a-y & b & 1 \\ 1 & 3 & 0 & a-y & b \\ 0 & 1 & 0 & 0 & a-y \end{pmatrix}$$

We can obtain this determinant with Sage:

```
R.<a,b,x,y> = QQ[]
f = x^3 + 3*x + 1
g = x^2 + b*x + (a-y)
S = matrix(R,[[1, 0, 1,   0,   0  ],
              [0, 1, b,   1,   0  ],
              [3, 0, a-y, b,   1  ],
              [1, 3, 0,   a-y, b  ],
              [0, 1, 0,   0,   a-y]])
R = S.det(); R
```

$a^3 + 3ab^2 - b^3 - 3a^2y - 3b^2y + 3ay^2 - y^3 - 6a^2 + 3ab + 12ay - 3by - 6y^2 + 9a - 3b - 9y + 1$

But it is more easy to call the method "resultant" to obtain the same result:

```
res = f.resultant(g,x);res
```

The list of coefficients of $-\mathrm{Res}_x(f,g)$ is given by

```
l =[-res.subs(y=0)] +  [-res.coefficient(y^k) for k in range(1,4)]
l
```

$$\left[-a^3-3ab^2+b^3+6a^2-3ab-9a+3b-1,\quad 3a^2+3b^2-12a+3b+9,\quad -3a+6,\quad 1\right]$$

We find the equation(13.26):

$$y^3+(6-3a)y^2+(9+3b+3b^2-12a+3a^2)y+P(a,b)=0,$$

where $P(a,b)=-a^3-3ab^2+b^3+6a^2-3ab-9a+3b-1$.

(b) The coefficient of y^2 vanishes if $a=2$, and then the coefficient of y vanishes if $0=9+3b+3b^2-12a+3a^2=3b+3b^2-3$:

$$b^2+b-1=0$$

so $b=\frac{\sqrt{5}-1}{2}$ is a solution.

If we pick $b=(\sqrt{5}-1)/2$, then the above cubic equation becomes

$$y^3+\frac{5}{2}\sqrt{5}-\frac{25}{2}=0,$$

so

$$\begin{aligned} y^3&=\frac{25-5\sqrt{5}}{2}\\ &=\sqrt{5}^3\,\frac{\sqrt{5}-1}{2}. \end{aligned}$$

By the property of the resultant, if y is evaluated to $y_0=\omega^k\sqrt{5}\sqrt[3]{\frac{\sqrt{5}-1}{2}}$, $k=0,1,2$, $\omega=e^{2i\pi/3}$, then there exists a common root of f and g in $\mathbb{C}$, where

$$\begin{aligned} f&=x^3+3x+1\\ g&=x^2+\frac{\sqrt{5}-1}{2}x+2-y_0. \end{aligned}$$

(c) The Euclidean division of f by g gives

$$x^3+3x+1=(x^2+bx+a-y)(x-b)+(y+b^2+3-a)x+1+ab-by.$$

If x_0 is a common root of f,g, then the remainder is 0, so

$$x_0=\frac{by_0-ab-1}{y_0+b^2+3-a},$$

and this gives a formula for the roots x_0 of f in terms of the roots y_0 of the resultant.

Since y_0 is an algebraic number of degree 3 over $\mathbb{Q}$, and $x_0\in\mathbb{Q}(\sqrt{5},y_0)$, there exists some polynomial p of degree 2 such that $x_0=p(y_0)$.

To find this more simple formula for x_0, we search the gcd of f,g in the field $\mathbb{Q}\left(\sqrt{5},\sqrt[3]{\frac{\sqrt{5}-1}{2}}\right)$ by the extended Euclid's algorithm.

This is obtained with the following Sage instructions:

```
K.<u>= QQ[sqrt(5)]
R.<z> = K[]
res = z^3 - (u-1)/2
L.<w> = K.extension(res)
A.<x> = L[]
f = x^3 + 3*x + 1
g = x^2 + (u - 1)/2 * x + (2 - u*w)
gcd(f,g)
```

$$x + \left(\frac{1}{2}\sqrt{5} + \frac{1}{2}\right) w^2 - w$$

We have obtained that

$$\gcd(f, g) = x + \frac{\sqrt{5}+1}{2}\, w^2 - w, \text{ where } w^3 = \frac{\sqrt{5}-1}{2}.$$

Since $w^3 = \frac{\sqrt{5}-1}{2}$, so $w^2 = \frac{\sqrt{5}-1}{2} w^{-1}$, $\frac{\sqrt{5}+1}{2} w^2 = w^{-1}$.

Therefore the roots of f are

$$\begin{aligned} x_0 &= w - \frac{\sqrt{5}+1}{2} w^2 \\ &= w - w^{-1} \end{aligned}$$

We can write $w = \omega^k \sqrt[3]{\frac{\sqrt{5}-1}{2}}, k = 0, 1, 2$, then $w^{-1} = \omega^{2k} \sqrt[3]{\frac{\sqrt{5}+1}{2}}$, where the cubic roots are chosen so that their product is real, equal to 1. Then

$$\omega^k \sqrt[3]{\frac{\sqrt{5}-1}{2}} - \omega^{2k} \sqrt[3]{\frac{\sqrt{5}+1}{2}}, \qquad k = 0, 1, 2$$

are the roots of f. This is identical to the formulas obtained with Cardano's formulas, with more sweat.

□

Ex. 13.2.18 *This exercise is concerned with the polynomials (13.28). As in the Historical Notes, we will assume that they lie in $\mathbb{Q}[x]$ and are irreducible.*

(a) Show that $\sqrt[5]{Q^2/P} + (P/Q)\sqrt[5]{Q^2/P}^2$ is a root of $x^5 - 5Px^2 - 5Qx - Q^2/P - P^3/Q$.

(b) Prove that the Galois group of $x^5 - 5Px^2 - 5Qx - Q^2/P - P^3/Q$ over $\mathbb{Q}$ is isomorphic to $AGL(1, \mathbb{F}_5)$.

(c) Prove that over $\mathbb{Q}(\sqrt{5})$, the first two polynomials of (13.28) have cyclic Galois group while the third has Galois group isomorphic to $\mathrm{AGL}(1, \mathbb{F}_5) \cap A_5$.

Proof. (a) As in Cardano's method, we substitute $u + v$ to x in

$$f(x) = x^5 - 5Px^2 - 5Qx - \frac{Q^2}{P} - \frac{P^3}{Q}.$$

We obtain

$$\begin{aligned} f(u+v) &= u^5 + 5\,u^4v + 10\,u^3v^2 + 10\,u^2v^3 + 5\,uv^4 + v^5 \\ &\quad - 5\,Pu^2 - 10\,Puv - 5\,Pv^2 - 5\,Qu - 5\,Qv - \frac{Q^2}{P} - \frac{P^3}{Q} \\ &= \left(u^5 + v^5 - \frac{Q^2}{P} - \frac{P^3}{Q}\right) + 5\left(u^3v + u^2v^2 + uv^3 - Pu - Pv - Q\right)(u+v). \end{aligned}$$

Then we verify that $u = \sqrt[5]{Q^2/P} = P^{-\frac{1}{5}}Q^{\frac{2}{5}}, v = (P/Q)\sqrt[5]{Q^2/P}^2 = P^{\frac{3}{5}}Q^{-\frac{1}{5}}$ is a solution of the system

$$\begin{aligned} 0 &= u^5 + v^5 - \frac{Q^2}{P} - \frac{P^3}{Q}, \\ 0 &= u^3v + u^2v^2 + uv^3 - Pu - Pv - Q. \end{aligned}$$

Indeed

$$u^5 + v^5 - \frac{Q^2}{P} - \frac{P^3}{Q} = \frac{Q^2}{P} + \frac{P^3}{Q} - \frac{Q^2}{P} - \frac{P^3}{Q} = 0,$$

and

$$\begin{aligned} &u^3v + u^2v^2 + uv^3 - Pu - Pv - Q \\ &= P^{-\frac{3}{5}}Q^{\frac{6}{5}}P^{\frac{3}{5}}Q^{-\frac{1}{5}} + P^{-\frac{2}{5}}Q^{\frac{4}{5}}P^{\frac{6}{5}}Q^{-\frac{2}{5}} + P^{-\frac{1}{5}}Q^{\frac{2}{5}}P^{\frac{9}{5}}Q^{-\frac{3}{5}} \\ &\qquad - P^{\frac{5}{5}}P^{-\frac{1}{5}}Q^{\frac{2}{5}} - P^{\frac{5}{5}}P^{\frac{3}{5}}Q^{-\frac{1}{5}} - Q \\ &= Q + P^{\frac{4}{5}}Q^{\frac{2}{5}} + P^{\frac{8}{5}}Q^{-\frac{1}{5}} - P^{\frac{4}{5}}Q^{\frac{2}{5}} - P^{\frac{8}{5}}Q^{-\frac{1}{5}} - Q \\ &= 0. \end{aligned}$$

Therefore

$$u + v = \sqrt[5]{Q^2/P} + (P/Q)\sqrt[5]{Q^2/P}^2$$

is a root of $x^5 - 5Px^2 - 5Qx - Q^2/P - P^3/Q$. If we replace $\sqrt[5]{Q}$ by another fifth root $\zeta^k\sqrt[5]{Q},\ k = 1, 2, 3, 4$, (where $\zeta = e^{2i\pi/5}$), in the equation

$$0 = u^3v + u^2v^2 + uv^3 - Pu - Pv - Q, \qquad \text{where } u = \sqrt[5]{P}^{-1}\sqrt[5]{Q}^2, v = \sqrt[5]{P}^3\sqrt[5]{Q}^{-1},$$

then u is replaced by $\zeta^{2k}u$, and v is replace by $\zeta^{4k}v$, therefore $\zeta^{2k}u, \zeta^{4k}v$ is a solution of the preceding system, so $\zeta^{2k}u + \zeta^{4k}v$ is also a root of f. So

$$u + v, \quad \zeta^2u + \zeta^4v, \quad \zeta^4u + \zeta^3v, \quad \zeta u + \zeta^2v, \quad \zeta^3u + \zeta v,$$

are roots of f, where

$$u = \sqrt[5]{Q^2/P}, \qquad v = (P/Q)\sqrt[5]{Q^2/P}^2.$$

These roots are the five roots of f, as proved in the following expansion:

$$\begin{aligned} &(x - (u+v))(x - (\zeta^2u + \zeta^4v))(x - (\zeta^4u + \zeta^3v))(x - (\zeta u + \zeta^2v))(x - (\zeta^3u + \zeta v)) \\ &= x^5 - 5uv^2x^2 - 5u^3vx - u^5 - v^5. \end{aligned}$$

Since

$$P = uv^2, \qquad Q = u^3 v,$$

we obtain

$$u^5 = \frac{Q^2}{P}, \qquad v^5 = \frac{P^3}{Q},$$

so

$$\begin{aligned} f &= x^5 - 5Px^2 - 5Qx - \frac{Q^2}{P} - \frac{P^3}{Q} \\ &= (x - (u+v))(x - (\zeta^2 u + \zeta^4 v))(x - (\zeta^4 u + \zeta^3 v))(x - (\zeta u + \zeta^2 v))(x - (\zeta^3 u + \zeta v)) \end{aligned}$$

Therefore the roots of f are $\zeta^{2k}u + \zeta^{4k}v,\ k = 0, 1, 2, 3, 4$.

This was perhaps the Euler's starting point.

(b) We obtain the discriminant of f with

```
R.<P,Q,u,v,e,x> = QQ[]
f = x^5 - 5*P*x^2 - 5*Q*x -e;
Delta = f.discriminant(x).subs(e = Q^2/P + P^3/Q).factor()
Delta
```

$$(3125) \cdot Q^{-4} \cdot P^{-4} \cdot (-P^8 - 11P^4Q^3 + Q^6)^2$$

so

$$\Delta = 5^5 \frac{(P^8 + 11P^4Q^3 - Q^6)^2}{P^4Q^4}.$$

Thus Δ is not a square in $\mathbb{Q}$.

Using the resolvent() function of Exercise 14, we obtain the sextic resolvent:

```
K.<P,Q,e> = QQ[]
S.<x> =PolynomialRing(K, order = 'degrevlex')
f = x^5 - 5*P*x^2 - 5*Q*x -e
theta = resolvent(f)[0]; theta.subs(e = Q^2/P + P^3/Q)
```

$$\begin{aligned} \theta_f(y) &= \left(100\,Qy^2 + y^3 + 2000\left(3\,Q^2 - \left(P^4 + Q^3\right)/Q\right)y\right)^2 - 1024\,\Delta(f)y \\ &= \left(y^3 + 100\,Qy^2 - 2000\left(\frac{P^4}{Q} - 2\,Q^2\right)y\right)^2 - 1024\,\Delta(f)y, \end{aligned}$$

which has root $0 \in \mathbb{Q}$ ($b_6 = 0$). By Section 13.2, the Galois group of f is $\mathrm{AGL}(1, \mathbb{F}_5)$ up to conjugacy.

(c) By Exercise 13, we know that the Galois group of $x^5 - 2$ over $\mathbb{Q}(\sqrt{5})$ is $\mathrm{AGL}(1, \mathbb{F}_5) \cap A_5$ which is not cyclic, so there is a misprint in the sentence.

- The polynomial $f = x^5 - D \in \mathbb{Q}(\sqrt{5})[x], D \in \mathbb{Q}$, is irreducible over $\mathbb{Q}$ by hypothesis. Let α be a root of f. Since $[\mathbb{Q}(\alpha) : \mathbb{Q}] = 5$ and $[\mathbb{Q}(\sqrt{5}) : \mathbb{Q}] = 2$, we obtain that $10 = 2 \times 5$ divides $[\mathbb{Q}(\sqrt{5}, \alpha) : \mathbb{Q}]$, where $[\mathbb{Q}(\sqrt{5}, \alpha) : \mathbb{Q}] \leq 10$. Therefore

$$10 = [\mathbb{Q}(\sqrt{5}, \alpha) : \mathbb{Q}] = [\mathbb{Q}(\sqrt{5}, \alpha) : \mathbb{Q}(\sqrt{5})][\mathbb{Q}(\sqrt{5}) : \mathbb{Q}],$$

so $[\mathbb{Q}(\sqrt{5},\alpha):\mathbb{Q}(\sqrt{5})]=5$. If p is the minimal polynomial of α over $\mathbb{Q}(\sqrt{5})$, then $\deg(p)=[\mathbb{Q}(\sqrt{5},\alpha):\mathbb{Q}(\sqrt{5})]=5$, and p divides f, therefore $p=f$, and we have proved that f is irreducible over $\mathbb{Q}(\sqrt{5})$.
Moreover

$$\Delta(f)=5^5D^4=(5^2D^2\sqrt{5})^2$$

is a square in $\mathbb{Q}(\sqrt{5})$, and $0\in\mathbb{Q}(\sqrt{5})$ is a root of the resolvent

$$\Theta_f(y)=y^6-\Delta(f)y.$$

If $\alpha=\sqrt[5]{D}\in\mathbb{R}$, and $\zeta=\zeta_5$, then

$$x^5-D=(x-\alpha)(x-\zeta\alpha)(x-\zeta^2\alpha)(x-\zeta^3\alpha)(x-\zeta^4\alpha).$$

But $\zeta\alpha\notin\mathbb{R}$, so $\zeta\alpha\notin\mathbb{Q}(\sqrt{5})(\alpha)\subset\mathbb{R}$. f doesn't split completely over $F(\alpha)$, where α is a root of f. By Theorem 13.2.6 and the table 13.2.C,

$$\mathrm{Gal}(L/\mathbb{Q}(\sqrt{5}))\simeq\mathrm{AGL}(1,\mathbb{F}_5)\cap A_5,$$

where L is the splitting field of x^5-D.

- Now $f=x^5-5Px^2-5Qx-\frac{Q^2}{P}-\frac{P^3}{Q}$, and f is assumed irreducible over $\mathbb{Q}$.
With the same proof as in the first bullet, f remains irreducible over $\mathbb{Q}(\sqrt{5})$.
By part (b),

$$\Delta=5^5\frac{(P^8+11P^4Q^3-Q^6)^2}{P^4Q^4}$$

is a square in $\mathbb{Q}(\sqrt{5})$, and

$$\theta_f(y)=\left(y^3+100\,Qy^2-2000\left(\frac{P^4}{Q}-2\,Q^2\right)y\right)^2-1024\,\Delta(f)y,$$

has root $0\in\mathbb{Q}(\sqrt{5})$. By part (a),

$$\begin{aligned}f&=x^5-5Px^2-5Qx-\frac{Q^2}{P}-\frac{P^3}{Q}\\&=(x-(u+v))(x-(\zeta^2u+\zeta^4v))(x-(\zeta^4u+\zeta^3v))(x-(\zeta u+\zeta^2v))(x-(\zeta^3u+\zeta v))\end{aligned}$$

We prove that $\alpha_2=\zeta^2u+\zeta^4v$ is not real. If $\alpha_2\in\mathbb{R}$, then $\alpha_2=\overline{\alpha_2}$, so $(\zeta^2-\zeta^3)u+(\zeta^4-\zeta)v=0$.
Then, using $\sqrt{5}=\zeta-\zeta^2-\zeta^3+\zeta^4$ and $\zeta^2+\zeta^3=\frac{-1-\sqrt{5}}{2}$,

$$\begin{aligned}\frac{v}{u}&=\frac{\zeta^2-\zeta^3}{\zeta-\zeta^4}=\frac{\zeta-\zeta^2}{1-\zeta^3}\\&=\frac{(\zeta-\zeta^2)(1-\zeta^2)}{(1-\zeta^3)(1-\zeta^2)}\\&=\frac{\zeta-\zeta^2-\zeta^3+\zeta^4}{1-(\zeta^2+\zeta^3)+\zeta^5}\\&=\frac{\sqrt{5}}{2+\frac{1+\sqrt{5}}{2}}\\&=\frac{\sqrt{5}-1}{2}\end{aligned}$$

Since $\frac{v}{u} = \frac{P}{Q}\sqrt[5]{\frac{Q^2}{P}}$, we would have $\sqrt[5]{\frac{Q^2}{P}} \in \mathbb{Q}(\sqrt{5})$, and then the root $\alpha_1 = u + v \in \mathbb{Q}(\sqrt{5})$, in contradiction with the irreducibility of f over $\mathbb{Q}(\sqrt{5})$.
So $\alpha_2 \notin \mathbb{R}$ is not in the field $\mathbb{Q}(\sqrt{5})(\alpha_1)$, and f doesn't split completely over $\mathbb{Q}(\sqrt{5})(\alpha_1)$.
By the table 13.2.C, $\mathrm{Gal}(L/\mathbb{Q}(\sqrt{5}) \simeq \mathrm{AGL}(1, \mathbb{F}_5) \cap A_5$.

- The third Euler's polynomial is $f = x^5 - 5Px^3 + 5P^2x - D$.
If $p = -5P, q = -D$, we obtain $f = x^5 + px^3 + \frac{1}{5}p^2x + q$, which is Example 13.2.10. We know from this example and from Exercise 14 that the Galois group of f over $\mathbb{Q}$ is $\mathrm{AGL}(1, \mathbb{F}_5)$.
With the same proof as in the first bullet, f remains irreducible over $\mathbb{Q}(\sqrt{5})$.
By Exercise 14,
$$\Delta(f) = \frac{1}{5^5} \cdot (4p^5 + 3125q^2)^2$$
is a square in the field $\mathbb{Q}(\sqrt{5})$, and
$$\theta_f(y) = \left(y^3 - 7\,p^2y^2 + 11p^4y + \frac{3}{25}\,p^6 + 4000\,pq^2\right)^2 - 2^{10}\Delta(f)y$$
has the root $5p^2 \in \mathbb{Q}$.
It remains to know if f splits completely over $\mathbb{Q}(\sqrt{5})$.
Note that
$$f(u+v) = u^5 + v^5 + q + (u^2 + uv + v^2 + \frac{p}{5})(5uv + p)(u + v).$$
Therefore if $uv = -\frac{p}{5}$, $f(u+v) = u^5 + v^5 + q$, so we find (see Exercise 14) that
$$f\left(z - \frac{p}{5z}\right) = z^5 - \frac{p^5}{5^5z^5} + q.$$
So $u + v$ is a root of f if
$$\begin{cases} u^5 + v^5 &= -q = D \\ uv &= -\frac{p}{5} = P \end{cases}$$
Therefore u^5, v^5 are the roots of $x^2 + qx - (\frac{p}{5})^5$ and satisfy $uv = -p/5 \in \mathbb{Q}$.
If $u + v$ is a root, so is $\zeta^k u + \zeta^{-k}v,\ k \in \mathbb{Z}$ (where $\zeta = \zeta_5 = e^{2i\pi/5}$).
Conversely
$$\begin{aligned}&(x - u - v)(x - \zeta u - \zeta^{-1}v)(x - \zeta^2 u - \zeta^{-2}v)(x - \zeta^3 u - \zeta^{-3}v)(x - \zeta^4 u - \zeta^{-4}v)\\ &= x^5 - 5uvx^3 + 5u^2v^2x - u^5 - v^5\\ &= x^5 - 5Px^3 + 5P^2x - D\end{aligned}$$
So the roots of $f = x^5 - 5Px^3 + 5P^2x - D = x^5 + px^3 + \frac{1}{5}p^2x + q$ are
$$u + v, \quad \zeta u + \zeta^{-1}v, \quad \zeta^2 u + \zeta^{-2}v, \quad \zeta^3 u + \zeta^{-3}v, \quad \zeta^4 u + \zeta^{-4}v,$$
where (u, v) is a solution of the system
$$uv = P = -\frac{p}{5}, \qquad u^5 + v^5 = D = -q,$$

so

$$u = \sqrt[5]{-\frac{q}{2} + \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{5}\right)^5}},$$
$$v = \sqrt[5]{-\frac{q}{2} - \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{5}\right)^5}},$$

(where we choose the real roots, so $u, v \in \mathbb{R}$).
We obtained the factorization of f:

$$\begin{aligned} f &= x^5 - 5Px^3 + 5P^2x - D \\ &= (x-u-v)(x-\zeta u-\zeta^{-1}v)(x-\zeta^2u-\zeta^{-2}v)(x-\zeta^3u-\zeta^{-3}v)(x-\zeta^4u-\zeta^{-4}v) \end{aligned}$$

If the root $\alpha_2 = \zeta u + \zeta^{-1}v$ is real, then $u = v$, so $\left(\frac{q}{2}\right)^2 + \left(\frac{p}{5}\right)^5 = 0$, and this imply $\Delta(f) = 0$, in contradiction with the assumed separability of f.
Therefore $\alpha_2 \notin \mathbb{Q}(\sqrt{5})(\alpha_1) \subset \mathbb{R}$, where $\alpha_1 = u + v \in \mathbb{R}$ is a root of f, so f doesn't split completely over $\mathbb{Q}(\sqrt{5})(\alpha_1)$.
$\mathrm{Gal}(L/\mathbb{Q}(\sqrt{5}))$ is isomorphic to $\mathrm{AGL}(1,\mathbb{F}_5) \cap A_5$.

We obtained the same Galois group for the 3 Euler's polynomials of (13.28).

□

Ex. 13.2.19 *Use the methods of this section to compute the Galois group over $\mathbb{Q}$ of each of the following polynomials. Be sure to check that they are irreducible. Remember that in Section 4.2 we learned how to factor polynomials over a finite extension of $\mathbb{Q}$.*

(a) $x^5 + x + 1$.

(b) $x^5 + 20x + 16$.

(c) $x^5 + 2$.

(d) $x^5 - 5x + 12$.

(e) $x^5 + x^4 - 4x^3 - 3x^2 + 3x + 1$.

Proof. We use the Exercise 14 resolvent() procedure to compute a sextic resolvent, and the additional procedure to verify that this resolvent has a rational root (the polynomials are supposed, as in this exercise, monic with integer coefficients, i.e., $f \in \mathbb{Z}[x]$, hence resolvent rational root is an integer):

```
def rational_root(theta):
    n = Integer(theta.subs(y=0))
    if n == 0:
        return True, 0
    for d in n.divisors():
        if theta.subs(y = d) == 0:
            return True, d
        if theta.subs(y = -d)== 0:
            return True, -d
    return False, None
```

(a)
```
R.<x> = QQ[]
f = x^5 + x + 1;f
```

$$x^5 + x + 1$$

```
f.is_irreducible()
```

False

```
f.factor()
```

$$f = (x^2 + x + 1) \cdot (x^3 - x^2 + 1)$$

The roots of $x^2 + x + 1$ are ω, ω^2. Write x_1, x_2, x_3 the roots of $x^3 - x^2 + 1$. Then $L = \mathbb{Q}(\omega, x_1, x_2, x_3)$ is the splitting field of f over $\mathbb{Q}$. As $L = \mathbb{Q}(\omega)(x_1, x_2, x_3)$, L is also the splitting field of $g = x^3 - x^2 + 1$ over $\mathbb{Q}(\omega)$. The discriminant of g is $\Delta(g) = -23$. We show that $-23 \notin \mathbb{Q}(\omega)^2$.

If $-23 = (a + b\omega)^2,\ a, b \in \mathbb{Q}$, then $-23 = a^2 - b^2 - \omega b(b - 2a)$. Therefore

$$\begin{cases} -23 &= a^2 - b^2, \\ 0 &= b(b - 2a). \end{cases}$$

Thus $b = 0$ or $b = 2a$. If $b = 0$, then $-23 = a^2$, which is impossible since $a \in \mathbb{Q}$, and $b = 2a$ gives $a^2 = 23/3$, which is also impossible. Therefore $\Delta(g)$ is not a square in $\mathbb{Q}(\omega)$, so the Galois group G_1 of g over $\mathbb{Q}(\omega)$ is S_3. This implies that $[L : \mathbb{Q}(\omega)] = |G_1| = 6$. Since $[\mathbb{Q}(\omega) : \mathbb{Q}] = 2$, $[L : \mathbb{Q}] = 12$, so the Galois group G of $f = x^5 + x + 1$ has order 12:

$$|G| = 12.$$

Since -23 is not a square of $\mathbb{Q}$, $\mathrm{Gal}(\mathbb{Q}(x_1, x_2, x_3)/\mathbb{Q}) \simeq S_3$.

Let

$$\varphi \begin{cases} \mathrm{Gal}(L/\mathbb{Q}) & \to & \mathrm{Gal}(\mathbb{Q}(x_1, x_2, x_3)/\mathbb{Q}) \times \mathrm{Gal}(\mathbb{Q}(\omega)/\mathbb{Q}) \\ \sigma & \mapsto & \left(\sigma|_{\mathbb{Q}(x_1, x_2, x_3)},\ \sigma|_{\mathbb{Q}(\omega)} \right). \end{cases}$$

Then φ is a group homomorphism, and the kernel of φ is $\{\mathrm{id}\}$, since every $\mathbb{Q}$-automorphism of L which fixes ω, x_1, x_2, x_3 is the identity of L. So φ is injective, and $|\mathrm{Gal}(L/\mathbb{Q})| = |\mathrm{Gal}(\mathbb{Q}(\omega)/\mathbb{Q}) \times \mathrm{Gal}(\mathbb{Q}(x_1, x_2, x_3)/\mathbb{Q})| = 12$, thefore φ is a group isomorphism.

$$\mathrm{Gal}(L/\mathbb{Q}) \simeq \mathrm{C}_2 \times S_3.$$

If we choose the numbering $x_1, x_2, x_3, x_4 = \omega, x_5 = \omega^2$ of the roots of f, then the Galois group G of f is

$$G = \mathrm{Gal}_{\mathbb{Q}}(f) = \langle (1\,2), (1\,2\,3), (4\,5) \rangle \simeq S_3 \times C_2.$$

(b)
```
R.<x> = QQ[]
f = x^5 + 20 *x + 16;f
```

$$x^5 + 20x + 16$$

```
f.is_irreducible()
```

True

```
theta = resolvent(f)[0]; theta.subs(Delta = f.discriminant()).expand()
```

$y^6-800\,y^5+352000\,y^4-71680000\,y^3+7168000000\,y^2-557056000000\,y+6553600000000$

```
res = rational_root(theta); res
```

(False, None)

```
f.discriminant().factor(),f.discriminant().is_square()
```

$(2^{16} \cdot 5^6, \text{True})$

Thus the Galois group of $f = x^5 + 20x + 16$ over $\mathbb{Q}$ is A_5.
Verification:

```
f.galois_group().gens()
```

$\langle (3,4,5), (1,2,3,4,5) \rangle$

(c)
```
R.<x> = QQ[]
f = x^5 + 2;f
```

$x^5 + 2$

```
f.is_irreducible()
```

True

```
theta = resolvent(f)[0]; theta.subs(Delta = f.discriminant()).expand()
```

$y^6 - 51200000\,y$

```
res = rational_root(theta); res
```

(True, 0)

```
f.discriminant().factor(),f.discriminant().is_square()
```

$(2^4 \cdot 5^5, \text{False})$

Thus the Galois group of $f = x^5 + 2$ over $\mathbb{Q}$ is $\mathrm{AGL}(1, \mathbb{F}_5)$, up to conjugacy.
Verification:

```
f.galois_group().gens()
```

$$\langle (1,2,3,4,5), (1,2,4,3) \rangle$$

(d)
```
R.<x> = QQ[]
f = x^5 -5*x + 12;f
```

$$x^5 - 5x + 12$$

```
f.is_irreducible()
```

True

```
theta = resolvent(f)[0]; theta.subs(Delta = f.discriminant()).expand()
```

$$y^6 + 200\,y^5 + 22000\,y^4 + 1120000\,y^3 + 28000000\,y^2 - 66016000000\,y + 1600000000$$

```
res = rational_root(theta); res
```

(True, 100)

```
f.discriminant().factor(),f.discriminant().is_square()
```

$$(2^{12} \cdot 5^6, \text{True})$$

```
K.<alpha> = NumberField(f)
S.<X> = K[]
g = f.change_ring(S)
g.factor()
```

$$(x-\alpha)\cdot\left(x^2+\left(\frac{1}{4}\alpha^4+\frac{1}{4}\alpha^3+\frac{1}{4}\alpha^2+\frac{1}{4}\alpha-1\right)x-\frac{1}{2}\alpha^3-\frac{1}{2}\alpha-1\right)$$
$$\cdot\left(x^2+\left(-\frac{1}{4}\alpha^4-\frac{1}{4}\alpha^3-\frac{1}{4}\alpha^2+\frac{3}{4}\alpha+1\right)x-\frac{1}{4}\alpha^4-\frac{1}{4}\alpha^3-\frac{1}{4}\alpha^2-\frac{5}{4}\alpha+2\right)$$

Thus the Galois group of $f = x^5 + 20x + 16$ over $\mathbb{Q}$ is $\mathrm{AGL}(1, \mathbb{F}_5) \cap A_5$, up to conjugacy.

Verification:

```
f.galois_group().gens()
```

$$\langle (1,2,3,4,5), (1,4)(2,3) \rangle$$

(e)
```
R.<x> = QQ[]
f = x^5 + x^4 - 4*x^3 - 3*x^2 + 3*x + 1
f
```

$$x^5 + x^4 - 4x^3 - 3x^2 + 3x + 1$$

```
f.is_irreducible()
```

True

```
theta = resolvent(f)[0]; theta.subs(Delta = f.discriminant()).expand()
```

$$y^6 - 264\,y^5 + 25168\,y^4 - 1022208\,y^3 + 14992384\,y^2 - 14992384\,y$$

```
res = rational_root(theta); res
```

(True, 0)

```
f.discriminant().factor(),f.discriminant().is_square()
```

$(11^4, \text{True})$

```
K.<alpha> = NumberField(f)
S.<X> = K[]
g = f.change_ring(S)
g.factor()
```

$$(x - \alpha) \cdot (x - \alpha^2 + 2) \cdot (x + \alpha^4 + \alpha^3 - 3\alpha^2 - 2\alpha + 1) \cdot (x - \alpha^3 + 3\alpha) \cdot (x - \alpha^4 + 4\alpha^2 - 2)$$

Thus the Galois group of $f = x^5 + 20x + 16$ over $\mathbb{Q}$ is $\langle (1\,2\,3\,4\,5) \rangle$, up to conjugacy. Verification:

```
f.galois_group().gens()
```

$$\langle (1, 2, 3, 4, 5) \rangle$$

□

Ex. 13.2.20 *In the Mathematical Notes to Section 10.3, we noted that the roots of the polynomial $x^5 - 4x^4 + 2x^3 + 4x^2 + 2x - 6 \in \mathbb{Q}[x]$ can be constructed using a marked ruler and compass. Show that this polynomial is not solvable by radicals over $\mathbb{Q}$.*

Proof. With the same procedures as in Exercise 19, we obtain

```
R.<x> = QQ[]
f = x^5 - 4*x^4 + 2*x^3 + 4 *x^2 + 2*x -6;f
```

$$x^5 - 4x^4 + 2x^3 + 4x^2 + 2x - 6$$

```
f.is_irreducible()
```

True

```
theta = resolvent(f)[0]; theta.subs(Delta = f.discriminant()).expand()
```

$$y^6 - 360\,y^5 + 47856\,y^4 - 3025152\,y^3 + 103474944\,y^2 - 1812875264\,y + 14770999296$$

```
res = rational_root(theta); res
```

(False, None)

```
f.discriminant().factor(),f.discriminant().is_square()
```

$$(-1 \cdot 2^4 \cdot 4003, \text{False})$$

So the Galois group of f is S_5, and f is not solvable by radicals over $\mathbb{Q}$.

Verification:

```
f.galois_group().gens()
```

$$\langle (1,2), (1,2,3,4,5) \rangle$$

□

13.3 Resolvents

Ex. 13.3.1 *Let $f(x) \in \mathbb{Q}[x]$.*

(a) Prove that there are $\lambda, \mu \in \mathbb{Q}^*$ such that $g(x) = \lambda f(\mu x) \in \mathbb{Z}[x]$ is monic.

(b) Prove that f and g have isomorphic Galois groups over $\mathbb{Q}$.

Proof.

(a) Let $f(x) = \frac{a_0}{b_0}x^n + \frac{a_1}{b_1}x^{n-1} + ... + \frac{a_n}{b_n} = \sum_{i=0}^n \frac{a_i}{b_i}x^{n-i}$, where $a_i, b_i \in \mathbb{Z}$, and $\nu = lcm(b_0, b_1, ..., b_n)$, then $f(x) = 1/\nu \sum_{i=0}^n \frac{a_i}{b_i}\nu x^{n-i} = \sum_{i=0}^n c_i x^{n-i}$, where $c_i \in \mathbb{Z}$.

After multiplication by c_0^{n-1} we have

$$c_0^{n-1}f(x) = \sum_{i=0}^n c_i c_0^{n-1}x^{n-i} = (c_0x)^n + \sum_{i=1}^n c_i c_0^{i-1}(c_0x)^{n-i} = g(c_0x),\ c_i c_0^{i-1} \in \mathbb{Z}.$$

Hence $g(x)$ is monic and $g(x) = \lambda f(\mu x) \in \mathbb{Z}[x]$, where $\lambda = c_0^{n-1}, \mu = 1/c_0, c_0 = \frac{a_0}{b_0}\nu$, i.e. $\lambda, \nu \in \mathbb{Q}^*$.

(b) Let $\alpha_i, i = 1, ..., n$ are the roots of $f(x)$, then $\beta_i = \alpha_i/\mu, i = 1, ..., n$ are the roots of $g(x)$. If σ, σ' are the elements of Galois groups of f and g over $\mathbb{Q}$ such that $\sigma(\alpha_i) = \alpha_j = \mu\beta_j = \mu\sigma'(\beta_i)$, then the bijection

$$\sigma' = \frac{1}{\mu}\sigma \iff \sigma(\alpha_i) = \alpha_j, \sigma'(\beta_i) = \beta_j,\ i, j = 1, ..., n$$

is an isomorphism of these groups. □

Ex. 13.3.2 *Let $f(x) = x^n - c_1x^{n-1} + ... + (-1)^n c_n \in \mathbb{Z}[x]$, and let $\Theta_f(y)$ be the resolvent built from $\varphi \in \mathbb{Z}[x_1, ..., x_n]$. Prove that $\Theta_f(y) \in \mathbb{Z}[y]$.*

Proof. Let G is the symmetry group of φ and $\tau_1, ..., \tau_l$ be representatives for the left cosets of G in S_n. The universal resolvent is $\Theta(y) = \prod_{i=1}^l (y - \tau_i\varphi(x_1, ..., x_n))$. Since $\varphi \in \mathbb{Z}[x_1, ..., x_n]$, for each $i, 1 \le i \le l$, $\tau_i\varphi(x_1, ..., x_n) \in \mathbb{Z}[x_1, ..., x_n]$, hence the coefficients of $\Theta(y)$ are in $\mathbb{Z}[x_1, ..., x_n]$, i.e., $\Theta(y) \in \mathbb{Z}[x_1, ..., x_n][y]$.

Suppose $\sigma \in S_n$, then $\sigma\Theta(y) = \prod_{i=1}^l (y - \sigma\tau_i\varphi(x_1, ..., x_n))$. But the set $\sigma\tau_1, ..., \sigma\tau_l$ is also a set of left coset representatives of G in S_n. Thus the application of σ has merely permuted the roots of $\Theta(y)$ leaving the coefficients fixed. It means that coefficients of $\Theta(y)$ are symmetric and are polynomials in $\sigma_1, ..., \sigma_n$ (cf. Ex.9.1.6), i.e., $\Theta(y) \in \mathbb{Z}[\sigma_1, ..., \sigma_n][y]$. The application of evaluation map $\sigma_i \mapsto c_i$ to $\Theta(y)$ gives $\Theta_f(y) \in \mathbb{Z}[c_1, ..., c_n][y] = \mathbb{Z}[y]$. □

Ex. 13.3.3 *In the proof of proposition 13.3.2, we asserted that*

$$\varphi(\alpha_1, ..., \alpha_n) = \varphi(\alpha_{\tau(1)}, ..., \alpha_{\tau(n)})$$

follows from $\beta = \varphi(\alpha_1, ..., \alpha_n) \in F$ *and* $\tau \in G_f$. *Prove this.*

Proof. Let $\sigma \in \mathrm{Gal}(L/F)$ corresponds to $\tau \in G_f$, and, since F is fixed for σ,

$$\sigma(\beta) = \sigma(\varphi(\alpha_1, ..., \alpha_n)) = \varphi(\alpha_1, ..., \alpha_n),$$

$$\sigma(\varphi(\alpha_1, ..., \alpha_n)) = \varphi(\alpha_{\tau(1)}, ..., \alpha_{\tau(n)}).$$

Therefore, $\varphi(\alpha_1, ..., \alpha_n) = \varphi(\alpha_{\tau(1)}, ..., \alpha_{\tau(n)})$ for $\varphi(\alpha_1, ..., \alpha_n) \in F$ and $\tau \in G_f$. □

Ex. 13.3.4 *As in Examples 13.3.3 and 13.3.4, let* $\varphi = \sqrt{\Delta}(x_1 + x_2 - x_3 - x_4)$.

(a) Show that the symmetry group of φ *is* $G = \langle(1324)\rangle \subset S_4$ *in characteristic* $\neq 2$.

(b) Show that in the universal case, φ *leads to the resolvent*

$$\Theta(y) = \prod_{i=1}^{3}(y^2 - \Delta(4y_i + \sigma_1^2 - 4\sigma_2)),$$

where $y_1 = x_1x_2 + x_3x_4, y_2 = x_1x_3 + x_2x_4, y_3 = x_1x_4 + x_2x_3$ *are the roots of the universal Ferrari resolvent* $\theta(y)$.

(c) Let $\Theta_f(y)$ *be obtained by specializing the resolvent* $\Theta(y)$ *of part (b) to* $f = x^4 + bx^2 + d$. *Show that*

$$\Theta_f(y) = y^2((y^2 + 4b\Delta(f))^2 - 2^6 d\Delta(f)^2).$$

Proof.

(a) Since $G = \langle(1324)\rangle = [(), (1,3,2,4), (1,3,2,4)^2, (1,3,2,4)^3]$, the direct calculation shows that $\varphi = (1,3,2,4)\varphi = (1,3,2,4)^2\varphi = (1,3,2,4)^3\varphi$.

Sage verification:

```
R.<x1,x2,x3,x4> = PolynomialRing(QQ, order = 'degrevlex')

Delta = (x1-x2)*(x1-x3)*(x1-x4)*(x2-x3)*(x2-x4)*(x3-x4)
D=Delta*(x1+x2-x3-x4);
(D==D.subs(x1=x3,x2=x4,x3=x2,x4=x1)
,D==D.subs(x1=x3,x2=x4,x3=x2,x4=x1).subs(x1=x3,x2=x4,x3=x2,x4=x1)
,D==D.subs(x1=x3,x2=x4,x3=x2,x4=x1).subs(x1=x3,x2=x4,x3=x2,x4=x1)
.subs(x1=x3,x2=x4,x3=x2,x4=x1))
```

```
(True, True, True)
```

Let $S_G = \{(), (1,2), (2,3), (1,4), (1,3), (2,4)\}$ is the complete system of coset representatives of G in S_4. Since $\varphi \neq \tau\varphi$ for all $\tau \in S_G, \tau \neq ()$, in characteristic $\neq 2$ only G is the symmetry group of φ. In case of characteristic 2, $\varphi = -(1,2)\varphi = (1,2)\varphi$ and the assertion of (a) is not valid.

Sage verification:

```
S4=SymmetricGroup(4);
H = S4.subgroup([S4("(1,3,2,4)")])
[S4.cosets(H)[i]  for i in range(S4.order()/H.order()) ]
```

$$[(), (1,2)(3,4), (1,3,2,4), (1,4,2,3)], [(3,4), (1,2), (1,4)(2,3), (1,3)(2,4)],$$
$$[(2,3), (1,3,4,2), (1,2,4), (1,4,3)], [(2,3,4), (1,3,2), (1,4), (1,2,4,3)],$$
$$[(2,4,3), (1,4,2), (1,2,3,4), (1,3)], [(2,4), (1,4,3,2), (1,3,4), (1,2,3)]$$

```
(D==D.subs(x1=x2,x2=x1),D==D.subs(x2=x3,x3=x2),
D==D.subs(x1=x4,x4=x1),D==D.subs(x1=x3,x3=x1),
D==D.subs(x2=x4,x4=x2))
```

```
  (False, False, False, False, False)
```

```
D==-D.subs(x1=x2,x2=x1)
```

```
  True
```

(b) By the definition $\Theta(y) = \prod_{\tau\in S_G}(y - \tau\varphi)$.
Since $(1,2)\varphi = -\varphi, (1,3)\varphi = -(2,4)\varphi$ and $(1,4)\varphi = -(2,3)\varphi$, we have

$$\Theta(y) = \prod_{\tau\in S'_G}(y - \tau\varphi)(y + \tau\varphi) = \prod_{\tau\in S'_G}(y^2 - \tau\varphi^2),$$

where $S'_G = \{(), (1,4), (1,3)\}$.

Since $(x_1 + x_2 - x_3 - x_4)^2 = (\sigma_1 - 2(x_3 + x_4))^2 = \sigma_1^2 - 4\sigma_1(x_3 + x_4) + 4(x_3 + x_4)^2 = \sigma_1^2 - 4(x_1 + x_2)(x_3 + x_4) = \sigma_1^2 - 4(\sigma_2 - x_1x_2 - x_3x_4) = 4y_1 + \sigma_1^2 - 4\sigma_2$ and $y_2 = (1,4)y_1, y_3 = (1,3)y_1$, we have

$$\Theta(y) = \prod_{\tau\in S'_G}(y^2 - \tau\Delta(4y_1 + \sigma_1^2 - 4\sigma_2)) = \prod_{i=1}^{3}(y^2 - \Delta(4y_i + \sigma_1^2 - 4\sigma_2))$$

(c) We have $\Sigma_3 y_1 = y_1 + y_2 + y_3 = \Sigma_4 x_1x_2 = \sigma_2$. The Sage calculation shows that:

```
R.<x1,x2,x3,x4,y1,y2,y3,y4> = PolynomialRing(QQ, order = 'degrevlex')
elt = SymmetricFunctions(QQ).e()
e = [elt([i]).expand(4).subs(x0=x1, x1=x2, x2=x3, x3 = x4) for i in range(5)]
J = R.ideal(e[1]-y1, e[2]-y2, e[3]-y3,e[4]-y4)
G = J.groebner_basis()
d1=x1*x2+x3*x4;d2=d1.subs(x2=x3,x3=x2);d3=d1.subs(x1=x3,x3=x1);
S2=d1*d2+d1*d3+d2*d3;S3=d1*d2*d3
var('sigma_1,sigma_2,sigma_3,sigma_4')
S2.reduce(G).subs(y1=sigma_1, y2 = sigma_2,y3=sigma_3,y4=sigma_4)
S3.reduce(G).subs(y1=sigma_1, y2 = sigma_2,y3=sigma_3,y4=sigma_4)
```

$$\Sigma_3 y_1y_2 = \sigma_1\sigma_3 - 4\sigma_4,\ y_1y_2y_3 = \sigma_1^2\sigma_4 + \sigma_3^2 - 4\sigma_2\sigma_4$$

Since $\sigma_1 = \sigma_3 = 0$, $\sigma_2 = b, \sigma_4 = d$ for $f = x^4 + bx^2 + d$,

$$\Sigma_3 y_1 = b,\ \Sigma_3 y_1y_2 = -4d,\ y_1y_2y_3 = -4bd.$$

Based on the formula for $\Theta(y)$ resolvent from (b), we have with $Z = y^2 + 4b\Delta(f)$:

$$\begin{aligned}
\Theta_f(y) &= \prod_{i=1}^{3}(y^2 - \Delta(f)(4y_i - 4b)) \\
&= \prod_{i=1}^{3}(Z - 4\Delta(f)y_i) \\
&= Z^3 - 4\Delta(f)\Sigma_3 y_1 Z^2 + 4^2\Delta(f)^2\Sigma_3 y_1 y_2 Z - 4^3\Delta(f)^3 y_1 y_2 y_3 \\
&= Z^3 - 4\Delta(f)bZ^2 + 4^2\Delta(f)^2(-4d)Z - 4^3\Delta(f)^3(-4bd) \\
&= (Z - 4b\Delta(f))Z^2 - (Z - 4b\Delta(f))4^3 d\Delta(f)^2 \\
&= y^2((y^2 + 4b\Delta(f))^2 - 2^6 d\Delta(f)^2).
\end{aligned}$$

□

Ex. 13.3.5 *This problem will sate and prove a relative version of Proposition 13.3.2. Fix a subgroup $H \subset S_n$ and suppose that $f \in F[x]$ is separable of degree n and that $G_f \subset H$. Now let $G \subset H$ be a subgroup. We want to know whether or not G_f lies in the smaller subgroup G. Let $\varphi \in F[x_1, ..., x_n]$ have G as its symmetry group and let $\varphi_1 = \varphi, \varphi_2, ..., \varphi_l$ be the ordit of H acting on φ. Then set*

$$\Theta^H(y) = \prod_{i=1}^{l}(y - \varphi_i) \in F[x_1, ..., x_n][y].$$

Finally, if $\alpha_1, ..., \alpha_n$ are the roots of f in the splitting field L, let

$$\Theta_f^H(y) = \prod_{i=1}^{l}(y - \varphi_i(\alpha_1, ..., \alpha_n)) \in L[y]$$

be the polynomial obtained by $x_i \mapsto \alpha_i$.

(a) Explain why the degree of $\Theta_f^H(y)$ is the index of G in H.

(b) Prove that $\Theta_f^H(y) \in F[y]$.

(c) Assume that G_f is conjugate within H to a subgroup of G (this means that $\tau G_f \tau^{-1} \subset G$ for some $\tau \in H$). Prove that $\Theta_f^H(y)$ has a root in F.

(d) Assume that $\Theta_f^H(y)$ has a simple root in F. Prove that G_f is conjugate within H to a subgroup of G.

Proof.

(a) G is the symmetry group of φ, therefore $G \subset H$ is the isotropy subgroup of φ and $|H : G| = |H \cdot \varphi| = l$ (cf. Theorem A.4.9). The degree of $\Theta_f^H(y)$ is equal to l, hence $deg(\Theta_f^H(y)) = |H : G|$.

(b) Let $\tau_1, ..., \tau_l$ be representatives for the left cosets of G in H. Then

$$\Theta_f^H(y) = \prod_{i=1}^{l}(y - \tau_i\varphi(\alpha_1, ..., \alpha_n)).$$

Since $\varphi \in F[x_1, ..., x_n]$, for each $i, 1 \leq i \leq l$, $\tau_i\varphi((\alpha_1, ..., \alpha_n)) \in F[\alpha_1, ..., \alpha_n]$, hence the coefficients of $\Theta_f^H(y)$ are in $F[\alpha_1, ..., \alpha_n]$.

Suppose $\sigma \in G_f$, then $\sigma \in H$ and $\sigma\Theta_f^H(y) = \prod_{i=1}^{l}(y - \sigma\tau_i\varphi(\alpha_1, ..., \alpha_n))$. But the set $\sigma\tau_1, ..., \sigma\tau_l$ is also a set of left coset representatives of G in H. Thus the application of σ has merely permuted the roots of $\Theta_f^H(y)$ leaving the coefficients fixed. It means that coefficients of $\Theta_f^H(y)$ are in F, i.e., $\Theta_f^H(y) \in F[y]$.

(c) Suppose $\tau G_f \tau^{-1} \subset G$ for some $\tau \in H$ and $\tau_1, ..., \tau_l$ be representatives for the left cosets of G in H. Then $\tau G_f \tau^{-1}(\varphi) = \varphi$ and $G_f(\tau^{-1}\varphi) = \tau^{-1}\varphi$, i.e., $\tau^{-1}\varphi$ is fixed under the action of Galois group. Therefore $\tau^{-1}\varphi(\alpha_1, ..., \alpha_n) \in F$.

Since $\tau^{-1} \in \tau_i G$ for some $i = 1, ..., l$, exists $g \in G$ such that $\tau^{-1}\varphi(\alpha_1, ..., \alpha_n) = \tau_i g\varphi(\alpha_1, ..., \alpha_n) = \tau_i\varphi(\alpha_1, ..., \alpha_n) \in F$, i.e., $\Theta_f^H(y)$ has a root in F.

(d) Suppose $\tau_i\varphi(\alpha_1, ..., \alpha_n) \in F$ is not repeated root of $\Theta_f^H(y)$. Then $G_f\tau_i\varphi(\alpha_1, ..., \alpha_n) = \tau_i\varphi(\alpha_1, ..., \alpha_n)$ and $\tau_i^{-1}G_f\tau_i\varphi(\alpha_1, ..., \alpha_n) = \varphi(\alpha_1, ..., \alpha_n)$. Hence $\tau_i^{-1}G_f\tau_i \subset G$, i.e., G_f is conjugate within H to a subgroup of G.

□

Ex. 13.3.6 *Let* $D = \sum_{\sigma \in A_4} \sigma \cdot x_1^3x_2^2x_3 \in F[x_1, x_2, x_3, x_4]$.

(a) Prove that $D = \frac{1}{2}(\sigma_1\sigma_2\sigma_3 - 3\sigma_1^2\sigma_4 - 3\sigma_3^2 + 4\sigma_2\sigma_4) + \frac{1}{2}\sqrt{\Delta}$ *in characteristic* $\neq 2$.

(b) Prove that $\sqrt{\Delta} = D - (12) \cdot D$ *in all characteristics.*

Proof.

(a) Since

$$\sum_{\sigma \in S_4} \sigma = \sum_{\sigma \in A_4} \sigma + \sum_{\sigma \in S_4 \setminus A_4} \sigma = \sum_{\sigma \in A_4} \sigma + (12) \cdot \sum_{\sigma \in A_4} \sigma$$

and based on results of Ex.2.2.3, we have:

$$\sum_{\sigma \in S_4} \sigma \cdot x_1^3x_2^2x_3 = D + (12) \cdot D = \sigma_1\sigma_2\sigma_3 - 3\sigma_1^2\sigma_4 - 3\sigma_3^2 + 4\sigma_2\sigma_4$$

Sage verification:

```
R.<x1,x2,x3,x4,y1,y2,y3,y4> = PolynomialRing(QQ, order = 'degrevlex')
elt = SymmetricFunctions(QQ).e()
e = [elt([i]).expand(4).subs(x0=x1, x1=x2, x2=x3, x3 = x4) for i in range(5)]
J = R.ideal(e[1]-y1, e[2]-y2, e[3]-y3,e[4]-y4)
G = J.groebner_basis()
D = x1^3*x2^2*x3;
D=D+D.subs(x1=x3,x2=x4,x3=x1)+D.subs(x1=x2,x2=x1,x3=x4)+D.subs(x1=x4,x2=x3,x3=x2)
D=D+D.subs(x1=x2,x2=x3,x3=x1)+D.subs(x1=x2,x2=x3,x3=x1).subs(x1=x2,x2=x3,x3=x1)
u=D+D.subs(x1=x2,x2=x1)
var('sigma_1,sigma_2,sigma_3,sigma_4')
u.reduce(G).subs(y1=sigma_1, y2 = sigma_2,y3=sigma_3,y4=sigma_4)
```

$$\sigma_1\sigma_2\sigma_3 - 3\sigma_1^2\sigma_4 - 3\sigma_3^2 + 4\sigma_2\sigma_4$$

Assuming that $D - (12) \cdot D = \sqrt{\Delta}$ is valid, then $2D \neq 0$ in characteristic $\neq 2$ and

$$D = \frac{1}{2}(\sigma_1\sigma_2\sigma_3 - 3\sigma_1^2\sigma_4 - 3\sigma_3^2 + 4\sigma_2\sigma_4) + \frac{1}{2}\sqrt{\Delta}.$$

(b) We use Sage to prove that $D - (12) \cdot D - \sqrt{\Delta} = 0$:

```
Delta = (x1-x2)*(x1-x3)*(x1-x4)*(x2-x3)*(x2-x4)*(x3-x4)
D-D.subs(x1=x2,x2=x1)-Delta==0
```

```
True
```

Hence in all characteristics

$$\sqrt{\Delta} = D - (12) \cdot D.$$

□

Ex. 13.3.7 *As in Example 13.3.7, let* $f = x^4 + (u+1)x^2 + ux + 1 \in F[x]$, *where* $F = \mathbb{F}_2(u)$.

(a) Use Gauss's Lemma and the Schönemann-Eisenstein criterion to show that f *is irreducible over* F. *(These results apply since* $\mathbb{F}_2[u]$ *is a PID.)*

(b) Verify the formulas for $D_f(y)$ *and* $\theta_f(y)$ *given in Example 13.3.7.*

(c) Show that $y^2 + uy + 1$ *is irreducible over the splitting field of* $D_f(y)$.

Proof.

(a) Let $\bar{f} = x^4 + x^2 + 1 \in F_u[x]$ is obtained from f by reducing all coefficients modulo u. Since $\bar{f}$ is irreducible (it is cyclotomic polynomial in $F_u[x^2]$) and since $F = \mathbb{F}_2[u]$ is a PID and $u \in F$ is prime, the Gauss's Lemma and Mod p test irreducibility is applicable, i.e., f is irreducible over F.

(b) For the given polynomial $c_1 = \sigma_1 = 0, c_2 = \sigma_2 = u+1, c_3 = \sigma_3 = -u, c_4 = \sigma_4 = 1$. For $D_f(y)$ we have (cf. Ex.13.3.6,13.3.4 and (13.3),(13.32)):

$$A = D + (12) \cdot D = \sigma_1\sigma_2\sigma_3 - 3\sigma_1^2\sigma_4 - 3\sigma_3^2 + 4\sigma_2\sigma_4 = \sigma_3^2 = c_3^2 = u^2 \ mod(2)$$
$$B = D \cdot (12)D = c_2^3c_3^2 + c_3^4 = (u+1)^3u^2 + u^4 = u^5 + u^3 + u^2 \ mod(2)$$
$$D_f(y) = y^2 - Ay + B = y^2 + u^2y + u^5 + u^3 + u^2$$

$$a = \sigma_2 = c_2 = u+1, \ b = \sigma_1\sigma_3 - 4\sigma_4 = c_1c_3 - 4c_4 = 0,$$
$$c = \sigma_1^2\sigma_4 + \sigma_3^2 - 4\sigma_2\sigma_4 = c_1^2c_4 + c_3^3 - 4c_2c_4 = u^2$$
$$\theta_f(y) = y^3 - ay^2 + by - c = y^3 + (u+1)y^2 + u^2 = (y+u)(y^2 + y + u)$$

Sage verification:

```
R.<x1,x2,x3,x4,y1,y2,y3,y4> = PolynomialRing(QQ, order = 'degrevlex')
elt = SymmetricFunctions(QQ).e()
e = [elt([i]).expand(4).subs(x0=x1, x1=x2, x2=x3, x3 = x4) for i in range(5)]
J = R.ideal(e[1]-y1, e[2]-y2, e[3]-y3,e[4]-y4)
G = J.groebner_basis()
D = x1^3*x2^2*x3;
D=D+D.subs(x1=x3,x2=x4,x3=x1)+D.subs(x1=x2,x2=x1,x3=x4)
  +D.subs(x1=x4,x2=x3,x3=x2)
D=D+D.subs(x1=x2,x2=x3,x3=x1)
  +D.subs(x1=x2,x2=x3,x3=x1).subs(x1=x2,x2=x3,x3=x1)
```

```
d1=x1*x2+x3*x4;d2=d1.subs(x2=x3,x3=x2);d3=d1.subs(x1=x3,x3=x1);
S1=d1+d2+d3; S2=d1*d2+d1*d3+d2*d3; S3=d1*d2*d3

S.<c1,c2,c3,c4,u> = PolynomialRing(ZZ, order = 'degrevlex')
A=(D+D.subs(x1=x2,x2=x1)).reduce(G).subs(y1=c1,y2=c2,y3=c3,y4=c4)
A=A.subs(c1=0,c2=u+1,c3=-u,c4=1).change_ring(GF(2));
B=(D*D.subs(x1=x2,x2=x1)).reduce(G).subs(y1=c1,y2=c2,y3=c3,y4=c4)
B=B.subs(c1=0,c2=u+1,c3=-u,c4=1).change_ring(GF(2));
a=S1.reduce(G).subs(y1=c1, y2 = c2,y3=c3,y4=c4)
a=a.subs(c1=0,c2=u+1,c3=-u,c4=1).change_ring(GF(2));
b=S2.reduce(G).subs(y1=c1, y2 = c2,y3=c3,y4=c4)
b=b.subs(c1=0,c2=u+1,c3=-u,c4=1).change_ring(GF(2));
c=S3.reduce(G).subs(y1=c1, y2 = c2,y3=c3,y4=c4)
c=c.subs(c1=0,c2=u+1,c3=-u,c4=1).change_ring(GF(2));
 (A,B);(a,b,c)
(y^3+a*y^2+b*y+c).factor()
```

(c) As per (b), $D_f(y) = y^2+u^2y+u^5+u^3+u^2 = (y+u)^2+u^2(y+u)+u^5 = u^4(Y^2+Y+u)$, where $Y = (y+u)/u^2$.

Then the splitting field of $D_f(y)$ is $F(D_f(y)) = \mathbb{F}_2(u,\alpha)$, where $\alpha^2+\alpha+u = 0$. Since $u = \alpha^2+\alpha$, $\mathbb{F}_2(u,\alpha) = \mathbb{F}_2(\alpha)$ and $g(y) = y^2+uy+1 = y^2+(\alpha^2+\alpha)y+1$. Then $g(y+1) = (y+1)^2+(\alpha^2+\alpha)(y+1)+1 = y^2+(\alpha^2+\alpha)y+(\alpha^2+\alpha)$ is irreducible in $\mathbb{F}_2(\alpha)$ by the Schönemann-Eisenstein criterion.

Thus y^2+uy+1 is irreducible over the splitting field of $D_f(y)$, therefore $D_f(y) \simeq D_8$ by Proposition 13.3.6. □