Solutions to David A.Cox "Galois Theory"

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9 Chapter 9 : CYCLOTOMIC EXTENSIONS

9.1 CYCLOTOMIC POLYNOMIALS

Ex. 9.1.1 *Prove that a congruence class* $[i] \in \mathbb{Z}/n\mathbb{Z}$ *has a multiplicative inverse if and only if* $\gcd(i, n) = 1$. *Conclude that* $(\mathbb{Z}/n\mathbb{Z})^*$ *has order* $\phi(n)$. *Be sure you understand what happens when* $n = 1$.

Proof. If $[i]$ has a multiplicative inverse in the ring $\mathbb{Z}/n\mathbb{Z}$, then there exists $[j] \in \mathbb{Z}/n\mathbb{Z}$ such that $[i][j] = [ij] = 1$, so $ij \equiv 1\ [n]$. Thus there exists $k \in \mathbb{Z}$ such that $ij - kn = 1$. This Bézout's relation between i and n shows that $i \wedge n = 1$.

Conversely, if $i \wedge n = 1$, by Bézout's Theorem, there exist integers j, k such that $ij - kn = 1$, so $[i][j] = [1]$, and $[i]$ has a multiplicative inverse $\mathbb{Z}/n\mathbb{Z}$.

$$[i] \in (\mathbb{Z}/n\mathbb{Z})^* \iff i \wedge n = 1.$$

The mapping

$$\left\{ \begin{array}{ccc} \{i \in \mathbb{N} \mid 0 \le i < n, i \wedge n = 1\} & \to & (\mathbb{Z}/n\mathbb{Z})^* \\ i & \mapsto & [i] \end{array} \right.$$

obtained by restriction of the bijection $[\![0, n[\![\to \mathbb{Z}/n\mathbb{Z},\ i \mapsto [i]$, is well defined, and this is a bijection.

Therefore

$$|(\mathbb{Z}/n\mathbb{Z})^*| = \mathrm{Card}(\{i \in \mathbb{N} \mid 0 \le i < n, i \wedge n = 1\} = \phi(n).$$

If $n = 1$, the ring $\mathbb{Z}/1\mathbb{Z}$ is the trivial ring $\{[0]\}$, where $[0] = [1]$, so the multiplicative group $(\mathbb{Z}/1\mathbb{Z})^* = \{[1]\}$ has one element, and the set of integers i such that $0 \le i < 1 = n$ is reduced to $\{0\}$, which satisfies $0 \wedge 1 = 1$, so $\phi(1) = 1$. □

Ex. 9.1.2 *Assume that* $\gcd(n, m) = 1$. *By Lemma A.5.2, we have a ring isomorphism* $\alpha : \mathbb{Z}/nm\mathbb{Z} \simeq \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z}$ *that sends* $[a]_{nm}$ *to* $([a_n], [a]_m)$. *Prove that* α *induces a group isomorphism* $(Z/nm\mathbb{Z})^* \simeq (\mathbb{Z}/nZ)^* \times (\mathbb{Z}/m\mathbb{Z})^*$.

Proof. Let A, B to commutative rings (with unity). Then

$$(A \times B)^* = A^* \times B^*.$$

Indeed, let $(a, b) \in A \times B$.

$$\begin{aligned}(a,b)\in(A\times B)^* &\iff \exists(c,d)\in A\times B,\ (a,b)(c,d)=(1,1)\\ &\iff \exists c\in A,\exists c\in B,\ ac=1, bd=1\\ &\iff a\in A^*, b\in B^*\\ &\iff (a,b)\in A^*\times B^*.\end{aligned}$$

Moreover, if $\varphi : A \to B$ is a ring isomorphism, then for all $a \in A$, $a \in A^* \Rightarrow \varphi(a) \in A^*$, since $ab = 1_A \Rightarrow \varphi(a)\varphi(b) = \varphi(1_A) = 1_B$. So we can define $\psi : A^* \to B^*$ by restriction with $a \mapsto \psi(a) = \varphi(a)$.

ψ is a group homomorphism: if $u, v \in A^*, \psi(uv) = \varphi(uv) = \varphi(u)\varphi(v) = \psi(u)\psi(v)$, and ψ is bijective:

- φ is injective, so its restriction ψ if also injective.
- If $b \in B^*$, then there exists $d \in B$ such that $bd = 1$. If we write $a = \varphi^{-1}(b), c = \varphi^{-1}(d)$, then $b = \varphi(a), d = \varphi(c), 1 = bd = \varphi(ac)$, so $ac = 1, a \in A^*$, thus $b = \psi(a)$, so ψ is surjective.

$$A \simeq B \Rightarrow A^* \simeq B^*.$$

If we apply these two results to the rings $\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/m\mathbb{Z}$, we obtain

$$(\mathbb{Z}/nm\mathbb{Z})^* \simeq (\mathbb{Z}/n\mathbb{Z} \times Z/m\mathbb{Z})^* = (\mathbb{Z}/nZ)^* \times (\mathbb{Z}/m\mathbb{Z})^*.$$

□

Ex. 9.1.3 *Let $\zeta_n = e^{2\pi i/n} \in \mathbb{C}$. Prove that ζ_n^i for $0 \le i < n$ and $\gcd(i, n) = 1$ are the primitive nth roots of unity in $\mathbb{C}$.*

Proof. Let $\mathbb{U}_n$ be the group of $n$throots of unity in $\mathbb{C}$. Then $\mathbb{U}_n = \langle \zeta_n \rangle$, where $\zeta_n = e^{2i\pi/n}$. Write $o(x)$ the order of an element $x \in G$. Then $o(x) = |\langle x \rangle|$.

Recall that if $d > 0$, $o(x) = d \iff (\forall k \in \mathbb{Z},\ x^k = e \iff d \mid k)$.

For all $i \in \mathbb{Z}$,

$$o(\zeta_n^i) = \frac{n}{n \wedge i}.$$

Indeed, for all $k \in \mathbb{Z}$,

$$(\zeta_n^i)^k = 1 \iff \zeta_n^{ik} = 1 \iff n \mid ik \iff \frac{n}{n \wedge i} \mid \frac{i}{n \wedge i} k \iff \frac{n}{n \wedge i} \mid k$$

(since $\frac{n}{n\wedge i} \wedge \frac{i}{n \wedge i} = 1$). So $o(\zeta_n^i) = \frac{n}{n \wedge i}$.

By definition, ζ is a primitive nth root of unity if and only if ζ is a generator of $\mathbb{U}_n$, if and only if $o(\zeta) = n$, so

$$\mathbb{U}_n = \langle \zeta_n^i \rangle \iff o(\zeta_n^i) = n \iff \frac{n}{n \wedge i} = n \iff n \wedge i = 1.$$

□

Ex. 9.1.4 *Let R be an integral domain, and let $f, g \in R[x]$, where $f \neq 0$. If K is the field of fractions of R, then we can divide g by f in $K[x]$ using the division algorithm of Theorem A.1.14. This gives $g = qf + r$, though $q, r \in K[x]$ need not lie in $R[x]$.*

(a) *Show that dividing x^2 by $2x+1$ in $\mathbb{Q}[x]$ gives $x^2 = q \cdot (2x+1) + r$, where $q, r \in \mathbb{Q}[x]$ are not in $\mathbb{Z}[x]$, even though x^2 and $2x + 1$ lie in $\mathbb{Z}[x]$.*

(b) *Show that if f is monic, then the division algorithm gives $g = qf + r$, where $q, r \in R[x]$. Hence the division algorithm works over R provides we divide by monic polynomials.*

Proof. (a) $x^2 = (\frac{1}{2}x - \frac{1}{4})(2x+1) + \frac{1}{4}$. The quotient $q(x) = \frac{1}{2}x - \frac{1}{4}$ is not in $\mathbb{Z}[x]$.

(b) Let $f = x^m + b_{m-1}x^{m-1} + \cdots + b_0$ be a fixed monic polynomial in $R[x]$.

We show by induction on the degree n the proposition

$P(n) : \forall g \in R[x], \deg(g) = n \Rightarrow \exists (q,r) \in R^2,\ g = qf + r,\ \deg(r) < \deg(f)$

(with the convention $\deg(0) = -\infty$).

We suppose that $P(k)$ is true for all $k < n$, and we prove $P(n)$. Let g any polynomial in $R[x]$.

• If $\deg(g) < m = \deg(f)$, then the pair $(q,r) = (0,g)$ is an answer.

• Suppose that $\deg(g) \geq m$. Write $g = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$, with $\deg(g) = n \geq m$ and $a_i \in R, i = 0, \ldots, n$.

The polynomial $g_1 = g - a_n x^{n-m} f \in R[x]$ satisfies $\deg(g_1) < n$. We can then apply to it the induction hypothesis:

$g_1 = q_1 f + r, q_1 \in R[x],\ r \in R[x],\ \deg(r) < \deg(f)$.

Then $g = (a_n x^{n-m} + q_1)f + r$.

If we write $q = a_n x^{n-m} + q_1$, then $q \in \mathbb{Z}[x]$ and $g = fq + r,\ (q,r) \in \mathbb{Z}[x]^2, \deg(r) < \deg(f)$. The pair (q,r) is an answer, and the induction is done.

In particular, if $g, f \in \mathbb{Z}[x]$, and $g = fq$, the unicity of the Euclidean division in $\mathbb{Q}[x]$ and the preceding result shows that $q \in \mathbb{Z}[x]$.

□

Ex. 9.1.5 *Verify the formula for $\Phi_{105}(x)$ given in Example 9.1.7.*

Proof. The factors of $105 = 3 \times 5 \times 7$ are $105, 35, 21, 15, 7, 5, 3, 1$, thus

$$x^{105} - 1 = \Phi_{105}\, \Phi_{35}\, \Phi_{21}\, \Phi_{15}\, \Phi_7\, \Phi_5\, \Phi_3\, \Phi_1.$$

As $x^{35} - 1 = \Phi_{35}\, \Phi_7\, \Phi_5\, \Phi_1$, we obtain

$$x^{105} - 1 = (x^{35} - 1)\Phi_{105}\, \Phi_{21}\, \Phi_{15}\, \Phi_3,$$

that is

$$x^{70} + x^{35} + 1 = \Phi_{105}\, \Phi_{21}\, \Phi_{15}\, \Phi_3.$$

Moreover $x^{21} - 1 = \Phi_{21}\, \Phi_7\, \Phi_3\, \Phi_1$, so

$$\begin{aligned}
\Phi_{21} &= (x^{21} - 1)\, \frac{x-1}{x^7 - 1}\, \frac{x-1}{x^3-1}\, \frac{1}{x-1} \\
&= \frac{x^{21} - 1}{(x^7 - 1)(x^2 + x + 1)} \\
&= \frac{x^{14} + x^7 + 1}{x^2 + x + 1} \\
&= x^{12} - x^{11} + x^9 - x^8 + x^6 - x^4 + x^3 - x + 1.
\end{aligned}$$

Similarly $x^{15} - 1 = \Phi_{15}\,\Phi_5\,\Phi_3\,\Phi_1$, so

$$\begin{aligned}\Phi_{15} &= \frac{x^{15}-1}{(x^5-1)(x^2+x+1)}\\ &= \frac{x^{10}+x^5+1}{x^2+x+1}\\ &= x^8 - x^7 + x^5 - x^4 + x^3 - x + 1.\end{aligned}$$

Therefore

$$\begin{aligned}\Phi_{105} =& \frac{x^{70}+x^{35}+1}{\Phi_{21}\Phi_{15}\Phi_3}\\ =& \frac{x^{70}+x^{35}+1}{x^{22}-x^{21}+x^{19}-x^{18}+x^{17}+x^{12}-x^{11}+x^{10}+x^5-x^4+x^3-x+1}\\ =&\quad x^{48}+x^{47}+x^{46}-x^{43}-x^{42}-2\,x^{41}-x^{40}-x^{39}+x^{36}+x^{35}+x^{34}+x^{33}+x^{32}\\ &+x^{31}-x^{28}-x^{26}-x^{24}-x^{22}-x^{20}+x^{17}+x^{16}+x^{15}+x^{14}+x^{13}+x^{12}-x^9-x^8\\ &-2\,x^7-x^6-x^5+x^2+x+1\end{aligned}$$

□

Ex. 9.1.6 *This exercise is concerned with the proof of Lemma 9.1.8.*

(a) Let $f \in \mathbb{Z}[x_1, \ldots, x_n]$ be symmetric. Prove that f is a polynomial in $\sigma_1, \ldots, \sigma_n$ with integer coefficients.

(b) Let p be prime and let $h \in \mathbb{F}_p[x_1, \ldots, x_n]$. Prove that $h(x_1, \ldots, x_n)^p = h(x_1^p, \ldots, x_n^p)$.

Proof. (a) The algorithm in the proof of Theorem 2.2.2 consists to replace the symmetric polynomial f, here with coefficients in $\mathbb{Z}$, by $f_1 = f - cg$, $f_2 = f - cg - c_1 g_1, \cdots$, until we obtain 0. The coefficient c is the leading coefficient of f, so it is an integer, and $g = \sigma_1^{a_1-a_2} \cdots \sigma_{n-1}^{a_{n-1}-a_n}\sigma_n^{a_n} \in \mathbb{Z}[\sigma_1, \ldots, \sigma_n]$, so $f_1 \in \mathbb{Z}[x_1, \ldots, x_n]$. The same reasoning applied to f_1 and to the following terms shows that $c_i \in \mathbb{Z}$ for all i. Therefore

$$f = cg + c_1 g_1 + \cdots + c_{m-1} g_{m-1} \in \mathbb{Z}[\sigma_1, \cdots, \sigma_n].$$

In particular, the symmetric polynomial $\sigma_i(x_1^p, \cdots, x_r^p) - \sigma_i(x_1, \cdots, x_r)^p$ is a polynomial in $\sigma_1, \ldots, \sigma_r$ with integer coefficients:

$$\sigma_i(x_1^p, \cdots, x_r^p) - \sigma_i(x_1, \cdots, x_r)^p = S(\sigma_1, \cdots, \sigma_r) \in \mathbb{Z}[\sigma_1, \cdots, \sigma_r].$$

(b) Let $h \in \mathbb{F}_p[x_1, \ldots, x_n]$. Write

$$h = \sum_{(i_1,\ldots,i_n)\in A} a_{i_1,\ldots,i_n} x_1^{i_1} \cdots x_n^{i_n},$$

where $A \subset \mathbb{N}^n$ is finite, and the coefficients $a_{i_1,\ldots,i_n} \in \mathbb{F}_p$. As the characteristic of the field $\mathbb{F}_p(x_1, \cdots, x_r)$ is p, using the Little Fermat's Theorem: $a^p = a$ for all

$a \in \mathbb{F}_p$,

$$\begin{aligned} f(x_1, \dots, x_n)^p &= \left(\sum_{(i_1,\dots,i_n) \in A} a_{i_1,\dots,i_n} x_1^{i_1} \cdots x_n^{i_n} \right)^p \\ &= \sum_{(i_1,\dots,i_n) \in A} a_{i_1,\dots,i_n}^p x_1^{pi_1} \cdots x_n^{pi_n} \\ &= \sum_{(i_1,\dots,i_n) \in A} a_{i_1,\dots,i_n} x_1^{pi_1} \cdots x_n^{pi_n} \\ &= f(x_1^p, \dots, x_n^p) \end{aligned}$$

In particular, write $\overline{\sigma}_i$ the projection of σ_i in $\mathbb{F}_p[x_1, \cdots, x_r]$, and $\overline{S}$ the projection of S. As the characteristic of the field $\mathbb{F}_p(x_1, \cdots, x_r)$ is p,

$$\begin{aligned} \overline{\sigma}_i(x_1, \cdots, x_r)^p &= \left(\sum_{1 \le j_1 < j_2 < \cdots < j_i \le r} x_{j_1} \cdots x_{j_i} \right)^p \\ &= \sum_{1 \le j_1 < j_2 < \cdots < j_i \le r} x_{j_1}^p \cdots x_{j_i}^p \\ &= \overline{\sigma}_i(x_1^p, \cdots, x_r^p) \end{aligned}$$

Hence $\overline{S}(\overline{\sigma}_1, \dots, \overline{\sigma}_r) = \overline{\sigma}_i(x_1^p, \cdots, x_r^p) - \overline{\sigma}_i^p = 0$. Since $\overline{\sigma}_1, \dots, \overline{\sigma}_r$ are algebraically independant over $\mathbb{F}_p$ (see Ex. 2.2.5), $\overline{S} = 0$, so $S \equiv 0 \pmod p$. Therefore p divides all the coefficients of S.

$\square$

Ex. 9.1.7 *This exercise is concerned with the proof of Theorem 9.1.9.*

(a) Let ζ be a primitive nth root of unity, and let i be relatively prime to n. Prove that ζ^i is a primitive nth root of unity and that every primitive nth root of unity is of this form.

(b) Let $\gamma_1, \dots, \gamma_r$ be distinct primitive nth roots of unity and let i be relatively prime to n. Prove that $\gamma_1^i, \dots, \gamma_r^i$ are distinct.

Proof. Let ζ be a primitive nth root of unity, so $o(\zeta) = n$ (where we write $o(x)$ the order of an element x in a group G). We have proved in Exercise 3 that for all $i \in \mathbb{Z}$,

$$o(\zeta^i) = \frac{n}{n \wedge i}$$

In particular, if i and n are relatively prime ($n \wedge i = 1$), then $o(\zeta^i) = n$, so ζ^i is a primitive nth root of unity.

If ξ is any primitive nth root of unity, as ζ is a generator of $\mathbb{U}_n$, $\xi = \zeta^i, 0 \le i < n$. As ζ^i is a primitive nth root of unity, $o(\zeta^i) = n = \frac{n}{n \wedge i}$, so $n \wedge i = 1$.

(b) Let $i \in \mathbb{Z}$ relatively prime to n. Consider

$$\varphi : \left\{ \begin{array}{ccc} \mathbb{U}_n & \to & \mathbb{U}_n \\ \lambda & \mapsto & \lambda^i \end{array} \right.$$

φ is a group homomorphism.

If $\lambda \in \ker(\varphi)$, then $\lambda = \zeta^k,\ k \in \mathbb{Z}$, and $1 = \lambda^i = \zeta^{ki}$, thus $n \mid ki$. Since $n \wedge i = 1$, $n \mid k$, hence $\lambda = \zeta^k = 1$, so $\ker(\varphi) = \{1\}$.

The group homomorphism φ is injective, so the images of the distinct $\gamma_1, \cdots, \gamma_r \in \mathbb{U}_n$ are distinct.

Conclusion: if $i \wedge n = 1$, $\zeta \mapsto \zeta^i$ is a bijection from the set of primitive nth roots of unity on itself.

□

Ex. 9.1.8 *This exercise will present an alternate proof of (9.8) that doesn't use symmetric polynomials.*

$$\text{(9.8)} \qquad \textit{If } \zeta \textit{ is a root of } f, \textit{ then so is } \zeta^p.$$

where f is an irreducible factor of Φ_n, and p a prime number such that $p \nmid n$..

Assume that ζ is a root of f such that $f(\zeta^p) \neq 0$. As in the text, $q(x) \in \mathbb{Z}[x]$ maps to the polynomial $\overline{q}(x) \in \mathbb{F}_p[x]$. Let $g(x)$ be as in (9.7), i.e. $\Phi_n(x) = f(x)g(x)$.

(a) *Prove that ζ is a root of $g(x^p)$, and conclude that $f(x) \mid g(x^p)$.*

(b) *Use Gauss's Lemma to explain why $f(x)$ divides $g(x^p)$ in $\mathbb{Z}[x]$, and conclude that $\overline{f}(x)$ divides $\overline{g}(x^p)$ in $\mathbb{F}_p[x]$.*

(c) *Use Exercise 7 to prove that $\overline{g}(x)^p = \overline{g}(x^p)$, and conclude that $\overline{f}(x)$ divides $\overline{g}(x)^p$.*

(d) *Now let $h(x) \in \mathbb{F}_p[x]$ be an irreducible factor of $\overline{f}(x)$. Show that $h(x)$ divides $\overline{g}(x)$, so that $h(x)^2$ divides $\overline{f}(x)\overline{g}(x)$.*

(e) *Conclude that $h(x)^2$ divides $x^n - 1 \in \mathbb{F}_p[x]$.*

(f) *Use separability to obtain a contradiction.*

Proof. As in the proof of Theorem 9.1.9, the Gauss's Lemma in the form of Corollary 4.2.1 allows us to assume that there exists a polynomial $f(x) \in \mathbb{Z}[x]$ of $\Phi_n(x)$ such that $\Phi_n(x) = f(x)g(x),\ f(x), g(x) \in \mathbb{Z}[x]$, where f is monic and irreducible over $\mathbb{Q}$. Let p be a prime number such that $p \nmid n$.

Reasoning by contradiction, we suppose that ζ is a root of f such that $f(\zeta^p) \neq 0$.

(a) As ζ is the root of f, where f divides Φ_n, ζ is a nth primitive root of unity. Since $p \nmid n, p \wedge n = 1$, hence ζ^p is also a nth primitive root of unity by Exercise 7(a), therefore $0 = \Phi(\zeta^p) = f(\zeta^p)g(\zeta^p)$. As $f(\zeta^p) \neq 0$, $g(\zeta^p) = 0$, so

$$\zeta \text{ is a root of } g(x^p).$$

As f is irreducible, f is the minimal polynomial of ζ over $\mathbb{Q}$, and ζ is a root of $g(x^p) \in \mathbb{Q}[x]$, hence

$$f(x) \mid g(x^p).$$

(b) As f is monic, the refined division algorithm of Exercise 4 show that the quotient $q(x)$ of $g(x^p)$ by $f(x)$ lies in $\mathbb{Z}[x]$, so $f(x)$ divides $g(x^p)$ in $\mathbb{Z}[x]$.

The projection homomorphism on $\mathbb{F}_p[x]$ gives $\overline{g}(x^p) = \overline{f}(x)\overline{q}(x)$, thus $\overline{f}(x)$ divides $\overline{g}(x^p)$ in $\mathbb{F}_p[x]$.

(c) As the characteristic of $\mathbb{F}_p(x)$ is p, writing $\overline{g}(x) = \sum_{i=0}^{r} a_i x^i \in \mathbb{F}_p[x]$, then (as in Exercise 7)

$$\overline{g}(x)^p = \left(\sum_{i=0}^{r} a_i x^i \right)^p = \sum_{i=0}^{r} a_i^p x^{ip} = \sum_{i=0}^{r} a_i x^{ip} = \overline{g}(x^p).$$

Therefore $\overline{f}(x)$ divides $\overline{g}(x)^p$ in $\mathbb{F}_p[x]$.

(d) Let $h(x) \in \mathbb{F}_p[x]$ an irreducible factor of $\overline{f}(x)$. Then $h(x) \mid \overline{g}(x)^p$. Since h is irreducible (hence prime) in $\mathbb{F}_p[x]$, then $h \mid g$.
$h(x) \mid \overline{f}(x), h(x) \mid \overline{g}(x)$, so $h(x)^2 \mid \overline{f}(x)\overline{g}(x)$.

(e) Therefore $h^2 \mid \overline{\Phi}_n$, and $\overline{\Phi}_n \mid x^n - 1$, thus $h^2 \mid x^n - 1 \in F_p[x]$.

(f) As $\deg(h) > 1$, every root of h in the splitting root of $x^n - 1 \in \mathbb{F}_p[x]$ is not a simple root, thus $x^n - 1$ would not be separable.
But n is relatively prime to p, so $(x^n - 1)' = nx^{n-1}$ is relatively prime to $x^n - 1$, and so $x^n - 1 \in \mathbb{F}_p[x]$ is separable: this is a contradiction, therefore.

$$f(\zeta) = 0 \Rightarrow f(\zeta^p) = 0.$$

We conclude that Φ_n is irreducible as in the conclusion of the proof of Theorem 9.1.9.

□

Ex. 9.1.9 *In proving Fermat's Little Theorem $a^p \equiv a \pmod p$, recall from the proof of Lemma 9.1.2 that we first proved $a^{p-1} \equiv 1 \pmod p$ when a is relatively prime to p. For general $n > 1$, Euler showed that $a^{\phi(n)} \equiv 1 \pmod n$ when a is relatively prime to n. Prove this. What basic fact from group theory do you use?*

Proof. If $a \wedge n = 1$, $[a] \in (\mathbb{Z}/n\mathbb{Z})^*$. By Lagrange Theorem, the order of $[a]$ divides the order of the group $(\mathbb{Z}/n\mathbb{Z})^*$, therefore the order of a divides $\phi(n) = |(\mathbb{Z}/n\mathbb{Z})^*|$, and so $[a]^{\phi(n)} = [1]$.

$$a \wedge n = 1 \Rightarrow a^{\phi(n)} \equiv 1 \ [n].$$

□

Ex. 9.1.10 *Prove that a cyclic group of order n has $\phi(n)$ generators.*

Proof. More generally, we prove that a cyclic group G of order n has $\phi(d)$ elements of order d if $d \mid n$ (0 otherwise !).

Let ζ a generator of G: $G = \langle \zeta \rangle$.

Every element $\alpha \in G$ is of the form $\zeta^k, 0 \leq k < n$. Recall (see Exercise 3), that

$$o(\zeta^k) = \frac{n}{n \wedge k}.$$

If $d \nmid n$, there is no element of order d by Lagrange's Theorem, and if $d \mid n$,

$$\begin{aligned} o(\zeta^k) = d &\iff \frac{n}{n \wedge k} = d \\ &\iff \frac{n}{d} = n \wedge k \\ &\iff \exists \lambda \in \mathbb{Z},\ k = \lambda \frac{n}{d},\ 0 \le \lambda < d,\ \lambda \wedge d = 1 \end{aligned}$$

Indeed, if $\delta = \frac{n}{d} = n \wedge k$, then there exists λ, μ, with $\lambda \wedge \mu = 1$, such that

$$\left\{ \begin{array}{lcl} n & = & \mu\delta \\ k & = & \lambda\delta \end{array} \right. .$$

$\mu = n/\delta = d$, so $\lambda \wedge d = 1$. As $0 \le k < n$, $0 \le \lambda < n/\delta = d$.
Conversely, if $k = \lambda \frac{n}{d}$, $\lambda \wedge d = 1$, then

$$n \wedge k = d\frac{n}{d} \wedge \lambda \frac{n}{d} = (d \wedge \lambda)\frac{n}{d} = \frac{n}{d}.$$

The elements of order d in G are so the elements ζ^k, where

$$k = \lambda \frac{n}{d},\ 0 \le \lambda < d,\ \lambda \wedge d = 1.$$

The mapping $\varphi : \{\lambda \in \mathbb{Z} \mid 0 \le \lambda < d, \lambda \wedge d = 1\} \to \{\alpha \in G \mid o(\alpha) = d\}$ defined by $\varphi(\lambda) = \zeta^{\lambda \frac{n}{d}}$ is so a bijection.

Hence there exist exactly $\phi(d)$ elements of order d in G, for every factor d of $n = |G|$. In particular, there exist $\phi(n)$ elements of order $n = |G|$ in G, hence $\phi(n)$ generators in a cyclic group G of order n. □

Ex. 9.1.11 *Prove that $n = \sum_{d \mid n} \phi(d)$.*

Proof. Let G a fixed cyclic group of order n, by example $G = \mathbb{U}_n$. If A_d is the set of elements of order d in G, then G is the disjoint union of the A_n, so $|G| = \sum_{d=0}^{n} |A_d|$.

By the proof of Exercise 10, $|A_d| = \phi(d)$ if $d \mid n$, and $|A_d| = 0$ if $d \nmid n$, so

$$n = \sum_{d \mid n} \phi(d).$$

Note: as an alternative proof, we can take the degrees in the formula $x^n - 1 = \prod_{d \mid n} \Phi_d(n)$. □

Ex. 9.1.12 *Here are some further properties of cyclotomic polynomials.*

(a) Given n, let $m = \prod_{d \mid n} p$. Prove that $\Phi_n(x) = \Phi_m(x^{n/m})$. This shows that we can reduce computing $\Phi_n(x)$ to the case when n is squarefree.

(b) Let $n > 1$ be an odd integer. Prove that $\Phi_{2n}(x) = \Phi_n(-x)$.

(c) Let p be a prime not dividing an integer $n > 1$. Prove that $\Phi_{pn}(x) = \Phi_n(x^p)/\Phi_n(x)$.

Lemma. *Let $f(x), g(x) \in \mathbb{C}[x]$ be two monic polynomials in $\mathbb{Q}[x]$, of same degree d, and f separable.*

If every root of f in $\mathbb{C}$ is a root of g, then $f = g$.

Proof of the Lemma. As $f(x)$ is monic separable of degree d, the decomposition in irreducible factors of $f(x)$ in $\mathbb{C}[x]$ is

$$f(x) = \prod_{\alpha \in S} (x - \alpha)$$

The hypothesis implies that for all $\alpha \in S$, $x - \alpha \mid g(x)$, hence $f(x) \mid g(x)$. As $\deg(f) = \deg(g)$, and as f, g are monic, then $f = g$. $\square$

Proof. (a) $n = p_1^{\nu_1} p_2^{\nu_2} \cdots p_r^{\nu_r}$. Write $m = p_1 \cdots p_r$. Then

$\deg(\Phi_n(x)) = \phi(n) = p_1^{\nu_1 - 1} p_2^{\nu_2 - 1} \cdots p_r^{\nu_r - 1}(p_1 - 1)(p_2 - 1) \cdots (p_r - 1)$.

$\deg(\Phi_m(x))) = \phi(p_1 p_2 \cdots p_n) = (p_1 - 1)(p_2 - 1) \cdots (p_r - 1)$, therefore

$\deg(\Phi_m(x^{n/m})) = p_1^{\nu_1 - 1} p_2^{\nu_2 - 1} \cdots p_r^{\nu_r - 1}(p_1 - 1)(p_2 - 1) \cdots (p_r - 1) = \deg(\Phi_n(x))$.

Moreover these polynomials are monic and Φ_n is separable. It remains to show that every root ζ of $\Phi_n(x)$ is a root of $\Phi_m(x^{n/m})$.

Such a root ζ has order $n = p_1^{\nu_1} p_2^{\nu_2} \cdots p_r^{\nu_r}$ in the group $\mathbb{C}^*$.

Write $\xi = \zeta^{n/m} = \zeta^{p_1^{\nu_1 - 1} p_2^{\nu_2 - 1} \cdots p_r^{\nu_r - 1}}$.

Then the order of ξ is $m = p_1 p_2 \cdots p_r$. Indeed, for all $k \in \mathbb{Z}$,

$\xi^k = 1 \iff \zeta^{k p_1^{\nu_1 - 1} p_2^{\nu_2 - 1} \cdots p_r^{\nu_r - 1}} = 1 \iff p_1^{\nu_1} p_2^{\nu_2} \cdots p_r^{\nu_r} \mid k p_1^{\nu_1 - 1} p_2^{\nu_2 - 1} \cdots p_r^{\nu_r - 1}$

$\iff p_1 p_2 \cdots p_r \mid k$.

Therefore, by definition of Φ_m, $\Phi_m(\zeta^{n/m}) = \Phi_m(\xi) = 0$.

The hypotheses of the lemma are satisfied, thus

$$\Phi_n(x) = \Phi_m(x^{n/m})$$

(b) We show that $\Phi_{2n}(x) = \Phi_n(-x)$ $(n > 1, n \text{ odd, so } n \geq 3)$.

Note first that $\deg(\Phi_{2n}(x)) = \phi(2n) = \phi(2)\phi(n) = \phi(n) = \deg(\Phi_n(-x))$.

If $n > 2$, then $\phi(n)$ is even. Indeed, we can group in pairs the elements of $(\mathbb{Z}/n\mathbb{Z})^*$, with the pairs $\{[d], -[d]\}$, where $d \wedge n = 1$ and $[d] \neq -[d]$ since $(n \mid 2d, d \wedge n = 1) \Rightarrow n \mid 2$, which is impossible if $n > 2$. Hence

$$(-1)^{\phi(n)} = 1 \qquad (n > 2).$$

$\Phi_{2n}(x)$ is monic by definition, and the leading coefficient of $\Phi_n(-x)$ est $(-1)^{\phi(n)} = 1$, so $\Phi_n(-x)$ is also monic.

Let α be any root of $\Phi_n(-x)$. Then $\alpha = -\zeta$, where ζ is a nth primitive root of unity , so ζ is an element of order n in the group $\mathbb{C}^*$.

Then the order of $\alpha = -\zeta$ is $2n$. Indeed, for all $k \in \mathbb{Z}$,

$(-\zeta)^k = 1$, that is $(-1)^k \zeta^k = 1$, implies $\zeta^{2k} = 1$, thus $n \mid 2k$, so $n \mid k$ (since n is odd), therefore $\zeta^k = 1, (-1)^k = 1$ and so $2 \mid k$.

As $n \wedge 2 = 1, 2n \mid k$.

Conversely, if $2n \mid k$, $(-\zeta)^{2n} = [(-1)^2]^n[\zeta^n]^2 = 1$.

Conclusion: $(-\zeta)^k = 1 \iff 2n \mid k$, so the order of $\alpha = -\zeta$ is $2n$, hence $x = -\zeta$ is a root of Φ_{2n}.

Every root of $\Phi_n(-x)$ in $\mathbb{C}$ is a root of $\Phi_{2n}(x)$. Moreover $\Phi_n(-x)$ is a separable polynomial, and $\deg(\Phi_{2n}(x)) = \deg(\Phi_n(-x))$. Then the lemma gives the conclusion, for all odd n, $n > 1$,

$$\Phi_{2n}(x) = \Phi_n(-x)$$

(c) We show first that $\Phi_n(x)$ divides $\Phi_n(x^p)$. As $\Phi_n(x)$ is separable, it is sufficient to verity that every root ζ of $\Phi_n(x)$ is a root of $\Phi_n(x^p)$. Such a root ζ is a nth primitive root of unity, so its order is n. Then the order of ζ^p is also n. Indeed, for all $k \in \mathbb{Z}$, as $n \wedge p = 1$,

$$(\zeta^p)^k = 1 \iff \zeta^{pk} = 1 \iff n \mid pk \iff n \mid k.$$

Therefore ζ^p is a root of Φ_n, so $\Phi_n(\zeta^p) = 0$ and ζ is a root of $\Phi_n(x^p)$.

$$\Phi_n(x) \mid \Phi_n(x^p) \quad (p \nmid n).$$

We compare the degrees:
$\deg(\Phi_{pn}(x)) = \phi(pn) = \phi(p)\phi(n) = (p-1)\phi(n)$,
$\deg(\Phi_n(x^p)/\Phi_n(x)) = p\phi(n) - \phi(n) = (p-1)\phi(n)$, thus

$$\deg(\Phi_n(x^p)/\Phi_n(x)) = \deg(\Phi_{pn}(x)).$$

Moreover, these two polynomials are monic, and Φ_{pn} is separable.

We show that every root ζ of $\Phi_{pn}(x)$ is a root of $\Phi_n(x^p)/\Phi_n(x)$.

If ζ is a root of $\Phi_{pn}(x)$, then $o(\zeta) = pn$, therefore $o(\zeta^p) = n$
(indeed, for all $k \in \mathbb{Z}$, $(\zeta^p)^k = 1 \iff \zeta^{pk} = 1 \iff pn \mid pk \iff n \mid k$).

So ζ^p is a root of $\Phi_n(x)$, which is equivalent to ζ is a root of $\Phi_n(x^p)$.

As $o(\zeta) = pn$, $\zeta^n \neq 1$, $\Phi_n(\zeta) \neq 0$, therefore ζ is a root of $\Phi_n(x^p)/\Phi_n(x)$.

The hypotheses of the lemma are so satisfied, so

$$\Phi_{pn}(x) = \Phi_n(x^p)/\Phi_n(x) \qquad (p \nmid n).$$

□

Ex. 9.1.13 *We know $\Phi_p(x)$ when p is prime. Use this and Exercise 12 to compute $\Phi_{15}(x)$ and $\Phi_{105}(x)$.*

Proof. (a) By Exercise 12(c),

$$\begin{aligned}\Phi_{15}(x) &= \frac{\Phi_3(x^5)}{\Phi_3(x)}\\ &= \frac{x^{10}+x^5+1}{x^2+x+1}\\ &= x^8 - x^7 + x^5 - x^4 + x^3 - x + 1.\end{aligned}$$

(b)

$$
\begin{aligned}
\Phi_{105}(x) &= \frac{\Phi_{15}(x^7)}{\Phi_{15}(x)}\\
&= \frac{x^{56} - x^{49} + x^{35} - x^{28} + x^{21} - x^7 + 1}{x^8 - x^7 + x^5 - x^4 + x^3 - x + 1}\\
&= \quad x^{48} + x^{47} + x^{46} - x^{43} - x^{42} - 2\,x^{41} - x^{40} - x^{39} + x^{36} + x^{35} + x^{34} + x^{33} + x^{32}\\
&\quad + x^{31} - x^{28} - x^{26} - x^{24} - x^{22} - x^{20} + x^{17} + x^{16} + x^{15} + x^{14} + x^{13} + x^{12} - x^9 - x^8\\
&\quad - 2\,x^7 - x^6 - x^5 + x^2 + x + 1
\end{aligned}
$$

□

Ex. 9.1.14 *The Möbius function is defined for integers $n \geq 1$ by*

$$
\mu(n) = \begin{cases} 1 & \text{if } n = 1,\\ (-1)^s, & \text{if } n = p_1 \cdots p_s \text{ for distinct primes } p_1, \ldots, p_s\\ 0, & \text{otherwise} \end{cases}
$$

Prove that $\sum_{d|n} \mu\left(\frac{n}{d}\right) = 0$ when $n > 1$.

Proof. Suppose $n > 1$. Write $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ its decomposition in prime factors. The factors d of n such that $\mu(d) \neq 0$ are the integers $d = p_1^{\beta_1} p_2^{\beta_2} \cdots p_k^{\beta_k}$ where $\beta_i = 0, 1$. If exactly r exponents β_i are non zero, then $\mu(d) = (-1)^r$, and there are $\binom{k}{r}$ such integers d.

Therefore

$$
\sum_{d|n} \mu(d) = \sum_{r=0}^{k} (-1)^r \binom{k}{r} = (1-1)^k = 0
$$

(since $k \neq 0$)

Conclusion: if $n > 1$,

$$
\sum_{d|n} \mu\left(\frac{n}{d}\right) = \sum_{d|n} \mu(d) = \begin{cases} 0, & \text{if } n > 1,\\ 1, & \text{if } n = 0. \end{cases}
$$

□

Ex. 9.1.15 *Let μ be the Möbius function defined in Exercise 14. Prove that*

$$
\Phi_n(x) = \prod_{d|n} (x^d - 1)^{\mu(n/d)}.
$$

Proof. Our starting point is

$$
F(n) = x^n - 1 = \prod_{d|n} \Phi_d \qquad (n \geq 1).
$$

It is sufficient to copy the proof of the Möbius Inversion Formula in multiplicative notations:

$$\begin{aligned}
\prod_{d|n}(x^d-1)^{\mu(\frac{n}{d})} &= \prod_{e|n}(x^{\frac{n}{e}}-1)^{\mu(e)}\\
&= \prod_{e|n}\prod_{d|\frac{n}{e}}\Phi_d^{\mu(e)}\\
&= \prod_{d|n}\prod_{e|\frac{n}{d}}\Phi_d^{\mu(e)} \qquad \text{(since } e|n \text{ and } d|\frac{n}{e} \iff d|n \text{ and } e|\frac{n}{d})\\
&= \prod_{d|n}\Phi_d^{\sum_{e|\frac{n}{d}}\mu(e)}\\
&= \Phi_n,
\end{aligned}$$

since by Exercise 14, $\sum_{e|\frac{n}{d}}\mu(e) \neq 0$ only if $\frac{n}{d}=1$, that is $d=n$, so the product is Φ_n.

Conclusion :

$$\Phi_n(x) = \prod_{d|n}(x^d-1)^{\mu(\frac{n}{d})} \quad (n \geq 1).$$

□

Ex. 9.1.16 *Let n and m be relatively prime positive integers.*

(a) Prove that $\mathbb{Q}(\zeta_n, \zeta_m) = \mathbb{Q}(\zeta_{nm})$.

(b) Prove that $\Phi_n(x)$ is irreducible over $\mathbb{Q}(\zeta_m)$.

Proof. Here we write $\zeta_k = e^{2i\pi/k}$ for all subscript k.

(a) $\zeta_n = (\zeta_{nm})^m \in \mathbb{Q}(\zeta_{nm})$, and $\zeta_m = (\zeta_{nm})^n \in \mathbb{Q}(\zeta_{nm})$,therefore

$$\mathbb{Q}(\zeta_n, \zeta_m) \subset \mathbb{Q}(\zeta_{nm}).$$

As $u \wedge v = 1$, there exists integers u, v such that $1 = un + vm$.
Therefore $\zeta_{nm} = (\zeta_{nm}^n)^u(\zeta_{nm}^m)^v = \zeta_m^u\zeta_n^v \in \mathbb{Q}(\zeta_n, \zeta_m)$, hence

$$\mathbb{Q}(\zeta_{nm}) \subset \mathbb{Q}(\zeta_n, \zeta_m)$$

We have proved

$$\mathbb{Q}(\zeta_{nm}) = \mathbb{Q}(\zeta_n, \zeta_m)$$

$$\begin{tikzcd}
 & \mathbb{Q}(\zeta_n, \zeta_m) & \\
\mathbb{Q}(\zeta_n) \arrow[ur] & & \mathbb{Q}(\zeta_m) \arrow[ul] \\
 & \mathbb{Q} \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

(b) By Corollary 9.1.10, $[\mathbb{Q}(\zeta_{nm}) : \mathbb{Q}] = \phi(nm)$. As $n \wedge m = 1, \phi(nm) = \phi(n)\phi(m)$ (Lemma 9.1.1), so $[\mathbb{Q}(\zeta_{nm}) : \mathbb{Q}] = \phi(n)\phi(m)$, and by part (a), this is equivalent to

$$[\mathbb{Q}(\zeta_n, \zeta_m) : \mathbb{Q}] = \phi(n)\phi(m).$$

Using the Tower Theorem,

$$\phi(n)\phi(m) = [\mathbb{Q}(\zeta_n, \zeta_m) : \mathbb{Q}] = [\mathbb{Q}(\zeta_n, \zeta_m) : \mathbb{Q}(\zeta_m)]\,[\mathbb{Q}(\zeta_m) : \mathbb{Q}] = \phi(m)[\mathbb{Q}(\zeta_n, \zeta_m) : \mathbb{Q}(\zeta_m)],$$

thus

$$\phi(n) = [\mathbb{Q}(\zeta_m)(\zeta_n) : \mathbb{Q}(\zeta_m)].$$

Let f be the minimal polynomial of ζ_n over $\mathbb{Q}(\zeta_m)$. Then

$$\deg(f) = [\mathbb{Q}(\zeta_m)(\zeta_n) : \mathbb{Q}(\zeta_m)] = \phi(n).$$

ζ_n is a root of $\Phi_n(x) \in \mathbb{Q}[x] \subset \mathbb{Q}(\zeta_m)[x]$, therefore $f \mid \Phi_n$ in $\mathbb{Q}(\zeta_m)[x]$. Moreover these two polynomials are monic of same degree $\phi(n)$, so they are identical. $\Phi_n = f$ is so irreducible over $\mathbb{Q}(\zeta_m)$.

□

9.2 GAUSS AND ROOTS OF UNITY (OPTIONAL)

Ex. 9.2.1 *Let G be a cyclic group of order n and let g be a generator of G.*

(a) Let f be a positive divisor of n and set $e = n/f$. Prove that $H_f = \langle g^e \rangle$ has order f and hence is the unique subgroup of order f.

(b) Let f and f' be positive divisors of $p-1$. Prove that $H_f \subset H_{f'}$ if and only if $f \mid f'$.

Proof. (a) • Let G be a cyclic group of order n and let g be a generator of G. If f is a positive divisor of n, write $e = n/f$, and $H = \langle g^e \rangle$.

The order of g is $n = ef$, hence the order of g^e is $\frac{n}{n \wedge e} = \frac{n}{e} = f$, therefore the set $A = \{(g^e)^0, \cdots, (g^e)^{f-1}\} \subset \langle g^e \rangle$ has f distinct elements: $|A| = f$.

Conversely, if $h \in \langle g^e \rangle$, then $h = (g^e)^k, k \in \mathbb{Z}$. The Euclidean division of k by f gives $k = qf + r, 0 \le r < f$, thus $h = (g^{ef})^q g^{er} = (g^e)^r, 0 \le r < f$, therefore $h \in A$.

Hence $H_f = \langle g^e \rangle = A$ has order f.

$$|H_f| = |\langle g^e \rangle| = f.$$

• Let K be any subgroup of order f. We must prove that $K = H$.

The set E of integers $m > 0$ such that $g^m \in K$ is non empty, since $g^n = e \in K$. Set

$$k = \min(E) = \min\{m \in \mathbb{N}^* \mid g^m \in K\},$$

so k is the least positive integer such that $g^k \in K$. We show that $K = \langle g^k \rangle$.

As $g^k \in K, \langle g^k \rangle \subset K$.

Conversely, if $h \in K$, then h is an element of G of the form $h = g^l,\ l \in \mathbb{Z}$. The Euclidean division of l by k gives $l = qk + r,\ 0 \le r < k$.

Then $g^r = g^l(g^k)^{-q} = h(g^k)^{-q} \in K$ and $0 \le r < k$. If r was not zero, it would lie in E and would be less than the minimum of E. This is a contradiction, so $r = 0$, and $h = g^l = (g^k)^q \in \langle g^k \rangle$. Therefore $K \subset \langle g^k \rangle$. Finally,

$$K = \langle g^k \rangle.$$

We show first that $k \mid n$. Write $d = k \wedge n$. There exist integers u, v such that $d = uk + vn$, therefore $g^d = (g^k)^u (g^n)^v = (g^k)^u \in K$, so $d \in E$, and $1 \le d \le k$, therefore $d = k$ by definition of $k = \min(E)$. So $k = k \wedge n$, hence $k \mid n$.

$K = \langle g^k \rangle$ is cyclic, and its cardinality is the order of g^k, $k \mid n$, so

$$|K| = \langle g^k \rangle = o(g^k) = \frac{n}{k},$$

by the first part of the proof.

By hypothesis the order of K is f, so $f = |K| = n/k$, and $k = n/f = e$.

$$K = \langle g^e \rangle = H.$$

Conclusion:

A cyclic group with generator g, of order $n = ef$, contains a unique subgroup of order f, written H_f, which is cyclic, generated by g^e.

(b) Let f, f' be positive divisors of $p - 1 = |(\mathbb{Z}/p\mathbb{Z})^*|$, and let g a generator of $(\mathbb{Z}/p\mathbb{Z})^*$. As in the text, write H_f the unique subgroup of $(\mathbb{Z}/p\mathbb{Z})^*$ of order f.

If $H_f \subset H'_f$, then H_f is a subgroup of H'_f. By Lagrange's Theorem $|H_f|$ divides $|H_{f'}|$, so $f \mid f'$.

Conversely, if $f \mid f'$, $f' = qf$, $q \in \mathbb{N}$. Moreover $H_f = \langle g^e \rangle, H_{f'} = \langle g^{e'} \rangle$, where $n = ef = e'f'$ by part (a). Therefore $e = e'q$, and $g^e = (g^{e'})^q \in H_{f'}$, hence $H_f = \langle g^e \rangle \subset H_{f'}$.

$$f \mid f' \iff H_f \subset H_{f'}.$$

□

Ex. 9.2.2 *Prove Proposition 9.2.1.*

Proof. Write $\tilde{H}_f$ the subgroup corresponding to H_f by the isomorphism $\mathrm{Gal}(\mathbb{Q}(\zeta_p)/\mathbb{Q}) \simeq (\mathbb{Z}/p\mathbb{Z})^*$. Then

$$\sigma \in \tilde{H}_f \iff \exists [i] \in H_f,\ \sigma(\zeta_p) = \zeta_p^i,$$

and

$$L_f = \{\alpha \in \mathbb{Q}(\zeta_p) \mid \forall \sigma \in \tilde{H}_f,\ \sigma(\alpha) = \alpha\}$$

is the fixed field of $\tilde{H}_f$, with $\mathbb{Q} \subset L_f \subset \mathbb{Q}(\zeta_p)$.

(a) As $G = \mathrm{Gal}(\mathbb{Q}(\zeta_p)/\mathbb{Q})$ is Abelian (G is cyclic since $(\mathbb{Z}/p\mathbb{Z})^* \simeq G$ is cyclic for prime p), so every subgroup of G is normal, therefore $\mathbb{Q} \subset L_f$ is a Galois extension (Theorem 7.3.2).

Moreover, by the Galois correspondence (Theorem 7.3.1), $[L_f : \mathbb{Q}] = (G : \tilde{H}_f)$, and $(G : \tilde{H}_f) = ((\mathbb{Z}/p\mathbb{Z})^* : H_f) = (p-1)/f = e$, so

$$[L_f : \mathbb{Q}] = e.$$

L_f is a Galois extension of $\mathbb{Q}$ of degree e.

(b) By Exercise 1, $f \mid f' \iff H_f \subset H_{f'}$. As the Galois correspondence is order reversing,

$$f \mid f' \iff H_f \subset H_{f'} \iff \tilde{H}_f \subset \tilde{H}_{f'} \iff L_f \supset L_{f'}.$$

(c) Let f, f' be positive divisors of $p-1$ such that $f \mid f'$. Since G is Abelian, $L_{f'} \subset L_f$ is a Galois extension, and by Theorem 7.3.2,

$$\mathrm{Gal}(L_f/L_{f'}) \simeq \mathrm{Gal}(\mathbb{Q}(\zeta_p)/L_{f'})/\mathrm{Gal}(\mathbb{Q}(\zeta_p)/L_f) = \tilde{H}_{f'}/\tilde{H}_f \simeq H_{f'}/H_f.$$

As $H_{f'}$ is cyclic of order f', the quotient group $H_{f'}/H_f$ is itself cyclic, of order f'/f.

Conclusion:

$$\mathrm{Gal}(L_f/L_{f'}) \text{ is cyclic of order } f'/f.$$

□

Ex. 9.2.3 *Let η_1, η_2, η_3 be as in Example 9.2.2.*

(a) We know that ζ_7 is a root of $x^6 + x^5 + x^4 + x^3 + x^2 + x + 1 = 0$. Dividing by x^3 gives

$$x^3 + x^2 + x + 1 + x^{-1} + x^{-2} + x^{-3} = 0.$$

Use this to show that η_1, η_2, η_3 are roots of $y^3 + y^2 - 2y - 1$.

(b) Prove that $[\mathbb{Q}(\eta_1) : \mathbb{Q}] = 3$, and conclude that $\mathbb{Q}(\eta_1)$ is the fixed field of the subgroup $\{e, \tau\} \subset \mathrm{Gal}(\mathbb{Q}(\zeta_7)/\mathbb{Q})$, where τ is the complex conjugation.

(c) Prove (9.10).

Proof. (a) Let ζ be any 7th primitive root of unity (i.e. $\zeta = \zeta_7^i,\ i = 1, \cdots, 6$). Then $1 + \zeta + \zeta^2 + \zeta^3 + \zeta^4 + \zeta^5 + \zeta^6 = 0$, and division by ζ^3 gives

$$\zeta^{-3} + \zeta^3 + \zeta^{-2} + \zeta^2 + \zeta + \zeta^{-1} + 1 = 0. \qquad (1)$$

Write $\eta = \zeta + \zeta^{-1}$. Then

$$\eta^2 = \zeta^2 + \zeta^{-2} + 2,$$
$$\eta^3 = \zeta^3 + \zeta^{-3} + 3(\zeta + \zeta^{-1}).$$

Therefore

$$\zeta^2 + \zeta^{-2} = \eta^2 - 2,$$
$$\zeta^3 + \zeta^{-3} = \eta^3 - 3\eta.$$

By (3), $(\eta^3 - 3\eta) + (\eta^2 - 2) + \eta + 1 = 0$, so

$$\eta^3 + \eta^2 - 2\eta - 1 = 0. \tag{2}$$

Applying the equality (2) to $\zeta_7, \zeta_7^2, \zeta_7^3$, we obtain that $\eta_1 = \zeta_7 + \zeta_7^{-1}, \eta_2 = \zeta_7^2 + \zeta_7^{-2}, \eta_3 = \zeta_7^3 + \zeta_7^{-3}$ are roots of

$$f = x^3 + x^2 - 2x - 1.$$

As the minimal polynomial of ζ_7 over $\mathbb{Q}$ is Φ_7 of degree 6, the list $(1, \zeta_7, \zeta_7^2, \zeta_7^3, \zeta_7^4, \zeta_7^5)$ is linearly independent over $\mathbb{Q}$, thus also the list obtained by multiplication by ζ_7, so $(\zeta_7, \zeta_7^2, \zeta_7^3, \zeta_7^4, \zeta_7^5, \zeta_7^6)$ is a linearly independent list, therefore $\eta_1 = \zeta_7 + \zeta_7^6, \eta_2 = \zeta_7^2 + \zeta_7^5, \eta_3 = \zeta_7^3 + \zeta_7^4$ are linearly independent, so are a fortiori distinct. Therefore

$$f = x^3 + x^2 - 2x - 1 = (x - \eta_1)(x - \eta_2)(x - \eta_3).$$

η_1, η_2, η_3 are the three distinct roots of f.

(b) f has no root in $\mathbb{Q}$. Indeed, if $\alpha = p/q, p \wedge q = 1$ was such a root, we would have the equality

$$p^3 + p^2 q - 2pq^2 - q^3 = 0,$$

which implies, since $p \wedge q = 1$, that $p \mid 1, q \mid 1$, so $\alpha = \pm 1$, but neither 1, nor -1 is a root of f.

Since f has no root in $\mathbb{Q}$ and $\deg(f) = 3$, f is irreducible over $\mathbb{Q}$. So f is the minimal polynomial of η_1 over $\mathbb{Q}$, and also of η_2, η_3, which are so conjugate of η_1 over $\mathbb{Q}$. Moreover

$$[\mathbb{Q}(\eta_1) : \mathbb{Q}] = \deg(f) = 3.$$

Let τ be the complex conjugation restricted to $\mathbb{Q}(\zeta_7)$. As $\tau(\zeta_7) = \overline{\zeta}_7 = \zeta_7^{-1} \in \mathbb{Q}(\zeta_7)$, τ is an automorphism of $\mathbb{Q}(\zeta_7)$ which fixes the elements of $\mathbb{Q}$, so $\tau \in \mathrm{Gal}(\mathbb{Q}(\zeta_7)/\mathbb{Q})$, and $\tau^2 = e$, therefore $\{e, \tau\} = \tilde{H}_2$ is the unique subgroup of $G = \mathrm{Gal}(\mathbb{Q}(\zeta_7)/\mathbb{Q})$ of order 2.

Let $L_2 = L_{\langle \tau \rangle}$ be the fixed field of $\tilde{H}_2$. By the Galois Correspondence (see Proposition 9.2.1 and Exercise 2),

$$[L_2 : \mathbb{Q}] = (G : H_2) = 3.$$

As $\eta_1 \in \mathbb{R}, \tau(\eta_1) = \eta_1$, hence $\eta_1 \in L_2$, and so $\mathbb{Q}(\eta_1) \subset L_2$.

Since $[L_2 : \mathbb{Q}] = [\mathbb{Q}(\eta_1) : \mathbb{Q}] = 3$, then $[L_2 : \mathbb{Q}(\eta_1)] = 1$, hence $L_2 = \mathbb{Q}(\eta_1)$.

The fixed field L_2 of $\tilde{H}_2 = \{e, \tau\}$ is $\mathbb{Q}(\eta_1)$.

(c) $\eta_1 = 2\cos(2\pi/7), \eta_2 = 2\cos(4\pi/7), \eta_3 = 4\cos(6\pi/7)$ are the roots of $f = x^3 + x^2 - 2x - 1$. We compute these roots with the Cardan's Formula.

The substitution $x = y - 1/3$ in f gives

$$\begin{aligned}
g(y) &= f\left(y - \frac{1}{3}\right) \\
&= \left(y - \frac{1}{3}\right)^3 + \left(y - \frac{1}{3}\right)^2 - 2\left(y - \frac{1}{3}\right) - 1 \\
&= y^3 - y^2 + \frac{1}{3}y - \frac{1}{27} + y^2 - \frac{2}{3}y + \frac{1}{9} - 2y + \frac{2}{3} - 1 \\
&= y^3 - \frac{7}{3}y - \frac{7}{27}
\end{aligned}$$

(Note: if Δ is the discriminant of f or g, then $\Delta = -4p^3 - 27q^2 = -4\left(-\frac{7}{3}\right)^3 - 27\left(\frac{7}{27}\right)^2 = \frac{1372}{27} - \frac{49}{27} = \frac{1323}{27} = 49 = 7^2$ is the square of an element of $\mathbb{Q}$, hence the Galois group of f is $A_3 \simeq \mathbb{Z}/3\mathbb{Z}$. This shows again that

$$|\mathrm{Gal}(\mathbb{Q}(\eta_1)/\mathbb{Q})| = [L_2 : \mathbb{Q}] = 3.)$$

Let α a root of g (that is to say $\alpha - 1/3$ is a root of f). There exist two complex numbers u, v such that $\alpha = u + v, uv = 7/9$. Then

$$\begin{aligned} 0 &= (u+v)^3 - \frac{7}{3}(u+v) - \frac{7}{27} \\ &= u^3 + v^3 + \left(3uv - \frac{7}{3}\right)(u+v) - \frac{7}{27} \\ &= u^3 + v^3 - \frac{7}{27} \end{aligned}$$

So (u, v), which satisfies the condition $uv = 7/9$, is a solution of the system

$$\begin{aligned} u^3 + v^3 &= \frac{7}{3^3} \\ u^3v^3 &= \frac{7^3}{3^6} \end{aligned}$$

u^3, v^3 are so the roots of the equation $x^2 - \frac{7}{3^3}x + \frac{7^3}{3^6}$, of discriminant

$$\delta = \frac{7^2}{3^6} - 4\frac{7^3}{3^6} = \frac{7^2(-27)}{3^6} = -\frac{7^2}{3^3} = -\frac{49}{27}.$$

$$\begin{aligned} u^3 &= \frac{1}{2}\left(\frac{7}{27} + i\sqrt{\frac{49}{27}}\right) = \frac{1}{27} \times \frac{7}{2}\left(1 + 3i\sqrt{3}\right) \\ v^3 &= \frac{1}{2}\left(\frac{7}{27} - i\sqrt{\frac{49}{27}}\right) = \frac{1}{27} \times \frac{7}{2}\left(1 - 3i\sqrt{3}\right) \end{aligned}$$

As $u^3 = \overline{v}^3$, and $uv = 7/9 \in \mathbb{R}$, then $v = \omega^k\overline{u},\ k = 0, 1, 2$, and so $uv = u\overline{u}\omega^k \in \mathbb{R}$, therefore $\omega^k \in \mathbb{R}$, so $k = 0$, which gives $v = \overline{u}$. The set $\{\eta_1, \eta_2, \eta_3\}$ of the three roots of f is so the set $\{-1/3 + u + \overline{u}, -1/3 + \omega u + \omega^2\overline{u}, -1/3 + \omega^2 u + \omega\overline{u}\}$.

To identify each root, we must define the determination of $3u = \sqrt[3]{\frac{7}{2}\left(1 + 3i\sqrt{3}\right)}$. Choose for this cubic root the one which lies in the first quadrant (there exists one and only one such a cubic root since $\mathrm{Arg}(1 + 3i\sqrt{3}) \in [0, \pi/2]$), and write $3\overline{u} = \sqrt[3]{\frac{7}{2}\left(1 - 3i\sqrt{3}\right)}$ its conjugate.

Then

$$\begin{aligned} -\frac{1}{3} + u + \overline{u} &= \frac{1}{3}(-1 + 3u + 3\overline{u}) \\ &= \frac{1}{3}\left(-1 + \sqrt[3]{\frac{7}{2}\left(1 + 3i\sqrt{3}\right)} + \sqrt[3]{\frac{7}{2}\left(1 - 3i\sqrt{3}\right)}\right) \end{aligned}$$

As $\left|\frac{7}{2}\left(1+3i\sqrt{3}\right)\right| = \frac{7}{2}\sqrt{28} = (\sqrt{7})^3$, then $|3u| = \sqrt{7}$, and $\mathrm{Arg}(3u) \in [0, \pi/6]$, therefore $\mathrm{Re}\,(3u) \geq \sqrt{7}\cos(\pi/6) = \sqrt{7}\sqrt{3}/2$, so $2\,\mathrm{Re}\,(3u) \geq \sqrt{21}$.

Therefore $\mathrm{Re}\,(-1 + 3u + 3\overline{u}) \geq \sqrt{21} - 1 > 0$.

As $\eta_1 = 2\cos(2\pi/7)$ is the only positive root of f,

$$\eta_1 = \zeta_7 + \zeta_7^{-1} = 2\cos(2\pi/7) = \frac{1}{3}\left(-1 + \sqrt[3]{\frac{7}{2}\left(1+3i\sqrt{3}\right)} + \sqrt[3]{\frac{7}{2}\left(1-3i\sqrt{3}\right)}\right)$$

where $\sqrt[3]{\frac{7}{2}\left(1+3i\sqrt{3}\right)}$ is chosen such that

$$\mathrm{Re}\left(\sqrt[3]{\frac{7}{2}\left(1+3i\sqrt{3}\right)}\right) > 0,\ \mathrm{Im}\left(\sqrt[3]{\frac{7}{2}\left(1+3i\sqrt{3}\right)}\right) > 0$$

and $\sqrt[3]{\frac{7}{2}\left(1-3i\sqrt{3}\right)}$ is its conjugate.

As ζ_7 is a root of $x^2 - \eta_1 x + 1$, with positive imaginary part, then $\zeta_7 = \frac{1}{2}\left(\eta_1 + i\sqrt{4 - \eta_1^2}\right)$, so

$$\begin{aligned}
\zeta_7 &= -\frac{1}{6} + \frac{1}{6}\sqrt[3]{\frac{7}{2}(1+3i\sqrt{3})} + \frac{1}{6}\sqrt[3]{\frac{7}{2}(1-3i\sqrt{3})} \\
&\quad + \frac{i}{2}\sqrt{4 - \left(\frac{1}{3} - \frac{1}{3}\sqrt[3]{\frac{7}{2}(1+3i\sqrt{3})} - \frac{1}{3}\sqrt[3]{\frac{7}{2}(1-3i\sqrt{3})}\right)^2} \\
&= -\frac{1}{6} + \frac{1}{6}\sqrt[3]{\frac{7}{2}(1+3i\sqrt{3})} + \frac{1}{6}\sqrt[3]{\frac{7}{2}(1-3i\sqrt{3})} \\
&\quad + i\sqrt{1 - \left(\frac{1}{6} - \frac{1}{6}\sqrt[3]{\frac{7}{2}(1+3i\sqrt{3})} - \frac{1}{6}\sqrt[3]{\frac{7}{2}(1-3i\sqrt{3})}\right)^2}
\end{aligned}$$

with the same cube roots.

(It seems that there is a misprint in (9.11)).

□

Ex. 9.2.4 *Let $A \subset B$ be subgroups of a group G, and assume that A has index d in B. Prove that every left coset of B in G is a disjoint union of d left cosets of A in G.*

Proof. Let $\{b_1 \cdots, b_d\}$ a complete system of representatives of left cosets of A in B, where $d = (B : A)$. Then

$$B = \biguplus_{1 \leq i \leq d} b_i A.$$

If $cB,\ c \in G$ is any left coset of B in G, then

$$cB = \biguplus_{1 \leq i \leq d} cb_i A.$$

Indeed,

- $b_iA \subset B$, thus $cb_iA \subset cB$, $i = 1, \dots, d$, therefore $\bigcup_{1\le i\le d} cb_iA \subset cB$.
- If $g \in cB$, then $g = ch,\ h \in B$, and $h \in b_iA$ for some $i,\ 1 \le i \le d$, so $h = b_ia, a \in A$, hence $g = cb_iA \in \bigcup_{1\le i\le d} cb_iA$. Therefore $cB \subset \bigcup_{1\le i\le d} cb_iA$.

$$cB = \bigcup_{1\le i\le d} cb_iA.$$

- The union is a disjoint union: if $g \in cb_iA$ and $g \in cb_jA$, then $c^{-1}g \in b_iA \cap b_jA$, which is possible only if $i = j$. Thus $i \ne j \Rightarrow cb_iA \cap cb_jA = \emptyset$.

Conclusion: every left coset of B in G is the disjoint union of $d = (B : A)$ left cosets of B in G. $\square$

Ex. 9.2.5 *Complete the proof of Proposition 9.2.8.*

Proof. By Exercise 4 , we obtain (9.12):

$$[\lambda]H_{f'} = [\lambda_1]H_f \cup \cdots \cup [\lambda_d]H_f, \qquad \lambda = \lambda_1.$$

We must prove that every period $(f, \lambda_j),\ j = 1, \dots, d$, is of the form $(f, \lambda_j) = \sigma(\eta) = \sigma((f, \lambda))$, where $\sigma \in \mathrm{Gal}(\mathbb{Q}(\zeta_p)/L_{f'})$.

Write $[i] = [\lambda]^{-1}[\lambda_j]$. As $[\lambda_j] \in [\lambda]H_{f'}$, then $[i] = [\lambda]^{-1}[\lambda_j] \in H_{f'}$.

Since $[\lambda_j] = [i\lambda]$,

$$(f, \lambda_j) = (f, i\lambda), \qquad i \in H_{f'}.$$

Let $\sigma \in \mathrm{Gal}(\mathbb{Q}(\zeta_p)/L_{f'})$ be defined by $\sigma(\zeta_p) = \zeta_p^i$, where $[i] \in H_{f'}$, so by Lemma 9.2.4(c),

$$(f, \lambda_j) = (f, i\lambda) = \sigma(\eta),\ \sigma \in \mathrm{Gal}(\mathbb{Q}(\zeta_p)/L_{f'}).$$

Every $(f, \lambda_j),\ j = 1, \dots, d$, is a conjugate of (f, λ) over $L_{f'}$. $\square$

Ex. 9.2.6 *Prove that the sum of the distinct f-periods equals -1.*

Proof. With a fixed divisor f of n, and $e = n/f$,

$$(\mathbb{Z}/p\mathbb{Z})^* = \biguplus_{1\le i\le e} \lambda_i H_f,$$

where $\lambda_1, \cdots, \lambda_e$ are distinct representatives of the cosets of H_f in $(\mathbb{Z}/p\mathbb{Z})^*$.

The e distinct f-periods are the $(f, \lambda_i),\ i = 1, \cdots, e$, thus

$$\sum_{i=1}^{e} (f, \lambda_i) = \sum_{i=1}^{e} \sum_{a\in[\lambda_i]H_f} \zeta_p^a = \sum_{a\in\bigcup_{1\le i\le e}[\lambda_i]H_f} \zeta_p^a = \sum_{a\in(\mathbb{Z}/p\mathbb{Z})^*} \zeta_p^a = -1,$$

since $\sum_{a\in(\mathbb{Z}/p\mathbb{Z})} \zeta_p^a = 0$. $\square$

Ex. 9.2.7 *This exercise is concerned with the details of Examples 9.2.10, 9.2.11, 9.2.12, and 9.2.13.*

(a) *Show that* 2 *is a primitive root modulo* 19.

(b) *Use the methods of Example 9.2.10 to obtain formulas for* $(6,2)^2$ *and* $(6,4)^2$.

(c) *Show that the formulas of part (b) follow from* $(6,1)^2 = 4 - (6,2)$ *and part (d) of Lemma 9.2.4.*

(d) *Prove (9.15) and use this and Exercise 6 to show that* $(6,1)(6,2)(6,4) = 7$.

(e) *Find the minimal polynomial of* $(3,2)$ *and* $(3,4)$ *over the field* L_6 *considered in Example 9.2.12.*

(f) *Show that (9.18) is the minimal polynomial of* ζ_{19} *over the field* L_3 *considered in Example 9.2.13.*

Proof. (a) $2^2 = 4 \not\equiv 1 \pmod{19}$, and $2^9 = 512 = 19 \times 26 + 18 \equiv -1 \pmod{19}$. Therefore the order of $[2]$ in $(\mathbb{Z}/19\mathbb{Z})^*$ is 18, so 2 is a primitive root modulo 19.

(b) In Example 9.2.10, we obtained

$$\begin{aligned} H_6 &= \{1,7,8,11,12,18\},\\ 2H_6 &= \{2,3,5,14,16,17\},\\ 4H_6 &= \{4,6,9,10,13,15\}. \end{aligned}$$

By Proposition 9.2.9,

$$(6,1)^2 = \sum_{\lambda' \in H_6} (6, \lambda' + 1),\ (6,2)^2 = \sum_{\lambda' \in 2H_6} (6, \lambda' + 2),\ (6,4)^2 = \sum_{\lambda' \in 4H_6} (6, \lambda' + 4).$$

$$\begin{aligned} (6,1)^2 &= (6,2) + (6,8) + (6,9) + (6,12) + (6,13) + 6\\ &= 2(6,1) + (6,2) + 2(6,4) + 6\\ &= (6,1) + (6,4) + 5\\ &= 4 - (6,2), \end{aligned}$$

$$\begin{aligned} (6,2)^2 &= (6,4) + (6,5) + (6,7) + (6,16) + (6,18) + 6\\ &= 2(6,1) + 2(6,2) + (6,4) + 6\\ &= (6,1) + (6,2) + 5\\ &= 4 - (6,4), \end{aligned}$$

$$\begin{aligned} (6,4)^2 &= (6,8) + (6,10) + (6,13) + (6,14) + (6,17) + 6\\ &= (6,1) + 2(6,2) + 2(6,4) + 6\\ &= (6,2) + (6,4) + 5\\ &= 4 - (6,1). \end{aligned}$$

$$(6,1)^2 = 4-(6,2),\ (6,2)^2 = 4-(6,4),\ (6,4)^2 = 4-(6,1).$$

If we write $\eta_1 = (6,1), \eta_2 = (6,2), \eta_3 = (6,4)$, then

$$\eta_1^2 = 4-\eta_2, \quad \eta_2^2 = 4-\eta_3, \quad \eta_3^2 = 4-\eta_1.$$

(c) The similarity of these results has an explanation. If $\sigma \in G = \mathrm{Gal}(\mathbb{Q}(\zeta_{19})/\mathbb{Q})$ is determined by $\sigma(\zeta_{19}) = \zeta_{19}^2$, then by Lemma 9.2.4(d), $\sigma((6,1)) = (6,2), \sigma((6,2)) = (6,4)$ and $\sigma((6,4)) = (6,8) = (6,1)$, so

$$\sigma(\eta_1) = \eta_2, \quad \sigma(\eta_2) = \eta_3, \quad \sigma(\eta_3) = \eta_1.$$

Therefore $\eta_1^2 = 4-\eta_2$ implies $\eta_2^2 = 4-\eta_3$ and $\eta_3^2 = 4-\eta_1$.

By Proposition 9.2.6 and Corollary 9.2.7, $L_6 = \mathbb{Q}(\eta_1) = \mathbb{Q}(\eta_1,\eta_2,\eta_3) = \mathrm{Vect}_{\mathbb{Q}}(\eta_1,\eta_2,\eta_3)$, and so σ sends L_6 on itself. The restriction $\tilde{\sigma}$ of σ to $\mathbb{Q}(\eta_1)$ is so a $\mathbb{Q}$-automorphism of $\mathbb{Q}(\eta_1)$ of order 3, since $\tilde{\sigma}^3(\eta_1) = \eta_1$. Moreover, the extension $\mathbb{Q} \subset \mathbb{Q}(\eta_1)$ is Galois (since $G = \mathrm{Gal}(\mathbb{Q}(\zeta_{19}/\mathbb{Q})$ is Abelian, every subgroup of G is normal), so

$$\mathrm{Gal}(\mathbb{Q}(\eta_1)/\mathbb{Q}) \simeq \mathrm{Gal}(\mathbb{Q}(\zeta_{19})/\mathbb{Q})/\mathrm{Gal}(\mathbb{Q}(\zeta_{19})/\mathbb{Q}(\eta_1)),$$

thus

$$|\mathrm{Gal}(\mathbb{Q}(\eta_1)/\mathbb{Q})| = [\mathbb{Q}(\eta_1):\mathbb{Q}] = (G:\tilde{H}_6) = ((\mathbb{Z}/19\mathbb{Z})^* : H_6) = 3,$$

therefore

$$\mathrm{Gal}(\mathbb{Q}(\eta_1)/\mathbb{Q}) \simeq \mathbb{Z}/3\mathbb{Z},\ \mathrm{Gal}(\mathbb{Q}(\eta_1)/\mathbb{Q}) = \langle \tilde{\sigma} \rangle.$$

(d)

$$\begin{aligned}
(6,1)(6,2) &= \sum_{\lambda' \in H_6} (6, \lambda'+2)\\
&= (6,3)+(6,9)+(6,10)+(6,13)+(6,14)+(6,1)\\
&= (6,2)+(6,4)+(6,4)+(6,4)+(6,2)+(6,1)\\
&= (6,1)+2(6,2)+3(6,4).
\end{aligned}$$

If we apply σ, σ^2 to this equality, we obtain (9.15) :

$$\begin{aligned}
(6,1)(6,2) &= (6,1)+2(6,2)+3(6,4),\\
(6,2)(6,4) &= 3(6,1)+(6,2)+2(6,4),\\
(6,4)(6,1) &= 2(6,1)+3(6,2)+(6,4).
\end{aligned}$$

It follows

$$\begin{aligned}
(6,1)(6,2)(6,4) &= (6,1)(3(6,1)+(6,2)+2(6,4))\\
&= 3(6,1)^2+(6,1)(6,2)+2(6,1)(6,4)\\
&= [12-3(6,2)]+[(6,1)+2(6,2)+3(6,4)]+2[2(6,1)+3(6,2)+(6,4)]\\
&= 12+5(6,1)+5(6,2)+5(6,4)\\
&= 7
\end{aligned}$$

We have so obtained

$$\eta_1 + \eta_2 + \eta_3 = -1,\ \ \eta_1\eta_2 + \eta_2\eta_3 + \eta_3\eta_1 = -6,\ \ \eta_1\eta_2\eta_3 = 7.$$

Hence the minimal polynomial of η_1 over $\mathbb{Q}$ (and also of η_2, η_3) is

$$f = (x - \eta_1)(x - \eta_2)(x - \eta_3) = x^3 + x^2 - 6x - 7.$$

The splitting field of f is $L_6 = \mathbb{Q}(\eta_1)$ generated by the 6-periods.

(e) Since

$$\begin{aligned} H_6 &= \{1, 7, 11\} \cup \{8, 12, 18\} = H_3 \cup 8H_3, \\ 2H_6 &= \{2, 3, 14\} \cup \{5, 16, 17\} = 2H_3 \cup 16H_3, \\ 4H_6 &= \{4, 6, 9\} \cup \{10, 13, 15\} = 4H_3 \cup 13H_3, \end{aligned}$$

we obtain

$$\begin{aligned} (6, 1) &= (3, 1) + (3, 8), \\ (6, 2) &= (3, 2) + (3, 16), \\ (6, 4) &= (3, 4) + (3, 13). \end{aligned}$$

In Example 9.2.12, we have proved that the minimal polynomial of $(3, 1)$ and $(3, 8)$ over L_6 est

$$(x - (3, 1))(x - (3, 8)) = x^2 - (6, 1)x + (6, 4) + 3 = x^2 - \eta_1 x + \eta_2 + 3.$$

If $\sigma \in \mathrm{Gal}(\mathbb{Q}(\zeta_p)/\mathbb{Q})$ is determined by $\sigma(\zeta_{19}) = \zeta_{19}^2$ then $\sigma((3, 1)) = (3, 2), \sigma((3, 8)) = (3, 16), \sigma((6, 1)) = (6, 2), \sigma(6, 4) = (6, 8) = (6, 1)$, so the minimal polynomial of $(3, 2)$ is

$$(x - (3, 2))(x - (3, 16)) = x^2 - (6, 2)x + (6, 1) + 3.$$

Similarly, applying σ^2, we obtain

$$(x - (3, 4))(x - (3, 13)) = x^2 - (6, 4)x + (6, 2) + 3.$$

(f) The extension $L_1/L_3 = \mathbb{Q}(\zeta_{19})/\mathbb{Q}((3, 1))$ is an extension of degree $d = 3$.
Here $[1]H_3 = \{[1], [7], [11]\} = [1]H_1 \cup [7]H_1 \cup [11]H_1$ (with $H_1 = \{1\}$). Proposition 9.2.8 shows that the minimal polynomial of ζ_{19} over L_3 is

$$(x - (1, 1))(x - (1, 7))(x - (1, 11)) = (x - \zeta_{19})(x - \zeta_{19}^7)(x - \zeta_{19}^{11}).$$

Without Proposition 9.2.8, note that $\mathrm{Gal}(L_1/L_3) = \tilde{H}_3 = \langle \sigma^6 \rangle = \{e, \sigma^6, \sigma^{12}\}$, where σ^6 takes ζ_{19} to $\zeta_{19}^{2^6} = \zeta_{19}^7$, so the minimal polynomial of ζ_{19} over L_3 is

$$(x - \zeta_{19})(x - \sigma^6(\zeta_{19}))(x - \sigma^{12}(\zeta_{19})) = (x - \zeta_{19})(x - \zeta_{19}^7)(x - \zeta_{19}^{11}).$$

As

$$\begin{aligned} \zeta_{19} + \zeta_{19}^7 + \zeta_{19}^{11} &= (3, 1), \\ \zeta_{19}\zeta_{19}^7\zeta_{19}^{11} &= \zeta_{19}^{19} = 1, \\ \zeta_{19}\zeta_{19}^7 + \zeta_{19}^7\zeta_{19}^{11} + \zeta_{19}\zeta_{19}^{11} &= \zeta_{19}^8 + \zeta_{19}^{18} + \zeta_{19}^{12} = (3, 8), \end{aligned}$$

we obtain that the minimal polynomial of ζ_{19} over L_3 is

$$(x - \zeta_{19})(x - \zeta_{19}^7)(x - \zeta_{19}^{11}) = x^3 - (3, 1)x^2 + (3, 8)x - 1.$$

□

Ex. 9.2.8 *In this exercise and the next, you will derive Gauss's radical formula (9.19) for* $\cos(2\pi/17)$.

(a) Show that 3 *is a primitive root modulo* 17.

(b) Show that

$$\begin{aligned} H_8 &= \{1, 2, 4, 8, 9, 13, 15, 16\}, \\ H_4 &= \{1, 4, 13, 16\}, \\ H_2 &= \{1, 16\}. \end{aligned}$$

(c) Use Propositions 9.2.8 and 9.2.9 to compute the following minimal polynomials:

Extension	Primitive Elements	Minimal Polynomial
$\mathbb{Q} \subset L_8$	$(8,1), (8,3)$	$x^2 + x - 4$
$L_8 \subset L_4$	$(4,1), (4,2)$	$x^2 - (8,1)x - 1$
	$(4,3), (4,6)$	$x^2 - (8,3)x - 1$
$L_4 \subset L_2$	$(2,1), (2,4)$	$x^2 - (4,1)x + (4,3)$

The resulting quadratic equations are easy to solve using quadratic formula. But how do the roots correspond to the periods? For example, the roots $(8,1), (8,3)$ *of* $x^2 + x - 4$ *are* $(-1 \pm \sqrt{17})/2$*. How do these match up? The answer will be given in the next exercise.*

Proof. (a) By Exercise 1, $3^8 \equiv 9^4 = 81^2 \equiv (-4)^2 \equiv -1 \not\equiv 1 \pmod{17}$, therefore the order of $[3]$ in $(\mathbb{Z}/17\,\mathbb{Z})^*$ is 16, so 3 is a primitive root modulo 17.

(b)

$$\begin{aligned} H_8 = \langle 3^2 \rangle &= \{1, 9, 9^2, 9^3, -1, -9, -9^2, -9^3\} \\ &= \{1, 9, -4, -2, -1, -9, 4, 2\} \\ &= \{1, 9, 13, 15, 16, 8, 4, 2\}, \end{aligned}$$

$H_4 = \langle 3^4 \rangle = \{1, 13, 16, 4\}$, and $H_2 = \langle 3^8 \rangle = \{1, 16\}$, so

$$\begin{aligned} H_8 &= \{1, 2, 4, 8, 9, 13, 15, 16\}, \\ H_4 &= \{1, 4, 13, 16\}, \\ H_2 &= \{1, 16\}. \end{aligned}$$

(c)

- Extension $\mathbb{Q} \subset L_8$.
  The cosets of H_8 in $(\mathbb{Z}/17\,\mathbb{Z})^*$ are

$$\begin{aligned} H_8 &= \{1, 2, 4, 8, 9, 13, 15, 16\}, \\ 3H_8 &= \{3, 6, 12, 7, 10, 5, 11, 14\}. \end{aligned}$$

  L_8 is generated over $\mathbb{Q}$ by the 8-periods $(8,1), (8,3)$, where $(8,1) + (8,3) = -1$, and

$$\begin{aligned} (8,1)(8,3) &= \sum_{\lambda \in H_8} (8, \lambda + 3) \\ &= (8,4) + (8,5) + (8,7) + (8,11) + (8,12) + (8,16) + (8,1) + (8,2) \\ &= 4(8,1) + 4(8,3) \\ &= -4. \end{aligned}$$

The minimal polynomial over $\mathbb{Q}$ of the 8-périods $(8,1),(8,3)$ is so

$$(x-(8,1))(x-(8,3)) = x^2+x-4.$$

- Extension $L_8 \subset L_4$.

$$\begin{aligned} H_8 &= \{1,4,13,16\} \cup \{2,8,9,15\} = H_4 \cup 2H_4, \\ 3H_8 &= \{3,5,12,14\} \cup \{6,7,10,11\} = 3H_4 \cup 6H_4. \end{aligned}$$

The 4-periods are so $(4,1),(4,2)$, and $(4,3),(4,6)$, where

$$\begin{aligned} (4,1)+(4,2) &= (8,1), \\ (4,1)\times(4,2) &= \sum_{\lambda \in H_4} (4,\lambda+2) \\ &= (4,3)+(4,7)+(4,15)+(4,1) \\ &= -1. \end{aligned}$$

The minimal polynomial of $(4,1)$ et $(4,2)$ over L_8 is so

$$(x-(4,1))(x-(4,2)) = x^2-(8,1)x-1.$$

Applying $\sigma : \zeta_{17} \mapsto \zeta_{17}^3$, we obtain the minimal polynomial of $(4,3)$ et $(4,6)$

$$(x-(4,3))(x-(4,6)) = x^2-(8,3)x-1.$$

- Extension $L_4 \subset L_2$.

$$\begin{aligned} H_4 &= \{1,16\} \cup \{4,13\} = H_2 \cup 4H_2, \\ 3H_4 &= \{3,14\} \cup \{5,12\} = 3H_2 \cup 5H_2, \\ &\cdots \end{aligned}$$

The 2-periods $(2,1),(2,4)$ satisfy

$$\begin{aligned} (2,1)+(2,4) &= (4,1), \\ (2,1)\times(2,4) &= \sum_{\lambda \in H_2} (2,\lambda+4) \\ &= (2,5)+(2,3) \\ &= (4,3). \end{aligned}$$

The minimal polynomial of $(2,1)$ and $(2,4)$ over L_4 is so

$$(x-(2,1))(x-(2,4)) = x^2-(4,1)x+(4,3).$$

$\square$

Ex. 9.2.9 *In this exercise, you will use numerical computations and the previous exercise to find radical expressions for various f-periods when $p = 17$.*

(a) Show that

$$\begin{aligned}
(8,1) &= 2\cos(2\pi/17) + 2\cos(4\pi/17) + 2\cos(8\pi/17) + 2\cos(16\pi/17)\\
(4,1) &= 2\cos(2\pi/17) + 2\cos(8\pi/17)\\
(4,3) &= 2\cos(6\pi/17) + 2\cos(10\pi/17)\\
(2,1) &= 2\cos(2\pi/17)
\end{aligned}$$

Then compute each of these periods to five decimal places.

(b) Use the numerical computations of part (a) and the quadratic polyomials of Exercice 8 to show that

$$\begin{aligned}
(8,1) &= \frac{1}{2}\left(-1+\sqrt{17}\right)\\
(8,3) &= \frac{1}{2}\left(-1-\sqrt{17}\right)\\
(4,1) &= \frac{1}{4}\left(-1+\sqrt{17}+\sqrt{34-2\sqrt{17}}\right)\\
(4,2) &= \frac{1}{4}\left(-1+\sqrt{17}-\sqrt{34-2\sqrt{17}}\right)\\
(4,3) &= \frac{1}{4}\left(-1-\sqrt{17}+\sqrt{34+2\sqrt{17}}\right)
\end{aligned}$$

(c) Use the quadratic polynomial $x^2 - (4,1)x + (4,3)$ and part (b) to derive (9.19).

Proof. Recall (see Exercise 8) that

$$\begin{aligned}
H_8 &= \{1,2,4,8,9,13,15,16\}\\
H_4 &= \{1,4,13,16\}\\
3H_4 &= \{3,5,12,14\}\\
H_2 &= \{1,16\}
\end{aligned}$$

Write $\zeta = \zeta_{17}$.

(a) Using these results, and $\zeta^{-k} = \zeta^{17-k}$, $k = 1,2,4,8$, and also $\zeta^k + \zeta^{-k} = 2\cos(2k\pi/17)$, we obtain

$$\begin{aligned}
(8,1) &= \sum_{[a]\in H_8} \zeta^a \\
&= \zeta + \zeta^2 + \zeta^4 + \zeta^8 + \zeta^9 + \zeta^{13} + \zeta^{15} + \zeta^{16} \\
&= (\zeta + \zeta^{-1}) + (\zeta^2 + \zeta^{-2}) + (\zeta^4 + \zeta^{-4}) + (\zeta^8 + \zeta^{-8}) \\
&= 2\cos(2\pi/17) + 2\cos(4\pi/17) + 2\cos(8\pi/17) + 2\cos(16\pi/17)
\end{aligned}$$

$$\begin{aligned}
(4,1) &= \sum_{[a]\in H_4} \zeta^a \\
&= \zeta + \zeta^4 + \zeta^{13} + \zeta^{16} \\
&= (\zeta + \zeta^{-1}) + (\zeta^4 + \zeta^{-4}) \\
&= 2\cos(2\pi/17) + 2\cos(8\pi/17) \\
(4,3) &= \sum_{[a]\in 3H_4} \zeta^a \\
&= \zeta^3 + \zeta^5 + \zeta^{12} + \zeta^{14} \\
&= (\zeta^3 + \zeta^{-3}) + (\zeta^5 + \zeta^{-5}) \\
&= 2\cos(6\pi/17) + 2\cos(10\pi/17)
\end{aligned}$$

$$\begin{aligned}
(2,1) &= \sum_{[a]\in H_2} \zeta^a \\
&= \zeta + \zeta^{16} \\
&= \zeta + \zeta^{-1} \\
&= 2\cos(2\pi/17)
\end{aligned}$$

$(2,1) = 2\cos(2\pi/17) \simeq 0.93247$,
$(4,1) \simeq 2.04948, (4,3) \simeq 0.34415$,
$(8,1) \simeq 1.56155$.
As $(4,1) + (4,2) = (8,1)$, we obtain $(4,2) \simeq -0.48792 < 0$.

(b) By Exercise 8, $(8,1),(8,3)$ are the roots of $x^2 + x - 4$, and by part (a) $(8,1) > 0$. The only positive root of $x^2 + x - 4$ is $(-1+\sqrt{17})/2$, therefore

$$\begin{aligned}
(8,1) &= \frac{1}{2}\left(-1+\sqrt{17}\right), \\
(8,3) &= \frac{1}{2}\left(-1-\sqrt{17}\right).
\end{aligned}$$

$(4,1),(4,2)$ are the roots of $x^2 - (8,1)x - 1$, with discriminant

$$\Delta = \frac{1}{4}(-1+\sqrt{17})^2 + 4 = \frac{1}{4}(34 - 2\sqrt{17}),$$

therefore

$$\{(4,1),(4,2)\} = \left\{\frac{1}{4}\left(-1+\sqrt{17}+\sqrt{34-2\sqrt{17}}\right), \frac{1}{4}\left(-1+\sqrt{17}-\sqrt{34-2\sqrt{17}}\right)\right\}.$$

By part (a) $(4,2) < 0 < (4,1)$, so

$$(4,1) = \frac{1}{4}\left(-1+\sqrt{17}+\sqrt{34-2\sqrt{17}}\right),$$
$$(4,2) = \frac{1}{4}\left(-1+\sqrt{17}-\sqrt{34-2\sqrt{17}}\right).$$

$(4,3),(4,6)$ are the roots of $x^2-(8,3)x-1$, with discriminant

$$\Delta = \frac{1}{4}(-1-\sqrt{17})^2+4 = \frac{1}{4}(34+2\sqrt{17}),$$

therefore

$$\{(4,3),(4,6)\} = \left\{\frac{1}{4}\left(-1-\sqrt{17}+\sqrt{34+2\sqrt{17}}\right), \frac{1}{4}\left(-1-\sqrt{17}-\sqrt{34+2\sqrt{17}}\right)\right\}.$$

As $(4,3) > 0$,

$$(4,3) = \frac{1}{4}\left(-1-\sqrt{17}+\sqrt{34+2\sqrt{17}}\right),$$
$$(4,6) = \frac{1}{4}\left(-1-\sqrt{17}-\sqrt{34+2\sqrt{17}}\right).$$

(c) $(2,1) = 2\cos(2\pi/17)$, and also $(2,4)$, is root of $x^2-(4,1)x+(4,3)$, with discriminant

$$\Delta = (4,1)^2 - 4(4,3).$$

As

$$\begin{aligned}(4,1)^2 &= \sum_{\lambda\in H_4}(4,\lambda+1)\\ &= (4,2)+(4,5)+(4,14)+4\\ &= (4,2)+2(4,3)+4,\end{aligned}$$

then

$$\begin{aligned}\Delta &= (4,2)-2(4,3)+4\\ &= \frac{1}{4}\left(-1+\sqrt{17}-\sqrt{34-2\sqrt{17}}-2\left(-1-\sqrt{17}+\sqrt{34+2\sqrt{17}}\right)+16\right)\\ &= \frac{1}{4}\left(17+3\sqrt{17}-\sqrt{34-2\sqrt{17}}-2\sqrt{34+2\sqrt{17}}\right).\end{aligned}$$

The roots of $x^2-(4,1)x+(4,3)$ are so $\frac{1}{2}((4,1)\pm\sqrt{\Delta})$

$= \frac{1}{8}\left(-1+\sqrt{17}+\sqrt{34-2\sqrt{17}}\right) \pm \frac{1}{4}\sqrt{17+3\sqrt{17}-\sqrt{34-2\sqrt{17}}-2\sqrt{34+2\sqrt{17}}}.$

As $(2,4) = 2\cos(4\pi/17) < 2\cos(2\pi/17) = (2,1)$, we can conclude that

$$\begin{aligned}\cos\left(\frac{2\pi}{17}\right) =& -\frac{1}{16}+\frac{1}{16}\sqrt{17}+\frac{1}{16}\sqrt{34-2\sqrt{17}}\\ &+\frac{1}{8}\sqrt{17+3\sqrt{17}-\sqrt{34-2\sqrt{17}}-2\sqrt{34+2\sqrt{17}}}.\end{aligned}$$

□

Ex. 9.2.10 *Let $p = 11$. Prove that $y^5+y^4-4y^3-3y^2+3y+1$ is the minimal polynomial of the 2-period* $(2,1) = 2\cos(2\pi/11)$.

Proof. Let $\zeta = \zeta_{11} = e^{2i\pi/11}$, et $\eta = (2,1) = \zeta + \zeta^{-1} = 2\cos(2\pi/11)$. The powers of 2 modulo 11 are $1, 2, 2^2 = 2, 2^3 = 8, 2^4 = 5, 2^5 = -1$, so the order of $[2]$ in $(\mathbb{Z}/11\,\mathbb{Z})^*$ is 10, so 2 is a primitive root modulo 11.

As $\Phi_{11}(\zeta) = 1+\zeta+\zeta^2+\zeta^3+\zeta^4+\zeta^5+\zeta^6+\zeta^7+\zeta^8+\zeta^9+\zeta^{10} = 0$, we obtain by multiplication by ζ^{-5} :

$$(\zeta^{-5}+\zeta^5)+(\zeta^{-4}+\zeta^4)+(\zeta^{-3}+\zeta^3)+(\zeta^{-2}+\zeta^2)+(\zeta^{-1}+\zeta)+1. \qquad (3)$$

Write $u_n = \zeta^n + \zeta^{-n}$. As

$$\zeta^{n+2}+\zeta^{-n-2} = (\zeta+\zeta^{-1})(\zeta^{n+1}+\zeta^{-n-1}) - (\zeta^n+\zeta^{-n}),$$

we obtain for all $n \in \mathbb{N}$

$$u_{n+2} = \eta\, u_{n+1} - u_n,\ u_0 = 2, u_1 = \eta.$$

Therefore

$$\begin{aligned}
\zeta+\zeta^{-1} &= \eta,\\
\zeta^2+\zeta^{-2} &= \eta^2-2,\\
\zeta^3+\zeta^{-3} &= \eta(\eta^2-2)-\eta = \eta^3-3\eta,\\
\zeta^4+\zeta^{-4} &= \eta(\eta^3-3\eta)-(\eta^2-2) = \eta^4-4\eta^2+2,\\
\zeta^5+\zeta^{-5} &= \eta(\eta^4-4\eta^2+2)-(\eta^3-3\eta) = \eta^5-5\eta^3+5\eta.
\end{aligned}$$

The equality (3) gives

$$\begin{aligned}
0 &= (\eta^5-5\eta^3+5\eta)+(\eta^4-4\eta^2+2)+(\eta^3-3\eta)+(\eta^2-2)+\eta+1\\
&= \eta^5+\eta^4-4\eta^3-3\eta^2+3\eta+1.
\end{aligned}$$

So η is a root of $f = x^5+x^4-4x^3-3x^2+3x+1 \in \mathbb{Q}[x]$.

By Proposition 9.2.6 (b), the fixed field L_2 of $\tilde{H}_2$ corresponding to $H_2 = \{-1,1\}$ is $L_2 = \mathbb{Q}(\eta)$, and $[L_2 : \mathbb{Q}] = 5$ by Proposition 9.2.1. (as $\mathbb{Q} \subset \mathbb{Q}(\zeta)$ is a Galois extension, $[\mathbb{Q}(\eta) : \mathbb{Q}] = |\mathrm{Gal}(\mathbb{Q}(\eta)/\mathbb{Q})| = (G : \tilde{H}_2) = ((\mathbb{Z}/11\mathbb{Z})^* : \{-1,1\}) = 5$).

The minimal polynomial g of η over $\mathbb{Q}$ divides f, and has degree 5, so $g = f$.

Using the other form of the minimal polynomial given in Proposition 9.2.6(a), we obtain that

$$\begin{aligned}
&(x-\zeta-\zeta^{-1})(x-\zeta^2-\zeta^{-2})(x-\zeta^3-\zeta^{-3})(x-\zeta^4-\zeta^{-4})(x-\zeta^5-\zeta^{-5})\\
&= x^5+x^4-4x^3-3x^2+3x+1
\end{aligned}$$

is the minimal polynomial of $\eta = \zeta_{11}+\zeta_{11}^{-1}$ over $\mathbb{Q}$. $\square$

Ex. 9.2.11 *Let $L_{fq} \subset L_f$ be the extension studied in Theorem 9.2.14. Thus f and fq divide $p-1$, and q is prime. As usual, $ef = p-1$ and g is a primitive root modulo p. Finally, let ω be a primitive qth root of unity.*

(a) Let $\tau \in \mathrm{Gal}(\mathbb{Q}(\zeta_p)/\mathbb{Q})$ satisfy $\tau(\zeta_p) = \zeta_p^{g^{e/q}}$, and let $\sigma' = \tau|_{L_f}$ be the restriction of τ to L_f. Prove that σ' generates $\mathrm{Gal}(L_f/L_{fq})$.

(b) *Prove that* $\mathrm{Gal}(L_f(\omega)/L_{fq}(\omega)) \simeq \mathrm{Gal}(L_f/L_{fq})$*, where the isomorphism is defined by restriction to* L_f*.*

(c) *Let* $\sigma \in \mathrm{Gal}(L_f(\omega)/L_{fq}(\omega))$ *map to the element* $\sigma' \in \mathrm{Gal}(L_f/L_{fq})$ *constructed in part (a). Prove that* σ *satisfies (9.21).*

(d) *Prove the coset decomposition of* H_{fq} *given in (9.23).*

Proof. (a) Let $f' = fq$, and $e' = n/f'$. Then $p-1 = ef = e'f'$, and $e = e'q$.

By section 9.2,

$$L_f \text{ is the fixed field of } \tilde{H}_f = \langle \sigma \rangle, \text{ where } \sigma(\zeta_p) = \zeta_p^{g^e}.$$

$\tilde{H}_f$ is the set of automorphisms ξ such that $\zeta_p \mapsto \xi(\zeta_p) = \zeta_p^i$, $i \in H_f = \{1, g^e, g^{2e}, \cdots, g^{(f-1)e}\}$.

This result applied to f' gives:

$$L_{fq} \text{ is the fixed field of } \tilde{H}_{fq} = \langle \tau \rangle, \text{ where } \tau(\zeta_p) = \zeta_p^{g^{e'}} = \zeta_p^{g^{e/q}}.$$

By the Galois correspondence, $\mathrm{Gal}(\mathbb{Q}(\zeta_p)/L_{fq}) = \tilde{H}_{fq} = \langle \tau \rangle$.

As $\mathbb{Q} \subset L_f$ is a Galois extension, $\tau L_f = L_f$ (Theorem 7.2.5).

If $\sigma' : L_f \to L_f$ is the restriction of τ to L_f, then $\sigma' \in \mathrm{Gal}(L_f/L_{fq})$.

The restriction mapping $\psi : \mathrm{Gal}(\mathbb{Q}(\zeta_p)/L_{fq}) \to \mathrm{Gal}(L_f/L_{fq})$ is a surjective mapping by the proof of Theorem 7.2.7, so every element of $\mathrm{Gal}(L_f/L_{fq})$ is of the form $\psi(\tau^k) = \sigma'^k$, $k \in \mathbb{Z}$, therefore

$$\mathrm{Gal}(L_f/L_{fq}) = \langle \sigma' \rangle.$$

Since $|\mathrm{Gal}(L_f/L_{fq})| = q$ (Proposition 9.2.1), the order of σ' is q.

Note: as $\tau(\zeta_p) = \zeta_p^{g^{e/q}}, \tau((f,\lambda)) = (f, g^{e/q}\lambda)$, for every period (f,λ) (Lemma 9.2.4(d)), and $(f,\lambda) \in L_f$, so

$$\sigma'((f,\lambda)) = (f, g^{e/q}\lambda).$$

(b) Since $q \mid p-1, p \wedge q = 1$, therefore $\Phi_q(x) = \frac{x^q-1}{x-1}$ is irreducible over $\mathbb{Q}(\zeta_p)$ by Exercise 9.1.16. Hence Φ_q is a fortiori irreducible over the subfields L_f, L_{fq} of $\mathbb{Q}(\zeta_p)$. Consequently

$$[L_f(\omega) : L_f] = [L_{fq}(\omega) : L_{fq}] = \deg(\Phi_q) = q-1.$$

$L_f(\omega)$ is the splitting field of Φ_q over L_f, $L_f \subset L_f(\omega)$ is so a Galois extension, and similarly $L_{fq} \subset L_{fq}(\omega)$ is Galois.

By Exercises 8.3.2 and 8.2.7, $L_f(\omega)$ is a Galois extension of L_{fq}, a fortiori of $L_{fq}(\omega)$.

Let

$$\varphi : \left\{ \begin{array}{ccc} \mathrm{Gal}(L_f(\omega)/L_{fq}(\omega)) & \to & \mathrm{Gal}(L_f/L_{fq}) \\ \sigma & \mapsto & \sigma|_{L_f} \end{array} \right.$$

This mapping is well defined since L_f is a normal extension of L_{fq}, so $\sigma L_f = L_f$, and σ fixes the elements of $L_{fq}(\omega)$, a fortiori the elements of L_{fq}.

φ is a group homomorphism, and φ is injective:

if $\sigma \in \ker(\varphi)$, then $\sigma(\omega) = \omega$, and σ is the identity on L_f, thus σ is the identity on $L_f(\omega)$, so $\sigma = e$, therefore $\ker(\varphi) = \{e\}$.

Moreover, $[L_f : L_{fq}] = q$ and $[L_f(\omega) : L_f] = [L_{fq}(\omega) : L_{fq}] = q - 1$, therefore, by the Tower Theorem, $[L_f(\omega) : L_{fq}(\omega)] = q$. Hence $|\mathrm{Gal}(L_f(\omega)/L_{fq}(\omega))| = |\mathrm{Gal}(L_f, L_{fq})| = q$, so φ is a group isomorphism.

(c) Let $\sigma = \varphi^{-1}(\sigma')$. Then σ is a generator of $\mathrm{Gal}(L_f(\omega)/L_{fq}(\omega))$, and $\varphi(\sigma) = \sigma'$.

As $\sigma|_{L_f} = \sigma'$, by the note in part (a),

$$\sigma((f, \lambda)) = \sigma'((f, \lambda)) = (f, g^{e/q}\lambda).$$

(d) $H_f = \langle g^e \rangle$, et $H_{fq} = \langle g^{e/q} \rangle$.

We show first that $g^{k(e/q)} \notin H_f$ if $1 \le k \le q - 1$. If not, there would exist an integer j such that $g^{k(e/q)} = g^{je}$. As the order of g is $p - 1 = ef$, $ef \mid k\frac{e}{q} - je$, so $\lambda efq = ke - jeq, \lambda \in \mathbb{Z}$, therefore $\lambda fq = k - jq$, and so $q \mid k$. It is impossible since $1 \le k \le q - 1$.

If $0 \le i < j \le q - 1$, by the preceding result, $(g^{i(e/q)})^{-1} g^{j(e/q)} = g^{(j-i)(e/q)} \notin H_f$, therefore $g^{i(e/q)} H_f \neq g^{j(e/q)} H_f$.

The q left cosets $H_f, g^{e/q}H_f, g^{2e/q}H_f, \cdots, g^{(q-1)e/q}H_f$ are so distinct. Since $(H_{fq} : H_f) = q$, the set of left cosets is reduced to these q cosets, which give a partition of H_{fq}:

$$H_{fq} = H_f \cup g^{e/q}H_f \cup g^{2e/q}H_f \cup \cdots \cup g^{(q-1)e/q}H_f.$$

$\square$

Ex. 9.2.12 *Let p be an odd prime, and let m be a positive integer relatively prime to p.*

(a) Prove that $1, \zeta_p, \ldots, \zeta_p^{p-2}$ are linearly independent over $\mathbb{Q}(\zeta_m)$.

(b) Explain why part (a) implies that $\zeta_p, \ldots, \zeta_p^{p-1}$ are linearly independent over $\mathbb{Q}(\zeta_m)$.

(c) Let $f \mid p - 1$. Prove that the f-periods are linearly independent over $\mathbb{Q}(\zeta_m)$.

Proof. (a) As $p \wedge m = 1$, $\Phi_p(x) = x^{p-1} + \cdots + x + 1$ is irreducible over $\mathbb{Q}(\zeta_m)$ by Exercise 9.1.16. Therefore the minimal polynomial of ζ_p over $\mathbb{Q}(\zeta_m)$ is $\Phi_p(x)$, of degree $p - 1$, so $1, \zeta_p, \zeta_p^2, \cdots, \zeta_p^{p-2}$ are linearly independent over $\mathbb{Q}(\zeta_m)$.

(b) If $a_1, \cdots, a_{p-1} \in \mathbb{Q}(\zeta_m)$, as $\zeta_p \neq 0$,

$$a_1\zeta_p + a_2\zeta_p^2 + \cdots + a_{p-1}\zeta_p^{p-1} = 0 \Rightarrow a_1 + a_2\zeta_p + \cdots + a_{p-1}\zeta_p^{p-1} = 0 \Rightarrow a_1 = a_2 = \cdots = a_{p-1} = 0,$$

so $\zeta_p, \zeta_p^2, \cdots, \zeta_p^{p-1}$ are linearly independent over $\mathbb{Q}(\zeta_m)$.

(c) Suppose that $\sum_{i=1}^{e} a_i(f, \lambda_i) = 0$, where $a_i \in \mathbb{Q}(\zeta_m)$. Let $\{[\lambda_1], \cdots, [\lambda_e]\}$ be a complete system of representatives of the cosets $[\lambda]H_f$, then

$$\sum_{i=1}^{e} a_i \sum_{a \in [\lambda_i]H_f} \zeta_p^a = 0.$$

As $(\lambda_i H_f)_{1\leq i\leq e}$ is a partition of $(\mathbb{Z}/p\mathbb{Z})^*$, this equality is equivalent to

$$\sum_{[k]\in(\mathbb{Z}/p\mathbb{Z})^*} b_k\zeta_p^k = \sum_{k=0}^{p-1} b_k\zeta_p^k = 0,$$

where b_k is a constant on every coset $[\lambda_i]H_f$, equal to a_i.

Since $\zeta_p, \zeta_p^2, \cdots, \zeta_p^{p-1}$ are linearly independent over $\mathbb{Q}(\zeta_m)$, all the b_k are zero, so $a_1 = \cdots = a_e = 0$.

Les f-périodes sont linéairement indépendantes sur $\mathbb{Q}(\zeta_m)$.

□

Ex. 9.2.13 *Prove (9.24):*

$$\sum_{a=0}^{17}\left(\frac{a}{17}\right)\zeta_{17}^a = \sqrt{17}.$$

Proof. By Exercise 8 (b), we have proved for $p = 17$, that

$$\begin{aligned}(8,1) &= \frac{1}{2}\left(-1+\sqrt{17}\right),\\ (8,3) &= \frac{1}{2}\left(-1-\sqrt{17}\right).\end{aligned}$$

So

$$\sqrt{17} = (8,1)-(8,3) = \sum_{a\in H_8}\zeta^a - \sum_{a\in 3H_8}\zeta^a.$$

Let

$$\varphi : \left\{\begin{array}{ccc}(\mathbb{Z}/p\mathbb{Z})^* & \to & (\mathbb{Z}/p\mathbb{Z})^*\\ x & \mapsto & x^2.\end{array}\right.$$

φ is a group homomorphism.

As $x^2 = 1 \iff (x-1)(x+1) = 0 \iff x \in \{-1,1\}$, $\ker(\varphi) = \{-1,1\} \subset (\mathbb{Z}/p\mathbb{Z})^*$. Write $C = \mathrm{im}(\varphi)$ the set of square elements in $(\mathbb{Z}/p\mathbb{Z})^*$. Then $\mathrm{im}(\varphi) \simeq (\mathbb{Z}/p\mathbb{Z})^*/\ker(\varphi)$, so $|C| = |\mathrm{im}(\varphi)| = (p-1)/2 = 8$. Moreover $H_8 = \langle 3^2\rangle$ (Exercise 1), so $H_8 \subset C$, and $|H_8| = 8 = |C|$, therefore $H_8 = C$ is the set of squares in $(\mathbb{Z}/17\,\mathbb{Z})^*$. Its complement $3H_8$ is the set of non squares in $(\mathbb{Z}/17\mathbb{Z})^*$.

Therefore, for all $a \in (\mathbb{Z}/17\mathbb{Z})^*$.

$$\left(\frac{a}{17}\right) = 1 \iff a \in H_8,$$

$$\left(\frac{a}{17}\right) = -1 \iff a \in 3H_8,$$

and $\left(\frac{a}{17}\right) = 0$ if $a = 0$ or $a = 17$ (where we write for all integer k, $\left(\frac{[k]}{17}\right) = \left(\frac{k}{17}\right)$). Hence

$$\sum_{a=0}^{17}\left(\frac{a}{17}\right)\zeta_{17}^a = \sqrt{17}.$$

□

Ex. 9.2.14 *Consider the quotation from Galois given at the end of the Historical Notes.*

(a) Show that the permutations obtained by mapping the first line in the displayed table to the other lines give a cyclic group of order $n-1$. Also explain how these permutations relate to the Galois group.

(b) Explain what Galois is saying in the last sentence of the quotation.

Proof. This group of permutations is generated by the cycle

$$(a, b, c, \cdots, k) = (r, r^g, r^{g^2}, \cdots, r^{g^{n-2}}).$$

It is a cyclic subgroup of order $n-1$ in the group of permutation of the $n-1$ roots of $\Phi_n(x)$. The Galois group of $\Phi_n(x)$, as a permutation group of the roots, is indeed a cyclic group of order $n-1$, if n is prime:

$$\mathrm{Gal}_{\mathbb{Q}}(\Phi_n) = \mathrm{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q}) \simeq (\mathbb{Z}/n\mathbb{Z})^* \simeq C_{n-1}.$$

For such a Galois extension,

$$|\mathrm{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q})| = [\mathbb{Q}(\zeta_n) : \mathbb{Q}] = n-1 = \deg(\Phi_n(x)).$$

(b) If all the roots are rational function of one fixed root α of f, then the extension $\mathbb{Q} \subset \mathbb{Q}(\alpha)$ is Galois, so the equality $|\mathrm{Gal}(\mathbb{Q}(\alpha)/\mathbb{Q})| = [\mathbb{Q}(\alpha) : \mathbb{Q}] = \deg(f)$ is true for the minimal polynomial f of α over $\mathbb{Q}$. □

Ex. 9.2.15 *What are the 1-periods?*

Proof. $H_1 = \{[1]\}$, and the coset of $[a] \in (\mathbb{Z}/p\mathbb{Z})^*$ is $[a]H_1 = \{[a]\}$, so the 1-périodes $(1, a)$ are the powers of ζ_p:

$$(1, a) = \zeta_p^a.$$

□

Ex. 9.2.16 *Redo Exercise 3 using periods.*

Proof. If $p = 7$, and $\zeta = e^{2i\pi/7}$, the 2-periods corresponding to $H_2 = \{-1, 1\} = \{1, 6\}$ are $(2, 1) = \zeta + \zeta^{-1}, (2, 2) = \zeta^2 + \zeta^{-2}, (2, 3) = \zeta^3 + \zeta^{-3}$. By Proposition 9.2.6, they are the roots of the irreducible polynomial

$$f = (x - (2, 1))(x - (2, 2))(x - (2, 3))$$

$$\begin{aligned}
&(2,1) + (2,2) + (2,3) = -1,\\
&(2,1)^2 = \sum_{\lambda \in H_2} (2, \lambda + 1) = (2,2) + 2,\\
&(2,1)(2,2) = \sum_{\lambda \in H_2} (2, \lambda + 2) = (2,3) + (2,1).
\end{aligned}$$

3 is a primitive root modulo 7. Let σ the $\mathbb{Q}$-automorphism determined by $\sigma(\zeta) = \zeta^3$. Then σ gives the chain $(2, 1) \mapsto (2, 3) \mapsto (2, 2) \mapsto (2, 1)$, so

$$(2,1)(2,2) = (2,3) + (2,1), \quad (2,3)(2,1) = (2,2) + (2,3), \quad (2,2)(2,3) = (2,1) + (2,2).$$

By sommation of these equalities,

$$(2,1)(2,2) + (2,3)(2,1) + (2,2)(2,3) = 2(2,3) + 2(2,1) + 2(2,2) = -2.$$

Finally

$$(2,1)(2,2)(2,3) = (2,1)[(2,1)+(2,2)] = (2,1)^2+(2,1)(2,2) = (2,2)+2+(2,3)+(2,1) = 1.$$

Therefore $f = x^3 + x^2 - 2x - 1$ is the minimal polynomial of $(2,1) = 2\cos(2\pi/7)$ over $\mathbb{Q}$ (and also of $(2,2), (2,3)$).

The fixed field L_2 of $\tilde{H}_2$ corresponding to H_2 is $\mathbb{Q}(\zeta + \zeta^{-1})$, of degree 3 over $\mathbb{Q}$, and $\tilde{H}_2 = \{e, \tau\}$, where $\tau(\zeta) = \zeta^{-1} = \overline{\zeta}$, so τ is the restriction of the complex conjugation to L_2. The end of the proof is the same as in Exercise 3. □

Ex. 9.2.17 *Let f be an even divisor of $p-1$ where p is an odd prime. Prove that every f-period (f, λ) lies in $\mathbb{R}$.*

Proof. As $2 \mid f$ is even, $H_2 \subset H_f$ (Exercise 1), so every coset $[\lambda]H_f$ is a disjoint union of $[\mu]H_2$ (cf Exercise 4), so

$$[\lambda]H_f = \bigcup_{[\mu]\in A} [\mu]H_2.$$

Therefore

$$(f, \lambda) = \sum_{a\in[\lambda]H_f} \zeta_p^a = \sum_{\mu\in A} \sum_{a\in[\mu]H_2} \zeta_p^a = \sum_{\mu\in A} (\zeta_p^\mu + \zeta_p^{-\mu}) \in \mathbb{R}.$$

□