14 Chapter 15 : THE LEMNISCATE

14.1 DIVISION POINTS AND ARC LENGTH

Ex. 15.1.1 *Prove that the numbers described in Abel's theorem at the beginning of the chapter are precisely those in Theorem 10.2.1, provided we replace "product of several numbers" with "product of distinct numbers" in Abel's statement of the theorem.*

Proof. The numbers described in Theorem 10.2.1 are the integers $n = 2^s p_1 \cdots p_r$, where $p_1, \dots, p_r$ are distinct Fermat primes, of the form $p_k = 2^{n_k} + 1$. Thus these numbers are the product of *distinct* numbers of the form 2^m, or $2^m + 1$, where $2^m + 1$ is prime, as described in the Theorem of Abel. $\square$

Ex. 15.1.2 *Show that in polar coordinates, the equation of the lemniscate is $r^2 = \cos(2\theta)$.*

Proof. By definition, a point $M = (x, y) \in \mathbb{R}^2$ is a point of the lemniscate L if and only if

$$(x^2 + y^2)^2 = x^2 - y^2.$$

If (r, θ) are polar coordinates of $M = M(r, \theta)$, then $x = r\cos\theta, y = r\sin\theta$, thus, using $\cos^2\theta + \sin^2\theta = 1$, and $\cos(2\theta) = \cos^2\theta) - \sin^2\theta$, we obtain

$$\begin{aligned} M(r,\theta) \in L &\iff (r^2\cos^2\theta + r^2\sin^2\theta)^2 = r^2\cos^2\theta - r^2\sin^2\theta \\ &\iff r^4 = r^2\cos(2\theta) \\ &\iff r^2 = \cos(2\theta). \end{aligned}$$

The equation of the lemniscate is $r^2 = \cos(2\theta)$. $\square$

Ex. 15.1.3 *Prove that the two improper integrals $\int_0^1 (1-t^4)^{-1/2} \mathrm{d}t$ and $\int_{-1}^0 (1-t^4)^{-1/2} \mathrm{d}t$ converge.*

Proof. The map $t \mapsto (1 - t^4)^{-1/2}$ is continuous on $[0, 1[$, thus $t \mapsto (1 - t^4)^{-1/2}\mathrm{d}t$ is summable on $[1, x]$ for all $x \in [0, 1]$.

Since $1 - t^4 = (1-t)(1+t+t^2+t^3)$, $(1-t^4)^{-1/2} \sim [4(1-t)]^{-1/2}$ in the neighborhood of 1. The Riemann Criterium shows that $\int_0^1 (1 - t)^{-\alpha}\mathrm{d}t$ converges if $\alpha < 1$, and here $\alpha = 1/2$. Since $(1-t^4)^{-1/2} > 0$, this is sufficient to prove that $\int_0^1 (1-t^4)^{-1/2}\mathrm{d}t$ converges.

Since $t \mapsto (1 - t^4)^{-1/2}$ is even, the same is true in the neighborhood of -1, thus $\int_{-1}^0 (1 - t^4)^{-1/2}\mathrm{d}t$ converges. $\square$

Ex. 15.1.4 *Prove the arc length formula stated in (15.6)*

Proof. Here the equation of th ellipse E is

$$x^2 + \frac{y^2}{b^2} = 1,$$

with eccentricity $k = \sqrt{1 - b^2}$.

We compute the arc length l of (E) between $x = u, y = v$ $(-1 < u < v < 1)$ on the upper part of the curve. Then

$$l = \int_u^v \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\, \mathrm{d}x,$$

where $y = f(x) = b\sqrt{1-x^2}$. Then $f'(x) = \frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{2x}{\sqrt{1-x^2}}$, thus

$$\begin{aligned} l &= \int_u^v \sqrt{1 + \left(\frac{bx}{\sqrt{1-x^2}}\right)^2}\, \mathrm{d}x \\ &= \int_u^v \sqrt{\frac{1 - x^2 + b^2x^2}{1-x^2}}\, \mathrm{d}x \\ &= \int_u^v \sqrt{\frac{1-k^2x^2}{1-x^2}}\, \mathrm{d}x \end{aligned}$$

We have proved

$$l = \int_u^v \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\, \mathrm{d}x = \int_u^v \sqrt{\frac{1-k^2x^2}{1-x^2}}\, \mathrm{d}x = \int_u^v \frac{\sqrt{(1-x^2)(1-k^2x^2)}}{1-x^2}\, \mathrm{d}x.$$

The arc length of the ellipse is given by an elliptic integral. □

Ex. 15.1.5 *Shows that (15.7) reduces to $(x^2+y^2)^2 = x^2 - y^2$ when $a = b = 1/\sqrt{2}$.*

Proof. If we take $a = b = 1/\sqrt{2}$ in the formula of the ovals of Cassini

$$((x-a)^2 + y^2)((x+a)^2 + y^2) = b^4,$$

we obtain

$$\begin{aligned} \frac{1}{4} &= \left[\left(x - \frac{1}{\sqrt{2}}\right)^2 + y^2\right]\left[\left(x + \frac{1}{\sqrt{2}}\right)^2 + y^2\right] \\ &= \left(x^2 + y^2 + \frac{1}{2} - \sqrt{2}\,x\right)\left(x^2 + y^2 + \frac{1}{2} + \sqrt{2}\,x\right) \\ &= \left((x^2 + y^2 + \frac{1}{2}\right)^2 - 2x^2 \\ &= (x^2+y^2)^2 + (x^2+y^2) - 2x^2 \\ &= (x^2+y^2)^2 + y^2 - x^2 + \frac{1}{4}. \end{aligned}$$

Therefore, for $a = b = 1/\sqrt{2}$, the equation $((x-a)^2+y^2)((x+a)^2+y^2) = b^4$ reduces to

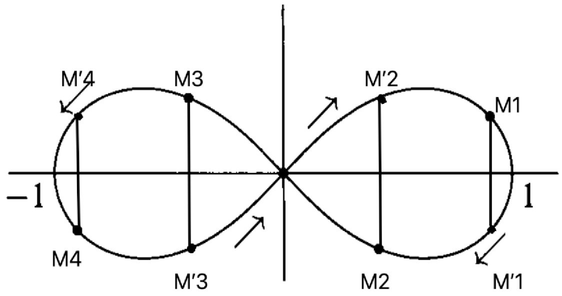
$$(x^2+y^2)^2 = x^2 - y^2,$$

which is the equation of the Lemniscate. □

Ex. 15.1.6 *Let $n > 0$ be an odd integer, and assume that the n-division points of the lemniscate can be constructed with straightedge and compass. prove that the same is true for the $2n$-division points. Your proof should include a picture.*

Proof. Let $M_0 = 0, \ldots, M_{n-1}$ the n-divisions points If n is odd, the middle segment $(M_{(n-1)/2}, M_{(n+1)/2})$ straddles the origin, The symmetry of the figure shows that the arc length between O and $M_{(n-1)/2}$ is the same as the arc length between O and $M_{(n-1)/2}$, so is half of the arc length M_kM_{k+1}. Therefore we can complete $M_0, M_1, \ldots, M_{n-1}$ by the symmetric points $M_0' = O, \ldots, M_{n-1}'$ relative to the x-axis to obtain the $2n$-division points (the point O is counted twice).

Figure for $n = 5:$ the 10-division points are $0, M_2', M_1, M_1', M_2, 0, M_3, M_3', M_4, M_3'$. □



Ex. 15.1.7 *Recall that in Greek geometry, the ellipse is defined to be the locus of all points whose* **sum** *of distances to two given points is constant. Suppose instead we consider the locus of all points whose* **product** *of distances to two given points is constant. Show that this leads to (15.7) when the given points are $(a, 0), (-a, 0)$ and the constant is b^4 (*).*

(*) Read b^2.

Proof. Let Γ the locus of all points whose product of distances to two points $(a, 0), (-a, 0)$ is the constant b^2. Then

$$\begin{aligned} M(x,y) \in \Gamma &\iff \sqrt{(x-a)^2+y^2}\sqrt{(x+a)^2+y^2} = b^2 \\ &\iff ((x-a)^2+y^2)((x-a)^2+y^2) = b^4. \end{aligned}$$

We obtain the formula of the ovals of Cassini. □

14.2 THE LEMNISCATIC FUNCTION

Ex. 15.2.1 *Give a careful proof of (15.9) using the hints given in the text.*

Proof. By section 15.2, we know that φ is 2ϖ periodic,

$$\varphi(s + 2\varpi) = \varphi(s), \qquad (s \in \mathbb{R}).$$

Moreover, for $-1 \le r \le 1$, and $\frac{-\varpi}{2} \le s \le \frac{\varpi}{2}$,

$$r = \varphi(s) \iff s = \int_0^r \frac{1}{\sqrt{1-t^4}} \mathrm{d}t.$$

Write $r' = \varphi(-s) \in [-1, 1]$. Then for every $s \in [-\frac{\varpi}{2}, \frac{\varpi}{2}]$,

$$\begin{aligned} r' = \varphi(-s) &\iff -s = \int_0^{r'} \frac{1}{\sqrt{1-t^4}} \mathrm{d}t \\ &\iff -s = -\int_0^{-r'} \frac{1}{\sqrt{1-\tau^4}} \mathrm{d}\tau \qquad (\tau = -t) \\ &\iff s = \int_0^{-r'} \frac{1}{\sqrt{1-\tau^4}} \mathrm{d}\tau \\ &\iff -r' = \varphi(s) \end{aligned}$$

This proves that

$$\varphi(-s) = -\varphi(s) \qquad \left(-\frac{\varpi}{2} \le s \le \frac{\varpi}{2}\right). \tag{1}$$

Write $M(s)$ the point on the lemniscate with signed arc length s. Consider $M' = M(s')$ the symmetric point of $M(s)$ about the origin. Since the lemniscate is symmetric about the origin, Consider first the case where $0 \le s \le \varpi$, then the signed arc length is the positive arc length. Let $M(s)$ the point on the lemniscate with arc length s. Then the symmetric point $M(s')$ about the x-axis is such that $r' = OM(s') = OM(s) = r$, thus, by definition of φ, $\varphi(s) = \varphi(s')$. The total arc length from $O = M(0)$ to $O = M(\varpi)$ in the first loop is ϖ, and the symmetry of the lemniscate about the x-axis implies that the arc length $\varpi - s$ between $M(s)$ and $O = M(\varpi)$ is equal to the arc length s' between $O = M(0)$ and $M(s')$, thus $s' = \varpi - s$. This proves

$$\varphi(\varpi - s) = \varphi(s) \qquad (0 \le s \le \varpi). \tag{2}$$

Now, if $\frac{\varpi}{2} \le s \le \varpi$, then $0 \le \varpi - s \le \frac{\varpi}{2}$, thus, using (1), (2), (3)

$$\begin{cases} \varphi(s) & = \varphi(\varpi - s) = -\varphi(s - \varpi) = -\varphi(s + \varpi), \\ \varphi(-s) & = \varphi(\varpi - (-s)) = \varphi(s + \varpi). \end{cases}$$

Therefore $\varphi(-s) = -\varphi(s)$ if $\frac{\varpi}{2} \le s \le \varpi$. Now, if we suppose $-\varpi \le s \le -\frac{\varpi}{2}$, then $\frac{\varpi}{2} \le -s \le \varpi$, so we can apply the last equality to $-s$: $\varphi(s) = \varphi(-(-s)) = -\varphi(-s)$. This proves

$$\varphi(-s) = \varphi(s) \qquad (-\varpi \le s \le \varpi). \tag{3}$$

Using the periodicity, if $s \in \mathbb{R}$, the is some $n \in \mathbb{Z}$ and $s' \in [-\varpi, \varpi[$ such that $s = 2n\varpi + s'$. Then

$$\varphi(-s) = \varphi(-s - 2n\varpi) = \varphi(-s') = -\varphi(s') = -\varphi(s - 2n\varphi) = -\varphi(s).$$

We have proved

$$\varphi(-s) = \varphi(s) \qquad (s \in \mathbb{R}).$$

We can now complete (2) to $-\varpi \le s \le 0$. Then $0 \le -s \le \varpi$, and by (2) applied to $-s$, $\varphi(s + \varpi) = \varphi(-s) = -\varphi(s)$, thus

$$\varphi(\varpi - s) = -\varphi(s - \varpi) = -\varphi(s + \varpi) = \varphi(s)$$

.

We have proved, for all $s \in \mathbb{R}$,

$$\begin{aligned} \varphi(-s) &= -\varphi(s) \\ \varphi(\varpi - s) &= \varphi(s). \end{aligned}$$

□

Ex. 15.2.2 *Supply the details needed to complete the proof of Proposition 15.2.1.*

Proof. The proof of Proposition 15.2.1 shows that

$$\varphi'(s) = \sqrt{1 - \varphi^4(s)}, \qquad 0 \le s \le \frac{\varpi}{2}.$$

By Exercise 3, parts (a) and (b), φ' is even and has period 2ϖ, and by part (c),

$$\varphi'(\varpi - s) = -\varphi'(s), \qquad s \in \mathbb{R}.$$

Therefore, if $-\frac{\varpi}{2} \leq s \leq 0$, then

$$\varphi'(s) = -\varphi'(-s) = -\sqrt{1-\varphi^4(-s)} = -\sqrt{1-\varphi^4(s)}.$$

Now, if $\frac{\varpi}{2} \leq s \leq \varpi$, then $0 \leq \varpi - s \leq \frac{\varpi}{2}$, thus

$$\varphi'(s) = -\varphi'(\varpi - s) = -\sqrt{1-\varphi^4(\varpi - s)} = -\sqrt{1-\varphi^4(s)}.$$

If $-\varpi \leq s \leq -\frac{\varpi}{2}$, then $\frac{\varpi}{2} \leq -s \leq \varpi$. Using the above equality, we obtain

$$\varphi'(s) = -\varphi(-s) = -\sqrt{1-\varphi^4(-s)} = -\sqrt{1-\varphi^4(s)}.$$

We have proved

$$\varphi'^2(s) = 1 - \varphi^4(s), \qquad -\varpi \leq s \leq \varpi.$$

Now if s is any real number, there is some $n \in \mathbb{Z}$ and $s' \in [-\varpi, \varpi[$ such that $s = 2n\varpi + s'$. Since 2ϖ is a period of φ and φ',

$$\varphi'^2(s) = \varphi'^2(s') = 1 - \varphi^4(s') = 1 - \varphi^4(s).$$

This complete the proof of Proposition 15.2.1. □

Ex. 15.2.3 *Here are some useful properties of φ'.*

(a) φ has period 2ϖ. Explain why this implies that the same is true for φ'.

(b) φ is an odd function by (15.9). Explain why this implies that φ' is even.

(c) Use (15.9) to prove that $\varphi'(\varpi - s) = -\varphi'(s)$.

(d) Use Proposition 15.2.1 to prove that $\varphi''(s) = -2\varphi^3(s)$.

Proof.

(a) For all $s \in \mathbb{R}$, $\varphi(s + 2\varpi) = \varphi(s)$. By differentiation, and the chain rule, we obtain

$$\varphi'(s + 2\varphi)(s) = \varphi(s).$$

φ' has period 2ϖ.

(b) Since $\varphi(-s) = -\varphi(s)$ for all $s \in \mathbb{R}$, the chain rule gives

$$-\varphi'(-s) = -\varphi'(s),$$

thus φ' is even.

(c) By (15.9), $\varphi(\varpi - s) = \varphi(s)$ for all $s \in \mathbb{R}$. Then the chain rule gives $-\varphi'(\varpi - s) = \varphi'(s)$, thus

$$\varphi'(\varpi - s) = -\varphi'(s), \qquad s \in \mathbb{R}.$$

(d) By differentiation of $\varphi'^2(s) = 1 - \varphi^4(s)$ $(s \in \mathbb{R})$, we obtain

$$2\varphi'(s)\varphi''(s) = -4\varphi^3(s)\varphi'(s).$$

If $s \neq \frac{\varpi}{2} + n\varpi, n \in \mathbb{Z}$, then $\varphi'(s) \neq 0$, so that

$$\varphi''(s) = -2\varphi^3(s), \qquad s \neq \frac{\varpi}{2} + n\varpi, n \in \mathbb{Z}.$$

If $s = \frac{\varpi}{2} + n\varpi$ for some integer $n \in \mathbb{Z}$, since φ is infinitely differentiable, φ'' is continuous, therefore

$$\varphi''(s) = \lim_{t \to s, t \neq s} \varphi''(t) = \lim_{t \to s, t \neq s} (-2\varphi^3(t)) = -2\varphi^3(s).$$

Therefore

$$\varphi''(s) = -2\varphi^3(s), \qquad s \in \mathbb{R}.$$

□

Ex. 15.2.4 *Suppose that we define* $\sin(x)$ *by* $y = \sin(x) \iff x = \int_0^y (1-t^2)^{-1/2}\mathrm{d}t$. *Then define* $\cos(x)$ *to be* $\sin'(x)$. *Use the method of Proposition 15.2.1 to prove the standard trigonometric identity* $\cos^2(x) = 1 - \sin^2(x)$.

Proof. We obtain the analog of (15.9) as in Exercise 1: for all $x \in \mathbb{R}$,

$$\sin(-x) = -\sin(x),$$
$$\sin(\pi - x) = \sin(x).$$

Now we use the definition of sin: for all $y \in [-1,1]$, for all $x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$,

$$y = \sin(x) \iff x = \int_0^y (1-t^2)^{-1/2}\mathrm{d}t,$$

where $\int_0^1 (1-t^2)^{-1/2}\mathrm{d}t$ and $\int_0^{-1}(1-t^2)^{-1/2}\mathrm{d}t$ converge.

If $x \in [0, \frac{\pi}{2}[$, differentiating each side of

$$s = \int_0^{\sin(x)} \frac{1}{\sqrt{1-t^2}}\mathrm{d}t,$$

we obtain

$$1 = \frac{1}{\sqrt{1-\sin^2(x)}}\sin'(x).$$

If $x = \frac{\pi}{2}$, then $\sin(x) = 1, \sin'(x) = 0$, thus $\sin'^2(x) = 1 - \sin^2(x)$. Therefore

$$\cos(x) = \sin'(x) = \sqrt{1-\sin^2(x)}, \qquad 0 \le x \le \frac{\pi}{2}.$$

We extend the equality $\sin^2(x) + \cos^2(x) = 1$ to all $x \in \mathbb{R}$ as in Exercise 2. $\square$

Ex. 15.2.5 *Here is Abel's proof of the addition law for* φ.

(a) *Let* $g(x,y)$ *be differentiable on* $\mathbb{R}^2$, *and set* $h(u,v) = g\left(\frac{1}{2}(u+v), \frac{1}{2}(u-v)\right)$. *Use the chain Rule to prove that*

$$\frac{\partial h}{\partial v}(u,v) = \frac{1}{2}\frac{\partial g}{\partial x}\left(\frac{1}{2}(u+v), \frac{1}{2}(u-v)\right) - \frac{1}{2}\frac{\partial g}{\partial y}\left(\frac{1}{2}(u+v), \frac{1}{2}(u-v).\right)$$

(b) *Use part (a) to show that* $g(x,y) = g(x+y, 0)$ *on,* $\mathbb{R}^2$ *if and only if* $\frac{\partial g}{\partial x} = \frac{\partial g}{\partial y}$ *on* $\mathbb{R}^2$.

(c) *Prove the addition law for* φ *by applying part (b) to*

$$g(x,y) = \frac{\varphi(x)\varphi'(y) + \varphi(y)\varphi'(x)}{1 + \varphi^2(x)\varphi^2(y)}.$$

Part (d) of Exercise 3 will be useful.

Proof.

(a) To apply the Chain Rule, we suppose that g is continuously differentiable ($g \in C_1(\mathbb{R}^2)$). Write $x, y : \mathbb{R}^2 \to \mathbb{R}^2$ the two maps defined by

$$x(u,v) = \frac{1}{2}(u+v), \qquad y(u,v) = \frac{1}{2}(u-v),$$

Then

$$\frac{\partial x}{\partial v}(u,v) = \frac{1}{2}, \qquad \frac{\partial y}{\partial v}(u,v) = -\frac{1}{2},$$

and

$$h(u,v) = g(x(u,v), y(u,v)), \qquad (u,v) \in \mathbb{R}^2.$$

The Chain Rule gives

$$\begin{aligned}\frac{\partial h}{\partial v}(u,v) &= \frac{\partial g}{\partial x}\left(x(u,v), y(u,v)\right)\frac{\partial x}{\partial v}(u,v) + \frac{\partial g}{\partial y}\left(x(u,v), y(u,v)\right)\frac{\partial y}{\partial v}(u,v)\\ &= \frac{1}{2}\frac{\partial g}{\partial x}\left(\frac{1}{2}(u+v), \frac{1}{2}(u-v)\right) - \frac{1}{2}\frac{\partial g}{\partial y}\left(\frac{1}{2}(u+v), \frac{1}{2}(u-v)\right)\end{aligned}$$

(b) Suppose that $g(x+y,0) = g(x,y)$ for all $x,y \in \mathbb{R}$. Write $f(x) = g(x,0)$. Then f is continuously differentiable, and $g(x,y) = f(x+y)$. By the Chain Rule, for all $(x,y) \in \mathbb{R}^2$,

$$\frac{\partial g}{\partial x}(x,y) = f'(x+y) = \frac{\partial g}{\partial y}(x,y),$$

therefore $\frac{\partial g}{\partial x} = \frac{\partial g}{\partial y}$ on $\mathbb{R}^2$.

Conversely, suppose that $\frac{\partial g}{\partial x} = \frac{\partial g}{\partial y}$ on $\mathbb{R}^2$. Then, for all $(u,v) \in \mathbb{R}^2$,

$$\frac{\partial h}{\partial v}(u,v) = \frac{1}{2}\frac{\partial g}{\partial x}\left(\frac{1}{2}(u+v), \frac{1}{2}(u-v)\right) - \frac{1}{2}\frac{\partial g}{\partial y}\left(\frac{1}{2}(u+v), \frac{1}{2}(u-v)\right) = 0.$$

This means that for every fixed $u_0 \in \mathbb{R}$, the map $v \mapsto h(u_0, v)$ has a null derivative, thus is constant: $h(u_0,v) = h(u_0,0)$ for all $v \in \mathbb{R}$. Since this is true for every u_0, we obtain

$$h(u,v) = h(u,0), \qquad \text{for all } u,v \in \mathbb{R}.$$

Write $f(u) = h(u,0)$ for all $u \in \mathbb{R}$. Then f is continuously differentiable, and for all $u,v \in \mathbb{R}$, $h(u,v) = f(u)$ depends only of u.

By definition of h, this means that, for all $u,v \in \mathbb{R}$,

$$g\left(\frac{1}{2}(u+v), \frac{1}{2}(u-v)\right) = f(u).$$

Taking $v = u$ in $g\left(\frac{1}{2}(u+v), \frac{1}{2}(u-v)\right) = h(u,v) = h(u,0)$, we obtain $g(u,0) = h(u,u) = h(u,0)$, therefore

$$g(u,0) = h(u,u) = h(u,0) = h(u,v) = g\left(\frac{1}{2}(u+v), \frac{1}{2}(u-v)\right),$$

thus

$$g(u,0) = g\left(\frac{1}{2}(u+v), \frac{1}{2}(u-v)\right), \qquad u,v \in \mathbb{R}.$$

If (x,y) is any pair in $\mathbb{R}^2$, there exists a unique pair $(u,v) \in \mathbb{R}^2$ such that $x = \frac{1}{2}(u+v), y = \frac{1}{2}(u-v)$, given by $u = x+y, v = x-y$. Therefore, the preceding equality implies that

$$g(x+y,0) = g(x,y), \qquad x,y \in \mathbb{R}.$$

(c) Define $g : \mathbb{R}^2 \to \mathbb{R}$ by

$$g(x,y) = \frac{\varphi(x)\varphi'(y) + \varphi(y)\varphi'(x)}{1+\varphi^2(x)\varphi^2(y)}.$$

The partial derivative of this quotient relative to the variable x gives, using $\varphi''(x) = -2\varphi^3(x)$ (see Exercise 3, part (d)), and $\varphi'(x)^2 = 1 - \varphi^4(x)$

$$\begin{aligned}
&\left(1+\varphi^2(x)\varphi^2(y)\right)\frac{\partial g}{\partial x}(x,y)\\
&= \left(\varphi'(x)\varphi'(y) + \varphi(y)\varphi''(x)\right)\left(1+\varphi^2(x)\varphi^2(y)\right) - 2\varphi(x)\varphi'(x)\varphi^2(y)\left(\varphi(x)\varphi'(y)+\varphi(y)\varphi'(x)\right)\\
&= \left(\varphi'(x)\varphi'(y) - 2\varphi(y)\varphi^3(x)\right)\left(1+\varphi^2(x)\varphi^2(y)\right) - 2\varphi(x)\varphi'(x)\varphi^2(y)\left(\varphi(x)\varphi'(y)+\varphi(y)\varphi'(x)\right)\\
&= \varphi'(x)\varphi'(y) + \varphi'(x)\varphi'(y)\varphi^2(x)\varphi^2(y) - 2\varphi(y)\varphi^3(x) - 2\varphi^3(y)\varphi^5(x)\\
&\quad - 2\varphi^2(x)\varphi^2(y)\varphi'(x)\varphi'(y) - 2\varphi(x)\varphi^3(y)\varphi'(x)^2\\
&= \varphi'(x)\varphi'(y) + \varphi'(x)\varphi'(y)\varphi^2(x)\varphi^2(y) - 2\varphi(y)\varphi^3(x) - 2\varphi^3(y)\varphi^5(x)\\
&\quad - 2\varphi^2(x)\varphi^2(y)\varphi'(x)\varphi'(y) - 2\varphi(x)\varphi^3(y)(1-\varphi^4(x))\\
&= \varphi'(x)\varphi'(y) + \varphi'(x)\varphi'(y)\varphi^2(x)\varphi^2(y) - 2\varphi(y)\varphi^3(x) - 2\varphi(x)\varphi^3(y)\\
&\quad - 2\varphi^2(x)\varphi^2(y)\varphi'(x)\varphi'(y).
\end{aligned}$$

This last expression is symmetric relatively to x, y, and also the denominator $1+\varphi^2(x)\varphi^2(y)$. Since $g(x,y) = g(y,x) = \frac{\varphi(y)\varphi'(x)+\varphi(x)\varphi'(y)}{1+\varphi^2(y)\varphi^2(x)}$, this proves that

$$\begin{aligned}
&\left(1+\varphi^2(y)\varphi^2(x)\right)\frac{\partial g}{\partial y}(x,y)\\
&= \varphi'(y)\varphi'(x) + \varphi'(y)\varphi'(x)\varphi^2(y)\varphi^2(x) - 2\varphi(x)\varphi^3(y) - 2\varphi(y)\varphi^3(x) - 2\varphi^2(y)\varphi^2(x)\varphi'(y)\varphi'(x)\\
&= \left(1+\varphi^2(x)\varphi^2(y)\right)\frac{\partial g}{\partial x}(x,y),
\end{aligned}$$

where $1+\varphi^2(y)\varphi^2(x) > 0$. Therefore $\frac{\partial g}{\partial x} = \frac{\partial g}{\partial y}$ on $\mathbb{R}^2$.

By part (b), $g(x,y) = g(x+y,0)$. Using $\varphi(0) = 0$, and $\varphi'(0) = \sqrt{1-\varphi^4(0)} = 1$,

$$\begin{aligned}
g(x,y) &= g(x+y,0)\\
&= \varphi'(0)\varphi(x+y)\\
&= \varphi(x+y).
\end{aligned}$$

We have proved the addition law for φ:

$$\varphi(x+y) = \frac{\varphi(x)\varphi'(y) + \varphi(y)\varphi'(x)}{1+\varphi^2(x)\varphi^2(y)}, \qquad x,y \in \mathbb{R}.$$

□