Ex. 14.4.13 *Let $G_0 \subset \mathrm{GL}(2, \mathbb{F}_p)$ be solvable. Prove that the subgroup generated by G_0 and $\mathbb{F}_p^* I_2$ is also solvable.*

Proof. The subgroup $N = \mathbb{F}_p^* I_2$ is a normal subgroup of $G = \mathrm{GL}(2, \mathbb{F}_p)$:

If $A \in G$, and $H = \lambda I_2$, then $AHA^{-1} = A\lambda I_p A^{-1} = \lambda AA^{-1} = \lambda I_2 \in H$.

Therefore $G_0 N$ is a subgroup of G. This implies that $G_0 N$ is the smallest subgroup containing G_0 and N, so that $G_0 N$ is the subgroup generated by G_0 and $\mathbb{F}_p^* I_2$.

By the Second Isomorphism Theorem (see the beginning of Exercise 12),

$$G_0 N/N \simeq G_0/N \cap G_0.$$

By hypothesis, G_0 is solvable, thus its subgroup $N \cap G_0$ is solvable, and the quotient group $G_0/N \cap G_0$ is solvable (Proposition 8.2.4).

N is cyclic, thus N is solvable. Using Proposition 8.2.4 anew, $G_0 N$ is solvable.

We have proved that if $G_0 \subset \mathrm{GL}(2, \mathbb{F}_p)$ is solvable, the subgroup generated by G_0 and $\mathbb{F}_p^* I_2$ is also solvable. $\square$