Solutions to David A.Cox "Galois Theory"

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7 Chapter 7 : THE GALOIS CORRESPONDENCE

7.1 GALOIS EXTENSIONS

Ex. 7.1.1 *Given a finite extension $F \subset L$, and a subgroup $H \subset \mathrm{Gal}(L/F)$, prove that $L_H = \{\alpha \in L \mid \forall \sigma \in H, \sigma(\alpha) = \alpha\}$ is a subfield of L containing F.*

Proof. Let $H \subset \mathrm{Gal}(L/F)$, and $L_H = \{\alpha \in L \mid \forall \sigma \in H,\, \sigma(\alpha) = \alpha\}$.

We show that L_H is a subfield of L containing F.

• By definition of $\mathrm{Gal}(L/F)$, every element σ of $H \subset \mathrm{Gal}(L/F)$ satisfies $\sigma(\alpha) = \alpha$ for all $\alpha \in F$, therefore $F \subset L_H$. In particular $1 \in F \subset L_H$, so $L_H \neq \emptyset$.

• If $\alpha, \beta \in L_H$, then

$$\sigma(\alpha - \beta) = \sigma(\alpha) - \sigma(\beta) = \alpha - \beta,$$
$$\sigma(\alpha\beta) = \sigma(\alpha)\sigma(\beta) = \alpha\beta,$$

thus $\alpha - \beta, \alpha\beta \in L_H$.

• If $\alpha \in L_H \setminus \{0\}$, $\sigma(\alpha) = \alpha$, thus $\sigma(\alpha^{-1}) = \sigma(\alpha)^{-1} = \alpha^{-1} : \alpha^{-1} \in L_H$.

Conclusion: L_H is a subfield of L containing F. □

Ex. 7.1.2 *In the proof of (c) $\Rightarrow$ (a) in Theorem 7.1.1, give the details of how the proof of Theorem 5.2.4 shows that L is the splitting field of f over F.*

Proof. By hypothesis, the extension $F \subset L$ is finite, normal and separable. As $F \subset L$ is finite, $L = F(\alpha_1, \cdots, \alpha_n)$, where $\alpha_i \in L$ has p_i as minimal polynomial over F. If $q_1, \ldots, q_r$ are the distinct elements in the set $\{p_1, \ldots, p_n\}$, then $f = q_1 \cdots q_r$ is a product of monic irreducible distinct polynomials (thus q_i is not associate to q_j if $i \neq j$). As in the text, we know by Lemma 5.3.4 that f is separable.

We show that L is the splitting field of f over F.

As $q_j = p_i$ for some i, $1 \leq i \leq n$ is the minimal polynomial of $\alpha_i \in L$ over F, and as $F \subset L$ is normal, then all the roots of p_i are in L, so q_j splits completely over L, thus $f = \prod_{j=1}^r q_j$ splits completely over L. Write $\beta_1, \ldots, \beta_m \in L$ the roots of f, and $L' = F(\beta_1, \ldots, \beta_m) \subset L$ the splitting field of f over F. As $F \subset L$, and $\beta_1, \ldots, \beta_m \in L$, we know that $L' \subset L$.

As every $\alpha_i, 1 \leq i \leq n$ is a root of a polynomial $p_i = q_j$, then α_i is a root of f, so $\alpha_i = \beta_k$ for some k, $1 \leq k \leq m$, thus $\alpha_i \in L'$. Consequently $\{\alpha_1, \ldots, \alpha_n\} \subset \{\beta_1, \ldots, \beta_m\}$ and

$$L = F(\alpha_1, \ldots, \alpha_n) \subset F(\beta_1, \ldots, \beta_m) = L' \subset L :$$

$L = L'$ is the splitting field of f over F.

Conclusion: if $F \subset L$ is a finite normal separable extension, L is the splitting field of a separable polynomial in $F[x]$. □

Ex. 7.1.3 *Suppose that $F \subset L$ and that $\alpha, \beta \in L$ are separable over F. Prove that $\alpha + \beta, \alpha\beta$, and α/β (assuming $\beta \neq 0$) are also separable over F.*

Proof. Let $\alpha, \beta \in L$ separable over F. By Proposition 7.1.6, $F \subset F(\alpha, \beta)$ is a separable extension.

$F(\alpha, \beta)$ being a field, $\alpha + \beta, \alpha\beta \in F(\alpha, \beta)$, and if $\beta \neq 0$, $\alpha/\beta \in F(\alpha, \beta)$, therefore $\alpha + \beta, \alpha\beta, \alpha/\beta$ are separable. □

Ex. 7.1.4 *Let $F \subset L$ be a finite extension, and assume F has characteristic p. Then consider the set $K = \{\alpha \in L \mid \alpha \text{ is separable over } F\}$.*

(a) Use Proposition 7.1.6 to show that K is a subfield of L containing F. Thus $F \subset K$ is a separable extension.

(b) Use part (c) of theorem 5.3.15 to show that $K \subset L$ is purely inseparable.

Proof. (a) Let $F \subset L$ a finite extension, where F has characteristic p, and let

$$K = \{\alpha \in L \mid \alpha \text{ is separable over } F\}.$$

By Theorem 7.1.6 and Exercise 3 (and 1, root of $x - 1$ is in K), K is a subfield of L. Moreover, every $\alpha \in F$ is root of the irreducible separable polynomial $x - \alpha \in F[x]$, so α is separable , thus $F \subset K$, and $F \subset K$ is a separable extension.

(b) We show that the extension $K \subset L$ is purely inseparable.

Let $\beta \in L \setminus K$.

If β was separable over K, then by Theorem 7.1.6, $K \subset K(\beta)$ would be a separable extension. But $F \subset K$ is also separable, thus by Theorem 5.3.15(c), $F \subset K(\beta)$ would be separable, and then β would be separable over F, that is $\beta \in K$: this is a contradiction. Every $\beta \in L \setminus K$ is inseparable, so the extension $K \subset L$ is purely inseparable.

□

Ex. 7.1.5 *Prove that the Galois closure of a finite separable extension $F \subset L$ is unique up to an isomorphism that is the identity on L.*

Proof. Let M, M' two Galois closures of the separable extension $F \subset L$. By Proposition 7.1.7, there exists a field homomorphism $\varphi : M \to M'$ that is identity on L.

As every field homomorphism, φ is injective, this is an embedding of M in M'. Moreover φ is the identity on L, so φ is a L-linear injective application between M and M' as L-vector spaces, thus $[M : L] \leq [M' : L]$. Exchanging M and M', we prove similarly that $[M' : L] \leq [M : L]$, thus $[M' : L] = [M : L]$. An injective linear application between two same dimensional vector spaces is bijective, thus φ is bijective. Therefore φ is a field isomorphism that is identity on L.

The Galois closure of a finite separable extension $F \subset L$ is unique up to an isomorphism that is the identity on L. □

Ex. 7.1.6 *In analogy with the Galois closure of a finite separable extension, every finite extension $F \subset L$ has a normal closure, which is essentially the smallest extension of L that is normal over F. State and prove the analog of Proposition 7.1.7 for normal closures.*

Proposition : *Let $F \subset L$ a finite extension. Then there is an extension $L \subset M$ such that:*

(a) $F \subset M$ is a finite normal extension.

(b) Given any other extension $L \subset M'$ such that M' is normal over F, there is a field homomorphism $\varphi : M \to M'$ that is identity on L.

Proof. $F \subset L$ is a finite extension, so $L = F(\alpha_1, \ldots, \alpha_n)$, where $\alpha_i \in L$ is algebraic over F, with minimal polynomial $p_i \in F[x]$.

Let $f = p_1 \cdots p_n$, and $M = L(\beta_1, \ldots, \beta_m)$ the splitting field of f over L, where $\beta_1, \ldots, \beta_m$ are the roots of f in M. As the α_i are roots of p_i, they are roots of f, so $\{\alpha_1, \ldots, \alpha_n\} \subset \{\beta_1, \ldots, \beta_m\}$. Therefore

$$L = F(\alpha_1, \ldots, \alpha_n) \subset F(\beta_1, \ldots, \beta_m) \subset L(\beta_1, \ldots, \beta_m) = M,$$

Thus $F(\beta_1, \ldots, \beta_m)$ contains L and $\beta_1, \ldots, \beta_m$, therefore $M = L(\beta_1, \ldots, \beta_m) \subset F(\beta_1, \ldots, \beta_m)$.

Therefore $M = F(\beta_1, \ldots, \beta_m)$ is the splitting field of f over F. Then , by Theorem 5.2.4, the extension $F \subset M$ is normal (and finite), so M satisfies (a).

Let $M' \supset L$ any normal extension of F. As $F \subset M'$ is normal, the p_i splits completely over M', thus also f. Let $\gamma_1, \ldots, \gamma_m \in M'$ the roots of f in M', and $M'' = F(\gamma_1, \ldots, \gamma_m) \subset M'$. As $\alpha_i \in L \subset M'$, the α_i are roots of f in M' : $\{\alpha_1, \ldots, \alpha_n\} \subset \{\gamma_1, \ldots, \gamma_m\}$, thus $L = F(\alpha_1, \ldots, \alpha_n) \subset F(\gamma_1, \ldots, \gamma_m) = M''$.

M'' and M are so two splitting fields of f over L. By the unicity of the splitting field (Corollary 5.1.7), there exist a field isomorphism of M in M'' that is identity on L. Since $M'' \subset M'$, we can regard this isomorphism as an injective field homomorphism $\varphi : M \to M'$. $\square$

Ex. 7.1.7 *Prove that the normal closure of a finite extension $F \subset L$ is unique up to an isomorphism that is the identity on L.*

Proof. Same proof as in Exercise 5.

Let M, M' two normal closures of the extension $F \subset L$. By Exercise 6, there exists a field homomorphism $\varphi : M \to M'$ that is identity on L.

As every field homomorphism, φ is injective, this is an embedding of M in M'. Moreover φ is the identity on L, so φ is a L-linear injective application between M and M' as L-vector spaces, thus $[M : L] \leq [M' : L]$. Exchanging M and M', we prove similarly that $[M' : L] \leq [M : L]$, thus $[M' : L] = [M : L]$. An injective linear application between two same dimensional vector spaces is bijective, thus φ is bijective. Therefore φ is a field isomorphism that is identity on L.

The normal closure of a finite extension $F \subset L$ is unique up to an isomorphism that is the identity on L. $\square$

Ex. 7.1.8 *Let h be the polynomial (7.1) used in the proof of $(b) \Rightarrow (c)$ from Theorem 7.1.1. Show that there is an integer m such that*

$$\prod_{\sigma \in \mathrm{Gal}(L/F)} (x - \sigma(\alpha)) = h^m.$$

Proof. Here, as in theorem 7.1.1, $F \subset L$ is a normal separable extension.

Let $\alpha \in L$, and h the minimal polynomial of α over F. As L is a normal extension of F, h splits completely over L, so $h = \prod_{i=1}^{r} (x - \alpha_i)$, where $\alpha_1, \ldots, \alpha_r \in L$, and the $\alpha_i, 1 \le i \le r$ are distinct since h is a separable polynomial.

The Galois group $G = \mathrm{Gal}(L/F)$ acts on the set $S = \{\alpha_1, \cdots, \alpha_r\}$, with the action defined by $\sigma \cdot \gamma = \sigma(\gamma), \sigma \in G, \gamma \in S$.

As h is irreducible over F, G acts transitively on S, so the orbit $\mathcal{O}_\alpha$ of α is S of cardinality r, and G_α, the stabilizer of α in G satisfies

$$r = |\mathcal{O}_\alpha| = (G : G_\alpha).$$

As $F \subset L$ is a Galois extension, the Galois group G has order $n = |G| = [L : F]$. Consequently $|G_\alpha| = n/r$, so G_α is a subgroup of G with index r and cardinality $m := n/r$.

(Note: For all $\sigma \in G$,

$$\sigma \in G_\alpha \iff \sigma(\alpha) = \alpha \iff \forall \gamma \in F(\alpha), \sigma(\gamma) = \gamma \iff \sigma \in \mathrm{Gal}(L/F(\alpha)).$$

Therefore

$$G_\alpha = \mathrm{Gal}(L/F(\alpha)).$$

As h is the minimal polynomial of α over F, $[F(\alpha) : F] = \deg(h) = r$.

As $F \subset L$ is a Galois extension, $F(\alpha) \subset L$ also, so we find again by the Tower Theorem:

$$|G_\alpha| = |\mathrm{Gal}(L/F(\alpha))| = [L : F(\alpha)] = [L : F] \,/\, [F(\alpha) : F] = n/r.)$$

Let $\sigma_1, \ldots, \sigma_r$ a complete system of representants of the left cosets $\sigma G_\alpha, \sigma \in G$. Then the $\sigma_i G_\alpha$ form a partition of G :

$$G = \bigcup_{i=1}^{r} \sigma_i G_\alpha,$$

$$i \neq j \Rightarrow \sigma_i G_\alpha \cap \sigma_j G_\alpha = \emptyset \ (1 \le i, j \le r).$$

If $\sigma \in \sigma_i G_\alpha$, then $\sigma = \sigma_i \tau, \tau \in G_\alpha$, thus $\sigma(\alpha) = \sigma_i(\tau(\alpha)) = \sigma_i(\alpha)$. Let $\gamma_i = \sigma_i(\alpha) \in S$. The image of α by all the elements of the left coset $\sigma_i G_\alpha$ is a constant equal to $\gamma_i = \sigma_i(\alpha)$. As $|\sigma_i G_\alpha| = |G_\alpha| = m$,

$$g = \prod_{\sigma \in G} (x - \sigma.\alpha) = \prod_{i=1}^{r} \prod_{\sigma \in \sigma_i G_\alpha} (x - \sigma.\alpha) = \prod_{i=1}^{r} (x - \gamma_i)^m.$$

Moreover $T := \{\gamma_1, \cdots, \gamma_r\} \subset \{\alpha_1, \cdots, \alpha_r\}$, and the $\gamma_i, 1 \le i \le r$, are distinct since

$$\sigma_i(\alpha) = \sigma_j(\alpha) \Rightarrow (\sigma_j^{-1}\sigma_i)(\alpha) = \alpha \Rightarrow \sigma_j^{-1}\sigma_i \in G_\alpha \Rightarrow \sigma_i G_\alpha = \sigma_j G_\alpha \Rightarrow i = j.$$

Moreover $T \subset S, |T| = |S| = r$, thus $T = S$.

Consequently $g = \prod_{i=1}^{r} (x - \gamma_i)^m = \prod_{i=1}^{r} (x - \alpha_i)^m = h^m$.

Conclusion: *if $F \subset L$ is a Galois extension, h the minimal polynomial of $\alpha \in L$ over F, and $g = \prod_{\sigma \in G} (x - \sigma.\alpha)$, then $g = h^m, m \in \mathbb{N}^*$ (where $m = [L : F(\alpha)]$).* □

Ex. 7.1.9 *For each of the following extensions, say whether it is a Galois extension. Be sure to say which of our four criteria (the three parts of Theorem 7.1.1 and part (c) of theorem 7.1.5) you are using.*

(a) $\mathbb{Q} \subset \mathbb{Q}(\sqrt{2}, \sqrt[3]{2})$.

(b) $\mathbb{Q} \subset \mathbb{Q}(\alpha, \beta)$, *$\alpha, \beta$ distinct roots of* $x^3 + x^2 + 2x + 1$.

(c) $\mathbb{F}_p(t^p) \subset \mathbb{F}_p(t)$, *$t$ a variable.*

(d) $\mathbb{C}(t + t^{-1}) \subset \mathbb{C}(t)$, *$t$ a variable.*

(e) $\mathbb{C}(t^n) \subset \mathbb{C}(t)$, *$t$ a variable, n a positive integer.*

Proof. (a) $f = x^3 - 2$ is irreducible over $\mathbb{Q}$, and has a root $\sqrt[3]{2}$ in $\mathbb{Q}(\sqrt{2}, \sqrt[3]{2})$, but $\omega\sqrt[3]{2}$ is a non real root of f, so is not in $\mathbb{Q}(\sqrt{2}, \sqrt[3]{2}) \subset \mathbb{R}$. Consequently, $\mathbb{Q} \subset \mathbb{Q}(\sqrt{2}, \sqrt[3]{2})$ is not a normal extension, so is not a Galois extension (Th. 7.1.1(c)).

(b) Let α, β, γ the roots of f, where we suppose $\alpha \neq \beta$ (in fact the discriminant of f is $-23 :$ the three roots of f are distinct). As $\alpha+\beta+\gamma = -1, \gamma = -1-\alpha-\beta \in \mathbb{Q}(\alpha, \beta)$, thus $\mathbb{Q}(\alpha, \beta) = \mathbb{Q}(\alpha, \beta, \gamma)$ is the splitting field of f, therefore $\mathbb{Q} \subset \mathbb{Q}(\alpha, \beta)$ is a normal extension. Moreover the characteristic of $\mathbb{Q}$ is 0, thus this extension is separable (Prop. 5.3.7).

$\mathbb{Q} \subset \mathbb{Q}(\alpha, \beta)$ is a normal and separable extension, so is a Galois extension (Th. 7.1.1(c)).

(c) t is a root of $f = x^p - t^p = (x - t)^p \in \mathbb{F}_p(t^p)$. The only root of f is t, and $t \notin \mathbb{F}_p(t^p)$, otherwise $t = u(t^p)/v(t^p)$, where $u, v \in \mathbb{F}_p[t], u \wedge v = 1$. Moreover $u(t)^p = (\sum_{i=0}^d a_i t^i)^p = \sum_{i=0}^d a_i^p t^{ip} = \sum_{i=0}^d a_i t^{ip} = u(t^p)$, and similarly for v.

Consequently, we would have $t = u(t)^p/v(t)^p, u \wedge v = 1$, which is impossible by Exercise 4.2.9.

The equation $f = x^p - t^p$ has so no root in $\mathbb{F}(t^p)$, where $p = \deg(f)$ is prime. By Proposition 4.2.6, f is irreducible over $\mathbb{F}(t^p)$: $f = (x - t)^p$ is so the minimal polynomial of t over $\mathbb{F}(t^p)$.

The minimal polynomial of $t \in \mathbb{F}(t)$ is not separable, so $\mathbb{F}_p(t)/\mathbb{F}_p(t^p)$ is not a Galois extension.

(d) Let $f = x^2 - \left(t + \frac{1}{t}\right) x + 1 \in \mathbb{C}(t + t^{-1})[x]$. Then t and t^{-1} are roots of f in $\mathbb{C}(t)$. Moreover $t^{-1} \in \mathbb{C}(t)$, therefore $\mathbb{C}(t) = \mathbb{C}(t, t^{-1})$ is the splitting field of f over $C(t + t^{-1})$. $\mathbb{C}(t + t^{-1}) \subset \mathbb{C}(t)$ is so a normal extension, and is separable since the characteristic of $\mathbb{C}$, and of $\mathbb{C}(t + t^{-1})$ is zero.

$\mathbb{C}(t + t^{-1}) \subset \mathbb{C}(t)$ is a Galois extension.

(e) t is a root of $x^n - t^n = (x-t)(x-\zeta t)\cdots(x-\zeta^{n-1}t) \in \mathbb{C}(t^n)[x]$, where $\zeta = e^{2i\pi}/n$. As $\zeta^k t \in \mathbb{C}(t), 0 \le k \le n-1$, $\mathbb{C}(t) = \mathbb{C}(t, \zeta t, \ldots, \zeta^{m-1}t)$ is the splitting field of the polynomial $x^n - t^n \in \mathbb{C}(t^n)[x]$, so $\mathbb{C}(t^n) \subset \mathbb{C}(t)$ is a normal extension. As the characteristic of $\mathbb{C}(t^n)$ is zero, this extension is also separable.

$\mathbb{C}(t^n) \subset \mathbb{C}(t)$ is a Galois extension.

□

Ex. 7.1.10 *Prove that $\mathbb{Q}(\omega, \sqrt[3]{2})$ is the Galois closure of $\mathbb{Q} \subset \mathbb{Q}(\sqrt[3]{2})$.*

Proof. The minimal polynomial of $\sqrt[3]{2}$ over $\mathbb{Q}$ is $f = x^3 - 2$. By sections 7.1.B, 7.1.C, the Galois closure of the extension $\mathbb{Q} \subset \mathbb{Q}(\omega, \sqrt[3]{2})$ is the splitting field of $f = x^3 - 2$ over $\mathbb{Q}$ (in $\mathbb{C}$), that is $\mathbb{Q}(\sqrt[3]{2}, \omega\sqrt[3]{2}, \omega^2\sqrt[3]{2}) = \mathbb{Q}(\omega, \sqrt[3]{2})$. □

Note: as a verification, note that the two parts of the definition of the Galois closure are satisfied.

- The extension $\mathbb{Q} \subset \mathbb{Q}(\omega, \sqrt[3]{2})$ is a Galois extension, since $\mathbb{Q}(\omega, \sqrt[3]{2})$ is the splitting field of the separable polynomial $x^3 - 2$.
- Let $M \supset \mathbb{Q}(\sqrt[3]{2})$ an extension such that M is a Galois extension of $\mathbb{Q}$. As $\sqrt[3]{2} \in M$ and as $\mathbb{Q} \subset M$ is normal, $x^3 - 2$ splits completely over M:

  $$x^3 - 2 = (x-\alpha)(x-\beta)(x-\gamma), \qquad \alpha, \beta, \gamma \in M,$$

  where $\alpha = \sqrt[3]{2} \in M$.

  $(\beta/\alpha)^3 = 1$, thus $\omega' = \beta/\alpha$ is a cube root of unity in M, with $\omega' \ne 1$ since $x^3 - 2$ is separable. So ω' is a root in M of $(x^3-1)/(x-1) = x^2 + x + 1$.

  x^2+x+1 has degree 2 and has no real root, so has no root in $\mathbb{Q}(\sqrt[3]{2})$, thus x^2+x+1 is irreducible over $\mathbb{Q}(\sqrt[3]{2})$. Therefore $\mathbb{Q}(\omega, \sqrt[3]{2}) \subset \mathbb{C}$ and $\mathbb{Q}(\omega', \sqrt[3]{2}) \subset M$ are two splitting fields of $x^2 + x + 1$ over $\mathbb{Q}(\sqrt[3]{2})$. Therefore there exists an isomorphism $\mathbb{Q}(\omega, \sqrt[3]{2}) \simeq \mathbb{Q}(\omega', \sqrt[3]{2})$ which is the identity on $\mathbb{Q}(\sqrt[3]{2})$, and which sends ω on ω', so there exists an embedding of $\mathbb{Q}(\omega, \sqrt[3]{2})$ in M which is the identity on $\mathbb{Q}(\sqrt[3]{2})$.

Ex. 7.1.11 *Construct the Galois closure of $\mathbb{Q} \subset \mathbb{Q}(\sqrt[4]{2})$.*

Proof. By sections 7.1.B,7.1.C, as the minimal polynomial of $\sqrt[4]{2}$ is $x^4 - 2$, a Galois closure of $\mathbb{Q} \subset \mathbb{Q}(\sqrt[4]{2})$ is the splitting field of $x^4 - 2$ over $\mathbb{Q}$, that is

$$\mathbb{Q}(\sqrt[4]{2}, i\sqrt[4]{2}, i^2\sqrt[4]{2}, i^3\sqrt[4]{2}) = \mathbb{Q}(i, \sqrt[4]{2}).$$

□

Ex. 7.1.12 *Let $F \subset L$ be an extension of degree 2, where F has characteristic $\ne 2$.*

(a) Show that $L = F(\alpha)$, where α is a root of an irreducible polynomial of degree 2.

(b) Show that the minimal polynomial of α over F is separable.

(c) Conclude that $F \subset L$ is a Galois extension with $\mathrm{Gal}(L/F) \simeq \mathbb{Z}/2\mathbb{Z}$.

(d) By completing the square, show that there is $\beta \in L$ such that $L = F(\beta)$ and $\beta^2 \in F$.

For β as in part (d), let $a = \beta^2 \in F$. Then we can write $\beta = \sqrt{a}$. This shows that if F has characteristic $\neq 2$, then every degree 2 extension of F is obtained by taking a square root.

Proof. Let $F \subset L$ be an extension of degree 2, where F has characteristic $\neq 2$. Then $[L : F] = 2, F \subset L, F \neq L$.

(a) Let $\alpha \in L \setminus F$. Then $(1, \alpha)$ is a linearly independent list, otherwise $\alpha \in F$. As $\dim_F(L) = 2$, $(1, \alpha)$ is a basis of of the F-vector space L.

Therefore there exists a pair $(a, b) \in F^2$ such that $\alpha^2 = a\alpha + b$, so α is a root of the polynomial $f = x^2 - ax - b \in F[x]$.

$$(x - \alpha)(x - (a - \alpha)) = x^2 - ax + \alpha(a - \alpha) = x^2 - ax - b = f.$$

The roots of f are so α and $\beta = a - \alpha$, both in L.

As $\alpha \notin F$, $1 < [F(\alpha) : F] \leq 2$, thus $[F(\alpha) : F] = 2 = [L : F]$ with $F[\alpha] \subset L$, therefore $L = F(\alpha)$.

The polynomial $f \in F[x]$ is irreducible over F since $\deg(f) = 2$ and the roots of f are $\alpha \notin F, a - \alpha \notin F$. So f is the minimal polynomial of α over F.

(b) The roots of f, minimal polynomial of α over F, are α, β, which are distinct, otherwise $\alpha = a - \alpha$, and then $\alpha = a/2 \in F$ (the characteristic is not equal to 2), which is false. The minimal polynomial of α over F is so separable.

(c) As $\beta = a - \alpha, a \in F$, $\beta \in F(\alpha)$, thus $L = F(\alpha) = F(\alpha, \beta)$ is the splitting field of the separable polynomial $f \in F[x]$. Therefore, by Theorem 7.1.1, $F \subset L$ is a Galois extension.

f being irreducible, there exists (Prop. 5.1.8) an isomorphism $\sigma : L \to L$ such that $\sigma(\alpha) = \beta$ and σ is the identity on F, so $\sigma \in \mathrm{Gal}(L/F)$.

(Explicitly, $\sigma : u + v\alpha \mapsto u + v\beta,\ u, v \in F$: we can verify directly that it is an isomorphism.)

Every $\tau \in \mathrm{Gal}(L/F)$ sends the root α of $f \in F[x]$ on a root of f, so $\tau(\alpha) = \alpha = 1_K(\alpha)$ or $\tau(\alpha) = \beta = \sigma(\alpha)$. As $L = F(\alpha)$, this F-automorphisme is uniquely determined by the image of α. Thus $\tau = \sigma$ or $\tau = 1_K = e$. Moreover $\sigma \neq e$, otherwise $\sigma(\alpha) = \alpha$, so $\beta = \alpha$, which is false by part (b). Consequently $G = \{e, \sigma\}$.

Every group of order 2 is isomorphic to $\mathbb{Z}/2\mathbb{Z}$, thus

$$G = \{e, \sigma\} \simeq \mathbb{Z}/2\mathbb{Z}.$$

(d) As the characteristic is not 2,

$$0 = \alpha^2 - a\alpha - b = \left(\alpha - \frac{a}{2}\right)^2 - \frac{a^2}{4} - b.$$

Therefore $\gamma = \alpha - \frac{a}{2}$ satisfies $\gamma^2 = \frac{a^2+4b^2}{4} \in F$.
As $\gamma = \alpha - \frac{a}{2}$ with $a \in F$, $F(\gamma) = F(\alpha) = L$. Write $c = \gamma^2 \in F$, and $\sqrt{c} = \gamma$, then

$$L = F(\gamma), \gamma^2 \in F, \qquad L = F(\sqrt{c}), c \in F$$

$\square$

7.2 NORMAL SUBGROUPS AND NORMAL EXTENSIONS

Ex. 7.2.1 *In the diagram (7.3), verify the following.*

(a) $\mathbb{Q}(\sqrt[3]{2})$ *has conjugate fields* $\mathbb{Q}(\sqrt[3]{2}), \mathbb{Q}(\omega\sqrt[3]{2})$, *and* $\mathbb{Q}(\omega^2\sqrt[3]{2})$.

(b) $\mathbb{Q}(\omega)$ *equals all of its conjugates.*

Proof. (a) By Section 6.4.A (or Exercises 6.2.2 and 6.3.1), there exists $\sigma, \tau \in \mathrm{Gal}(L/\mathbb{Q})$ uniquely determined by

$$\sigma(\omega) = \omega,\ \sigma(\sqrt[3]{2}) = \omega\sqrt[3]{2},$$

$$\tau(\omega) = \omega^2, \tau(\sqrt[3]{2}) = \sqrt[3]{2},$$

and $G = \mathrm{Gal}(L/\mathbb{Q}) = \langle \sigma, \tau \rangle$.

Let $K = \mathbb{Q}(\sqrt[3]{2})$. We show that $\sigma K = \mathbb{Q}(\omega\sqrt[3]{2})$.

If $\beta \in \sigma K, \beta = \sigma(\alpha), \alpha \in K = \mathbb{Q}[\sqrt[3]{2}]$, thus $\alpha = p(\sqrt[3]{2}), p \in \mathbb{Q}[x]$, $\beta = \sigma(p(\sqrt[3]{2})) = p(\sigma(\sqrt[3]{2})) = p(\omega\sqrt[3]{2}) \in \mathbb{Q}(\omega\sqrt[3]{2})$, consequently $\sigma K \subset \mathbb{Q}(\omega\sqrt[3]{2})$.

Conversely, if $\beta \in \mathbb{Q}(\omega\sqrt[3]{2}) = \mathbb{Q}[\omega\sqrt[3]{2}]$, $\beta = p(\omega\sqrt[3]{2})$, $p \in \mathbb{Q}[x]$, then $\beta = \sigma(p(\sqrt[3]{2})) = \sigma(\alpha)$, where $\alpha = p(\sqrt[3]{2}) \in \mathbb{Q}(\sqrt[3]{2})$, consequently $\mathbb{Q}(\omega\sqrt[3]{2}) \subset \sigma K$.

$$\sigma K = \mathbb{Q}(\omega\sqrt[3]{2}).$$

As $\sigma^2(\sqrt[3]{2}) = \omega^2\sqrt[3]{2}$, we obtain similarly

$$\sigma^2 K = \mathbb{Q}(\omega^2\sqrt[3]{2}),$$

and of course, $eK = K$. So $\mathbb{Q}(\sqrt[3]{2}), \mathbb{Q}(\omega\sqrt[3]{2}), \mathbb{Q}(\omega^2\sqrt[3]{2})$ are conjugates fields of K over $\mathbb{Q}$.

As $\tau K = K$, and $G = \langle \sigma, \tau \rangle$, they are the only ones.

Conclusion:

the conjugate fields of $\mathbb{Q}(\sqrt[3]{2})$ in the extension $\mathbb{Q} \subset \mathbb{Q}(\sqrt[3]{2})$ are $\mathbb{Q}(\sqrt[3]{2}), \mathbb{Q}(\omega\sqrt[3]{2}), \mathbb{Q}(\omega^2\sqrt[3]{2})$.

(b) As $\sigma(\omega) = \omega$ and as σ is the identity on $\mathbb{Q}$, $\sigma\mathbb{Q}(\omega) = \mathbb{Q}(\omega)$. Moreover $\tau\mathbb{Q}(\omega) = \mathbb{Q}(\omega^2)$. Since $\omega^2 = -1 - \omega$, $\mathbb{Q}(\omega^2) = \mathbb{Q}(\omega)$. As $\sigma\mathbb{Q}(\omega) = \mathbb{Q}(\omega), \tau\mathbb{Q}(\omega) = \mathbb{Q}(\omega)$, and as $G = \langle \sigma, \tau \rangle$, $\lambda\mathbb{Q}(\omega) = \mathbb{Q}(\omega)$ for all $\lambda \in \mathrm{Gal}(L/F)$.

The only conjugate field of $\mathbb{Q}(\omega)$ is so $\mathbb{Q}(\omega)$.

Note: As $\mathbb{Q} \subset \mathbb{Q}(\omega)$ is a quadratic extension, thus a normal extension (Ex. 7.1.12), by Theorem 7.2.5, $K = \sigma K$ for all $\sigma \in \mathrm{Gal}(\mathbb{Q}(\omega, \sqrt[3]{2})/\mathbb{Q})$. We find again that the only conjugate field of $\mathbb{Q}(\omega)$ is $\mathbb{Q}(\omega)$.

□

Ex. 7.2.2 *Complete the proof of Lemma 7.2.4 by showing that*

$$\mathrm{Gal}(L/\sigma K) \subset \sigma\mathrm{Gal}(L/K)\sigma^{-1}.$$

Proof. $F \subset K \subset L$.

Let $\tau \in \mathrm{Gal}(L/\sigma K)$. Then $\tau : L \to L$ is an automorphism of L, and $\tau(\gamma) = \gamma$ for all $\gamma \in \sigma K$, thus $\tau(\sigma(\alpha)) = \sigma(\alpha)$ for all $\alpha \in K$.

Let $\lambda = \sigma^{-1}\tau\sigma \in \mathrm{Gal}(L/F)$. For all $\alpha \in K$,

$$\begin{aligned}\lambda(\alpha) &= \sigma^{-1}(\tau(\sigma(\alpha)) \\ &= \sigma^{-1}(\sigma(\alpha)) \\ &= \alpha.\end{aligned}$$

Thus $\lambda = \sigma^{-1}\tau\sigma \in \mathrm{Gal}(L/K)$, so $\tau = \sigma\lambda\sigma^{-1} \in \sigma\mathrm{Gal}(L/K)\sigma^{-1}$:

$$\mathrm{Gal}(L/\sigma K) \subset \sigma\mathrm{Gal}(L/K)\sigma^{-1}.$$

As the converse inclusion is proved in section 7.2.A,

$$\mathrm{Gal}(L/\sigma K) = \sigma\mathrm{Gal}(L/K)\sigma^{-1}.$$

□

Ex. 7.2.3 *Prove (7.6).*

Proof. We prove that $K_1 \subset K_2 \subset L \Rightarrow \mathrm{Gal}(L/K_1) \supset \mathrm{Gal}(L/K_2)$.

Suppose that $K_1 \subset K_2 \subset L$. Let $\sigma \in \mathrm{Gal}(L/K_2)$. Then $\sigma : L \to L$ is an automorphism of L and for all $\alpha \in K_2, \sigma(\alpha) = \alpha$. As $K_1 \subset K_2$, a fortiori $\sigma(\alpha) = \alpha$ for all $\alpha \in K_1$. Consequently, $\sigma \in \mathrm{Gal}(L/K_1)$. □

Ex. 7.2.4 *Verify that applying $K \mapsto \mathrm{Gal}(L/K)$ to (7.3) gives (7.7). Don't forget to include the extreme cases $K = \mathbb{Q}$ and $K = L$.*

Proof.

$$\begin{tikzcd}
 & & & \mathbb{Q}(\omega,\sqrt[3]{2}) & & & \\
\mathbb{Q}(\omega) \arrow[urrr] & & \mathbb{Q}(\sqrt[3]{2}) \arrow[ur] & & \mathbb{Q}(\omega\sqrt[3]{2}) \arrow[ul] & & \mathbb{Q}(\omega^2\sqrt[3]{2}) \arrow[ulll] \\
 & & & \mathbb{Q} \arrow[ulll] \arrow[ul] \arrow[ur] \arrow[urrr] & & &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & & & \{e\} \arrow[dlll] \arrow[dl] \arrow[dr] \arrow[drrr] & & & \\
\langle\sigma\rangle \arrow[drrr] & & \langle\tau\rangle \arrow[dr] & & \langle\sigma^2\tau\rangle \arrow[dl] & & \langle\sigma\tau\rangle \arrow[dlll] \\
 & & & \mathrm{Gal}(L/\mathbb{Q}) & & &
\end{tikzcd}$$

Here σ, τ are the elements of $G = \mathrm{Gal}(L/\mathbb{Q})$, where $L = \mathbb{Q}(\omega, \sqrt[3]{2})$, determined by

$$\sigma(\omega) = \omega,\ \sigma(\sqrt[3]{2}) = \omega\sqrt[3]{2},$$

$$\tau(\omega) = \omega^2, \tau(\sqrt[3]{2}) = \sqrt[3]{2}.$$

We show that the map $K \mapsto \mathrm{Gal}(L/\mathbb{Q})$ applies the left diagram on the right diagram, the inclusion arrows are opposite by Exercise 3.

• If $K = L$, $\mathrm{Gal}(L/L) = \{e\}$, and if $K = \mathbb{Q}$, $\mathrm{Gal}(L/K) = \mathrm{Gal}(L/\mathbb{Q}) = G$.

• If $K = \mathbb{Q}(\omega)$, note that $\sigma(\omega) = \omega$, thus $\sigma(\alpha) = \alpha$ for all $\alpha \in \mathbb{Q}(\omega)$, so $\sigma \in \mathrm{Gal}(L/\mathbb{Q}(\omega))$. Therefore

$$\langle\sigma\rangle = \{e, \sigma, \sigma^2\} \subset \mathrm{Gal}(L/\mathbb{Q}(\omega)).$$

Moreover, as $\mathbb{Q} \subset L$ is a Galois extension, then $K \subset L$ is also Galois for all intermediate fields K, therefore $|\mathrm{Gal}(L/\mathbb{Q}(\omega))| = [L : \mathbb{Q}(\omega)] = 3$. Consequently

$$\langle \sigma \rangle = \{e, \sigma, \sigma^2\} = \mathrm{Gal}(L/\mathbb{Q}(\omega)).$$

- If $K = \mathbb{Q}(\sqrt[3]{2})$, then $[L : K] = 2 = |\mathrm{Gal}(L/K)|$, and $\tau \in \mathrm{Gal}(L/K)$, thus

$$\langle \tau \rangle = \{e, \tau\} = \mathrm{Gal}(L/\mathbb{Q}(\sqrt[3]{2})).$$

- If $K = \mathbb{Q}(\omega \sqrt[3]{2})$, with the same reasoning, as $\sigma^2\tau$ has order 2 and $(\sigma^2\tau)(\omega\sqrt[3]{2}) = \sigma^2(\omega^2 \sqrt[3]{2}) = \omega^4 \sqrt[3]{2} = \omega \sqrt[3]{2}$,

$$\langle \sigma^2\tau \rangle = \{e, \sigma^2\tau\} = \mathrm{Gal}(L/\mathbb{Q}(\omega \sqrt[3]{2})).$$

- If $K = \mathbb{Q}(\omega \sqrt[3]{2})$, we have a similar result, by exchanging ω with $\overline{\omega} = \omega^2$:

$$\langle \sigma\tau \rangle = \{e, \sigma\tau\} = \mathrm{Gal}(L/\mathbb{Q}(\omega^2 \sqrt[3]{2})).$$

$\square$

Ex. 7.2.5 *Prove (7.9) in the proof of Theorem 7.2.7.*

Proof. In the context of the proof of Theorem 7.2.7, $F \subset K \subset L$, L/F and K/F are Galois extensions, and $\sigma, \tau \in \mathrm{Gal}(L/F)$.

$\sigma K = K$ by Theorem 7.2.5, thus for all $\alpha \in K$, $\sigma(\alpha) \in K$.

We write here $\sigma|_K : K \to K$ the restriction (and corestriction) of σ to K, defined by $\sigma|_K(\alpha) = \sigma(\alpha)$.

For all $\alpha \in K$,

$$(\sigma|_K \circ \tau|_K)(\alpha) = \sigma|_K(\tau|_K(\alpha)) = \sigma(\tau(\alpha)) = (\sigma \circ \tau)(\alpha) = (\sigma \circ \tau)|_K(\alpha).$$

Therefore $\sigma\tau|_K = (\sigma \circ \tau)|_K = \sigma|_K \circ \tau|_K = \sigma|_K\, \tau|_K$: the map

$$\Psi : \left\{ \begin{array}{ccc} \mathrm{Gal}(L/F) & \to & \mathrm{Gal}(L/K) \\ \sigma & \mapsto & \sigma|_K \end{array} \right.$$

is a group homomorphism. $\square$

Ex. 7.2.6 *For the extension* $\mathbb{Q} \subset L = \mathbb{Q}(\omega, \sqrt[3]{2})$*, we listed some subgroups of* $\mathrm{Gal}(L/\mathbb{Q})$ *in diagram (7.7). Prove that this gives all subgroups of* $\mathrm{Gal}(L/\mathbb{Q})$.

Proof. $\langle \sigma \rangle, \langle \tau \rangle, \langle \sigma\tau \rangle, \langle \sigma^2\tau \rangle, \{e\}, G$ are subgroups of $G = \mathrm{Gal}(\mathbb{Q}(\omega, \sqrt[3]{2})/\mathbb{Q}) \simeq S_3$, corresponding to the subgroups of S_3 given by $\langle (1,2,3) \rangle, \langle (1,2) \rangle, \langle (2,3) \rangle, \langle (1,3) \rangle, \{()\}, S_3$. We show that S_3 has no other subgroup.

The order of a subgroup H of S_3 divides 6. If $|H| = 1, H = \{()\}$, if $|H| = 6, H = S_3$. If $|H| = 3$, H is cyclic of order 3. As the only elements of order 3 of S_3 are $\tilde{\sigma} = (1,2,3)$ and $(1,3,2) = \tilde{\sigma}^{-1}$, $H = \langle \tilde{\sigma} \rangle$.

If $|H| = 2$, is cyclic of order 2. The only elements of S_3 of order 2 are the three transpositions $(1,2), (2,3), (1,3)$. S_3, so $H \in \{\langle (1,2) \rangle, \langle (2,3) \rangle, \langle (1,3) \rangle\}$. S_3 has exactly 6 subgroups, therefore $\mathrm{Gal}(\mathbb{Q}(\omega, \sqrt[3]{2})/\mathbb{Q}) \simeq S_3$ has exactly six subgroups given in diagram (7.7). $\square$

Ex. 7.2.7 *Suppose that $F \subset K \subset L$, where L is Galois over F, and let $\sigma \in \mathrm{Gal}(L/F)$. Show that*

$$K = \sigma K \iff \mathrm{Gal}(L/K) = \sigma \mathrm{Gal}(L/K)\sigma^{-1},\ \sigma \text{ in } \mathrm{Gal}(L/F).$$

Proof. If $\sigma \in \mathrm{Gal}(L/F)$ satisfies $K = \sigma K$, then by Lemma 7.2.4,

$$\sigma \mathrm{Gal}(L/K)\sigma^{-1} = \mathrm{Gal}(L/\sigma K) = \mathrm{Gal}(L/K).$$

Conversely, if $\sigma \in \mathrm{Gal}(L/F)$ satisfies $\sigma \mathrm{Gal}(L/K)\sigma^{-1} = \mathrm{Gal}(L/K)$, then by the same Lemma, $\mathrm{Gal}(L/K) = \mathrm{Gal}(L/\sigma K)$. As $F \subset L$ is a Galois extension, so are $K \subset L$ and $\sigma K \subset L$, the fixed field of $\mathrm{Gal}(L/K)$ is K, and the fixed field of $\mathrm{Gal}(L/\sigma K)$ is σK. As these two groups are identical, $K = \sigma K$.

$$\forall \sigma \in \mathrm{Gal}(L/F),\ (K = \sigma K \iff \mathrm{Gal}(L/K) = \sigma \mathrm{Gal}(L/K)\sigma^{-1}).$$

(Consequently

$$(\forall \sigma \in \mathrm{Gal}(L/F), \sigma K = K) \iff \mathrm{Gal}(L/K) \lhd \mathrm{Gal}(L/F)).$$

□

Ex. 7.2.8 *Let H be a subgroup of a group G, and let $N_G(H) = \{g \in G \mid gHg^{-1} = H\}$ be the normalizer of H in G, as defined in the Mathematical Notes.*

(a) Prove that $N_G(H)$ is a subgroup of G containing H.

(b) Prove that H is normal in $N_G(H)$.

(c) Let N be a subgroup of G containing H. Prove that H is normal in N if and only if $N \subset N_G(H)$. Do you see why this shows that $N_G(H)$ is the largest subgroup of G in which H is normal?

(d) Prove that H is normal in G if and only if $N_G(H) = G$.

Proof. (a) If $x \in H, xHx^{-1} = H$, so $H \subset N_G(H)$.

- $eHe^{-1} = H$, thus $e \in N_G(H) \neq \emptyset$.
- If $x, y \in N_G(H)$, then $(xy)H(xy)^{-1} = x(yHy^{-1})x^{-1} = xHx^{-1} = H$, thus $xy \in N_G(H)$.
- If $x \in N_G(H)$, then $xHx^{-1} = H$, thus $xH = Hx$, and $H = x^{-1}Hx$: $x^{-1} \in N_G(H)$.

$N_G(H)$ is a subgroup of G.

(b) For all $g \in N_G(H)$, $gHg^{-1} = H$, so $H \lhd N_G(H)$.

(c) Let N be a subgroup of G, $H \subset N \subset G$.
$H \lhd N \iff \forall g \in N,\ gHg^{-1} = H \iff \forall g \in N, g \in N_G(H) \iff N \subset N_G(H)$.
H is normal in $N_G(H)$, and every subgroup G in which H is normal is contained in $N_G(H)$, so $N_G(H)$ is the largest subgroup of G in which H is normal.

(d) :

- If H is normal in G, then every element of G is in the normalizer of H in G, therefore $G \subset N_G(H)$. As $N_G(H) \subset G$, $N_G(H) = G$.
- If $G = N_G(H)$, then every element $g \in G$ is in $N_G(H)$, and so satisfies $gHg^{-1} = H$, so H is a normal subgroup of G.

$$G = N_G(H) \iff H \lhd G.$$

□

Ex. 7.2.9 *Let $F \subset L$ be Galois, and suppose that $F \subset K \subset L$ is an intermediate field. The goal of this exercise is to show that the number of conjugates of K in L is*

$$[\mathrm{Gal}(L/F) : N] = \frac{|\mathrm{Gal}(L/F)|}{|N|},$$

where N is the normalizer of $\mathrm{Gal}(L/K)$ in $\mathrm{Gal}(L/F)$. More precisely, suppose that the distinct conjugates of K are

$$K = \sigma_1 K, \sigma_2 K, \ldots, \sigma_r K,$$

where $\sigma_1 = e$. Then we need to show that $r = [\mathrm{Gal}(L/F) : N]$.

(a) Show that $\mathrm{Gal}(L/F)$ acts on the set of conjugates $\{\sigma_1 K, \sigma_2 K, \ldots, \sigma_r K\}$.

(b) Show that the isotropy subgroup of K is the normalizer subgroup N.

(c) Explain how $r = [\mathrm{Gal}(L/F) : N]$ follows from the Fundamental Theorem of Group Actions (Theorem A.4.9 from Appendix A).

Proof. (a) Write $O = \{\sigma_1 K, \sigma_2 K, \cdots, \sigma_r K\}$ the set of conjugate fields of K and $r = |O|$.

If $\sigma \in \mathrm{Gal}(L/F)$, and $M = K_j = \sigma_j K \in O, 1 \le j \le r$, write $\sigma \cdot M = \sigma M = \sigma K_j$:

$$\sigma \cdot K_j = \sigma \cdot (\sigma_j K) = (\sigma \circ \sigma_j) K.$$

Therefore $\sigma \cdot M = \sigma \cdot K_j$ is a conjugate field of K, so

$$M \in O \Rightarrow \sigma \cdot M \in O.$$

Moreover, for all $M \in O, e \cdot M = eM = M$, and if $\sigma, \tau \in \mathrm{Gal}(L/F), \sigma \cdot (\tau \cdot M) = \sigma(\tau M) = (\sigma \circ \tau) M = (\sigma \circ \tau) \cdot M$.

So $G = \mathrm{Gal}(L/F)$ acts on the set $O = \{\sigma_1 K, \sigma_2 K, \cdots, \sigma_r K\}$ of the conjugate fields of K, the action being defined by $\sigma \cdot M = \sigma M$ ($\sigma \in \mathrm{Gal}(L/F), M \in O$).

(b) Let G_K the stabilizer of K for this action : $G_K = \{\sigma \in G \mid \sigma K = K\}$.

By Exercise 7, for all $\sigma \in G = \mathrm{Gal}(L/F)$,

$$\sigma K = K \iff \mathrm{Gal}(L/K) = \sigma \mathrm{Gal}(L/K) \sigma^{-1} \iff \sigma \in N.$$

Thus $G_K = N$.

(c) The orbit $\mathcal{O}_K$ of K for the action of $G = \mathrm{Gal}(L/F)$ on O is the whole O, since O is by definition the set of conjugate fields of K: $\mathcal{O}_K = O$. the Fundamental Theorem of Group Actions gives then the equality

$$r = |\mathcal{O}_K| = [G : G_K] = [\mathrm{Gal}(L/F) : N].$$

The number of distinct conjugate fields of K is so the index $[G : N_G(H)]$ of the normalizer of $H = \mathrm{Gal}(L/K)$ in $G = \mathrm{Gal}(L/F)$.

□

Ex. 7.2.10 *In (7.5), explain why τ is complex conjugation restricted to $\mathbb{Q}(\omega, \sqrt[3]{2})$.*

Proof. Let $L = \mathbb{Q}(\omega, \sqrt[3]{2})$.

τ is the unique $\mathbb{Q}$-automorphism of $G = \mathrm{Gal}(L/\mathbb{Q})$ such as

$$\tau(\omega) = \omega^2, \tau(\sqrt[3]{2}) = \sqrt[3]{2}.$$

If $z \in \mathbb{C}$ is element of L, then $z = p(\omega, \sqrt[3]{2})$, where $p(x, y) \in \mathbb{Q}[x, y]$, thus $\overline{z} = p(\overline{\omega}, \sqrt[3]{2}) = p(-1 - \omega, \sqrt[3]{2}) \in L$. Let $\lambda : L \to L, z \mapsto \overline{z}$ the restriction (and corestriction) of the conjugation in $\mathbb{C}$. Then λ is an involutive ring homomorphism, thus an automorphism of the field L, which is the identity on $\mathbb{Q}$: $\lambda \in \mathrm{Gal}(L/\mathbb{Q})$. As

$$\lambda(\omega) = \omega^2, \lambda(\sqrt[3]{2}) = \sqrt[3]{2},$$

and as a $\mathbb{Q}$-automorphism of $L = \mathbb{Q}(\omega, \sqrt[3]{2})$ is uniquely determined by the images of $\omega, \sqrt[3]{2}$, $\tau = \lambda$, so τ is the complex conjugation restricted to $\mathbb{Q}(\omega, \sqrt[3]{2})$. $\square$

Ex. 7.2.11 *Consider the extension $\mathbb{Q} \subset L = \mathbb{Q}(\sqrt{2}, \sqrt{3})$.*

(a) Show that $\mathrm{Gal}(L/\mathbb{Q}) = \{e, \sigma, \tau, \sigma\tau\}$, where

$$\begin{aligned} \sigma(\sqrt{2}) &= \sqrt{2}, & \sigma(\sqrt{3}) &= -\sqrt{3}, \\ \tau(\sqrt{2}) &= -\sqrt{2}, & \tau(\sqrt{3}) &= \sqrt{3}. \end{aligned}$$

(b) Find all subgroups of $\mathrm{Gal}(L/\mathbb{Q})$, and use this to draw a picture similar to (7.7).

(c) For each subgroup of part (b), determine the corresponding subfield of L and use this to draw a picture similar to (7.3).

(d) Explain why all of the subgroups in part (b) are normal. What does this imply about the subfields in part (c)?

Proof. (a) We have proved in Exercise 6.1.2 that $|\mathrm{Gal}(L/\mathbb{Q})| = 4$, and

$$\mathrm{Gal}(\mathbb{Q}(\sqrt{2}, \sqrt{3})/\mathbb{Q}) = \{1_L, \sigma, \tau, \sigma\tau\}.$$

where

$$\begin{aligned} \sigma(\sqrt{2}) &= \sqrt{2}, & \sigma(\sqrt{3}) &= -\sqrt{3}, \\ \tau(\sqrt{2}) &= -\sqrt{2}, & \tau(\sqrt{3}) &= \sqrt{3}. \end{aligned}$$

and (Ex. 6.2.1) that $G = \mathrm{Gal}(L/\mathbb{Q}) \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2/Z$.

(b) The subgroups of $G = \mathrm{Gal}(L/\mathbb{Q})$ are $\{e\}, G, \langle\sigma\rangle = \{e, \sigma\}, \langle\tau\rangle = \{e, \tau\}, \langle\sigma\tau\rangle = \{e, \sigma\tau\}$.

$$\begin{tikzcd} & \{e\} \arrow[dl] \arrow[d] \arrow[dr] & \\ \langle\sigma\rangle \arrow[dr] & \langle\sigma\tau\rangle \arrow[d] & \langle\tau\rangle \arrow[dl] \\ & \mathrm{Gal}(L/\mathbb{Q}) & \end{tikzcd} \qquad \begin{tikzcd} & \mathbb{Q}(\sqrt{2}, \sqrt{3}) & \\ \mathbb{Q}(\sqrt{2}) \arrow[ur] & \mathbb{Q}(\sqrt{6}) \arrow[u] & \mathbb{Q}(\sqrt{3}) \arrow[ul] \\ & \mathbb{Q} \arrow[ul] \arrow[u] \arrow[ur] & \end{tikzcd}$$

(c) We obtain the right diagram from the left diagram by the map $H \mapsto L_H$. Explicitely:

$L_{\{e\}} = L$, and as $\mathbb{Q} \subset L$ is Galois, $L_G = \mathbb{Q}$.

As $(1, \sqrt{3})$ is a basis of L over $\mathbb{Q}(\sqrt{2})$, a basis of the $\mathbb{Q}$-vector space L is $(1, \sqrt{2}, \sqrt{3}, \sqrt{6})$. Let $\alpha = a + b\sqrt{2} + c\sqrt{3} + d\sqrt{6}$ $(a, b, c, d \in \mathbb{Q})$ any element of L. Then

$$\begin{aligned} \sigma(\alpha) = \alpha &\iff a + b\sqrt{2} - c\sqrt{3} - d\sqrt{6} = a + b\sqrt{2} + c\sqrt{3} + d\sqrt{6} \\ &\iff c = d = 0 \\ &\iff \alpha \in \mathbb{Q}(\sqrt{2}) \end{aligned}$$

thus $L_{\langle\sigma\rangle} = \mathbb{Q}(\sqrt{2})$. We verify similarly $L_{\langle\tau\rangle} = \mathbb{Q}(\sqrt{3})$.

We compute $L_{\langle\sigma\tau\rangle}$:

$$\begin{aligned} (\sigma\tau)(\alpha) = \alpha &\iff a - b\sqrt{2} - c\sqrt{3} + d\sqrt{6} = a + b\sqrt{2} + c\sqrt{3} + d\sqrt{6} \\ &\iff b = c = 0 \\ &\iff \alpha \in \mathbb{Q}(\sqrt{6}) \end{aligned}$$

We obtain the left diagram from the right diagram by the map $K \mapsto \mathrm{Gal}(L/K)$. By example, the only elements of G who fix $\mathbb{Q}(\sqrt{2})$ are e and σ.

(d) G is Abelian, so all its subgroups are normal.

This implies (Theorem 7.2.5) that $\mathbb{Q}(\sqrt{2})$ equals all of its conjugates and so is a normal extension of $\mathbb{Q}$. Same conclusion for $\mathbb{Q}(\sqrt{3}), \mathbb{Q}(\sqrt{6})$.

□

7.3 THE FUNDAMENTAL THEOREM OF GALOIS THEORY

Ex. 7.3.1 *Complete the proof of Theorem 7.3.1 by showing that* $[\mathrm{Gal}(L/F) : H] = [L_H : F]$ *for all subgroups* $H \subset \mathrm{Gal}(L/F)$.

Proof. By hypothesis, $F \subset L$ is a Galois extension, and H is a subgroup of $\mathrm{Gal}(L/F)$. The proof of Theorem 7.3.1 shows that $L_H \subset L$ is Galois and $H = \mathrm{Gal}(L/L_H)$, thus $|H| = |\mathrm{Gal}(L/L_H)| = [L : L_H]$.

Since $F \subset L$ is a Galois extension,

$$|\mathrm{Gal}(L/F)| = [L : F] = [L : L_H]\,[L_H : F] = |H|\,[L_H : F],$$

therefore

$$[\mathrm{Gal}(L/F) : H] = |\mathrm{Gal}(L/F)|/|H| = [L_H : F].$$

□

Ex. 7.3.2 *Same as Ex. 6.3.2(b).*

Proof. The Exercise 6.3.2(b) proves in details that $\mathrm{Gal}(\mathbb{Q}(i, \sqrt[4]{2})/\mathbb{Q}) = \langle\sigma, \tau\rangle \simeq D_8$, where $\sigma(i) = i, \sigma(\sqrt[4]{2}) = i\sqrt[4]{2}$ and $\tau(i) = -i, \tau(\sqrt[4]{2}) = \sqrt[4]{2}$ (τ is the complex conjugation restricted to $\mathbb{Q}(i, \sqrt[4]{2})$). □

Ex. 7.3.3 *Let $L = \mathbb{Q}(i\sqrt[4]{2})$ and $\sigma, \tau \in \mathrm{Gal}(L/\mathbb{Q})$ be as in Exercise 2 and Example 7.3.4.*

(a) Show that all subgroups of $\mathrm{Gal}(L/\mathbb{Q})$ are given by (7.13).

(b) Show that the corresponding fixed fiels are given by (7.14).

(c) Determine which subgroups in part (a) are normal in $\mathrm{Gal}(L/\mathbb{Q})$, and for those that are normal, construct a polynomial whose splitting field is the corresponding fixed field.

(d) For the subfields in part (b) that are not Galois over $\mathbb{Q}$, find all of their conjugates fields. Also describe the conjugates of their corresponding groups.

$$\begin{tikzcd}
 & & \{e\} \arrow[dll] \arrow[dl] \arrow[d] \arrow[dr] \arrow[drr] & & \\
\langle \tau \rangle \arrow[dr] & \langle \sigma^2\tau \rangle \arrow[d] & \langle \sigma^2 \rangle \arrow[dl] \arrow[d] \arrow[dr] & \langle \sigma\tau \rangle \arrow[d] & \langle \sigma^3\tau \rangle \arrow[dl] \\
 & \langle \sigma^2, \tau \rangle \arrow[dr] & \langle \sigma \rangle \arrow[d] & \langle \sigma^2, \sigma\tau \rangle \arrow[dl] & \\
 & & G = \langle \sigma, \tau \rangle & &
\end{tikzcd}$$

$$\begin{tikzcd}
 & & \mathbb{Q}(i, \sqrt[4]{2}) & & \\
\mathbb{Q}(\sqrt[4]{2}) \arrow[urr] & \mathbb{Q}(i\sqrt[4]{2}) \arrow[ur] & \mathbb{Q}(i, \sqrt{2}) \arrow[u] & \mathbb{Q}((1+i)\sqrt[4]{2}) \arrow[ul] & \mathbb{Q}((1-i)\sqrt[4]{2}) \arrow[ull] \\
 & \mathbb{Q}(\sqrt{2}) \arrow[ul] \arrow[u] \arrow[ur] & \mathbb{Q}(i) \arrow[u] & \mathbb{Q}(i\sqrt{2}) \arrow[ul] \arrow[u] \arrow[ur] & \\
 & & \mathbb{Q} \arrow[ul] \arrow[u] \arrow[ur] & &
\end{tikzcd}$$

Proof. (a) We obtain the subgroups of D_8 and their inclusions with the following GAP instructions:

```
S:=Group((1,2,3,4),(1,3));
T:=Group(());
L:=IntermediateSubgroups(S,T).subgroups;
i:=1;
```

```
for H in L do
   Print(i, " : \t",StructureDescription(H),"\t",Order(H),"\t", H,"\t","\n");
   i:=i+1;
od;
Print("inclusions : \n",IntermediateSubgroups(S,T).inclusions);
```

On obtient :

```
1 :  C2 2 Group( [ (1,3)(2,4) ] )
2 :  C2 2 Group( [ (2,4) ] )
3 :  C2 2 Group( [ (1,3) ] )
4 :  C2 2 Group( [ (1,2)(3,4) ] )
5 :  C2 2 Group( [ (1,4)(2,3) ] )
6 :  C2 x C2 4 Group( [ (1,3)(2,4), (2,4) ] )
7 :  C4 4 Group( [ (1,3)(2,4), (1,2,3,4) ] )
8 :  C2 x C2 4 Group( [ (1,3)(2,4), (1,2)(3,4) ] )
inclusions :
[ [ 0, 1 ], [ 0, 2 ], [ 0, 3 ], [ 0, 4 ], [ 0, 5 ], [ 1, 6 ], [ 2, 6 ],
[ 3, 6 ], [ 1, 7 ], [ 1, 8 ], [ 4, 8 ], [ 5, 8 ], [ 6, 9 ], [ 7, 9 ],
 [ 8, 9 ] ]
```

This corresponds to the lattice of subgroups of G written in the first diagram (the node (1) corresponding to the subgroup generated by $\sigma^2 = (1,3)(2,4)$).

We find again these results directly without computer in

$$D_8 = \langle \sigma, \tau \rangle = \{e, \sigma, \sigma^2, \sigma^3, \tau, \sigma\tau, \sigma^2\tau, \sigma^3\tau\} \simeq G,$$

where $\sigma = (1,2,3,4), \tau = (1,3)$ (cf Ex. 6.3.2(b)). Here the numbering of the roots is

$$z_1 = i\sqrt[4]{2}, z_2 = -\sqrt[4]{2}, z_3 = -i\sqrt[4]{2}, z_4 = \sqrt[4]{2},$$

so τ, which exchanges z_1, z_3 corresponds to the transposition $(1,3)$, and σ to the 4-cycle $(1,2,3,4)$.

σ is of order 4 and generates $H = \langle \sigma \rangle = \{e, \sigma, \sigma^2, \sigma^3\}$, τ is of order 2, and $\sigma\tau = \tau\sigma^{-1} = (1,4)(2,3)$:

$$\sigma^4 = \tau^2 = e, \qquad \sigma\tau = \tau\sigma^{-1}.$$

Note that $\tau\sigma = \sigma^{-1}\tau$ and $\tau\sigma^k = \sigma^{-k}\tau \Rightarrow \tau\sigma^{k+1} = \sigma^{-k}\tau\sigma = \sigma^{-k-1}\tau$. This induction proves that $\tau\sigma^k = \sigma^{-k}\tau$ for all $k \in \mathbb{N}$. Moreover $(\sigma^k\tau)^2 = \sigma^k\tau\sigma^k\tau = \sigma^k\sigma^{-k}\tau\tau = e$, so all the elements of the right coset $H\tau$ are of order 2.

We find all the subgroups of order 2 by checking the elements of order 2 in D_8. They are the elements of $H.\tau = \{\tau, \sigma\tau, \sigma^2\tau, \sigma^3\tau\}$, and also $\sigma^2 \in H$: this gives all the subgroups of level 2 in the first diagram.

We know a subgroup of G of order 4, the subgroup $H = \langle \sigma \rangle$.

Let M any subgroup of G of order 4. If M is cyclic, it is generated by an element of order 4, so $M = H = \langle \sigma \rangle = \langle \sigma^3 \rangle$.

Otherwise M is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, generated by to distinct elements of order 2 in $D_8 \simeq G$. If one of these elements is σ^2, we obtain the two subgroups

$$\begin{aligned} H_1 &= \langle \sigma^2, \tau \rangle = \{e, \sigma^2, \tau, \sigma^2\tau\} = \langle \sigma^2, \sigma^2\tau \rangle \\ H_2 &= \langle \sigma^2, \sigma\tau \rangle = \{e, \sigma^2, \sigma\tau, \sigma^3\tau\} = \langle \sigma^2, \sigma^3\tau \rangle. \end{aligned}$$

Otherwise $M = \langle \sigma^k\tau, \sigma^l\tau \rangle,\ 1 \leq k, l \leq 3, k \neq l$. As $\sigma^k\tau\sigma^l\tau = \sigma^{k-l} \in H$ is of order 2, $\sigma^{k-l} = \sigma^2$ and so

$$M = \{e, \sigma^k\tau, \sigma^l\tau, \sigma^2\}.$$

Since $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ is generated by any pair of elements not equal to e, $M = \langle \sigma^2, \sigma^k\tau \rangle$, and so $M = H_1$ where $M = H_2$. We find again the subgroups of diagram 1.

(b) With find the fixed fields L_M corresponding with the subgroups M of G. Consider the chain of fields going from $\mathbb{Q}$ to L :

$$\mathbb{Q} \subset \mathbb{Q}(\sqrt{2}) \subset \mathbb{Q}(i\sqrt[4]{2}) \subset \mathbb{Q}(i\sqrt[4]{2}, \sqrt[4]{2}) = \mathbb{Q}(i, \sqrt[4]{2}) = L,$$

where each field is a quadratic extension of the preceding field. Write

$$\alpha = \sqrt{2}, \beta = -i\sqrt[4]{2}, \gamma = \sqrt[4]{2}$$

(the symbol $-$ for β is intended for obtaining $\sigma(\beta) = \gamma$. If we number the roots of $x^4 - 2$ by $x_1 = \beta, x_2 = \gamma, x_3 = -\beta, x_4 = -\gamma$, the permutations corresponding to σ, τ are $\tilde{\sigma} = (1, 2, 3, 4), \tilde{\tau} = (2, 4)$, with $D_8 = \langle (1, 2, 3, 4), (2, 4) \rangle$).

Then $(1, \alpha)$ is a basis $\mathbb{Q}(\sqrt{2})$ over $\mathbb{Q}$, $(1, \beta)$ a basis of $\mathbb{Q}(i\sqrt[4]{2})$ over $\mathbb{Q}(\sqrt{2})$, and $(1, \gamma)$ a basis of $\mathbb{Q}(i\sqrt[4]{2}, \sqrt[4]{2})$ over $\mathbb{Q}(i\sqrt[4]{2})$, thus

$$\mathcal{B} = (1, \alpha, \beta, \gamma, \alpha\beta, \alpha\gamma, \beta\gamma, \alpha\beta\gamma)$$

is a basis of L over $\mathbb{Q}$.

Recall that (see Ex. 6.3.2(b))

$$\sigma(i) = i, \sigma(\sqrt[4]{2}) = i\sqrt[4]{2},$$

$$\tau(i) = -i, \tau(\sqrt[4]{2}) = \sqrt[4]{2}.$$

Consequently, $\sigma(\sqrt{2}) = (\sigma(\sqrt[4]{2}))^2 = -\sqrt{2}$,

$$\begin{aligned} &\sigma(\alpha) = -\alpha, \sigma(\beta) = \gamma, \sigma(\gamma) = -\beta, \\ &\tau(\alpha) = \alpha, \tau(\beta) = -\beta, \tau(\gamma) = \gamma. \end{aligned}$$

Every element $z \in L$ spans on the basis $\mathcal{B}$ under the form

$$z = a_1 + a_2\alpha + a_3\beta + a_4\gamma + a_5\alpha\beta + a_6\beta\gamma + a_7\alpha\gamma + a_8\alpha\beta\gamma$$

(where $a_i \in \mathbb{Q}$)

- Computation of $L_{\langle\sigma\rangle}$

$$\begin{aligned} z &= a_1 + a_2\alpha + a_3\beta + a_4\gamma + a_5\alpha\beta + a_6\beta\gamma + a_7\alpha\gamma + a_8\alpha\beta\gamma \\ \sigma(z) &= a_1 - a_2\alpha + a_3\gamma - a_4\beta - a_5\alpha\gamma - a_6\beta\gamma + a_7\alpha\beta + a_8\alpha\beta\gamma \end{aligned}$$

$$\begin{aligned} z \in L_{\langle\sigma\rangle} &\iff 0 = z - \sigma(z) \\ &\iff 0 = 2a_2\alpha + (a_3 + a_4)\beta + (-a_3 + a_4)\gamma + (a_5 - a_7)\alpha\beta + (a_7 + a_5)\alpha\gamma + 2a_6\beta\gamma \\ &\iff a_2 = a_3 = a_4 = a_5 = a_6 = a_7 = 0 \\ &\iff z = a_1 + a_8\alpha\beta\gamma, \quad a_1, a_8 \in \mathbb{Q} \\ &\iff z \in \mathbb{Q}[\alpha\beta\gamma] \end{aligned}$$

$$L_{\langle\sigma\rangle} = \mathbb{Q}(\alpha\beta\gamma) = \mathbb{Q}(i)$$

As expected, this is a quadratic extension of $\mathbb{Q}$, corresponding with a subgroup of index 2 in G.

- Computation of $L_{\langle\tau\rangle}$

$$\begin{aligned} z &= a_1 + a_2\alpha + a_3\beta + a_4\gamma + a_5\alpha\beta + a_6\beta\gamma + a_7\alpha\gamma + a_8\alpha\beta\gamma \\ \tau(z) &= a_1 + a_2\alpha - a_3\beta + a_4\gamma - a_5\alpha\beta - a_6\beta\gamma + a_7\alpha\gamma - a_8\alpha\beta\gamma \end{aligned}$$

$$\begin{aligned} z \in L_{\langle\tau\rangle} &\iff 0 = z - \tau(z) \\ &\iff 0 = a_3 = a_5 = a_6 = a_8 = 0 \\ &\iff z = a_1 + a_2\alpha + a_4\gamma + a_7\alpha\gamma \ (a_i \in \mathbb{Q}) \\ &\iff z \in \mathbb{Q}(\alpha, \gamma) \\ &\iff z \in \mathbb{Q}(\gamma) \end{aligned}$$

(indeed $\alpha \in \mathbb{Q}(\gamma)$).

$$L_{\langle\tau\rangle} = \mathbb{Q}(\gamma) = \mathbb{Q}(\sqrt[4]{2}).$$

- Computation of $L_{\langle\sigma^2\rangle}$

$$\sigma^2(\alpha) = \alpha, \sigma^2(\beta) = -\beta, \sigma^2(\gamma) = -\gamma.$$

$$\begin{aligned} z &= a_1 + a_2\alpha + a_3\beta + a_4\gamma + a_5\alpha\beta + a_6\beta\gamma + a_7\alpha\gamma + a_8\alpha\beta\gamma \\ \sigma^2(z) &= a_1 + a_2\alpha - a_3\beta - a_4\gamma - a_5\alpha\beta + a_6\beta\gamma - a_7\alpha\gamma + a_8\alpha\beta\gamma \end{aligned}$$

$$\begin{aligned} z \in L_{\langle\sigma^2\rangle} &\iff 0 = z - \sigma^2(z) \\ &\iff 0 = a_3 = a_4 = a_5 = a_7 \\ &\iff z = a_1 + a_2\alpha + a_6\beta\gamma + a_8\alpha\beta\gamma \ (a_i \in \mathbb{Q}) \\ &\iff z \in \mathbb{Q}(\alpha, \beta\gamma) \end{aligned}$$

$$L_{\langle\sigma^2\rangle} = \mathbb{Q}(\alpha, \beta\gamma) = \mathbb{Q}(\sqrt{2}, i\sqrt{2}) = \mathbb{Q}(i, \sqrt{2})$$

- Computation of $L_{\langle\sigma^2\tau\rangle}$

$$(\sigma^2\tau)(\alpha) = \alpha, (\sigma^2\tau)(\beta) = \beta, (\sigma^2\tau)(\gamma) = -\gamma$$

$$\begin{aligned} z &= a_1 + a_2\alpha + a_3\beta + a_4\gamma + a_5\alpha\beta + a_6\beta\gamma + a_7\alpha\gamma + a_8\alpha\beta\gamma \\ (\sigma^2\tau)(z) &= a_1 + a_2\alpha + a_3\beta - a_4\gamma + a_5\alpha\beta - a_6\beta\gamma - a_7\alpha\gamma - a_8\alpha\beta\gamma \end{aligned}$$

$$\begin{aligned} z \in L_{\langle\sigma^2\tau\rangle} &\iff 0 = z - (\sigma^2\tau)(z) \\ &\iff a_4 = a_6 = a_7 = a_8 = 0 \\ &\iff z = a_1 + a_2\alpha + a_3\beta + a_5\alpha\beta \ (a_i \in \mathbb{Q}) \\ &\iff z \in \mathbb{Q}(\alpha, \beta) \\ &\iff z \in \mathbb{Q}(\beta) \end{aligned}$$

$$L_{\langle\sigma^2\tau\rangle} = \mathbb{Q}(\beta) = \mathbb{Q}(i\sqrt[4]{2}).$$

- Computation of $L_{\langle\sigma^2,\tau\rangle}$

$$\begin{aligned} z \in L_{\langle\sigma^2,\tau\rangle} &\iff z = \sigma^2(z) \text{ et } z = \tau(z) \\ &\iff a_3 = a_4 = a_5 = a_7 = 0 \text{ et } a_3 = a_5 = a_6 = a_8 = 0 \\ &\iff a_3 = a_4 = a_5 = a_6 = a_7 = a_8 = 0 \\ &\iff z = a_1 + a_2\alpha, \ (a_1, a_2 \in \mathbb{Q}) \\ &\iff z \in \mathbb{Q}(\alpha) \end{aligned}$$

$$L_{\langle\sigma^2,\tau\rangle} = \mathbb{Q}(\alpha) = \mathbb{Q}(\sqrt{2}).$$

- Computation of $L_{\langle\sigma^3\tau\rangle}$

$$(\sigma^3\tau)(\alpha) = -\alpha, (\sigma^3\tau)(\beta) = \gamma, (\sigma^3\tau)(\gamma) = \beta$$

$$\begin{aligned} z &= a_1 + a_2\alpha + a_3\beta + a_4\gamma + a_5\alpha\beta + a_6\beta\gamma + a_7\alpha\gamma + a_8\alpha\beta\gamma \\ (\sigma^3\tau)(z) &= a_1 - a_2\alpha + a_3\gamma + a_4\beta - a_5\alpha\gamma + a_6\beta\gamma - a_7\alpha\beta - a_8\alpha\beta\gamma \end{aligned}$$

$$\begin{aligned} z \in L_{\langle\sigma^3\tau\rangle} &\iff 0 = z - (\sigma^3\tau)(z) \\ &\iff 2a_2\alpha + (a_3 - a_4)\beta + (a_4 - a_3)\gamma + (a_5 + a_7)\alpha\beta + (a_7 + a_5)\alpha\gamma + 2a_8\alpha\beta\gamma \\ &\iff a_2 = a_8 = 0 \text{ et } a_3 = a_4 \text{ et } a_7 = -a_5 \\ &\iff z = a_1 + a_3(\beta + \gamma) + a_5\alpha(\beta - \gamma) + a_6\beta\gamma \ (a_i \in \mathbb{Q}) \\ &\iff z \in \mathbb{Q}(\beta + \gamma) \end{aligned}$$

We justify this last equivalence:
$(\beta + \gamma)[\alpha(\beta - \gamma)] = -4 \in \mathbb{Q}^*$, thus $\alpha(\beta - \gamma) \in \mathbb{Q}(\beta + \gamma)$, and $(\beta + \gamma)^2 = \beta^2 + \gamma^2 + 2\beta\gamma = 2\beta\gamma$, so $\beta\gamma \in \mathbb{Q}(\beta + \gamma)$.

$$L_{\langle\sigma^3\tau\rangle} \subset \mathbb{Q}(\beta+\gamma).$$

Conversely $L_{\langle\sigma^3\tau\rangle}$ is a field (fixed field of $\langle\sigma^3\tau\rangle$), extension of $\mathbb{Q}$ containing $\beta+\gamma$. So it contains also $\mathbb{Q}(\beta+\gamma)$.

$$L_{\langle\sigma^3\tau\rangle} \supset \mathbb{Q}(\beta+\gamma).$$

Conclusion:

$$L_{\langle\sigma^3\tau\rangle} = \mathbb{Q}(\beta+\gamma) = \mathbb{Q}((1-i)\sqrt[4]{2}).$$

- Computation of $L_{\langle\sigma\tau\rangle}$

$$(\sigma\tau)(\alpha) = -\alpha, (\sigma\tau)(\beta) = -\gamma, (\sigma\tau)(\gamma) = -\beta$$

$$\begin{aligned} z &= a_1 + a_2\alpha + a_3\beta + a_4\gamma + a_5\alpha\beta + a_6\beta\gamma + a_7\alpha\gamma + a_8\alpha\beta\gamma \\ (\tau\circ\sigma^3)(z) &= a_1 - a_2\alpha - a_3\gamma - a_4\beta + a_5\alpha\gamma + a_6\beta\gamma + a_7\alpha\beta - a_8\alpha\beta\gamma \end{aligned}$$

$$\begin{aligned} z \in L_{\langle\sigma\tau\rangle} &\iff 0 = z - (\sigma\tau)(z) \\ &\iff 2a_2\alpha + (a_3+a_4)\beta + (a_4+a_3)\gamma + (a_5-a_7)\alpha\beta + (a_7-a_5)\alpha\gamma + 2a_8\alpha\beta\gamma \\ &\iff a_2 = a_8 = 0 \text{ et } a_3 = -a_4 \text{ et } a_7 = a_5 \\ &\iff z = a_1 + a_3(\beta-\gamma) + a_5\alpha(\beta+\gamma) + a_6\beta\gamma \ (a_i \in \mathbb{Q}) \\ &\iff z \in \mathbb{Q}(\beta-\gamma) \end{aligned}$$

(with a similar justification, by exchanging γ and $-\gamma$)
Conclusion:

$$L_{\langle\sigma\tau\rangle} = \mathbb{Q}(\beta-\gamma) = \mathbb{Q}((1-i)\sqrt[4]{2})$$

- Computation of $L_{\langle\sigma^2,\sigma\tau\rangle}$

$$\begin{aligned} z \in L_{\langle\sigma^2,\sigma\tau\rangle} &\iff z = \sigma^2(z) \text{ et } z = (\sigma\tau)(z) \\ &\iff a_3 = a_4 = a_5 = a_7 = 0 \text{ et } a_2 = a_8 = 0 \\ &\iff z = a_1 + a_6\beta\gamma,\ (a_1, a_6 \in \mathbb{Q}) \\ &\iff z \in \mathbb{Q}(\beta\gamma) \end{aligned}$$

$$L_{\langle\sigma^2,\sigma\tau\rangle} = \mathbb{Q}(\beta\gamma) = \mathbb{Q}(i\sqrt{2}).$$

We obtain so all the fields of the second diagram.

(c) The three subgroups of order 4 have the index 2 in G, therefore are normal subgroups of G. They correspond with three quadratic extensions of $\mathbb{Q}$, which are Galois extensions as every quadratic extension of $\mathbb{Q}$.

$\mathbb{Q}(\sqrt{2})$ is the splitting field of x^2-2 over $\mathbb{Q}$, $\mathbb{Q}(i)$ the splitting field of x^2+1, and $\mathbb{Q}(i\sqrt{2})$ the splitting field of x^2+2.

The subgroup $H = \langle\sigma^2\rangle$ is normal in $G = \langle\sigma,\tau\rangle$, since

$$\tilde{\tau}\tilde{\sigma}^2\tilde{\tau}^{-1} = (2,4)(1,3)(2,4)(2,4) = (2,4)(1,3) = (1,3)(2,4) = \tilde{\sigma}^2,$$

thus $\tau\sigma^2\tau^{-1} = \sigma^2 \in H$ (and of course $\sigma\sigma^2\sigma^{-1} = \sigma^2 \in H$). H corresponds with $\mathbb{Q}(i, \sqrt{2})$, which is so a Galois extension of $\mathbb{Q}$. $\mathbb{Q}(i, \sqrt{2})$ is the splitting field of the irreducible polynomial

$$x^4 - 2x^2 + 9 = (x - i - \sqrt{2})(x - i + \sqrt{2})(x + i - \sqrt{2})(x + i + \sqrt{2})$$

(or of the reducible polynomial $(x^2 - 2)(x^2 + 1)$).

These are the only normal subgroups of G, as we will see in part (d).

(d) As $\tau\sigma^{-1} = \sigma\tau$, then $\sigma\tau\sigma^{-1} = \sigma^2\tau$, so the subgroups $\langle\tau\rangle$ and $\langle\sigma^2\tau\rangle$ are conjugate, thus are not normal subgroups of G.

Similarly $\sigma^3\tau = \sigma^{-1}\tau = \tau\sigma = \sigma^{-1}(\sigma)\tau)\sigma$, so the subgroups $\langle\sigma^3\tau\rangle$ and $\langle\sigma\tau\rangle$ are conjugate, and are not normal subgroups.

The subgroups $\langle\tau\rangle$ and $\langle\sigma\tau\rangle$ are not conjugate, since τ corresponds to $(2,4)$, and $\sigma\tau$ to the permutation $(1,2)(3,4)$ which are not conjugate, since the conjugate of a transposition is a transposition.

So the corresponding extensions $\mathbb{Q}(\sqrt[4]{2}), \mathbb{Q}(i\sqrt[4]{2}), \mathbb{Q}((1+i)\sqrt[4]{2}), \mathbb{Q}((1-i)\sqrt[4]{2})$ are not Galois extensions of $\mathbb{Q}$.

□

Ex. 7.3.4 *Prove that the extension $F \subset L$ of Example 7.3.6 has* $\mathrm{Gal}(L/F) = \{1_L\}$.

Proof. In Example 7.3.6, k has characteristic p, and the extension L of $F = k(t, u)$ is the splitting field of $f = (x^p - t)(x^p - u) \in F[x]$.

We showed in Exercise 5.4.4 that $F \subset L$ is purely inseparable, and $L = F(\alpha, \beta)$, where $\alpha^p = t, \beta^p = u$. Moreover the intermediate fields $F \subset F(\alpha + \lambda\beta) \subset L$ are distinct.

Now we show that $\mathrm{Gal}(L/F) = \{1_L\}$.

α is a root $x^p - t \in F[x]$, thus $\sigma(\alpha)$ is also a root. Since $x^p - t = (x - \alpha)^p$ has the only root α, $\sigma(\alpha) = \alpha$.

Similarly β is the only root of $x^p - u = (x - \beta)^p$, thus $\sigma(\beta) = \beta$.

Moreover $L = F(\alpha, \beta)$, so an element $\sigma \in \mathrm{Gal}(L/F)$ is uniquely determined by the images of α, β, therefore $\sigma = 1_L$.

$$\mathrm{Gal}(L/F) = \{1_L\}.$$

□

Ex. 7.3.5 *Consider the extension $F = \mathbb{C}(t^4) \subset L = \mathbb{C}(t)$, where t is a variable.*

(a) Show that L is the splitting field of $x^4 - t^4 \in F[x]$ over F.

(b) Show that $x^4 - t^4$ is irreducible over F.

(c) Show that $\mathrm{Gal}(L/F) \simeq \mathbb{Z}/4\mathbb{Z}$.

(d) Similar to what you did in Exercise 3, determine all subgroups of $\mathrm{Gal}(L/F)$ *and the corresponding intermediate fields between F and L.*

Proof. Consider the extension $F \subset \mathbb{C}(t^4) \subset L = \mathbb{C}(t)$, where t is a variable.

(a) $t^4 \in F$, thus $f = x^4 - t^4 \in F[x]$, and $f = (x-t)(x+t)(x-it)(x+it)$.

f splits completely on L, and the roots of f in L are $t, it, -t, -it$. The splitting field of f over $\mathbb{C}(t^4)$ is so $\mathbb{C}(t^4)(t, it, -t, -it) = \mathbb{C}(t^4, t) = \mathbb{C}(t)$, since $t^4 \in \mathbb{C}(t)$.

$\mathbb{C}(t)$ being the splitting field of the separable polynomial f over $\mathbb{C}(t^4)$, $\mathbb{C}(t^4) \subset \mathbb{C}(t)$ is a Galois extension.

(b) $t \notin \mathbb{C}(t^4)$, otherwise $t = u(t^4)/v(t^4),\ u, v \in F[x], v \neq 0$, where t is transcendental over $\mathbb{C}$, and the identity $u(t^4) - tv(t^4) = 0$ is impossible, since all the monomials in $u(t^4)$ have even degree, and all the monomial in $tv(t^4) \neq 0$ have odd degree. Consequently the other roots of f in L, which are $-t, it, -it$, are not in $\mathbb{C}(t^4)$.

If f was reducible over F, f would be the product of two polynomials $p, q \in F[x]$ of degree 2, each gathering two factors of the form $x - i^k t$:

$$p = (x - i^k t)(x - i^l t) \in F[x],\ 0 \leq k, l \leq 3.$$

But then the coefficient of degree 0 in x, which is $i^{k+l}t^2$ is in F, therefore $t^2 \in F$:

$$t^2 = \frac{u(t^4)}{v(t^4)},\ u, v \in F[x], v \neq 0, u \wedge v = 1.$$

Then $s = t^2$ is transcendental over $\mathbb{C}$ (otherwise t would be algebraic over $\mathbb{C}$) and satisfies

$$s = \frac{u(s^2)}{v(s^2)}.$$

The identity $u(s^2) - sv(s^2) = 0$, with s transcendental, implies $u(x^2) - xv(x^2)$ where x is a variable, so is impossible, since all the monomials in $u(x^2)$ have even degree, and all the monomial in $xv(x^2) \neq 0$ have odd degree.

$f = x^4 - t^4$ is so irreducible over $\mathbb{C}(t^4)$.

(c) $\deg(f) = 4$, and f is monic irreducible, so is the minimal polynomial of t over F. Therefore

$$|\mathrm{Gal}(L/F)| = [L : F] = [\mathbb{C}(t^4, t) : \mathbb{C}(t^4)] = \deg(f) = 4.$$

Let $\sigma \in G = \mathrm{Gal}(L/F)$. As t is a root of $f \in F[x]$, $\sigma(t)$ is a root of f, thus $\sigma(t) \in \{t, it, i^2t, i^3t\}$. Moreover $L = \mathbb{C}(t) = \mathbb{C}(t^4)(t)$, so σ is uniquely determined by the image of t. As $|G| = 4$, these four possibilities occur and correspond to an element of G: if $0 \leq k \leq 3$, there exists one and only one $\sigma_k \in G$ such that

$$\sigma_k(t) = i^k t.$$

Let $\sigma = \sigma_1 : t \mapsto it$. Then $\sigma^k(t) = i^k t = \sigma_k(t)$, so $\sigma^k = \sigma_k$, and $G = \langle \sigma \rangle$ is cyclic.

$$G = \{e, \sigma, \sigma^2, \sigma^3\} \simeq \mathbb{Z}/4\mathbb{Z}.$$

(d) The only non trivial subgroup of G is $H = \langle \sigma^2 \rangle = \{e, \sigma^2\}$,where G is an Abelian group, so H is normal in G. Let L_H its fixed field. By the Fundamental Theorem of Galois Theory, there exists so a unique intermediate field distinct of $\mathbb{C}(t^4)$ and $\mathbb{C}(t)$, which is so $\mathbb{C}(t^2)$:

$$L_H = \mathbb{C}(t^2).$$

The Galois correspondence is between the two chains:

$$\mathbb{C}(t^4) \subset \mathbb{C}(t^2) \subset \mathbb{C}(t)$$

$$G = \langle \sigma \rangle \supset \langle \sigma^2 \rangle \supset \{e\}.$$

□

Ex. 7.3.7 *Let $\zeta_7 = e^{2\pi i/7}$, and consider the extension $\mathbb{Q} \subset L = \mathbb{Q}(\zeta_7)$.*

(a) Show that L is the splitting field of $f = x^6 + x^5 + x^4 + x^3 + x^2 + x + 1$ over $\mathbb{Q}$ and that f is the minimal polynomial of ζ_7.

(b) Let $(\mathbb{Z}/7\mathbb{Z})^*$ be the group of non zero congruence classes modulo 7 under multiplication. By Exercise 4 of section 6.2 there is a group isomorphism $\mathrm{Gal}(L/Q) \simeq$ *$(\mathbb{Z}/7\mathbb{Z})^*$. Let $H \subset (\mathbb{Z}/7\mathbb{Z})^*$ be the subgroup generated by the congruence class of -1. Prove that $\mathbb{Q}(\zeta_7 + \zeta_7^{-1})$ is the fixed field of the subgroup of* $\mathrm{Gal}(L/\mathbb{Q})$ *corresponding to H.*

Proof. (a) Proposition 4.2.5, with $p = 7$ prime, shows that

$$f = \Phi_7 = x^6 + x^5 + x^4 + x^3 + x^2 + x + 1$$

is irreducible over $\mathbb{Q}$. $\zeta = \zeta_7 = e^{2i\pi/7}$ being a root of $\Phi_7 = (x^7 - 1)/(x - 1)$, $f = \Phi_7$ is the minimal polynomial of ζ over $\mathbb{Q}$.

The roots of f are the roots of $x^7 - 1$ distinct of 1, they are $\zeta, \zeta^2, \cdots, \zeta^6$. The splitting field of f is so $\mathbb{Q}(\zeta, \zeta^2, \cdots, \zeta^6) = \mathbb{Q}(\zeta)$, since $\zeta^k \in \mathbb{Q}(\zeta)$ for all integers k.

Conclusion: $L = \mathbb{Q}(\zeta)$, where $\zeta = e^{2i\pi/7}$, is the splitting field of $f = x^6 + x^5 + x^4 + x^3 + x^2 + x + 1$, and f is the minimal polynomial of ζ.

Therefore $\mathbb{Q} \subset \mathbb{Q}(\zeta)$ is a Galois extension, and

$$|\mathrm{Gal}(\mathbb{Q}(\zeta)/\mathbb{Q})| = [\mathbb{Q}(\zeta) : \mathbb{Q}] = \deg(f) = 6.$$

(b) The Exercice 6.2.4(f) shows that $G = \mathrm{Gal}(L/\mathbb{Q}) \simeq (\mathbb{Z}/7\mathbb{Z})^*$, the isomorphism φ being defined by

$$\varphi : \left\{ \begin{array}{ccc} \mathrm{Gal}(\mathbb{Q}(\zeta)/\mathbb{Q}) & \to & (\mathbb{Z}/7\mathbb{Z})^* \\ \sigma & \mapsto & [k] : \ \sigma(\zeta) = \zeta_n^k \end{array} \right.$$

Let $\tilde{H} = \{-\overline{1}, +\overline{1}\} \subset (\mathbb{Z}/7\mathbb{Z})^*$, and $H \subset G$ the corresponding subgroup. We compute its fixed field L_H.

Write τ the unique element of G such that $\tau(\zeta) = \zeta^{-1}$. We prove that $H = \{e, \tau\}$. As $\overline{\zeta} = \zeta^6 = \zeta^{-1} \in \mathbb{Q}(\zeta)$, then $\chi : L \to L, z \mapsto \overline{z}$ is an automorphism of L which is the identity on $\mathbb{Q}$, consequently $\chi \in \mathrm{Gal}(L/\mathbb{Q})$. Since $\chi(\zeta) = \overline{\zeta} = \zeta^{-1} = \tau(\zeta)$, $\tau = \chi$ is the complex conjugation restricted to L, $\varphi(\tau) = [-1]$, and $H = \{e, \tau\}$.

For all $z \in L$,

$$z \in L_H \iff \overline{z} = z \iff z \in L \cap \mathbb{R}$$

$$L_H = \mathbb{Q}(\zeta) \cap \mathbb{R}.$$

$\zeta + \zeta^{-1} = 2\cos(2\pi/7) \in \mathbb{Q}(\zeta) \cap \mathbb{R}$, thus

$$\mathbb{Q}(\zeta + \zeta^{-1}) \subset \mathbb{Q}(\zeta) \cap \mathbb{R} = L_H \tag{1}$$

Write $\alpha = \zeta + \zeta^{-1}$. Then

$$\zeta^2 + \zeta^{-2} = (\zeta + \zeta^{-1})^2 - 2 = \alpha^2 - 2 \in \mathbb{Q}(\alpha).$$
$$\zeta^3 + \zeta^{-3} = (\zeta^2 + \zeta^{-2})(\zeta + \zeta^{-1}) - (\zeta + \zeta^{-1}) = (\alpha^2 - 2)\alpha - \alpha = \alpha^3 - 3\alpha \in \mathbb{Q}(\alpha).$$

As f is irreducible over $\mathbb{Q}$, a basis of L over $\mathbb{Q}$ is $(1, \zeta, \ldots, \zeta^6)$.

Let $z = a_0 + a_1\zeta + a_2\zeta^2 + a_3\zeta^3 + a_4\zeta^4 + a_5\zeta^5 + a_6\zeta^6,\ a_i \in \mathbb{Q},\ 0 \le i \le 6$, any element of L.

If $z \in L_H$, then $z = \tau(z)$, so

$$a_0 + a_1\zeta + a_2\zeta^2 + a_3\zeta^3 + a_4\zeta^4 + a_5\zeta^5 + a_6\zeta^6 = a_0 + a_1\zeta^6 + a_2\zeta^5 + a_3\zeta^4 + a_4\zeta^3 + a_5\zeta^2 + a_6\zeta,$$

therefore $a_1 = a_6, a_2 = a_5, a_3 = a_4$, so

$$z = a_0 + a_1(\zeta + \zeta^{-1}) + a_2(\zeta^2 + \zeta^{-2}) + a_3(\zeta^3 + \zeta^{-3}) \in \mathbb{Q}(\zeta + \zeta^{-1}),$$

thus $L_H \subset \mathbb{Q}(\zeta + \zeta^{-1})$, which gives, with the inclusion (1),

$$L_H = \mathbb{Q}(\zeta + \zeta^{-1}) = \mathbb{Q}(\zeta) \cap \mathbb{R}.$$

□

Ex. 7.3.8 *Let $\alpha = \zeta_7 + \zeta_7^{-1}$, where $\zeta_7 = e^{2\pi i/7}$.*

(a) Show that the minimal polynomial of α over $\mathbb{Q}$ is $x^3 + x^2 - 2x - 1$.

(b) Use Exercise 7 to show that the splitting field of $x^3 + x^2 - 2x - 1$ over $\mathbb{Q}$ is a Galois extension of degree 3 with Galois group isomorphic to $\mathbb{Z}/3\mathbb{Z}$.

Proof. (a) Let $\zeta = \zeta_7$ and $\alpha = \zeta + \zeta^{-1}$. We compute the minimal polynomial of α.

We have shown in Exercice 7 that

$$\begin{aligned} \zeta + \zeta^{-1} &= \alpha \\ \zeta^2 + \zeta^{-2} &= \alpha^2 - 2 \\ \zeta^3 + \zeta^{-3} &= \alpha^3 - 3\alpha. \end{aligned}$$

Thus

$$\begin{aligned} 0 &= 1 + \zeta + \zeta^2 + \zeta^3 + \zeta^4 + \zeta^5 + \zeta^6 \\ &= 1 + (\zeta + \zeta^{-1}) + (\zeta^2 + \zeta^{-2}) + (\zeta^3 + \zeta^{-3}) \\ &= 1 + \alpha + (\alpha^2 - 2) + (\alpha^3 - 3\alpha) \\ &= \alpha^3 + \alpha^2 - 2\alpha - 1 \end{aligned}$$

α is so a root of $p = x^3 + x^2 - 2x - 1$.

We could verify directly the irreducibility of p, but it is more simple to proceed so:

• As $p(\alpha) = 0$, the minimal polynomial of q of α over $\mathbb{Q}$ divides p: $q \mid p$,

• $\mathbb{Q}(\alpha) = L_H$ is the fixed field of $H = \{e, \sigma\}$ (Exercice 6). Then $\mathrm{Gal}(L/L_H) = H$, and $[L : L_H] = |H| = 2$, so

$$[\mathbb{Q}(\alpha) : \mathbb{Q}] = [L_H : \mathbb{Q}] = [L : \mathbb{Q}]/[L : L_H] = [L : \mathbb{Q}]/|H| = 6/2 = 3,$$

thus $\deg(q) = [\mathbb{Q}(\alpha) : \mathbb{Q}] = 3 = \deg(p)$,

• Moreover p, q are monic. Consequently $p = q$, and so p is irreducible over $\mathbb{Q}$, and α is a root of p.

Conclusion: $p = x^3 + x^2 - 2x - 1$ is the minimal polynomial of $\alpha = \zeta + \zeta^{-1}$ over $\mathbb{Q}$.

Note: as an alternative method, to find the minimal polynomial of α, we can use the Lagrange's construction described in the proof of Theorem 7.1.1:

3 is a generator of the cyclic group $(\mathbb{Z}/7\mathbb{Z})^*$ ($3^2 = 2, 3^3 = -1$), so $\mathrm{Gal}(L/\mathbb{Q}) = \{e, \sigma, \sigma^2, \sigma^3, \sigma^4, \sigma^5\}$, where σ is characterized by $\sigma(\zeta) = \zeta^3$ (then $\sigma^k(\zeta) = \zeta^{3^k} = \zeta, \zeta^3, \zeta^2, \zeta^{-1}, \zeta^{-3}, \zeta^{-2}$ for $k = 0, 1, 2, 3, 4, 5$). The *distinct* images of α by the automorphisms of G are so $\zeta + \zeta^{-1}, \zeta^2 + \zeta^{-2}, \zeta^3 + \zeta^{-3}$, so the minimal polynomial of α over $\mathbb{Q}$ is (see the proof of Th. 7.1.1)

$$(x - \zeta - \zeta^{-1})(x - \zeta^2 - \zeta^{-2})(x - \zeta^3 - \zeta^{-3}).$$

To expand this polynomial, we use the following Sage instructions:

```
K.<zeta> = NumberField(1+x+x^2+x^3+x^4+x^5+x^6)
R.<t> = PolynomialRing(QQ)
f = (t-zeta - zeta^(-1))*(t-zeta^2-zeta^(-2))*(t-zeta^3-zeta^(-3));f
```

$$t^3 + t^2 - 2t - 1$$

which gives the minimal polynomial

$$\begin{aligned} p &= x^3 + x^2 - 2x - 1 \\ &= (x - \zeta - \zeta^{-1})(x - \zeta^2 - \zeta^{-2})(x - \zeta^3 - \zeta^{-3}) \\ &= (x - 2\cos(2\pi/7))(x - 2\cos(4\pi/7))(x - 2\cos(6\pi/7)) \end{aligned}$$

(b) By Exercise 6, $\mathbb{Q}(\alpha) = L_H$ is associate to H of order 2 in the Galois correspondence.

As $G = \mathrm{Gal}(L/F) \simeq (\mathbb{Z}/7\mathbb{Z})^*$ is Abelian, H is normal in G, so $\mathbb{Q} \subset L_H$ is a Galois extension.

Consequently all the roots α, β, γ of p are in $\mathbb{Q}(\alpha)$, thus $\mathbb{Q}(\alpha) = \mathbb{Q}(\alpha, \beta, \gamma)$ is the splitting field of p over $\mathbb{Q}$.

(In fact, the other roots of p are $\sigma(\zeta + \zeta^{-1}) = \zeta^3 + \zeta^{-3} = \alpha^3 - 3\alpha$, and $\sigma^2(\zeta + \zeta^{-1}) = \zeta^2 + \zeta^{-2} = \alpha^2 - 2$, and are all in $\mathbb{Q}(\zeta + \zeta^{-1})$.)

Conclusion: the splitting field of $p = x^3 + x^2 - 2x - 1$ over $\mathbb{Q}$ is $E = \mathbb{Q}(\zeta_7) \cap \mathbb{R} = \mathbb{Q}(\zeta_7 + \zeta_7^{-1})$, and $\mathbb{Q} \subset E$ is a Galois extension of degree 3.

Moreover (Theorem 7.2.7), $\mathrm{Gal}(E/\mathbb{Q}) \simeq \mathrm{Gal}(L/\mathbb{Q})/\mathrm{Gal}(L/E) = G/H$.

As G is cyclic and $|H| = 2$, G/H is the quotient group of a cyclic group, so is cyclic, of order 3, isomorphic to $\mathbb{Z}/3\mathbb{Z}$.

$$\mathrm{Gal}(\mathbb{Q}(\zeta_7 + \zeta_7^{-1})/\mathbb{Q}) \simeq \mathbb{Z}/3\mathbb{Z}.$$

□

Ex. 7.3.9 *Let F be a field of characteristic different from 2, and let $F \subset L$ be a finite extension. Prove that the following are equivalent:*

(a) L is a Galois extension of F with $\mathrm{Gal}(L/F) \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

(b) L is the splitting field of a polynomial of the form $(x^2-a)(x^2-b)$, where $a, b \in F$ but $\sqrt{a}, \sqrt{b}, \sqrt{ab}$ do not lie in F.

Proof. • Suppose (b): L is the splitting field of $f = (x^2-a)(x^2-b)$, where $a, b \in F$, but $\sqrt{a}, \sqrt{b}, \sqrt{ab}$ do not lie in F.

The splitting field of f is $F(\sqrt{a}, -\sqrt{a}, \sqrt{b}, -\sqrt{b}) = F(\sqrt{a}, \sqrt{b})$:

$$L = F(\sqrt{a}, \sqrt{b}).$$

Consider the ascending chain of fields:

$$F \subset F(\sqrt{a}) \subset F(\sqrt{a}, \sqrt{b}).$$

As $\sqrt{a} \notin F, [F(\sqrt{a}) : F] \neq 1$, and $[F(\sqrt{a}) : F] \leq 2$ since $\sqrt{a}$ is a root of $x^2 - a \in F[x]$, thus $[F(\sqrt{a}) : F] = 2$.

With a reduction ad absurdum, suppose that $\sqrt{b} \in F(\sqrt{a})$, then

$$\sqrt{b} = u + v\sqrt{a},\ u, v \in F.$$

By squaring this equality, $b = u^2 + av^2 + 2uv\sqrt{a}$.

If $uv \neq 0$, then $\sqrt{b} = \frac{b-u^2-av^2}{2uv} \in F$, in contradiction with the hypothesis, so $uv = 0$.

If $v = 0$, $\sqrt{b} = u \in F$: this is excluded.

If $u = 0$, $\sqrt{b} = v\sqrt{a}$, so $\sqrt{ab} = va \in F$: this is also excluded.

This proves that $\sqrt{b} \notin F(\sqrt{a})$, and $\sqrt{b}$ is a root of $x^2 - b \in F(\sqrt{a})[x]$, thus

$$[F(\sqrt{a}, \sqrt{b}) : F(\sqrt{a})] = 2.$$

Finally

$$[L : F] = [F(\sqrt{a}, \sqrt{b}) : F] = [F(\sqrt{a}, \sqrt{b}) : F(\sqrt{a})]\ [F(\sqrt{a}) : F] = 4.$$

As the characteristic of F is different from 2, $\sqrt{a} \neq -\sqrt{a}$, otherwise $\sqrt{a} = 0 \in F$, and the same is true for b. Moreover $\sqrt{a} \neq \pm\sqrt{b}$, otherwise $\sqrt{ab} = \pm a \in F$, so

$$f = (x - \sqrt{a})(x + \sqrt{a})(x - \sqrt{b})(x + \sqrt{b})$$

is a separable polynomial, and the splitting field L of the separable polynomial $f \in F[x]$ is a Galois extension of F. Therefore,

$$|\mathrm{Gal}(L/F) = [L : F] = 4.$$

If $\sigma \in G = \mathrm{Gal}(L/F)$, as a is a root of $x^2 - a \in F[x]$, $\sigma(a)$ also, thus $\sigma(\sqrt{a}) = (-1)^k\sqrt{a},\ 0 \leq k \leq 1$. Similarly $\sigma(\sqrt{b}) = (-1)^l\sqrt{b},\ 0 \leq l \leq 1$. As σ is uniquely determined by the images of $\sqrt{a}, \sqrt{b}$, there are at most 4 F-automorphisms of L.

As $|\mathrm{Gal}(L/F)| = 4$, these 4 possibilities occur, and give an element of the Galois group $\mathrm{Gal}(L/F)$, otherwise this group would have less than 4 elements.

Then $G = \{e, \sigma, \tau, \zeta\}$, where

$$\begin{aligned} \sigma(\sqrt{a}) &= -\sqrt{a}, & \sigma(\sqrt{b}) &= \sqrt{b}, \\ \tau(\sqrt{a}) &= \sqrt{a}, & \tau(\sqrt{b}) &= -\sqrt{b}, \\ \zeta(\sqrt{a}) &= -\sqrt{a}, & \zeta(\sqrt{b}) &= -\sqrt{b}. \end{aligned}$$

As σ, τ, ζ are of order 2,

$$\mathrm{Gal}(L/F) \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}.$$

- Conversely, suppose (a):

$$L/F \text{ is a Galois extension of } F, \text{ and } \mathrm{Gal}(L/F) \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}.$$

Then

$$[L : F] = \mathrm{Gal}(L/F) = 4.$$

Write e, σ, τ, ζ the elements of $G = \mathrm{Gal}(L/F)$, where e the identity of G. As $G = \{e, \sigma, \tau, \zeta\} \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, $\zeta = \sigma \circ \tau$ and all the elements different from e are of order 2.

The only non trivial subgroups have cardinality 2: they are $\langle \sigma \rangle = \{e, \sigma\}, \langle \tau \rangle = \{e, \tau\}, \langle \zeta \rangle = \{e, \zeta\}$.

$$\begin{tikzcd} & \{e\} \arrow[dl] \arrow[d] \arrow[dr] & \\ \langle \sigma \rangle \arrow[dr] & \langle \sigma \circ \tau \rangle \arrow[d] & \langle \tau \rangle \arrow[dl] \\ & \mathrm{Gal}(L/F) & \end{tikzcd} \qquad \begin{tikzcd} & L & \\ K_\sigma \arrow[ur] & K_\zeta \arrow[u] & K_\tau \arrow[ul] \\ & F \arrow[ul] \arrow[u] \arrow[ur] & \end{tikzcd}$$

The intermediate field corresponding with these subgroups are the fixed fields

$$K_\sigma = L_{\langle \sigma \rangle}, K_\tau = L_{\langle \tau \rangle}, K_\zeta = L_{\langle \sigma \circ \tau \rangle}.$$

As the index in G of these three subgroups is 2, $K_\sigma, K_\tau, K_\zeta$ are quadratic extensions of F (by Theorem 7.3.1$[L_H : F] = [\mathrm{Gal}(L/F) : H]$). Since $[L : F] = \mathrm{Gal}(L/F) = 4$, L is a quadratic extension of each of them.

As the characteristic of F is different from 2, the Exercise 7.1.12 shows that $K_\sigma = F(\alpha)$, where $a = \alpha^2 \in F, \alpha \notin F$. Write $\alpha = \sqrt{a}$, then $K_\sigma = F(\sqrt{a}), a \in F, \sqrt{a} \notin F$. Similarly $K_\tau = F(\sqrt{b}), b \in F, \beta = \sqrt{b} \notin F$.

$\alpha = \sqrt{a} \in L_{\langle \sigma \rangle}$, so $\sigma(\sqrt{a}) = \sqrt{a}$.

$K_\sigma \cap K_\tau = L_{\langle \sigma \rangle} \cap L_{\langle \tau \rangle} = L_{\langle \sigma, \tau \rangle} = L_G = F$ by Theorem 7.1.1(b), so

$$K_\sigma \cap K_\tau = F.$$

Since $\sqrt{a} \in K_\sigma \setminus F$, $\sqrt{a} \notin K_\tau$, thus $\tau(\sqrt{a}) \neq \sqrt{a}$.

Moreover $\sqrt{a}$ is a root of $x^2 - a \in F[x]$, thus $\tau(\sqrt{a}) \in \{\sqrt{a}, -\sqrt{a}\}$. Consequently $\tau(\sqrt{a}) = -\sqrt{a}$.

$$\sigma(\sqrt{a}) = \sqrt{a}, \qquad \tau(\sqrt{a}) = -\sqrt{a},$$

and similarly

$$\sigma(\sqrt{b}) = -\sqrt{b}, \qquad \tau(\sqrt{b}) = \sqrt{b}.$$

As $(\alpha\beta)^2 = ab$, write $\alpha\beta = \sqrt{ab} = \sqrt{a}\sqrt{b}$. Then

$$\sigma(\sqrt{ab}) = -\sqrt{ab}, \qquad \tau(\sqrt{ab}) = -\sqrt{ab}.$$

Thus $\sqrt{ab}$ lies not in the fixed field of G, so $\sqrt{ab} \notin F$.

The intermediate extension $E = F(\sqrt{a}, \sqrt{b})$ contains $K_\sigma = F(\sqrt{a})$ and $K_\tau = F(\sqrt{b})$, so $E \supset L_{\langle\sigma\rangle}, E \supset L_{\langle\tau\rangle}$. Therefore, by the Galois correspondence, $\mathrm{Gal}(L/E) \subset \mathrm{Gal}(L/L_{\langle\sigma\rangle}) = \langle\sigma\rangle$ and $\mathrm{Gal}(L/E) \subset \langle\tau\rangle$, thus $\mathrm{Gal}(L/E) \subset \langle\sigma\rangle \cap \langle\tau\rangle = \{e\}$. Thus $\mathrm{Gal}(L/E) = \{e\}$, and so $E = L$.

$$L = F(\sqrt{a}, \sqrt{b}).$$

As $f = (x^2 - a)(x^2 - b) = (x - \sqrt{a})(x + \sqrt{a})(x - \sqrt{b})(x + \sqrt{b}) \in F[x]$ splits completely in L, the splitting field of f is $F(\sqrt{a}, \sqrt{b}) = L$.

The equivalence (a) $\iff$ (b) is proved. □

Ex. 7.3.10 *Suppose that $\alpha, \beta \in \mathbb{C}$ are algebraic of degree 2 over $\mathbb{Q}$ (i.e., they are both roots of irreducible quadratic polynomials in $\mathbb{Q}[x]$). Prove that the following are equivalent:*

(a) $\mathbb{Q}(\alpha) = \mathbb{Q}(\beta)$.

(b) $\alpha = a + b\beta$ *for some* $a, b \in \mathbb{Q}, b \neq 0$.

(c) $\alpha + \beta$ *is the root of a quadratic polynomial in* $\mathbb{Q}[x]$.

Proof. (a)$\Rightarrow$(b):

If $\mathbb{Q}(\alpha) = \mathbb{Q}(\beta)$, then $\alpha \in \mathbb{Q}(\beta)$. Since $[\mathbb{Q}(\beta) : \mathbb{Q}] = 2$, $\beta \notin \mathbb{Q} = \mathrm{Vect}_{\mathbb{Q}}(1)$, then $(1, \beta)$ is a linearly independent list with 2 elements in a 2-dimensional vector space, so is a basis of $\mathbb{Q}(\beta)$ over $\mathbb{Q}$. Then α spans on this basis under the form

$$\alpha = a + b\beta, \ a, b \in \mathbb{Q}.$$

Moreover, $b \neq 0$, otherwise $\alpha \in \mathbb{Q}$, and α would not be of degree 2 over $\mathbb{Q}$.

(b)$\Rightarrow$(a):

If $\alpha = a + b\beta, \ a, b \in \mathbb{Q}, \ b \neq 0$, then $\alpha \in \mathbb{Q}(\beta)$, thus $\mathbb{Q}(\alpha) \subset \mathbb{Q}(\beta)$.

Moreover $\beta = b^{-1}(\alpha - a) \in \mathbb{Q}(\alpha)$, so $\mathbb{Q}(\beta) \subset \mathbb{Q}(\alpha)$.

$$\mathbb{Q}(\alpha) = \mathbb{Q}(\beta).$$

(b)$\Rightarrow$(c):

$\delta = \alpha + \beta = a + (b + 1)\beta \in \mathbb{Q}(\beta)$. Therefore the list $(1, \delta, \delta^2)$ of 3 vectors in a 2-dimensional vector space is linearly dependent over $\mathbb{Q}$, so there exist $(u, v, w) \in \mathbb{Q}^3 \setminus \{(0, 0, 0)\}$ such that $u\delta^2 + v\delta + w = 0$.

Let $f(x) = ux^2 + vx + w \in \mathbb{Q}[x]$. Then $f(\alpha + \beta) = 0$, with $f \neq 0, \deg(f) \leq 2$. If $\deg(f) = 2$, (c) is proved.

But $\deg(f) < 2$ is a possibility, for instance if $\beta = -\alpha$. As $f \neq 0$, then $\deg(p) = 0$ is in contradiction with $f(\delta) = 0$, so in this case $\deg(f) = 1$: $f(x) = vx + w, \ v \neq 0$. Then $\delta = \alpha + \beta$ is a root of the polynomial of degree 2 $x(vx + w)$. In both cases,

$\alpha + \beta$ is the root of a quadratic polynomial in $\mathbb{Q}[x]$.

(c)$\Rightarrow$(a): Suppose that $\alpha + \beta$ is a root of a quadratic polynomial, and suppose on the contrary that $\mathbb{Q}(\alpha) \neq \mathbb{Q}(\beta)$. By assumption, $[\mathbb{Q}(\alpha) : \mathbb{Q}] = [\mathbb{Q}(\beta) : \mathbb{Q}] = 2$. Therefore $[\mathbb{Q}(\alpha, \beta) : \mathbb{Q}(\beta)] \leq 2$. If $[\mathbb{Q}(\alpha, \beta) : \mathbb{Q}(\beta)] = 1$, then $\alpha \in \mathbb{Q}(\beta)$, so $\alpha = a + b\beta$ for some $a, b \in \mathbb{Q}$, and $b \neq 0$ otherwise $\alpha \in \mathbb{Q}$.

The implication (b) $\Rightarrow$ (a) shows that $\mathbb{Q}(\alpha) = \mathbb{Q}(\beta)$, and this is a contradiction. Therefore $[\mathbb{Q}(\alpha, \beta) : \mathbb{Q}(\beta)] = 2$, and

$$[\mathbb{Q}(\alpha, \beta) : \mathbb{Q}] = [\mathbb{Q}(\alpha, \beta) : \mathbb{Q}(\beta)][\mathbb{Q}(\beta) : \mathbb{Q}] = 4.$$

Write $L = \mathbb{Q}(\alpha, \beta)$. Then $[L : \mathbb{Q}] = 4$.

Let $f = x^2 + rx + s, g = x^2 + r'x + s' \in \mathbb{Q}[x]$ be the minimal polynomials of α, β over $\mathbb{Q}$, and write α, α' the roots of f, β, β' the root of g.

As $\alpha + \alpha' = -r \in \mathbb{Q}$, $\alpha' \in \mathbb{Q}(\alpha)$, and similarly $\beta' \in \mathbb{Q}(\beta)$. Therefore the splitting field of fg is $\mathbb{Q}(\alpha, \alpha', \beta, \beta') = \mathbb{Q}(\alpha, \beta)$. This shows that $\mathbb{Q} \subset \mathbb{Q}(\alpha, \beta)$ is a normal extension, and also separable since the characteristic of $\mathbb{Q}$ is 0. So $\mathbb{Q} \subset L$ is a Galois extension, therefore

$$|\mathrm{Gal}(L/\mathbb{Q})| = [L : \mathbb{Q}] = 4.$$

Consequently, if we write $G = \mathrm{Gal}(L/\mathbb{Q})$,

$$G \simeq \mathbb{Z}/4\mathbb{Z} \text{ or } G \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}.$$

• If $G \simeq \mathbb{Z}/4\mathbb{Z}$, as $\mathbb{Z}/4\mathbb{Z}$ has a unique subgroup H of index 2 in G, there exists a unique quadratic extension of $\mathbb{Q}$ included in L, and so $\mathbb{Q}(\alpha) = \mathbb{Q}(\beta) = L_H$, in contradiction with the hypothesis.

• We suppose now that $G \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Then by the Galois correspondence, the extension $\mathbb{Q}(\alpha)$ corresponds to a subgroup H of index 2 in G, thus of order $4/2 = 2$. So $H = \{e, \sigma\}$, and $\mathbb{Q}(\alpha) = L_H$ is the fixed field of σ. Similarly there exists $\tau \in G, \tau \neq \sigma$, such that $\mathbb{Q}(\beta)$ is the fixed field of $K = \{e, \tau\}$. There exist exactly 3 subgroups of G of index 2 : $\langle\sigma\rangle, \langle\tau\rangle, \langle\zeta\rangle$, where $\zeta = \sigma\tau = \tau\sigma$, in correspondence with 3 quadratic sub-extensions of $\mathbb{Q} \subset L$, two of them being $\mathbb{Q}(\alpha), \mathbb{Q}(\beta)$. As every quadratic extension of $\mathbb{Q}$, the third is of the form $\mathbb{Q}(\gamma), \gamma \in L$, fixed field of $\{e, \zeta\}$.

$$\begin{tikzcd}
& \{e\} \arrow[dl] \arrow[d] \arrow[dr] & \\
\langle\sigma\rangle \arrow[dr] & \langle\tau\rangle \arrow[d] & \langle\zeta\rangle \arrow[dl] \\
& G &
\end{tikzcd}
\qquad
\begin{tikzcd}
& \mathbb{Q}(\alpha, \beta) & \\
\mathbb{Q}(\alpha) \arrow[ur] & \mathbb{Q}(\beta) \arrow[u] & \mathbb{Q}(\gamma) \arrow[ul] \\
& \mathbb{Q} \arrow[ul] \arrow[u] \arrow[ur] &
\end{tikzcd}$$

We show that $\mathbb{Q}(\alpha + \beta) = \mathbb{Q}(\gamma)$. $\alpha + \beta \notin \mathbb{Q}$, otherwise $\mathbb{Q}(\alpha) = \mathbb{Q}(\beta)$, so $\mathbb{Q}(\alpha + \beta)$ is a quadratic extension of $\mathbb{Q}$, therefore is equal to $\mathbb{Q}(\alpha), \mathbb{Q}(\beta)$ or $\mathbb{Q}(\gamma)$.

$\mathbb{Q}(\alpha + \beta) \neq \mathbb{Q}(\beta)$, otherwise $\beta \in \mathbb{Q}(\beta)$, and so $\mathbb{Q}(\alpha) = \mathbb{Q}(\beta)$. Similarly $\mathbb{Q}(\alpha + \beta) \neq \mathbb{Q}(\alpha)$. It remains only the possibility $\mathbb{Q}(\alpha + \beta) = \mathbb{Q}(\gamma)$, fixed field of $\zeta = \sigma \circ \tau$.

Note that $\tau(\alpha) \neq \alpha$, otherwise $\alpha \in L_{\langle\tau\rangle} = \mathbb{Q}(\beta)$, which is excluded.

As $\alpha+\beta \in \mathbb{Q}(\gamma) = L_{\langle \sigma\tau \rangle}$,

$$(\sigma\tau)(\alpha+\beta) = \alpha+\beta.$$

But, since G is commutative, we have also

$$(\sigma\tau)(\alpha+\beta) = (\sigma\tau)(\alpha) + (\sigma\tau)(\beta) = (\tau\sigma)(\alpha) + (\sigma\tau)(\beta) = \tau(\alpha) + \sigma(\beta).$$

Therefore $\alpha+\beta = \tau(\alpha)+\sigma(\beta)$, thus $\alpha - \tau(\alpha) = \sigma(\beta) - \beta$.

As $\mathbb{Q}(\alpha)$ is a normal extension, $\tau(\alpha) \in \mathbb{Q}(\alpha)$, and similarly $\sigma(\beta) \in \mathbb{Q}(\beta)$. Therefore

$$\alpha - \tau(\alpha) = \sigma(\beta) - \beta \in \mathbb{Q}(\alpha) \cap \mathbb{Q}(\beta) = \mathbb{Q},$$

Thus $\sigma(\beta) = \beta + c, \tau(\alpha) = \alpha - c, c \in \mathbb{Q}^*$.

Then $(\sigma\tau)(\alpha+\beta) = \alpha+\beta$, so the orbit of $\alpha+\beta$ under the action of $G = \{e, \sigma, \tau, \sigma\tau\}$ is $\mathcal{O}_{\alpha+\beta} = \{\alpha+\beta, \alpha+\beta+c, \alpha+\beta-c\}$ has exactly 3 elements. As the cardinality of the orbit is the index of the stabilizer of $\alpha+\beta$ in G, so divides the order of G, we would have $3 \mid 4 = |G|$: this is a contradiction, obtained under the hypothesis $\mathbb{Q}(\alpha) \neq \mathbb{Q}(\beta)$, so

$$\mathbb{Q}(\alpha) = \mathbb{Q}(\beta).$$

(c)$\Rightarrow$(a) is proved.

□

Ex. 7.3.11 *Let $F \subset L$ be a Galois extension, and let $F \subset K \subset L$ be an intermediate field. Then let N be the normalizer of $\mathrm{Gal}(L/K) \subset \mathrm{Gal}(L/F)$. Prove that the fixed field L_N is the smallest subfield of K such that K is Galois over the subfield.*

Proof. As $N = N_G(H)$ is the largest subgroup of $G = \mathrm{Gal}(L/F)$ such that $H = \mathrm{Gal}(L/K)$ is normal in N, since the Galois correspondance reverse inclusions, L_N is the smallest subfield of $K = L_H$ such that the extension $L_N \subset K$ is normal. We give the details.

• Write $H = \mathrm{Gal}(L/K)$. Then $L_H = K$. Since $H \subset N$, then $L_H \supset L_N$, so L_N is a subfield of K.

$$L_N \subset K,$$

• H is a normal subgroup of $N = N_G(H)$. Therefore the extension $L_N \subset L_H = K$ is normal (Theorem 7.3.2).

$$L_N \subset K \text{ is a Galois extension.}$$

• Let $F \subset M \subset K$ an intermediate field, such that $M \subset K$ is a Galois extension.

Let $S = \mathrm{Gal}(L/M)$. S is a subgroup of $G = \mathrm{Gal}(L/F)$ since $F \subset M \subset K$.

The extension $M \subset K$ is normal. Therefore the subgroup $H = \mathrm{Gal}(L/K)$ is normal in $S = \mathrm{Gal}(L/M)$ (Theorem 7.3.2). Since the normalizer $N = N_G(H)$ is the largest subgroup of G with this property, we conclude $S = \mathrm{Gal}(L/M) \subset N$, therefore $M = L_S \supset L_N$.

Conclusion: L_N is the smallest subfield of K such that K is Galois over the subfield.

□

Ex. 7.3.12 *Let H be a subgroup of a group G, and let $N = \bigcap_{g \in G} gHg^{-1}$.*

(a) Show that N is a normal subgroup of G.

(b) Show that N is the largest normal subgroup of G contained in H.

Proof. (a) Let $k \in G$. Then

$$kNk^{-1} = k \left(\bigcap_{g \in G} gHg^{-1} \right) k^{-1} = \bigcap_{g \in G} (kg)H(kg)^{-1} = \bigcap_{u \in G} uHu^{-1} = N,$$

thus $N \lhd G$.

(b) $H = eHe^{-1} \supset \bigcap\limits_{g \in G} gHg^{-1} = N$, so $N \subset H \subset G$.

If any subgroup M of H is normal in G, then for all $g \in G$, $gMg^{-1} = M$, therefore $M = \bigcap\limits_{g \in G} gMg^{-1} \subset \bigcap\limits_{g \in G} gHg^{-1} = N$.

Conclusion: $\mathrm{Core}_G(H) = \bigcap\limits_{g \in G} gHg^{-1}$ is the largest subgroup of H normal in G.

∎

Ex. 7.3.13 *Let $F \subset L$ be a Galois extension, and let $F \subset K \subset L$ be an intermediate field. If we apply the construction of Exercise 12 to $\mathrm{Gal}(L/K) \subset \mathrm{Gal}(L/F)$, then we obtain a normal subgroup $N \subset \mathrm{Gal}(L/F)$. Prove that the fixed field L_N is the Galois closure of K.*

Proof. Let $F \subset L$ a Galois extension, $F \subset K \subset L$ an intermediate field, $G = \mathrm{Gal}(L/F), H = \mathrm{Gal}(L/K)$, $N = \mathrm{Core}_G(H)$, and $M = L_N$ the fixed field of N. We show that $M = L_N$ is the Galois closure of K over F.

Since $N \subset H, L_N \supset L_H = K$, so K is a subfield of L_N.

- As N is normal in G, $M = L_N$ is a Galois extension of F.
- Let M' an extension of K such that M' is Galois over F, and suppose first that $M' \subset L$. We call $S = \mathrm{Gal}(L/M')$.

  As $F \subset M'$ is a Galois extension, $S = \mathrm{Gal}(L/M')$ is normal in G, and since $K \subset M'$, $H = \mathrm{Gal}(L/K) \supset \mathrm{Gal}(L/M') = S$. So S is a subgroup of H, and S is normal in G. By exercise 12, $S \subset N = \mathrm{Core}_G(H)$, thus $M = L_N \subset L_S = M'$.

  $M = L_N$ is so the smallest intermediate field of the extension $F \subset L$ which contains K and is a Galois extension of F.

  Let M_0 be any Galois closure of K over F. As $F \subset M$ is a Galois extension, there exists by proposition 7.1.7 an embedding ψ of M_0 in M that is the identity on K. Then $K \subset \psi(M_0) \subset M \subset L$, and since $M_0 \simeq \psi(M_0)$, $\psi(M_0)$ is a Galois extension of F. But M is the smallest intermediate field of the extension $F \subset L$ which contains K and is a Galois extension of F, therefore $\psi(M_0) = M$, so $\psi : M_0 \to M$ is an isomorphism.

  If M'' is any extension of K which is Galois over F, by the definition of a Galois closure, there exists an field homomorphism $\varphi : M_0 \to M''$ that is the identity on K, so $\varphi \circ \psi^{-1}$ is an embedding from M to M'' that is the identity on L, so $M = L_N$ is a Galois closure of K.

Note: this exercise shows that there exists always a Galois closure of an intermediate field K of a Galois extension $F \subset L$ that is included in L. Moreover it is characterized by the fact that it is the smallest intermediate field of $F \subset L$ containing K that is a Galois extension of F. Such a subfield of L is unique (not only up to an isomorphism). $\square$

Ex. 7.3.14 *Prove the implication (b) $\Rightarrow$ (a) of Theorem 6.5.5.*

(a) $\mathbb{Q} \subset L$ *is normal and* $\mathrm{Gal}(L/Q)$ *is Abelian.*

(b) There is a root of unity $\zeta_n = e^{2i\pi/n}$ *such that* $L \subset \mathbb{Q}(\zeta_n)$.

Proof. Suppose that $L \subset \mathbb{Q}(\zeta_n)$, where $\zeta_n = e^{2\pi i/n}$. The Exercise 6.2.4 prove the existence of an injective group homomorphism, given by

$$\varphi : \left\{ \begin{array}{ccc} \mathrm{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q}) & \to & (\mathbb{Z}/n\mathbb{Z})^* \\ \sigma & \mapsto & [k] : \sigma(\zeta_n) = \zeta_n^k. \end{array} \right.$$

Consequently $G = \mathrm{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q})$ is isomorphic to a subgroup of $(\mathbb{Z}/n\mathbb{Z})^*$, so G is Abelian. As all subgroups of an Abelian group are normal, $H = \mathrm{Gal}(\mathbb{Q}(\zeta_n)/L)$ is a normal subgroup of G, therefore (Theorem 7.2.5) $\mathbb{Q} \subset L$ is a Galois extension, a fortiori a normal extension, and $\mathrm{Gal}(L/\mathbb{Q}) \simeq \mathrm{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q})/\mathrm{Gal}(\mathbb{Q}(\zeta_n)/L)$ is isomorphic to a quotient group of an Abelian group, so is Abelian: the implication (b)$\Rightarrow$(a) of Theorem 6.5.5 is proved. $\square$

Ex. 7.3.15 *Let p be prime. Consider the extension $\mathbb{Q} \subset L = \mathbb{Q}(\zeta_p, \sqrt[p]{2})$ discussed in section 6.4. There, we showed that* $\mathrm{Gal}(L/Q) \simeq \mathrm{AGL}(1, \mathbb{F}_p)$. *The group* $\mathrm{AGL}(1, \mathbb{F}_p)$ *has two subgroups defined as follows:*

$$T = \{\gamma_{1,b} \,|\, b \in \mathbb{F}_p\} \quad \text{and} \quad D = \{\gamma_{a,0} \,|\, a \in \mathbb{F}_p^*\},$$

where $\gamma_{a,b}(u) = au+b, u \in \mathbb{F}_p$. Let T' and D' be the corresponding subgroups of $\mathrm{Gal}(L/\mathbb{Q})$.

(a) Show that the fixed field of T' is $\mathbb{Q}(\zeta_p)$.

(b) What is the fixed field of D'? What are the conjugates of this fixed field?

Proof. let $L = \mathbb{Q}(\zeta_p, \sqrt[p]{2})$.

By the isomorphism $\psi : \mathrm{Gal}(L/\mathbb{Q}) \to \mathrm{AGL}(1, \mathbb{F}_p)$, $\gamma_{a,b} = \psi(\sigma_{a,b})$ corresponds to $\sigma_{a,b}$ uniquely determined by (see section 6.4)

$$\sigma_{a,b}(\zeta_p) = \zeta_p^a, \sigma_{a,b}(\sqrt[p]{2}) = \zeta_p^b \sqrt[p]{2}.$$

(a) T' is so the set of the $\sigma_{1,b},\ b \in \mathbb{F}_p$, where $\sigma_{1,b}(\zeta_p) = \zeta_p$. Therefore

$$\mathbb{Q}(\zeta_p) \subset L_{T'}.$$

$T' = \mathrm{Gal}(L/L_{T'})$, thus $p = |T'| = [L : L_{T'}]$.

Moreover, $[\mathbb{Q}(\zeta_p, \sqrt[p]{2}) : \mathbb{Q}(\zeta_p)] = p$, since $p - 1 = [\mathbb{Q}(\zeta_p) : \mathbb{Q}]$ and $[\mathbb{Q}(\sqrt[p]{2}) : \mathbb{Q}] = p$ are relatively prime.

Thus $[L : L_{T'}] = [L : \mathbb{Q}(\zeta_p)]$, so $[L_{T'} : \mathbb{Q}] = [\mathbb{Q}(\zeta_p) : \mathbb{Q}]$, with $\mathbb{Q}(\zeta_p) \subset L_{T'}$, therefore

$$\mathbb{Q}(\zeta_p) = L_{T'}.$$

(b) D' is the set of $\sigma_{a,0}$, where $\sigma_{a,0}(\sqrt[p]{2}) = \sqrt[p]{2}$. Therefore

$$\mathbb{Q}(\sqrt[p]{2}) \subset L_{D'}.$$

By Theorem 7.3.1(b), $[L : L_{D'}] = |D'| = p - 1 = [L : \mathbb{Q}(\sqrt[p]{2})]$, so we can conclude

$$\mathbb{Q}(\sqrt[p]{2}) = L_{D'}.$$

As $\sigma_{a,b}(\sqrt[p]{2}) = \zeta_p^b \sqrt[p]{2}$, the conjugate fields of $L_{D'}$ are the fields

$$\mathbb{Q}(\zeta_p^b \sqrt[p]{2}), \qquad b = 0, \cdots, p - 1.$$

□

7.4 FIRST APPLICATIONS

Ex. 7.4.1 *Give a detailed proof of Proposition 7.4.2:*

Let $f \in F[x]$ be a monic irreducible separable cubic, where F has characteristic $\neq 2$. If L is the splitting field of f over F, then

$$\mathrm{Gal}(L/F) \simeq \begin{cases} \mathbb{Z}/3\mathbb{Z}, & \text{if } \Delta(f) \text{ is a square in } F, \\ S_3, & \text{otherwise.} \end{cases}$$

Proof. Since L is the splitting field of the separable polynomial f, $F \subset L$ is a Galois extension.

By Exercise 6.2.6, f being irreducible and separable, $n = |\mathrm{Gal}(L/F)|$ is a multiple of $3 = \deg(f)$. Moreover $\mathrm{Gal}(L/F)$ is isomorphic to a subgroup H of S_3, so $n \mid 6$: $n = 3$ or $n = 6$. Since S_3 has a unique subgroup of cardinality 3, namely $A_3 \simeq \mathbb{Z}/3\mathbb{Z}$,

$$\mathrm{Gal}(L/F) \simeq A_3 \text{ or } \mathrm{Gal}(L/F) \simeq S_3.$$

By Theorem 7.4.1, since the characteristic of F is different from 2, $\mathrm{Gal}(L/F) \simeq H \subset A_3$ if and only if $\sqrt{\Delta} \in F$, therefore

$$\mathrm{Gal}(L/F) \simeq \begin{cases} \mathbb{Z}/3\mathbb{Z}, & \text{if } \Delta(f) \text{ is a square in } F, \\ S_3, & \text{otherwise.} \end{cases}$$

□

Ex. 7.4.2 *Compute the Galois groups of the following cubic polynomials:*

(a) $x^3 - 4x + 2$ *over* $\mathbb{Q}$.

(b) $x^3 - 4x + 2$ *over* $\mathbb{Q}(\sqrt{37})$.

(c) $x^3 - 3x + 1$ *over* $\mathbb{Q}$.

(d) $x^3 - t$ *over* $\mathbb{C}(t)$, *t a variable.*

(e) $x^3 - t$ *over* $\mathbb{Q}(t)$, *t a variable.*

Proof. (a) $f = x^3 - 4x + 2$.

f is irreducible by the Sch$\neg \ddot{o}nemann - Eisenstein Criterion with$p $= 2. \Delta(f) = -4p^3 - 27q^2 = -4(-4)^3 - 27(2)^2 = 256 - 108 = 148 = 2^2 \times 37. As \Delta(f) \neq 0$, f is separable, so Proposition 7.4.2 applies to f.

Recall that an integer $k \in \mathbb{Z}$ is a square in $\mathbb{Q}$ if and only if it is a square in $\mathbb{Z}$. As 37 is not a square, $\Delta(f)$ is not a square in $\mathbb{Q}$, so

$$\mathrm{Gal}_{\mathbb{Q}}(x^3 - 4x + 2) = S_3.$$

(b) $f = x^3 - 4x + 2$ has discriminant $\Delta(f) = 148 = (2\sqrt{37})^2$, which is a square in $\mathbb{Q}(\sqrt{37})$, thus

$$\mathrm{Gal}_{\mathbb{Q}(\sqrt{37})}(x^3 - 4x + 2) = A_3.$$

(c) $f = x^3 - 3x + 1$.

If $\alpha = p/q,\ p \wedge q = 1$ is a root of f in $\mathbb{Q}$, then $p^3 - 3pq^2 + q^3 = 0$, thus $p \mid q, q \mid p$ with $p \wedge q = 1$, therefore $\alpha = \pm 1$, but nor 1 neither -1 is a root of f, thus f has no rational root. As $\deg(f) = 3$, f is irreducible over $\mathbb{Q}$. $\Delta(f) = -4(-3)^3 - 27 = 81 = 9^2$, thus $\Delta(f) \neq 0$ and so f is separable. Moreover $\Delta(f) = 9^2$ is a square in $\mathbb{Q}$. By Proposition 7.4.2,

$$\mathrm{Gal}_{\mathbb{Q}}(x^3 - 3x + 1) = A_3.$$

(d) Let u a root of $f = x^3 - t \in \mathbb{C}(t)$ in a splitting field of f over $\mathbb{C}(t)$. Then

$$f = (x - u)(x - \omega u)(x - \omega^2 u).$$

We have proved in Exercise 4.2.9 that f has no root in $\mathbb{C}(t)$, and that f is irreducible over $\mathbb{C}(t)$ (Proposition 4.2.6). Moreover f is separable.

$\Delta(f) = -27t^2 = (i\sqrt{27}\,t)^2$ is a square in $\mathbb{C}(t)$, thus

$$\mathrm{Gal}_{\mathbb{C}(t)}(x^3 - t) = A_3.$$

(e) If $\Delta(f) = -27t^2$ was the square of an element $\alpha = p(t)/q(t)$ in $\mathbb{Q}(t)$, then

$$-27 = \left(\frac{p(t)}{tq(t)}\right)^2, \qquad p, q \in \mathbb{Q}[t].$$

Applying the evaluation homomorphism defined by $t \mapsto t_0$, were $t_0 \in \mathbb{Q}, t_0 \neq 0$ and t_0 is not a root of $q(t)$, we obtain that -27 is a square in $\mathbb{Q}$: this is false, thus $\Delta(f)$ is not the square of an element in $\mathbb{Q}(t)$. Therefore

$$\mathrm{Gal}_{\mathbb{Q}(t)}(x^3 - t) = S_3.$$

□

Ex. 7.4.3 *This exercise will study part (b) of Theorem 7.4.4 when f is a polynomial in $x_1, \dots, x_n$ that is invariant under A_n. The theorem implies that $f = A + B\sqrt{\Delta}$ for some $A, B \in F(\sigma_1, \dots, \sigma_n)$. You will prove that A and B are polynomials in the σ_i. Recall that F is a field of characteristic $\neq 2$.*

(a) Show that $f + (1\,2) \cdot f = 2A$.

(b) In part (a), the left-hand side is a polynomial while the right-hand side is a symmetric rational function. Use theorem 2.2.2 to conclude that A is a polynomial in the σ_i.

(c) Let P denote the product of $f - A$ and $(1\,2) \cdot (f - A)$. Show that $P = -B^2\Delta$.

(d) Let $B = u/v$, where $u, v \in F[\sigma_1, \dots, \sigma_n]$ are relatively prime (recall that $F[\sigma_1, \dots, \sigma_n]$ is a UFD). In Exercise 8 of section 2.4 you showed that Δ is irreducible in $F[\sigma_1, \dots, \sigma_n]$. Use this and the equation $v^2P = -u^2\Delta$ to show that v must be constant. This will prove that $B \in F[\sigma_1, \dots, \sigma_n]$.

Proof. Let $f = f(x_1, \cdots, x_n) \in F[x_1, \cdots, x_n]$ that is invariant under A_n. By Theorem 7.4.4, $f = A + B\sqrt{\Delta},\ A, B \in F(\sigma_1, \cdots, \sigma_n)$.

(a) Let $\tau = (12)$.
By (7.16), $\tau \cdot \sqrt{\Delta} = \mathrm{sgn}(\tau)\sqrt{\Delta} = -\sqrt{\Delta}$.
As τ fixes $A, B \in F(\sigma_1, \cdots, \sigma_n)$, $\tau \cdot f = A - B\sqrt{\Delta}$, thus

$$f + \tau \cdot f = 2A.$$

(b) The polynomial $A = \frac{1}{2}(f + \tau \cdot f) \in F[x_1, \cdots, x_n]$ satisfies $\sigma \cdot A = A$. By Theorem 2.2.2, $A = h(\sigma_1, \cdots, \sigma_n)$, where h is a polynomial.

(c) Let $P = (f - A)(\tau \cdot (f - A))$.
Then $P = (B\sqrt{\Delta})(-B\sqrt{\Delta}) = -B^2\Delta$.

(d) Let $B = u/v,\ u, v \in F[\sigma_1, \cdots, \sigma_n]$, where u, v are relatively prime. Then $v^2P = -u^2\Delta$.

As $\tau \cdot P = (\tau \cdot (f - A))(\tau \cdot (\tau \cdot (f - A))) = (\tau \cdot (f - A))(f - A) = P$, P is invariant under A_n and also invariant under τ, thus is invariant under S_n, and P is a polynomial in $x_1, \cdots, x_n$, since $f, A \in F[x_1, \dots, x_n]$. Therefore there exists a polynomial g such that $P = g(\sigma_1, \cdots, \sigma_n)$, and $v^2g = -u^2\Delta$ is an equality in $F[\sigma_1, \cdots \sigma_n]$: $u, v, g, \Delta \in F[\sigma_1, \cdots, \sigma_n]$.

By Exercise 2.4.8, Δ is irreducible in $F[\sigma_1, \cdots, \sigma_n]$. Moreover v^2 divides $u^2\Delta$ and is relatively prime with u^2, thus v^2 divides Δ, where Δ is irreducible. This is impossible, unless v is a constant $\lambda \in F^*$. Therefore $B = \lambda^{-1}u$ is a polynomial in $\sigma_1, \cdots, \sigma_n$.

Conclusion: if $f = f(x_1, \cdots, x_n) \in F[x_1, \cdots, x_n]$ is invariant under A_n, where the characteristic of F is not 2, then

$$f = A + B\sqrt{\Delta},\ A, B \in F[\sigma_1, \cdots, \sigma_n].$$

□

Ex. 7.4.4 *Let G be a group of order n, and fix $g \in G$.*

(a) Show that the map $G \to G$ defined by $h \mapsto gh$ is one-to-one and onto.

(b) Explain why part (a) implies that each row of the Cayley table of G is a permutation of the elements of G.

(c) Write $G = \{g_1, \ldots, g_n\}$, and fix $g_i \in G$. Use part (a) to show the existence of $\sigma_i \in S_n$ satisfying $g_i g_j = g_{\sigma_i(j)}$ as in (7.19).

Proof. (a) Let

$$\varphi_g : \left\{ \begin{array}{ccc} G & \to & G \\ h & \mapsto & gh \end{array} \right.$$

- φ_g is injective: let $h, k \in G$.

  If $\varphi_g(h) = \varphi_g(k)$, then $gh = gk$, therefore $g^{-1}gh = g^{-1}gk$, $h = k$.

$$\forall h \in G,\ \forall k \in G,\ \varphi_g(h) = \varphi_g(k) \Rightarrow h = k.$$

- φ_g is surjective: let k be any element in G.

  Put $h = g^{-1}k$. Then $\varphi_g(h) = g(g^{-1}k) = (gg^{-1})k = ek = k$.

$$\forall k \in G,\ \exists h \in G, \varphi_g(h) = k.$$

(b) A row of the Cayley table of G corresponding to the element $g \in G$ is the list of the $\varphi_g(g_i) = gg_i$, where g_i traces the list of the elements of G in an arbitrary fixed order. Since φ_g is bijective, we find all the elements of G once and only once. This defines a permutation of G.

(c) Write $S(G)$ the group of bijections of G in G, and S_n the group of bijections of $[\![1, n]\!]$ in $[\![1, n]\!]$ (where $[\![1, n]\!] = \{1, 2, \cdots, n\}$).

The map $\varphi : G \to S(G),\ g \mapsto \varphi_g = \varphi(g)$ is an injective group homomorphism.

Indeed, for all $g, h, k \in G$,

$(\varphi(g) \circ \varphi(h))(k) = \varphi_g(\varphi_h(k)) = g(hk) = (gh)k = \varphi(gh)(k)$, thus

$$\varphi(g) \circ \varphi(h) = \varphi(gh).$$

If $\varphi(g) = 1_G$, $\varphi(g)(e) = ge = g$, therefore $g = e$: $\ker(\varphi) = \{e\}$.

Moreover, if $f : [\![1, n]\!] \to G, i \mapsto g_i$ is the bijection representing the chosen numbering of G, we can associate to it the isomorphism

$$\psi : \left\{ \begin{array}{ccc} S(G) & \to & S_n \\ u & \mapsto & f^{-1} \circ u \circ f \end{array} \right.$$

where $\psi(u) = f^{-1} \circ u \circ f$ is indeed a permutation of $[\![1, n]\!]$.

If $u, v \in S(G)$, $\psi(u) \circ \psi(v) = f^{-1} \circ u \circ f \circ f^{-1} \circ v \circ f = f^{-1} \circ (u \circ v) \circ f = \psi(u \circ v)$, so ψ is a group homomorphism.

If $\psi(u) = e$, then $f^{-1} \circ u \circ f = e$, thus $u = f \circ f^{-1} = e$. Therefore $\ker(\psi) = \{e\}$.

Let σ any permutation in S_n. Put $u = f \circ \sigma \circ f^{-1}$.

Then $\psi(u) = f^{-1} \circ f \circ \sigma \circ f^{-1} \circ f = \sigma$, thus ψ is surjective. ψ is a group isomorphism (depending of the chosen numbering).

Thus $\chi = \psi \circ \varphi : G \to S_n$ is an injective group homomorphism.

For each $g_i \in G$, we associate to it $\sigma_i = \chi(g_i)$.

Let $k \in [1, n]$ defined by $g_i g_j = g_k$, which is equivalent to $k = f^{-1}(g_i g_j)$.

$$\begin{aligned} g_i g_j &= \varphi_{g_i}(g_j) \\ &= (\varphi_{g_i} \circ f)(j), \end{aligned}$$

therefore

$$\begin{aligned} k &= f^{-1}(g_i g_j) \\ &= (f^{-1} \circ \varphi_{g_i} \circ f)(j) \\ &= [\psi(\varphi_{g_i})](j) \\ &= [(\psi \circ \varphi)(g_i)](j) \\ &= \sigma_i(j). \end{aligned}$$

IIf $\sigma_I = \chi(g_i)$, we have so proved that for all $i, j \in [1, n]$,

$$g_i g_j = g_{\sigma_i(j)}.$$

□

Ex. 7.4.5 *Label the elements of S_3 as $g_1 = e, g_2 = (1\,2\,3), g_3 = (1\,3\,2), g_4 = (1\,2), g_5 = (1\,3)$, and $g_6 = (2\,3)$. Write down the six permutations $\sigma_i \in S_6$ defined by the rows of the Cayley table (7.18).*

Proof. The numbering of S_3 is given by

$$g_1 = e, g_2 = (123), g_3 = (132), g_4 = (12), g_5 = (13), g_6 = (23).$$

Write σ_i the permutation defined by $g_i g_j = g_{\sigma_i(j)}$, $1 \le i, j \le n$. The Cayley table of the group gives

$$\begin{bmatrix} g_1 & g_2 & g_3 & g_4 & g_5 & g_6 \\ g_2 & g_3 & g_1 & g_5 & g_6 & g_4 \\ g_3 & g_1 & g_2 & g_6 & g_4 & g_5 \\ g_4 & g_6 & g_5 & g_1 & g_3 & g_2 \\ g_5 & g_4 & g_6 & g_2 & g_1 & g_3 \\ g_6 & g_5 & g_4 & g_3 & g_2 & g_1 \end{bmatrix}$$

where the element of the ith row, jth column is $g_i \circ g_j = g_i g_j = g_{\sigma_i(j)}$.

Thus

$$\begin{aligned} &\sigma_1 = (), \\ &\sigma_2 = (123)(456), \\ &\sigma_3 = (132)(465) = \sigma_2^2, \\ &\sigma_4 = (14)(26)(35), \\ &\sigma_5 = (15)(24)(36), \\ &\sigma_6 = (16)(25)(34). \end{aligned}$$

□

Ex. 7.4.6 *In the situation of Exercise 4, let $G = \{g_1, \ldots, g_n\}$, and assume that $g_i g_j = g_k$. Let $\sigma_i, \sigma_j, \sigma_k \in S_n$ be the corresponding permutations determined by (7.19).*

(a) Prove that $\sigma_i \sigma_j = \sigma_k$.

(b) Prove that the map $G \to S_n$ defined by $g_i \mapsto \sigma_i$ is a one-to-one group homomorphism.

Proof. We have carefully proved in Exercise 4 that $\chi = \psi \circ \phi : G \to S_n, g_i \mapsto \sigma_i$ is an injective group homomorphism (so if $g_k = g_i g_j$, $\sigma_k = \sigma_i \circ \sigma_j$).

□

Ex. 7.4.7 *Let f and $F \subset L$ satisfy the hypothesis of Proposition 7.4.2, and assume that $\sqrt{\Delta(f)} \notin F$. Prove that* $\mathrm{Gal}\left(L/F\left(\sqrt{\Delta(f)}\right)\right) = \mathbb{Z}/3\mathbb{Z}$ *and that f is irreducible over $F\left(\sqrt{\Delta(f)}\right)$.*

Proof. By hypothesis, $f \in F[x]$ is a monic irreducible separable polynomial of degree 3, the characteristic of F is not 2, and L is the splitting field of f over F.

We suppose here that $\Delta = \Delta(f)$ is not a square in F. Theorem 7.4.2 give then the result

$$\mathrm{Gal}(L/F) \simeq S_3.$$

Therefore $[L : F] = |\mathrm{Gal}(L/F)| = 6$. Since $\sqrt{\Delta} \notin F$, $[F(\sqrt{\Delta}) : F] = 2$, and so $[L : F(\sqrt{\Delta})] = 3$.

By the Galois correspondence, the extension $F(\sqrt{\Delta})$ of degree 2 over F corresponds to the subgroup $H = \mathrm{Gal}(L/F(\sqrt{\Delta}))$ of $G = \mathrm{Gal}(L/F)$, of index 2 in $G \simeq S_3$. As S_3 has a unique subgroup of index 2, which is $A_3 \simeq \mathbb{Z}/3\mathbb{Z}$, we can conclude

$$\mathrm{Gal}(L/F(\sqrt{\Delta})) \simeq \mathbb{Z}/3\mathbb{Z}.$$

Let $\alpha \in L$ a root of f. Since f is irreducible over F, f is the minimal polynomial of α over F. Let $p \in F(\sqrt{\Delta})[x]$ the minimal polynomial of α over $F(\sqrt{\Delta})$. As α is a root of $f \in F[x] \subset F(\sqrt{\Delta})[x]$, p divides f in $F(\sqrt{\Delta})[x]$. Moreover $\deg(p) = [L : F(\sqrt{\Delta})] = 3 = \deg(f)$.

$p \mid f, \deg(p) = \deg(f)$, and f, p are monic, thus $f = p$. Therefore f is irreducible over $F(\sqrt{\Delta})$. □

7.5 AUTOMORPHISMS AND GEOMETRY (OPTIONAL)

Ex. 7.5.1 *Let $P, Q \in F[x, y]$ be polynomials such that $P \mid Q$ and $P \in F[x]$, and write $Q = a_0(x) + a_1(x)y + a_(x)y^2 + \cdots + a_m(x)y^m$. Prove that $P \mid a_i$ for $i = 0, \ldots, m$.*

Proof. By hypothesis, $P \in F[x]$ divides in $F[x, y]$ the polynomial

$$Q = Q(x, y) = a_0(x) + a_1(x)y + \cdots + a_m(x)y^m,$$

so $Q(x, y) = P(x)S(x, y), S \in F[x, y]$. The evaluation $y \mapsto 0$ gives

$$a_0(x) = Q(x, 0) = P(x)S(x, 0),$$

thus $P \mid a_0$.

By induction, we suppose that $P \mid a_i,\ 0 \leq i < k$, where $k \leq m$.

Then P divides $a_k(x)y^k + \cdots a_m(x)y^m = y^k(a_k(x) + \cdots + a_m(x)y^{m-k})$.

In the UFD $F[x,y]$, every irreducible factor of y^k is associate to y, and y doesn't divide $P(x)$. Therefore $P(x)$ and y^k are relatively prime, so P divides $a_k(x) + \cdots + a_m(x)y^{m-k}$. The same evaluation $y \mapsto 0$ gives then $p \mid a_k$, so the induction is done. Consequently

$$P \mid a_i, \ 0 \leq i \leq m.$$

$\square$

Ex. 7.5.2 *In the prof of Proposition 7.5.5, we showed that $a(x) - yb(x)$ is irreducible in $F[x,y]$ and we want to conclude that it is also irreducible in $F(y)[x]$. Prove this using the version of Gauss's Lemma stated in Theorem A.5.8.*

Proof. Suppose that $f(x,y)$ is irreducible $F[x,y]$. We prove that it is irreducible in $F(y)[x]$, using Gauss's Lemma:

Theorem A.5.8 : *Let R be an UFD with field of fractions K. Suppose that $f \in R[x]$ is non constant and that $f = gh$, where $g, h \in K[x]$. There is a nonzero $\delta \in K$ such that $\tilde{g} = \delta g$ and $\tilde{h} = \delta^{-1}h$ have coefficients in R. Thus $f = \tilde{g}\tilde{h} \in R[x]$.*

In the context of the Exercise 7.5.2, take $R = F[y]$, whose field of fractions is $F(y)$.

Suppose that $f = gh$, where $g, h \in F(y)[x]$. By Theorem A.5.8, there exists $\delta \in F(y), \delta \neq 0$, such that $\tilde{g} = \delta g \in F[y][x] = F[x,y]$ and $\tilde{h} = \delta^{-1}h \in F[x,y]$. Then $f = \tilde{g}\tilde{h}$, where $f, \tilde{g}, \tilde{h} \in F[x,y]$. As f is irreducible in $F[x,y]$, $\tilde{g} \in F^*$ or $\tilde{f} \in F^*$. Then $g \in F(y)$ or $h \in F(y)$, which proves the irreducibility of f in $F(y)[x]$.

In particular, $p(x) - yq(x)$, irreducible in $F[x,y]$, is so irreducible in $F(y)[x]$. $\square$

Ex. 7.5.3 *The proof of Proposition 7.5.5 shows that $a(x) - yb(x)$ is irreducible in $F[x,y]$. In this exercise, you will give an elementary proof that $a(x) - yb(x)$ is irreducible over $F(y)[x]$. Suppose that*

$$a(x) - yb(x) = AB, \qquad A, B \in F(y)[x].$$

You need to prove that A or B is constant, which in this case means that A or B lies in $F(y)$.

(a) *Show that there are nonzero polynomials $g(y), h(y) \in F[y]$ that clear the denominators of A and B, i.e, $g(y)A = A_1$ and $h(y)B = B_1$ for some $A_1, B_1 \in F[x,y]$.*

(b) *Show that $g(y)h(y)(a(x) - yb(x)) = A_1B_1$ in $F[x,y]$ and explain why $a(x) - yb(x)$ must divide either A_1 or B_1 in $F[x,y]$.*

(c) *Assume that $A_1 = (a(x) - yb(x))A_2$, where $A_2 \in F[x,y]$. Show that this implies that $g(y)h(y) = A_2B_1$, and then conclude that $B_1 \in F[y]$.*

(d) *Show that $B \in F(y)$.*

Proof. We give another proof of Exercise 2, knowing that $f(x,y) = a(x) - yb(x)$ is irreducible in $F[x,y]$. We must prove that a factorization

$$a(x) - yb(x) = AB, \ A, B \in F(y)[x],$$

implies $A \in F(y)$ or $B \in F(y)$.

(a) A is expressed by

$$A(x,y) = \frac{a_0(y)}{b_0(y)} + \frac{a_1(y)}{b_1(y)}x + \cdots + \frac{a_m(y)}{b_m(y)}x^m.$$

If we take $g(y) = b_0(y)\cdots b_m(y) \in F[y]$ the product of the b_i (or the lcm of the b_i), then $g(y)\frac{a_i(y)}{b_i(y)} \in F[y]$, thus $A_1 = g(y)A \in F[x,y]$. Similarly, there is $h \in F[y]$ such that $B_1 = h(y)B \in F[x,y]$.

(b) Therefore, $g(y)h(y)(a(x) - yb(x)) = A_1B_1 \in F[x,y]$, where g, h, f, A_1, B_1 are in $F[x,y]$. As f is irreducible and divides A_1, B_1 in the UFD $F[x,y]$, f divides A_1 or f divides B_1.

(c) Suppose by example that f divides A_1 (the other case is similar):

$$A_1 = (a(x) - yb(x))A_2, \ \ A_2 \in F[x,y].$$

Then, dividing the equality in (b) by $a(x) - yb(x) \neq 0$, we obtain

$$g(y)h(y) = A_2B_1.$$

The degree of x in A_2B_1 is zero, thus the degree of x in B_1 is also 0, so $B_1 \in F[y]$.

(d) Consequently $B = B_1(y)/h(y) \in F(y)$. In the other case, we obtain $A \in F(y)$. So $a(x) - yb(x)$ is irreducible in $F(y)[x]$.

$\square$

Ex. 7.5.4 *Prove that the map $\Phi : \mathrm{GL}(2,F) \to \mathrm{Gal}(F(t)/F)$ defined in the proof of Theorem 7.5.7 is a group homomorphism.*

Proof. Let $\Phi : \left\{ \begin{array}{ccc} \mathrm{GL}(2,F) & \to & \mathrm{Gal}(F(t)/F) \\ \gamma & \mapsto & \sigma_{\gamma^{-1}}. \end{array} \right.$

Let $\delta = \begin{pmatrix} e & f \\ g & h \end{pmatrix}, \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in $\mathrm{GL}(2,F)$. Then $\delta\gamma = \begin{pmatrix} ea+fc & eb+fd \\ ga+hc & gb+hd \end{pmatrix}$.
For all $\alpha \in F(t)$, define $\beta = \sigma_\delta(\alpha) = \alpha\left(\frac{et+f}{gt+h}\right)$.

$$\begin{aligned}
(\sigma_\gamma \circ \sigma_\delta)(\alpha) &= \sigma_\gamma[\sigma_\delta(\alpha)] \\
&= \sigma_\gamma(\beta) \\
&= \beta\left(\frac{at+b}{ct+d}\right) \\
&= \alpha\left(\frac{e\left(\frac{at+b}{ct+d}\right)+f}{g\left(\frac{at+b}{ct+d}\right)+h}\right) \\
&= \alpha\left(\frac{(ea+fc)t+(eb+fd)}{(ga+hc)t+(gb+hd)}\right) \\
&= \sigma_{\delta\gamma}(\alpha)
\end{aligned}$$

Therefore

$$\sigma_\gamma \circ \sigma_\delta = \sigma_{\delta\gamma}.$$

Applying this equality to δ^{-1}, γ^{-1}, we obtain

$$\Phi(\delta) \circ \Phi(\gamma) = \sigma_{\delta^{-1}} \circ \sigma_{\gamma^{-1}} = \sigma_{\gamma^{-1}\delta^{-1}} = \sigma_{(\delta\gamma)^{-1}} = \Phi(\delta\gamma).$$

For all $\delta, \gamma \in \mathrm{GL}(2, F)$,

$$\Phi(\delta) \circ \Phi(\gamma) = \Phi(\delta\gamma).$$

Φ is so a group homomorphism.

Note: in terms of group actions, if we write $\alpha^\gamma = \alpha(\frac{at+b}{ct+d})$, the preceding calculation proves that $(\alpha^\delta)^\gamma = \alpha^{\delta\gamma}$, so $\gamma \mapsto \alpha^\gamma = \alpha \cdot \gamma$ defines a right action, and this is equivalent to the fact that $\Phi : GL(2, F) \to F(t)$ defined by $\gamma \mapsto \alpha^{\gamma^{-1}}$ is a group homomorphism :

$$[\Phi(\delta) \circ \Phi(\gamma)](\alpha) = (\alpha^{\gamma^{-1}})^{\delta^{-1}} = \alpha^{\gamma^{-1}\delta^{-1}} = \alpha^{(\delta\gamma)^{-1}} = \Phi(\delta\gamma)(\alpha).$$

$\square$

Ex. 7.5.5 *Prove (7.26):* $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(2, F)$

Proof. If $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \notin \mathrm{GL}(2, F)$, then the two rows $(a, b), (c, d)$ are linearly dependent.

Moreover $(c, d) \neq 0$, otherwise $B(t) = at + b$ is zero, in contradiction with $\sigma(t) = A(t)/B(t) \in F(t)$.

So there exists $\lambda \in F$ such that $(a, b) = \lambda(c, d)$, and then $\sigma(t) = A(t)/B(t) = \lambda \in F$. As $\sigma^{-1} \in \mathrm{Gal}(F(t)/F)$, $t = \sigma^{-1}(\lambda) = \lambda \in F$, which is impossible since t is transcendental over F.

Conclusion: $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(2, F)$ $\square$

Ex. 7.5.6 *In this exercise, you will prove that* $\mathrm{PGL}(2, F)$ *acts on* $\hat{F} = F \cup \{\infty\}$.

(a) First show that

$$\gamma \cdot \alpha = \frac{a\alpha + b}{c\alpha + d}, \qquad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

defines an action of $\mathrm{GL}(2, F)$ *on* $\hat{F}$. *Explain carefully what happens when* $\alpha = \infty$.

(b) Show that nonzero multiples of the identity matix act trivially on $\hat{F}$, *and use this to give a carefull proof that (7.27) gives a well-defined action of* $PGL(2, F)$ *on* $\hat{F}$.

Proof. (a) The group $GL(2, F)$ of the matrix $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $ad - bc \neq 0$ acts on F^2, identified to the matrix columns of order 2, by the action defined by

$$(x', y') = M \cdot (x, y) \iff \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}$$

Indeed, if we write $X = \begin{pmatrix} x \\ y \end{pmatrix}$, then $I \cdot X = X$, $M \cdot (N \cdot X) = (MN) \cdot X$.

The relation $\mathcal{R}$ defined on $F^2 \setminus \{(0, 0)\}$ by

$$(x, y)\mathcal{R}(x', y') \iff \exists \lambda \in F^*, x' = \lambda x, y' = \lambda y$$

is an equivalence relation. The quotient set is the projective line $\mathbb{P}_1(F)$. Write $[x, y]$ the class of (x, y) for the relation $\mathcal{R}$, in other words the projective point with homogen coordinates (x, y).

If $(x, y)\,\mathcal{R}\,(x', y')$, then $M \cdot (x, y)\ \mathcal{R}\ M \cdot (x', y')$. Moreover $M \cdot (x, y) \neq (0, 0)$ if $(x, y) \neq (0, 0)$, so we can define the action on a projective point $P = [x, y]$ by $M \cdot [x, y] = M \cdot (x, y)$, where (x, y) is any representative of the class P. This is again an action of the group $GL(2, F)$ on the set $\mathbb{P}_1(F)$.

The map $f : \mathbb{P}_1(F) \to \hat{F} = F \cup \{\infty\}$, defined for $X = [x, y]$ by $f([x, y]) = x/y$ if $y \neq 0$, $f([x, 0]) = \infty$ otherwise, is well defined, and this is a bijection, whose inverse $f^{-1} = g$ is defined by $g(x) = [x, 1], g(\infty) = [1, 0]$.

By representing the projective point by its coordinate $z \in F \cup \{\infty\}$, we define for $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $M \cdot z = f(M \cdot f^{-1}(z))$. Explicitly, for $z \in F \setminus \{-d/c\}$

$$M \cdot z = f\left(M \cdot [z, 1]\right) = f([az + b, cz + d]) = \frac{az + b}{cz + d}$$

and also

$$M \cdot (-d/c) = \infty, M \cdot \infty = a/c$$

The group $GL(2, F)$ acts on $\hat{F}$: for all $z \in \hat{F}$, and all $M, N \in GL(2, F)$, $I.z = z$ and

$$\begin{aligned} M \cdot (N \cdot z) &= f(M \cdot f^{-1}(f(N \cdot f^{-1}(z))) \\ &= f(M \cdot (N \cdot f^{-1}(z))) = f(MN \cdot f^{-1}(z)) \\ &= (MN) \cdot z \end{aligned}$$

We resume this in the following proposition:

Proposition. *The action defined for every* $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2, F)$ *and for every* $z \in \hat{F} = F \cup \{\infty\}$ *by*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az + b}{cz + d} \qquad (z \in F \setminus \{-d/c\})$$

$$M \cdot (-d/c) = \infty, \quad M \cdot \infty = a/c \qquad (\text{if } c \neq 0),$$

$$M \cdot \infty = \infty \qquad (\text{if } c = 0),$$

is a (left) action of the group $GL(2, F)$ *on* $F \cup \{\infty\}$*: for all* $z \in \hat{F}$*, and for all* $M, N \in GL(2, F)$*,*

(i) $I \cdot z = z$

(ii) $M \cdot (N \cdot z) = (MN) \cdot z$

(b) If $\lambda \in F^*$, and $z \in F$, $(\lambda I) \cdot z = \frac{\lambda z + 0}{0.z + \lambda} = z$, so $(\lambda I) \cdot \infty = \infty$. The elements of $C = F^* I_2$ act trivially on $\hat{F}$. The quotient group $PGL(2, F) = GL(2, F)/C$, where $C = F^* I_2 = \{\lambda I, \lambda \in F^*\}$, acts on $\hat{F}$.

Indeed the action is well defined: two elements M, N of a same class modulo C satisty $M = \lambda N, \lambda \in F^*$, thus $M \cdot z = (\lambda N) \cdot z = N \cdot ((\lambda I_2) \cdot z) = N \cdot z$. We can so define the action by $[M] \cdot z = M \cdot z$, where $[M]$ is the class of M in $\mathrm{PGL}(2, F)$. Then the relations (i)(ii) are always true

(i) $[I] \cdot z = z$
(ii) $[M] \cdot ([N] \cdot z) = ([M][N]) \cdot z$

$\square$

Ex. 7.5.7 *Proposition 7.5.8 asserts that we can map any triple of distinct points of $\hat{F}$ to any other such triple via a unique element $[\gamma] \in \mathrm{PGL}(2, F)$. We will defer the proof of existence of $[\gamma]$ until Exercise 24 in Section 14.3. In this exercise, we will prove the uniqueness part of the proposition, since this is what is used in Example 7.5.10.*

(a) First suppose that $[\gamma] \in \mathrm{PGL}(2, F)$ fixes ∞ and also fixes two points $\alpha \neq \beta$ of F. Prove that γ is a nonzero multiple of the identity matrix.

(b) Now suppose that $[\gamma] \in \mathrm{PGL}(2, F)$ fixes three distinct points of F, and let α be one of these points. Show that there is $[\delta] \in \mathrm{PGL}(2, F)$ such that $[\delta] \cdot \alpha = \infty$. Then prove that γ is a nonzero multiple of the identity matrix by applying part (a) to $[\delta\gamma\delta^{-1}]$.

(c) Show that the desired uniqueness follows from parts (a) and (b).

Proof. (a) Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(2, F)$, and suppose that $[\gamma] \in \mathrm{PGL}(2, F)$ fixes ∞, and also two distinct points α, β of F.

If $c \neq 0$, then $[\gamma] \cdot \infty = a/c \neq \infty$, which is in contradiction with the fact that $[\gamma]$ fixes ∞. Therefore $c = 0$.

If $z \in F$, using $c = 0$,

$$\gamma \cdot z = z \iff \frac{az + b}{cz + d} = z \iff cz^2 + (d - a)z - b = 0 \iff (d - a)z - b = 0.$$

This equation is satisfied by α and β. The polynomial $(d - a)x - b$ has degree at most 1 and has two distinct roots $\alpha \neq \beta$, thus is the null polynomial. This implies $c = b = 0, a = d$, so $\gamma \in F^* I_2$, and $[\gamma] = e$ is the identity of the group $\mathrm{PGL}(2, F)$.

(b) Suppose now that $[\gamma]$ fixes three distinct points α, β, ξ de F.

Let $\delta = \begin{pmatrix} 0 & 1 \\ -1 & \alpha \end{pmatrix}$. Then $\det(\delta) = 1 : [\delta] \in \mathrm{PGL}(2, F)$, and

$$[\delta] \cdot \alpha = \infty, \qquad \forall z \neq \alpha,\ [\delta] \cdot z = \frac{1}{\alpha - z}.$$

As β, ξ are two distinct elements of F, then $\delta(\beta), \delta(\xi)$ are two distinct points of F since δ is a bijection of $\hat{F}$.

Moreover $\eta = \delta\gamma\delta^{-1}$ satisfies

$$[\eta] \cdot \infty = [\delta\gamma\delta^{-1}] \cdot \infty = [\delta\gamma] \cdot \alpha = [\delta] \cdot \alpha = \infty$$
$$[\eta] \cdot ([\delta] \cdot \beta) = [\delta\gamma\delta^{-1}] \cdot ([\delta] \cdot \beta) = [\delta\gamma] \cdot \beta = [\delta] \cdot \beta$$
$$[\eta] \cdot ([\delta] \cdot \xi) = [\delta\gamma\delta^{-1}] \cdot ([\delta] \cdot \xi) = [\delta\gamma] \cdot \xi = [\delta] \cdot \xi$$

So η fixes the three points $\infty, [\delta] \cdot \beta, [\delta] \cdot \xi$, where $[\delta] \cdot \beta, [\delta] \cdot \xi$ are two distinct points of F. By part (a), $\eta = \lambda I_2, \lambda \in F^*$. Therefore $\gamma = \delta^{-1}\eta\delta = \lambda\delta^{-1}\delta = \lambda I_2$, so $[\gamma] = e$ is the identity of $\mathrm{PGL}(2, F)$.

(c) By parts (a) and (b), if $[\gamma]$ fixes three points of $\hat{F}$, then $[\gamma] = e$.

If γ, γ' satisfy $[\gamma] \cdot \alpha_i = [\gamma'] \cdot \alpha_i$, $i = 1, 2, 3$, then $[\gamma' \gamma^{-1}]$ fixes three points of $\hat{F}$. Therefore $[\gamma' \gamma^{-1}] = [\gamma'][\gamma]^{-1} = e$, so $[\gamma'] = [\gamma]$: the uniqueness is proved.

□

Ex. 7.5.8 *Prove the formula (7.28) for stereographic projection.*

Proof. Let $P = (a, b, c) \neq (0, 0, 1)$ be a point of S_2, so $a^2 + b^2 + c^2 = 1$. Then $c \neq 1$. Write $N = (0, 0, 1)$ the north pole. Any point $M = (x, y, z)$ lies on the line (NP), if and only if $\overrightarrow{NM} = \lambda \overrightarrow{NP}$, $\lambda \in \mathbb{R}^*$, which gives the parametric system of equations

$$\begin{aligned} x &= \lambda a \\ y &= \lambda b \\ z &= \lambda(c-1) + 1 \end{aligned}$$

The intersection with the equatorial plane is given by $z = 0$, so $\lambda = 1/(1-c)$, which gives $x = a/(1-c), y = b/(1-c)$:

$$\pi(a, b, c) = \left(\frac{a}{1-c}, \frac{b}{1-c}, 0 \right) = \frac{a}{1-c} + i \frac{b}{1-c},$$

(where the points $(x, y, 0)$ are identified with the complex numbers $x + iy$.) □

Ex. 7.5.9 *In Example 7.5.10, we consider rotations r_1, r_2, r_3 of the octahedron and defined matrices $\gamma_1, \gamma_2, \gamma_3 \in \mathrm{GL}(2, \mathbb{C})$. We also proved carefully that r_1 corresponds to $[\gamma_1]$ under the homomorphism of Theorem 7.5.9. In a similar way, prove that r_2 corresponds to $[\gamma_2]$ and r_3 corresponds to $[\gamma_3]$.*

Proof. • The text proves that the isomorphism $r \mapsto [\gamma]$ sends $r_1 = \mathrm{Rot}(\pi, \overrightarrow{e_1})$ on γ_1, where

$$[\gamma_1] \cdot z = \frac{1}{z}.$$

• Let $\gamma_2 = \begin{pmatrix} i & 0 \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}(2, F)$. The homography $[\gamma_2]$ satisfies $[\gamma_2] \cdot z = iz$ for all $z \in \mathbb{C}$, and $[\gamma_2] \cdot \infty = \infty$. So

$$[\gamma_2] \cdot \infty = \infty, \quad [\gamma_2] \cdot i = -1, \quad [\gamma_2] \cdot 1 = i.$$

The rotation $r_2 = \mathrm{Rot}(\pi/2, \overrightarrow{e_3})$ satisfies

$$\begin{aligned} &r_2(\hat{\pi}^{-1}(\infty)) = r_2(N) = N = \hat{\pi}^{-1}(\infty), \\ &r_2(\hat{\pi}^{-1}(i)) = r_2(0, 1, 0) = (-1, 0, 0) = \hat{\pi}^{-1}(-1), \\ &r_2(\pi^{-1}(1)) = r_2(1, 0, 0) = (0, 1, 0) = \hat{\pi}^{-1}(i). \end{aligned}$$

Thus

$$[\hat{\pi} \circ r_2 \circ \hat{\pi}^{-1}] \cdot \infty = \infty, \; [\hat{\pi} \circ r_2 \circ \hat{\pi}^{-1}] \cdot i = -1, \; [\hat{\pi} \circ r_2 \circ \hat{\pi}^{-1}] \cdot 1 = i.$$

By the uniqueness proved in Exercise 8, $[\hat{\pi} \circ r_2 \circ \hat{\pi}^{-1}] = [\gamma_2]$.

In other words, the isomorphism $r \mapsto [\gamma]$ sends $r_2 = \text{Rot}(\pi/2, \overrightarrow{e_3})$ on γ_2, where

$$[\gamma_2] \cdot z = iz.$$

- Let $\gamma_3 = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \in \text{GL}(2, F)$.

The homography $[\gamma_3]$ satisfies $[\gamma_3] \cdot z = \frac{z-1}{z+1}$ for all $z \in \mathbb{C}$, and $[\gamma_3] \cdot \infty = 1$. So

$$[\gamma_3] \cdot i = i, \quad [\gamma_3] \cdot (-i) = -i, \quad [\gamma_3] \cdot \infty = 1.$$

The rotation $r_3 = \text{Rot}(\pi/2, \overrightarrow{e_2})$ satisfies

$$\begin{aligned} r_3(\hat{\pi}^{-1}(i)) &= r_3(0,1,0) = (0,1,0) = \hat{\pi}^{-1}(i), \\ r_3(\pi^{-1}(-i)) &= r_3(0,-1,0) = (0,-1,0) = \hat{\pi}^{-1}(-i) \\ r_3(\hat{\pi}^{-1}(\infty)) &= r_3(0,0,1) = (1,0,0) = \hat{\pi}^{-1}(1), \end{aligned}$$

thus

$$[\hat{\pi} \circ r_3 \circ \hat{\pi}^{-1}] \cdot i = i, \; [\hat{\pi} \circ r_2 3 \circ \hat{\pi}^{-1}] \cdot (-i) = -i, \; [\hat{\pi} \circ r_3 \circ \hat{\pi}^{-1}] \cdot \infty = 1$$

By the same uniqueness property, $[\hat{\pi} \circ r_3 \circ \hat{\pi}^{-1}] = [\gamma_3]$.

In other words, the isomorphism $r \mapsto [\gamma]$ sends $r_3 = \text{Rot}(\pi/2, \overrightarrow{e_3})$ on γ_3, where

$$[\gamma_3] \cdot z = \frac{z-1}{z+1}.$$

□

Ex. 7.5.10 *The goal of this exercise is to prove that the symmetry group G of the octahedron is isomorphic to S_4. By symmetry group, we mean the group of rotations that carry the octahedron to itself. We think of G as acting on the octahedron.*

(a) Let ν be a vertex of the octahedron. Use the action of G on ν and the Fundamental Theorem of Group Actions to prove that $|G| = 24$.

(b) The eight face centers of the octahedron form the vertices of an inscribed cube. Explain why the octahedron and its inscribed cube have the same symmetry group.

(c) The cube has four long diagonals that connect a vertex to an opposite vertex. Explain why the action of G on these diagonals gives a group homomorphism $G \to S_4$.

(d) Let $r_1, r_2, r_3 \in G$ be the rotations described in Example 7.5.1. Explain how each rotation acts on the inscribed cube and describe its corresponding permutation in S_4.

(e) Prove that the three permutations constructed in part (d) generates S_4.

(f) Use part (a) and (c) to show that $G \simeq S_4$. Also prove that G is generated by r_1, r_2, r_3.

See Section 14.4 for a different approach to proving that a group is isomorphic to S_4.

Proof. (a) Write S the set of the 6 vertices of the octahedron, with coordinates

$$(\pm 1, 0, 0), (0, \pm 1, 0), (0, 0, \pm 1),$$

and G the group of rotations that carry S to itself. G acts transitively on S, we can go from a vertex to a near vertex by a rotation of angle $\pm\pi/2$, the axe being orthogonal to the plane containing these two summits and O. The orbit $\mathcal{O}_\nu$ of a fixed vertex ν is so the whole octahedron S:

$$|\mathcal{O}_\nu| = 6.$$

Write G_ν the stabilizer in G of the vertex ν.

Every rotation $r \in G$ gives a permutation of the 6 vertices of the octahedron, thus fixes their gravity center $O = (0, 0, 0)$. If $r \in G_\nu \setminus \{e\}$, r fixes O and ν, so is a rotation of axis 0ν. Thus, if P is the orthogonal plane of the axe 0ν, r sends P on itself. The restriction of r to this plane is so a rotation that carry the square of vertices of S which lie in this plane to itself. So it is a rotation of angle $k\pi/2, k = 0, 1, 2, 3$. As the rotation r of axis $O\nu$ is uniquely determined by this restriction, G_ν is so the set of 4 rotations of axis $O\nu$, and of angle $k\pi/2,\ k = 0, 1, 2, 3$.

$$|G_\nu| = 4.$$

The Fundamental Theorem of Group Actions gives then $|O_\nu| = [G : G_\nu]$, thus

$$|G| = |O_\nu| \times |G_\nu| = 6 \times 4 = 24.$$

(b) As a rotation $r \in G$ is an isometry, r sends the 3 points of a face of the octahedron on the three points of a face of the same octahedron, so sends the gravity center of a face on the gravity center of the image. The cube C whose vertices are the center of the faces of the octahedron is so invariant by G. Conversely if a rotation r let invariant the cube C, it let invariant the octahedron whose vertices are the centers of the 6 faces of the cube de S, this octahedron is a dilatation of S, thus $r \in G$. So G is the symmetry group of C.

(c) As $r \in G$ is an isometry, r sends a long diagonal on a long diagonal, and two distinct long diagonal have not the same image. r gives then a permutation of the 4 diagonals, numbered 1,2,3,4, and so induces a permutation of S_4. The composition of two rotations corresponds to the composition of two permutations. So we obtain a group homomorphism

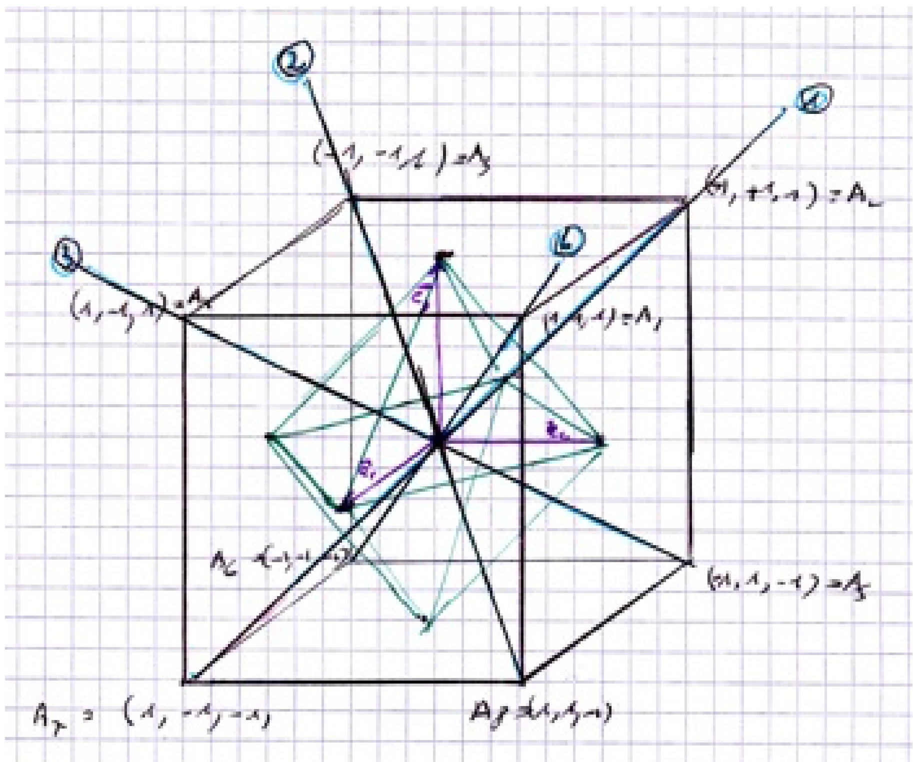
$$\varphi : G \to S_4.$$

The cube of the centers of the faces of S is a dilatation of a cube whose vertices are the points $(\pm 1, \pm 1, \pm 1)$,

$$A_1 = (1, 1, 1), A_2 = (-1, 1, 1), A_3 = (-1, -1, 1), A_4 = (1, -1, 1),$$
$$A_5 = (-1, 1, -1), A_6 = (-1, -1, -1), A_7 = (1, -1, -1), A_8 = (1, 1, -1).$$

We give an arbitrary numbering of the four long diagonals:

$$D_1 = A_2A_7, D_2 = A_3A_8, D_3 = A_4A_5, D_4 = A_1A_6.$$

(d) 

The rotation $r_1 = \mathrm{Rot}(\pi, \overrightarrow{e_1})$ exchanges A_4 and A_8, and also A_2 and A_6, thus exchanges D_3 with D_2, D_1 with D_4 :

$$\varphi(r_1) = (14)(23).$$

$r_2 = \mathrm{Rot}(\pi/2, \overrightarrow{e_3})$ gives the cycle $A_1 \mapsto A_2 \mapsto A_3 \mapsto A_4 \mapsto A_1$, thus $D_1 \mapsto D_2 \mapsto D_3 \mapsto D_4 \mapsto D_1$:

$$\varphi(r_2) = (1234).$$

$r_3 = \mathrm{Rot}(\pi/2, \overrightarrow{e_2})$ gives $A_1 \mapsto A_8 \mapsto A_5 \mapsto A_2 \mapsto A_1$, thus $D_1 \mapsto D_4 \mapsto D_2 \mapsto D_3 \mapsto D_1$:

$$\varphi(r_3) = (1423).$$

(e) Let $H = \langle (14)(23), (1234), (1423) \rangle \subset S_4$.

H contains $[(14)(23)] \circ (1234) = (13)$. Moreover the two permutations $(13) = (31), (1423) = (3142)$ generate S_4, since $(a_1 a_2), (a_1 a_2 \cdots a_n)$ generate S_n generally. Thus $H = G$:

$$S_4 = \langle \varphi(r_1), \varphi(r_2), \varphi(r_3) \rangle.$$

(f) As the subgroup $\varphi(G)$ contains $\varphi(r_1), \varphi(r_2), \varphi(r_3)$, it contains $S_4 = \langle \varphi(r_1), \varphi(r_2), \varphi(r_3) \rangle$. Therefore

$$\varphi(G) = S_4.$$

So $\varphi : G \to S_4$ is surjective. Moreover $|G| = |S_4| = 24$, so φ is bijective, $\varphi : G \to S_4$ is so an isomorphism.

$$G \simeq S_4.$$

As $\varphi(G) = \langle \varphi(r_1), \varphi(r_2), \varphi(r_3) \rangle$, where φ is an isomorphism,

$$G = \langle r_1, r_2, r_3 \rangle.$$

$\square$

Ex. 7.5.11 *In this exercise, you will represent* $\mathrm{AGL}(1, F)$ *as a subgroup of* $\mathrm{PGL}(2, F)$.

(a) *Show that the map* $\gamma_{a,b} \mapsto \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}$

defines a one-to-one group homomorphism

$$\mathrm{AGL}(1, F) \to \mathrm{PGL}(2, F).$$

(b) *Consider the action of* $\mathrm{PGL}(2, F)$ *on* $\hat{F}$. *Show that the isotropy subgroup of* $\mathrm{PGL}(2, F)$ *acting on* ∞ *is the image of the homomorphism of part (a).*

Proof. (a) Write $\gamma_{a,b} : F \to F,\ \alpha \mapsto \gamma_{a,b}(\alpha) = a\alpha + b$. For all $\alpha \in F$,

$(\gamma_{a,b} \circ \gamma_{c,d})(\alpha) = a(c\alpha + d) + b = ac\alpha + ad + b = \gamma_{ac,ad+b}(\alpha)$,

thus

$$\gamma_{a,b} \circ \gamma_{c,d} = \gamma_{ac,ad+b}.$$

Let

$$\varphi : \left\{ \begin{array}{ccc} \mathrm{AGL}(1, F) & \to & \mathrm{PGL}(2, F) \\ \gamma_{a,b} & \mapsto & \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix} \end{array} \right. .$$

$$\begin{aligned} \varphi(\gamma_{a,b})\varphi(\gamma_{c,d}) &= \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix} \begin{bmatrix} c & d \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} ac & ad+b \\ 0 & 1 \end{bmatrix} \\ &= \varphi(\gamma_{ac,ad+b}) \\ &= \varphi(\gamma_{a,b} \circ \gamma_{c,d}). \end{aligned}$$

$\varphi : \mathrm{AGL}(1, F) \to \mathrm{PGL}(2, F)$ is so a group homomorphism.

$$\begin{aligned} \gamma_{a,b} \in \ker(\varphi) &\iff \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix} = [I_2] \\ &\iff \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix},\ \lambda \in F^* \\ &\iff a = 1, b = 0 \\ &\iff \gamma_{a,b} = 1_F \end{aligned}$$

$\ker(\varphi) = \{1_F\}$, thus φ is an injective group homomorphism, which embeds $\mathrm{AGL}(1, F)$ in $\mathrm{PGL}(2, F)$.

(b) Write G_∞ the stabilizer of ∞ in $\mathrm{PGL}(2, F)$.

$$\begin{aligned} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in G_\infty &\iff \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \lambda \begin{pmatrix} 1 \\ 0 \end{pmatrix},\ \lambda \in F^* \\ &\iff c = 0 \end{aligned}$$

Let $\gamma = \begin{pmatrix} r & s \\ t & u \end{pmatrix} \in \mathrm{GL}(2,F)$. If $[\gamma] \in \varphi(\mathrm{AGL}(1,F))$, then $[\gamma] = \varphi(\gamma_{a,b}) = \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}$, thus $[\gamma] \in G_\infty$ by the preceding equivalence.

Conversely, if $[\gamma] \in G_\infty$, then $t = 0$, therefore $\det(\gamma) = ru \neq 0$, so $u \neq 0$.

$\begin{pmatrix} r & s \\ t & u \end{pmatrix} = u \begin{pmatrix} r/u & s/u \\ 0 & 1 \end{pmatrix}$, $u \in F^*$, thus $[\gamma] = \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}$, where $a = r/u, b = s/u$: $[\gamma] = \varphi(\gamma_{a,b}) \in \varphi(\mathrm{AGL}(1,F))$.

$$G_\infty = \varphi(\mathrm{AGL}(1,F)).$$

So $\mathrm{AGL}(1,F)$ is identified with the stabilizer of ∞ in $\mathrm{PGL}(2,F)$ and is isomorphic to a subgroup of $\mathrm{PGL}(2,F)$.

□

Ex. 7.5.12 *In this exercise, you will construct polyhedra whose symmetry groups are isomorphic to C_n and D_{2n}. For D_{2n}, consider the polyhedron whose vertices are the north and south poles of S^2 together with the nth roots of unity along the equator (see picture in [D.Cox]). Note that to obtain a three dimensional object, we must assume $n \geq 3$.*

(a) Show that the symmetry group of this polyhedron is isomorphic to D_{2n} when $n \neq 4$, and S_4 when $n = 4$.

(b) Now take the vertices on the equator and move them up in S_2 so that they become the vertices of a regular n-gone lying in the plane $z = c$, where $c > 0$ is small. Prove that the symmetry group of this polyhedron is isomorphic to C_n.

(c) Find polyhedra inscribed in S^2 whose symmetry groups are C_1 (the trivial group), C_2, D_4 (the Klein four-group), and D_8 respectively.

Proof. (a) Write N, S the north and south poles, and A_k the point of complex coordinate ζ_n^k in the equatorial plane. The polyhedron P is the set of vertices

$$P = \{N, S, A_0, A_1, \cdots, A_{n-1}\}.$$

The group of symmetry of this polyhedron is

$$G = G_P = \{r \in \mathrm{SO}(3) \mid r(P) = P\}.$$

G contains the rotation $r = \mathrm{Rot}(\vec{e}_3, 2\pi/n)$ of axis $(O, \vec{e}_3)$ and angle $2\pi/n$, and also the rotation $s = \mathrm{Rot}(\vec{e}_1, \pi)$. So it contains the set $H = \{e, r, \cdots, r^{n-1}, s, rs, \cdots, r^{n-1}s\}$.

$$G \supset H = \{e, r, \cdots, r^{n-1}, s, rs, \cdots, r^{n-1}s\}.$$

The rotations $r^k = \mathrm{Rot}(\vec{e}_3, 2k\pi/n),\ k = 0, \cdots, n-1$ send A_0 on A_k. So they are distinct and they fix the poles, and the rotations $r^{n-1} \circ s$ are distinct and exchange the two poles, so are distinct of the r^k. Consequently $|H| = 2n$.

$$\begin{aligned}
(r \circ s)(O) &= O = (s \circ r^{-1})(O), \\
(r \circ s)(P) &= S = (s \circ r^{-1})(P), \\
(r \circ s)(A_0) &= A_1 = (s \circ r^{-1})(A_0), \\
(r \circ s)(A_1) &= A_0 = (s \circ r^{-1})(A_1).
\end{aligned}$$

The 4 points O, P, A_0, A_1 are not coplanar, so form an affine frame, so the two rotations $r \circ s, s \circ r^{-1}$ gives the same image to the points of this frame are identical.

As r is of order n, as s is of order 2, and as $r \circ s = s \circ r^{-1}$, H is a subgroup of G isomorphic to the diedral group D_{2n}.

We show that if $n \neq 4, n \geq 3$, this inclusion is an equality.

Let $\rho \in G, \rho \neq e$, a rotation in G. ρ fixes the gravity center O of P, thus $\rho(O) = O$.

Write $A'_k = \rho(A_k)$. As ρ is an isometry, $A'_0A'_1 = A_0A_1 = |\zeta_n - 1|$.

Moreover

$$\begin{aligned}
A'_0A'_1 = A_0A_1 = |\zeta_n - 1| &= |e^{2i\pi/n} - 1| \\
&= \sqrt{\left(\cos\frac{2\pi}{n} - 1\right)^2 + \sin^2\frac{2\pi}{n}} \\
&= \sqrt{2\left(1 - \cos\frac{2\pi}{n}\right)} \\
&= 2\sin\frac{\pi}{n}
\end{aligned}$$

Note that $PA_k = SA_k = \sqrt{2}$.

As $n \geq 3$,

$2\sin\frac{\pi}{n} = 2$ is impossible since $\sin\frac{\pi}{n} \leq \sin\frac{\pi}{3} < 1$.

$2\sin\frac{\pi}{n} = \sqrt{2} \iff \sin\frac{\pi}{n} = \sin\frac{\pi}{4} \iff n = 4$

Suppose that $n \neq 4$. With a reductio ad absurdum, if A'_0 was a pole, then $A'_0A'_1 = \sqrt{2}$ (if $A'_1 = A_k$) or $A'_0A'_1 = 2$ (if A'_1 is the opposite pole). In both cases, this is impossible, as previously proved.

Consequently $\rho(A_0) = A_k,\ k = 0, 1, \cdots, n-1$. The same argument proves that the image of A_i is in the polygon $\{A_0, , \cdots, A_{n-1}\}$, so ρ is a permutation of the vertices of this polygon, thus sends $\{P, S\}$ over $\{P, S\}$. Therefore ρ fixes these two poles, or exchanges them.

- case 1: if $\rho(P) = P$, then since $\rho(O) = O, \rho(A_0) = A_k$, ρ is the rotation $r^k = \mathrm{Rot}(\vec{e}_3, 2k\pi/n)$ of axis OP ($1 \leq k \leq n-1$), or the identity ($k = 0$).
- case 2: if $\rho(P) = S$, then $(\rho \circ s)(P) = P$, and by case 1, $\rho \circ s = r^j, j = 0, \cdots, n-1$, that is $\rho = r^j \circ s$.

In both cases, $\rho \in H$, therefore

$$G = H = \{e, r, \cdots, r^{n-1}, s, rs, \cdots, r^{n-1}s\} \simeq D_{2n}.$$

In the case $n = 4$, we have proved in Exercise 10 that $G \simeq S_4$.

(b) The modification of the polyhedron given in the text (or more simply the suppression of the south pole S) implies then $\rho(N) = N$, so it remains only the case 1 in the preceding discussion, so $G = \langle r \rangle \simeq C_n$.

(c) • The irregular tetrahedron $T = \{(1,0,0),(0,1,0),(0,0,1),(\frac{1}{9},\frac{4}{9},\frac{8}{9})\}$, set of four non coplanar points 4, inscribed in S_2 (since $\left(\frac{1}{9}\right)^2 + \left(\frac{4}{9}\right)^2 + \left(\frac{8}{9}\right)^2 = 1$) and its symmetry group ist $G_T = \{e\} \simeq C_1$.

• Complete an isosceles non equilateral triangle in the equatorial plane by the two poles:

$P = \{(1,0,0),(\frac{4}{5},\frac{3}{5},0),(\frac{4}{5},-\frac{3}{5},0),(0,0,1),(0,0,-1)\}$ has the symmetry group $G_P = \langle s\rangle \simeq C_2$, where $s = \mathrm{Rot}(\vec{e}_1,\pi)$.

• The rectangle in the equatorial plane is completed by the two poles:

$R = \{(\frac{4}{5},\frac{3}{5},0),(\frac{4}{5},-\frac{3}{5},0),(-\frac{4}{5},-\frac{3}{5},0),(-\frac{4}{5},\frac{3}{5},0),(0,0,1),(0,0,-1)\}$ has the symmetry group $G_R = \langle \sigma,\tau,\xi\rangle$, where σ,τ,ξ are the rotations of angles π and axes $\vec{e}_1,\vec{e}_2,\vec{e}_3$. $G_R \simeq D_4$ is the Klein four-group.

• Let C be a rectangular parallelepiped with square basis inscribed in the sphere S_2 :

$C = \{(\frac{\sqrt{40}}{9},\frac{\sqrt{40}}{9},\frac{1}{9}),(-\frac{\sqrt{40}}{9},\frac{\sqrt{40}}{9},\frac{1}{9}),(-\frac{\sqrt{40}}{9},-\frac{\sqrt{40}}{9},\frac{1}{9}),(\frac{\sqrt{40}}{9},-\frac{\sqrt{40}}{9},\frac{1}{9}),$
$(\frac{\sqrt{40}}{9},\frac{\sqrt{40}}{9},-\frac{1}{9}),(-\frac{\sqrt{40}}{9},\frac{\sqrt{40}}{9},-\frac{1}{9}),(-\frac{\sqrt{40}}{9},-\frac{\sqrt{40}}{9},-\frac{1}{9}),(\frac{\sqrt{40}}{9},-\frac{\sqrt{40}}{9},-\frac{1}{9})\}$.

The symmetry group of C is $G_C = \langle r,s\rangle = \{e,r,r^2,r^3,s,rs,r^2s,r^3s\}$, where $r = \mathrm{Rot}(\vec{e}_3,\pi/2), s = \mathrm{Rot}(\vec{e}_1,\pi)$, so $G_C \simeq D_8$.

□

Ex. 7.5.13 *Consider the automorphism of $L = \mathbb{C}(t)$ defined by $\alpha(t) \mapsto \alpha(\zeta_n t)$. This generates a cyclic group G of automorphisms such that $|G| = n$. Adapt the methods of example 7.5.6 to show that $L_G = \mathbb{C}(t^n)$ and conclude that $\mathbb{C}(t^n) \subset \mathbb{C}(t)$ is a Galois extension whose Galois group is cyclic of order n.*

Proof. Let σ the automorphism of $L = \mathbb{C}(t)$ defined by $\alpha(t) \mapsto \alpha(\zeta_n t)$.

For all $\alpha \in \mathbb{C}(t)$, for all $k \in \mathbb{N}$, $\sigma^k(\alpha(t)) = \alpha(\zeta_n^k t)$. Then $\sigma^n = e$, and for $\alpha(t) = t, 1 \le k \le n-1$, $\sigma^k(t) = \zeta_n^k t \ne t$, so $\sigma^k \ne e$. Therefore the order of σ is n, and $G = \langle\sigma\rangle$ is a cyclic group of order n.

By Theorem 7.5.3, the extension $L_G \subset \mathbb{C}(t)$ is a Galois extension of degree n, with Galois group $G = \langle\sigma\rangle$.

We like to specify the field L_G.

$\sigma(t^n) = (\zeta_n t)^n = t^n$, thus $t^n \in L_G$ and so $\mathbb{C}(t^n) \subset L_G \subset \mathbb{C}(t)$.

By Theorem 7.5.5(c), the extension $\mathbb{C}(t^n) \subset \mathbb{C}(t)$ has degree n, so

$$n = [\mathbb{C}(t) : \mathbb{C}(t^n)] = [\mathbb{C}(t) : L_G]\,[L_G : \mathbb{C}(t^n)] = n\,[L_G : \mathbb{C}(t^n)],$$

therefore $[L_G : \mathbb{C}(t^n)] = 1$:

$$L_G = \mathbb{C}(t^n).$$

Conclusion:

$\mathbb{C}(t^n) \subset \mathbb{C}(t)$ is a Galois extension whose Galois group G is cyclic of order n. □

Ex. 7.5.14 *Consider the automorphisms of $L = F(t)$ defined by*

$$\sigma\left(\alpha(t)\right) = \alpha(t^{-1}) \quad \text{and} \quad \tau\left(\alpha(t)\right) = \alpha(1-t).$$

(a) Prove that σ and τ generate a group G of automorphisms of $F(t)$ isomorphic to S_3.

(b) Show that G corresponds to the subgroup of $\mathrm{PGL}(2,F)$ *consisting of all elements that map the subset $\{0,1,\infty\} \subset \hat{F}$ to itself.*

(c) Prove that

$$L_G = F\left(\frac{(t^2-t+1)^3}{t^2(t-1)^2}\right),$$

and conclude that

$$F\left(\frac{(t^2-t+1)^3}{t^2(t-1)^2}\right) \subset F(t)$$

is a Galois extension with Galois group $G \simeq S_3$.

Proof. Consider the automorphisms of $L = F(t)$ defined by

$$\begin{aligned}\sigma(\alpha(t)) &= \alpha(t^{-1}),\\ \tau(\alpha(t)) &= \alpha(1-t),\end{aligned}$$

and $G = \langle \sigma, \tau \rangle$.

(a) Note that $\sigma^2 = \tau^2 = e$, σ and τ have order 2. Let $\rho = \sigma \circ \tau = \sigma\tau$. For all $\alpha(t) \in F(t)$,

$$\begin{aligned}\rho(\alpha(t)) &= \sigma(\alpha(1-t))\\ &= \alpha\left(1-\frac{1}{t}\right),\\ \rho^2(\alpha(t)) &= \alpha\left(1-\frac{1}{1-\frac{1}{t}}\right)\\ &= \alpha\left(\frac{1}{1-t}\right),\\ \rho^3(\alpha(t)) &= \alpha\left(\frac{1}{1-(1-\frac{1}{t})}\right)\\ &= \alpha(t).\end{aligned}$$

Thus ρ is of order 3, and as $\tau = \sigma\rho$, $G = \langle \sigma, \rho \rangle$.

Moreover, for all $\alpha(t) \in F(t)$,

$$\begin{aligned}(\rho\sigma)(\alpha(t)) &= \rho\left(\alpha\left(\frac{1}{t}\right)\right)\\ &= \alpha\left(\frac{t}{t-1}\right),\\ (\sigma\rho^{-1})(\alpha(t)) &= (\sigma\rho^2)(\alpha(t))\\ &= \sigma\left(\alpha\left(\frac{1}{1-t}\right)\right)\\ &= \alpha\left(\frac{1}{1-\frac{1}{t}}\right)\\ &= \alpha\left(\frac{t}{t-1}\right).\end{aligned}$$

Thus $\rho\sigma = \sigma\rho^{-1}$.

To summarise, $G = \langle \sigma, \rho \rangle$, with $\sigma^2 = \rho^3 = e$ $(\sigma \neq e, \rho \neq e)$, $\rho\sigma = \sigma\rho^{-1}$, therefore

$$G \simeq D_6 \simeq S_3.$$

(b) By the isomorphism $\mathrm{PGL}(2, F) \simeq \mathrm{Gal}(F(t)/F)$ described in Section 7.5.C, σ corresponds to $[\gamma] \in \mathrm{PGL}(2, F)$, where $\gamma^{-1} = \gamma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and τ corresponds to $[\delta]$, where $\delta^{-1} = \delta = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}$, and $\rho = \sigma\tau$ to $[\varepsilon], \varepsilon = (\gamma\delta)^{-1} = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$,

so $[\varepsilon] \in \mathrm{PGL}(2, F)$ is of order 3 (but not $\varepsilon \in \mathrm{GL}(2, F) : \varepsilon^3 = -I_2$).

By $[\gamma], [\delta]$ acting on $\hat{F}$,

$$[\gamma] \cdot 0 = \infty, [\gamma] \cdot 1 = 1, [\gamma] \cdot \infty = 0.$$

$$[\delta] \cdot 0 = 1, [\delta] \cdot 1 = 0, [\delta] \cdot \infty = \infty.$$

Thus $\hat{G} = \langle [\gamma], [\delta] \rangle \simeq G$ maps $\{0, 1, \infty\}$ on itself.

Conversely, let $[\xi] = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{PGL}(2, F)$ mapping $A = \{0, 1, \infty\}$ on itself. We show that $[\xi]$ lies in $\hat{G}$.

We know by Exercise 7, which proves uniqueness in Theorem 7.5.8, that there exists at most an element in $\mathrm{PGL}(2, F)$ sending $0, 1, \infty$ on three fixed points in $\hat{F}$. As the elements of $\hat{G}$ map $\{0, 1, \infty\}$ on itself, the group homomorphism sending G on the group S_A of permutions of A is injective by this uniqueness. Moreover $|\hat{G}| = 6 = |S_A|$, so this is a group isomorphism, so the elements of $\hat{G}$ give the 6 possible permutations of $\{0, 1, \infty\}$. As $[\xi]$ has the same images for the elements of A that an element $[\zeta]$ of $\hat{G}$ and as $[\xi]$ is uniquely determined by these images, $[\xi] = [\zeta]$, so $[\xi] \in \hat{G}$.

Conclusion: G corresponds to the subgroup of $\mathrm{PGL}(2, F)$ consisting of all elements that map the subset $\{0, 1, \infty\} \subset \hat{F}$ to itself.

(c) We verify that $\alpha(t) = \frac{(t^2-t+1)^3}{t^2(t-1)^2} \in L_G$:

$$\begin{aligned}
\sigma(\alpha(t)) &= \frac{((\frac{1}{t})^2 - (\frac{1}{t}) + 1)^3}{(\frac{1}{t})^2((\frac{1}{t}) - 1)^2} \\
&= \frac{(1 - t + t^2)^3}{t^2(1-t)^2} \\
&= \alpha(t), \\
\tau(\alpha(t) &= \frac{((1-t)^2 - (1-t) + 1)^3}{(1-t)^2((1-t) - 1)^2} \\
&= \frac{(t^2 - t + 1)^3}{(t-1)^2 t^2} \\
&= \alpha(t).
\end{aligned}$$

As $G = \langle \sigma, \tau \rangle, \alpha(t) \in L_G$, thus

$$F\left(\frac{(t^2 - t + 1)^3}{t^2(t-1)^2}\right) \subset L_G \subset F(t).$$

By Theorem 7.5.3, $L_G \subset F(t)$ is a Galois extension and $[F(t) : L_G] = |G| = 6$, and by Theorem 7.5.5,

$$\left[F(t) : F\left(\frac{(t^2-t+1)^3}{t^2(t-1)^2}\right)\right] = \max(\deg(t^2-t+1)^3), \deg(t^2(t-1)^2)) = 6,$$

so $\left[L_G : F\left(\frac{(t^2-t+1)^3}{t^2(t-1)^2}\right)\right] = 1$, therefore

$$L_G = F\left(\frac{(t^2-t+1)^3}{t^2(t-1)^2}\right).$$

Conclusion: $F\left(\frac{(t^2-t+1)^3}{t^2(t-1)^2}\right) \subset F(t)$ is a Galois extension of Galois group $G \simeq S_3$.

$\square$