

14 Chapter 15 : THE LEMNISCATE

14.1 DIVISION POINTS AND ARC LENGTH

Ex. 15.1.1 Prove that the numbers described in Abel's theorem at the beginning of the chapter are precisely those in Theorem 10.2.1, provided we replace "product of several numbers" with "product of distinct numbers" in Abel's statement of the theorem.

Proof. The numbers described in Theorem 10.2.1 are the integers $n = 2^s p_1 \cdots p_r$, where p_1, \dots, p_r are distinct Fermat primes, of the form $p_k = 2^{n_k} + 1$. Thus these numbers are the product of *distinct* numbers of the form 2^m , or $2^m + 1$, where $2^m + 1$ is prime, as described in the Theorem of Abel. \square

Ex. 15.1.2 Show that in polar coordinates, the equation of the lemniscate is $r^2 = \cos(2\theta)$.

Proof. By definition, a point $M = (x, y) \in \mathbb{R}^2$ is a point of the lemniscate L if and only if

$$(x^2 + y^2)^2 = x^2 - y^2.$$

If (r, θ) are polar coordinates of $M = M(r, \theta)$, then $x = r \cos \theta, y = r \sin \theta$, thus, using $\cos^2 \theta + \sin^2 \theta = 1$, and $\cos(2\theta) = \cos^2 \theta - \sin^2 \theta$, we obtain

$$\begin{aligned} M(r, \theta) \in L &\iff (r^2 \cos^2 \theta + r^2 \sin^2 \theta)^2 = r^2 \cos^2 \theta - r^2 \sin^2 \theta \\ &\iff r^4 = r^2 \cos(2\theta) \\ &\iff r^2 = \cos(2\theta). \end{aligned}$$

The equation of the lemniscate is $r^2 = \cos(2\theta)$. \square

Ex. 15.1.3 Prove that the two improper integrals $\int_0^1 (1-t^4)^{-1/2} dt$ and $\int_{-1}^0 (1-t^4)^{-1/2} dt$ converge.

Proof. The map $t \mapsto (1-t^4)^{-1/2}$ is continuous on $[0, 1[$, thus $t \mapsto (1-t^4)^{-1/2} dt$ is summable on $[1, x]$ for all $x \in [0, 1]$.

Since $1-t^4 = (1-t)(1+t+t^2+t^3)$, $(1-t^4)^{-1/2} \sim [4(1-t)]^{-1/2}$ in the neighborhood of 1. The Riemann Criterion shows that $\int_0^1 (1-t)^{-\alpha} dt$ converges if $\alpha < 1$, and here $\alpha = 1/2$. Since $(1-t^4)^{-1/2} > 0$, this is sufficient to prove that $\int_0^1 (1-t^4)^{-1/2} dt$ converges.

Since $t \mapsto (1-t^4)^{-1/2}$ is even, the same is true in the neighborhood of -1 , thus $\int_{-1}^0 (1-t^4)^{-1/2} dt$ converges. \square

Ex. 15.1.4 Prove the arc length formula stated in (15.6)

Proof. Here the equation of the ellipse E is

$$x^2 + \frac{y^2}{b^2} = 1,$$

with eccentricity $k = \sqrt{1-b^2}$.

We compute the arc length l of (E) between $x = u, y = v$ ($-1 < u < v < 1$) on the upper part of the curve. Then

$$l = \int_u^v \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx,$$

where $y = f(x) = b\sqrt{1-x^2}$. Then $f'(x) = \frac{dy}{dx} = -\frac{2x}{\sqrt{1-x^2}}$, thus

$$\begin{aligned} l &= \int_u^v \sqrt{1 + \left(\frac{bx}{\sqrt{1-x^2}}\right)^2} dx \\ &= \int_u^v \sqrt{\frac{1-x^2+b^2x^2}{1-x^2}} dx \\ &= \int_u^v \sqrt{\frac{1-k^2x^2}{1-x^2}} dx \end{aligned}$$

We have proved

$$l = \int_u^v \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx = \int_u^v \sqrt{\frac{1-k^2x^2}{1-x^2}} dx = \int_u^v \frac{\sqrt{(1-x^2)(1-k^2x^2)}}{1-x^2} dx.$$

The arc length of the ellipse is given by an elliptic integral. \square

Ex. 15.1.5 Shows that (15.7) reduces to $(x^2 + y^2)^2 = x^2 - y^2$ when $a = b = 1/\sqrt{2}$.

Proof. If we take $a = b = 1/\sqrt{2}$ in the formula of the ovals of Cassini

$$((x-a)^2 + y^2)((x+a)^2 + y^2) = b^4,$$

we obtain

$$\begin{aligned} \frac{1}{4} &= \left[\left(x - \frac{1}{\sqrt{2}} \right)^2 + y^2 \right] \left[\left(x + \frac{1}{\sqrt{2}} \right)^2 + y^2 \right] \\ &= \left(x^2 + y^2 + \frac{1}{2} - \sqrt{2}x \right) \left(x^2 + y^2 + \frac{1}{2} + \sqrt{2}x \right) \\ &= \left(x^2 + y^2 + \frac{1}{2} \right)^2 - 2x^2 \\ &= (x^2 + y^2)^2 + (x^2 + y^2) - 2x^2 \\ &= (x^2 + y^2)^2 + y^2 - x^2 + \frac{1}{4}. \end{aligned}$$

Therefore, for $a = b = 1/\sqrt{2}$, the equation $((x-a)^2 + y^2)((x+a)^2 + y^2) = b^4$ reduces to

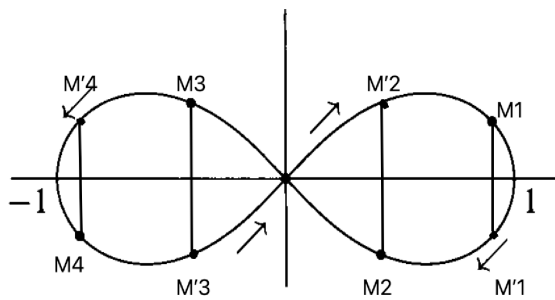
$$(x^2 + y^2)^2 = x^2 - y^2,$$

which is the equation of the Lemniscate. \square

Ex. 15.1.6 Let $n > 0$ be an odd integer, and assume that the n -division points of the lemniscate can be constructed with straightedge and compass. prove that the same is true for the $2n$ -division points. Your proof should include a picture.

Proof. Let $M_0 = 0, \dots, M_{n-1}$ the n -divisions points. If n is odd, the middle segment $(M_{(n-1)/2}, M_{(n+1)/2})$ straddles the origin, The symmetry of the figure shows that the arc length between O and $M_{(n-1)/2}$ is the same as the arc length between O and $M_{(n+1)/2}$, so is half of the arc length $M_k M_{k+1}$. Therefore we can complete M_0, M_1, \dots, M_{n-1} by the symmetric points $M'_0 = O, \dots, M'_{n-1}$ relative to the x -axis to obtain the $2n$ -division points (the point O is counted twice).

Figure for $n = 5$: the 10-division points are $0, M'_2, M_1, M'_1, M_2, 0, M_3, M'_3, M_4, M'_4$. \square



Ex. 15.1.7 Recall that in Greek geometry, the ellipse is defined to be the locus of all points whose **sum** of distances to two given points is constant. Suppose instead we consider the locus of all points whose **product** of distances to two given points is constant. Show that this leads to (15.7) when the given points are $(a, 0), (-a, 0)$ and the constant is b^4 (*).

(*) Read b^2 .

Proof. Let Γ the locus of all points whose product of distances to two points $(a, 0), (-a, 0)$ is the constant b^2 . Then

$$\begin{aligned} M(x, y) \in \Gamma &\iff \sqrt{(x-a)^2 + y^2} \sqrt{(x+a)^2 + y^2} = b^2 \\ &\iff ((x-a)^2 + y^2)((x+a)^2 + y^2) = b^4. \end{aligned}$$

We obtain the formula of the ovals of Cassini. □