# PHYS 306 Lecture Notes & Worksheets

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## Contents

# 1 Lecture 1

## 1.1 Lecture Notes - Lagrangian Mechanics Part 1

### 1.1.1 Newtonian Mechanics to Lagrangian Mechanics

Recall the Newtonian formulation of classical mechanics; given the forces $\mathbf{F}_1(t), \cdots \mathbf{F}_n(t)$ on a particle, we can solve Newton's second law (a second order differential equation):

$$\sum_{i=1}^{n} \mathbf{F}_i(t) = m\ddot{\mathbf{r}}(t)$$

To obtain the trajectory $\mathbf{r}(t)$, which is uniquely determined by the initial conditions $\mathbf{r}(t_0)$ and $\dot{\mathbf{r}}(t_0)$.

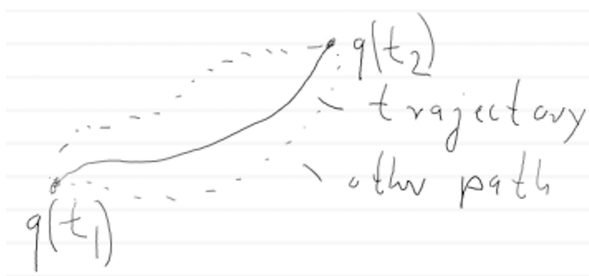
In this course, we will begin by looking at the Lagrangian formulation of classical mechanics. While this formulation contains no new physics compared to the Newtonian formulation, there are two distinct benefits:

(a) We can obtain EOM that do not depend on the coordinate system

(b) It is easier to treat constrained systems.

### 1.1.2 The Variational Principle Setup

To do this, we will use a new approach, known as the **Variational principle**. To set this up, let us consider the trajectory (as well as some "wrong" paths between the same two endpoints) travelled by a particle:




The trajectory from time $t_1$ to $t_2$ is parametrized by the generalized coordinate $q$. Examples of these are $x_1, y_1, \theta_1$.

### 1.1.3 Generalized Coordinates

For $N$ particles, a generalized coordinate $q_i$ depends on the positions of the $N$ particles:

$$q_i = q_i(\mathbf{r}_1, \cdots, \mathbf{r}_N)$$

It shares inverse relationship with position vector of each particle: $\mathbf{r}_i = \mathbf{r}_i(q_1, q_2, \ldots, q_n)$. In Cartesian coordinates, we have $3N$ coordinates but these are not necessarily independent. In general, we may have constraint functions $f_\alpha(\mathbf{r}, \dot{\mathbf{r}}, t) = 0$ where $\alpha = 1, \cdots, k$ (i.e. $k$ constraints). We then have $q_i$ independent coordinates, where $i = 1, \cdots n$ where $i = 3N - k$. In other words, for a system of $N$ particles, we have $3N - k$ independent/generalized coordinates. Generalized coordinates allow us to avoid worrying about the constrained parameters.

For example, consider motion of a pendulum of length $L$. The position of the blob can be represented by vector $\mathbf{r} = (x, y)$ where $x = L\cos\theta$ and $y = L\sin\theta$. $\theta$ is the only generalized coordinate here. $\theta$ is related to position $\mathbf{r}$ by the following relation: $\theta(\mathbf{r}) = \arccos\left(\frac{\hat{\mathbf{x}}\cdot\mathbf{r}}{L}\right)$.

### 1.1.4 Hamilton's Principle and the Lagrangian

Let us consider assigning to each generalized coordinate $q(t)$ a number/value:

$$q(t) \mapsto S[q(t)] \in \mathbb{R}$$

This is a functional (as denoted by the square brackets), as it takes in a function as an argument. Now, let us consider **Hamilton's principle**:

*The actual path of a particle between times $t_1$ and $t_2$ is such that the line integral:*

$$S[q] = \int_{t_1}^{t_2} \mathcal{L}(q_1, \dot{q}_1, t) dt$$

*is stationary.*

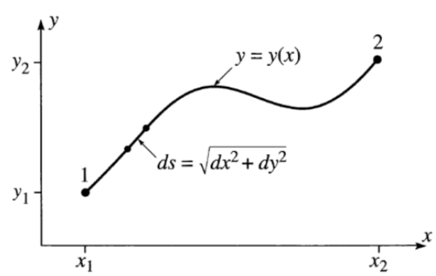
Though the principle saysthe integral is stationary, often this correponds to a minimum (though not always). The function $\mathcal{L}$ is defined as:

$$\mathcal{L} = T - U$$

where $T$ is the kinetic energy and $U$ is the potential energy. This is called the **Lagrange Function** or **Lagrangian**. $S$ is called the **action**. Though we have not shown it explicitly, this Lagrange function gives the correct trajectory. Note that we have in essence replaced a second order ODE (Newton's second law) with an integral of $\mathcal{L}$. Having two endpoints $t_1$ and $t_2$ is consistent with the two initial conditions for a second order ODE.

## 1.2 Worksheet - Shortest Distance and Variational Principle

**Problem 1.1.** *We wish to find the shortest path between two points $(x_1, y_1)$ and $(x_2, y_2)$ in the plane. First, write out the distance between these two points in terms of a general function $y(x)$ connecting points $(x_1, y_1)$ and $(x_2, y_2)$, and possibly it's derivative $y'(x)$, in terms of an independent variable $x$.*




*Solution.* We can define an infinitesimal path element of the trajectory from $(x_1, y_1)$ to $(x_2, y_2)$ as:

$$ds = \sqrt{dx^2 + dy^2}$$

We can use the identity:

$$dy \equiv \frac{\mathrm{d}y}{\mathrm{d}x} dx = y'(x) dx$$

Substituting this into the path element equation above, we obtain:

$$ds = \sqrt{dx^2 + \left(y'(x) dx\right)^2}$$

We may integrate the path element from the initial state to the final state to obtain the length of the path:

$$L(y(x), y'(x)) = \int_{t_1}^{t_2} ds = \int_{x_1}^{x_2} \sqrt{dx^2 + \left(y'(x)dx\right)^2} = \int_{x_1}^{x_2} \sqrt{1 + y'(x)^2}dx$$

Notice here we have the functional $L[y(x), y'(x)]$ (here only depends on $y'(x)$).

□

**Problem 1.2.** *If light were travelling between points $(x_1, y_1)$ and $(x_2, y_2)$, we would expect it to follow a straight line, but if the index of refraction $n = n(x, y)$ is not constant, the speed of light in the medium is $v = c/n$ and the path is not a straight line but follows the trajectory of minimum time (Fermat, 1662).Find an integral expression for the time taken by a trajectory $y(x)$ through a medium $n(x, y)$.*

*Solution.* The infinitesimal time element $dt$ to travel a distance $ds$ is given by:

$$dt = \frac{ds}{v} = \frac{n(x, y)ds}{c} = \frac{1}{c}n(x, y)\sqrt{1 + y'(x)^2}dx$$

Where in the second inequality we use the identity $v = c/n$. Again we can integrate from an initial time to a final time to obtain the total time the light takes to travel along the path:

$$T = \int_{t_1}^{t_2} dt = \int_{x_1}^{x_2} \frac{1}{c}n(x, y)\sqrt{1 + y'(x)^2}dx = \frac{1}{c}\int_{x_1}^{x_2} n(x, y)\sqrt{1 + y'(x)^2}dx$$

Where again we have a functional $T[y(x), y'(x)]$. □

**Problem 1.3.** *In our variational treatment, if the "wrong" or "varied" curves from the minimum pass through the points 1 and 2, write the condition on the deviations from the correct curve, $\eta(x)$.*

*Solution.* $\eta(x_1) = \eta(x_2) = 0$ as the endpoints of the wrong path must match that of the correct path. □

**Problem 1.4.** *If $S(\alpha) = \int_{x_1}^{x_2} f\left(y + \alpha\eta, y' + \alpha\eta', x\right) dx$, use the chain rule to write $dS(\alpha)/d\alpha$ in terms of an integral of a function that contains derivatives on $y$ and $y'$, and $\eta$ and $\eta'$.*

*Solution.* As the integral is with respect to x, we may interchange the order of integration and differentiation:

$$\left.\frac{\mathrm{d}S(\alpha)}{\mathrm{d}\alpha}\right|_{\alpha=0} = \left.\frac{\mathrm{d}}{\mathrm{d}\alpha}\int_{x_1}^{x_2} f\left(y + \alpha\eta, y' + \alpha\eta', x\right) dx\right|_{\alpha=0} = \int_{x_1}^{x_2}\left(\left.\frac{\partial}{\partial\alpha} f\left(z_1, z_2, z_3\right)\right|_{\alpha=0}\right) dx$$

Then by the multivariable chain rule, we have:

$$\int_{x_1}^{x_2}\left(\left.\frac{\partial}{\partial\alpha} f\left(z_1, z_2, z_3\right)\right|_{\alpha=0}\right) dx = \int_{x_1}^{x_2}\left(\frac{\partial f}{\partial z_1}\frac{\partial z_1}{\partial\alpha} + \frac{\partial f}{\partial z_2}\frac{\partial z_2}{\partial\alpha} + \frac{\partial f}{\partial z_3}\frac{\partial z_3}{\partial\alpha}\right) dx =$$

$$\int_{x_1}^{x_2}\left(\frac{\partial f}{\partial z_1}\eta(x) + \frac{\partial f}{\partial z_2}\eta'(x) + \frac{\partial f}{\partial z_3}\cdot 0\right) dx = \int_{x_1}^{x_2}\left(\frac{\partial f}{\partial z_1}\eta + \frac{\partial f}{\partial z_2}\eta'\right) dx =$$

$\frac{\partial f}{\partial y}$ and $\frac{\partial f}{\partial y'}$ in terms of total differentials are:

$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial z_1}\frac{\partial z_1}{\partial y} + \frac{\partial f}{\partial z_2}\frac{\partial z_2}{\partial y} + \frac{\partial f}{\partial z_3}\frac{\partial z_3}{\partial y} = \frac{\partial f}{\partial z_1}\cdot 1 + \frac{\partial f}{\partial z_2}\cdot 0 + \frac{\partial f}{\partial z_3}\cdot 0 = \frac{\partial f}{\partial z_1}$$

$$\frac{\partial f}{\partial y'} = \frac{\partial f}{\partial z_1}\frac{\partial z_1}{\partial y'} + \frac{\partial f}{\partial z_2}\frac{\partial z_2}{\partial y'} + \frac{\partial f}{\partial z_3}\frac{\partial z_3}{\partial y'} = \frac{\partial f}{\partial z_1}\cdot 0 + \frac{\partial f}{\partial z_2}\cdot 1 + \frac{\partial f}{\partial z_3}\cdot 0 = \frac{\partial f}{\partial z_2}$$

Finally,

$$\left.\frac{\mathrm{d}S(\alpha)}{\mathrm{d}\alpha}\right|_{\alpha=0} = \int_{x_1}^{x_2}\left(\eta\frac{\partial f}{\partial y} + \eta'\frac{\partial f}{\partial y'}\right) dx = \int_{x_1}^{x_2}\eta\frac{\partial f}{\partial y}\, dx + \int_{x_1}^{x_2}\eta'\frac{\partial f}{\partial y'}\, dx$$

□

**Problem 1.5.** *Rewrite the* $2^{nd}$ *term* $\int_{x1}^{x2} \eta' \frac{\partial f}{\partial y'} dx$ *by integrating by parts. Note* $\int v du = [uv] - \int u dv$ *is equivalent to* $\int u' v dx = [uv] - \int u v' dx$. *After simplifying, write the integral expression for* $\frac{\partial S}{\partial \alpha} = 0$. *l.e.* $\frac{\partial S}{\partial \alpha} = \int_{x1}^{x2} \eta(x)[\ldots]dx = 0$. *This condition ensures that* $S(\alpha)$ *has a minimum at* $\alpha = 0$ *and* $y$ *is the curve that extremizes* $S$.

*Solution.* Carrying out integration by parts on the second term, we have:

$$\left.\frac{\mathrm{d}S(\alpha)}{\mathrm{d}\alpha}\right|_{\alpha=0} = \int_{x_1}^{x_2} \eta \frac{\partial f}{\partial y} \, dx + \eta \left.\frac{\partial f}{\partial y'}\right|_{x_1}^{x_2} - \int_{x_1}^{x_2} \eta \left( \frac{\mathrm{d}}{\mathrm{d}x} \frac{\partial f}{\partial y'} \right) dx$$

From problem 3, we know that $\eta(x_1) = \eta(x_2) = 0$ and so the second term evaluates to zero, leaving us with:

$$\left.\frac{\mathrm{d}S(\alpha)}{\mathrm{d}\alpha}\right|_{\alpha=0} = \int_{x_1}^{x_2} \left( \eta \frac{\partial f}{\partial y} - \eta \frac{\mathrm{d}}{\mathrm{d}x} \frac{\partial f}{\partial y'} \right) dx$$

and we set this to zero:

$$\left.\frac{\mathrm{d}S(\alpha)}{\mathrm{d}\alpha}\right|_{\alpha=0} = \int_{x_1}^{x_2} \eta \left( \frac{\partial f}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x} \frac{\partial f}{\partial y'} \right) dx = 0$$

We require that this holds for **all** possible deviations $\eta(x)$. The only way this could hold is if the term in brackets is zero; this is exactly the Euler-Lagrange equation! □

**Problem 1.6.** *Write the general form of the Euler-Lagrange Equation. What is the function* $f$ *for the distance between two points? What does this say about* $\frac{\partial f}{\partial y}$ *in the E-L equation?*

*Solution.* The general form of the Euler-Lagrange equation is given by:

$$\frac{\partial f}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x} \frac{\partial f}{\partial y'} = 0$$

The function for the distance between two points is $f = \sqrt{1 + y'^2}$ as derived above. Hence, $\dfrac{\partial f}{\partial y} = 0$, and therefore by the Euler Lagrange equation:

$$\frac{\partial f}{\partial y'} = \text{constant}$$

□

**Problem 1.7.** *Solve the above equation for* $y'^2$ *and hence* $y'$ *and* $y$.

*Solution.* We have that:

$$\frac{\partial f}{\partial y'} = \frac{y'}{\sqrt{1 + y'^2}} = \text{Constant}$$

Therefore:

$$y'^2 = c^2(1 + y'^2)$$

Or alternatively:

$$y'^2 = \frac{c^2}{1 - c^2} = c' \implies y'(x) = \sqrt{c'} = m$$

So integrating, we have that:

$$y(x) = mx + b$$

as we knew already! □

# 2 Lecture 2

## 2.1 Lecture Notes - Lagrangian Mechanics Part 2

### 2.1.1 Review of Variational Principle

Variational principle. We want to minimize a functional (that takes in a function as an argument), by finding stationary points. This is reminiscent of the process of finding extremum in elementary/single variable calculus, where if we move slightly away from the point, we don't change the function very much. Now, we generalize this notion to a function space, where if we change the function slightly, we do not change the functional very much.
We want to derive that the shortest path between two points is a straight line (no objections general relativists!). We consider the variation $y(x) + \alpha\eta(x)$ around the true path $y(x)$. Look at worksheet one for the solutions/walkthrough of this.

## 2.2 Worksheet - Variational Principle Applied to Free Fall

As a second example for the variational principle, let's consider the (one-dimensional) free fall of a particle of mass $m$ due to gravity starting at zero and from rest for a duration T.

**Problem 2.1.** *Write down the Lagrange-Function $\mathcal{L} = T - U$ using a generalized coordinate $q$.*

*Solution.* Let us use the height of particle as our generalized coordinate $q$. In this case the velocity of the particle is simply $\dot{q}$, so we have that:

$$T = \frac{1}{2}m\dot{q}^2, \quad U = mgq$$

Hence the Lagrangian is:

$$\mathcal{L} = T - U = \frac{1}{2}m\dot{q}^2 - mgq$$

□

**Problem 2.2.** *Using Newton's $2^{nd}$ law, write down the equation of motion for $q$ and the solution $q(t)$.*

*Solution.* By Newton's second law, we have that there is only a gravitational force $-mg$ acting on the particle, and hence it reads:

$$\sum F = -mg = m\ddot{q} \implies -g = \ddot{q}$$

We solve this second order differential equation by integrating twice, which gives us:

$$q(t) = -\frac{1}{2}gt^2 + \dot{q}(t=0)t + q(t=0)$$

Since the particle starts at rest, we have that $\dot{q}(0) = 0$ and hence:

$$q(t) = -\frac{1}{2}gt^2 + q(t=0)$$

□

**Problem 2.3.** *Write down the "boundary conditions" $q(t=0)$ and $q(t=T)$.*

*Solution.* We define our coordinate system such that $q(t=0) = 0$, and then by the solution above, $q(t = T) = -\frac{1}{2}gT^2$. □

**Problem 2.4.** *A possible 'trial trajectory' could be the from $q_{trial\ ,\alpha}(t) = -\frac{1}{2}gt^2 + \alpha\sin\left(\frac{\pi t}{T}\right)$. (This is how variational calculus is carried out!) Check that this function satisfies the boundary conditions at* t = 0 *and* t = T.

*Solution.* We see that:

$$q(t=0) = -\frac{1}{2}g(0)^2 + \alpha \sin\left(\frac{\pi(0)}{T}\right) = 0$$

and

$$q(t=T) = -\frac{1}{2}gT^2 + \alpha \sin\left(\frac{\pi T}{T}\right) = -\frac{1}{2}gT^2$$

as desired. □

**Problem 2.5.** *Instead of considering all possible trajectories that obey the boundary conditions, we consider here only a specific family parameterized by* $\alpha$. *Thus* $S\left[q_{trial,\alpha}(t)\right] = S(\alpha)$ *is a function of* $\alpha$ *Write down* $S\left[q_{trial,\alpha}(t)\right]$ *and show that it can be written as*

$$S\left[q_{trial,\alpha}(t)\right] = S[q(t)] + m\alpha^2\pi^2/4T$$

*In the above form it is evident that the true trajectory* ($\alpha = 0$) *minimizes* $S$.

*Solution.* Plugging in the trial trajectory into the Lagrangian determined in question 1, we have:

$$\begin{aligned}
S\left[q_{\text{trial},\alpha}(t)\right] &= \int_0^T \mathcal{L}(q, \dot{q}, t)dt \\
&= \int_0^T \frac{1}{2}m\left(\frac{\mathrm{d}}{\mathrm{d}t}\left(-\frac{1}{2}gt^2 + \alpha \sin\left(\frac{\pi t}{T}\right)\right)\right)^2 - mg\left(-\frac{1}{2}gt^2 + \alpha \sin\left(\frac{\pi t}{T}\right)\right)dt \\
&= \int_0^T \frac{1}{2}m\left(-gt + \frac{\alpha\pi}{T}\cos\left(\frac{\pi t}{T}\right)\right)^2 + \frac{mg^2}{2}t^2 - mg\alpha \sin\left(\frac{\pi t}{T}\right)dt \\
&= \int_0^T \frac{1}{2}m\left(g^2t^2 - 2\frac{gt\alpha\pi}{T}\cos\left(\frac{\pi t}{T}\right) + \frac{\alpha^2\pi^2}{T^2}\cos^2\left(\frac{\pi t}{T}\right)\right) + \frac{mg^2}{2}t^2 - mg\alpha \sin\left(\frac{\pi t}{T}\right)dt \\
&= \int_0^T mg^2t^2 + \frac{m\alpha^2\pi^2}{2T^2}\cos^2\left(\frac{\pi t}{T}\right)dt \text{ (Integrals over full period of sine/cosine are zero)} \\
&= \int_0^T mg^2t^2 + \frac{m\alpha^2\pi^2}{4T} \\
&= S[q(t)] + \frac{m\alpha^2\pi^2}{4T}
\end{aligned}$$

Clearly this is minimized for $\alpha = 0$, and for any $\alpha > 0$ the action is larger. The one the action that makes the action minimal is the true physical path. □

**Problem 2.6.** *Show that if* $S = \int_{t_1}^{t_2} f(x, y, \dot{x}, \dot{y})dt$ *that there are* ***two*** *Euler-Lagrange equations for the stationary curves* $x(t), y(t)$. *Write the Euler-Lagrange equations.*

*Solution.* Let the correct path be given by $x = x(u)$ and by $y = y(u)$, and the wrong/varied path be given by $x = x(u) + \alpha\xi(u)$ and $y = y(u) + \beta\eta(u)$. By a very similar process to worksheet 1, we impose the requirement that $\left.\frac{\mathrm{d}S}{\mathrm{d}\alpha}\right|_{\alpha=0} = 0$ and $\left.\frac{\mathrm{d}S}{\mathrm{d}\beta}\right|_{\beta=0} = 0$. This leads to the two Euler Lagrange equations:

$$\frac{\partial f}{\partial x} = \frac{\mathrm{d}}{\mathrm{d}u}\frac{\partial f}{\partial x'} \text{ and } \frac{\partial f}{\partial y} = \frac{\mathrm{d}}{\mathrm{d}u}\frac{\partial f}{\partial y'}$$

□

**Problem 2.7.** *Construct Lagrange's equations for a particle in a two-dimensional potential* $U(x, y)$, *and show that these are equivalent to a particle obeying Newton's equations.*

*Solution.* We have that the Lagrangian is given by:

$$\mathcal{L} = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) - U(x,y)$$

So by the two Euler-Lagrange equations above:

$$\frac{\partial L}{\partial x} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{x}} \implies -\frac{\mathrm{d}U(x,y)}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}t}m\dot{x} \implies F_x = m\ddot{x}$$
$$\frac{\partial L}{\partial y} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{y}} \implies -\frac{\mathrm{d}U(x,y)}{\mathrm{d}y} = \frac{\mathrm{d}}{\mathrm{d}t}m\dot{y} \implies F_y = m\ddot{y}$$

So we recover Newton's second law of:

$$\mathbf{F} = m\ddot{\mathbf{r}}$$

□

# 3 Lecture 3

## 3.1 Lecture Notes - Lagrangian Mechanics Part 3

### 3.1.1 Review of Variations

Recall **Hamilton's Principle**, where we consider a trajectory between $t_1$, $t_2$, and variations from that trajectory $\eta(t)$. The true trajectory taken by the physical system is given by $\bar{q}(t)$. We parameterize the variations with an $\alpha$ term, where $q(t) = \bar{q}(t) + \alpha\eta(t)$. $\bar{q}(t)$ minimizes the action functional $S[q]$ if $S[\bar{q}] < S[q]$ for any other trajectories $q$. Our functional $S[q]$ is given by:

$$S[q] = \int_{t_1}^{t_2} L(q, \dot{q}, t)dt$$

We found a method to find $\bar{q}$ by taking the derivative of the action (i.e. a stationary point):

$$\left.\frac{\mathrm{d}S[\bar{q} + \alpha\eta]}{\mathrm{d}\alpha}\right|_{\alpha=0} = 0$$

or alternatively, the statement that the (first order) variation must vanish:

$$\delta S[\bar{q}] = \lim_{\alpha\to 0}\frac{1}{\alpha}\left(S[\bar{q} + \alpha\eta] - S[\bar{q}]\right) = 0$$

We will explore this more on Monday when we look at constraint forces.

### 3.1.2 Invariance, The Functional Derivative, and Multiple Variables

(a) Form invariance of Euler-Lagrange equations. Going from generalized coordinates $q \to \tilde{q}$, then we have that:

$$\mathcal{L}(\tilde{q}, \dot{\tilde{q}}, t) = \mathcal{L}(q, \dot{q}, t)$$

by the definition of the Lagrangian. Alternatively, see that the action is invariant of the choice of general coordinates:

$$\tilde{S}[\tilde{q}] = \int_{t_1}^{t_2} \tilde{\mathcal{L}}\left(\tilde{q}, \dot{\tilde{q}}, t\right)dt = \int_{t_1}^{t_2} \mathcal{L}(q, \dot{q}, t)dt = S[q]$$

(b) One can take a functional derivative and set this to zero. The variation can be written using a functional derivative (indicated by lowercase $\delta$):

$$\delta S[q] = \int \frac{\delta S}{\delta q}\eta(t)dt$$

What is the definition of the functional derivative? It's very similar to the conventional derivative:

$$\frac{\delta S}{\delta q(t)} = \lim_{\alpha=0} \frac{S[q+\alpha\delta(t)] - S[q]}{\alpha} = \frac{\partial \mathcal{L}}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{q}}$$

Therefore, $\frac{\delta S}{\delta q} = 0$ implies the Euler Lagrange equations. The way you can think about it is like the partial derivative. We are familiar with the total differential of a function (the sum of the partial differentials, see HW1 as an example). Now, imagine we have a function that depends continuously on a function $q$; its like a partial derivative with an index that is continuous (rather than discrete).

(c) If we have multiple variables, the generalization is straightforwards; for $n$ variables, the action functional becomes:

$$S = \int_{t_1}^{t_2} \mathcal{L}(q_1, q_2, \cdots, q_n, \dot{q}_1, \dot{q}_2, \cdots, \dot{q}_n, t)dt$$

So we would get $n$ Euler-Lagrange equations:

$$\frac{\partial \mathcal{L}}{\partial q_i} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{q}_i}$$

Where $\dfrac{\partial \mathcal{L}}{\partial q_i}$ is the generalized force, and $\dfrac{\partial \mathcal{L}}{\partial \dot{q}_i}$ is the generalized momentum.

### 3.1.3 Torque, Angular Momentum, and Generalized Momentum Conservation

See derivation in Worksheet three for the relation:

$$\frac{\partial \mathcal{L}}{\partial \phi} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{\phi}}$$

Which corresponds to:

$$\Gamma = \frac{\mathrm{d}L}{\mathrm{d}t}$$

So if there is zero torque, we have conservation of angular momentum. Generally, if $\mathcal{L}$ is independent of $q_i$, then the generalized momentum $\frac{\partial \mathcal{L}}{\partial \dot{q}_i}$ is conserved. We have a conservation law that comes from a certain property in the Lagrangian. We call variables that do not appear in the Lagrangian as a "cyclic" or "ignorable" variable.

## 3.2 Worksheet - Lagrangians and Coordinate Transformations

**Problem 3.1.** *If we transform $x, y$ into polar coordinates, what happens to the principle of least action? What do Lagrange's equations become for a particle in a two-dimensional potential $U(x, y)$, now using polar coordinates. What are the generalized forces and generalized momenta?*

*Solution.* Since the Lagrangian is invariant of the choice of coordinates, nothing happens; the principle of least action still holds (the Lagrangian and action integral are equivalent between the two coordinate systems):

$$S[r,\theta] = \int_{t_1}^{t_2} \mathcal{L}(r, \dot{r}, \theta, \dot{\theta}, t)dt = \int_{t_1}^{t_2} \mathcal{L}(x, \dot{x}, y, \dot{y}, t)dt = S[x, y]$$

Using the fact that $\mathbf{r} = r\hat{\mathbf{r}}$ and $\dot{\mathbf{r}} = \dot{r}\hat{\mathbf{r}} + r\dot{\phi}\hat{\boldsymbol{\phi}}$ The Lagrangian becomes:

$$\mathcal{L} = T - U = \frac{1}{2}m\dot{\mathbf{r}}^2 - U(x, y) = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2) - U(r, \theta)$$

So Lagrange's equations become:

$$\frac{\partial \mathcal{L}}{\partial r} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{r}} \implies -\frac{\partial U}{\partial r} + mr\dot{\phi}^2 = m\ddot{r}$$

So the generalized force is $-\dfrac{\partial U}{\partial r} + mr\dot{\phi}^2$ and the generalized momentum is $m\dot{r}$.

$$\frac{\partial \mathcal{L}}{\partial \phi} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{\phi}} \implies -\frac{\partial U}{\partial \phi} = mr^2\ddot{\phi}$$

So the generalized force is $-\dfrac{\partial U}{\partial \phi}$ (which is just the torque!) and the generalized momentum is $mr^2\dot{\phi}$ (which is the angular momentum!). □

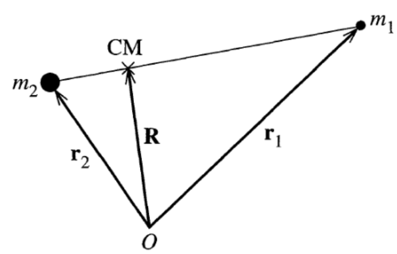
**Problem 3.2.** *Write the Lagrangian for two particles interacting through a potential $U(\mathbf{r}_1, \mathbf{r}_2)$, using the "lab frame" coordinates $\mathbf{r}_1, \mathbf{r}_2$. How does the potential simplify if it is translationally-invariant? How does it simplify if it is orientationally-invariant (i.e. central)?*

*Solution.* The Lagrangian is given by:

$$\mathcal{L} = \frac{1}{2}m_1\dot{\mathbf{r}}_1^2 + \frac{1}{2}m_2\dot{\mathbf{r}}_2^2 - U(x, y)$$

If the potential is translationally invariant, then there is no difference if we move the entire system through space; hence, it can only depend on the difference, i.e. $U(\mathbf{r}_1 - \mathbf{r}_2)$. If it is orientationally-invariant (central), then it only depends on the magnitude of the the difference, i.e. $U(|\mathbf{r}_1 - \mathbf{r}_2|)$. □

**Problem 3.3.** *Rewrite the kinetic energy in terms of the centre of mass (CM) and relative coordinates, $\mathbf{R}, \mathbf{r}$. What are the Lagrange equations in these coordinates?*




*Solution.* We have that the relative coordinate is $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$ and $U = U(|\mathbf{r}|)$. The CM position is given by:

$$\mathbf{R} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1 + m_2}$$

Let us also define the combined mass $M = m_1 + m_2$, we then have that:

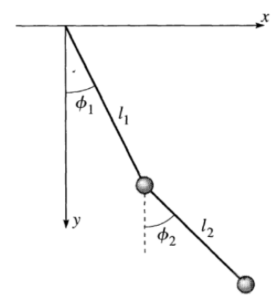
$$T = \frac{1}{2}\left(m_1\dot{\mathbf{r}}_1^2 + m_2\dot{\mathbf{r}}_2^2\right) = \frac{1}{2}\left(m_1\left(\dot{\mathbf{R}} + \frac{m_2}{M}\dot{\mathbf{r}}\right)^2 + m_2\left(\dot{\mathbf{R}} - \frac{m_1}{M}\dot{\mathbf{r}}\right)^2\right) = \frac{1}{2}\left(M\dot{\mathbf{R}}^2 + \frac{m_1m_2}{M}\dot{\mathbf{r}}^2\right)$$

So defining the reduced mass $\mu = \frac{m_1 m_2}{M}$ we have:

$$\mathcal{L} = T - U = \frac{M}{2}\dot{\mathbf{R}}^2 + \left(\frac{\mu}{2}\dot{\mathbf{r}}^2 - U(r)\right)$$

We end up with a Lagrangian that has essentially two independent terms; a COM motion term (which is trivial, just a particle of mass $M$) and a relative position term which is equivalent to a particle of mass $\mu$ subject to potential $U(r)$. We have essentially converted a two particle problem into what is effectively a one particle problem. This makes the two-body problem analytically (somewhat) easy to solve with this method. □

**Problem 3.4.** *Write the expressions for $\mathbf{r}_1$ and $\mathbf{r}_2$ in terms of appropriate generalized coordinates, for the double pendulum.*




*Solution.* Trigonometry to get the position vectors:

$$\mathbf{r}_1 = l_1 \sin\phi_1 \hat{\mathbf{x}} + l_1 \cos\phi_1 \hat{\mathbf{y}}$$

$$\mathbf{r}_2 = (l_1 \sin\phi_1 + l_2 \sin\phi_2)\,\hat{\mathbf{x}} + (l_1 \cos\phi_1 + l_2 \cos\phi_2)\,\hat{\mathbf{y}}$$

This problem has four degrees of freedom (motion in the plane) but two constraints that $|\mathbf{r}_1| = l_1$ and $|\mathbf{r}_2 - \mathbf{r}_1| = l_2$. This leaves to two generalized coordinates. Note that the constraints $f(\mathbf{r}_1, \cdots, \mathbf{r}_n, t) = 0$ are called "holonomic" and are in general nice for solving problems. □

# 4 Lecture 4

## 4.1 Lecture Notes - Lagrangian Mechanics Part 4

### 4.1.1 Proof of Lagrange Equations for Holonomic Systems

We consider a holonomic system with $f(\mathbf{r}_i, t) = 0$ constraints. Consider a true path $\mathbf{r}^*(t)$, and a general path with variation of $\mathbf{r}(t) = \mathbf{r}^*(t) + \alpha\delta\mathbf{r}(t)$ Note that $\mathbf{r}$ is not necessarily generalized coordinate. $\mathbf{r}(t)$ and $\mathbf{r}^*(t)$ are on a surface to which the particle is constrained; this implies that the variation is also constrained. Now, we study the action:

$$S[\mathbf{r}^* + \alpha\delta\mathbf{r}] = \int_{t_1}^{t_2} \mathcal{L}[\mathbf{r}^* + \alpha\delta\mathbf{r}, \dot{\mathbf{r}}^* + \alpha\delta\dot{\mathbf{r}}, t]dt$$

Now, we do a Taylor expansion around the true path:

$$S[\mathbf{r}^* + \alpha\delta\mathbf{r}] = \int_{t_1}^{t_2} dt\left(\mathcal{L}(\mathbf{r}^*, \dot{\mathbf{r}}^*, t) + \alpha\delta\mathbf{r}\,\frac{\partial\mathcal{L}}{\partial\mathbf{r}} + \alpha\delta\dot{\mathbf{r}}\,\frac{\partial\mathcal{L}}{\partial\dot{\mathbf{r}}} + \delta(\alpha^2)\right) \quad \text{First term is unperturbed action, so we can write as:}$$

$$= S[\mathbf{r}^*, \dot{\mathbf{r}}^*, t] + \alpha\int_{t_1}^{t_2} dt\,\frac{\partial\mathcal{L}}{\partial\mathbf{r}}\,\delta\mathbf{r} + \alpha\int_{t_1}^{t_2} dt\,\frac{\partial\mathcal{L}}{\partial\dot{\mathbf{r}}}\,\frac{\mathrm{d}}{\mathrm{d}}\,t + \delta(\alpha^2) \quad \text{Integrating by parts, we have:}$$

$$= S[\mathbf{r}^*, \dot{\mathbf{r}}^*, t] + \alpha\int_{t_1}^{t_2} dt\,\frac{\partial\mathcal{L}}{\partial\mathbf{r}}\,\delta\mathbf{r} + \frac{\partial\mathcal{L}}{\partial\dot{\mathbf{r}}}\,\delta\mathbf{r}\bigg|_{t_1}^{t_2} - \int_{t_1}^{t_2} dt\left(\frac{\mathrm{d}}{\mathrm{d}t}\,\frac{\partial\mathcal{L}}{\partial\dot{\mathbf{r}}}\right)\delta\mathbf{r} \quad \text{Boundary term is zero, so:}$$

$$= S[\mathbf{r}^*, \dot{\mathbf{r}}^*, t] + \alpha\int_{t_1}^{t_2} dt\left(\frac{\partial\mathcal{L}}{\partial\mathbf{r}} - \frac{\mathrm{d}}{\mathrm{d}t}\,\frac{\partial\mathcal{L}}{\partial\dot{\mathbf{r}}}\right)\delta\mathbf{r} + \delta(\alpha^2)$$

This is a first order variation (as we can tell from the fact that we expanded around $\alpha$ to first order). This is given by:

$$\delta S = \lim_{\alpha\to 0}\frac{1}{\alpha}\left(S[\mathbf{r}^* + \alpha\delta\mathbf{r}] - S[\mathbf{r}^*]\right) = \int_{t_1}^{t_2} dt\left(\frac{\partial\mathcal{L}}{\partial\mathbf{r}} - \frac{\mathrm{d}}{\mathrm{d}t}\,\frac{\partial\mathcal{L}}{\partial\dot{\mathbf{r}}}\right)\delta\mathbf{r}$$

We have that the first term $\dfrac{\partial\mathcal{L}}{\partial\mathbf{r}} = -\boldsymbol{\nabla}U$ are conservative forces. Now considering the kinetic energy as $T = \frac{m}{2}\dot{\mathbf{r}}^2$, then the second term is $\dfrac{\mathrm{d}}{\mathrm{d}t}\,\dfrac{\partial\mathcal{L}}{\partial\dot{\mathbf{r}}} = m\ddot{\mathbf{r}} = \mathbf{F}_{tot} = \mathbf{F}_{constraint} + \mathbf{F}_{conservative}$. The contribution to the action integral for the constraint forces are given by $-\int_{t_1}^{t_2}\delta\mathbf{r}\cdot\mathbf{F}_{constaint} = 0$ as the constraint force is perpendicular to any direction for which we can actually very the path! Hence, the first order variation is given by:

$$\delta S = -\int_{t_1}^{t_2}\delta\mathbf{r}\cdot\left(m\ddot{\mathbf{r}} + \boldsymbol{\nabla}U\right)dt$$

If the particle follows Newton's laws of motion, then $m\ddot{\mathbf{r}} + \boldsymbol{\nabla}U = 0$ and hence the first order variation is $\delta S = 0$! The beauty of this formulation is the constraint forces do not contribute. Here, $\delta\mathbf{r}$ must be consistent with the constraint. So for the generalized coordinate, any variation works, and $\delta S[q] = 0$.

#### 4.1.2 Note on the Sign of Gravitational Potential Energy

The sign in gravitational potential energy $U = mgy$ depends on whether the $y$-axis goes up or down. If you decide that the $y$-axis goes down then use $U = -mgy$, otherwise use $U = mgy$. To understand why, suppose we calculate the work done by gravity when $y$ goes up. This gives $W = \int_{y_1}^{y_2} -mgdy = +mgy_1 - mgy_2$, so we get the positive result. On the other hand, when $y$ goes down our work becomes $W = -mgy_1 + mgy_2$, so we get the negative result.
As an example, let's calculate $U$ for a pendulum blob. If $y$ goes up, the $y$ position of the blob is given by $y = -l\cos\theta$ and the gravitational potential is $U = mgy = -mgl\cos\theta$. If $y$ goes down, the $y$ position of the blob is given by $y = l\cos\theta$ and the gravitational potential is $U = -mgy = -mgl\cos\theta$. The result for $U$ is the same!

### 4.2 Worksheet - Lagrangian Mechanics with Dissipative Forces & Constrained Systems

**Problem 4.1.** *Write down Newton'sequation of motion for a mass $m$ hanging on a spring with spring constant $k$, equilibrium length $x_0$, and damping coefficient $b$,and subject to a vertical forcing function $F(t)$.*

*Solution.* In one dimension, we have ($x$ goes up):

$$\sum F = -k(x - x_0) - b\dot{x} - mg + F(t) = m\ddot{x}$$

Where the first term is the spring force, the second term is the air friction, the third term is the gravitational force, and the fourth term is the forcing function. We may rearrange this to say:

$$m\ddot{x} + b\dot{x} + k(x - x_0) + mg = F(t)$$

□

**Problem 4.2.** *In the previous problem, there were nonconservative forces that are not included in our Lagrange formalism. Compare the equation of motion above to the Langrange equation for a mass on a spring without damping and forcing, and suggest how the Lagrange equations should be modified to include friction.*

*Solution.* We consider that Lagrange's equation of motion $\frac{\partial \mathcal{L}}{\partial x} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{x}}$ does **not** account for the nonconservative forces (i.e. the forcing function $F(t)$ and the air friction term $-b\dot{x}$), as the Lagrangian is given by $\mathcal{L} = T - U = \frac{1}{2}m\dot{x}^2 - mgx - \frac{1}{2}k(x - x_0)^2$ and clearly this does not account for the damping or forcing terms. Without these terms, we have that:

$$\frac{\partial \mathcal{L}}{\partial x} = -k(x - x_0) - mg = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{x}} = m\ddot{x}$$

But with these terms, we have that:

$$\frac{\partial \mathcal{L}}{\partial x} = -k(x - x_0) - mg - b\dot{x} + F(t)$$

Rearranging the equation gives:

$$m\ddot{x} + b\dot{x} + k(x - x_0) + mg = F(t)$$

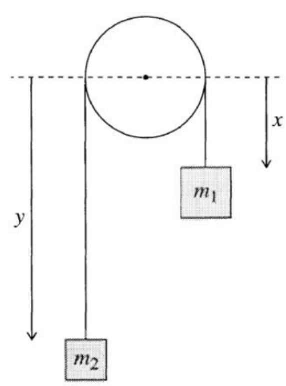
We get the same equation as before! So the "fix" for nonconservative forces to the Lagrange equations of motion are:

$$\frac{\partial \mathcal{L}}{\partial x} + F_{noncons} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{x}}$$

You have to add these in manually because there is no principle of least action for dissipative forces.

□

**Problem 4.3.** *Write down the Lagrangian for the Atwood machine. How many degrees of freedom are there? Pick a generalized coordinate, find the Lagrange equation of motion, and solve it for the acceleration.*




*Solution.* There is only one degree of freedom as the motion of one of the masses is completely determined by the other. Let our generalized coordinate be $x$ (goes down), and we can define the height of the other mass as $y = l - x$ where $l$ is some constant representing the length of the string. The potential energy of the system is given by:

$$U = -m_1 g x - m_2 g(l - x) = xg(m_2 - m_1) - m_2 l g$$

The kinetic energy of the system is given by:

$$T = \frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}m_2(-\dot{x})^2 = \frac{1}{2}(m_1 + m_2)\dot{x}^2$$

Hence the Lagrangian of our system is given by:

$$\mathcal{L} = T - U = \frac{1}{2}(m_1 + m_2)\dot{x}^2 - xg(m_2 - m_1) + m_2 l g$$

Solving for the equation of motion:

$$\frac{\partial \mathcal{L}}{\partial x} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{x}}$$
$$g(m_1 - m_2) = \frac{\mathrm{d}}{\mathrm{d}t}(m_1 + m_2)\dot{x}$$
$$g(m_1 - m_2) = (m_1 + m_2)\ddot{x}$$

Hence solving for the acceleration, we get:

$$\ddot{x} = g\frac{m_1 - m_2}{m_1 + m_2}$$

The beauty here is that we really can ignore the constraint forces; if we want to know the motion of the particles, we can use directly the Euler-Lagrange equations of motion. With Newton's laws, we have to keep track of them explicitly; this is why the Lagrangian formulation is often easier. □

**Problem 4.4.** *A particle of mass $m$ is constrained to move on a frictionless cylinder of radius $R$, given by the equation $\rho = R$ in cylindrical polar coordinates $(\rho,\ \phi,\ z)$. Besides the force of constraint (the normal force of the cylinder), the only force on the mass is a force $F = -kr$ directed toward the origin. Using $z$ and $\phi$ as generalized coordinates, find the Lagrangian $\mathcal{L}$ Write down and solve Lagrange's equations and describe the motion.*

*Solution.* The "spring" potential energy from the force directed towards the origin is given by $U_{spr} = \frac{1}{2}k(R^2 + z^2)$. The kinetic energy of the particle is given as $T = \frac{1}{2}m\dot{\mathbf{r}}^2 = \frac{1}{2}m(\dot{z}^2 + R^2\dot{\phi}^2)$. The Lagrangian is therefore given by:

$$\mathcal{L} = T - U = \frac{1}{2}m(\dot{z}^2 + R^2\dot{\phi}^2) - \frac{1}{2}k(R^2 + z^2)$$

Now we use the EL equations to find the equations of motion for $z$ and $phi$. Starting with $z$:

$$\frac{\partial \mathcal{L}}{\partial z} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{z}}$$
$$-kz = m\ddot{z}$$

The solution to this differential equation is simple harmonic motion with frequency $\omega = \sqrt{\frac{k}{m}}$:

$$z(t) = A\cos(\omega t) + B\sin(\omega t)$$

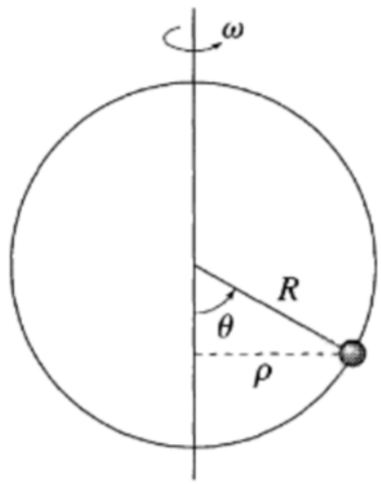
Next for $\phi$:

$$\frac{\partial \mathcal{L}}{\partial \phi} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{\phi}}$$
$$0 = \frac{\mathrm{d}}{\mathrm{d}t}\left(mR^2\dot{\phi}\right)$$
$$0 = mR^2\ddot{\phi}$$

Hence (as we knew already from angular momentum conservation) we find that $\ddot{\phi}$ is conserved and hence $\dot{\phi}$ is constant. □

# 5 Lecture 5

## 5.1 Worksheet - Bead on a spinning circular wire




A bead of mass $m$ is threaded on a frictionless circular wire hoop of radius $R$. The hoop lies in a vertical plane, which is forced to rotate about the hoop's vertical diameter with constant angular velocity $\dot{\phi} = \omega$. The bead's position on the hoop is specified by the angle $\theta$.

**Problem 5.1.** *Write an expression for the potential energy of the system.*

*Solution.* The height of the bead is determined (via trigonometry) to be $h = 1 - \cos\theta$, so the gravitational potential energy is given by:

$$U = mgh = mg(1 - \cos\theta)$$

□

**Problem 5.2.** *Write an expression for the kinetic energy of the system. Thus you can now write out the Lagrangian $\mathcal{L}$.*

*Solution.*

$$T = \frac{1}{2}\frac{m}{v}\dot{\mathbf{r}}^2 = \frac{m}{2}\left(R\omega\sin\theta + R\dot{\theta}\right)^2 = \frac{m}{2}\left(R^2\omega^2\sin^2\theta + R^2\dot{\theta}^2\right)$$

Where the first term is the normal velocity (out of the page in the diagram) and the second term is the tangential velocity (velocity in the frame of the page in the diagram). Hence the Lagrangian is:

$$\mathcal{L} = T - U = \frac{m}{2}\left(R^2\omega^2\sin^2\theta + R^2\dot{\theta}^2\right) + mg(\cos\theta - 1)$$

□

**Problem 5.3.** *From the Lagrangian, what is the torque acting on the angle $\theta$ of the bead? What is the equation of motion for the bead?*

*Solution.* To get the torque, we look at the generalized force for $\theta$:

$$\frac{\partial\mathcal{L}}{\partial\theta} = -mgR\sin\theta + mR^2\omega^2\sin\theta\cos\theta$$

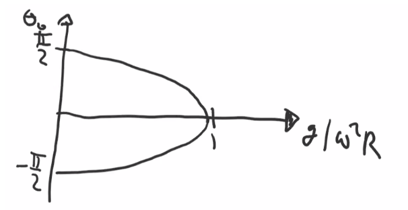
By the EL equation, we obtian the equation of motion for $\theta$:

$$-mgR\sin\theta + mR^2\omega^2\sin\theta\cos\theta = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathcal{L}}{\partial\dot{\theta}} = mR^2\ddot{\theta}$$

□

**Problem 5.4.** *Sketch your best estimate for where the equilibrium point(s) of the bead are, i.e. sketch the equilibrium value(s) of $\theta_0$ versus $\omega$.*

*Solution.* There is an equilibrium point at the bottom of the ring (somewhat intuitively), and when the ring spins fast enough, there are additional equilibrium points symmetrically across the axis:




The derivation of this graph and other equilibrium points mathematically is given in the next problem. We can see that as $\omega \to \infty$ that the bead will be perfectly horizontal (which lines up with our intuition somewhat). □

**Problem 5.5.** *Convince yourself that an equilibrium point has $\dot{\theta} = \ddot{\theta} = 0$. Find all equilibrium points of the bead. Where is the equilibrium point when $0 < \omega^2 < g/R$?*

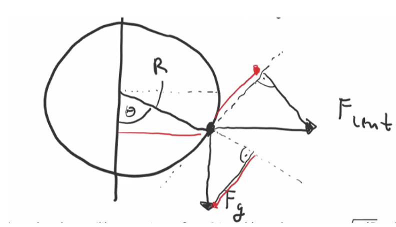
*Solution.* If we want equilibrium, we require the condition of $\ddot{\theta} = 0$ (as the bead should not feel any net torque and start with no angular velocity if it is to remain at rest), which results in

$$\left(\omega^2\cos\theta_0 - \frac{g}{R}\right)\sin\theta_0 = 0$$

This has solutions of $\theta_0 = 0$ or $\theta_0 = \pi$ (the sine term is zero) or $\cos\theta_0 = \frac{g}{\omega^2 R}$ or $\theta_0 = \pm\arccos\left(\frac{g}{\omega^2 R}\right)$. Since cos only varies between 1 and $-1$ this arccos solution only has a solution for $\omega^2 \geq \frac{g}{R}$. □

**Problem 5.6.** *Show that at the equilibrium points defined by $\omega^2\cos\theta - g/R = 0$, the tangential components of the gravitational and centrifugal forces (in the non-inertial frame of the hoop) cancel. Show that for any points with $\theta > \pi/2$ including near the top, the above two forces are in the same direction.*

*Solution.* Consider the balance of forces at these equilibrium points. There is the centrifugal force that throws the particle outwards, and the gravitational force that pulls the particle downwards. The centrifugal force is given by $|\mathbf{F}_{cent}| = m\omega^2 r = m\omega^2 R\sin\theta$ that throws the particle outwards and the gravitational force $|\mathbf{F}_g| = -mg$ downwards. We decompose these forces into the radial and tangential components.



For the angle to remain constant, we require that the tangential components (pictured in red above) must cancel. This is given by:

$$F_{tan} = -mg\sin\theta + m\omega^2 R\sin\theta\cos\theta = \sin\theta\left(\omega^2 R\cos\theta - g\sin\theta\right)$$

Which we see is 0 whenever $\omega^2\cos\theta - g/R = 0$ and hence we get the same answer in two different ways. If $\theta > \frac{\pi}{2}$, then we have that $\sin\theta > 0$ and $\cos\theta < 0$ and hence we can see from the above expression of the tangential force above that the gravitational tangent force and the centrifugal force point in the same direction; there is no hope of having an equilibrium point on the upper half of the metal loop! □

**Problem 5.7.** *Show that the equilibrium point at $\theta_0 = 0$ is stable, so long as $\omega < \sqrt{g/R}$. What is the oscillation frequency about the equilibrium point?*

*Solution.* When $\theta$ is small, we can do a small angle approximation and so $\cos\theta \sim 1$ and $\sin\theta \sim \theta$. Going back to our equation of motion, we see that:

$$mR\ddot{\theta} = -mgR\theta + mR^2\omega^2\theta$$

and therefore:

$$\ddot{\theta} = \left(\omega^2 - \frac{g}{R}\right)\theta$$

Which is a nice equation we can solve analytically. We see that the RHS is negative when $\omega < \sqrt{\frac{g}{R}}$ and positive when $\omega > \sqrt{\frac{g}{R}}$. In the first case, the solutions to the differential equations are sines/cosines; i.e. simple harmonic oscillation with frequency $\Omega = \sqrt{\frac{g}{R} - \omega^2}$:

$$\ddot{\theta} = -\Omega^2\theta$$

$$\theta(t) = A\cos(\Omega t) + B\sin(\Omega t)$$

Which is a stable equilibrium; it oscillates about the $\theta = 0$ point when perturbed from it. □

**Problem 5.8.** *Find the oscillation frequencies about the equilibrium points $\theta_0$ when $\omega > \sqrt{g/R}$. What is the stability condition for these oscillations? What is the oscillation frequency? Now sketch the equilibrium values of $\theta_0$, as a function of $\omega$, over the full range $\omega > 0$.*

*Solution.* Conversely, when $\omega > \sqrt{g/R}$, consider a small perturbation $\theta_0 + \epsilon$ from the equilibrium. For a small $\epsilon$, we have:

$$\cos(\theta_0 + \epsilon) \approx \cos\theta_0 - \epsilon\sin\theta_0$$

$$\sin(\theta_0 + \epsilon) \approx \sin\theta_0 + \epsilon\sin\theta_0$$

The equation of motion then becomes:

$$\ddot{\theta} = \left[\omega^2\cos(\theta_0 + \epsilon) - \frac{g}{R}\right]\sin(\theta_0 + \epsilon)$$

$$\ddot{\theta} \approx \left[\omega^2\cos\theta_0 - \epsilon\omega^2\sin\theta_0 - \frac{g}{R}\right][\sin\theta_0 + \epsilon\cos\theta_0]$$

Now the $\omega^2\cos\theta_0$ and $-g/R$ terms cancel, and neglecting terms in $\epsilon^2$, we have:

$$\ddot{\theta} = \ddot{\epsilon} = -\epsilon\omega^2\sin^2\theta_0 = -\Omega'\epsilon$$

So we can see that if we perturb around $\theta_0$ that we have oscillation around $\theta_0$ which corresponds to a stable equilibrium. □

In conclusion, we see that this is quite a rich problem with respect to different equilibrium points and their behavior. There is also a discontinuous jump between certain equilibria $\omega = \sqrt{g/R}$, which is known as **bifurcation**, which can lead to chaotic motion.

# 6 Lecture 6

## 6.1 Lecture Notes - Charged Particles in EM Fields

### 6.1.1 Newton's Law in the Presence of EM Fields

Newton says:

$$m\ddot{\mathbf{r}} = q(\mathbf{E} + \dot{\mathbf{r}} \times \mathbf{B})$$

for an electric field $\mathbf{E}$ and magnetic field $\mathbf{B}$. This is just the sum of the Coloumb force and Lorentz force.

### 6.1.2 Scalar and Vector Potentials

We will now apply the ideas of vector potential (for the magnetic field) from PHYS 301 to this situation. Recall that we defined the vector potential $\mathbf{A}$ as

$$\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}$$

. We also recall Faraday's Law, which states (combined with the definition of the potential above) that:

$$\boldsymbol{\nabla} \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} = -\frac{\partial \boldsymbol{\nabla} \times \mathbf{A}}{\partial t}$$

Interchanging the order of taking the curl and taking the time derivative, we see that:

$$\boldsymbol{\nabla} \times \left(\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t}\right) = \mathbf{0}$$

So there exists a scalar potential $V$ such that:

$$-\boldsymbol{\nabla} V = \mathbf{E} + \frac{\partial \mathbf{A}}{\partial t}$$

Which comes from the fact that a curl of a gradient is zero. We can rewrite this to say that:

$$\mathbf{E} = -\boldsymbol{\nabla} V - \frac{\partial \mathbf{A}}{\partial t}$$

Now, we construct $\mathcal{L}$ such that we get the Lorentz Force. Generalize $U = qV$ to get:

$$U' = qV - q\dot{\mathbf{r}} \cdot \mathbf{A}$$

And we use $\mathcal{L}' = T - U'$ as our Lagrangian.

### 6.1.3 Conservation of Angular Momentum in Spherically Symmetric Lagrangian

This central force Lagrangian $\mathcal{L} = \frac{1}{2}\mu|\dot{\mathbf{r}}|^2 - U(r)$ is spherically symmetric (ie. doesn't change when the coordinate system rotates). This implies energy is not always conserved but angular momentum is conserved.

To see why, we will study rotational symmetry around z-axis. When a vector in 3D space $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ is rotated about z by $\theta$, the new vector is

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Consider infinitesimal rotation by $\delta\theta$ and using small angle approximations $\cos\theta \approx 1$ and $\sin\theta \approx \theta$,

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} 1 & -\delta\theta & 0 \\ \delta\theta & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

The Lagrangian becomes

$$\mathcal{L}'(x', y', z', \dot{x}', \dot{y}', \dot{z}') = \mathcal{L}(x - \delta\theta y, y + \delta\theta x, z, \dot{x} - \delta\theta \dot{y}, \dot{y} + \delta\theta \dot{x}, \dot{z})$$

Applying the first order multivariable Taylor expansion about $(x, y, z, \dot{x}, \dot{y}, \dot{z})$ gives

$$\mathcal{L}'(x', y', z', \dot{x}', \dot{y}', \dot{z}') = \mathcal{L}(x, y, z, \dot{x}, \dot{y}, \dot{z}) + \frac{\partial \mathcal{L}}{\partial x'}(x' - x) + \frac{\partial \mathcal{L}}{\partial y'}(y' - y) + \frac{\partial \mathcal{L}}{\partial z'}(z' - z) + \frac{\partial \mathcal{L}}{\partial \dot{x}'}(\dot{x}' - \dot{x}) + \frac{\partial \mathcal{L}}{\partial \dot{y}'}(\dot{y}' - \dot{y}) + \frac{\partial \mathcal{L}}{\partial \dot{z}'}(\dot{z}' - \dot{z})$$

To calculate a x-term, we know that $x' = x - \delta\theta y$. This gives $x' - x = -\delta\theta y$ and $\frac{\partial \mathcal{L}}{\partial x'} = \frac{\partial \mathcal{L}}{\partial x}\frac{\partial x}{\partial x'} = \frac{\partial \mathcal{L}}{\partial x}$. The other terms can be calculated the similar way. All of these gives

$$\mathcal{L}'(x', y', z', \dot{x}', \dot{y}', \dot{z}') = \mathcal{L}(x, y, z, \dot{x}, \dot{y}, \dot{z}) - \delta\theta\, y\, \frac{\partial \mathcal{L}}{\partial x} + \delta\theta\, x\, \frac{\partial \mathcal{L}}{\partial y} - \delta\theta\, \dot{y}\, \frac{\partial \mathcal{L}}{\partial \dot{x}} + \delta\theta \dot{x}\, \frac{\partial \mathcal{L}}{\partial \dot{y}}$$

$$\mathcal{L}'(x', y', z', \dot{x}', \dot{y}', \dot{z}') = \mathcal{L}(x, y, z, \dot{x}, \dot{y}, \dot{z}) + \delta\theta \left[ \left( x \frac{\partial \mathcal{L}}{\partial y} - y \frac{\partial \mathcal{L}}{\partial x} \right) + \left( \dot{x} \frac{\partial \mathcal{L}}{\partial \dot{y}} - \dot{y} \frac{\partial \mathcal{L}}{\partial \dot{x}} \right) \right]$$

Because of rotational symmetry, Lagrangian should not change at all (ie. $\mathcal{L}' - \mathcal{L} = 0$). And this means

$$\left( x \frac{\partial \mathcal{L}}{\partial y} - y \frac{\partial \mathcal{L}}{\partial x} \right) + \left( \dot{x} \frac{\partial \mathcal{L}}{\partial \dot{y}} - \dot{y} \frac{\partial \mathcal{L}}{\partial \dot{x}} \right) = 0$$

Substituting EL equations $\frac{\partial \mathcal{L}}{\partial x} = \frac{\mathrm{d}}{\mathrm{d}t}\left( \frac{\partial \mathcal{L}}{\partial \dot{x}} \right)$ and $\frac{\partial \mathcal{L}}{\partial y} = \frac{\mathrm{d}}{\mathrm{d}t}\left( \frac{\partial \mathcal{L}}{\partial \dot{y}} \right)$,

$$x \frac{\mathrm{d}}{\mathrm{d}t}\left( \frac{\partial \mathcal{L}}{\partial \dot{y}} \right) + \frac{\mathrm{d}x}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{y}} - y \frac{\mathrm{d}}{\mathrm{d}t}\left( \frac{\partial \mathcal{L}}{\partial \dot{x}} \right) - \frac{\mathrm{d}y}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{x}} = 0$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left( x \frac{\partial \mathcal{L}}{\partial \dot{y}} - y \frac{\partial \mathcal{L}}{\partial \dot{x}} \right) = 0$$

Denote $\frac{\partial \mathcal{L}}{\partial \dot{y}} = p_y$ and $\frac{\partial \mathcal{L}}{\partial \dot{x}} = p_x$

$$\frac{\mathrm{d}}{\mathrm{d}t}(x p_y - y p_x) = 0$$

Notice that $x p_y - y p_x$ is z-component of angular momentum $\mathbf{L}$. Therefore

$$\frac{\mathrm{d}L_z}{\mathrm{d}t} = 0$$

So angular momentum is conserved under rotational symmetry. We will discuss more about this next class.

## 6.2 Worksheet - Lagrangians for EM fields, Uniqueness

**Problem 6.1.** *From the form of the Lagrangian for a charged particle in an electromagnetic field, find the generalized momentum* $\mathbf{p}$.

*Solution.* As discussed in the lecture, we generalize the potential

$$U' = qV - q\dot{\mathbf{r}} \cdot \mathbf{A}$$

and hence construct the Lagrangian:

$$\mathcal{L} = T - U' = \frac{1}{2}m\dot{\mathbf{r}}^2 - qV + q\dot{\mathbf{r}} \cdot \mathbf{A}$$

The generalized momentum is then given by:

$$\mathbf{p} = \frac{\partial \mathcal{L}}{\partial \dot{\mathbf{r}}} = m\dot{\mathbf{r}} + q\mathbf{A}$$

□

**Problem 6.2.** *Find the generalized force for the above system in the x-direction, and the total time derivative of the generalized momentum,* $\frac{\mathrm{d}\mathbf{p}}{\mathrm{dt}}$ *(Recall that* $\mathbf{A} = \mathbf{A}(x, y, z, t)$*)).*

*Solution.* The generalized force in the x direction is given by:

$$F_x = \frac{\partial \mathcal{L}}{\partial x} = -q\left(\frac{\partial V}{\partial x} - \dot{x}\,\frac{\partial A_x}{\partial x} - \dot{y}\,\frac{\partial A_y}{\partial x} - \dot{z}\,\frac{\partial A_z}{\partial x}\right)$$

The total time derivative of the generalized momentum is given by:

$$\frac{\mathrm{d}}{\mathrm{dt}}\,p_x = m\ddot{x} + q\left(\dot{x}\,\frac{\partial A_x}{\partial x} + \dot{y}\,\frac{\partial A_x}{\partial y} + \dot{z}\,\frac{\partial A_x}{\partial z} + \frac{\partial A_x}{\partial t}\right)$$

Where we have used the chain rule. □

**Problem 6.3.** *Write the equations of motion in terms of the electric and magnetic fields.*

*Solution.* Combining the two equations above (EL equation), we have:

$$m\ddot{x} = -q\left(\frac{\partial V}{\partial x} + \frac{\partial A_x}{\partial t}\right) + q\dot{y}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) + q\dot{z}\left(\frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z}\right)$$

Recognizing the first term as the electric field term and the second/third terms as $B_z$ and $-B_y$ respectively, we recognize this as the x component of $\dot{\mathbf{r}} \times (\boldsymbol{\nabla} \times \mathbf{A})$, or the Lorentz Force! The y and z components follow similarly and we recover:

$$m\ddot{\mathbf{r}} = q(\mathbf{E} + \dot{\mathbf{r}} \times \mathbf{B})$$

Which matches up with Newton's Law. Hence, we have been able to extend the Lagrangian formalism to charged particles in electromagnetic fields. □

**Problem 6.4.** *Consider a charged, relativistic particle in an electric field* $\mathbf{E}$*. Show that the Lagrangian* $\mathcal{L} = -mc^2\sqrt{1 - v^2/c^2} - qV + q\mathbf{v}\cdot\mathbf{A}$*, with* $m$ *given by the rest mass, gives the correct (relativistic) equations of motion.*

*Solution.* If we recall from PHYS 200, the momentum generalized to the relativistic case has a Lorentz factor of $\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$, given by:

$$\mathbf{p} = m_0\gamma\mathbf{v}$$

So we require that we get this back from the Lagrangian expression. Checking the generalized momentum, we see:

$$\mathbf{p} = \frac{\partial \mathcal{L}}{\partial \mathbf{v}} = -m_0c^2\frac{1}{2}\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\left(-2\frac{\mathbf{v}}{c^2}\right) = m_0\gamma\mathbf{v}$$

Which lines up with our expectation. □

**Problem 6.5.** *Show that* $\mathcal{L} = T - U + x^2\dot{x}$ *also gives the Newton's equations of motion. What is the implication for the uniqueness of the Lagrangian then?*

*Solution.* Showing that this satisfies Newton's equations of motion:

$$\frac{\partial \mathcal{L}'}{\partial x} = -\frac{\partial U}{\partial x} + 2x\dot{x}$$

$$\frac{\partial \mathcal{L}'}{\partial \dot{x}} = m\dot{x} + x^2 \implies \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}'}{\partial \dot{x}} = m\ddot{x} + 2x\dot{x}$$

Therefore by the EL equation:

$$\frac{\partial \mathcal{L}'}{\partial x} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{x}}$$

$$-\frac{\partial U}{\partial x} + 2x\dot{x} = m\ddot{x} + 2x\dot{x}$$

So we recover Newton's law:

$$-\frac{\partial U}{\partial x} = m\ddot{x}$$

This shows that the Lagrangian is not unique. In general, we can always add a total derivative of the form:

$$\frac{\mathrm{d}}{\mathrm{d}t} G(q,t)$$

to $\mathcal{L}$ without changing the equations of motion. To see that this is the case, consider the modified Lagrangian:

$$\mathcal{L}'(q,\dot{q},t) = \mathcal{L}(q,\dot{q},t) + \frac{\mathrm{d}}{\mathrm{d}t} G(q,t)$$

And now computing the action, we have:

$$S'(q,\dot{q},t) = \int_{t_1}^{t_2} \mathcal{L}(q,\dot{q},t)dt + \int_{t_1}^{t_2} \frac{\mathrm{d}}{\mathrm{d}t} G(q,t)dt$$

The second term we compute to be $G(q_2,t_2) - G(q_1,t_1)$ which are independent of the trajectory (as the start and endpoints are the same). So this does not affect the overall action, or the trajectory that minimizes the action, or in other words:

$$\delta S' = \delta S$$

□

# 7 Lecture 7

## 7.1 Lecture Notes - Symmetries and Conservation Laws

### 7.1.1 Noether's Theorem

Idea: Certain symmetries we observe in nature are associated with conservation laws. E.g. momentum with translational symmetry. Formally, we consider the following:
Consider a Lagrangian $\mathcal{L}(q,\dot{q},t)$ is invariant under a coordinate transformation $(q_1, \cdots, q_n) \to (\tilde{q}_1(\alpha), \cdots, \tilde{q}_n(\alpha))$ with $\tilde{q}_i(\alpha) = q_i + \alpha h_i(q,t) + \delta(\alpha^2)$. Consider taking the derivative with respect to $\alpha$ and set $\alpha = 0$:

$$\frac{\mathrm{d}}{\mathrm{d}\alpha} \mathcal{L}(\tilde{q}(\alpha), \dot{\tilde{q}}(\alpha), t)\bigg|_{\alpha=0} = 0$$

Where the expression is zero as $\mathcal{L}$ is invariant of $\alpha$. Now, by the chain rule, we can say:

$$0 = \sum_{j=1}^{n} \left( \frac{\partial \mathcal{L}}{\partial \tilde{q}_j}\frac{\partial \tilde{q}_j}{\partial \alpha} + \frac{\partial \mathcal{L}}{\partial \dot{\tilde{q}}_j}\frac{\partial \dot{\tilde{q}}_j}{\partial \alpha} \right)\bigg|_{\alpha=0}$$

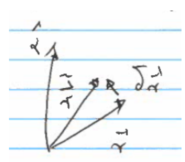
Now, we can say that $\frac{\partial \dot{\tilde{q}}_j}{\partial \alpha} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \tilde{q}_j}{\partial \alpha}$ my equality of mixed partials. Using the EL equations, we can also say that $\frac{\partial \mathcal{L}}{\partial \tilde{q}_j} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{\tilde{q}}_j}$. We can then write the whole expression as:

$$0 = \frac{\mathrm{d}}{\mathrm{d}t}\sum_{j=1}^{n}\left(\frac{\partial \mathcal{L}}{\partial \dot{\tilde{q}}_j}\frac{\partial \tilde{q}_j}{\partial \alpha}\right)\Bigg|_{\alpha=0}$$

By the chain rule. We have just recollapsed the sum using the chain rule multiple times. Now, call this sum $I(q, \dot{q}, t)$. This object is conserved as its time derivative is zero, i.e. $\frac{\mathrm{d}}{\mathrm{d}t} I(q, \dot{q}, t) = 0$. Let us apply this to make this more concrete.

### 7.1.2 Translational and Rotational Symmetry

Suppose our Lagrangian is $\mathcal{L} = \frac{1}{2}\sum_j m_j \dot{\mathbf{r}}_j^2 - U$. The first symmetry we will consider is the homoegeneity of space. This can be mathematically represented as $\mathbf{r}_i \mapsto \tilde{\mathbf{r}}_i + \alpha\hat{\mathbf{e}}$ (where $\hat{\mathbf{e}}$ is some unit vector) and where $\dot{\mathbf{r}}_i = \dot{\tilde{\mathbf{r}}}_i$.
The next symmetry we can consider is the isotropy of space. This is represented as $\mathbf{r}_i \mapsto \tilde{\mathbf{r}}_i = \mathbf{r}_i + \hat{\boldsymbol{\alpha}} \times \mathbf{r}_i$ (where $\hat{\boldsymbol{\alpha}}$ is the axis of rotation).



Now, what does Noether tell us about these symmetries? Assuming that the Lagrangians are invariant under the transformations, then $I$ is conserved, and hence:

$$I = \sum_{j=1}^{n}\left(\frac{\partial \mathcal{L}}{\partial \dot{\tilde{q}}_j}\frac{\partial \tilde{q}_j}{\partial \alpha}\right)\Bigg|_{\alpha=0} = \begin{cases}\text{(Translation) } \sum_{j=1}^{n} m_j \dot{\mathbf{r}}_j \cdot \hat{\mathbf{e}} \\ \text{(Rotation) } \sum_{j=1}^{n} m_j \dot{\mathbf{r}}_j \cdot (\hat{\boldsymbol{\alpha}} \times \mathbf{r}_j) = \hat{\boldsymbol{\alpha}} \cdot \sum_{j=1}^{n} \mathbf{r}_j \times m_j \dot{\mathbf{r}}_j = \hat{\boldsymbol{\alpha}} \cdot \mathbf{L}\end{cases}$$

(For rotation, recall that $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a})$.) In the first case (with translational symmetry) we have conservation of linear momentum along $\hat{\mathbf{e}}$. In the second case (with rotational symmetry) we have the conservation of angular momentum along $\hat{\boldsymbol{\alpha}}$.

### 7.1.3 Time Symmetry and the Hamiltonian

We next consider a scenario where we have homogeneity of time; In other words, where $\mathcal{L}$ is unchanged by $t$. $t \mapsto t + \epsilon$, and $\frac{\partial \mathcal{L}}{\partial t} = 0$. Expanding out the total time derivative of the Lagrangian, we have:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{L}(q, \dot{q}, t) = \sum_{j=1}^{n}\left(\frac{\partial \mathcal{L}}{\partial q_j}\frac{\partial q_j}{\partial t} + \frac{\partial \mathcal{L}}{\partial \dot{q}_j}\frac{\partial}{\partial t}\frac{\partial q_j}{\partial t}\right)$$

We recall that $\frac{\partial q_j}{\partial t} = \dot{q}_j$, $\frac{\partial \mathcal{L}}{\partial \dot{q}_j} = p_j$ (generalized momentum) and $\frac{\partial \mathcal{L}}{\partial q_j} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{q}_j}$ (generalized force/time derivative of generalized momentum). Again by taking out the time differential operator (by product rule) out of the sum, we have:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{L}(q, \dot{q}, t) = \frac{\mathrm{d}}{\mathrm{d}t}\sum_j p_j \dot{q}_j$$

where $p_j$ is the generalized momentum of the coordinate $q_j$. We can write this as:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_j p_j \dot{q}_j - \mathcal{L}\right) = 0$$

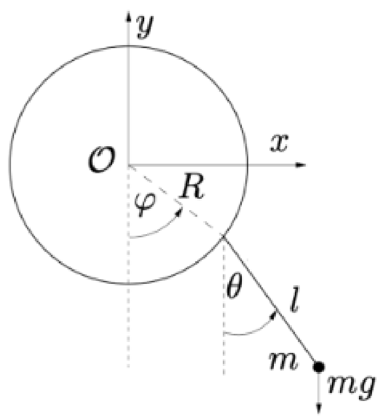
And we call the term in the brakets to be the **Hamiltonian**:

$$\mathcal{H} = \sum_j p_j \dot{q}_j - \mathcal{L}$$

which is a conserved quantity.

### 7.1.4 Remarks on Hamiltonian

Hamiltonian is not always conserved $\frac{\mathrm{d}\mathcal{H}}{\mathrm{d}t} \neq 0$. It happens when Lagrangian explicitly depends on time. For example, consider a pendulum of length $l$ and blob of mass $m$ subjected to gravity with its attachment point moving at constant angular velocity $\omega$ on a circle of radius $R$.




The Lagrangian for this system is (derivation is left as an exercise):

$$\mathcal{L} = \frac{m}{2}\left[R^2\omega^2 + l^2\dot{\theta}^2 + 2R\omega l\dot{\theta}\cos(\omega t - \theta)\right] + mg(R\cos(\omega t) + l\cos\theta)$$

Since it depends on time, Hamiltonian is not conserved.
Hamiltonian of a system is equal to the system's total energy if coordinates are natural (ie. no time dependence):

$$\mathbf{r}_i = \mathbf{r}_i(q_1, q_2, \ldots, q_n)$$

Textbook example 7.6 (bead on spinning hoop) is one of the physical situations when Hamiltonian is not equal to the total energy. The coordinate for the bead is:

$$\mathbf{r} = (R\sin\theta\cos(\omega t), R\sin\theta\sin(\omega t), -R\cos\theta)$$

Since it depends on time, the Hamiltonian is not equal to the total energy.

## 7.2 Worksheet - Energy and the Hamiltonian

**Problem 7.1.** *Show that if a coordinate transformation is "natural" (no explicit time-dependence, $\mathbf{r} = \mathbf{r}(q_1, \cdots, q_n)$) that the kinetic energy is a homogenous quadratic function of the velocities (a function is homogenous of degree $k$ if $f(\alpha x) = \alpha^k f(x)$. Find this quadratic function.*

*Solution.* Note that any function that is a power law will satisfy this property; it turns out that kinetic energy has this property (as we know). To solve the problem, we consider $\dot{\mathbf{r}}_j$:

$$\dot{\mathbf{r}}_j = \sum_{i=1}^{n} \frac{\partial \mathbf{r}_j}{\partial q_i} \dot{q}_i$$

To calculate $\dot{\mathbf{r}}_j^2$, we multiply the two sums:

$$\dot{\mathbf{r}}_j^2 = \left( \sum_i \frac{\partial \mathbf{r}_j}{\partial q_i} \dot{q}_i \right) \left( \sum_k \frac{\partial \mathbf{r}_j}{\partial q_k} \dot{q}_k \right)$$

This gives us:

$$T = \frac{1}{2} \sum_j m_j \dot{\mathbf{r}}_j^2 = \frac{1}{2} \sum_{i,k} \dot{q}_i \dot{q}_j \left( \sum_j m_j \frac{\partial \mathbf{r}_j}{\partial q_i} \frac{\partial \mathbf{r}_j}{\partial q_k} \right)$$

This term on the right is a matrix which we can call $A_{ik}$. Once we sum over these components, we get the total energy. Now we can see that this is homogenous, quadratic in $\dot{\mathbf{r}}_j$. □

**Problem 7.2.** *Show then that the conjugate momenta have the form $p_i = \sum_j A_{ij} \dot{q}_j$, and thus that the hamiltonian $\mathcal{H} = E$, the total energy. Thus, under the above conditions, symmetry under time translation is equivalent to conservation of energy.*

*Solution.* Using the result from above:

$$p_i = \frac{\partial \mathcal{L}}{\partial \dot{q}_i} = \frac{\partial T}{\partial \dot{q}_i} = \sum_j A_{ij} \dot{q}_j$$

Therefore:

$$\sum_i p_i \dot{q}_i = \sum_i \left( \sum_j A_{ij} \dot{q}_j \right) \dot{q}_i = \sum_{i,j} A_{ij} \dot{q}_i \dot{q}_j = 2T$$

Hence (using the defintion of $\mathcal{H}$ from lecture):

$$\mathcal{H} = 2T - \mathcal{L} = 2T - (T - U) = T + U = E$$

Which is just the total energy. □

**Problem 7.3.** *Show that the same derivation follows directly by using the property that the kinetic energy is a homogenous function of degree 2 in the generalized velocities (use Euler's homogeneity theorem: $f$ is positively homogenous of degree $k$ if and only if $\mathbf{x} \cdot \boldsymbol{\nabla} f(\mathbf{x} = k f(\mathbf{x})$.*

*Solution.* We have (letting $\mathbf{x} = \dot{\mathbf{q}}$:

$$\dot{\mathbf{q}} \cdot \frac{\partial T}{\partial \dot{\mathbf{q}}} = \sum_i p_i \dot{q}_i = 2T$$

Where the last line follows from the homogeneity theorem.

□

# 8 Lecture 8

## 8.1 Lecture Notes - Lagrange Multipliers

### 8.1.1 Motivation

How do you deal with a problem with constraints. Before, we had a method with generalized coordinates, obtain an EoM for each of them. But sometimes we might be interested in the constraint force itself. For this, we turn to the method of Lagrange multipliers (which have much broader use than mechanics!)

## 8.2 Worksheet - Lagrange Multipliers & Atwood Machine

**Problem 8.1.** *Write the constraint equations $f(x, y) = const.$ for $x$ and $y$ for the simple plane pendulum and the Atwood Machine.*

*Solution.* The constraint equation is $f(x, y) = \sqrt{x^2 + y^2} = L$ for the simple pendulum and $f(x, y) = x + y = L$ for the Atwood machine (constrained by length of rope). □

**Problem 8.2.** *Write down the Hamilton's principle $\delta S(x, y) = 0$ for the two constrained variables $x$ and $y$. If we try to make the action stationary, how would the deviations $\delta x$ and $\delta y$ have to be constrained?*

*Solution.*

$$\delta S + \int \left( \frac{\partial \mathcal{L}}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial \mathcal{L}}{\partial \dot{x}} \right) \delta x dt + \int \left( \frac{\partial \mathcal{L}}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}} \frac{\partial \mathcal{L}}{\partial \dot{y}} \right) \delta y dt = 0$$

But, these variations must be constrained as $x$ and $y$ are not independent of one another. We note that $\delta x, \delta y$ obey the constraints. □

**Problem 8.3.** *For deviations that meet the conditions of the question above, write down the variations of the constraint $\delta f$. Multiply $\delta f$ by an unknown function $\lambda(t)$ and add to $\delta S$.*

*Solution.*

$$\delta f = \frac{\partial f}{\partial x} \delta x + \frac{\partial f}{\partial y} \delta y = 0$$

As the displacement must leave the constraint unchanged. Hence we may add a 0 to the total variation of $S$ without changing anything. We therefore write:

$$\delta S = \int \left( \frac{\partial \mathcal{L}}{\partial x} + \lambda(t) \frac{\partial f}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial \mathcal{L}}{\partial \dot{x}} \right) \delta x \delta t + \int \left( \frac{\partial \mathcal{L}}{\partial y} + \lambda(t) \frac{\partial f}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial \mathcal{L}}{\partial \dot{y}} \right) \delta y dt = 0$$

□

**Problem 8.4.** *Can $\delta x$ and $\delta y$ be independently varied? The multiplying function of the constraint $\lambda(t)$ is so far undetermined and can be chosen as we wish. What special choice of $\lambda(t)$ makes all of the arguments $\delta x$, $\delta y$, $\delta z$... vanish? How have the Lagrange equations been modified when the dependent variables are constrained?*

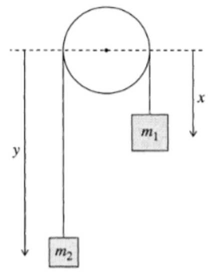
*Solution.* Unlike the case with generalized coordinates, the variations are not independent. Consider that we can pick $\lambda(t)$ (Lagrange multiplier) to be whatever we like to make the first integral vanish. However, this then means that the second term must vanish as well, as the sum has to be zero (and if the term is to be zero for all variations, then the integrand must be zero)! In general, if we have a function of (not necessarily generalized coordinates) $\mathcal{L}(x_k, \dot{x}_k, t)$ with $f_i(x_k, t) = 0$ ($m$ holonomic constraints), we can consider the modified Lagrangian:

$$\tilde{L}(x_k, \dot{x}_k, t) = \mathcal{L}(x_k, \dot{x}_k, t) + \sum_{i=1}^{m} \lambda_i f_i(x_k, t)$$

And if we require that the variation vanishes $\delta S = 0$, then we get a set of generalized Lagrange equations:

$$\frac{\partial \mathcal{L}}{\partial x_k} + \sum_{i=1}^{m} \lambda_i \frac{\partial f_i}{\partial x_k} = \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial \mathcal{L}}{\partial \dot{x}_k}$$

□




**Problem 8.5.** *What are Lagrange's equations for $x$ and $y$ of the Atwood Machine?*

*Solution.* The Lagrangian is given by:

$$\tilde{\mathcal{L}} = \mathcal{L} + \lambda f(x,y) = \frac{m_1}{2}\dot{x}^2 + \frac{m_2}{2}\dot{y}^2 + m_1 g x + m_2 g y + \lambda(x + y - L)$$

Hence the equations of motions are:

$$m_1\ddot{x} = m_1 g + \lambda$$
$$m_2\ddot{y} = m_2 g + \lambda$$

Where $\lambda$ is the Lagrange multiplier. □

**Problem 8.6.** *Using the constraint equation for $x$ and $y$, eliminate the Lagrange multiplier and solveagain for the acceleration of $x$.*

*Solution.* From the constraint equation (taking two time derivatives of it) we obtain that $\ddot{x} = -\ddot{y}$. We therefore have that:

$$\ddot{x} = \frac{(m_1 - m_2)}{(m_1 + m_2)}g$$

□

**Problem 8.7.** *Now solve the equations for the Lagrange multiplier.*

*Solution.* Solving for $\lambda$, we have:

$$\lambda = m_1\ddot{x} - m_1 g$$

We note that $m_1\ddot{x}$ is the total force of the system, so

$$m\ddot{x} = -\frac{\partial U}{\partial x} - F_T$$

Where the first term is the conservative (gravitational) force from potential and $F_T$ is the tension/constraint force. Therefore, we have that:

$$\lambda = -F_T$$

And we have found the lagrange multiplier to be the constraint/tension force! □

**Problem 8.8.** *Show that $\lambda\frac{\partial f}{\partial x}$ s the constraint force on mass $m_1$.*

*Solution.* Here, $\dfrac{\partial f}{\partial x} = 1$ so $\lambda\dfrac{\partial f}{\partial x} = \lambda = -F_T$ which is the expected result. □

# 9 Lecture 9

## 9.1 Worksheet - Review of Damped & Driven Harmonic Oscillators

**Problem 9.1.** *An undamped harmonic oscillator has general solution $x(t) = B_1 \cos(\omega t) + B_2 \sin(\omega t)$. Show this can be recast as $x(t) = A\cos(\omega t - \delta)$, where $A = \sqrt{A^2 + B^2}$.*

*Solution.* Consider a right-angle triangle with side lengths $B_1$, $B_2$ and hypotenuse $A + \sqrt{B_1^2 + B_2^2}$. Then, let $\cos\delta = \frac{B_1}{A}$ and $\sin\delta = \frac{B_2}{A}$. Looking at the equation for $x(t)$ above, we can multiply it by one in a clever way:

$$x(t) = A\left[\frac{B_1}{A}\cos\omega t + \frac{B_2}{A}\sin\omega t\right] = A\left[\cos\delta\cos\omega t + \sin\delta\sin\omega t\right] = A\cos(\omega t - \delta)$$

Where the last equality follows by a trigonometric identity. □

**Problem 9.2.** *Show that the kinetic and potential energy of a simple undamped oscillator have the same amplitude but are out of phase, such that the total energy is conserved.*

*Solution.* Using our equation for $x(t)$ above (where $x(t)$ is the displacement from equilibrium), the total energy is given by:

$$E = U + T = \frac{1}{2}kx^2 + \frac{1}{2}m\dot{x}^2 = \frac{1}{2}kA^2\cos^2(\omega t - \delta) + \frac{1}{2}mA^2\omega^2\sin^2(\omega t - \delta)$$

We have that $\omega^2 = \frac{k}{m}$ so:

$$\frac{k}{2}A^2\cos^2(\omega t - \delta) + \frac{k}{2}A^2\sin^2(\omega t - \delta) = \frac{kA^2}{2}$$

And hence the total energy is a constant; we can therefore see that $\frac{\mathrm{d}E}{\mathrm{d}t} = 0$ and that the total energy is conserved. □

**Problem 9.3.** *Damped oscillatons are described by the ODE $\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0$. Show that for overdamped motion, the decay "constant" (decay parameter) for a damped oscillator decreases with increasing friction. Sketch the decay parameters vs. $\beta$ for the whole range of $\beta$.*

*Solution.* We guess a solution of the form $x(t) = \exp(rt)$. We can therefore generate a characteristic equation for $r$, by substituting in the solution into the ODE and then cancelling out the exponential terms (as these can never be zero). Hence, we have:

$$r^2 + 2\beta r + \omega_0^2 = 0$$

This is a quadratic equation with solutions:

$$r = -\beta \pm \sqrt{\beta^2 - \omega_0^2}$$

There are now different possibilities for the damping depending on the ratio of $\beta$ and $\omega_0$. For $\beta > \omega_0$, the oscillator is overdamped. For $\beta < \omega_0$, the oscillator is underdamped. For $\beta = \omega_0$, the oscillator is critically damped. Let us now consider the overdamped case. Then, we have that the discriminant in the equation above $\beta^2 - \omega_0^2$ is positive, and the general solution is the sum of decaying exponentials:

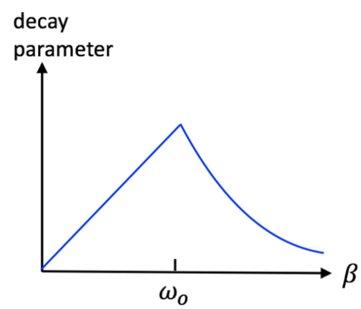
$$x(t) = C_1\exp(r_1 t) + C_2\exp(r_2 t)$$

where $r_1 = -\beta + \sqrt{\beta^2 - \omega^2}, r_2 = -\beta - \sqrt{\beta^2 - \omega^2}$. The dominant term will be the $\exp(r_1 t)$ term as this decays more slowly. Therefore, the decay parameter is:

$$-\beta + \sqrt{\beta^2 - \omega_0^2}$$

Which decreases with increasing $\beta$. We will also have a decay parameter for the underdamped case. In this case, the decay parameter is just $\beta$, as the discriminant is negative, and hence the solution $x(t)$ is composed of an oscillating part (the imaginary exponential part from the square root) and a real exponentially decaying part ($\exp(-\beta t)$). Hence the decay parameter increases linearly in this regime. Overall, we obtain a plot that looks like the follows:

Where the decay parameter is maximized when $\beta = \omega_0$, where we have critical damping.




□

**Problem 9.4.** *Show that for critical damping, the solution $x(t) = te^{rt}$ solves the ODE $\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0$. Find $r$. Write the general solution.*

*Solution.* For critical damping, we have that $\beta = \omega_0$ and we can calculate $r$ to be:

$$r = -\beta \pm \sqrt{\beta^2 - \beta^2} = -\beta$$

Hence one solution is given by:

$$x_1(t) = C_1 \exp(-\beta t)$$

And we check the Ansatz provided in the question to verify it as a second solution. The first time derivative is given by:

$$\dot{x} = \exp(rt) + rt\exp(rt)$$

And the second time derivative as:

$$\ddot{x} = r\exp(rt) + r\exp(rt) + r^2 t\exp(rt) = 2r\exp(rt) + r^2 t\exp(rt)$$

So substituting this into the ODE, with $r = -\beta$:

$$-2\beta\exp(-\beta t) + \beta^2 t\exp(-\beta t) + 2\beta\left(\exp(-\beta t) - \beta t\exp(-\beta t)\right) + \beta^2 t\exp(-\beta t) = 0$$

So we can see that this is indeed a solution! Hence the general solution is given by:

$$x(t) = C_1\exp(-\beta t) + C_2 t\exp(-\beta t)$$

□

**Problem 9.5.** *Find the condition on $C$ for $z(t) = Ce^{i\omega t}$ to be a solution to $\ddot{z} + 2\beta\dot{z} + \omega_0^2 z = f_0 e^{i\omega t}$. Express the coefficient $C$ as $Ae^{-i\delta}$ and find $A$ and $\delta$.*

*Solution.* Let us plug in $z(t)$ into the ODE:

$$(-\omega^2 + 2i\beta\omega + \omega_0^2)C\exp(i\omega t) = f_0\exp(i\omega t)$$

We may cancel out the exponentials on both sides:

$$(-\omega^2 + 2i\beta\omega + \omega_0^2)C = f_0$$

Hence we find:

$$C = \frac{f_0}{-\omega^2 + 2i\beta\omega + \omega_0^2} = A\exp(-i\delta)$$

We have that $A^2 = CC^*$ and so:

$$A^2 = \frac{f_0}{-\omega^2 + 2i\beta\omega + \omega_0^2}\frac{f_0}{-\omega^2 - 2i\beta\omega + \omega_0^2} = \frac{f_0^2}{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}$$

To get the phase, we see that:

$$f_0\exp(i\delta) = A(\omega_0^2 - \omega^2 + 2i\beta\omega)$$

And solving (using some knowledge of complex numbers and extracting their phase) we get:

$$\delta = \arctan\left(\frac{\text{Im}}{\text{Re}}\right) = \arctan\left(\frac{2\beta\omega}{\omega_0^2 - \omega^2}\right)$$

□

**Problem 9.6.** *Write the general solution to $\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = f_0\cos(\omega t)$ for the underdamped case. What are the two undetermined constants?*

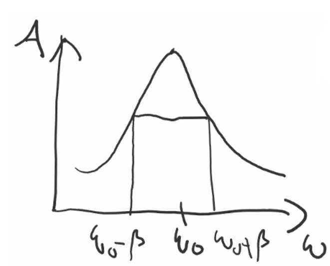
*Solution.* Writing the general solution, we have:

$$x(t) = A\cos(\omega t - \delta) + C_1\exp(r_1 t) + C_2\exp(r_2 t)$$

Where the first term is the particular solution (comes from the driving) and the exponentials come from the homogenous solution. The former is the long-term oscillatory behavior, the latter is the transient solution (the exponentials decay away with time). □

**Problem 9.7.** *Show that the Q-factor given by the ratio of the width to the mean of the resonance curve is equal to $\pi$ times the number of oscillations in one decay time.*

*Solution.* When $\omega \sim \omega_0$ (when the driving frequency approaches the natural frequency), we notice a resonance phenomenon, where the amplitude of the driven oscillator gets very large (technically it is slightly off from $\omega = \omega_0$, but to good approximation it is at the natural frequency of the system). This is pictured below:




We can also consider the width of the resonance peak. It is defined as:

$$Q = \frac{\omega_0}{2\beta}$$

Stronger damping implies a smaller quality factor and a narrower peak, and weak damping implies a larger quality factor and a broadened peak. To prove the claim provided in the question, we consider that the decay time is given by $\tau = \frac{1}{\beta}$ and the period is $T = \frac{2\pi}{\omega_0}$, so plugging this in we can immediately see that:

$$Q = \frac{\omega_0}{2\beta} = \frac{\frac{1}{\beta}\pi}{\frac{2\pi}{\omega_0}} = \pi\frac{\tau}{T}$$

which is the desired result. □

**Problem 9.8.** *Find the phase shift at resonance when the driving frequency $\omega$ is varied. Sketch the phase shift $\delta$ vs. $\omega$.*

*Solution.* The phase shift is given by $\delta = \frac{\pi}{2}$ (perfectly out of phase). We can see this as the phase shift is given above by:

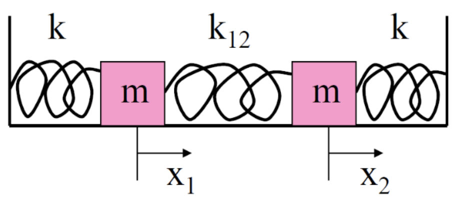
$$\delta = \arctan\left(\frac{2\beta\omega}{\omega_0^2 - \omega^2}\right)$$

And at resonance we have $\omega \sim \omega_0$, so therefore the argument fo the arctan goes to infinity, and hence the value of $\delta$ goes to $\frac{\pi}{2}$. □

# 10 Lecture 10

## 10.1 Lecture Notes - Intro to Coupled Oscillators

### 10.1.1 Analysis with Newtonian Mechanics




A system of two coupled harmonic oscillators has spring constants $k$ (left spring), $k_{12}$ (middle spring) and $k$ (right spring). The displacements of the blocks are measured from equilibrium. The forces on the blocks are therefore given by:

$$F_1 = -kx_1 - k_{12}(x_1 - x_2)$$
$$F_2 = -kx_2 - k_{12}(x_2 - x_1)$$

The Lagrangian formulation gives the same result; see worksheet!

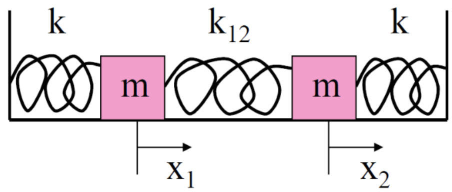
## 10.2 Worksheet - Intro to Coupled Oscillators

**Problem 10.1.** *Give some examples of coupled oscillators.*

*Solution.*

- Masses coupled together by springs
- A crystal lattice, which can be approximated as atoms being connected by springs in a lattice structure
- Molecules (e.g. CO2) which we can treat as a carbon atom connected linearly to two oxygen atoms.

□




**Problem 10.2.** *Find the Lagrangian for the two coupled oscillators(two masses connected by springs). Find the equations of motion.*

*Solution.* The Lagrangian is given by:

$$\mathcal{L} = T - U = \frac{m}{2}\left(\dot{x}_1^2 + \dot{x}_2^2\right) - \left[\frac{k}{2}x_1^2 + \frac{k_{12}}{2}(x_1 - x_2)^2 + \frac{k}{2}x_2^2\right]$$

To obtain the equations of motion, we use the EL equations:

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{x}_1} = m\ddot{x}_1 = \frac{\partial \mathcal{L}}{\partial x_1} = -kx_1 - k_{12}x_1 + k_{12}x_2$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{x}_2} = m\ddot{x}_2 = \frac{\partial \mathcal{L}}{\partial x_2} = -kx_2 - k_{12}x_2 + k_{12}x_1$$

Which agrees with our analysis using the Newtonian formulation. This is clearly a coupled system of ODEs. To start solving this, we start by writing this in a nicer form, defining a column vector $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. This transforms things into a matrix equation:

$$\begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} = -\begin{pmatrix} k + k_{12} & -k_{12} \\ -k_{12} & k + k_{12} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Which we can write as:

$$\mathbb{M}\ddot{\mathbf{x}} = -\mathbb{K}\mathbf{x}$$

□

**Problem 10.3.** *Using the complex quantity* $\mathbf{z} = \mathbf{a}\exp(i\omega t)$*, show that the equation for the normal modes can be written as* $(\mathbb{K} - \omega^2\mathbb{M})\mathbf{a} = 0$.

*Solution.* We define the complex quantity $\mathbf{z}$ as above (such that $\mathbf{x} = \mathrm{Re}\,\mathbf{z}$, and then we have:

$$\mathbb{M}\ddot{\mathbf{z}} = \mathbb{M}\mathbf{a}(-\omega^2)\exp(i\omega t) = -\mathbb{K}\mathbf{a}\exp(i\omega t)$$

Then cancelling out the exponentials, we have:

$$\mathbb{K}\mathbf{a} = \mathbb{M}\omega^2\mathbf{a}$$

Rearranging, we have:

$$(\mathbb{K} - \omega^2\mathbb{M})\mathbf{a} = 0$$

This is an **eigenvalue problem**, which we are familiar with from linear algebra. We note that here, $\mathbb{M}$ is just an identity matrix multiplied by $m$. □

**Problem 10.4.** *Write out the characteristic equation. What are the roots? (i.e. the eigenfrequencies).*

*Solution.* To solve this eigenvalue problem (i.e. for the system to have nontrivial solutions) we require that $\det(\mathbb{K} - \omega^2\mathbb{M}) = 0$. This is a polynomial/characteristic equation, for which the solution are the eigenvalues. For $\mathbb{M}$, $\mathbb{K}$ as we have defined them, this looks like:

$$\det\Big(\mathbb{K} - \omega^2\mathbb{M}\Big) = (k + k_{12} - m\omega^2)^2 - k_{12}^2 = 0$$

Where the first term is the product of the diagonals and the second term is the product of the off diagonals. Factoring, we have:

$$(k - m\omega^2)(k + 2_{k12} - m\omega^2) = 0$$

So the characterstic equation hence has the two roots of:

$$\omega_1 = \sqrt{\frac{k + 2k_{12}}{m}}, \omega_2 = \sqrt{\frac{k}{m}}$$

The types of motion described by these two eigenfrequencies are as follows. For $\omega_1$, the blocks move with the same frequency and exactly out of phase. For $\omega_2$, the blocks move with the same frequency, and exactly in phase. We will see why this is in the last two question, by solving for the amplitudes $a_1, a_2$. □

**Problem 10.5.** *Find the normal mode corresponding to the eigenfrequency* $\sqrt{k/m}$.

*Solution.* To find the normal mode, we solve for the eigenvector corresponding to the above eigenfrequency/eigenvalue:

$$(\mathbb{K} - \omega_2\mathbb{M})\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} k_{12} & -k_{12} \\ -k_{12} & k_{12} \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = 0$$

Or equivalently:

$$k_{12}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = 0$$

From which we can see that the restriction on $a_1, a_2$ (remeber that they are complex) is that:

$$a_1 = a_2 = A\exp(-i\delta)$$

Hence solving for $\mathbf{z}$:

$$\mathbf{z} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}\exp(i\omega_2 t) = \begin{bmatrix} A \\ A \end{bmatrix}\exp\big(i(\omega_2 t - \delta)\big)$$

Therefore, finding $\mathbf{x}$ we have:

$$\mathbf{x}_{II} = \operatorname{Re}\mathbf{z} = \begin{bmatrix} A \\ A \end{bmatrix}\cos(\omega_2 t - \delta)$$

This is an eigenmode, at the lower frequency. We can see from this that the two masses oscillate in phase with each other. □

**Problem 10.6.** *Find the normal mode corresponding to the eigenfrequency* $\sqrt{(k + 2k_{12})/m}$.

*Solution.* Similarly solving for the eigenvector, we have:

$$(\mathbb{K} - \omega^2\mathbb{M})\mathbf{a} = \begin{bmatrix} -k_{12} & -k_{12} \\ -k_{12} & -k_{12} \end{bmatrix}\mathbf{a} = -k_{12}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = 0$$

Therefore we obtain the requirement that $a_1 = -a_2$, and hence:

$$\mathbf{x}_I = \begin{bmatrix} A \\ -A \end{bmatrix}\cos(\omega_1 t - \delta)$$

□

**Problem 10.7.** *What is the general solution?*

*Solution.* The general solution is the sum of the eigenmodes:

$$\mathbf{x}(t) = \mathbf{x}_I(t) + \mathbf{x}_{II}(t)$$

□

**Problem 10.8.** *If block 1 oscillates while block 2 is held fixed, what is the frequency of oscillations?*

*Solution.* The frequency of oscillations of the first block would be given by $\omega_0 = \sqrt{\frac{k+k_{12}}{m}}$; the reasoning for this is the block feels a restoring force $-kx$ from the left, restoring force $k_{12}x$ from the right, which leads to an effective spring constant $k + k_{12}$ and hence leads to the solution as stated. □

**Problem 10.9.** *How does the uncoupled frequency above compare to the two eigenfrequencies?*

*Solution.*

$$\omega_2 < \omega_0 < \omega_1$$

□

$\omega_2$ is the frequency of the lowest mode, the blocks are in phase. The middle frequency is where we fix one block and just let the other oscillate. $\omega_1$ corresponds to the higher eigenmode, with the blocks oscillating out of phase, at the highest frequency. Next day, we will look at coupled pendulums and normal coordinates (which is changing bases to diagonalize our matrices), where we obtain the useful result that the normal coordinates are independent of one another.

**Problem 10.10.** *If the coupling is weak ($k_{12} \ll k$), what is an approximate expression for the frequency $\omega_0 = \sqrt{\frac{k+k_{12}}{m}}$?*

*Solution.* We apply Taylor expansion:

$$\omega_0 = \sqrt{\frac{k}{m}\left(1 + \frac{k_{12}}{k}\right)} \approx \sqrt{\frac{k}{m}}\left(1 + \frac{k_{12}}{2k}\right)$$

□

# 11 Lecture 11

## 11.1 Lecture Notes - Normal Coordinates

### 11.1.1 Normal Coordinates of Spring-Mass system

From last day, we recall the general solution of the coupled spring mass system:

$$x_1(t) = A_1 \exp(i\omega_1 t) + A_2 \exp(i\omega_2 t)$$

$$x_2(t) = -A_1 \exp(i\omega_1 t) + A_2 \exp(i\omega_2 t)$$

We would like a transformation such that for each mode, there is only one non-zero coefficient. The first normal coordinate is given by:

$$\xi_1(t) = \frac{1}{2}(x_1 - x_2)$$

and the second by:

$$\xi_2(t) = \frac{1}{2}(x_1 + x_2)$$

For mode I ($A_1 = A$ and $A_2 = 0$), we have that $\xi_1(t) = A\cos(\omega t - \delta)$, and $\xi_2(t) = 0$. For mode II ($A_1 = 0$ and $A_2 = A$), we have that $\xi_2(t) = A\cos(\omega t - \delta)$ and $\xi_1(t) = 0$. Why is this useful? Because each normal mode describes an oscillation at a single frequency.

### 11.1.2 Generalization

The motion of a system of coupled oscillators can be described with normal coordinates:

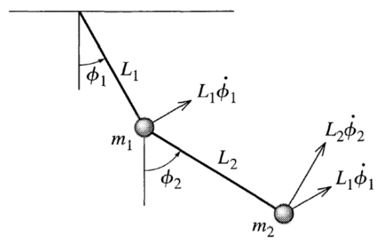
$$\mathbf{q}(t) = \sum_{i=1}^{n} \mathbf{a}_i \xi_i(t)$$

where each normal coordinate $\xi_i(t)$ satisfies:

$$\ddot{\xi}_i + \omega_i^2 \xi_i = 0$$

as each of the $\xi_i$s are an independent harmonic oscillator with a certain frequency. We will return to this point on Friday.

## 11.2 Worksheet - Double Pendulum




**Problem 11.1.** *Find the potential energy of the double pendulum.*

*Solution.* The height of the first mass is given by $L_1(1-\cos\phi_1)$, and the height of the second mass is given by $L_1(1-\cos\phi_1) + L_2(1-\cos\phi_2)$ by trigonometry (the second mass is just the height of the first + the relative height from the first mass). Hence the total potential energy is given by:

$$U = U_1 + U_2 = m_1 g L_1(1-\cos\phi_1) + m_2 g L_1(1-\cos\phi_1) + m_2 g L_2(1-\cos\phi_2)$$

□

**Problem 11.2.** *Write the kinetic energy in terms of $\dot{\phi}_1^2$ and $\dot{\phi}_2^2$*

*Solution.* The first mass is easy; it is just $T_1 = \frac{1}{2} m_1 L^2 \dot{\phi}_1^2$. The second term is slightly harder as the position vector is given by the sum of the vector from the fixed point to $m_1$ plus the vector from $m_1$ to $m_1$. Expanding this out, this yields total kinetic energy:

$$T = \frac{m_1}{2} L_1 \dot{\phi}_1^2 + \frac{m_2}{2}\left(L_1^2\dot{\phi}_1^2 + L_2^2\dot{\phi}_2^2 + 2L_1L_2\dot{\phi}_1\dot{\phi}_2\cos(\phi_1-\phi_2)\right)$$

Note the cross term we get at the end. □

**Problem 11.3.** *Write the kinetic and potential energies in the simplifying case where the angles $\phi_i$ and their derivatives $\dot{\phi}_i$ are small.Find the two equations of motion in this case.*

*Solution.* Time for the small angle approximation! In the limit of small angles, $\cos\phi \approx 1 - \frac{\phi^2}{2}$ and so the potential energy reduces to:

$$U = \frac{m_1 + m_2}{g} L_1 \phi_1^2 + \frac{m_2}{2} g L_2 \phi_2^2$$

For the kinetic term, the only cosine that shows up is the last term. We do have to be slightly careful here as expanding this out, we already get terms that are nonlinear (e.g. $(\phi_1-\phi_2)^2$). We only want terms of order 2 and lower in our expression, (e.g. $\phi^2, \phi\dot{\phi}, \dot{\phi}^2$) so in our expansion of $\cos(\phi_1 - \phi_2)$ we only keep the highest order term (e.g. just 1)! Thus the kinetic energy reduces to:

$$T = \frac{m_1 + m_2}{2} L_1^2 \dot{\phi}_1^2 + m_2 L_1 L_2 \dot{\phi}_1 \dot{\phi}_2 + \frac{m_2}{2} L_2^2 \dot{\phi}_2^2$$

□

**Problem 11.4.** *From the two linearized equations of motion for the double pendulum, find the corresponding matrices $\mathbb{M}$ and $\mathbb{K}$.*

*Solution.* The EL equations yield:

$$(m_1 + m_2) L_1^2 \ddot{\phi}_1 + m_2 L_1 L_2 \ddot{\phi}_2 = -(m_1 + m_2) g L_1 \phi_1$$

$$m_2 L_1 L_2 \ddot{\phi}_1 + m_2 L_2^2 \ddot{\phi}_2 = -m_2 g L_2 \phi_2$$

Which we can see are (fortunately) linear from our small angle simplification above. We can write these equations in matrix form:

$$\mathbb{M}\ddot{\boldsymbol{\phi}} = -\mathbb{K}\boldsymbol{\phi}$$

Writing out these matrices, we have:

$$\mathbb{M} = \begin{bmatrix} (m_1 + m_2) L_1^2 & m_2 L_1 L_2 \\ m_2 L_1 L_2 & m_2 L_2^2 \end{bmatrix}$$

$$\mathbb{K} = \begin{bmatrix} (m_1 + m_2) g L_1 & 0 \\ 0 & m_2 g L_2 \end{bmatrix}$$

□

**Problem 11.5.** *What are the eigenfrequencies and eigenvectors (the normal modes) when $m_1 = m_2 = m$ and $l_1 = l_2 = l$?*

*Solution.* In the case that all the masses and lengths are the same, we may write these matrices as:

$$\mathbb{M} = mL^2 \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

$$\mathbb{K} = mL^2 \begin{bmatrix} 2\omega_0^2 & 0 \\ 0 & \omega_0^2 \end{bmatrix}$$

Where we introduce $\omega_0 = \sqrt{\frac{g}{l}}$. Exactly as we did last day, we make an Ansatz:

$$\mathbf{z} = \mathbf{a} \exp(i\omega t)$$

This generates a characteristic equation for the eigenvalues:

$$\det\Big(\mathbb{K} - \omega^2 \mathbb{M}\Big) = 0$$

Expanding this out, we have:

$$\det\left( mL^2 \begin{bmatrix} 2(\omega_0^2 - \omega^2) & -\omega^2 \\ -\omega^2 & (\omega_0^2 - \omega^2) \end{bmatrix} \right) = 0$$

Which yields the equation:

$$2(\omega_0^2 - \omega^2)^2 - \omega^4 = \omega^4 - 4\omega_0^2\omega^2 + 2\omega_0^4 = 0$$

This is a quadratic equation in $\omega^2$, which has roots:

$$\omega_{1/2}^2 = 2\omega_0^2 \pm \sqrt{\frac{16\omega_0^2 - 8\omega_0^2}{4}} = \omega_0^2\left(2 \pm \sqrt{2}\right)$$

Now solving for the normal modes (e.g. the eigenvectors), we get solutions:

$$\phi_I(t) = A\begin{bmatrix} 1 \\ \sqrt{2} \end{bmatrix}\cos(\omega_1 t - \delta)$$

$$\phi_I I(t) = A\begin{bmatrix} 1 \\ -\sqrt{2} \end{bmatrix}\cos(\omega_2 t - \delta)$$

We see that this is similar to the results of last day (one mode with the two masses in phase, one mode with the two masses out of phase), but with a difference that the lower mass has a greater amplitude.
*Remark:* After linearization, the problem solving method reduces to what we saw last day; without linearization, the double pendulum system is chaotic! Perhaps to be revisited at the end of this class. But for small deviations from local minima in the potential, this method is generally feasible. □

# 12 Lecture 12

## 12.1 Lecture Notes - Weakly Coupled Oscillators

Recall from our work two lectures ago that for the 2-mass 3-springs problem, the eigenfrequencies were determined to be:

$$\omega_1 = \sqrt{\frac{k + 2k_{12}}{m}}, \quad \omega_2 = \sqrt{\frac{k}{m}}$$

Now, we make the assumption that $k_{12} \ll k$, i.e. that the middle spring joining the masses is quite weak. In this limit, we can clearly see that $\omega_1 \approx \omega_2$. This generates an interesting physical case; here we can define a "middle frequency" which is the average of the two eigenfrequencies above:

$$\omega_0 = \frac{\omega_1 + \omega_2}{2}$$

We could then rewrite $\omega_1, \omega_2$ in terms of $\omega_0$ and a small parameter $\epsilon$:

$$\omega_1 = \omega_0 + \epsilon$$

$$\omega_2 = \omega_0 - \epsilon$$

The general solution could then be written as:

$$\mathbf{z}(t) = C_1\begin{bmatrix} 1 \\ -1 \end{bmatrix}\exp\bigl(i(\omega_0 + \epsilon)t\bigr) + C_2\begin{bmatrix} 1 \\ 1 \end{bmatrix}\exp\bigl(i(\omega_0 - \epsilon)t\bigr)$$

We can factor this expression, recognizing common terms:

$$\mathbf{z}(t) = \left(C_1\begin{bmatrix} 1 \\ -1 \end{bmatrix}\exp(i\epsilon t) + C_2\begin{bmatrix} 1 \\ 1 \end{bmatrix}\exp(-i\epsilon t)\right)\exp(i\omega_0 t)$$

Now, suppose that $C_1 = C_2 = \frac{A}{2}$ (i.e. the two modes are excited to be the same amplitude). We then have that:

$$\mathbf{z}(t) = \frac{A}{T}\begin{bmatrix} \exp(i\epsilon t) + \exp(-i\epsilon t) \\ \exp(-i\epsilon t) - \exp(i\epsilon t) \end{bmatrix}\exp(i\omega_0 t)$$

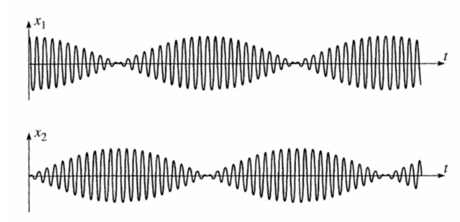
Then using Euler's formula:

$$\mathbf{z}(t) = A \begin{bmatrix} \cos(\epsilon t) \\ -i\sin(\epsilon t) \end{bmatrix} \exp(i\omega_0 t)$$

Now taking the real part of this:

$$\operatorname{Re}\mathbf{z}(t) = \mathbf{x}(t) = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} A\cos(\epsilon t)\cos(\omega_0 t) \\ A\sin(\epsilon t)\sin(\omega_0 t) \end{bmatrix}$$

This result corresponds to a fast oscillation (e.g. at frequency $\omega_0$) modulated by a slower oscillation (at frequency $\epsilon$ which is small by assumption). When graphed, this looks like:




Where we see "beats"! Exactly like with AM radio waves, we see the frequency stay constant but the amplitude going up and down with time.

## 12.2 Worksheet - Generalized Coupled Oscillators

**Problem 12.1.** *Show that for any system undergoing small oscillations, the kinetic and potential energies can be written as quadratic forms of the generalized velocities and coordinates, respectively.*

*Solution.* Our vector of $n$ generalized coordinates is given by:

$$\mathbf{q} = (q_1, \cdots, q_n)$$

Which may be related to the Cartesian coordinates $\mathbf{r}_i = \mathbf{r}_i(q_1, \cdots, q_n)$. Previously, we solved for the kinetic energy:

$$T(\mathbf{q}, \dot{\mathbf{q}}) = \frac{1}{2}\sum_{j,k} A_{jk}(q)\dot{q}_j\dot{q}_k$$

Which is a quadratic form. When we actually evaluate this, we only keep the constant terms from $A_{jk}$ so we get the familiar kinetic energy:

$$T(\dot{\mathbf{q}}) = \frac{1}{2}\sum_{j,k} M_{jk}\dot{q}_j\dot{q}_k = \frac{1}{2}\dot{\mathbf{q}}^T\mathbb{M}\dot{\mathbf{q}}$$

Where things have been written as a matrix product in the last line (replacing the double sum). The potential energy we can do a similar way. In general, the potential energy depends on all of the positions of the particles. We can taylor expand this around an equilibrium value, where $\mathbf{q} = \mathbf{0}$:

$$U(\mathbf{q}) = U(\mathbf{0}) + \sum_j \frac{\partial U}{\partial q_j} q_j + \frac{1}{2}\sum_{j,k}\frac{\partial^2 U}{\partial q_j \partial q_k} q_j q_k + \cdots$$

By the fact that we take $\mathbf{0}$ as a minimum, the second term vanishes. We can also neglect the constant term, so:

$$U(\mathbf{q}) = \frac{1}{2}\sum_{j,k} K_{jk} q_j q_j = \frac{1}{2}\mathbf{q}^T\mathbb{K}\mathbf{q}$$

Hence, we have obtained quadratic forms for both the kinetic and potential energies! This is just a general description that we could do for any system. $\square$

**Problem 12.2.** *Prove that the equations of motion for the eigenfrequencies and eigenvectors are* $\det\{\mathbb{K} - \omega^2\mathbb{M}\} = 0$ *and* $(\mathbb{K} - \omega^2\mathbb{M})\mathbf{a} = 0$.

*Solution.* Given our system, we have $m$ Lagrange equations in general. These are given by:

$$\sum_{j=1}^{m} M_{ij}\ddot{q}_j = \frac{\partial U}{\partial q_i} = -\sum_{j=1}^{m} K_{ij}q_j$$

Which we may rewrite as:

$$\mathbb{M}\ddot{\mathbf{q}} = -\mathbb{K}\mathbf{q}$$

As done previously, let $\mathbf{q}(t) = \operatorname{Re}\mathbf{z}(t)$, where $\mathbf{z}(t) = \mathbf{a}\exp(i\omega t)$. Plugging this in and cancelling out the time derivatives, we obtain an algebraic set of equations:

$$(\mathbb{K} - \omega^2\mathbb{M})\mathbf{a} = \mathbf{0}$$

Which we can get the eigenvalues from:

$$\det\Big(\mathbb{K} - \omega^2\mathbb{M}\Big) = 0$$

We know that these matrices are positive semidefinite, so the eigenvalues will be positive (or perhaps zero). □

**Problem 12.3.** *Show generally that each normal coordinate* $\xi_i$ *oscillates at its own normal frequency* $\omega_i$*, uncoupled to the other normal coordinates.*

*Solution.* Normal modes are given by:

$$\mathbf{q}_i(t) = \mathbf{a}_i\cos(\omega_i t - \delta_i)$$

I.e. each have an oscillatory solution with characterstic frequency $\omega_i$. It also satisfies the eigenvalue problem from above:

$$\mathbb{K}\mathbf{a}_i = \omega_i^2\mathbb{M}\mathbf{a}_i$$

The normal mode expansion is then given by:

$$\mathbf{q}(t) = \sum_{i=1}^{m}\xi_i(t)\mathbf{a}_i$$

Where $\xi_i(t)$ are the weights of the modes in the general solution representation. We note that $\mathbf{a}_i$ forms a basis in which we can expand the solution, which is nice because in this basis we can solve the equations of motion for the normal coordinates trivially. We have that $\mathbf{q}(t)$ satisfies:

$$\mathbb{M}\ddot{\mathbf{q}} = -\mathbb{K}\mathbf{q}$$

Let us plug in this normal mode expansion into the equation of motion. This yields:

$$\sum_{i=1}^{m}\ddot{\xi}_i(t)\mathbb{M}\mathbf{a}_i = -\sum_{i=1}^{m}\xi_i(t)\mathbb{K}\mathbf{a}_i$$

But $\mathbb{K}\mathbf{a}_i = \omega_i^2\mathbb{M}\mathbf{a}_i$ (eigenvalue) so:

$$\sum_{i=1}^{m}\ddot{\xi}_i(t)\mathbb{M}\mathbf{a}_i = -\sum_{i=1}^{m}\xi_i(t)(-\omega_i^2)\mathbb{M}\mathbf{a}_i$$

This must hold for the components individually, i.e. this must hold for each value of $i$. We therefore obtain the identity:

$$\ddot{\xi}_i(t) = -\omega_i^2\xi_i(t)$$

From which we recover the fact that the normal coordinates satisfy equations for simple harmonic motion. □

**Problem 12.4.** *Check for the 2-mass-3-springs problem that the normal coordinates are the coefficients of the eigenvector expansion for the displacements* $\binom{x_1}{x_2}$.

*Solution.* The eigenvectors on Monday were given by:

$$\mathbf{a}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \mathbf{a}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

The normal mode expansion is then:

$$\mathbf{q} = \xi_1 \mathbf{a}_1 + \xi_2 \mathbf{a}_2$$

So therefore:

$$\mathbf{q} = \begin{bmatrix} \xi_1 + \xi_2 \\ \xi_1 - \xi_2 \end{bmatrix} = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$$

From which we obtain the normal coordinates:

$$\xi_1 = \frac{q_1 + q_2}{2}, \quad \xi_2 = \frac{q_1 - q_2}{2}$$

□

**Problem 12.5.** *Using the expansion of* $(x_1, x_2)$ *in terms of eigenvectors,find the normal coordinates for the double pendulum.*

*Solution.* For the double pendulum, we had the general solution:

$$\phi_1(t) = A_1 \exp(i\omega_1 t) + A_2 \exp(i\omega_2 t), \quad \phi_2(t) = A_1\sqrt{2}\exp(i\omega_1 t) - A_2\sqrt{2}\exp(i\omega_2 t)$$

This corresponds to eigenvectors:

$$\mathbf{a}_1 = \begin{bmatrix} 1 \\ \sqrt{2} \end{bmatrix}, \mathbf{a}_2 \begin{bmatrix} 1 \\ -\sqrt{2} \end{bmatrix}$$

The normal mode expansion is given by:

$$\boldsymbol{\phi} = \xi_1 \mathbf{a}_1 + \xi_2 \mathbf{a}_2 = \begin{bmatrix} \xi_1 + \xi_2 \\ \sqrt{2}\xi_1 - \sqrt{2}\xi_2 \end{bmatrix} = \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}$$

Solving for $\xi_1, \xi_2$ we then solve the system:

$$\xi_1 + \xi_2 = \phi_1, \quad \xi_1 - \xi_2 = \frac{\phi}{\sqrt{2}}$$

Adding and subtracting these equations, we get:

$$\xi_1 = \frac{\left(\phi_1 + \frac{\phi_2}{\sqrt{2}}\right)}{2}$$

$$\xi_2 = \frac{\left(\phi_1 - \frac{\phi_2}{\sqrt{2}}\right)}{2}$$

□

# 13 Lecture 13

## 13.1 Worksheet - Review of Weeks 1-4

**Problem 13.1.** *Which of these problems must be solved using the calculus of variations?*
*1. Find the period of small oscillations for a particle sliding (without friction) on the inside of a sphere.*
*2. Find the surface with fixed area that encloses the maximum volume.*
*3. Find the path between two points that minimizes the time for a particle to slide (without friction) between the points.*
*4. Find the path of a projectile (with no air resistance) that leads to the maximum range.*

*Solution.* Exactly two. For 1, we looked at the equations of motion and Taylor expanded around the minimum for small $\phi$. So, we aren't looking for the path that minimizes some functional in this case, hence its not particularly a variational problem. For 2, we have an optimization problem; we are trying to extremize/maximize a volume. We can write an expression for the surface area, and add in a constraint (e.g. we could use Lagrange multipliers for example). We could optimize this with the Calculus of variations. 3 is the brachistochrone problem, obviously yes. 4 does not require varations; this is just a question of initial conditions of the trajectory. □

**Problem 13.2.** *A calculus of variations problem requires minimizing*

$$J[y(x)] = \int_{x_1}^{x_2} f\left[y(x); y'(x); x\right] dx$$

*When we solve Euler's equation*

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} = 0$$

*what do we learn?*

*Solution.* We learn of the path $y(x)$ that minimizes $J[y(x)]$; the EL equation gives a differential equation for the path $y(x)$ which minimizes $J[y(x)]$. □

**Problem 13.3.** *What is the Lagrangian of a particle of mass $m$ attached to a spring with spring constant $k$?*

*Solution.* $\mathcal{L} = T - U = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2$ (the kinetic energy term minus the potential energy term). □

**Problem 13.4.** *What is the Lagrangian of a pendulum of mass $m$, length $l$? Assume the potential energy is zero when $\theta$ is zero.*

*Solution.* $\mathcal{L}(\theta, \dot{\theta}, t) = \frac{1}{2}ml^2\dot{\theta}^2 - mgl(1 - \cos\theta)$ □

**Problem 13.5.** *For which of these systems could you use Lagrange's equations of motion?*
*1. A double pendulum: a pendulum (mass* m*, length 1 ) has a second pendulum (mass* m*, length l) connected to its bob.*
*2. A projectile moves in two dimensions with gravity and air resistance.*
*3. A bead slides without friction on a circular, rotating wire.*

*Solution.* Everything but 2 works; with 2, we have friction/a non-conservative force and hence the Lagrange equations no longer apply (though we can add in a correction to account for this). □

**Problem 13.6.** *A particle moves in one dimension with Lagrangian $\mathcal{L} = T - U$. Suppose we shift the potential energy $U$ by a constant $C$. What changes?*

*Solution.* The value of $S$ changes ($S$ depends on the Lagrangian) but the physical path $x(t)$ taken by the particle remains invariant (that is, the path that gives $\delta S = 0$); one way of seeing this is the equations of motion are given by derivatives of the Lagrangian, which would remove the effects of any constants. □

**Problem 13.7.** *A bead of mass $m$ slides on a circular wire of radius $R$, which rotates about a vertical axis with angular velocity $\Omega$. The equation of motion of the bead is*

$$\ddot{\theta} + \frac{g}{R}\sin\theta - \Omega^2 \sin\theta\cos\theta = 0$$

*What are the equilibrium values of $\theta$?*

*Solution.* $\theta = 0, \pi$ and $\theta = \pm\arccos\left(\frac{g}{R\Omega^2}\right)$. □

**Problem 13.8.** *The equation of motion for small motions about the equilibrium $\theta = \arccos\left(\frac{g}{R\Omega^2}\right)$ above is given by:*

$$\ddot{\theta} + \Omega^2 \sin_0^\theta \theta = 0$$

*What is the oscillation frequency of the bead?*

*Solution.* $\omega = \Omega \sin\theta_0$ □

**Problem 13.9.** *Which of these systems are holonomic? E.g. which of these systems have a constraint that can be written as $f(q_1, \ldots, q_n, t) = 0$?*
*1. The double pendulum, but with the lower mass attached by a spring instead of a spring.*
*2. The motion of a hockey puck around a frictionless air hockey table*
*3. A bead moving frictionless on a circular wire hoop spinning at fixed angular velocity.*

*Solution.* A and C. With C, this is obviously possible (we constrain the radius). With B, the constraint is an inequality (e.g. the normal force is just such that the mass stays at the level of the table), which makes it non-holonomic. With A, we have that the lower mass can move more freely; the system is still holonomic, we just got rid of the constraint on the second mass. Note that if forces are dissipative, then we can also not write constrains as holonomic. □

**Problem 13.10.** *What is the constraint equation?*

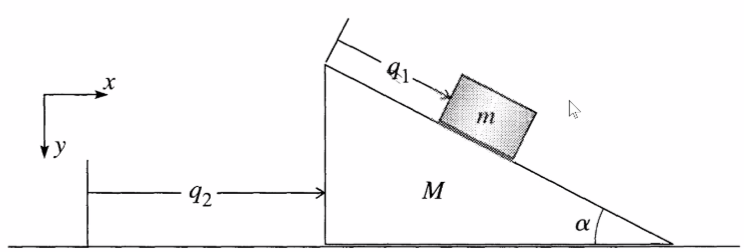



Figure 7.8 A block of mass $m$ slides down a wedge of mass $M$, which is free to slide over the horizontal table.

*Solution.* $f(x,y) = \frac{y}{x} - \tan(\alpha) = 0$. A typical holonomic constraint. □

**Problem 13.11.** *A particle of mass $m$ slides on the outside of a cylinder of radius $a$. A good choice of generalized coordinates is $(r, \theta)$. What is the constraint equation?*

*Solution.* $f(r,\theta) = r - a = 0$ □

**Problem 13.12.** *What can you conclude from the fact that $\dfrac{\partial \mathcal{L}}{\partial \dot{x}_i}$ is constant for all $i$?*

*Solution.* Momentum is conserved. □

**Problem 13.13.** *How can we rewrite:*

$$\int_{t_1}^{t_2} p_j \frac{\mathrm{d}}{\mathrm{d}t} \delta q_j dt$$

*Solution.* By integrating by parts, we get:

$$-\int_{t_1}^{t_2} \dot{p}_j \delta q_j dt$$

(the variation vanishes at the boundary, so it is discarded). □

# 14 Lecture 14

## 14.1 Lecture Notes - Review of Inertial Frames

### 14.1.1 Definition & Newton's Law in an accelerating Frame

By definition, an inertial frame $S_0$ is a frame with no acceleration or rotation. We now consider a constantly accelerating frame $S$ with $\mathbf{A} = \dot{\mathbf{v}}$. Newton's law holds in the inertial frame, so:

$$m\ddot{\mathbf{r}}_0 = \mathbf{F}$$

The velocity $\dot{\mathbf{r}}_0$ (in the non-inertial frame) and $\dot{\mathbf{r}}$ (in the inertial frame) are related by:

$$\dot{\mathbf{r}}_0 = \dot{\mathbf{r}} + \mathbf{v}$$

Taking the time derivative:

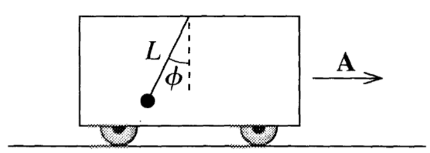
$$\ddot{\mathbf{r}} = \ddot{\mathbf{r}}_0 - \mathbf{A}$$

So applying Newton's law:

$$m\ddot{\mathbf{r}} = \mathbf{F} - m\mathbf{A} = \mathbf{F} + \mathbf{F}_{inertial}$$

We get an extra term accounting for the fictitous force.

## 14.2 Worksheet - Review of Noninertial Frames




**Problem 14.1.** *A pendulum is inside of a railcar that is accelerating in the x-direction with acceleration A Find the equilibrium value of the angle $\phi$.*

*Solution.* Apply Newton's law in the non-inertial frame:

$$m\ddot{\mathbf{r}} = \mathbf{F} - m\mathbf{A}$$

We have that the forces in the inertial frame are given by $\mathbf{F} = m\mathbf{g} + \mathbf{T}$, (gravity and tension) so:

$$m\ddot{\mathbf{r}} = m\mathbf{g} + \mathbf{T} - m\mathbf{A}$$

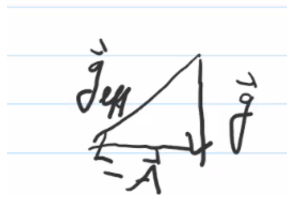
Grouping terms:

$$m\ddot{\mathbf{r}} = \mathbf{T} + m(\mathbf{g} - \mathbf{A})$$

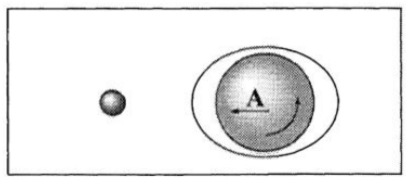
Where we may call $\mathbf{g} - \mathbf{A}$ the effective acceleration, $\mathbf{g}_{eff}$. By trigonometry, the equilibrium angle would be given by:

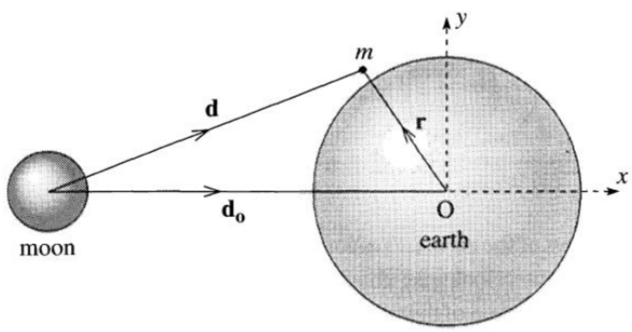
$$\tan \phi_{eq} = \frac{A}{g}$$

As can be seen from the diagram below:




$\square$




**Problem 14.2.** *The earth and moon orbit each other, while both earth and moon frames of reference are accelerating. Find the acceleration in each frame of reference.*

*Solution.* In the inertial frame, the forces on the test mass on the surface of the Earth is given by:

$$\mathbf{F} = m\mathbf{g} - gM_m m \frac{\hat{\mathbf{d}}}{d^2}$$

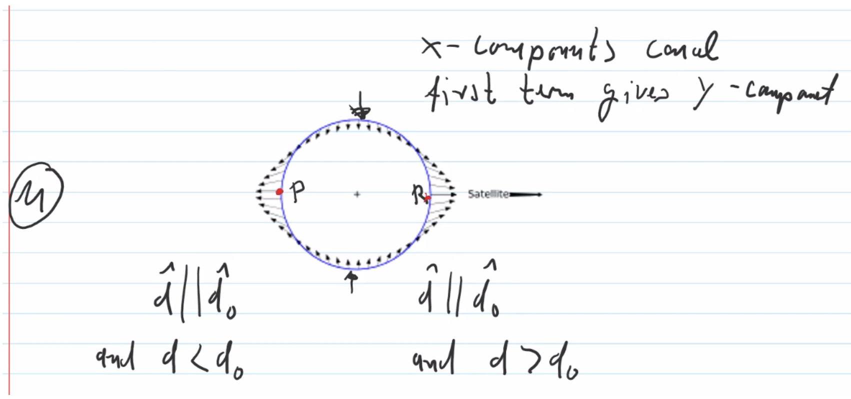
Where $M_m$ is the mass of the moon. The Earth is not an inertial system, so it experiences some acceleartion. The acceleration of the center of the mass of the earth is given by:

$$\mathbf{A} = -GM_m \frac{\hat{\mathbf{d}}_0}{d_0^2}$$

Hence we have:

$$m\ddot{\mathbf{r}} = m\mathbf{g} - GM_m m \left( \frac{\hat{\mathbf{d}}}{d^2} - \frac{\hat{\mathbf{d}}_0}{d_0^2} \right) = m\mathbf{g} - \mathbf{F}_{tidal}$$

The tidal force is the vector difference between if the mass is at the surface of earth vs. at the center of mass of the earth. This results in tidal effects at either side of the earth, with the same magnitude and in the opposite direction ($\hat{\mathbf{d}}$ and $\hat{\mathbf{d}}_0$ are parallel at these two points, so the effect is maximal). We end up with a bulge on both sides of the Earth. At the top and bottom, we have that the x components cancel by symmetry, so we only have the y component (weaker effect, inwards pointing). This is shown in the diagram below:




□

**Problem 14.3.** *Find the equation of motion for a particle on the surface of the ocean of the earth, in the earth's frame of reference.*

*Solution.* □

# 15 Lecture 15

## 15.1 Lecture Notes - Rotating Frame Clickers

Which of the following motion leads to fictitious forces?

1. The frame moves at a constant velocity with respect to an inertial reference frame.

2. The frame rotates at a constant angular velocity with respect to an inertial reference frame.

3. The frame moves at a constant acceleration with respect to an inertial reference frame.

4. The frame rotates at a constant angular acceleration with respect to an inertial reference frame.

*Solution.* Three of these frames have fictituous forces (1 does not, its inertial). 2. We saw last day, 3/4 lead to fictituous forces as we will see today! □

The coriolis and centrifugal "forces" are

$$\begin{aligned} \mathbf{F}_{\text{coriolis}} &= -2\ \mathrm{m}\boldsymbol{\Omega} \times \mathbf{v} \\ \mathbf{F}_{\text{centifitigal}} &= -\mathrm{m}\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}) \end{aligned}$$

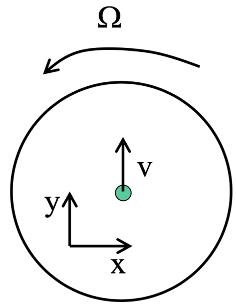
Which force is more important in the limit of slow velocity (in the rotating frame)?

*Solution.* In the limit of slow velocity, the centrifugal force (independent of velocity) dominates (the Coriolois force is linear in velocity). □

A disk drive typically rotates at 3600rpm, or 360 radians per second. For a dust particle at radius $r = 5$cm, how fast must the particle be moving (in the rotating frame) for the Coriolis and the centrifugal forces to have approximately equal magnitude?

*Solution.* Equating the two expressions and solving for $|\mathbf{v}|$, we find that $|\mathbf{v}| = 900$cm/s. ☐

A hockey puck slides from the center towards the edge of a frictionless, rotating merry-go-round. The merry-go-round has angular velocity $\Omega$ and rotates CCW when viewed from above. In the rotating frame, the initial velocity is in the positive y direction. In the **inertial** frame, which way does the path of the puck bend?

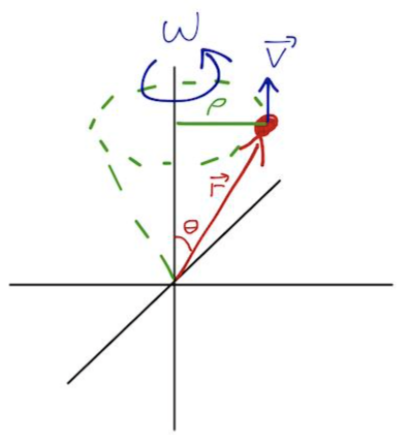



*Solution.* The path of the puck does not bend in the inertial frame; it retains a straight trajectory (there is no force acting on it!) ☐

In the rotating frame, which way does the path of the puck bend?

*Solution.* Applying the RHR, we see that the path curves to the right when viewed from above (towards positive $x$). ☐

## 15.2 Worksheet - Rotating Coordinate Systems




**Problem 15.1.** *What is the time rate of change,* $\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} = \mathbf{v}$ *of a vector* $\mathbf{r}$ *to a point in a body that is rotating about an axis* $O$ *with angular velocity* $\omega$*?*

*Solution.* We define the rotation vector:

$$\boldsymbol{\omega} = \omega\hat{\mathbf{u}}$$

Where $\hat{\mathbf{u}}$ points along the rotation axis (the sign is determined by the RHR). Geometry tells us that the distance from the rotation axis $\rho$ is given by:

$$\rho = r\sin\theta$$

Hence the velocity of this point is given by:

$$v = \rho\omega = r\sin\theta\omega$$

Generalizing this to the vector form, we have:

$$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r} = \dot{\mathbf{r}}$$

□

**Problem 15.2.** *Consider a body rotating in a reference frame that is itself rotating with respect to a fixed reference frame. Show that angular velocities add just like linear velocities.*

*Solution.* This is a consequence of the linearity of the cross product. Suppose we have:

$$\mathbf{v}_{31} = \boldsymbol{\omega}_{31} \times \mathbf{r} = \mathbf{v}_{32} + \mathbf{v}_{21} = \boldsymbol{\omega}_{32} \times \mathbf{r} + \boldsymbol{\omega}_{21} \times \mathbf{r} = (\boldsymbol{\omega}_{32} + \boldsymbol{\omega}_{21}) \times \mathbf{r}$$

Hence:

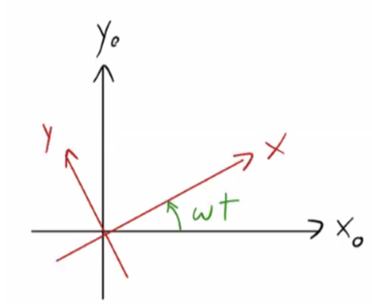
$$\boldsymbol{\omega}_{31} = \boldsymbol{\omega}_{32} + \boldsymbol{\omega}_{21}$$

So we can see that we can add angular velocities just like linear velocities. □

**Problem 15.3.** *Consider a vector* $\mathbf{Q}$*, which may be a position, velocity, or force vector. Show that the time rate of change of* $\mathbf{Q}$*,* $\frac{\mathrm{d}\mathbf{Q}}{\mathrm{d}t}$ *in a fixed frame of reference* $S_0$ *is related to* $\frac{\mathrm{d}\mathbf{Q}}{\mathrm{d}t}$ *in a rotating frame of reference* $S$ *by:*

$$\left(\frac{\mathrm{d}\mathbf{Q}}{\mathrm{d}t}\right)_{S_0} = \left(\frac{\mathrm{d}\mathbf{Q}}{\mathrm{d}t}\right)_S + \boldsymbol{\Omega} \times \mathbf{Q}$$

*Solution.*




We express the unit vectors in the rotating frame by transforming the unit vectors in the fixed frame:

$$\hat{\mathbf{x}} = \hat{\mathbf{x}}_0 \cos\omega t + \hat{\mathbf{y}}_0 \sin\omega t$$

$$\hat{\mathbf{y}} = \hat{\mathbf{y}}_0 \cos\omega t - \hat{\mathbf{x}}_0 \sin\omega t$$

So expressing an arbitrary vector $\mathbf{Q}$ in the rotating frame, we have:

$$\mathbf{Q} = Q_x\hat{\mathbf{x}} + Q_y\hat{\mathbf{y}} + Q_z\hat{\mathbf{z}}$$

Now expressing this in terms of the unit vectors in the stationary frame, we have:

$$\mathbf{Q} = (Q_x\cos\omega t - Q_y\sin\omega t)\hat{\mathbf{x}}_0 + (Q_x\sin\omega t + Q_y\cos\omega t)\hat{\mathbf{y}}_0 + Q_z\hat{\mathbf{z}}_0$$

We pick $\hat{\mathbf{z}}$ to be the rotation axis, so $\hat{\mathbf{z}} = \hat{\mathbf{z}}_0$ by our choice of coordinate system. Looking at the rate of change w.r.t the lab frame, we have:

$$\left(\frac{\mathrm{d}\mathbf{Q}}{\mathrm{d}t}\right)_{S_0} = \big(-\omega Q_x\sin\omega t - \omega Q_y\cos\omega t\big)\,\hat{\mathbf{x}}_0 + \big(\omega Q_x\cos\omega t - \omega Q_y\sin\omega t\big)\,\hat{\mathbf{y}}_0$$

Where the unit vectors do not have time dependence, only the $\cos\omega t$ and $\sin\omega t$ terms do. Looking at this expression, we can see that this can be written as the expression:

$$\left(\frac{\mathrm{d}\mathbf{Q}}{\mathrm{d}t}\right)_{S_0} = \omega(\hat{\mathbf{z}}_0 \times \mathbf{Q}) = \boldsymbol{\omega} \times \mathbf{Q}$$

Now, we take the derivative in the rotating frame:

$$\left(\frac{\mathrm{d}\mathbf{Q}}{\mathrm{d}t}\right)_{S} = \dot{Q}_x\hat{\mathbf{x}} + \dot{Q}_y\hat{\mathbf{y}} + \dot{Q}_z\hat{\mathbf{z}}$$

Now, we consider what is $\left(\frac{\mathrm{d}Q}{\mathrm{d}t}\right)_{S_0,x}$. By the chain rule:

$$\left(\frac{\mathrm{d}Q}{\mathrm{d}t}\right)_{S_0,x} = \frac{\mathrm{d}}{\mathrm{d}t}\left(Q_x\cos\omega t - Q_y\sin\omega t\right)\hat{\mathbf{x}}_0$$

$$\left(\frac{\mathrm{d}Q}{\mathrm{d}t}\right)_{S_0,x} = \left[\dot{Q}_x\cos\omega t - \dot{Q}_y\sin\omega t) + (-\omega Q_x\sin\omega t - \omega Q_y\cos\omega t)\right]\hat{\mathbf{x}}_0$$

Doing the same for the $x$ and $y$ components, we get the simple formula:

$$\left(\frac{\mathrm{d}\mathbf{Q}}{\mathrm{d}t}\right)_{S_0} = \left(\frac{\mathrm{d}\mathbf{Q}}{\mathrm{d}t}\right)_{S} + \boldsymbol{\omega} \times \mathbf{Q}$$

□

**Problem 15.4.** *Now find the acceleration in a fixed frame of reference, and thus write the modified Newton's law for a constantly rotating frame of reference, as we'd experience on the earth for example.*

*Solution.* The goal is to find the second derivative of the position vector in the $S$ (rotating) frame. First calculating $\left(\frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2}\right)_{S_0}$, we have:

$$\left(\frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2}\right)_{S_0} = \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)_{S_0}\left(\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right)_{S_0}$$

So applying the result from above twice, we have:

$$\left(\frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2}\right)_{S_0} = \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)_{S_0}\left(\left(\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right)_{S} + \boldsymbol{\Omega}\times\mathbf{r}\right)$$

$$\left(\frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2}\right)_{S_0} = \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)_{S}\left(\left(\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right)_{S} + \boldsymbol{\Omega}\times\mathbf{r}\right) + \boldsymbol{\Omega}\times\left(\left(\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right)_{S} + \boldsymbol{\Omega}\times\mathbf{r}\right)$$

We now simplify this expression. Henceforth, we will use the dot notation to refer to a derivative in $S$.

$$\left(\frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2}\right)_{S_0} = \ddot{\mathbf{r}} + \dot{\boldsymbol{\Omega}}\times\mathbf{r} + \boldsymbol{\Omega}\times\dot{\mathbf{r}} + \boldsymbol{\Omega}\times\dot{\mathbf{r}} + \boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\mathbf{r})$$

Multiplying both sides by $m$, and simplifying using the antisymmetry of the cross product ($\mathbf{a}\times\mathbf{b} = -\mathbf{b}\times\mathbf{a}$) have:

$$m\left(\frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2}\right)_{S_0} = \mathbf{F} = m\ddot{\mathbf{r}} - 2m\dot{\mathbf{r}}\times\boldsymbol{\Omega} - m(\dot{\mathbf{r}}\times\dot{\boldsymbol{\Omega}}) - m(\boldsymbol{\Omega}\times\mathbf{r})\times\boldsymbol{\Omega}$$

Where we use the identity that the LHS is just the total force on the system by Newton's second law. Rearranging this, we have:

$$m\ddot{\mathbf{r}} = \mathbf{F} + 2m\dot{\mathbf{r}} \times \mathbf{\Omega} + m(\mathbf{\Omega} \times \mathbf{r}) \times \mathbf{\Omega} + m(\mathbf{r} \times \dot{\mathbf{\Omega}})$$

Where the first term is the sum of the (real) forces on the system, the second term is the coriolois force, the third term is the centrifugal force, and the fourth term (which is nonzero only if the angular velocity is changing in time) is the Euler force. We may write this as:

$$m\ddot{\mathbf{r}} = \mathbf{F} + \mathbf{F}_{cor} + \mathbf{F}_{cent} + \mathbf{F}_{euler}$$

We note that we can also derive this using the Lagrangian approach! If we write the Lagrangian with the correct velocity, i.e.:

$$\mathcal{L} = \frac{m}{2}\left|\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)_{S_0}\right|^2 - U(\mathbf{r})$$

Then we will find that we recover the same result.

□

# 16 Lecture 16

## 16.1 Lecture Notes - Focault Pendulum

### 16.1.1 Rotating Coordinate System EOM

$$\left(\frac{d\mathbf{Q}}{dt}\right)_{S_0} = \left(\frac{d\mathbf{Q}}{dt}\right)_S + \vec{\omega} \times \mathbf{Q}$$

$$m\ddot{\mathbf{r}} = \vec{F} + 2m\dot{\mathbf{r}} \times \mathbf{\Omega} + m(\mathbf{\Omega} \times \mathbf{r}) \times \mathbf{\Omega} + m\mathbf{r} \times \dot{\vec{\Omega}} = \mathbf{F} + \mathbf{F}_{\text{coriolis}} + \mathbf{F}_{\text{centrifugal}} + \mathbf{F}_{\text{Euler}}$$

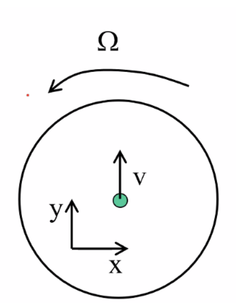
### 16.1.2 Review Questions

A bead rests on a wire that extends from the origin at an angle $\theta$ to the vertical. The wire rotates with angular velocity $\Omega$ about the vertical. In the frame rotating with the wire, what is the magnitude and direction of the centrifugal force when the bead is a distance $r$ from the origin?

*Solution.* By trigonometry, the bead is $r \sin\theta$ away from the rotation axis, so there is a centrifugal force of $m\Omega^2 r \sin\theta$ away from the rotation axis. □

A bucket of water spins about its central axis. After a relaxation time, the shape of the water surface reaches a steady state. Where is the water surface highest?

*Solution.* At the edge of the bucket, as the centrifugal force pulls the water towards the edge. The surface of the water is a parabola. □



A puck slides from the center towards the edge of a frictionless, rotating merry-go-round. The merry-go-round has angular velocity $\Omega$ and rotates CCW when viewed from above. In the rotating frame, the initial velocity is in the positive $y$ direction. What effect does the Coriolis force have on the velocity of the puck?

*Solution.* The coriolis force changes the direction of the velocity (deflected left) but does not change the magnitude (we can see from the $\mathbf{v} \times \boldsymbol{\Omega}$ form that the force does no work). □

Consider the same scenario as the previous question. How many rotations does the merry-go-round make before the puck slides off of the edge?

*Solution.* #rotations $= \frac{a\Omega}{2\pi v}$. First consider that the time to reach the edge is simply the distance $a$ (radius) divided by the velocity $v$ of the puck. Then, we may divide this time by the time per rotation (the period), which is $T = \frac{2\pi}{\Omega}$. This yields:

$$\#\text{rotations} = \frac{\Delta t}{T} = \frac{a\Omega}{2\pi v}$$

□

At which of these points will a person's measured weight be the largest (equator, 30, 40, 60 degrees latitude, or north pole)

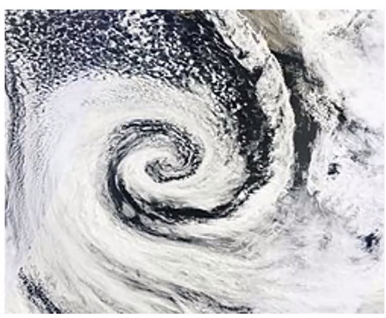
*Solution.* At the north pole; there we have no centrifugal force there (which acts against the gravitational force and decreases the weight of the person). □

In the northern hemisphere, which directions are winds from the north and south deflected by the Coriolis force?

*Solution.* Winds from the N are deflected E and winds from the S are deflected W. □

Where are these low pressure areas?

L



R

*Solution.* The left image rotates clockwise, the right one rotates counterclockwise. For a low pressure system, we have air coming in, and in the northern hemisphere, we have right deflection (and vise versa for the southern hemisphere) so we would expect a counterclockwise motion for the northern hemisphere and clockwise for the southern hemisphere. □

### 16.1.3 The Foucault Pendulum

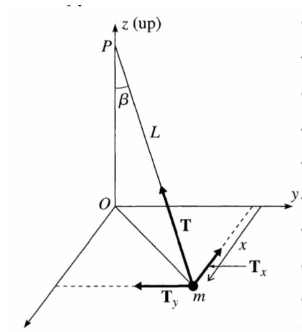
We start with our general expression:

$$m\ddot{\mathbf{r}} = \mathbf{F} + 2m\dot{\mathbf{r}} \times \boldsymbol{\Omega} + m(\boldsymbol{\Omega} \times \mathbf{r}) \times \boldsymbol{\Omega} + m\mathbf{r} \times \dot{\boldsymbol{\Omega}}$$

The third term can be neglected as the Earth spins at a constant rate, and the second term can be neglected as $\Omega$ is small. Define $x$ to be north south, $y$ to be east west. $\mathbf{F}$ is the sum of the tension and the gravitational force, that is:

$$\mathbf{F} = \mathbf{T} + m\mathbf{g}$$

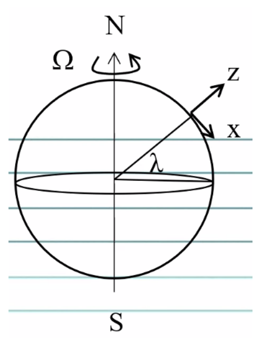
Now we consider the given picture:




By similar triangles:

$$T_x = -T\frac{x}{L}$$
$$T_y = -T\frac{y}{L}$$
$$T_z = -T\frac{z-L}{L}$$

But for the $T_z$, we can consider that we do small amplitudes, so $z \approx 0$, and $\dot{z} \approx 0$. Hence, $T_z \approx T \approx mg$. Hence, we have dealt with the tension. Now, we think about $\mathbf{\Omega}$.




If our latitude is $\lambda$ and we use coordinates such that $x$ points towards the equator, $y$ points parallel to the latitude line, and $z$ points away from the center of the earth, what is the rotation vector in this coordinate system?

*Solution.* We know that $\mathbf{\Omega}_{earth}$ points straight upwards. Projecting this, we get:

$$\mathbf{\Omega} = \omega \begin{bmatrix} -\cos\lambda \\ 0 \\ \sin\lambda \end{bmatrix}$$

□

Next, what is $\mathbf{\Omega} \times \mathbf{v}$?

*Solution.* Using that $\mathbf{v} = \begin{bmatrix} \dot{x} \\ \dot{y} \\ 0 \end{bmatrix}$ (Assume $\dot{z}$ is negligible) and $\mathbf{\Omega}$ from above, we get:

$$\mathbf{\Omega} \times \mathbf{v} = \begin{bmatrix} -\dot{y}\Omega\sin\lambda \\ \dot{x}\Omega\sin\lambda \\ -\dot{y}\Omega\cos\lambda \end{bmatrix}$$

□

Now, putting together the equations of motion for $x$ and $y$ we get:

$$m\ddot{x} = -mg\frac{x}{L} + 2m\Omega\sin\lambda\dot{y}$$

$$m\ddot{y} = -mg\frac{y}{L} - 2m\Omega\sin\lambda\dot{x}$$

We have a system of coupled equations. As a trick, multiply both equations by $i$ and add them together, and define $s = x + iy$. We then have:

$$\ddot{s} + 2i\alpha\dot{s} + k^2\dot{s} = 0$$

Where $\alpha = \Omega\sin\lambda$, $k^2 = \frac{g}{L}$. To solve this differential equation, we guess $s(t) = c\exp(\gamma t)$. This yields a characteristic equation:

$$\gamma^2 + 2i\alpha\gamma + k^2 = 0$$

Solving this, we get:

$$\gamma_{1/2} = -i\alpha \pm i\sqrt{\alpha^2 + k^2}$$

The general solution is the sum of these two:

$$s(t) = C_1\exp(\gamma_1 t) + C_2\exp(\gamma_2 t)$$

We assume initial conditions of $s(0) = \hat{\mathbf{x}}, \dot{s}(0) = 0$ (elongation along $x$, with no initial velocity). We may then solve for the coefficients $C_1, C_2$ (homework!). Then, taking the real part of the complex solution, we get the final result:

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} \cos(\alpha t) & \sin(\alpha t) \\ -\sin(\alpha t) & \cos(\alpha t) \end{bmatrix} \begin{bmatrix} \hat{\mathbf{x}}\cos\left(\sqrt{\alpha^2 + k^2}t\right) \\ \hat{\mathbf{x}}\frac{\sin\left(\sqrt{\alpha^2+k^2}t\right)\alpha}{\sqrt{\alpha^2+k^2}} \end{bmatrix}$$

Question: We have this result. What is the effect of multiplying through by this matrix?

*Solution.* We recognize this just as a rotation matrix, rotating by a time dependent angle $\Omega_z t = \alpha t$. □

This is a characteristic feature of the Focault pendulum, indeed that it precesses.

# 17 Lecture 17

## 17.1 Lecture Notes - Rotational Motion of Rigid Bodies

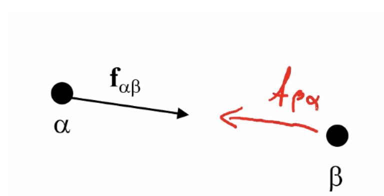
### 17.1.1 Review of Center of Mass - Clickers

A system of $n$ particles is described by the masses and positions of each particle, $m_\alpha, \mathbf{r}_\alpha$. The total mass is $M = \sum_\alpha m_\alpha$. What is the center of mass of the system?

*Solution.* The center of mass is the weighted positional average of the position vectors (weighted by the masses, normalized by the total mass). It is given by $\mathbf{R} = \frac{1}{M}\sum_\alpha m_\alpha \mathbf{r}_\alpha$ □

Consider particles $\alpha$ and $\beta$, and the internal force $\mathbf{F}_{\alpha\beta}$ (the force on particle $\alpha$ due to particle $\beta$). How does Newton's third law relate the internal forces?

*Solution.* $\mathbf{F}_{\alpha\beta} = -\mathbf{F}_{\beta\alpha}$ □



Suppose a system of particles experiences only internal forces (no external forces). What can we say of the linear momentum of the system?

*Solution.* The momentum of the COM is constant; the momentum of individual particles can change due to internal forces, but the lack of external forces means the center of mass will have conserved momentum. □

#### 17.1.2 Linear and Angular Momentum of COM

We have that the center of the mass position vector is:

$$\mathbf{R} = \frac{1}{M}\sum_{\alpha} m_\alpha \mathbf{r}_\alpha$$

Then the momentum of the center of mass is:

$$\mathbf{P} = M\dot{\mathbf{R}} = \sum_{\alpha} m_\alpha \dot{\mathbf{r}}_\alpha$$

The external force is given by:

$$\mathbf{F}_{ext} = M\ddot{\mathbf{R}}$$

We can resolve the motion of the body into the motion of the center of mass and motion relative to the center of mass. Define $\mathbf{r}_\alpha$ (the position vector of one of the point masses that makes up the body) as:

$$\mathbf{r}_\alpha = \mathbf{R} + \mathbf{r}'_\alpha$$

Where the primed coordinates are in the COM frame. The total angular momentum is given by:

$$\mathbf{L} = \sum_{\alpha} \mathbf{l}_\alpha = \sum_{\alpha} \mathbf{r}_\alpha \times m_\alpha \dot{\mathbf{r}}_\alpha$$

We substitue $\mathbf{r}_\alpha$ with our expression above, which yields four terms:

$$\mathbf{L} = \sum_{\alpha} \mathbf{R} \times m_\alpha \dot{\mathbf{R}} + \sum_{\alpha} \mathbf{R} \times m_\alpha \dot{\mathbf{r}}'_\alpha + \sum_{\alpha} \mathbf{r}'_\alpha \times m_\alpha \dot{\mathbf{R}} + \sum_{\alpha} \mathbf{r}'_\alpha + m_\alpha \dot{\mathbf{r}}'$$

What can we say of the quantity $\sum_\alpha m_\alpha \mathbf{r}'_\alpha$?

*Solution.* Evidently, $\sum_\alpha m_\alpha \mathbf{r}'_\alpha = \mathbf{0}$ as the sum of the relative positions to the COM weighted by the masses would be zero by definition of the COM! □

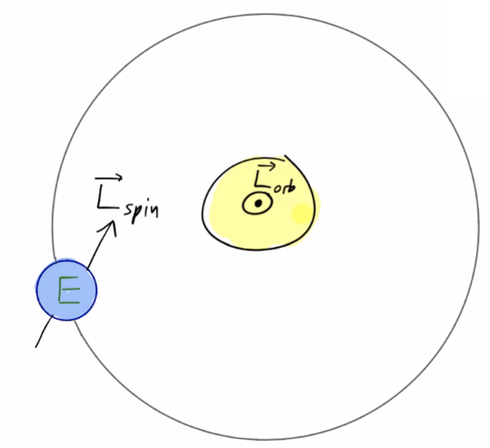
This makes two of our terms in the above sum vanish (namely, the second and third terms). This leaves us with:

$$\mathbf{L} = \sum_{\alpha} \mathbf{R} \times m_\alpha \dot{\mathbf{R}} + \sum_{\alpha} \mathbf{r}'_\alpha + m_\alpha \dot{\mathbf{r}}'$$

The first term we may rewrite as $\mathbf{R} \times M\dot{\mathbf{R}} = \mathbf{R} \times \mathbf{P}$, i.e. the angular momentum of the center of mass. The other term is the angular momentum relative to the center of mass. This is a nice decomposition; any angular momentum we can write as:

$$\mathbf{L} = \mathbf{L}_{orbital} + \mathbf{L}_{spin}$$

For example, with the Earth-sun system, we have:



### 17.1.3 Potential and Kinetic Energy of a Rigid Body

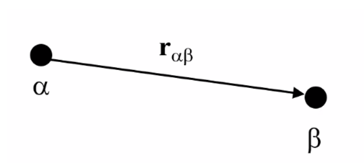
We can also decompose the potential energy of a rigid body:

$$U = U_{ext} + U_{int}$$

Where:

$$U_{int} = \sum_{i<j} U_{ij}(r_{ij})$$

But since in a rigid body all particle distances $r_{ij}$ are fixed, the internal potential energy $U_{int}$ must be constant!
Consider particles $\alpha$ and $\beta$; the internal force $\mathbf{f}_{\alpha\beta}$ = the force on particle $\alpha$ due to particle $\beta$; the position vector $\mathbf{r}_{\alpha\beta}$ = the vector from particle $\alpha$ to particle $\beta$. If the force between the two particles is central, what can you say about $\mathbf{r}_{\alpha\beta} \times \mathbf{f}_{\alpha\beta}$?




*Solution.* Since the vectors would be parallel in the case of central forces, $\mathbf{r}_{\beta\alpha} \times \mathbf{f}_{\alpha\beta} = \mathbf{0}$, that is internal central forces do not change the angular momentum. □

A system of n particles is described by the masses and positions of each particle, relative to the center of mass: $\mathrm{m}_\alpha, \mathbf{r}'_\alpha$ The squared velocity of each particle is therefore

$$\mathrm{v}_\alpha^2 = \dot{\mathbf{r}}_\alpha'^2 + 2\dot{\mathbf{r}}'_\alpha \cdot \dot{\mathbf{R}} + \dot{\mathbf{R}}^2 = \mathrm{v}_\alpha'^2 + 2\dot{\mathbf{r}}'_\alpha \cdot \dot{\mathbf{R}} + \mathrm{V}^2$$

What is the total kinetic energy of the system?

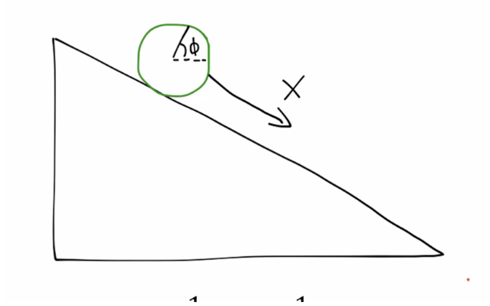
*Solution.* $\mathrm{T} = \frac{1}{2}\sum_\alpha \mathrm{m}_\alpha \mathrm{v}_\alpha'^2 + \sum_\alpha m_\alpha \mathbf{v}'_\alpha \cdot \mathbf{V} + \frac{1}{2}\mathrm{M}\mathrm{V}^2$ But the second term vanishes by the same previous argument, so: $\mathrm{T} = \frac{1}{2}\sum_\alpha \mathrm{m}_\alpha \mathrm{v}_\alpha'^2 + \frac{1}{2}\mathrm{M}\mathrm{V}^2$ Where we can see that the kinetic energy can also be decomposed to the kinetic energy of the center of mass, and the kinetic energy relative to the center of mass. □

Note that for the kinetic energy about an instantaneous axis of rotation of a rigid body, we have

$$T = \frac{1}{2}\sum_\alpha m_\alpha \dot{\mathbf{r}}_\alpha'^2$$

Where we have chosen the center of mass $\mathbf{R}$ to be a point of the body at rest (and hence $\dot{\mathbf{R}} = 0$).

### 17.1.4 Example: Rolling Disk

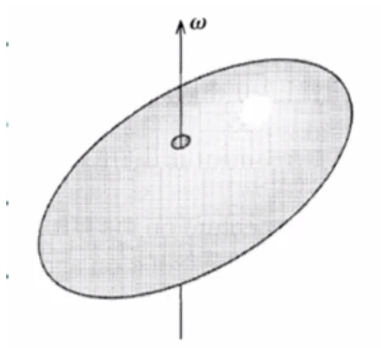



The kinetic energy of a disk rolling down an incline is given by:

$$T = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}I\dot{\phi}^2$$

Where $I$ is the moment of inertia about the central axis.

### 17.1.5 Rotation about the z axis




In this case, the angular momentum and kinetic energy are given by:

$$\mathbf{L} = \sum_{\alpha} \mathbf{r}_\alpha \times m_\alpha \mathbf{v}_\alpha$$

Where:

$$\mathbf{v}_\alpha = \omega \times \mathbf{r}_\alpha = \begin{bmatrix} 0 \\ 0 \\ \omega \end{bmatrix} \times \begin{bmatrix} x_\alpha \\ y_\alpha \\ z_\alpha \end{bmatrix} = \begin{bmatrix} -\omega y_\alpha \\ \omega x_\alpha \\ 0 \end{bmatrix}$$

Hence:

$$m_\alpha \mathbf{r}_\alpha \times \mathbf{v}_\alpha = m_\alpha \begin{bmatrix} x_\alpha \\ y_\alpha \\ z_\alpha \end{bmatrix} \times \begin{bmatrix} -\omega y_\alpha \\ \omega x_\alpha \\ 0 \end{bmatrix} = m_\alpha \omega \begin{bmatrix} -z_\alpha x_\alpha \\ -z_\alpha y_\alpha \\ x_\alpha^2 + y_\alpha^2 \end{bmatrix}$$

$x_\alpha^2 + y_\alpha^2$ is just the distance from the rotation axis, so identify it as $\rho_\alpha^2$. Then, the z component of the angular momentum is given by:

$$L_z = \sum_{\alpha} m_\alpha \rho_\alpha^2 \omega = I_{zz}\omega$$

The double $zz$ notation will become clear in a moment.
Next, let's look at the kinetic energy $T = \frac{1}{2}\sum_\alpha m_\alpha v_\alpha^2$. Since we are doing circular motion about a rotational axis, $v_\alpha = \omega \rho_\alpha$. Hence:

$$T = \frac{1}{2}\sum_{\alpha} m_\alpha \omega^2 \rho_\alpha^2 = \frac{1}{2} I_{zz}\omega^2$$

Next looking at the other components of the angular momentum, we have

$$L_x = -\sum_\alpha m_\alpha x_\alpha z_\alpha \omega, \quad L_y = -\sum_\alpha m_\alpha y_\alpha z_\alpha \omega$$

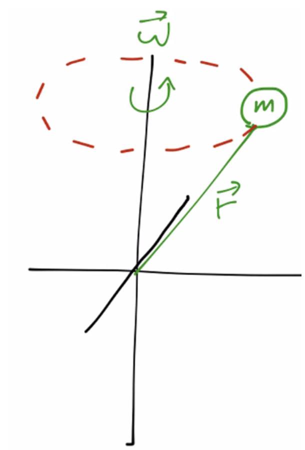
We now define some new quantities:

$$L_x = -I_{xz}\omega, \quad L_y = -I_{yz}\omega$$

We see that $\omega$ only has one component (in the z direction), but evidently, the angular momentum has more than one component, and hence is not parallel to omega! In the past (e.g. first year physics) we have only studied problems where $\mathbf{L}$ is parallel to $\omega$ but here this is definitely not the case.
We note that $I_{ij}$ are the products of inertia. This can be generalized to three dimensions, by introducing the inertia matrix/tensor:

$$\mathbf{L} = \mathbb{I}\boldsymbol{\omega}$$

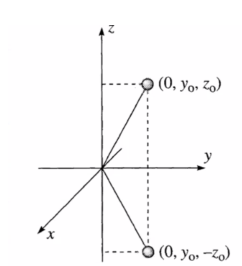
It is no longer sufficient to treat the moment of inertia as a single number.
What is the direction of $\mathbf{L}$ of $m$ at this moment pictured?




*Solution.* The velocity is into the page, and the position vector points up and right, so evaluating $\mathbf{l} = \mathbf{r} \times \mathbf{v}$ (with the RHR), we get that the direction is up and left. □

Now considering the same setup as above, what happens to $\mathbf{L}$ and the moment of inertia $\mathbb{I}$ as the mass spins around?

*Solution.* As the mass rotates around the axis, the direction of the angular momentum will change. Since $\boldsymbol{\omega}$ is constant in time, we must have that $\mathbb{I}$ changes in time by $\mathbf{L} = \mathbb{I}\boldsymbol{\omega}$. Hence, $\mathbf{L}$ and $\mathbb{I}$ both change, and there is a net torque on the mass. There will only be some special axis for which the moment of inertial will not change (e.g. the wheel rolling down the inclined plane). But in general, $\mathbf{L}$, $\mathbb{I}$ are not invariant in time. □



For the shown configuration, what are the products of inertia for rotation about the z axis? Using the formulas we derived previously, we have:

$$I_{zz} = \sum_\alpha m_\alpha (x_\alpha^2 + y_\alpha^2) = m(0 + y_0^2 + 0 + y_0^2) = 2my_0^2$$

$$I_{xz} = -\sum_\alpha m_\alpha x_\alpha z_\alpha = 0$$

$$I_{yz} = -\sum_\alpha m_\alpha y_\alpha z_\alpha = -m(y_0 z_0 + y_0(-z_0)) = 0$$

So in this case, we have that $\boldsymbol{\omega} \parallel \mathbf{L}$. Next day, we will look at the general inertia tensor, and perhaps the principle axes of rotation (for which the inertia tensor is simple/diagonal).

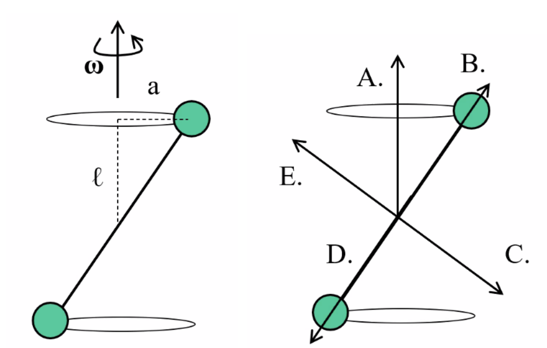
# 18 Lecture 18

## 18.1 Lecture Notes - Moment of Inertia Tensor

### 18.1.1 Review of Last Day's Results

$$\mathbf{L} = \mathbf{L}_{CM} + \mathbf{L}_{rel}$$
$$T = T_{CM} + T_{rel}$$
$$U = U_{CM} + U_{rel}$$

We can decompose the angular momentum and energies into the center of mass term and the relative to the COM term. We note that for rigid bodies, $U_{rel}$ is constant.
Last time (see 17.1.5) we observed that the angular momentum vector and the rotation vector are, in general, not parallel. Let us solve a question with a similar idea. Suppose we have a rotating dumbbell of two masses $m$ which move in circles (radius $a$) at a z displacement $l$ and $-l$, joined by a massless rod. The Angular velocity vector is given by $\boldsymbol{\omega} = \omega\hat{\mathbf{z}}$. What is the direction of $\mathbf{L}$?




*Solution.* Direction E. Use the right hand rule with $\mathbf{L} = \mathbf{r} \times \mathbf{v}$ and the add the angular momenta of the two terms. □

Follow-up question; consider the body frame where the position of the masses are $(0, a, l)$, $(0, -a, -l)$. What is the $I_{zz}$ component of the inertia tensor?

*Solution.* Recall that $I_{zz} = \sum_\alpha m_\alpha (x_\alpha^2 + y_\alpha^2)$. Applying the formula, we have that $I_z z = ma^2 + m(-a)^2 = 2ma^2$. □

Next, what is the $I_{xz}$ component of the inertia tensor?

*Solution.* Recall that $I_{xz} = -\sum_\alpha x_\alpha z_\alpha$. We see that $x = 0$ for both masses so hence $I_{xz} = 0$. $\square$

What's $I_{yz}$?

*Solution.* Recall that $I_{yz} = -\sum_\alpha m_\alpha y_\alpha z_\alpha$. We therefore have that $I_{yz} = -mal - m(-a)(-l) = -2mal$. $\square$

What is the kinetic energy of the system?

*Solution.* We use that $T = \frac{1}{2} I_{zz}\omega^2 = \frac{1}{2}(2ma^2)\omega^2 = ma^2\omega^2$. $\square$

*Remark:* We can genrealize this to be $T = \frac{1}{2}\boldsymbol{\omega} \cdot \mathbf{L}$.

### 18.1.2 Angular momentum for rigid body with angular velocity along arbitrary direction

We have that:

$$\boldsymbol{\omega} = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}$$

As well as that:

$$\mathbf{L} = \sum_\alpha m_\alpha \left( \mathbf{r}_\alpha \times (\boldsymbol{\omega} \times \mathbf{v}_\alpha) \right)$$

We apply the BAC-CAB rule, that is:

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$$

Using this, the above expression for the angular momentum becomes:

$$\mathbf{L} = \begin{bmatrix} L_x \\ L_y \\ L_z \end{bmatrix} = \sum_\alpha m_\alpha \begin{bmatrix} (y_\alpha^2 + z_\alpha^2)\omega_x & -x_\alpha y_\alpha \omega_y & -x_\alpha z_\alpha \omega_z \\ -y_\alpha x_\alpha \omega_x & (z_\alpha^2 + x_\alpha^2)\omega_y & -y_\alpha z_\alpha \omega_z \\ -z_\alpha x_\alpha \omega_x & -z_\alpha y_\alpha \omega_y & (x_\alpha^2 + y_\alpha^2)\omega_z \end{bmatrix}$$

We may pull out these coefficients and define a moment of inertia matrix/tensor:

$$\mathbb{I} = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix}$$

Where $\mathbf{L} = \mathbb{I}\boldsymbol{\omega}$. Note that this matrix is both real symmetric, as $I_{ij} = I_{ji}$, and hence contains 6 independent elements. We also note that this means $\mathbb{I}^T = \mathbb{I}$ (equal to its transpose). For example, using the definition, we can say that:

$$I_{xx} = \sum_\alpha m_\alpha (y_\alpha^2 + z_\alpha^2)$$

$$I_{xy} = -\sum_\alpha m_\alpha x_\alpha y_\alpha = -\sum_\alpha m_\alpha y_\alpha x_\alpha = I_{yx}$$

We can extend this notion to continuous mass distributions:

$$\mathbb{I} = \int dV \rho(x, y, z) \begin{bmatrix} y^2 + z^2 & -xy & -xz \\ -xy & x^2 + z^2 & -yz \\ -xz & -yz & x^2 + y^2 \end{bmatrix}$$

### 18.1.3 Index notation

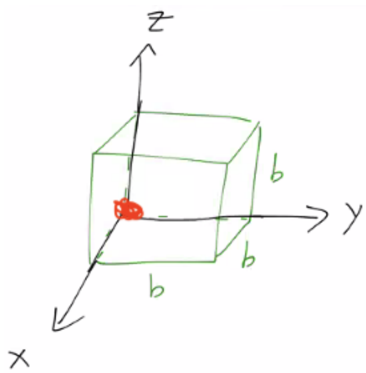
Note that when we write $\mathbf{L} = \mathbb{I}\boldsymbol{\omega}$, this is equivalent to $L_i = \sum_j I_{ij}\omega_j$. We can also write this compactly by introducing the Kronecker Delta notation:

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

Hence we could write the above expression for the inertia tensor more compactly as:

$$I_{ij} = \int dV \rho(x, y, z) \left(\mathbf{r}^2 \delta_{ij} - r_i r_j\right)$$

### 18.1.4 Example: Components of Inertia Tensor for rotation of cube about corner




We have a uniform solid cube of mass $M$ and side $b$, rotating about a corner (the origin). We assume a constant $\rho$ of $\rho = \frac{M}{b^3}$. Calculating $I_{xx}$, we have:

$$I_{xx} = \rho \int_0^b dx \int_0^b dy \int_0^b dz (y^2 + z^2) = \ldots = \frac{2}{3} M b^2$$

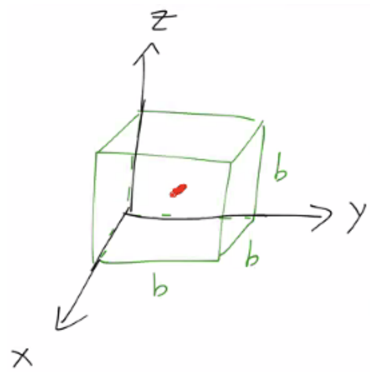
Note we have pulled out $\rho$ from the integration as these are constant. By symmetry of the object, $I_{xx} = I_{yy} = I_{zz}$. What about the off diagonal terms? Calculating $I_{xy}$ we have:

$$I_{xy} = \rho \int_0^b dx \int_0^b dy \int_0^b dz (-xy) = -\frac{M}{4} b^4$$

And we would expect the other off diagonal elements to again be identical by symmetry. Writing the total inertia tensor, we then have:

$$\mathbb{I} = M b^2 \begin{bmatrix} 2/3 & -1/4 & -1/4 \\ -1/4 & 2/3 & -1/4 \\ -1/4 & -1/4 & 2/3 \end{bmatrix}$$

### 18.1.5 Example: Components of Inertia Tensor for rotation of cube about COM




Our bounds of integration will change compared to the last case. Calculating $I_{xx}$, we have:

$$I_{xx} = \rho \int_{-b/2}^{b/2} dz \int_{-b/2}^{b/2} dy \int_{-b/2}^{b/2} dz (y^2 + z^2) = \frac{Mb^2}{6}$$

Again by symmetry, $I_{xx} = I_{yy} = I_{zz}$. We note that this is different from before! Calculating $I_{xy}$, we have:

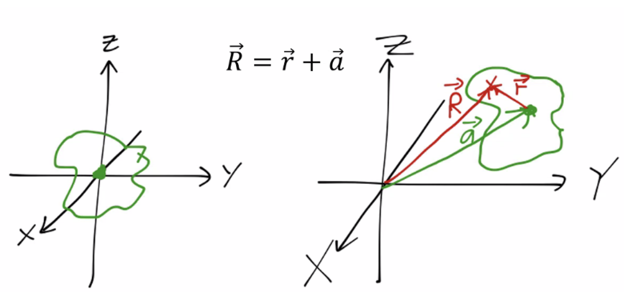
$$I_{xy} = \rho \int_{-b/2}^{b/2} dz \int_{-b/2}^{b/2} dy \int_{-b/2}^{b/2} dz (-xy) = 0$$

The integral is immediately zero by the fact that the integrand is odd. The same goes for the other off diagonal elements, which yields the final inertia tensor:

$$\mathbb{I} = \frac{Mb^2}{6} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Which is diagonal! This happens to be the case because this is a principal axis of rotation, where the inertia tensor has a particularly simple (diagonal) form.

### 18.1.6 Parallel Axis Theorem




If $I_{ij}$ is the inertia tensor calculated in the CM coordinates, and $J_{ij}$ is the tensor element in the displaced coordinates (where $\mathbf{R} = \mathbf{r} + \mathbf{a}$), then:

$$J_{ij} = I_{ij} + M(a^2 \delta_{ij} - a_i a_j)$$

#### 18.1.7 Example: Applying the Parallel Axis Theorem to the Cube

For the displacement of coordinates from the center of the mass of the cube to the corner of the cube, we have that:

$$\mathbf{a} = \begin{bmatrix} -b/2 \\ -b/2 \\ -b/2 \end{bmatrix}, \quad a^2 = |\mathbf{a}|^2 = \frac{3}{4}b^2$$

Then, for the diagonal elements we have:

$$M(a^2\delta_{ii} - a_i a_i) = M\left[\frac{3}{4}b^2 - \frac{-b}{2}\cdot\frac{-b}{2}\right] = \frac{Mb^2}{2}$$

And for the off diagonals:

$$M(a^2\delta_{ij} - a_i a_j) = M\left[-\frac{-b}{2}\frac{-b}{2}\right] = -\frac{Mb^2}{4}$$

Hence calculating the inertia tensor about the corner of the cube (in the displaced coordinates) we get:

$$\mathbb{J} = Mb^2\begin{bmatrix} 1/6 & 0 & 0 \\ 0 & 1/6 & 0 \\ 0 & 0 & 1/6 \end{bmatrix} + Mb^2\begin{bmatrix} 1/2 & -1/4 & -1/4 \\ -1/4 & 1/2 & -1/4 \\ -1/4 & -1/4 & 1/2 \end{bmatrix} = Mb^2\begin{bmatrix} 2/3 & -1/4 & -1/4 \\ -1/4 & 2/3 & -1/4 \\ -1/4 & -1/4 & 2/3 \end{bmatrix}$$

Which agrees with the result obtained from the direct calculation.

#### 18.1.8 Principal axes

If $\mathbf{L} = \lambda\boldsymbol{\omega}$ for a scalar $\lambda$, the body rotates around one of its principal axes. $\lambda$ is the "Moment of inertia" about that axis. So if:

$$\mathbb{I} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

Then the chosen axes are the principal axes and $\lambda_i$ are the principal moments. For any rigid body and any point $O$, there are three perpendicular axes with respect to which the inertia tensor is diagonal! This is a consquence of the fact that the moment of inertia tensor has real entries and is symmetric.
Which of the following statements are a consequence of the fact that the inertia tensor is a 3x3 matrix with real positive eigenvalues and orthogonal eigenvectors?

(a) The matrix can be diagonalized

(b) The matrix of eigenvectors is an orthogonal matrix

(c) The matrix of eigenvectors is a rotation matrix (if properly normalized)

(d) In the coordinate system aligned with the eigenvectors, the tensor is diagonal.

*Solution.* All 4 are correct. □

# 19 Lecture 19

## 19.1 Lecture Notes - Principle Axes of Inertia & Euler's Equations

### 19.1.1 Motivation Clickers

Suppose we calculate a diagonal inertia matrix for an objecct in some coordinate system, with $\lambda_1 > \lambda_2 > \lambda_3$. What are the principle axes?

*Solution.* $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$. If we calculate the inertia matrix to be diagonal in our coordinate system, then it is just the x/y/z axis that are the principal axes. In general, the principal axes are just the coordinate axes if $\mathbb{I}$ is diagonal! □

Next, suppose we calculate the moment of inertia of a second object, and it is not diagonal. What are the principle axes in this case?

*Solution.* $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ where $\mathbf{e}_i$ are the eigenvectors of $\mathbb{I}$. □

### 19.1.2 Linear algebra review

A real symmetric (nxn) matrix can be diagonalized, i.e.:

$$S^{-1}AS = \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}$$

The matrix A has eigenvectors $\mathbf{v}_1, \cdots \mathbf{v}_n$ corresponding to eigenvalues $\lambda_1, \cdots \lambda_n$.

The matrix $S$ has the eigenvectors as its columns, that is, $S = [\mathbf{v}_1, \cdots \mathbf{v}_n]$.

The eigenvectors obey the equation $A\mathbf{v}_i = \lambda_i \mathbf{v}_i$.

The eigenvectors form an orthonormal basis.

Note that $S$ here is an orthogonal matrix, that is, $S^T = S^{-1}$, or $SS^T = SS^{-1} = I$. So we could write the above equivalently as $S^T AS$.

Another property is that $\det S = \pm 1$.

A group theoretic property is that Orthogonal matrices with $\det(S) = 1$ form the $SO(n)$ group.
Multiplication of a vector by a matrix $A$ is a linear transformation $\mathbf{v} \mapsto A\mathbf{v}$. What happens to an eigenvector of $A$ under this linear transformation?

*Solution.* The magnitude of the vector can change (in fact, it is rescaled by the corresponding eigenvalue) but the direction does not change. □

The eigenvalue equation can be written as $(A - \lambda I)\mathbf{v} = \mathbf{0}$. What condition must be satisfied for $\lambda$ to be an eigenvalue of $A$?

*Solution.* $\det(A - \lambda I) = 0$ □

The matrix A has eigenvectors $\mathbf{v}_1, \cdots \mathbf{v}_n$ corresponding to eigenvalues $\lambda_1, \cdots \lambda_n$. The matrix $S$ has the eigenvectors as its columns, that is, $S = [\mathbf{v}_1, \cdots \mathbf{v}_n]$. What is the product $AS$?

*Solution.* Recall that:

$$S^{-1}AS = \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}$$

Hence:

$$AS = S \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}$$

Therefore the answer is:

$$AS = [\lambda_1 \mathbf{v}_1 \cdots \lambda_n \mathbf{v}_n]$$

□

(Question from last day) Which of the following statements are a consequence of the fact that the inertia tensor is a 3x3 matrix with real positive eigenvalues and orthogonal eigenvectors?

(a) The matrix can be diagonalized

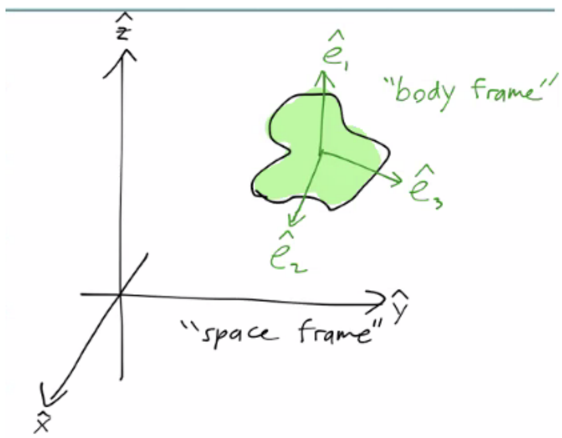
(b) The matrix of eigenvectors is an orthogonal matrix

(c) The matrix of eigenvectors is a rotation matrix (if properly normalized)

(d) In the coordinate system aligned with the eigenvectors, the tensor is diagonal.

*Solution.* All 4 are correct (for (c), recall that when we diagonalize a matrix, we rotate our axes/coordinate system). □

### 19.1.3 Rotational motion and principal axes




We have that the torque is defined as:

$$\mathbf{\Gamma} = \dot{\mathbf{L}}$$

Where $\mathbf{\Gamma} = \mathbf{r} \times \mathbf{F}$. We can expand out the time derivative of the angular momentum with:

$$\dot{\mathbf{L}} = \dot{\mathbb{I}}\boldsymbol{\omega} + \mathbb{I}\dot{\boldsymbol{\omega}}$$

But it is quite nasty to work in a frame where the inertia tensor is constantly changing. Hence, let us work in a coordinate system where $\mathbb{I}$ is fixed and hence $\dot{\mathbb{I}} = 0$.
How is the time derivative of a vector $\mathbf{v}$ in an inertial frame I related to the time derivative of the vector in a rotating frame R, which rotates with angular velocity vector $\boldsymbol{\omega}$?

*Solution.* As we derived previously,

$$\left.\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t}\right|_I = \left.\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t}\right|_R + \boldsymbol{\omega} \times \mathbf{v}$$

□

So, we work in the body frame where $\mathbb{I}$ is fixed. Then, expressing the torque in this frame, we have:

$$\mathbf{\Gamma} = \left.\frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t}\right|_{S_0} = \left.\frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t}\right|_{Sbod} + \boldsymbol{\omega} \times \mathbf{L} = \mathbb{I}\dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times \mathbf{L}$$

Choosing our principal axes such that $\mathbb{I}$ is diagonal, we have that:

$$\boldsymbol{\omega} \times \mathbf{L} = \boldsymbol{\omega} \times \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} = \det \begin{bmatrix} \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3 \\ \omega_1 & \omega_2 & \omega_3 \\ \lambda_1\omega_1 & \lambda_2\omega_2 & \lambda_3\omega_3 \end{bmatrix}$$

Hence, Euler's equations in the body frame (expanding out our expression for $\mathbf{\Gamma}$ above are:

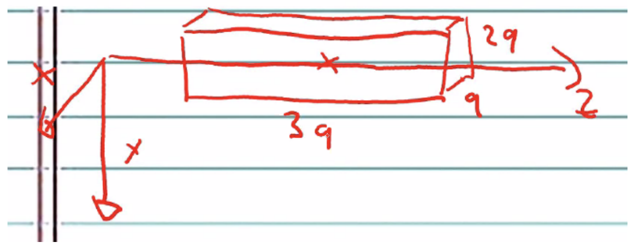
$$\Gamma_1 = \lambda_1 \dot{\omega}_1 - (\lambda_2 - \lambda_3)\omega_2\omega_3$$

$$\Gamma_2 = \lambda_2 \dot{\omega}_2 - (\lambda_3 - \lambda_1)\omega_1\omega_3$$

$$\Gamma_3 = \lambda_3 \dot{\omega}_3 - (\lambda_1 - \lambda_2)\omega_1\omega_2$$

### 19.1.4 Torque free tumbling

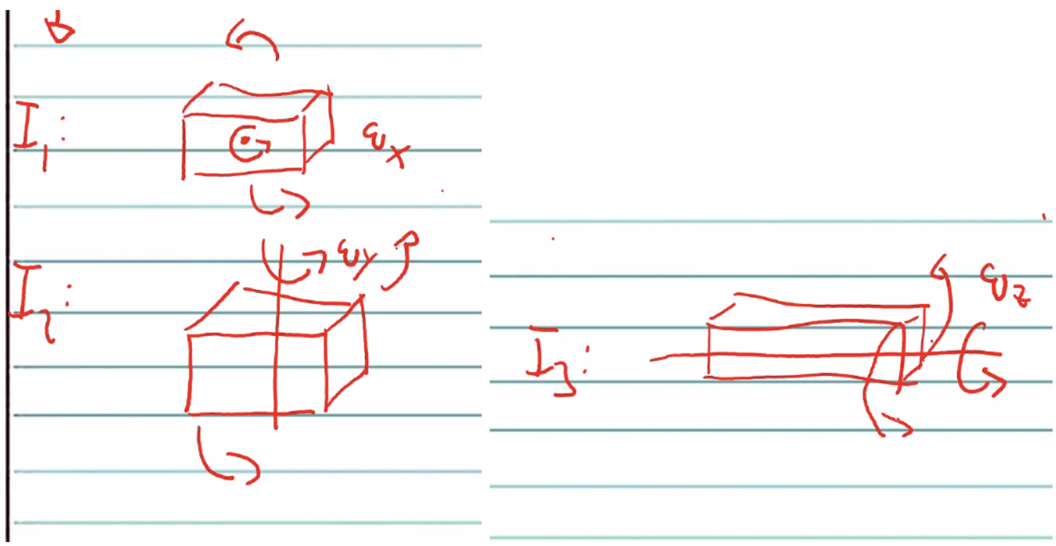
Consider a box of side lengths $a, 2a, 3a$ aligned as follows:




We can write the inertia tensor to be diagonal:

$$\mathbb{I} = \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix}$$

Where $I_1 > I_2 > I_3$ We can then have rotations around three possible axes:




We can now apply Euler's equations to learn something about this tumbling motion. Pick arbitrary axis (3) such that $\omega_3 \gg \omega_2 = \omega_1 \approx 0$, i.e. the body is rotating very fast about one axis and not the others. The Euler equations then give us:

$$\lambda_3 \dot{\omega}_3 = (\lambda_1 - \lambda_2)\omega_1\omega_2 \approx 0$$

$$\lambda_1 \dot{\omega}_1 = (\lambda_2 - \lambda_3)\omega_3\omega_2$$

$$\lambda_2 \dot{\omega}_2 = (\lambda_3 - \lambda_1)\omega_3\omega_1$$

The first equation tells us that $\dot{\omega}_3 \approx 0$ and the angular velocity stays relatively constant. Let us take the time derivative of the second equation to learn more information:

$$\lambda_1 \ddot{\omega}_1 = (\lambda_2 - \lambda_3)(\dot{\omega}_3 \omega_2 + \omega_3 \dot{\omega}_2)$$

By the first equation, $\dot{\omega}_3 \omega_2 \approx 0$. We can therefore use the third equation to eliminate $\dot{\omega}_2$, which gives us:

$$\lambda_1 \ddot{\omega}_1 \approx (\lambda_2 - \lambda_3)\omega_3 \frac{(\lambda_3 - \lambda_1)\omega_3 \omega_1}{\lambda_2}$$

We can rewrite this as:

$$\ddot{\omega}_1 = -k\omega_1$$

Where

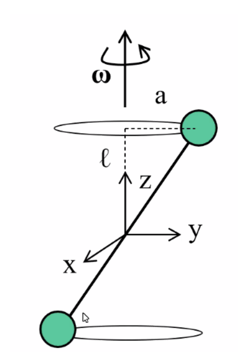
$$k = \frac{(\lambda_3 - \lambda_2)(\lambda_3 - \lambda_1)}{\lambda_1 \lambda_2}\omega_3$$

This is a very familiar differential equation. The sign of $k$ will tell us about the resulting motion.
(1) In the first case, suppose $\lambda_3 > \lambda_2$ and $\lambda_3 > \lambda_1$, or $\lambda_3 < \lambda_2$ and $\lambda_3 < \lambda_2$. In these two cases, we have that $k > 0$, and the above equation of motion for $\omega_1$ corresponds to simple harmonic motion (stable + oscillatory).
(2) In the second case, suppose $\lambda_1 < \lambda_3 < \lambda_2$ or $\lambda_2 < \lambda_3 < \lambda_1$, then $k$ is negative, and the above equation of motion for $\omega_1$ corresponds to real exponential solutions (unstable!)
Our conclusion is that the rotation around the middle axis is unstable. This is a mathematical explanation for the famous intermediate axis theorem/tennis racket theorem, which we can see realized here:
`https://www.youtube.com/watch?v=1n-HMSCDYtM`

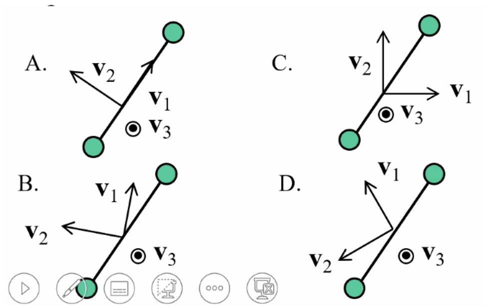
# 20 Lecture 20

## 20.1 Lecture Notes - Free Rotation of Spinning Top & Euler Angles

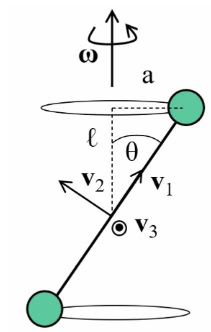
### 20.1.1 Euler Equations Review




A rotating dumbbell consists of two masses $m$ which move in circles at a $z$ displacement $l$ and $-l$, joined by a massless rod. The angular velocity vector is $\boldsymbol{\omega} = \omega\hat{\mathbf{z}}$. Consider the body frame where the positions of the masses are $(0, a, l)$ and $(0, -a, -l)$. What are the principle axes of inertia?




*Solution.* A). The principle axes of inertia are aligned with the symmetries of the body. If there is a symmetry axes, we can expect this to correspond to a principle axis. ☐




Followup: What is the components of the angular velocity vector in this frame? (where $\theta$ is the angle formed by the z-axis/rotation axis and the mass?)

*Solution.* $\boldsymbol{\omega} = \begin{bmatrix} \omega\cos\theta \\ \omega\sin\theta \\ 0 \end{bmatrix}$ by trigonometry. ☐

Consider the body frame aligned with the principle axes of inertia (as sketched). What are Euler's equations in this frame?

*Solution.* We first recall the three moments of inertia:

$$\lambda_1 = 0$$

$$\lambda_2 = 2m(a^2 + l^2)$$

$$\lambda_3 = 2m(a^2 + l^2)$$

We also recall the Euler equations:

$$\Gamma_1 = \lambda_1\dot{\omega}_1 - (\lambda_2 - \lambda_3)\omega_2\omega_3$$

$$\Gamma_2 = \lambda_2\dot{\omega}_2 - (\lambda_3 - \lambda_1)\omega_1\omega_3$$

$$\Gamma_3 = \lambda_3\dot{\omega}_3 - (\lambda_1 - \lambda_2)\omega_1\omega_2$$

In this case, we have that $\omega_3$ along $\mathbf{v}_3$ is zero (from the previous problem) and that the time derivatives of all of the $\omega_i$s are zero (as the dumbell rotates at constant velocity. From this we get:

$$\Gamma_1 = 0$$

$$\Gamma_2 = 0$$

$$\Gamma_3 = 2m(a^2 + l^2)\omega^2\sin\theta\cos\theta$$

☐

Cosnider the body frame aligned with the principle axes of inertia. In this frame, the torque is constant in the 3 directions (out of the page). How can you describe the torque in the space frame?

*Solution.* Since $\mathbf{L}$ is rotating and $\mathbf{\Gamma}$ is perpendicular to this and rotating with it (in the lab frame), we therefore have that $|\mathbf{\Gamma}|$ is constant and it is rotating about the z axis. □

What is the angular momentum in the body frame?

*Solution.* $\mathbf{L} = \mathbb{I}\boldsymbol{\omega}$, and since $\mathbb{I}$ is diagonal in the body frame, we have:

$$\mathbf{L} = \mathbb{I}\boldsymbol{\omega} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2m(a^2+l^2) & 0 \\ 0 & 0 & 2m(a^2+l^2) \end{bmatrix} \begin{bmatrix} \omega\cos\theta \\ \omega\sin\theta \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 2m(a^2+l^2)\omega\sin\theta \\ 0 \end{bmatrix}$$

□

### 20.1.2 Free Rotation of symmetric top

Here, we study the motion of a symmetric top. This means that $\lambda_1 = \lambda_2$. In addition, no torque, so $\mathbf{\Gamma} = \mathbf{0}$. Writing down the Euler equations (where the LHS will be zero), we then have:

$$0 = \lambda_1\dot{\omega}_1 - (\lambda_2 - \lambda_3)\omega_2\omega_3$$

$$0 = \lambda_2\dot{\omega}_2 - (\lambda_3 - \lambda_1)\omega_1\omega_3$$

$$0 = \lambda_3\dot{\omega}_3 - (\lambda_1 - \lambda_2)\omega_1\omega_2$$

Since $\lambda_1 - \lambda_2 = 0$, we therefore find that $\dot{\omega}_3 = 0$ and hence $\omega_3$ is constant (as lines up with our experience. Writing the other equations down (making the substitution that $\lambda_2 = \lambda_1$, we have:

$$\dot{\omega}_1 = \frac{\lambda_1 - \lambda_3}{\lambda_1}\omega_2\omega_3$$

$$\dot{\omega}_2 = -\frac{\lambda_1 - \lambda_3}{\lambda_1}\omega_1\omega_3$$

Let us define $\Omega_b = \frac{\lambda_1 - \lambda_3}{\lambda_1}\omega_3$, then we have:

$$\dot{\omega}_1 = \Omega_b\omega_2$$

$$\dot{\omega}_2 = -\Omega_b\omega 1$$

Let us add $i$ times the second equation to the first equation. Then, define $\eta = \omega_1 + i\omega_2$. We then have:

$$\dot{\omega}_1 + i\dot{\omega}_2 = \dot{\eta} = \Omega_b(\omega_2 - i\omega_1) = -i\Omega_b\eta$$

This has a complex exponential solution:

$$\eta(t) = \eta_0\exp(-i\Omega_b t)$$

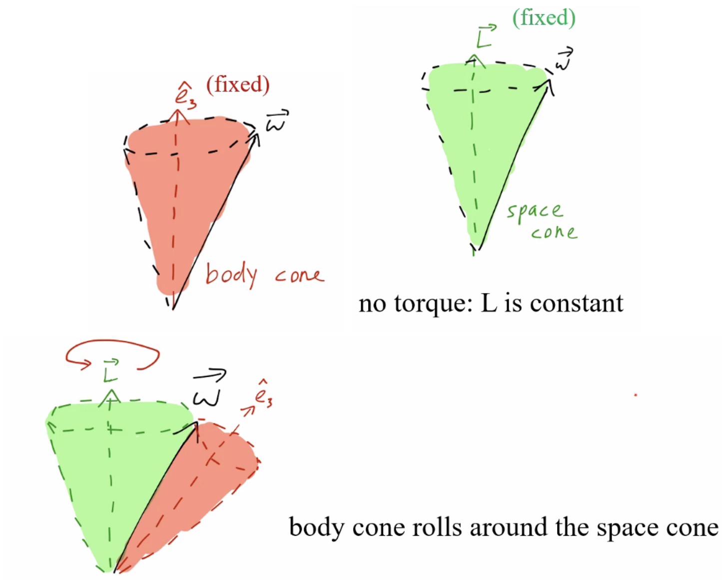
Suppose $\eta_0 = \omega_0$. then:

$$\eta(t) = \omega_0\exp(-i\Omega_b t)$$

Takign the real and imaginary parts to recover $\omega_1$ and $\omega_2$, we get:

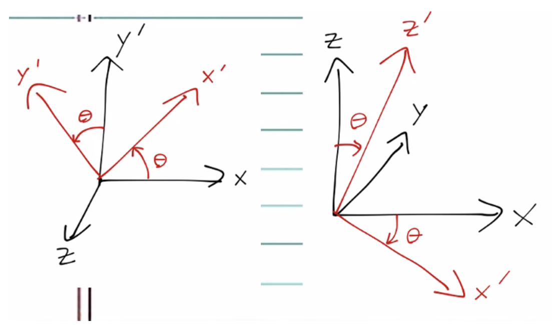
$$\boldsymbol{\omega} = \begin{bmatrix} \omega_0\cos(\Omega_b t) \\ -\omega_0\sin(\Omega_b t) \\ \omega_3 \end{bmatrix}$$

From which we can see tha the free top undergoes precession. We can check that $\dot{bm\omega} = \mathbf{\Omega}_b \times \boldsymbol{\omega}$ which would indeed correspond to rotation. We note that $|\boldsymbol{\omega}|$ is a constant here. Visually, we could think of these as follows:




### 20.1.3 Rotation Matrices

We need to establish a more systematic way to go from body to lab frame (i.e. rotating the coordinate system).




These rotations are performed through rotation matrices. The first picture is a rotation around the z axis. The matrix that does this is

$$R_z = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The second picture is a rotation around the y axis. The matrix that does this is:

$$R_y = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}$$

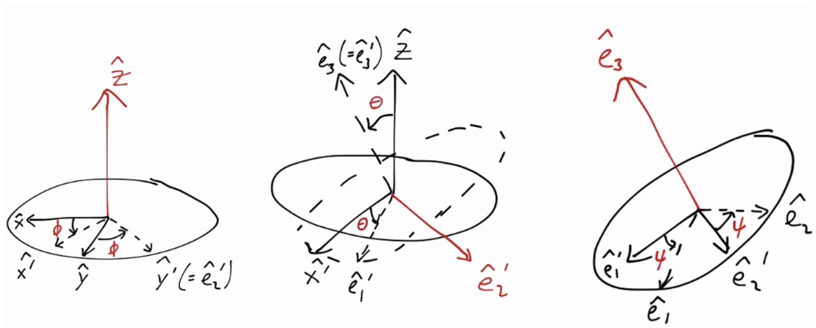
Where the minus sign is switched in order to preserve the handedness of the coordinate system. X is the same, with:

$$R_x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix}$$

In general, to go from one coordinate system to another, we can decompose the rotations into rotations about x, y, and z. But note that the **order matters**! This brings us to discussion of Euler angles.

### 20.1.4 Euler Angles

The Euler angles gives us a convention for the order of rotations.



1. First, we rotate around the z-axis by $\phi$.
2. Next, we rotate around the **new** y axis by $\theta$
3. Finally, we rotate around the **new** z-axis by $\psi$.

Next day: A general Lagrangian for rigid systems, and then applying this to a spinning top with Torque applied to it. Then, we get into Hamiltonian mechanics!

# 21 Lecture 21

## 21.1 Lecture Notes - Lagrangian for Rigid Body, Spinning Top with Torque

### 21.1.1 Review

If $\mathbf{v}$ is a vector in one frame and $\mathbf{v}'$ is the same vector in a rotated frame, we showed that $\mathbf{v}' = A\mathbf{v}$, where $A$ is a rotation matrix (matrix of direction cosines). This determines $\mathbf{v}'$ given $A$ and $\mathbf{v}$. If instead we know $\mathbf{v}'$ and $A$, how can we find $\mathbf{v}$?

*Solution.* $\mathbf{v} = A^T\mathbf{v}'$. For a rotation, $A^T = A^{-1}$ so this corresponds to multiplying by the inverse. □

For a rotation matrix, $A^TA = I$. What does this imply about the determinant of $A$?

*Solution.* $\det A = 1$, as $\det I = 1$ so $\det A^TA = \det A^T \det A = 1$ and hence $\pm 1$ is possible, but, $\det A = -1$ corresponds to a reflection (Rather than a rotation). So for a rotation, we have unit determinant. □

What is the correct matrix for rotation about the x-axis?

*Solution.* The correct matrix is:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix}$$

□

Using Euler angles, we wish to construct a rotation matrix that rotates first by angle $\varphi$ about the z-axis, then $\theta$ about the x' axis, then $\psi$ about the z" axis. How should we multiply the three rotation matrices to get the total/final rotation matrix $A(\varphi, \theta, \psi)$?

*Solution.* Recalling order of matrix application (right to left) we have:

$$A(\varphi, \theta, \psi) = A_\psi A_\theta A_\varphi$$

□

What is the matrix $A^{-1}$ that reverses this series of rotations?

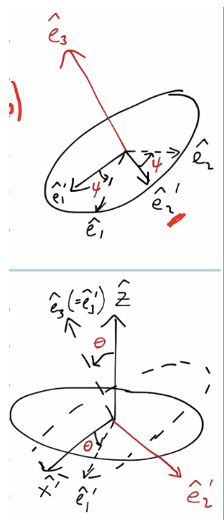
*Solution.* The inverse is:

$$A^{-1} = A^T(\varphi, \theta, \psi) = A_\varphi^T A_\theta^T A_\psi^T]$$

□

Last lecture, we looked at free rotation of a symmetric top. Given $\boldsymbol{\omega}$ and $\mathbf{L}$, we found that both precess/rotate about the symmetry axis with angular velocity $\Omega_B = \frac{\lambda_1 - \lambda_3}{\lambda_1}\omega_3$ which on earth is $\omega_3/300$. In the lab frame, there is no torque, so $\mathbf{L}$ is constant and hence we found the result that the body cone "rolls around" the space cone, and that $\boldsymbol{\omega}$ and $\hat{\mathbf{e}}_3$ precess about $\mathbf{L}$ with $\Omega_S = L/\lambda_1$.

### 21.1.2 Lagrangian of Rigid Body



We will use the Euler angles to get the general Lagrangian. Recall that $T = \frac{1}{2}\boldsymbol{\omega} \cdot \mathbf{L}$. Using Euler Angles, we have:

$$\boldsymbol{\omega} = \dot{\phi}\hat{\mathbf{z}} + \dot{\theta}\hat{\mathbf{e}}_2' + \dot{\psi}\hat{\mathbf{e}}_3$$

this is the vector sum of these three components, each of which are written in a different coordinate system (related by rotations). We will now expand this in the body frame. We know that:

$$\hat{\mathbf{e}}_2' = R_{e3}\hat{\mathbf{e}}_2 = \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \sin\psi\hat{\mathbf{e}}_1 + \cos\psi\hat{\mathbf{e}}_2$$

In a similar way, we get:

$$\hat{\mathbf{e}}_1' = \cos\psi\hat{\mathbf{e}}_1 - \sin\psi\hat{\mathbf{e}}_2$$

Note that this is a passive transformation. Next, we determine:

$$\hat{\mathbf{z}} = R_{e2'}\hat{\mathbf{e}}_3 = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = -\sin\theta\hat{\mathbf{e}}_1' + \cos\theta\hat{\mathbf{e}}_3' = -\sin\theta\cos\psi\hat{\mathbf{e}}_1 + \sin\theta\sin\psi\hat{\mathbf{e}}_2 + \cos\theta\hat{\mathbf{e}}_3$$

Therefore plugging back into the original equation and collecting the terms, we get:

$$\boldsymbol{\omega} = (\dot{\theta}\sin\psi - \dot{\phi}\sin\theta\cos\psi)\hat{\mathbf{e}}_1 + (\dot{\theta}\cos\psi + \dot{\phi}\sin\theta\sin\psi)\hat{\mathbf{e}}_2 + (\dot{\phi}\cos\theta + \dot{\psi})\hat{\mathbf{e}}_3$$

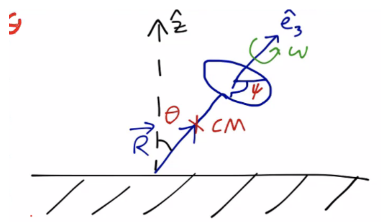
Therefore, in the body frame, the kinetic energy is easy to calculate (as the inertia tensor is diagonal) so:

$$T = \frac{1}{2}\left(\lambda_1\omega_1^2 + \lambda_2\omega_2^2 + \lambda_3\omega_3^2\right) = \frac{\lambda_1}{2}(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta) + \frac{\lambda_3}{2}(\dot{\psi} + \dot{\phi}\cos\theta)^2$$

Where we suppose that $\lambda_1 = \lambda_2$ as the top is symmetric.
Question: How does the kinetic energy in the body frame $T = \frac{1}{2}\boldsymbol{\omega}\cdot\mathbb{I}\boldsymbol{\omega}$ compare to the kinetic energy in the space frame?

*Solution.* We will come back to this on Wednesday. Try a conversion between the two frames by a rotation and see what you get. □

### 21.1.3 Symmetry Spinning Top with Torque (Gravity)




Now, we consider a spinning top with gravity. We have that the Lagrangian gives:

$$\mathcal{L} = \frac{1}{2}\left[\lambda_3\left(\dot{\psi} + \dot{\phi}\cos\theta\right)^2 + \lambda_1\left(\dot{\phi}^2\sin^2\theta + \dot{\theta}^2\right)\right] - mgR\cos\theta$$

From which we can see that the total energy, the generalized momentum associated with $\psi$, and the generalized momentum associated with $\phi$ are conserved (The Lagrangian does not depend on $\psi$ or $\phi$, so they are cyclic and conserved). Calculating $p_\psi$, we get:

$$p_\psi = \frac{\partial\mathcal{L}}{\partial\dot{\psi}} = \lambda_3(\dot{\psi} + \dot{\phi}\cos\theta) = \lambda_3\omega_3 = L_3 = C$$

Which is just the angular momentum in the e3 direction in the body frame. Doing the same for $p_\phi$, we have:

$$p_\phi = \frac{\partial\mathcal{L}}{\partial\dot{\phi}} = \lambda_1\dot{\phi}\sin^2\theta + \lambda_3(\dot{\phi}\cos\theta + \dot{\psi})\cos\theta = L_z = C$$

Which is the z-component of the angular momentum in the space frame, which makes sense if we think about the effects of gravity. Finally, we calculate the equation for $\theta$ (which is the only quantity that can evolve:

$$\lambda_1\ddot{\theta} = \lambda_1\dot{\phi}^2\sin\theta\cos\theta - \lambda_3\dot{\phi}\sin\theta(\dot{\psi} + \dot{\phi}\cos\theta) + MgR\sin\theta$$

Now, we consider the simple situation where $\theta$ is constant. Then, $\dot{\theta} = \ddot{\theta} = 0$, and from the equations for $\dot{\phi}$ and $\dot{\psi}$ we see that the also are constant. Then, let us call $\dot{\phi} = \Omega$. Writing the EOM for $\theta$, we have:

$$0 = \lambda_1\Omega^2\cos\theta - \lambda_3\Omega\omega_3 + MgR$$

This is a quadratic equation for $\Omega$, from which we can solve for it. For $\omega_3$ large, we get two solutions:

$$\Omega_1 \approx \frac{\lambda_3\omega_3}{\lambda_1\cos\theta}$$

Which is actually independent of $g$! It is the same precession we saw for the free top. The other frequency is given by:

$$\Omega_2 \approx \frac{MgR}{\lambda_3\omega_3}$$

Which is precession due to the gravitational torque.

# 22 Lecture 22

## 22.1 Lecture Notes - Spinning Top with Torque, Nutation, Intro to Hamiltonian Mechanics

### 22.1.1 Returning to a Question

We determined the scalar product $\mathbf{L}^T\mathbf{L}$ in the body frame. How does this compare to the scalar product in the space frame?

*Solution.* The scalar product should be the same under rotation (invaraince under rotation). Going from the body frame to the space frame is just a rotation, and rotation preserves the angle between and the magnitude of two vectors. Another way to think about it:

$$\mathbf{L}^T\mathbf{L} = \mathbf{L}'^T R^T R \mathbf{L}' = \mathbf{L}'^T \mathbf{L}'$$

□

How does the rotational kinetic energy in the body frame $T = \frac{1}{2}\boldsymbol{\omega}^T \mathbb{I} \boldsymbol{\omega}$ compare to the rotational kinetic energy in the space frame.

*Solution.* As above, the kinetic energy in the body frame and space frames should be equivalent. Mathematically:

$$\frac{1}{2}\boldsymbol{\omega}^T \mathbb{I} \boldsymbol{\omega} = \frac{1}{2}\boldsymbol{\omega}^{T'} R R^T \mathbb{I} R R^T \boldsymbol{\omega}' = \frac{1}{2}\boldsymbol{\omega}^{T'} \mathbb{I} \boldsymbol{\omega}'$$

□

### 22.1.2 Spinning top with gravity/Nutation

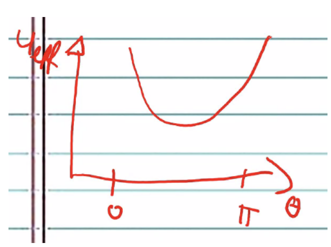
Last time, we derived an equation of motion for $\theta$ of a spinning top, and solved the equations for the case where $\theta$ was held constant. From this, we obtained the precession of the spinning top under gravity. We now consider the more general case where $\theta$ varies with time (is not constant). We will consider explicitly $\theta(t)$, and find "nutation"; that is, nodding repeatedly. Let us consider the energy:

$$E = T + U = \frac{\lambda_1}{2}\dot{\theta}^2 + U_{eff}(\theta)$$

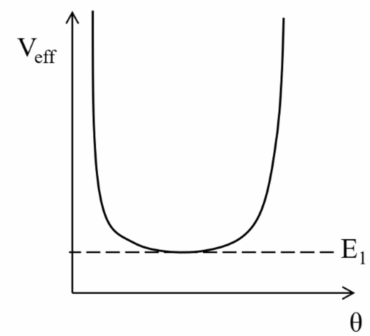
Where:

$$U_{eff}(\theta) = \frac{(p_\phi - p_\psi \cos\theta)^2}{2\lambda_1 \sin^2\theta} + \frac{p_\psi^2}{2\lambda_3} + MgR\cos\theta$$

(This was obtained just by replacing terms of $\dot{\phi}$ and $\dot{\psi}$ with the generalized momenta). This looks like the two body problem reduced to a one body problem with effective potential. We first notice that $MgR\cos\theta$ is negative for $\theta > \frac{\pi}{2}$ and $\frac{(p_\phi - p_\psi\cos\theta)^2}{2\lambda_1 \sin^2\theta} + \frac{p_\psi^2}{2\lambda_3}$ is positive. At $\theta = 0, \pi$ we have that $U_{eff} \to \infty$. Graphically:



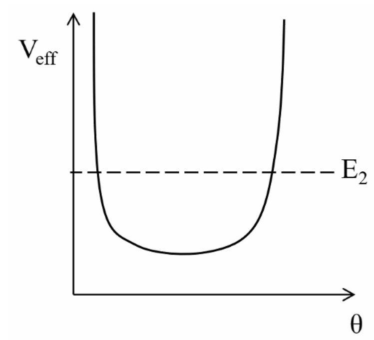

Now consider:




If a symmetric top has total effective energy $E_1$, then what type of motion does this top undergo?

*Solution.* The top undergoes constant precession about the vertical axis, at a constant $\theta$; we have not excited the one dimensional system, and $\theta$ must hence be constant (stay in the minimum) □

If we now give it some energy:



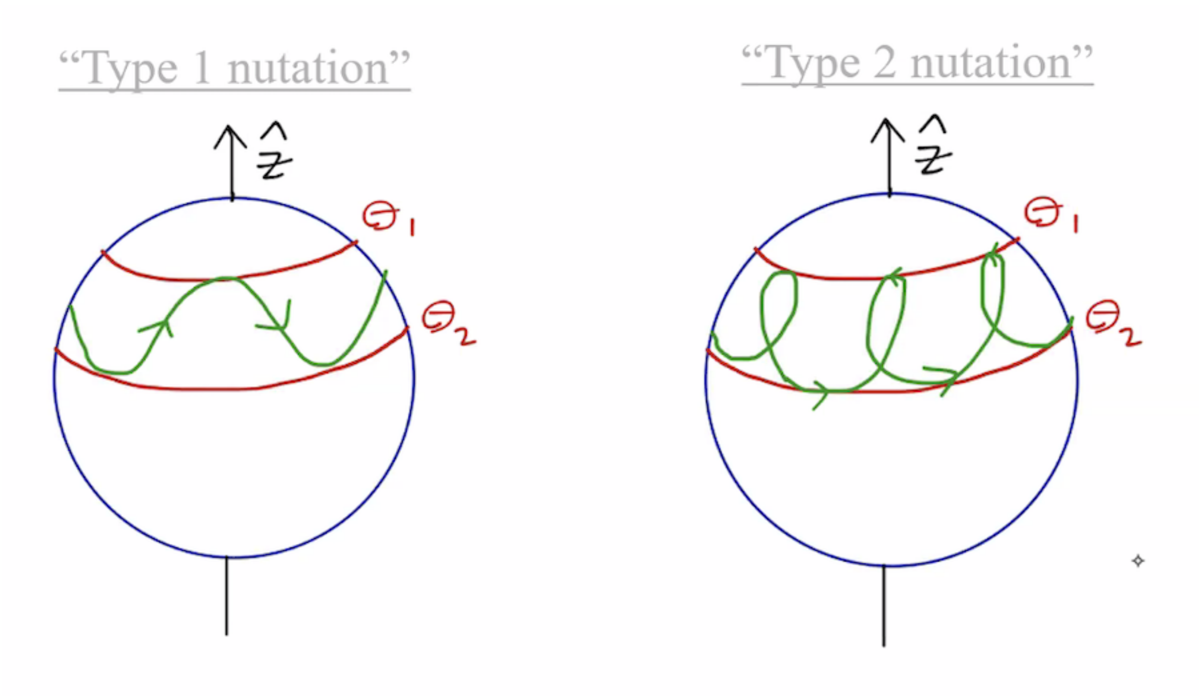

What is the motion?

*Solution.* The top will now undergo precession combined with nutation, as $\theta$ oscillates about the minimum. □

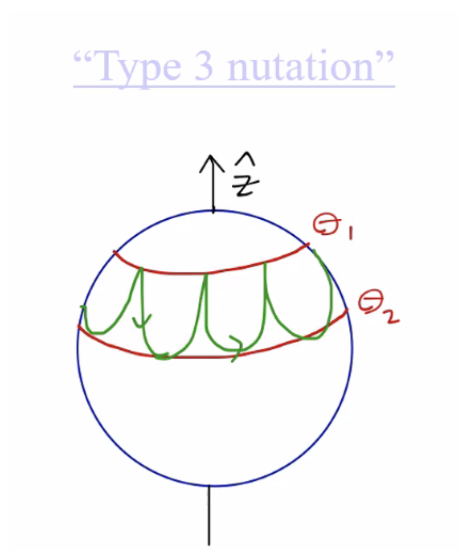
There are several qualitatively different scenarios that can occur. We have the equation:

$$\dot{\phi} = \frac{p_\phi - p_\psi \cos\theta}{\lambda_1 \sin^2\theta}$$

Two possible scenarios are:




The first scenario occurs where $p_\phi > p_\psi$ and $\dot{\phi} > 0$. The second scenario occurs where $p_\phi \approx p_\psi$, then we have $p_\phi < p_\psi \cos\theta$ and $p_\phi > p_\psi \cos\theta_2$ and hence we have that $\dot{\phi}$ changes direction (which is quite cool!) There is also a third case, where we get cusps:




Where $p_\phi \approx p_\psi$ and we can have $p_\phi = p_\psi \cos\theta_1$ or $p_\phi = p_\psi \cos\theta_2$.

#### 22.1.3 Intro to Hamiltonian Mechanics - Motivation

Why do we need another form of mechanics? Newton was nice and intuitive, and Lagrange is much easier if we have constraints in our system and want to solve problems without having to explicitly worry about the constraints. Why can't we just be happy with Lagrange and leave it as it is? Reason 1 is that its cool/beautiful. From a practical perspective, it doesn't give us much of an advantage over Lagrange, but it will allow us to give a richer understanding of dynamical systems and conserved quantities. There is also an advantage when we want to link this to the most important theory of 20th century physics, that is, quantum mechanics.

#### 22.1.4 What was the Hamiltonian, Again?

As a reminder, we have the Lagrangian:

$$\mathcal{L} = \mathcal{L}(q_1, q_2, \ldots, q_n, \dot{q}_1, \dot{q}_2, \cdots, \dot{q}_n)$$

We also introduced the generalized, or canonical momentum:

$$p_i = \frac{\partial \mathcal{L}}{\partial \dot{q}_i}$$

We then defined the Hamiltonian:

$$\mathcal{H} = \sum_{i=1}^{m} p_i \dot{q}_i - \mathcal{L}$$

When the transformation from cartesian to generalized coordinates was natural (did not depend on time), then $\mathcal{H}$ was just the total energy. For further applications in this course, we will generally assume this is the case. The Hamiltonian is a function:

$$\mathcal{H} = \mathcal{H}(q_1, q_2, \ldots, q_n, \dot{q}_1, \dot{q}_2, \cdots, \dot{q}_n, t)$$

The main difference is that we trade between the generalized velocities (Lagrangian) with the generalized momentum (Hamiltonian). How does this occur? We will see in a moment, but let us do a basic example first.

#### 22.1.5 The Harmonic Oscillator

The Lagrangian for the Harmonic oscillator is:

$$\mathcal{L} = \frac{m}{2}\dot{x}^2 - k\frac{x^2}{2}$$

So the generalized momentum is:

$$\frac{\partial \mathcal{L}}{\partial \dot{x}} = m\dot{x} = p$$

So calculating $\mathcal{H}$, we have:

$$\mathcal{H} = m\dot{x}^2 - \mathcal{L} = \frac{m}{2}\dot{x}^2 + k\frac{x^2}{2} = \frac{p^2}{2m} + k\frac{x^2}{2} = T + U$$

Calculating $\mathcal{H}$ is pretty straightforward.

#### 22.1.6 The Hamiltonian as the Legendre Transform of L

Consider a function $f(x, y)$. The total differential is then given by:

$$df = \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy = udx + vdy$$

Now, we want to "trade" the differentials, $x \mapsto u$. We can do this via a function $g = f - ux$. Let's check:

$$dg = df - udx - xdu$$

But we've already calculated $df$, so:

$$df = udx + vdy - udx - xdu = vdy - xdu$$

So as advertised, $g$ is the Legendre transform of $f$, and it depends explictly on the variables $u$ and $y$:

$$g = g(u, y)$$

To recap, we look at the total differential, we want to trade one function against the other, we then construct a function such that the independent variables are swapped. This has connections to thermodynamics; we recall that the energy of a thermodynamic system is dependent on the natural variables of number of particles, volume, and entropy, that is $E = E(N, V, S)$. But we had other relevant quantities, such as the Hemholtz free energy; $F = E - TS = F(N, V, T)$. Hence, the varaibles $S$ and $T$ are conjugate to each other. In many cases, its more convenient to use $F$ as it is easier to control the temperature rather than the entropy. In classical mechanics, it might be more convenient to work with the momenta rather than velocities, and easier to work with $\mathcal{H}$ rather than $\mathcal{L}$.

#### 22.1.7 Hamilton Equations of Motion

We need to see what kind of equations of motion come out of this formulation of classical mechanics. Let's write the hamiltonian as:

$$\mathcal{H} = \sum_i p_i \dot{q}_i - \mathcal{L}$$

Now, consider the total differential of $\mathcal{H}$. We have:

$$d\mathcal{H} = \sum_i (\dot{q}_i dp_i + p_i d\dot{q}_i) - \frac{\partial \mathcal{L}}{\partial q_i} dq_i - \frac{\partial \mathcal{L}}{\partial \dot{q}_i} d\dot{q}_i - \frac{\partial \mathcal{L}}{\partial t} dt$$

If we recall the definitions, the $\frac{\partial \mathcal{L}}{\partial \dot{q}_i}$ term is just $p_i$, and $\frac{\partial \mathcal{L}}{\partial q_i}$ is just $\dot{p}_i$ if we use the Euler-Lagrange equation. Hence the above becomes:

$$d\mathcal{H} = \sum_i (-\dot{p}_i dq_i + \dot{q}_i dp_i) - \frac{\partial \mathcal{L}}{\partial t} dt$$

Hence we can read off the differential:

$$\boxed{\frac{\partial \mathcal{H}}{\partial p_i} = \dot{q}_i, \frac{\partial \mathcal{H}}{\partial q_i} = -\dot{p}_i}$$

And these are Hamilton's equations! We will look at these more closely on Friday.

# 23 Lecture 23

## 23.1 Lecture Notes - Hamiltonian Mechanics

### 23.1.1 Hamilton's Equations & Properties

A particle slides on a helical wire defined by $z = k\theta$, $r = R$. If we use $\theta$ as the generalized coordinate, the generalized momentum is $p = m(r^2 + k^2)\dot{\theta}$. What is the Hamiltonian of the system?

*Solution.* Using the fact that here the Hamiltonian is equal to the energy:

$$\mathcal{H} = T + U = \frac{p^2}{2m} + mgk\theta$$

□

Last lecture, we derived Hamilton's equations of motion:

$$\boxed{\frac{\partial \mathcal{H}}{\partial p_i} = \dot{q}_i, \quad \frac{\partial \mathcal{H}}{\partial q_i} = -\dot{p}_i}$$

This was done by starting with the Hamiltonian (Legendre transform of $\mathcal{L}$), expanding out in the total differential of $\mathcal{H}$, then using the EL equation to solve for Hamilton's equation (see last day's notes for more detail). We notice that Hamilton's equations are two first order equations, rather than a single second order equation. In general, if we have $n$ degrees of freedom, the Lagrangian gives us $n$ 2nd order differential equations, while Hamilton gives us $2n$ 1st order differential equations.
Now suppose that the Hamiltonian is independent of $p_i$. What can we say about the system?

*Solution.* $q_i$ is constant by Hamilton's equations. □

Now, what do Hamilton's equations tell us about the derivatives $\frac{\partial \dot{q}_i}{\partial q_i} + \frac{\partial \dot{p}_i}{\partial p_i}$?

*Solution.* Again using Hamilton's equation, these are equal to:

$$\frac{\partial \mathcal{H}}{\partial q_i p_i} - \frac{\partial \mathcal{H}}{\partial p_i q_i} = 0$$

By the equality of mixed partials. □

### 23.1.2 The Variational Principle, Revisited

From Lagrangian mechanics, we have that the variation in the action is zero:

$$\delta S = \delta \int_{t_1}^{t_2} \mathcal{L} dt = \delta \int_{t_1}^{t_2} [\sum p_i \dot{q}_i - \mathcal{H}] dt = 0$$

Expanding this, we have:

$$\int_{t_1}^{t_2} \sum_i \left[ -\left(\dot{p}_i + \frac{\partial \mathcal{H}}{\partial q_i}\right) \delta q_i + \left(\dot{q}_i - \frac{\partial \mathcal{H}}{\partial p_i}\right) \delta p_i \right] dt = 0$$

What does this imply about the terms in parentheses?

*Solution.* Since $q$ and $p$ can be varied independently, both $\dot{p}_i + \frac{\partial \mathcal{H}}{\partial q_i}$ and $\dot{q}_i - \frac{\partial \mathcal{H}}{\partial p_i}$ must vanish independently; these are exactly Hamilton's equations of motion! □

### 23.1.3 Hamiltonian Time Dependence

Let us now look at the time derivative of the Hamiltonian:

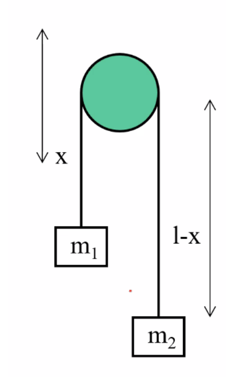
$$\frac{\mathrm{d}\mathcal{H}}{\mathrm{d}t} = \sum_i \left( \frac{\partial \mathcal{H}}{\partial q_i} \dot{q}_i + \frac{\partial \mathcal{H}}{\partial p_i} \dot{p}_i \right) + \frac{\partial \mathcal{H}}{\partial t}$$

Assuming the trajectory obeys Hamilton's equations, we have:

$$\frac{\mathrm{d}\mathcal{H}}{\mathrm{d}t} = \sum_i (-\dot{p}_i \dot{q}_i + \dot{q}_i \dot{p}_i) + \frac{\partial \mathcal{H}}{\partial t} == \frac{\partial \mathcal{H}}{\partial t} = -\frac{\partial \mathcal{L}}{\partial t}$$

Even though $\mathcal{H}$ is a function of $q, p, t$, the only explicit time dependence comes from the Hamiltonian itself! $\mathcal{H}$ is constant if $\mathcal{L}$ is independent of time.

### 23.1.4 The Atwood Machine, Again




The Atwood machine has Hamiltonian:

$$\mathcal{H} = \frac{p^2}{2(m_1 + m_2)} - (m_1 - m_2)gx - m_2 gl$$

What are the equations of motion?

*Solution.* By Hamilton's equations:

$$\dot{x} = \frac{p}{m_1 + m_2}$$
$$\dot{p} = (m_1 - m_2)g$$

□

How does the system move?

*Solution.* From above, we can see that $x$ moves with constant gravitational acceleration with effective mass

$$\frac{m_1 - m_2}{m_1 + m_2}$$

Which we can see from:

$$\ddot{x} = \frac{m_1 - m_2}{m_1 + m_2} g$$

We can also reason this from limits of $m_1 \ll m_2$, $m_1 \gg m_2$. □

### 23.1.5 Phase Space of the Harmonic Oscillator

Phase space is the set of points $\{q_i, p_i\}$ i.e. possible combinations of (generalized) position/momenta. We can write the phase space vector as $\mathbf{z} = (\mathbf{q}, \mathbf{p})$. For the Harmonic oscillator in one dimension (spring constant $k$, mass $m$ distance from equlibrium $x$), we have the Hamiltonian:

$$\mathcal{H} = \frac{p_x^2}{2m} + \frac{k}{2} x^2$$

Hamilton's equations give:

$$\dot{p}_x = -\frac{\partial \mathcal{H}}{\partial x} = -kx$$
$$\dot{x} = \frac{\partial \mathcal{H}}{\partial p_x} = \frac{p_x}{m}$$

Of course, we may combine these to get:

$$\ddot{x} = \frac{p_x}{m} = -\frac{k}{m}x$$

We may write Hamilton's equations as a vector:

$$\begin{bmatrix} \dot{x} \\ \dot{p}_x \end{bmatrix} = \begin{bmatrix} p_x/m \\ -kx \end{bmatrix}$$

We know that this will have solutions:

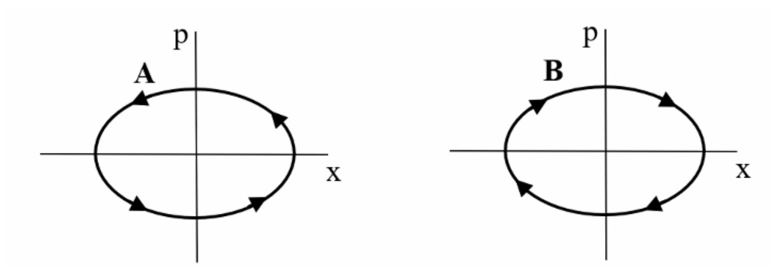
$$x = x_0 \cos\left(\sqrt{\frac{k}{m}}t - \delta\right)$$

$$p_x = m\dot{x} = -mx_0\sqrt{\frac{k}{m}} \sin\left(\sqrt{\frac{k}{m}}t - \delta\right)$$

Now, we observe:

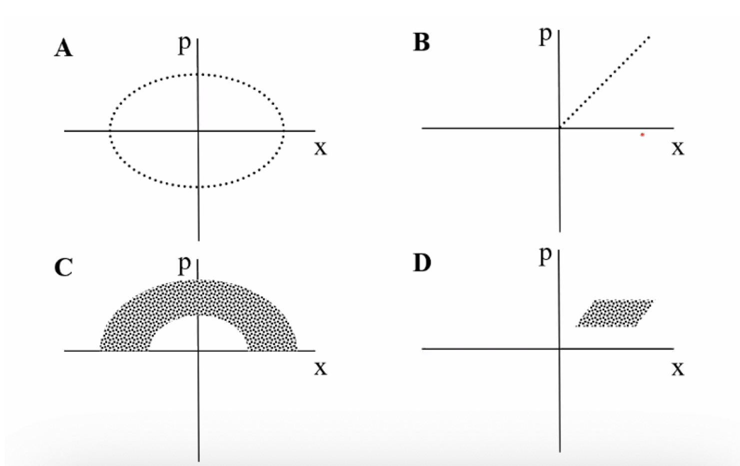
$$\frac{x^2}{x_0^2} + \frac{p_x^2}{mx_0^2 k} = \sin^2(x) + \cos^2(x) = 1$$

But this of course is just the equation for an ellipse, that tells us that the trajectory in phase space of the harmonic oscillator will be an ellipse! In particular, which of the two directions (if both) are possible?



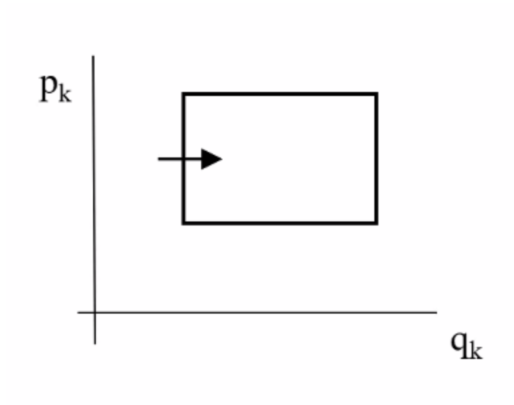

*Solution.* Only B; consider that when we sit at the rightmost point in the ellipse, we get pulled back towards the origin, hence the magnitude of the momentum increases, in the negative direction. Hence only the second trajectory makes sense. □

Now, consider a collection of harmonic oscillators which have the same energy but different relative phases. Which collection of phase points represents this system at a given instant in time?



*Solution.* For a given fixed energy, we stay on the ellipse, and the phase angle just tells you where you are on the ellipse. So, A. □

Consider an area element in phase space. What physically must happen for a particle to move into the area across the left boundary?




*Solution.* The position $q_k$ increases but the momentum $p_k$ is constant. □

### 23.1.6 Particle in a Central Force Field

The kinetic energy for a particle of mass $m$ in a central force field is given by:

$$T = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\phi}^2\right)$$

Then we have the generalized momenta:

$$p_r = \frac{\partial T}{\partial \dot{r}} = m\dot{r} \implies \dot{r} = \frac{p_r}{m}$$

$$p_\phi = \frac{\partial T}{\partial \dot{\phi}} = mr^2\dot{\phi} \implies \dot{\phi} = \frac{p_\phi}{mr^2}$$

Hence the Hamiltonian has the form:

$$\mathcal{H} = T + U = \frac{1}{2m}\left(p_r^2 + \frac{p_\phi^2}{r^2}\right) + U(r)$$

Hence Hamilton's equations yield:

$$\dot{r} = \frac{\partial \mathcal{H}}{\partial p_r} = \frac{p_r}{m}, \quad \dot{p}_r = -\frac{\partial \mathcal{H}}{\partial r} = \frac{p_\phi^2}{mr^3} - \frac{\mathrm{d}U}{\mathrm{d}r}$$

$$\dot{\phi} = \frac{\partial \mathcal{H}}{\partial p_\phi} = \frac{p_\phi}{mr^2}, \quad p_\phi = -\frac{\partial \mathcal{H}}{\partial \phi} = 0$$

### 23.1.7 General Procedure for setting up Hamilton's equations

1. Choose suitable generalized coordinates, $q_1, \cdots, q_n$.

2. Write down the kinetic and potential energies, $T$ and $U$, in terms of the $q$ 's and $\dot{q}$ 's.

3. Find the generalized momenta $p_1, \cdots, p_n$. (We are now assuming our system is conservative, so $U$ is independent of $\dot{q}_i$ and we can use $p_i = \partial T/\partial \dot{q}_i$. In general, one must use $p_i = \partial \mathcal{L}/\partial \dot{q}_i$.
4. Solve for the $\dot{q}$ 's in terms of the $p$ 's and $q$ 's.
5. Write down the Hamiltonian $\mathcal{H}$ as a function of the $p$ 's and $q$ 's. [Provided our coordinates are "natural" (relation between generalized coordinates and underlying Cartesians is independent of time), $\mathcal{H}$ is just the total energy $\mathcal{H} = T + U$, but when in doubt, use $\mathcal{H} = \sum p_i \dot{q}_i - \mathcal{L}$. See Problems 13.11 and 13.12.].
6. Write down Hamilton's equations (13.25).

We will illustrate the advantages of using Hamilton's equations rather than Lagrange next week. One advantage is that we get first order versus second order equations, which can be easier sometimes (we have more techniques). A deeper advantage is theoretical, which we will see. We will also see how to construct a quantum theory with this framework.

# 24 Lecture 24

## 24.1 Lecture Notes - Phase Space & Canonical Transformations

### 24.1.1 Cyclic/Ignorable Coordinates

If $\mathcal{L}$ is independent of a generalized coordinate $q_i$, then for the associated generalized momentum we have that $\dot{p}_i = 0$ and hence the generalized momentum is conserved. Equivalently, if $\mathcal{H}$ is independent of of a generalized coordinate $q_i$, the same conclusion holds (as $\frac{\partial \mathcal{L}}{\partial q_i} = -\frac{\partial \mathcal{H}}{\partial q_i}$.) With Hamiltonian dynamics, you can do another thing; consider $\mathcal{H} = \mathcal{H}(q_1, p_1, p_2)$ and with $p_2 = \text{Const.} = K$, then we can reduce the degrees of freedom by one explicitly; that is, $\mathcal{H}(q, p_1, K)$. Hence we have one fewer equations of motion directly. We have reduced the complexity by 1 DoF. It's not necessarily easier to solve Hamilton's equations than Lagrange's, but we may be able to reduce the number of coordinates by using Hamilton's formalism.

### 24.1.2 Phase Space Vectors

In the Lagrangian formulation, the equations of motion are of the form $f(\ddot{q}, \dot{q}, q) = 0$ (2nd order DEs). As we have said in this course before (and has been covered in differential equations courses), we can convert this second order DE into a system of two differential equations. If we define $s = \dot{q}, \dot{s} = \ddot{q}$, then we have:

$$f(\dot{s}, s, q) = 0$$

Hence we have the two first order differential equations:

$$\dot{s} = \ddot{q}, f(\dot{s}, s, q) = 0$$

Hamilton's equations of motion read:

$$\dot{q}_i = \frac{\partial \mathcal{H}}{\partial p_i} = f_i(\mathbf{q}, \mathbf{p})$$

$$\dot{p}_i = -\frac{\partial \mathcal{H}}{\partial q_i} = g_i(\mathbf{q}, \mathbf{p})$$

Both p and q are independent variables, so we might as well combine them into a single vector, which is the *phase space vector*:

$$\mathbf{z} = (\mathbf{q}, \mathbf{p})$$

Hence we may write the equations as:

$$\dot{\mathbf{z}} = \mathbf{f}(\mathbf{z})$$

So we have a system of first order DEs. This may be beneficial as we have many mathematical techniques to solve first order DEs. Note that in particular we may write:

$$\dot{\mathbf{z}} = J \cdot \nabla_{\mathbf{z}} \mathcal{H}(z, t)$$

What is the form of the matrix $J$?

*Solution.*

$$J = \begin{bmatrix} 0 & \cdots & I \\ \vdots & \ddots & \vdots \\ -I & \cdots & 0 \end{bmatrix}$$

Where the 0s and $I$s represent blocks of the matrices. Note that $\dot{q}_i = \frac{\partial \mathcal{H}}{\partial p_i}$ and $\dot{p}_i = -\frac{\partial \mathcal{H}}{\partial q_i}$. Note that $J$ is known as the "metric of phase space". □

#### 24.1.3 Canonical Transformations

In Lagrangian mechanics, we can take the generalized coordinates $\mathbf{q} = \mathbf{q}(q_1, \ldots, q_n)$ that can be replaced by $Q = (Q_1, \ldots, Q_n)$ with $\mathbf{Q} = \mathbf{Q}(\mathbf{q})$ and for which the Euler-Lagrange equations are still valid. The question becomes: Can we do something similar with Hamilton's equations? Can we transform $q_i \mapsto Q_i = Q_i(q, p, t)$ and $p_i \mapsto P_i = P_i(q, p, t)$ so that:

$$\dot{Q}_i = \frac{\partial K}{\partial P_i}$$

$$\dot{P}_i = -\frac{\partial K}{\partial Q_i}$$

with a new hamiltonian $K$? The answer turns out to be not in general.
As a counterexample, we can have $Q = p, P = q$ with $H = K$. The transform of phase space is a wider class than the transformation of just the coordinate space, so there will only be a subset of the many possible transformation that leave the Hamilton's equation of motion form invariant. Such a subset is known as the canonical transformations.
Recall in Lecture 6 that we talked about how general the Lagrangian was, and we showed that one can add a total derivative $\frac{\mathrm{d}F}{\mathrm{d}t}$ to $\mathcal{L}$ and not change the equations of motion. This is because adding $\frac{\mathrm{d}F}{\mathrm{d}t}$ does not contribute to the variation, that is:

$$\delta \int_{t_1}^{t_2} \frac{\mathrm{d}F}{\mathrm{d}t}\, dt = \delta F\Big|_{t_1}^{t_2} = 0$$

Where in the last equality we use the fact that the variation vanishes at the endpoints. Hence adding the total derivative does not contribute to the variation. This $F$ is known as the generating function. In particular, we can set:

$$\sum_i \dot{q}_i p_i - \mathcal{H} = \sum_i \dot{Q}_i P_i - K + \frac{\mathrm{d}F}{\mathrm{d}t}$$

And I know that these new variables will satisfy the same equations. We can use this $F$ to generate such tranformations. In full, we have $4n$ variables of $q, p, Q, P$ but only $2n$ are independent; hence $F$ should depend on $2n$ independent variables. Hence, there are 4 possible ways we could construct a generating function. These would be:

$$F_1 = F_1(q, Q, t)$$

$$F_2 = F_2(q, P, t)$$

$$F_3 = F_3(p, Q, t)$$

$$F_4 = F_4(p, P, t)$$

For now, we will look at the first case, $F_1(q, Q, t)$. In this case,

$$\sum_i \dot{q}_i p_i - \mathcal{H} = \sum_i \dot{Q}_i P_i - K + \frac{\mathrm{d}F_1}{\mathrm{d}t} = \sum_i \dot{Q}_i P_i + \sum_i \frac{\partial F_1}{\partial q_i}\,\dot{q}_i + \sum_i \frac{\partial F_1}{\partial Q_i}\,\dot{Q}_i + \frac{\partial F_1}{\partial t}$$

We now collect some terms (i.e. the terms that depend commonly on $\dot{q}_i$, $\dot{Q}_i$). Doing so we have:

$$\sum_i \dot{q}_i \left(p_i - \frac{\partial F_1}{\partial q_i}\right) - \sum_i \dot{Q}_i \left(P_i + \frac{\partial F_1}{\partial Q_i}\right) + K - \mathcal{H} - \frac{\partial F_1}{\partial t} = 0$$

From this equation, it is nice to read off what the transformed variables must look like. From the first term, $p_i = \dfrac{\partial F_1}{\partial q_i}$, $P_i = -\dfrac{\partial F}{\partial Q_i}$, and the new Hamiltonian is $K = \mathcal{H} + \dfrac{\partial F_1}{\partial t}$.

This is all quite abstract, so let us consider an example. Let $F_1 = F_1(q, Q, t) = \sum_i q_i Q_i$. We therefore have that $p_i = \frac{\partial F_1}{\partial q_i} = Q_i$, $P_i = \frac{\partial F_1}{\partial Q_i} = -q_i$, and $K = \mathcal{H}$ as the Hamiltonian is not explicitly time dependent. This is just an interchange of coordinates. This shows that in the Hamiltonian description, $p$ and $q$ are compeltely symmetric and can be completely interchanged.

We can derive very very similar transformation laws for the other three general cases above.

Whas is all of this good for? Can we do something more with it? Indeed we can, but we will see it on Wednesday. The first thing we shall do is to introduce a criterion for when a transform is canonical, and in doing so we will introduce something known as a Poisson bracket. We will also show that a time evolution itself is a canonical transformation. We will also show conservation laws associated with canonical transformations, and this will get us into Liouville's theorem.

# 25 Lecture 25

## 25.1 Lecture Notes - The Poisson Brackeet and Liouville's Theorem

### 25.1.1 Review

On Monday, we talked about canonical transformations. In Lagrangian picture, we found that Lagrange's equation of motion were invaraint under coordinate transformations. In Hamilton's picture, $p$ and $q$ play the same role. In the phase space spanned by $p$ and $q$, there are a wider class of possible transformations that can be made. Not all of these transformations leave Hamilton's equations invariant. The subclass that do is called canaonical. Adding he total time derivative of a function $F$ is what could accomplish this.

### 25.1.2 Poisson Bracket

Consider the phase space function $F(q, p, t)$ (any function of the variables of phase space, could depend on time, could be energy, angular momentum etc.). Let us write the total time derivative of this. As usual, we expand this total derivative to get:

$$\frac{\mathrm{d}F}{\mathrm{d}t} = \sum_j \frac{\partial F}{\partial q_j}\,\dot{q}_j + \sum_j \frac{\partial F}{\partial p_j}\,\dot{p}_j + \frac{\partial F}{\partial t}$$

Using Hamilton's equations to replace $\dot{q}_j$ and $\dot{p}_j$, we get that:

$$\frac{\mathrm{d}F}{\mathrm{d}t} = \sum_j \frac{\partial F}{\partial q_j}\frac{\partial \mathcal{H}}{\partial p_j} - \sum_j \frac{\partial F}{\partial p_j}\frac{\partial \mathcal{H}}{\partial q_j} + \frac{\partial F}{\partial t}$$

We may write this as:

$$[F, H] + \frac{\partial F}{\partial t}$$

This is the **Poisson Bracket**. It's a shorthand for $\sum_j \frac{\partial F}{\partial q_j}\frac{\partial \mathcal{H}}{\partial p_j} - \sum_j \frac{\partial F}{\partial p_j}\frac{\partial \mathcal{H}}{\partial q_j}$. As a note of notation, it can also be written with curly brackets $\{\ \}$. If $F$ is conserved, an equivalent statement is that $\frac{\mathrm{d}F}{\mathrm{d}t} = 0$, and equivalent to this is that $[F, H] = 0$.

### 25.1.3 Properties of the Poisson Bracket

(a) Anti-symmetry $[F, G] = -[G, H]$, from which we obtain $[F, F] = 0$.

(b) Bilinearity $[aF + bG, H] = a[F, H] + b[G, H]$ and $[H, aF + bG] = a[H, F] + b[H, G]$

(c) Leibniz' Rule $[FG, H] = [F, H]G + F[G, H]$

(d) Jacobi Identity $[F, [G, H]] + [G, [H, F]] + [H, [F, G]] = 0$

### 25.1.4 Poisson Bracket and Canonical Transformations

We test if the following transformation is canonical:

$$q_i \mapsto Q_i(p, q), \quad p_i \mapsto P_i(p, q), \quad \frac{\partial F}{\partial t} = 0, \quad K = \mathcal{H}$$

We claim that this is the case if the following identities hold for the fundamental Poisson brackets:

$$[Q_i, Q_l] = 0, \quad [P_i, P_l] = 0, \quad [Q_i, P_l] = \delta_{il}$$

are obeyed. As a remark, these identities are *quite* similar to the canonical commutation relations that we see in quantum mechanics. This shows that the structure in phase space is ready to be quantized. But, we return to this on Friday. For now, we return to the proof. Taking the total time derivative of $F$, we have:

$$\frac{\mathrm{d}F}{\mathrm{d}t} = \ldots = \sum_l \frac{\partial K}{\partial Q_l}[F, Q_l] + \sum_l \frac{\partial K}{\partial P_l}[F, P_l]$$

These identities were just obtained by the chain rule and reordering. We now consider some cases. In the first case, consider $F = Q_i$. In this case,

$$\dot{Q}_i = \sum_l \frac{\partial K}{\partial Q_l}[Q_i, Q_l] + \sum_l \frac{\partial K}{\partial P_l}[Q_i, P_l]$$

In order for Hamilton's equation to be satisfied, we require that this expression is equal to $\frac{\partial K}{\partial P_i}$. From this, we require that $[Q_i, Q_l] = 0$, and $[Q_i, P_l] = \delta_{il}$. In the second case, consider $F = P_i$. Then,

$$\dot{P}_i = \sum_l \frac{\partial K}{\partial Q_l}[P_i, Q_l] + \sum_l \frac{\partial K}{\partial P_l}[P_i, P_l]$$

Again, in order for Hamilton's equations to be satisfied, we require that this is equal to $-\frac{\partial K}{\partial Q_i}$. From this we obtain that $[P_i, Q_l] = -\delta_{il}$, and $[P_i, P_l] = 0$. This completes the proof of the claim.
We also see that

$$\dot{q}_i = [q_i, \mathcal{H}], \quad \dot{p}_i = [p_i, \mathcal{H}]$$

Since

$$\dot{q}_i = [q_i, \mathcal{H}] = \sum_k \underbrace{\frac{\partial q_i}{\partial q_k}}_{=\delta_{ik}} \frac{\partial \mathcal{H}}{\partial p_k} - \sum_k \underbrace{\frac{\partial q_i}{\partial p_k}}_{=0} \frac{\partial \mathcal{H}}{\partial q_k} = \frac{\partial \mathcal{H}}{\partial p_i}$$

$$\dot{p}_i = [p_i, \mathcal{H}] = \sum_k \underbrace{\frac{\partial p_i}{\partial q_k}}_{=0} \frac{\partial \mathcal{H}}{\partial p_k} - \sum_k \underbrace{\frac{\partial p_i}{\partial p_k}}_{=\delta_{ik}} \frac{\partial \mathcal{H}}{\partial q_k} = -\frac{\partial \mathcal{H}}{\partial q_i}$$

### 25.1.5 Hamiltonian Flow

Recall our discussion of the phase space vector:

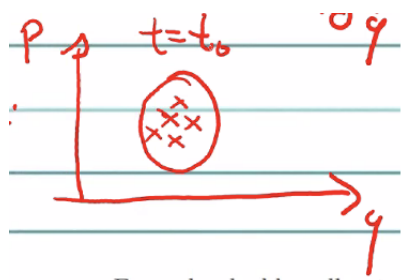
$$\mathbf{z} = \begin{bmatrix} \mathbf{q} \\ \mathbf{p} \end{bmatrix}$$

Which in general has $2n$ elements for $n$ degrees of freedom (recall the 2d phase space vector for the 1-dimensional harmonic oscillator). Taking the time derivative of this, we get the phase space velocity vector:

$$\dot{\mathbf{z}} = \begin{bmatrix} \dot{\mathbf{q}} \\ \dot{\mathbf{p}} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial \mathcal{H}}{\partial \mathbf{p}} \\ -\dfrac{\partial \mathcal{H}}{\partial \mathbf{q}} \end{bmatrix} = \mathbf{v}(\mathbf{z})$$

The idea is that the phase space trajectories depend on the total energy of the system. For example with the double well potential:


Now, let us consider a bunch of systems that are some time close together (in phase space):




What happens to this cloud of points as it goes through a time evolution? These trajectories are unique and deterministic, so at some later point in time, this cloud may have moved in phase space. It also does not have to have the same shape. But, the points inside the cloud have to stay inside the cloud, as trajectories cannot cross in Hamiltonian dynamics. Points inside stay on the inside.

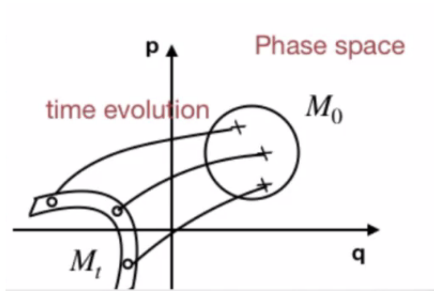

In general, there is a hyper-volume (the cloud) in phase space, which moves through the space with time; this is the picture we want to have when thinking about Hamiltonian flow.

#### 25.1.6 Liouville's Theorem

We hence consider a map:

$$\mathbf{z}_0 \mapsto \mathbf{z}(t)$$

which is the definition of the Hamiltonian/phase space flow. The statement of Liouville's theorem is that the area initially occupied by the cloud in phase space is the same as the area occupied at some point later (for higher dimensions, replace area with hyper-volume).




We will make this statement more precise as we go on. In order to see this, we again look at this Hamiltonian flow. Hamilton's equations are a description/map of these positions in phase space to a later time. Let us examine the properties of this map for a small time interval:

$$q_i \mapsto Q_i = q_i + \dot{q}_i dt$$

$$p_i \mapsto P_i = p_i + \dot{p}_i dt$$

This is of course just a first-order Taylor expansion/linear approximation. Of course we can rewrite these expressions using Hamilton's equations:

$$q_i \mapsto q_i + \frac{\partial \mathcal{H}}{\partial p_i} dt$$

$$p_i \mapsto p_i - \frac{\partial \mathcal{H}}{\partial q_i} dt$$

The statement of Liouville's theorem can then be made to say:

$$V_{pq} = d\mathbf{q}d\mathbf{p} = V_{PQ} = d\mathbf{P}d\mathbf{Q}$$

What happens if we apply this transformation and compare? This is just a statement about a change of variables. Let us recall what we did for integration. When switching integration variables $x = x(u,v)$, $y = y(u,v)$, which of the following is correct?

*Solution.* We recall how we did coordinate transformations using the Jacobian:

$$\iint f(x,y)dxdy = \iint f(x(u,v),y(u,v))\left|\frac{\partial(x,y)}{\partial(u,v)}\right|dudv$$

Note that here $\left|\frac{\partial(x,y)}{\partial(u,v)}\right|$ is the determinant of the 2x2 matrix containing the partial derivatives (the determinant of the Jacobian), e.g.:

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix}$$

Hence, returning to our discussion of the conserved volumes; we can write $d\mathbf{P}d\mathbf{Q}$ as:

$$V_{pq} = d\mathbf{q}d\mathbf{p} = V_{PQ} = d\mathbf{P}d\mathbf{Q} = \left|\frac{\partial \mathbf{z}_t}{\partial \mathbf{z}_0}\right| d\mathbf{p}d\mathbf{q}$$

We can write this Jacobian as:

$$\frac{\partial \mathbf{z}_t}{\partial \mathbf{z}_0} = \begin{bmatrix} \frac{\partial Q}{\partial q} & \frac{\partial Q}{\partial p} \\ \frac{\partial P}{\partial q} & \frac{\partial P}{\partial p} \end{bmatrix} = \begin{bmatrix} 1 + \frac{\partial^2 \mathcal{H}}{\partial p \partial q} dt & \frac{\partial^2 \mathcal{H}}{\partial p^2} dt \\ -\frac{\partial^2 \mathcal{H}}{\partial p^2} dt & 1 - \frac{\partial^2 \mathcal{H}}{\partial p \partial q} dt \end{bmatrix}$$

We have done this many times in the past, e.g. when transforming from cartesian to spherical coordinates. The factor $r^2 \sin\theta$ in this coordinate transformation from cartesian to spherical is the determinant of the Jacobian in that case. Here, we are just transforming from a system at time $t = 0$ to time $t = t$. Hence taking the determinant, we have:

$$\left|\frac{\partial \mathbf{z}_t}{\partial \mathbf{z}_0}\right| = 1 - \left(\frac{\partial^2 \mathcal{H}}{\partial p \partial q}\right)^2 dt^2 + \frac{\partial^2 \mathcal{H}}{\partial p^2}\frac{\partial^2 \mathcal{H}}{\partial q^2} dt^2 + \cdots$$

We note that there are no linear terms in $dt$. Hence, the time derivative of this object is zero. Hence, the phase space volume does not change. Equivalently, $\frac{\mathrm{d}V}{\mathrm{d}t} = 0$. The fluid of phase space moves like an incompressable fluid, keeping its volume. $\square$

# 26 Lecture 26

## 26.1 Lecture Notes - Liouville's Theorem, Path to QM (Canonical Quantization)

### 26.1.1 Review - Liouville's Theorem

- The volume $V$ of a region of phase space does not change in time.
- The phase space density $N/V$ moving with the slow lines remains constant.
- The phase space density moves like an incompressible fluid.
- $V_t = \int d\mathbf{z}_t = \int \left|\frac{\partial \mathbf{z}_t}{\partial \mathbf{z}_0}\right| d\mathbf{z}_0 = V_0$, $abs \frac{\partial \mathbf{z}_t}{\partial \mathbf{z}_0} = 1$

### 26.1.2 Liouville's Theorem from Gauss's (Divergence) Theorem

Recall the divergence theorem; for a vector field $\mathbf{v}$, it holds that:

$$\int_V \boldsymbol{\nabla} \cdot \mathbf{v} dV = \int_S \mathbf{v} \cdot \mathbf{n} dA$$

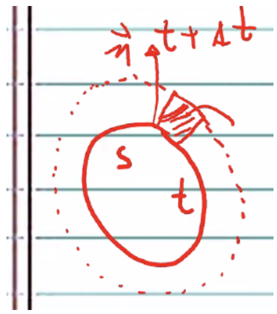
Now we consider the divergence of the phase space velocity, $\mathbf{v}$:

$$\mathbf{v} = \begin{bmatrix} \dot{\mathbf{q}} \\ \dot{\mathbf{p}} \end{bmatrix} = \begin{bmatrix} \frac{\partial \mathcal{H}}{\partial \mathbf{p}} \\ -\frac{\partial \mathcal{H}}{\partial \mathbf{q}} \end{bmatrix}$$

The divergence is given by:

$$\nabla \cdot \mathbf{v} = \frac{\partial \dot{\mathbf{q}}}{\partial \mathbf{q}} + \frac{\partial \dot{\mathbf{p}}}{\partial \mathbf{p}} = \frac{\partial}{\partial \mathbf{q}}\left(\frac{\partial \mathcal{H}}{\partial \mathbf{p}}\right) - \frac{\partial}{\partial \mathbf{p}}\left(\frac{\partial \mathcal{H}}{\partial \mathbf{q}}\right) = 0$$

Where we use Hamilton's equations and the equality of mixed partials. How do we interpret this? Consider a surface in phase space that moves out with time. As the surface expands there will be a change in volume covered by this propagating surface.




Giving this a surface normal $\mathbf{n}$, calculating the change in volume w.r.t. time we have $\delta V = \mathbf{n} \cdot \mathbf{v} \delta t dA$ and if we integrate over the entire surface, we get teh total change. This yields $\Delta V = \int_S \mathbf{n} \cdot \mathbf{v} \Delta t dA$. And hence:

$$\frac{\mathrm{d}V}{\mathrm{d}t} = \int_S \mathbf{n} \cdot \mathbf{v} dA = 0$$

Where the RHS we set to zero by the divergence theorem. This concludes the proof, and is another way of proving the theorem.

### 26.1.3 Poisson Brackets Revisited & The path to QM

Recall we could write:

$$\frac{\mathrm{d}F}{\mathrm{d}t} = [F, \mathcal{H}] + \frac{\partial F}{\partial t}$$

where $F$ is some function on phase space. By definition, we had:

$$[F, G] = \sum_i \frac{\partial F}{\partial q_i}\frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q_i}$$

The fundamental poisson brackets of position and momentum were given by:

$$[q_i, q_j] = [p_i, p_j] = 0, \quad [q_i, p_j] = \delta_{ij}$$

We now connect this to QM, one of the defining theories of the 20th century.
Recall that in QM, measurements are operations. We perturb systems by measuring/observing them (e.g. measure position of ball by shining light on it, but light carries momentum, so it will interact with the ball in doing so.). Also recall that in qm, $pq$ and $qm$ are different, that is that operators do not (in general) commute.

### 26.1.4 Non-Commutative Structure of Phase Space

Phase space functions become non-commutative operators (matrices). Consider four phase space functions, $F_1, F_2, G_1, G_2$. Now, consider the object $[F_1F_2, G_1G_2]$. We pull this apart using the Leibniz product rule:

$$[F_1F_2, G_1G_2] = [F_1, G_1G_2]F_2 + F_1[F_2, G_1G_2]$$

we can also apply it on the second argument:

$$[F_1F_2, G_1G_2] = [F_1F_2, G_1]G_2 + G_1[F_1F_2, G_2]$$

Hence, we get the following expression if we apply hte product rule again:

$$[F_1F_2, G_1G_2] = [F_1, G_1](F_2G_2 - G_2F_2) = (F_1G_1 - G_1F_1)[F_2, G_2]$$

If we have classical observables, the $F, G$s are just numbers, they commute and the relation is obviously satisfied (both sides are just zero!). But in QM, we want to consider arbitrary, non-commuting operators. Hence, the condition for the above equation to be satified for arbitrary non-commuting operators is that:

$$[F_1, G_1] = F_1G_1 - G_1F_1$$

That is, we want the Poisson bracket to be a commutator. Furthermore, we will embellish this with an imaginary unit $i$:

$$i[F_1, G_1] = F_1G_1 - G_1F_1$$

Which is there to ensure that the bracket of two Hermitian operators is again Hermitian. Recall that a Hermitian operator $O$ satisfies $O^\dagger = O$. Finally, we put a number $\hbar$ in front of the commutator as well:

$$i\hbar[F_1, G_1] = F_1G_1 - G_1F_1$$

This now gives the quantum mechanics we know. This process is known as **canonical quantization**.

### 26.1.5 Canonical Quantization

There are two steps of going from classical mechanics to quantum mechanics:

(a) Phase space functions (numbers) turn into operators (non-commuting in general).

(b) $[F, G]_P \mapsto \frac{1}{i\hbar}[F, G]$, that is, we embellish the poisson bracket with a factor of $i\hbar$ and turn it into a commutator.

We can check the canonical commutation relations of quantum mechanics:

$$[q_i, q_j] = [p_i, p_j] = 0$$

$$[p_i, p_j] = i\hbar\delta_{ij}$$

From these relations, we can derive many fundamental results of quantum mechanics, such as the Heisenberg uncertainty principle, that is, $(\Delta q)(\Delta p) \geq \frac{\hbar}{2}$. By making the Poisson bracket the commutator, we get this quantized theory with uncertainty built into it. We may also look at the time evolution of operators. The classical evolution equation is:

$$\frac{\mathrm{d}F}{\mathrm{d}t} = [F, H]_P + \frac{\partial F}{\partial t}$$

the quantum version of this replaces the poisson bracket with a commutator; for a quantum mechanical operator $O$, the time evolution of it is given by:

$$\boxed{\frac{\mathrm{d}O}{\mathrm{d}t} = -\frac{i}{\hbar}[O, H] + \frac{\partial O}{\partial t}}$$

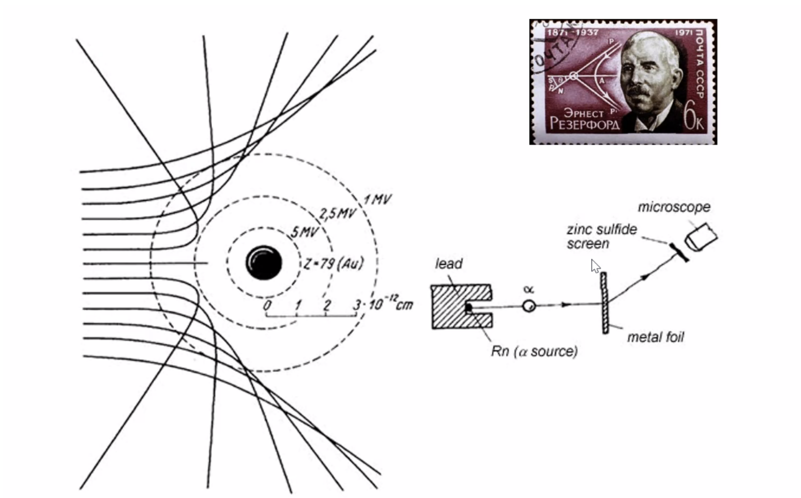
Which is known as the Heisenberg equation of motion. This is the Heisenberg picture of QM, where you move the idea of time-evolution from the states (the Schrodinger picture) to the operators. The canonical quantization process takes us into the Heisenberg picture. Note that one can also do this on fields and get Quantum Field Theory, but that is a story for another course.

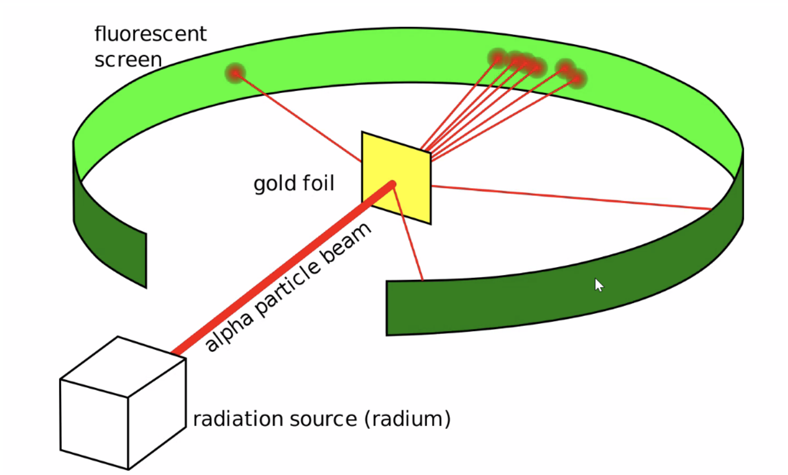
# 27 Lecture 27

## 27.1 Lecture Notes - Scattering Theory

### 27.1.1 Motivation

Scattering Theory is very useful for probing information on atomic scales, in condensed matter/nuclear/atomic physics. A familiar example is Rutherford scattering:

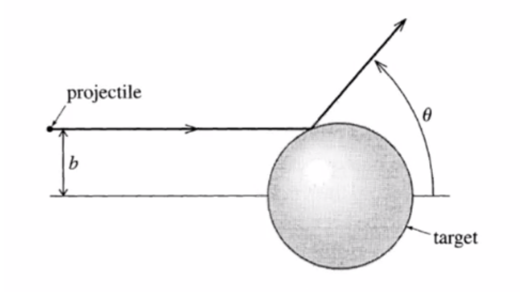


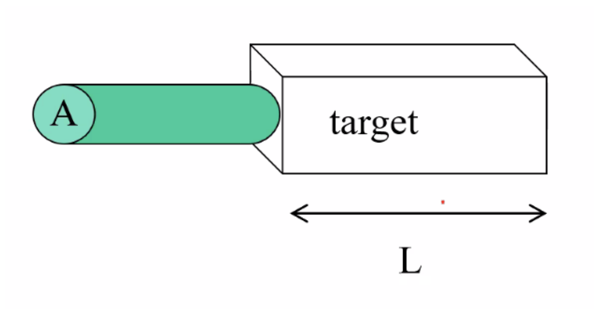

Where $\alpha$ particles (Helium nuclei) were scattered off of gold atoms, and large deflections were observed.

### 27.1.2 Fundamental parameters

There are a couple quantities that are of relevance to consider. First, we have the impact parameter $b$, which is the perpendicular distance from the incoming trajectory to the parallel axis through the center of the target. Then, we have the scattering angle $\theta$, which is the angle between the initial and final velocities. The simplest possible interaction is the hard-sphere interaction:

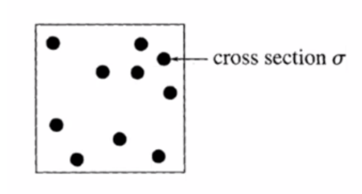



Now, lets consider a beam of area $A$ passing through a target of length $L$ and number density of particles $n$. Assume the target is larger in the cross sectional area than in the beam. What is the total number of target particles in the beam?




*Solution.* Since $n$ gives the volume number density, we have that $AL$ gives the volume of the beam in the target and hence the total number of target particles is given by $nAL$. □

Next, the **cross-section** $\sigma$ is defined as the effective area of target for interacting with the particle. For hard spheres of radius $R$, we have that $\sigma = \pi R^2$ (the cross-sectional area of a circle).




Now, given this cross section $\sigma$ of a single target particle, what is the probability that any one projectile makes a hit (assume the same scenario above with the beam of area $A$, the target of length $L$?)

*Solution.* By dimensional analysis, since probability is dimensionless, since $n$ has units of inverse volume, then $L$ has units of length and $\sigma$ has units of area and hence this works out. This makes sense as the probability should scale with the cross-sectional area of a target particle, the length of the target, and the number density of the target. A way to see that the area $A$ of the beam drops out to see is that:

$$P(\text{hit}) = \frac{\text{Area of all targets}}{A} = \frac{n\sigma AL}{A} = n\sigma L$$

□

Given the beam has an incident rate $R_{inc}$ of incoming particles per unit time, what is the scattered rate (number of scattered particles per unit time?)

*Solution.* The rate would just be $R_{inc}\sigma nL$, just multiply the scattering probability by the incoming rate. □

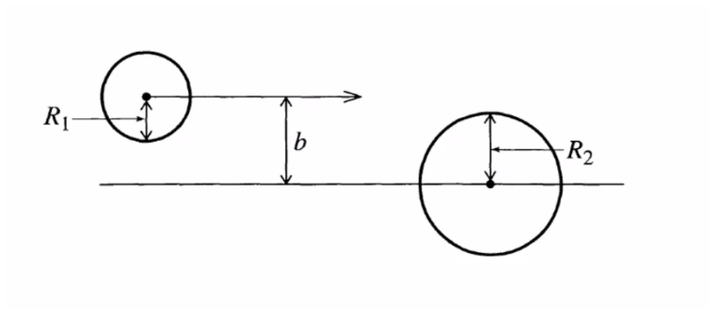
### 27.1.3 Example: Scattering Neutrons on Aluminum Foil

Take $N_{\text{inc}} = 10000$. the alumnimum foil has thickness of 0.1mm. For neutrons, we have that $\sigma = 1.5 \cdot 10^{-28}\text{m}^2$. Since this is such a common unit in nuclear physics, this is often denoted with a new unit, the *barn* (1barn = $1 \times 10^{-28}\text{m}^2$; i.e. $\sigma = 1.5$barns. For alumnimum, we have mass density $\rho_{Al} = 2.7 \times 10^3\text{kg/m}^3$ and we know that $m_{Al} = 27u$. Hence, the scattered number of particles is given by:

$$N_{\text{scatter}} = N_{\text{inc}} \frac{\rho_{\text{Al}}}{m_{Al}} L\sigma = 9$$

### 27.1.4 Example: Scattering of Two Hard Spheres

In this case, we have effective scattering area of $\pi(R_1 + R_2)^2$.




### 27.1.5 Example: Mean free path of air molecule

Air molecules can be approximated as hard spheres with $R = 0.15$nm. Define the quantity **mean free path** $\lambda$ as the average distance between two collisions. For sigma, we take (from the formula above):

$$\sigma = \pi(2R)^2 = 4\pi R^2$$

We have number density:

$$n = \frac{N}{V}$$

The probability of a collision when travelling a distance $dx$ is given by:

$$P(\text{coll in dx}) = n\sigma dx$$

Hence the probability of having a first collision in $x$ and $x + dx$ as:

$$P(\text{first coll between in x and x + dx}) = P(\text{no coll in x}) \cdot n\sigma dx$$

We can write this in another way as:

$$P(\text{first coll between in x and x + dx}) = P(\text{no coll in x}) - P(\text{no coll in x + dx})$$

Turning this small $dx$ into a differential, we have:

$$P(\text{first coll between in x and x + dx}) = -\frac{\mathrm{d}}{\mathrm{d}x} P(\text{no coll in x})dx$$

We call $P(\text{no coll in x})$ as $P(x)$ for notations. Setting the two expressions equal to each other, we have:

$$\frac{\mathrm{d}}{\mathrm{d}x} P(x) = -\frac{N\sigma}{V} P(x)$$

This has solution:

$$P(x) \propto \exp\left(-\frac{N\sigma}{V}x\right)$$

Then the mean free path can be calculated as the average value of x given this probaility distribution:

$$\lambda = \langle x \rangle = \int_0^\infty xP(x)dx = \int_0^\infty xn\sigma \exp\left(-\frac{N\sigma}{V}x\right)$$

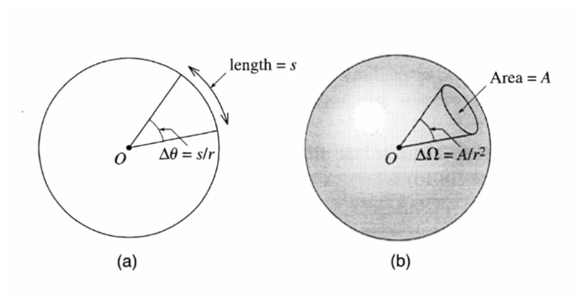
Note that the $n\sigma$ is there as a normalization factor for the distribution. Taking this integral, we have:

$$\lambda = \frac{1}{n\sigma}$$

This makes sense intuitively; the larger the cross section and the larger the number density, the smaller the mean free path between collisions. Dimensionally, this also has units of length, which is good! At STP, we can calculate what this would be numerically:

$$\lambda = \frac{V_a}{N_a(4\pi R^2)} \approx 130\text{nm}$$

#### 27.1.6 Solid Angle




We are familiar with the normal angle, $\Delta\theta = \frac{s}{r}$, the ratio of the arc length to the radius. Generalizing this to 3D, we have the solid angle, $\Delta\Omega = \frac{A}{r^2}$ where $A$ is the "arc area" and $r$ the radius. For a cone with polar angles $\theta, \theta + d\theta$, $\phi, \phi + d\phi$. The expression is therefore given by:

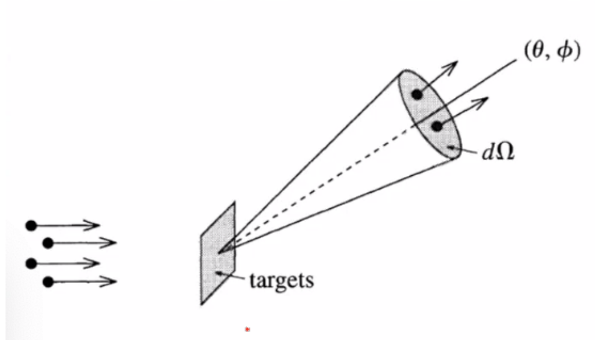
$$d\Omega = \sin\theta d\theta d\phi$$

What is the integral of the solid angle increment $d\Omega$ over all possible solid angles (over the surface of a sphere?)

*Solution.* $\int d\Omega = 4\pi$ (surface area of unit sphere). We could actually do the integral, or we could just recognize that the surface area of a sphere is given as $4\pi r^2$ and divide this by $r^2$. □

#### 27.1.7 Differential Cross Section

Typical scenario is we have that a detector covers some portion of a sphere around our target.



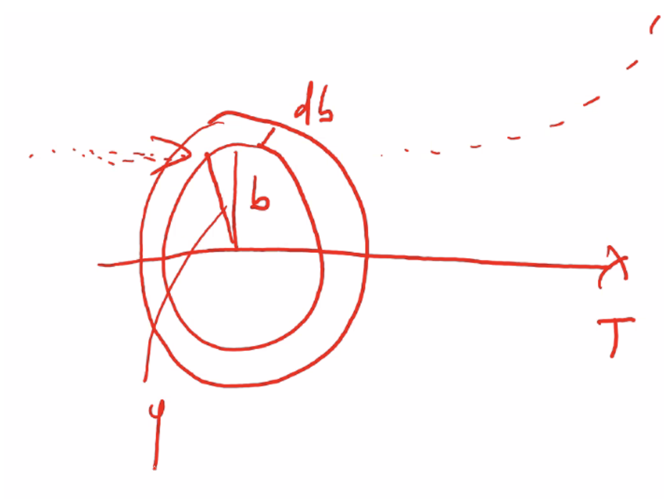

We must have that:

$$N_{\text{scatter}}(\text{into d}\Omega) = N_{inc}n_{target}d\sigma(\text{into d}\Omega) = N_{inc}n_{target}\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}(\theta,\phi)\right)d\Omega$$

The term $\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}(\theta,\phi)$ is the differential cross section. This can be measured in experiment, or predicted in theory. We can also obtain the total cross section, which is the integral over all differnetial cross sections.

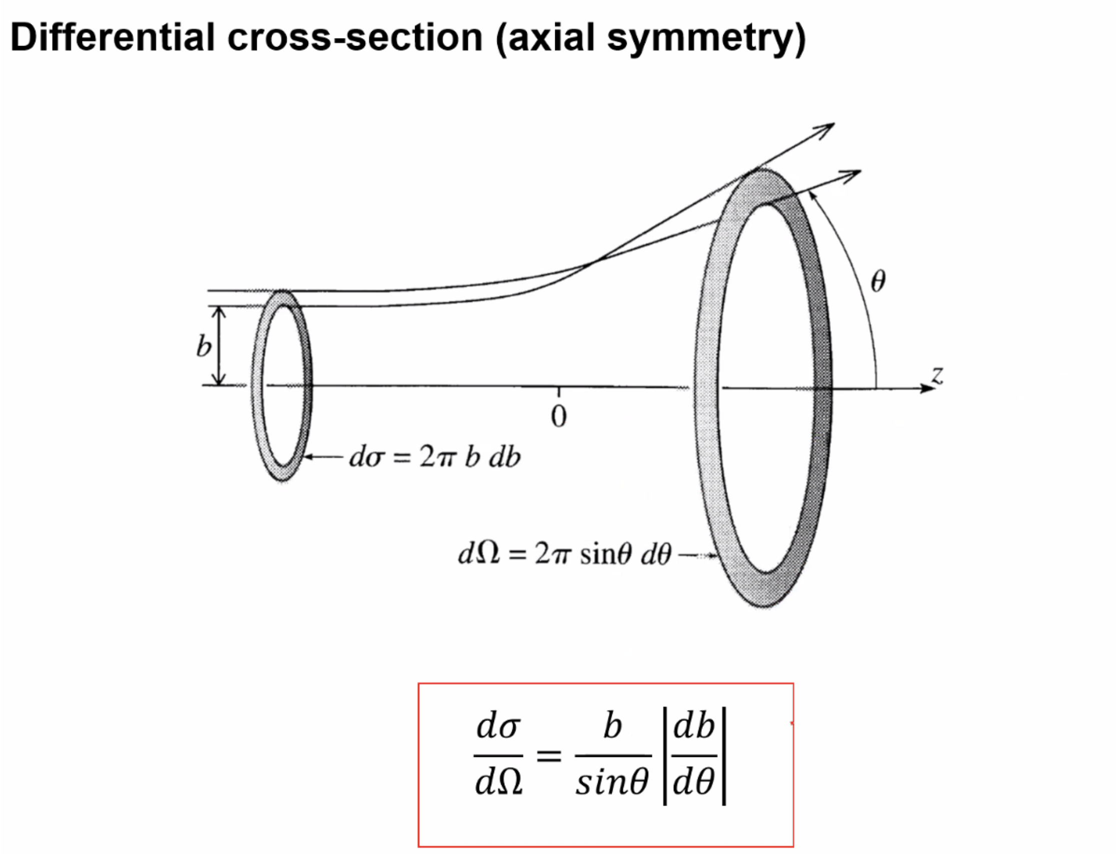
$$\sigma_{tot} = \int \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}(\theta,\phi)d\Omega = \int_0^{\pi}\sin\theta d\theta\int_0^{2\pi}d\phi\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}$$

For a beam with area $A$ and total number of particles $N$, what is the total number of particles $dN$ that passes through the segment between $b$ and $b+db$ and $\phi$ and $\phi+d\phi$?




*Solution.* We know that the area of the segment is given by $bdbd\phi$, and then dividing this by the total area $A$ we get the fraction of particles that would hit that area. Hence, $dN = \frac{N}{A}bdbd\phi$. □

Note that for the case with axial symmetry, we have:




But we will continue this discussion next day.

# 28 Lecture 28

## 28.1 Lecture Notes - Differential Cross Section

### 28.1.1 Review

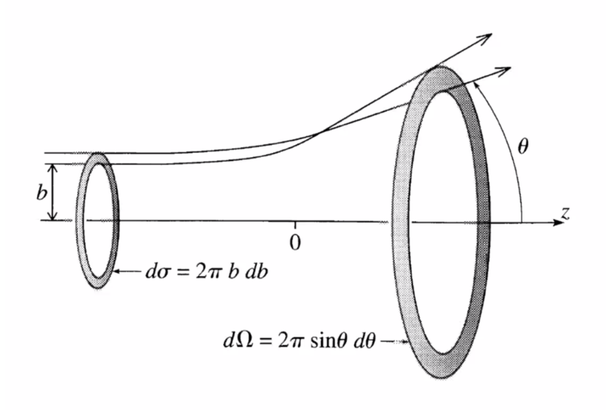
Cross section = area of ring of radius $b$ and width $db$. Particles hitting the ring between $b$ and $b + db$ are scattered by an angle between $\theta$ and $\theta + d\theta$. Scattered onto a larger ring on a sphere with scallering nucleus in center. The solid angle of the entire ring is:

$$d\Omega = \frac{2\pi R \sin\theta R d\theta}{R^2} = 2\pi \sin\theta d\theta$$

And the solid angle of a small area is given by:

$$d\Omega = \frac{d\phi R \sin\theta R d\theta}{R^2} = \sin\theta d\phi d\theta$$

This notion is useful as often our particle detectors cover a certain fraction of the area.




The ratio between the $d\sigma$ of the first ring and the $d\Omega$ of the second ring is the the differential cross section, which is:

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{b}{\sin\theta}\left|\frac{\mathrm{d}b}{\mathrm{d}\theta}\right|$$

Where $d\sigma = 2\pi b db$ and $d\Omega = 2\pi \sin\theta d\theta$. The task today is then to obtain $b$ in terms of $\theta$ ($b(\theta)$). Today, we will consider the case where our target is fixed (i.e. a heavy target) and try to describe the trajectory of this scenario. This is just the two body central force problem (as covered in PHYS 216), with the only difference that the orbits are open and not closed.

### 28.1.2 Calculation of the differential cross-section - The Kepler Approach

We first remark that this chapter is interesting as not only is it highly relevant to research fields (e.g. particle physics) but also puts something that we know (the two-body problem) into a new context. We now move onto solving this two-body central force problem. The total energy of the system in polar coordinates is given by:

$$E = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\phi}^2\right) + U(r) = \text{Const}$$

Where $U(r)$ is the scattering potential. Here the energy is conserved. Another quantity that is conserved is the angular momentum:

$$L = mr^2\dot{\phi} = \text{Const}$$

From this, we can immediately write:

$$r^2\dot{\phi}^2 = \frac{L^2}{m^2r^2}$$

Hence sustituting this into the energy expression to obtain everything in terms of $r$, we have:

$$E = \frac{m}{2}\left(\dot{r}^2 + \frac{L^2}{m^2r^2}\right) + U(r)$$

We can therefore solve for $|\dot{r}|$:

$$|\dot{r}| = \sqrt{\frac{2E}{m} - \frac{2U(r)}{m} - \frac{L^2}{m^2r^2}}$$

We have a two body problem, but it is effectively a one body problem. We may introduce an "effective potential":

$$U_{eff} = U(r) + \frac{L^2}{2mr^2}$$

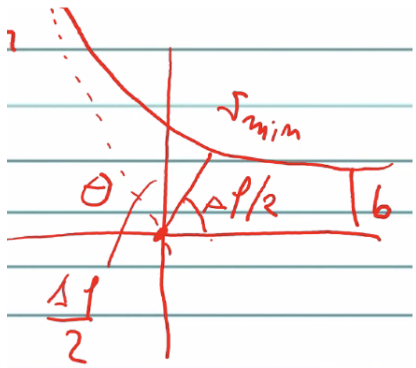
Since we know that $\phi(t)$ is monotonic, we may write:

$$\left|\dot{\phi}\right| = \frac{|L|}{mr^2}$$

Dividing the $\dot{\phi}$ equation by the $\dot{r}$ equation, we get:

$$\frac{\left|\dot{\phi}\right|}{|\dot{r}|} = \left|\frac{\mathrm{d}\phi}{\mathrm{d}r}\right| = \frac{\frac{|L|}{mr^2}}{\sqrt{\frac{2E}{m} - \frac{2U(r)}{m} - \frac{L^2}{m^2r^2}}}$$

We can from this expression obtain the trajectory in the following way. The particle is incoming with impact parameter $b$, and deflects off by angle $\theta$. There is some closest point of approach $r_{min}$ that divides the trajectory into two pieces, that is, the trajectory is symmetric about $r_{min}$. Call the angle at this point $\Delta\phi/2$. This can be visualized as follows:




We therefore have that:

$$\Delta\phi(r) = 2\int_{r_{min}}^{\infty} \frac{\frac{L}{r^2}}{\sqrt{2m(E-U) - \frac{L^2}{r^2}}}$$

Note that we obtain $r_{min}$ by solving for the turnaround point, i.e the energy equals the effective potential (the kinetic energy is zero) so:

$$2m(E-U) = \frac{L^2}{r_{min}^2}$$

However, all of our equations are in terms of $L$ and $E$; should recast these in terms of physical parameters. Set:

$$E = T_\infty = \frac{m}{2}v_\infty^2$$

and

$$L = |\mathbf{r} \times \mathbf{p}_\infty| = mbv_\infty$$

Hence we may write:

$$\Delta\phi = 2b\int_{r_{min}}^{\infty} \frac{dr}{r^2\sqrt{1 - \frac{2U(r)}{mv_\infty^2} - \frac{b^2}{r^2}}}$$

So we have expressed the integral purely in terms of the mass, velocity, and impact parameter, which are all measureable variables. Note that this solution isn't really scattering; this is a very general solution to the two-body problem. We can now apply it to some relevant scattering scenarios to obtain the differential cross section.

### 28.1.3 Example - Hard Sphere Differential Cross-Section

In this case, $r_{min} = R$ (closest point of approach is just the surface of the sphere), and $V(r) = 0$ for $r > R$ (the particles don't see each other). For this case, we have:

$$\Delta\phi = 2b \int_R^\infty \frac{dr}{r^2\sqrt{1 - \frac{b^2}{r^2}}}$$

The middle term is zero as the potential is zero in the region of interest. This integral is solvable analytically, but we may very well just look this up instead of undergoing a tedious calculation:

$$\Delta\phi = 2\arcsin\left(\frac{b}{R}\right)$$

Rearranging, we get:

$$b = R\sin\left(\frac{\Delta\phi}{2}\right)$$

According to the picture we have drawn, $\Delta\phi + \theta = \pi$, so we can convert this to be in terms of $\theta$, which works out to be:

$$b = R\cos\left(\frac{\theta}{2}\right) = b(\theta)$$

Now, all is left to do is to calculate the differential cross section:

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{b}{\sin\theta}\left|\frac{\mathrm{d}b}{\mathrm{d}\theta}\right| = \frac{R\cos\left(\frac{\theta}{2}\right)}{\sin\theta}\left|-\frac{R}{2}\sin\left(\frac{\theta}{2}\right)\right| = \frac{R^2}{2\sin\theta}\cos\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right) = \frac{R^2}{2\sin\theta}\frac{1}{2}\sin(\theta) = \frac{R^2}{4}$$

Hence calculating $\sigma_{tot}$ (total cross section) we have:

$$\sigma_{tot} = \int \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\, d\Omega = \int \frac{R^2}{4} d\Omega = \pi R^2$$

Which is a good sanity check, as it agrees with what we found on Monday!

### 28.1.4 Example - Coloumb Potential Differential Cross-Section

In this case, we have that:

$$U = -\frac{\beta}{r}$$

Where $\beta = -kqQ < 0$ (repulsive). The procedure is exactly the same, but the integral is just harder.

$$\Delta\phi = 2b \int_{r_{min}}^\infty \frac{dr}{r\sqrt{r^2 + \frac{2\beta r}{mv_\infty} - b^2}}$$

$$\Delta\phi = \; 2\arccos\left(\frac{1 - \frac{mv_\infty^2 b^2}{\beta}}{\sqrt{1 + \left(\frac{mv_\infty^2 b}{\beta}\right)^2}}\right)\Bigg|_{r_{min}}^\infty$$

Evaluating at the bounds, we get:

$$\Delta\phi = 2\arccos\left(\frac{1}{\sqrt{1 + \left(\frac{mv_\infty b}{\beta}\right)^2}}\right)$$

(this is the term at infinity, the term at $r_{min}$ vanishes, as the value is $2\arccos(-1) = 2\pi$ and $\phi = \phi + 2\pi$). Hence solving for $b(\theta)$ we have:

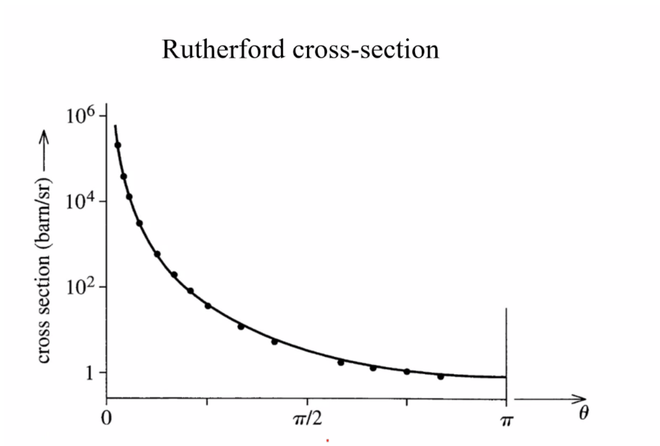
$$b(\theta) = \frac{1}{mv_\infty^2}\sqrt{\frac{1}{\sin^2(\frac{\theta}{2})} - 1}$$

Using this, we find the differential cross section:

$$\frac{d\sigma}{d\Omega} = \frac{k^2 q_1^2 q_2^2}{16E^2}\frac{1}{\sin^4(\frac{\theta}{2})}$$

Plotting the Rutherford cross section, we find:

Where we have a characteristic divergence at the origin and a rapid decrease, with a value of 1 at $\pi$.




Question: What is the total cross section?

*Solution.* $\sigma = \int \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\, d\Omega = \infty$ due to the divergence at the origin. This can be traced back to the $\frac{1}{r}$ type behavior of the Columb potential. □

A remark: One could state that this (classical treatment) is nonsense and we require a quantum treatment. Luckily, the classical treatment actually does work out in this case (due to the Coulomb potential in particular) but for a full description we require a quantum formulation, which is something we can expect to see in Graduate school QM. Next day we will look at the center of mass lab frame, and how we treat scattering off of nuclei that are not infinitely heavy.

# 29 Lecture 29

## 29.1 Lecture Notes - Cross Sections in Different Frames

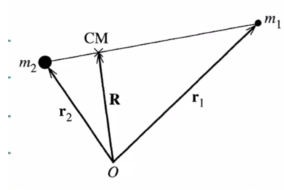
### 29.1.1 Main Result from Last Day

Rutherford Cross section of:

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{q_1^2 q_2^2}{16E^2}\frac{1}{\sin^6 \frac{\theta}{2}}$$

But so far, we have assumed that the scatterer is infinitely heavy/immobile. This is in general not the case, the target can of course move if $m_1$ is on the same mass scale as $m_2$. Today we deal with this scenario.

### 29.1.2 CM vs Lab Frames




We have two coordinates in the COM frame:

$$\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2 \quad \text{(Relative coordinate)}$$

The relative coordinate moves like a single particle with reduced mass $\mu = \frac{m_1 m_2}{M}$ where $M = m_1 + m_2$. The Lagrangian in this frame is given by:

$$\mathcal{L} = T_{COM} + T_{rel} - U$$
$$\mathcal{L} = \frac{M}{2}\dot{\mathbf{R}}_{COM}^2 + \frac{\mu}{2}\dot{\mathbf{r}}^2 - U(\mathbf{r})$$

The generalized momentum $\mathbf{p}$ is given as:

$$\mathbf{p} = \mu\dot{\mathbf{r}}$$

A question: What can we say of the momenta of the two particles in the COM frame?

*Solution.* In the COM frame, it must hold that $\mathbf{p}_1 = -\mathbf{p}_2$ as of course the total momentum must be zero. $\square$

We can also see the above fact if we write:

$$\mathbf{r}_1 = \mathbf{R} + \frac{m_2}{M}\mathbf{r}, \quad \mathbf{r}_2 = \mathbf{R} - \frac{m_1}{M}\dot{\mathbf{r}}$$

In the CM, $\dot{\mathbf{R}} = 0$, as the total momentum must be zero. Hence, taking the derivative of the above equations:

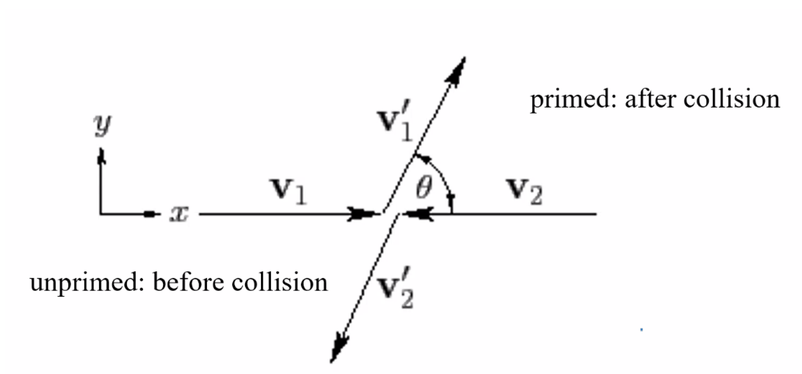
$$\dot{\mathbf{r}}_1 = \frac{m_2}{M}\dot{\mathbf{r}} \implies m_1\mathbf{r}_1 = \frac{m_1 m_2}{M}\mathbf{r} = \mathbf{p}$$
$$\dot{\mathbf{r}}_2 = -\frac{m_1}{M}\dot{\mathbf{r}} \implies m_2\dot{\mathbf{r}}_2 = -\frac{m_1 m_2}{M}\dot{\mathbf{r}} = -\mathbf{p}$$

What can we say about the momenta of two particles in the COM frame before (unprimed) and after (primed) an elastic collision?

*Solution.* After the collision, since energy is conserved, the length of these vectors are conserved; hence,

$$|\mathbf{p}_1| = |\mathbf{p}_1'|, \quad |\mathbf{p}_2| = |\mathbf{p}_2'|$$

Velocities rotate, but remain colinear by momentum/energy conservation.




$\square$

### 29.1.3 Differential Cross Section in COM Frame

In the COM frame, we just find $\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}$ as if a single particle of mass $\mu$ scatters off a fixed target. This is the beauty of working in the COM frame. We then have $\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{COM}$ which we must convert to the laboratory cross section. How do we do this? Consider the number of scattered particles in the center of mass frame:

$$N_{sc}^{COM} = N_{inc}^{COM} \cdot \sigma_{CM}^{tot} \cdot \frac{N_{target}}{A}$$

This has to be the same in the lab frame, so:

$$N_{sc}^{COM} = N_{sc}^{lab} = N_{inc}^{lab} \cdot \sigma_{lab}^{tot} \cdot \frac{N_{target}}{A}$$

$\frac{N_{target}}{A}$ is also equal between the two expressions. Hence, we conclude that the total cross sections must be the same, that is:

$$\sigma_{COM}^{tot} = \sigma_{lab}^{tot}$$

What about the differential cross section? What must be true for the number of scattered particles in a given solid angle to be the same in the COM and lab frames?

*Solution.* We require that $\sigma_{COM}^{tot} = \sigma_{lab}^{tot}$ so hence:

$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{COM} d\Omega_{COM} = \left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{lab} d\Omega_{lab}$$

This can be justified as:

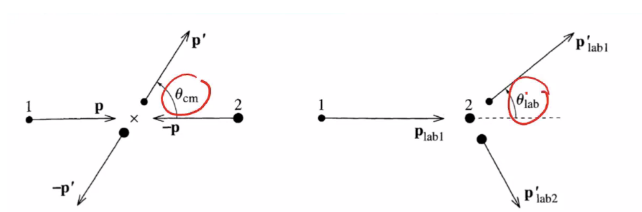
$$N_{sc}(\text{into } d\Omega) = N_{inc}\frac{N_{target}}{A}\boxed{\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\,d\Omega}$$

Where the boxed expression must be the same between two cases. □

Hence we may write:

$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{lab} = \left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{COM} \frac{d\Omega_{COM}}{d\Omega_{lab}} = \left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{COM} \left|\frac{d\cos\theta_{COM}}{d\cos\theta_{lab}}\right|$$

Depicitions of the collision in the two frames is given by:



(a) CM frame (b) lab frame

And we want to find the relationship between the two given angles. We start with:

$$\mathbf{p}_{COM1} = -\mathbf{p}_{COM2} = \mathbf{p}$$

$$\mathbf{p}'_{COM1} = -\mathbf{p}'_{COM2} = \mathbf{p}'$$

By the transformation from lab to COM coordinates:

$$\mathbf{r}_1 = \mathbf{R} + \frac{m_2}{M}\mathbf{r}, \quad \mathbf{r}_2 = \mathbf{R} - \frac{m_1}{M}\dot{\mathbf{r}}$$

In the lab frame, particle 2 is at rest, so:

$$\mathbf{R} = \frac{m_1}{M}\dot{\mathbf{r}} = \frac{\mu}{m_2}\dot{\mathbf{r}} = \frac{\mathbf{p}}{m_2}$$

Hence we can write:

$$\mathbf{p}_{lab1} = m_1\dot{\mathbf{r}}_1 = m_1\dot{\mathbf{R}} = \mu\dot{\mathbf{r}} = \frac{m_1}{m_2}\mathbf{p} + \mathbf{p}$$

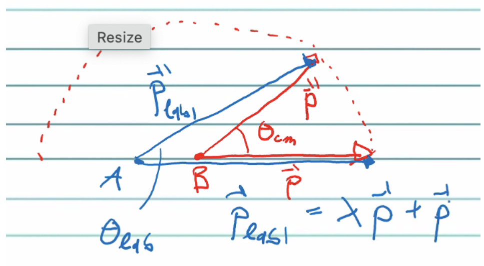
Calling $\lambda = \frac{m_1}{m_2}$, we have that:

$$\mathbf{p}_{lab1} = \lambda\mathbf{p} + \mathbf{p}$$

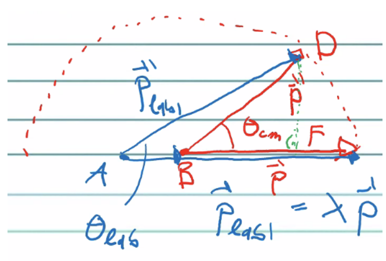
After the scattering:

$$\mathbf{p}'_{lab1} = \lambda\mathbf{p}' + \mathbf{p}'$$

Drawing a picture, we know that $\mathbf{p}'$ is just the original vector rotated by $\theta_{COM}$ on a circle. But, $\mathbf{p}_{lab1}$ is of course larger than $\mathbf{p}_{COM}$ (see the expression above):




Then using some geometry, we draw a right triangle $AFD$:




Hence we can write (using trigonometry):

$$\tan\theta_{lab} = \frac{DF}{AF} = \frac{p\sin\theta_{COM}}{\lambda p + p\cos\theta_{COM}}$$

$p$ cancels, giving us:

$$\tan\theta_{lab} = \frac{\sin\theta_{COM}}{\lambda + \cos\theta_{COM}}$$

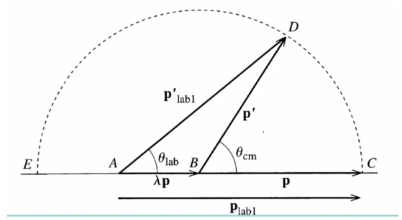
In the case where $m_1 = m_2$ and hence $\lambda_{=}1$, we have that:

$$\theta_{lab} = \frac{1}{2}\theta_{COM}$$

Which, returning back to the problem we wished to solve:

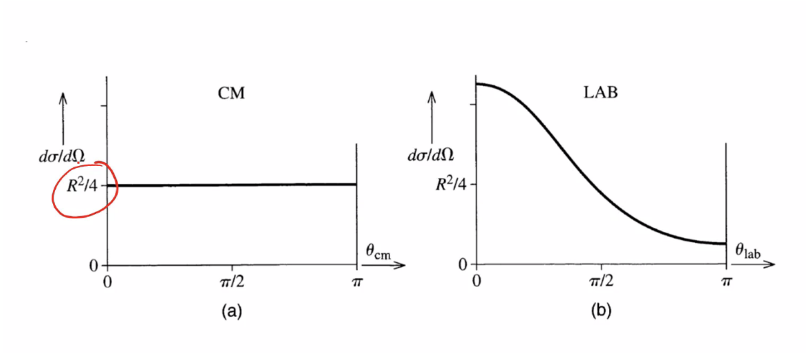
$$\left|\frac{d(\cos\theta_{COM})}{d(\cos\theta_{lab})}\right| = \frac{(1 + 2\lambda\cos\theta_{COM} + \lambda^2)^{3/2}}{|1 + \lambda\cos\theta_{COM}|}$$

The full diagram is given here:




### 29.1.4 Example: Hard spheres

Consider the example with hard spheres with $m_2 = 2m_1$. The differential cross sections can be calculated as follows:




This concludes our discussion of scattering! It was interesting to see how much a conversion to the COM frame can simplify our lives.

### 29.1.5 Coming up

Next week, we move onto continuum mechanics, going from the discrete (photons) to waves. We recall from the HW that we could model a rope as a chain of photons, write out the coupling matrices etc. But obviously this gets very tedious and complicated if our system gets very large (we do not want to write down a coupling matrix for 1000s of masses!) Instead, we can step back and just consider the mass as a continuum, and think about the displacement of the string in space and time (the wave equation). Next week we will see how we can make the jump from the discrete masses to the continuum limit of matter, where we no longer talk about particles but macroscopic objects described by displacement fields. In doing so, we will talk about wavelike motion/standing waves, as well as other ideas that come out of this framework. It turns out that there is a lot of fundamental physics we can get to without having to know lots of microscopic information. Finally, we will end the course with nonlinear dynamics after that.

# 30 Lecture 30

## 30.1 Lecture Notes - Continuum Mechanics

### 30.1.1 Setting Up The Continuum Limit

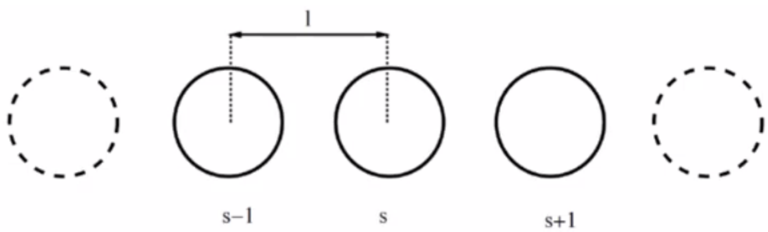
Recall HW5, where we considered an infinitely long chain of particles. Then, defining $u_s$ to be the displacement of the $s$th photon from the equilibrium position, we solved (by Newton's law):

$$m\ddot{u}_s = m\omega_0^2(u_{s+1} + u_{s-1} - 2u_s)$$

Where $\omega_0^2 = \frac{k}{m}$. The forces depend on the neighbouring interactions and displacements from equilibrium. Now, it becomes a bit of a subtle issue to how we take limits to infinity without getting things to blow up. We start with $N$ masses which are separated by a distance $l$ (chain of SHOs).

- We want to take $N \to \infty$ and $l \to 0$ at the same time. We do this in such a way that the total length $L = Nl$ of the string remains constant.
- As we increase the number of particles, they also get lighter, so we take $m \to 0$ and $l \to 0$ such that the line mass density $\lambda = \frac{m}{l}$ of the string also is a constant.

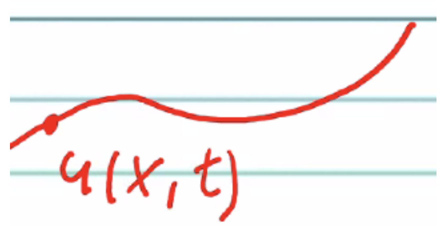
Question: When we double the number of masses and reduce the equilibrium length by a factor of 2, what must happen with the spring constant $k$?




*Solution.* Since $F = kx$, if $x \mapsto \frac{x}{2}$, then $k \mapsto 2k$ in order to give the same force between the particles as the original configuration. □

- So another condition on our limit; we will take $k \to \infty$ and $l \to 0$ with $kl$ held constant.

The idea is we will go from the position of a single particle $u_s(t)$ and take it to the continuum limit of $u(x,t)$, now considering a field.




### 30.1.2 Deriving the Wave Equation

Replacing the discrete positions with a continuous variable in the Newton's law equation we had in the section above, we have:

$$\frac{\partial^2 u(x,t)}{\partial t^2} = \omega_0^2 \big(u(x+l,t) + u(x-l,t) - 2u(x,t)\big)$$

But there is an issue; $\omega_0^2 = \frac{k}{m}$ diverges as $k \to \infty$ and $m \to 0$! How do we rescue this? We multiply the above equation by 1 in a clever way, multiplying and dividing by $l^2$:

$$\frac{\partial^2 u(x,t)}{\partial t^2} = \omega_0^2 l^2 \left(\frac{u(x+l,t) + u(x-l,t) - 2u(x,t)}{l}\right)$$

But this is quite convenient, as $\omega_0^2 l^2 = kl\frac{l}{m} = \frac{kl}{\lambda}$ which is constant! Let us therefore denote $\omega_0^2 l^2 = c^2$ where $c$ si the speed of sound. We can therefore write the above equation in the following way:

$$\frac{\partial^2 u(x,t)}{\partial t^2} = \frac{c^2}{l}\left[\frac{u(x+l) - u(x,t)}{l} - \frac{u(x,t) - u(x-l,t)}{l}\right]$$

We recognize that the two terms in the bracket are two first order spatial derivatives. Further taking another derivative by considering the difference of the two first order derivatives by taking $l \to 0$, the RHS becomes a single second order spatial derivative:

$$\boxed{\frac{\partial^2 u(x,t)}{\partial t^2} = c^2 \frac{\partial^2 u(x,t)}{\partial x^2}}$$

Which we recognize is a wave equation!

#### 30.1.3 Wave equation Solutions and Dispersion Relation

One prime example of a solution to the wave equation is:

$$u(x,t) = A\sin(kx - \omega t)$$

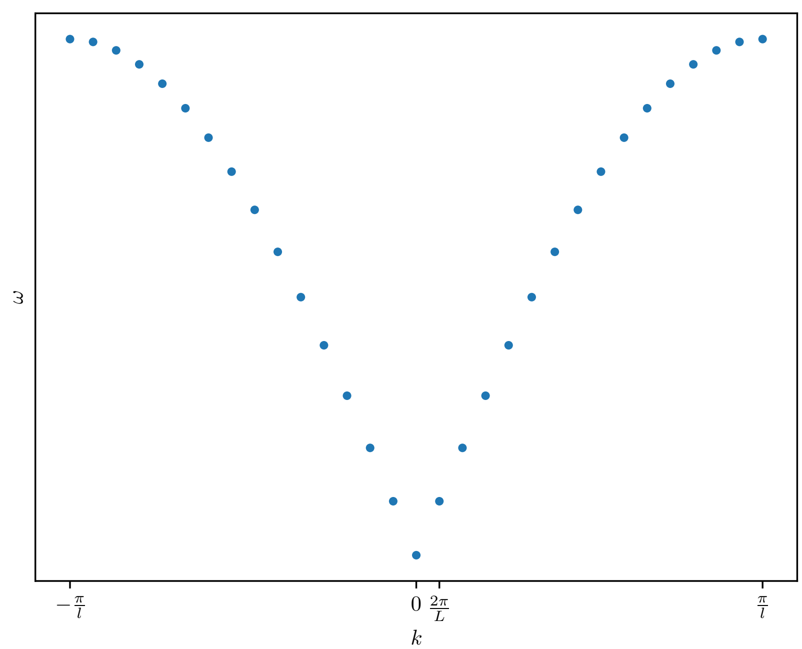
Where $k$ is the wavevector and $\omega$ the frequency. Plugging this into the wave equation, we immediately obtain the (familiar) dispersion relation:

$$\omega = ck$$

Recall that in HW5 we found a relation that was slightly more complicated, and was given as:

$$\omega = 2\omega_0 \left|\sin\left(\frac{kl}{2}\right)\right|$$

Which graphically looks like:




So if we consider the limit of this dispersion for $kl \ll 1$, we can Taylor expand the sine and get:

$$\omega = 2\omega_0 \frac{kl}{2} = \omega_0 kl = ck$$

So we therefore have that the slope of the above graph for small $k$ is equal to $c$. We then see that we can recover the linear dispersion relationship that we obtained in the continuum limit!

### 30.1.4 Generalization to 3D

Suppose we work in three dimensions, and instead have $P = p(x, y, z)$ (pressure, or some other variable). We then have that:

$$\frac{\partial^2 p}{\partial t^2} = c^2 \left( \frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} + \frac{\partial^2 p}{\partial z^2} \right) = c^2 \nabla^2 p$$

Where $\nabla^2$ is the laplacian.

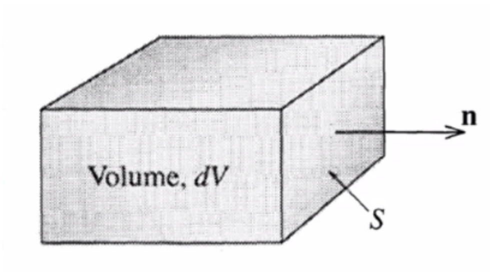
### 30.1.5 Volume and Surface Forces

Since we now work with materials with a finite extension, we have more types of forces to consider. We first have **volume forces**, which are proportional to $dV$. Typical examples are gravity and electrostatic forces:

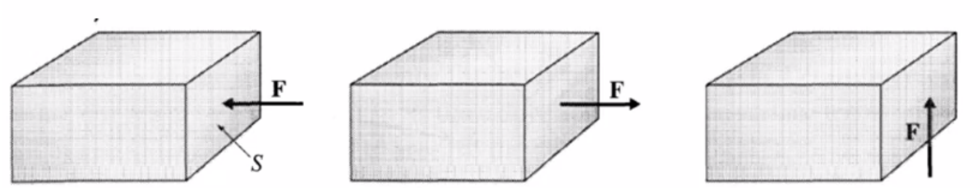
$$\mathbf{F}_g = \rho_0 \mathbf{g} dV$$

$$\mathbf{F}_e = \rho_e \mathbf{E} dV$$

$\rho_0, \rho_e$ represent mass and charge densities repectively. These are in a sense "boring" and just come from external fields.




Slightly more interesting are surface forces, which are proportional to dA. We then have Pressure, Tension, and Shear forces, as pictured below. Already things become more complicated as they depend not just on the magnitude of the force as well as the orientation of the body.




(a) Pressure (b) Tension (c) Shear

We could of course build up to something like torsion from combinations of the above "elementary" surface forces.
A question might be is where does this resistance to these surface forces come from? The answer is the intermolecular forces between the atoms in the material.

### 30.1.6 Stress & Strain - Basic Definitions

- Stress = Force / Area = Pressure (Fluid)
- Stress = Tension / Area (Wire)
- Stress = Shear Force / Area (Shear)

- Strain = dV/V (Fluid)
- Strain = dl/l (Wire)
- Strain = dy/dx (Shear)

One can think of the stress as a pressure, and the Strain as a "fractional deformation"/relative change.

#### 30.1.7 Hooke's Law for Solids (Linear Elasticity)

For small deformations, materials will also follow Hooke's law. We start with the wire case. If we change the wire by length $dl$, we have that the force $dF$ is given by:

$$dF = k \cdot dl$$

However, it is more useful to divide both sides by the area $A$ and add a factor of $l$ on the RHS, which gives:

$$\frac{dF}{A} = \frac{kl}{A}\frac{dl}{l}$$

Therein, the term on the LHS is the tensile stress, the rightmost term is the tensile strain (could be compressive if $dl < 0$, but let us assume for now that $dl > 0$) and define:

$$\frac{lk}{A} = Y$$

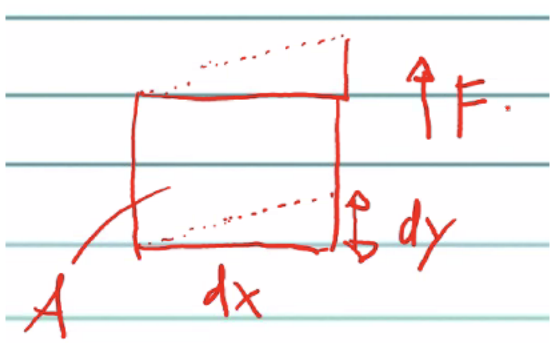
Which is Young's modulus, which is a characteristic of the material/material property.
Next, we look at a change of pressure in a fluid. We then have:

$$dp = -B\frac{dV}{V}$$

Where if we pull/expand a fluid, the pressure decreases. The $B$ is known as the bulk modulus, which tells us the response to a volumetric change (the proportionality constant to the strain). Note that in general liquids are non-compressible and hence $B$ for something like water is extremely high (but not infinite!)
Finally, considering a shear, we have

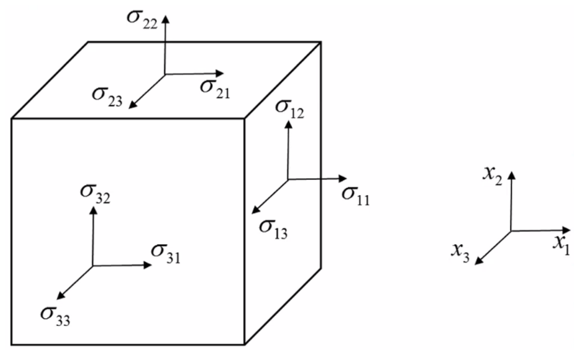
$$\frac{F}{A} = G\frac{dy}{dx}$$

Where $G$ is the shear modulus. As a visual, consider shearing a material by displacing it. The amount by which we shear is $dy$. The amount of tilt is the shear strain, and the coefficient of the force that fights back is the shear modulus.



### 30.1.8 The Stress Tensor

When we think of solids, we worry about two things; we both worry about the force (which has three components) as well as the orientation of the surface in which we apply the force, as the material acts differently depending along which surface we apply the force. The stress is specified by the direction of action and the orientation of the surface. This yields $3 \cdot 3 = 9$ possible components, yielding the stress tensor!




Question: A fluid is an isotropic molecule that cannot sustain any shearing forces in equilibrium. What must be true for the stress tensor of the fluid?

*Solution.* The tensor must be digonal (no shear forces) with one unique component (isotropic). The idea is if we push, we feel the same pressure on all sides. There is only one scalar value which quantifies the response. For the fluid, the stress tensor will have a simple form:

$$\sigma = \begin{bmatrix} -P & 0 & 0 \\ 0 & -P & 0 \\ 0 & 0 & -P \end{bmatrix}, \quad \sigma_{ij} = -P\delta_{ij}$$

Note that this is not true of all liquids, e.g. with viscous liquids like honey we can expect shear forces when we stir. □

Next day we will look at the counterpart of the stress tensor, the strain tensor, and hopefully get to equations of motion that describe the motion of objects with finite extension.

# 31 Lecture 31

## 31.1 Lecture Notes - The Strain Tensor and Hooke's Law for Solids

### 31.1.1 The Stress Tensor

Recall we can write the vector area element as $d\mathbf{A} = \hat{\mathbf{n}}dA$. What is then the stress tensor? We consider the force on a surface element $\mathbf{F}(d\mathbf{A})$, which we can express using the stress tensor:

$$\mathbf{F}(d\mathbf{A}) = \sigma d\mathbf{A}$$

Where $\sigma$ is the stress tensor, which is a 3x3 matrix. We could alternatively write this as:

$$F_i(d\mathbf{A}) = \sum_{j=1}^{3} \sigma_{ij} dA_j$$

#### 31.1.2 Stress Tensor Elements

Let

$$F_1(\text{On area dA normal to}\hat{\mathbf{e}}_1) = \sigma_{11} dA$$

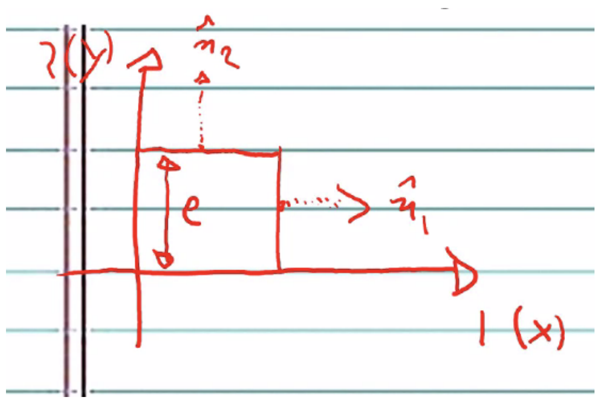
This is of course true for $\sigma_{22}$ and $\sigma_{33}$. $\sigma_{ii}$ is therefore ith component of the force $\perp$ to the i-axis. These are volumetric forces, and this tells us that the tensile and compressive forces correspond to the diagonal entries of the stress tensor. Next, consider

$$F_2(\text{on area dA normal to } \hat{\mathbf{e}}_1) = \sigma_{21} dA$$

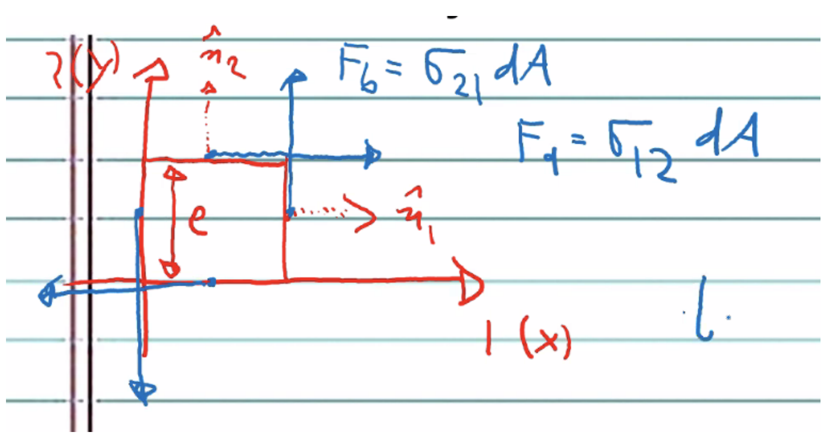
And similarly for $\sigma_{31}$. These are clearly shearing forces, forces that act perpendicular to the plane.

#### 31.1.3 Symmetry of the Stress Tensor

We said above that the stress tensor has 9 entries (3x3) but there is good news; it turns out that the tensor is actually symmetric (i.e. just 6 components to worry about!). To see this, consider the following geometry:




We now apply a force in the 1 direction, perpendicular to the 2 direction, i.e. $F_a = \sigma_{12} dA$. We can also apply a force in the 2 direction, perpendicular to the 1 direction, which gives $F_b = \sigma_{21} dA$. It is clear to see that these forces induce a torque. WE can also apply the same forces at the opposite corner, on the opposite direction (of equal magnitude):




The torque is then given by:

$$\tau = F_b l - F_q l = (\sigma_{21} - \sigma_{12}) l dA = \Gamma_3$$

(this torque is in the 3-direction). Furthermore, we have that:

$$\Gamma_3 = \frac{\mathrm{d}L_3}{\mathrm{d}t}$$

We now argue that $\Gamma_3$ is zero (and hence that $\sigma_{21} = \sigma_{12}$. To see that this is the case, consider shrinking all sides of the square by a factor $\lambda$. If I change $l$ by $\lambda l$, I pick up a factor of $\lambda$, and the area picks up a factor of $\lambda^2$, for a total scaling of:

$$\Gamma_3 \mapsto \lambda_3 \Gamma_3$$

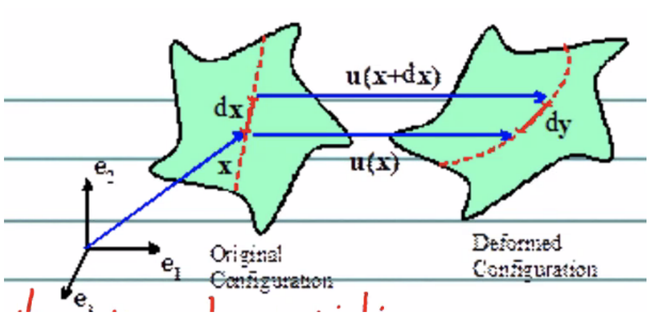
Then, what happens to the angular momentum? We pick up $\lambda^2$ from the $\mathbf{r} \times \mathbf{p}$, and then integrating over the plane, we have that we pick up a $\lambda^2$, so it follows that:

$$\lambda^3 \Gamma_3 = \lambda^4 \frac{\mathrm{d}L_3}{\mathrm{d}t}$$

But this is ture for all $\lambda$, so it follows that $\Gamma_3$ must be zero, and hence $\sigma_{21} = \sigma_{12}$. An identical argument shows that $\sigma_{ij} = \sigma ji$ in general.

### 31.1.4 Displacements

We now have a measure of force on the system, but we also need a measure of deformation. In general, we can write down a vector $\mathbf{r}$ from the origin to any position in the original configuration, and we change this $\mathbf{r}$ to a new vector $\mathbf{r} + \mathbf{u}(\mathbf{r})$ where $\mathbf{u}$ is the displacement from the reference to the current position.




This displacement vector in general depends on the position, not all points in the object will move the same amount. One might ask why do we want $\mathbf{r} + \mathbf{u}(\mathbf{r})$ and not just $\mathbf{u}$ by itself; consider that $\mathbf{u}$ itself would change during a constant translation ($\mathbf{u}(\mathbf{r}) = \mathbf{u}_0$) of the entire object, and hence is not a good measure of the strain. We need to look at **distortions**. A general way to write down/pick up these distortions:

$$du_i = \sum_j \frac{\partial u_i}{\partial r_j} dr_j$$

Or we can write this vectorially:

$$d\mathbf{u} = \mathbb{D} d\mathbf{r}$$

Where:

$$\mathbb{D} = \begin{bmatrix} \frac{\partial u_1}{\partial r_1} & \frac{\partial u_1}{\partial r_2} & \frac{\partial u_1}{\partial r_3} \\ \frac{\partial u_2}{\partial r_1} & \frac{\partial u_2}{\partial r_2} & \frac{\partial u_2}{\partial r_3} \\ \frac{\partial u_3}{\partial r_1} & \frac{\partial u_3}{\partial r_2} & \frac{\partial u_3}{\partial r_3} \end{bmatrix}$$

And this matrix contains the rate of change of the displacement. This is nice, as evidently this is now insensitive to any constant translations. The gradient of the constant translation will be zero, which is what we want as we should not have to pay any energy just by moving our rigid body back and forth.

### 31.1.5 The Strain Tensor

But there is another wrinkle to consider; rotating the body should also not change the energy of the body/the energy should not depend on the orientation. What do we then do about rotations? COnsider that for a small rotation:

$$\boldsymbol{\theta} = \theta \mathbf{u}$$

About an axis $\mathbf{u}$. To see this,

$$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$$

So we can use this to write:

$$\mathbf{u}(\mathbf{r}) = \mathbf{v} dt = \boldsymbol{\omega} dt \times \mathbf{r} = \boldsymbol{\theta} \times \mathbf{r}$$

Then we have that the displacement gradient has the form:

$$\mathbb{D} = \begin{bmatrix} 0 & \theta_3 & -\theta_3 \\ -\theta_3 & 0 & \theta_1 \\ \theta_2 & -\theta_1 & 0 \end{bmatrix}$$

Which we can see is an antisymmetric matrix. The antisymmetry means that:

$$\mathbb{D}^T = -\mathbb{D}$$

We need to get rid of this; we dont want a measure that picks up these rotations. We can construct this by remembering that any matrix can be decomposed into a symmetric and antisymmetric part. So, we write:

$$\mathbb{D} = \frac{1}{2}(\mathbb{D} - \mathbb{D}^T) + \frac{1}{2}(\mathbb{D} + \mathbb{D}^T)$$

Where the first term is by construction anti-symmetric, and the second term is by construction symmetric. Hence, we will just keep the second term, and use this as the measure of strain. The first term corresponds to the vorticity/curl part, but here we only want to keep the symmetric part. Hence, we can define the small-strain tensor as:

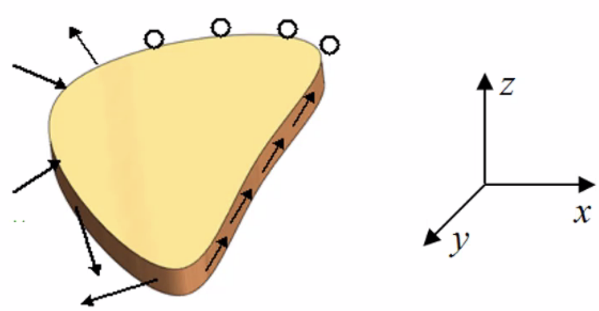
$$\epsilon = \frac{1}{2}\left(\mathbb{D} + \mathbb{D}^T\right)$$

Which we can see is symmetric by construction;

$$\epsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial r_j} + \frac{\partial u_j}{\partial r_i}\right)$$

### 31.1.6 Example: Thin/thick plate in xy plane

Consider a thin plate subject to in plane (xy) tensile, compressive, or shear forces. Are $\sigma_{zz}$ and $\epsilon_{zz}$ zero? nonzero?



*Solution.* $\sigma_{zz} = 0$ and $\epsilon_{zz} \neq 0$. For the first point, we can recognize that pulling on the plate in the xy plane induces no stress on the plane in the z direction. For the second point, we consider a sheet of rubber which changes in thickness as we pull it. □

What if we ask the same question, but this time the plate is thick?

*Solution.* If we compare the thick to the thing plate, any length change from the contraction effect would be very very small as the rod is tall. Hence, if we elongate it a little bit, then to first order, there is no strain. But, there can be a stress, as the system would like to contract. □

The first case (thin plate) corresponds to a plane stress condition. There is no stress in the z-axis, but if we pull, we get an appreciable change in the thickness of the plate and hence a nonzreo strain. On the other hand, we have the plane strain condition, where there is no strain in the z axis (rod is so tall such that the strain in the z-direction is negligeble/the rod does not change thickness when we pull) but we could still have a stress in the z-axis.

#### 31.1.7 Hooke's Law for Isotropic and Homogenous Solids

Consider a decomposition of strain tensor. Consider the quantity of average dilation:

$$e = \frac{1}{3}(\epsilon_{11} + \epsilon_{22} + \epsilon_{33}) = \frac{1}{3}\operatorname{Tr}(\epsilon)$$

Which is a measure of how much the system is compressed/pulled. The last equality we just write the expression as the trace of the strain tensor. Then, decomposing we have:

$$\epsilon = e\mathbb{I} + \epsilon^{dev} = \operatorname{Vol}(\epsilon) + \operatorname{Dev}(\epsilon)$$

Where the first term is the spherical term (the term that couples to the volume changes) and the second term is the deviatoric part (everything else, e.g. shear). Then, Hooke's Law says that:

$$\sigma = f(\epsilon)$$

Where $f$ is a linear function. Then, we write (Without proof) that in the linear case, we can use this decomposition to obtain:

$$\sigma = 3B\operatorname{Vol}(\epsilon) + 2G\operatorname{Dev}(\epsilon)$$

Where $B$, $G$ are the bulk and shear moduli. We can alternatively write this as:

$$\sigma = 2\mu\epsilon + \lambda\operatorname{Tr}(\epsilon)\mathbb{I}$$

Where $\mu = G$ and $\lambda = B - \frac{2}{3}\mu$. What is important to realize is that this is true for an isotropic and homoegnous solid, in which case only two elastic moduli are sufficient to characterize this response (we only need to know the bulk and shear moduli). Of course in something like an anisotropic metal, this would be more complicated.

# 32 Lecture 32

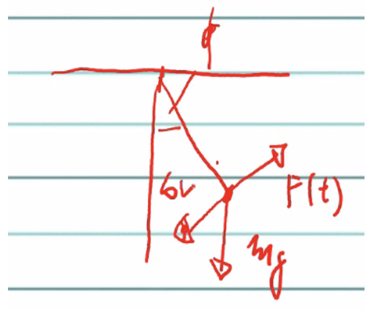
## 32.1 Chaos and Nonlinear Dynamics

### 32.1.1 What is chaos?

Chaotic systems can be described as systems with extreme sensitivity to initial conditions. Nonlinearity is a necessary condition, but not sufficient (not all nonlinear systems are chaotic).

### 32.1.2 Driven Damped Pendulum

We recall the setup of the driven damped pendulum:




This has the equation of motion:

$$mL^2\ddot{\phi} = -bL^2\dot{\phi} - mgL\sin\phi + LF(t)$$

Where $F(t) = F_0\cos(\omega t)$. Our standard technique for this course has been to linearize the equation, but here we want to look at the full behavior. This is where numerical simulations can come in useful. Rewriting this equation of motion slightly, we have:

$$\ddot{\phi} = -\frac{b}{m}\dot{\phi} - \frac{g}{L}\sin\phi + \frac{F_0}{mL}\cos(\omega t)$$

Define the damping parameter $2\beta = \frac{b}{m}$, the natural frequency $\omega_0^2 = \frac{g}{L}$, and the driving force $\gamma = \frac{F_0}{mL\omega_0^2}$. This yields the equation:

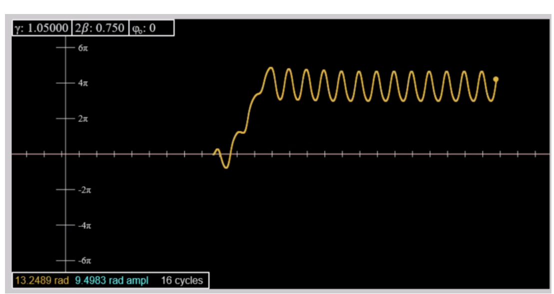
$$\ddot{\phi} + 2\beta\dot{\phi} + \omega_0\sin\phi = \gamma\omega_0^2\cos(\omega t)$$

Our analytical analysis of this equation ends here (as it is nonlinear), so we turn to simulations to study the behavior.

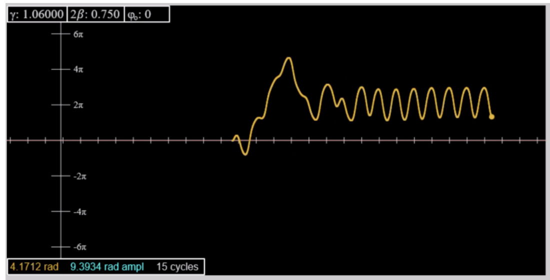
A simulation can be found here:

`http://galileoandeinstein.physics.virginia.edu/more_stuff/Applets/DampedDrivenPendulum/dampdrivPend__1.h`

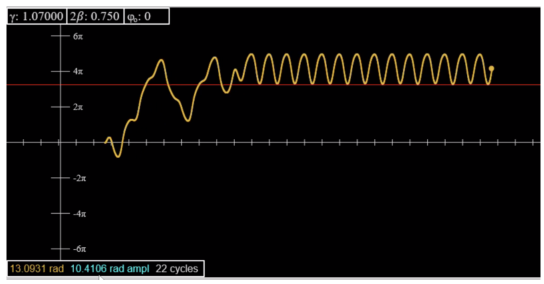
For a driving strength of 0.9, looking at $\phi$ it looks quite smooth/sinusoidal. However, looking at $\dot{\phi}$, the graph is no longer sinusoidal, but quite spiky (sawtooth). Since the system is chaotic, the solutions are not just simple sine functions. Let's then move to a stronger driving of 1.05. Then, we see an initial very strange jump in the motion, before the pendulum reaches a steady state. As we drive more strongly, there is an initial transience, which eventually settles to more regular looking periodic motion.



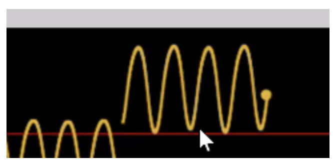

Increasing the driving to 1.06, we see that the transient period increases even further:



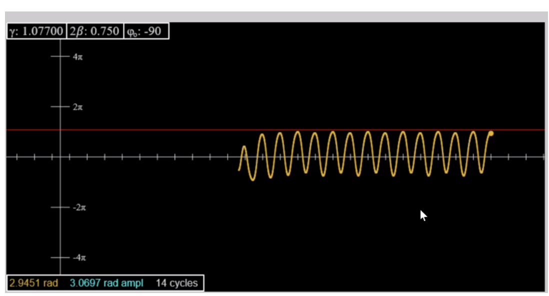

Further increasing it to 1.07, this transient period continues to increase. If one follows the values of the high/low amplitudes (or by tracking the red line of fixed amplitude, we notice that they do not immediately repeat:



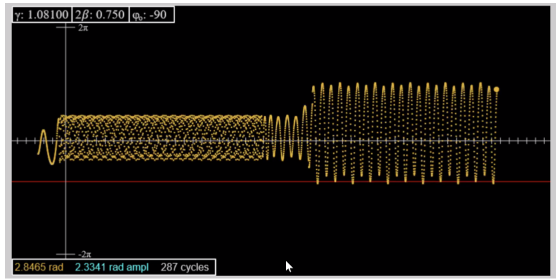

In particular, we can see that the period doubles; period doubling is a prelude to chaos. This is **not** higher harmonics, but rather subharmonics.



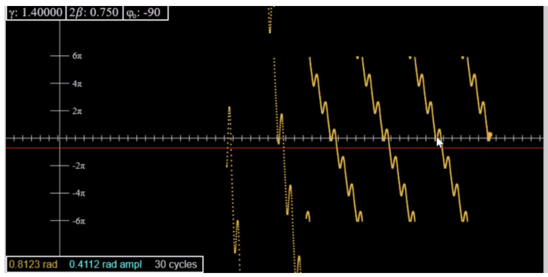
Going further to a driving strength of 1.077, the motion really does not look particularly sinusoidal any longer (the strange looking motion we noticed in the transient part for weaker strengths keeps going). But by changing the initial condition, the motion changes drastically:




Now, we increase the playing time and fast forward the system, to see when the transients die out, and we get a sequence of period doubling:




We could also drive the pendulum very hard (1.4) and we see that hte pendulum goes wild; it swings over the top frequently. But, the motion becomes periodic; we dont see the period doubling phenomena:




Note that in general, chaos is not random (all of this has been deterministic), it just never repeats.

### 32.1.3 Period Doubling and Bifurcations

| $n$ | period | $\gamma_m$ |
|---|---|---|
| 1 | 1 to 2 | 1.0663 |
| 2 | 2 to 4 | 1.0793 |
| 3 | 4 to 8 | 1.0821 |
| 3 | 8 to 16 | 1.0827 |

We see the spacings between period doubling gets narrower. Note as $\gamma_m \to \gamma_l = 1.0829$, we have chaos. Note that this relates to a constant;

$$\gamma_{n+1} - \gamma_n = \frac{1}{\delta}(\gamma_n - \gamma_{n-1})$$

And taking $n \to \infty$ we have:

$$\delta = 4.6692016$$

Which is Feigenbaum's number. This is a unviersal constant for chaotic systems. We will explore chaos in further detail Friday.

# 33 Lecture 33

## 33.1 Lecture Notes - Lyapunov Exponents, Bifurcation Diagrams, State-Space orbits, and Poincare Sections

### 33.1.1 Period doubling cascade - "Route to Chaos"

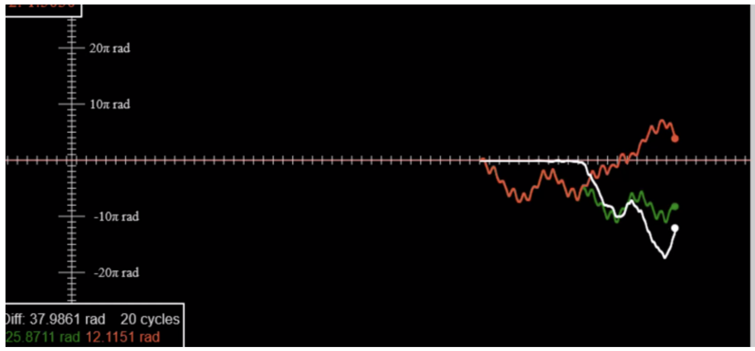
Period doubles each time the driving strength of the driven damped pendulum is increased past $\gamma_n$:

$$\delta = \lim_{n\to\infty} \frac{\gamma_{n-1} - \gamma_{n-2}}{\gamma_n - \gamma_{n-1}} = 4.6692016\ldots$$

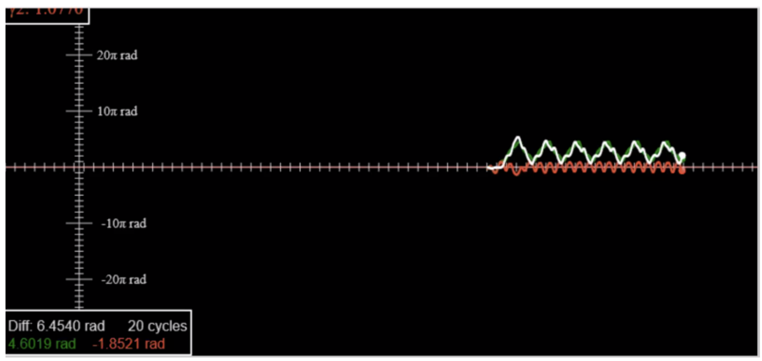
This is the universal "Feigenbaum number".

### 33.1.2 The driven damped pendulum revisited

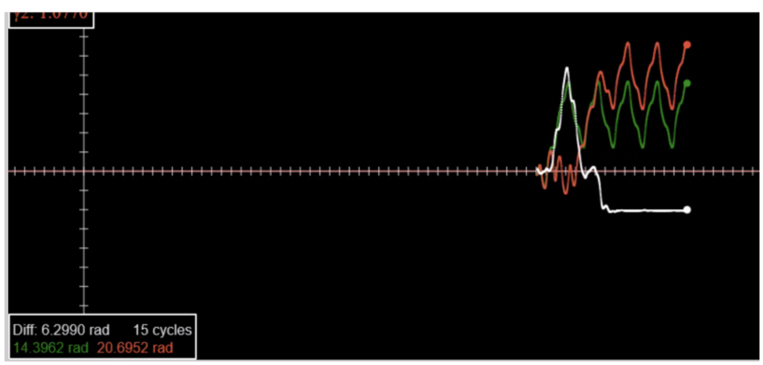
We return to the driven damped pendlum from last day. Giving two pendulums driving strengths of $\gamma = 1.503$, and one with initial phase of $\phi_0 = 0$ and the other with $\phi_0 = 0.005$, we can see that after time, the trajectories begin to diverge significantly.



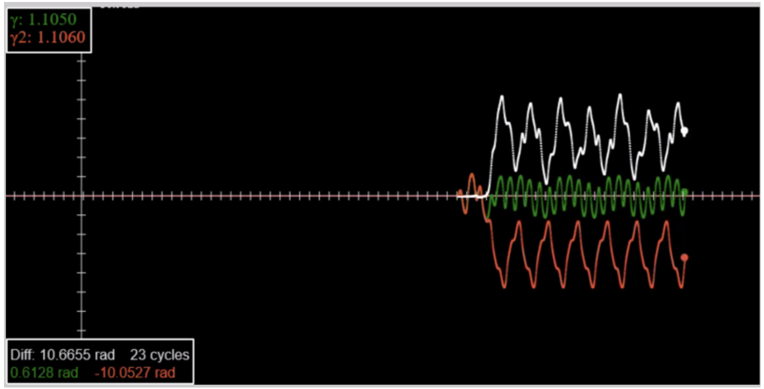

With a driving strength of 1.077, and an initial phase shift of $-28$, we see that there is definitely a divergence in the two trajectories, but it looks periodic.




With $\Delta\phi = -27$, the difference is different from $\Delta\phi = -28$ but still periodic. With $\Delta\phi = -29$, we see that we actually reach a constant difference:




With a slightly different driving strength, we see:




We recognize that we are slightly further into the chaotic regime; looking at the difference between the two pendulums, we see that there is no repeated pattern (there are slight differneces between cycles, so there doesn't appear to be periodic/perfect repetition).

### 33.1.3 Sensitivity to Initial Conditions & Lyapunov Exponents

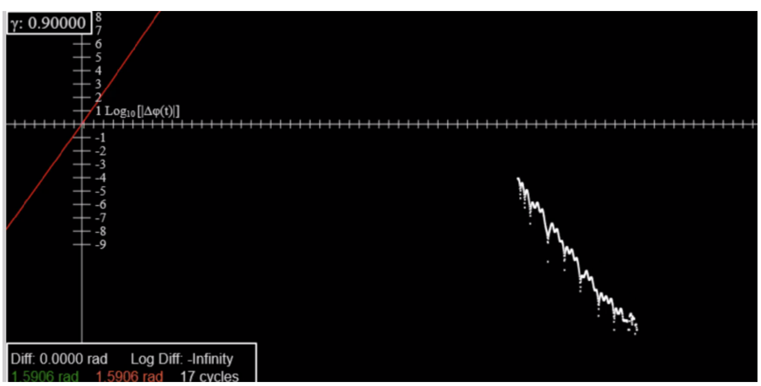
$\Phi(t) = \Phi_2(t) - \Phi_1(t)$ os the difference between two solutions with slightly different initial conditions. For linear oscillations,

$$\Delta\Phi(t) = D\exp(-\beta t)\cos(\omega t - \delta)$$

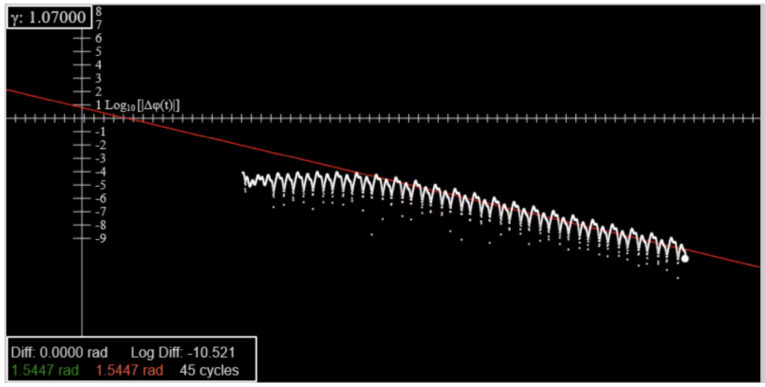
In general:

$$\Delta\Phi(t) \sim K\exp(\lambda t)$$

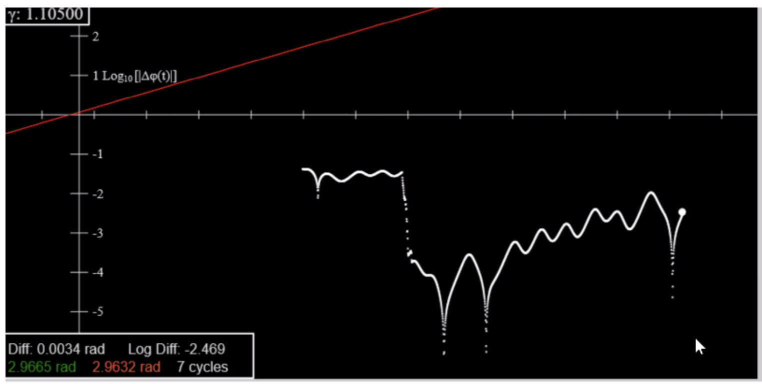
$\lambda$ is the Lyapunov exponent, with periodic motion when it is negative and chaotic motion when it is positive. It is often best to plot $\log\left|\Delta\Phi(t)\right| \sim \lambda t + \text{Const.}$ to see what happens with time.Doing so, we should either see a line with a positive or negative slope, depending on whether the trajectory is divergent (chaotic) or convergent (periodic) respectively. Plotting this for our two driven damped pendulums, we see:



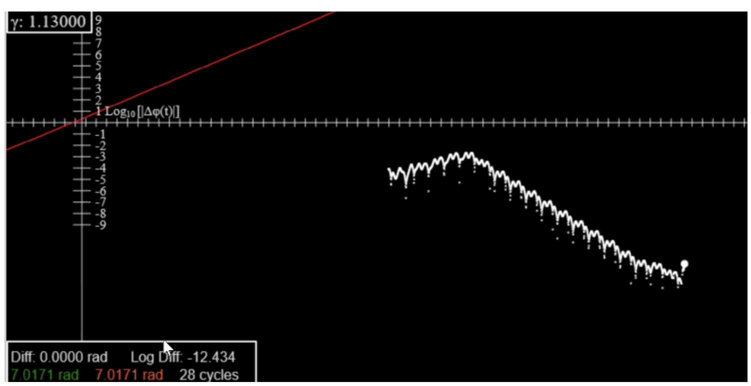

The peaks follow the linear trend, and the dips correspond to when $\Delta\Phi$ becomes negative. For driving of 1.07 (higher), we se that we still have convergence, but the Lyapunov exponent is less negative; it takes longer for the difference between the two osicllators to vanish.




Ramping it up to a driving strength of 1.105, we get into the Chaotic regime, where the overall slope is positive:

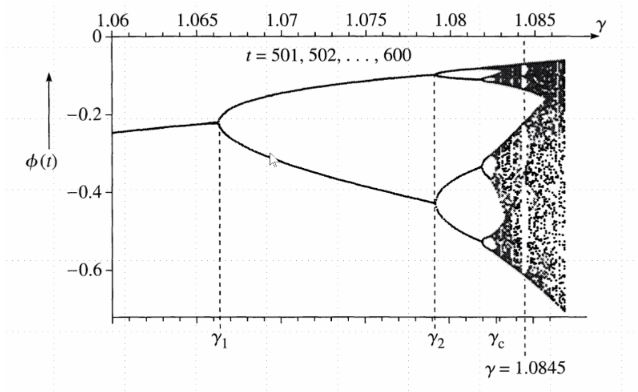



But bringing it up to 1.13, we actually go back to the non-chaotic regime:

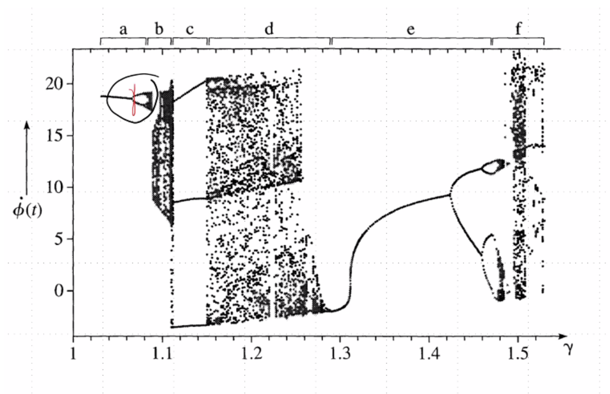



### 33.1.4 Bifurcation Diagrams

It gets quite confusing as to when the motion is chaotic, or not! A nice way of visualizing this is with a bifurcation diagram. We plot the driving strength on the $x$ axis and $\phi(t)$ on the y axis:



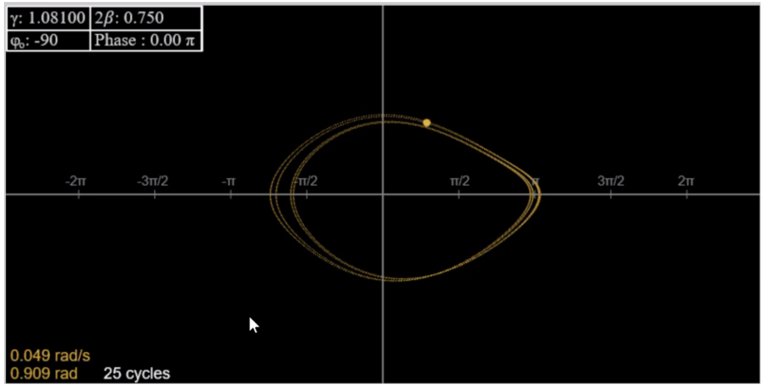

Here, we only plot the phase at specific times. If the motion is periodic and we take pictures at specific intervals, we always end up at the same point (e.g. before $\gamma_1$). However, past $\gamma_1$, we get period doubling and hence taking snapshots of the pendulum at the original period, we will now see two periods. Past $\gamma_2$, we see 4 different values, and so on. Past $\gamma = 1.0845$ we get into the chaotic regime. There is one more subtlety; once we increase the driving strength past a certain point, the pendulum can roll over the top, so the angle can go to infinity; this is a bit inconvenient! Although we could make the phase $2\pi$ periodic, another fix is to just plot the velocity as a function of the driving strength:



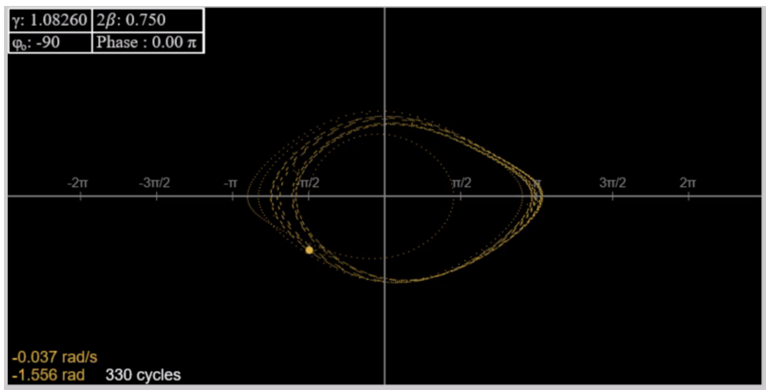

We can see distint regions of periodicity and chaos; the circled part of $a$ was the period doubling we were studying earlier, $b$ is chaotic, then $c$ goes back to regular/periodic motion, then it gets chaotic for a while $d$, then we have regular motion for a while $e$ and so on. These diagrams can be quite useful to see these regions of chaos and regularity.

### 33.1.5 State Space Orbits

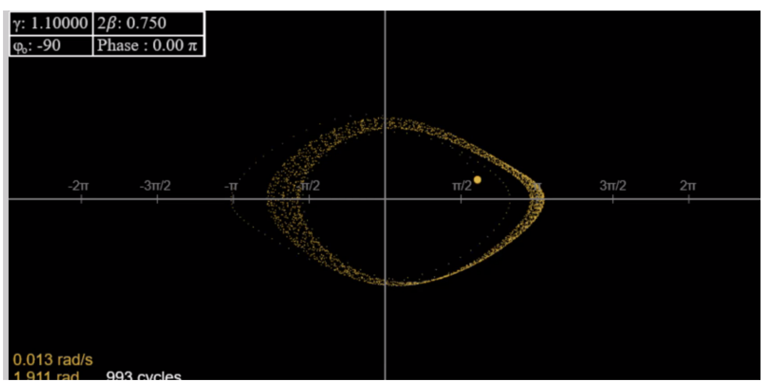
Very similar to phase space diagrams we did with Hamiltonian mechanics, but now we plot $\dot{\phi}$ versus $\phi$. Looking at this plot letting it run for a little while, we can see that there are four different cycles. (4 different trajectories); we are in a period 4 scenario.



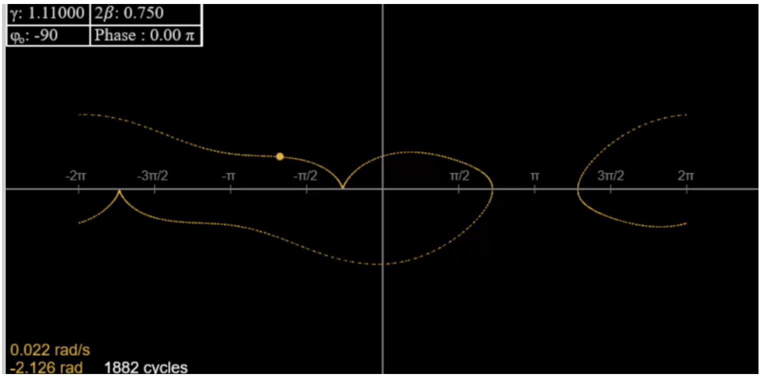

Going to 1.0826, we see that after the initial transience, looking very carefully we have a period 8 scenario:



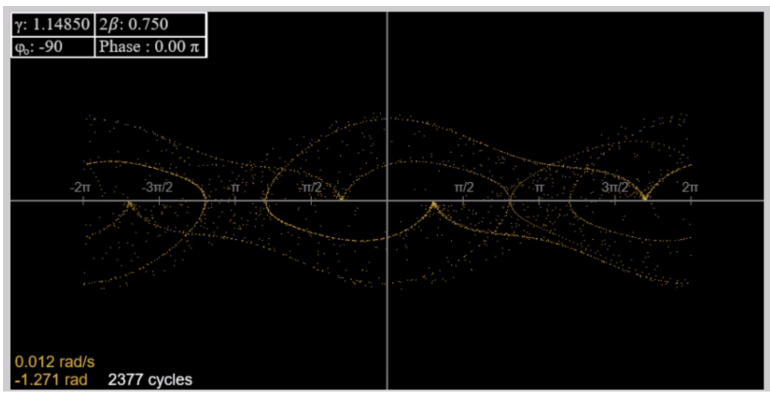

And for 1.087 we get period 16 and so on.




As we increase the driving strength even further, the motion becomes truly aperiodic.But, increasing it back to 1.1, we get back to a periodic region (where the pendulum starts to roll over):

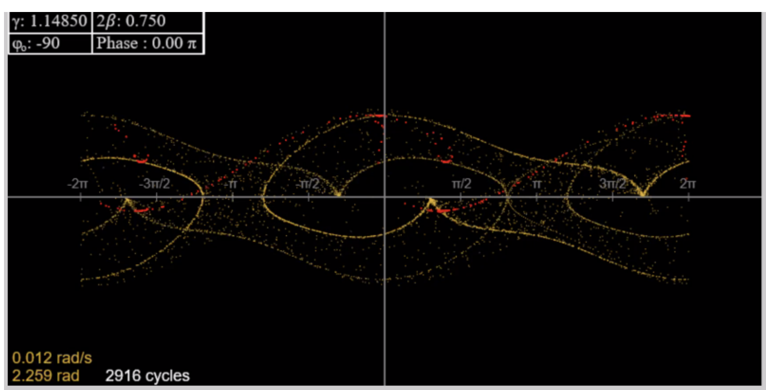


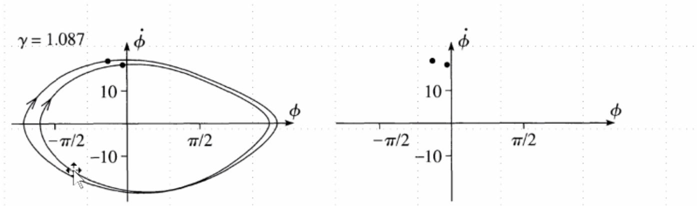

And increasing it further, the motion gets chaotic again:




### 33.1.6 Poincare Sections

We can plot a point per cycle to get a Poincare section:

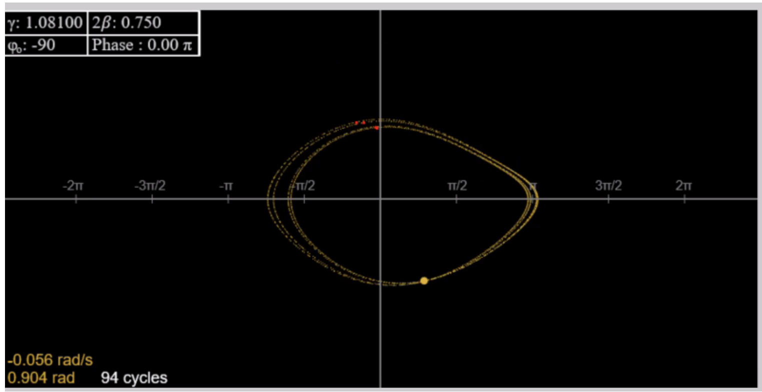


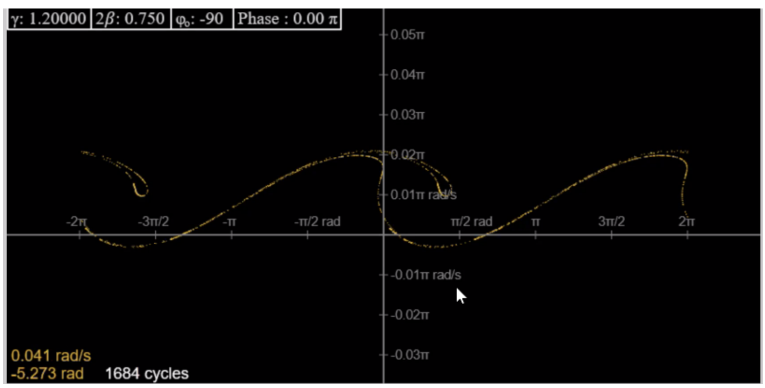

(a) State-Space Orbit

(a) Poincare Section

So for the 4-period case, we expect to see 4 dots, which is indeed the case (though two are close together in the below plot):



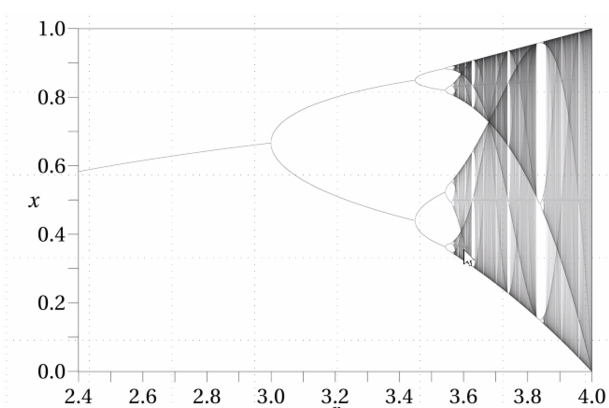

Pictured below is a strange attractor:




This is actually a fractal; fractals have scale invariance/self-similarity.

### 33.1.7 The Logistic map

The logistic map is defined as $x_{t+1} \mapsto rx_t(1-x_t)$, describing the reproduction and starvation of a population. Looking a the bifurcation diagram as we vary $r$, we see a similar development of chaotic behavior as the pendulum, due to the nonlinearity:



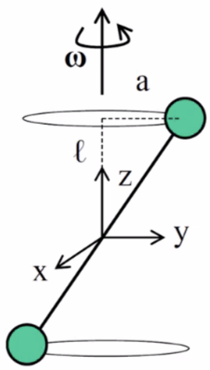
# 34 Lecture 34

## 34.1 Lecture Notes - Course Review I

### 34.1.1 Quiz II Recap

**Question 1** 1 pts

What happens to $\vec{L}$ and the moments of inertia I as the masses spin around?




- $\vec{L}$ changes because I changes; no net torque.
- $\vec{L}$ and I are both constant; no net torque.
- $\vec{L}$ is constant, but I changes; no net torque.
- $\vec{L}$ and I both change; there is a net torque.
- $\vec{L}$ changes but I is constant; there is a net torque.

*Solution.* $\mathbf{L}$ and and $\mathbb{I}$ both change; there is a net torque. $\mathbf{L}$ is parallel to $\boldsymbol{\omega}$ when the rotation is about a principal axis, which is not the case here. Here, there are off diagonal elements in the inertia tensor. $\mathbf{L}$ changes in time ($\mathbf{r} \times \mathbf{p}$ changes in time) and hence there must be a net torque $\mathbf{\Gamma}$ and the inertia tensor must change as well. $\square$

**Question 2** 1 pts

Select ALL correct statements about the Hamiltonian of a system:

- ☐ $H = \sum_i p_i \dot{q}_i - L$
- ☐ $\frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}$
- ☐ $\frac{dH}{dt} = \frac{\partial H}{\partial t}$
- ☐ $\dot{p}_i = -\frac{\partial H}{\partial q_i}$
- ☐ $\dot{q}_i = \frac{\partial H}{\partial p_i}$

*Solution.* All of them! The first is the definition. The last two are Hamilton's equations. The last two are less obvious, but can be checked by differentiating the expression. Taking the derivative, we have:

$$\frac{\mathrm{d}\mathcal{H}}{\mathrm{dt}} = \sum_i \frac{\partial \mathcal{H}}{\partial q_i}\dot{q}_i + \frac{\partial \mathcal{H}}{\partial p_i}\dot{p}_i + \frac{\partial \mathcal{H}}{\partial t}$$

Where the first two terms are zero by Hamilton's equations. We also have that:

$$\mathcal{H} = \sum_i p_i \dot{q}_i(q, p, t) - \mathcal{L}$$

So taking the derivative:

$$\frac{\partial \mathcal{H}}{\partial t} = p\frac{\partial \dot{q}}{\partial t} - \frac{\partial \mathcal{L}}{\partial \dot{q}}\frac{\partial \dot{q}}{\partial t} - \frac{\partial \mathcal{L}}{\partial t}$$

And the first two terms cancel using the Euler-Lagrange equation. $\square$

**Question 3** 1 pts

Select ALL correct statements about the components $I_{ij}$ and the principal moments $\lambda_i$ of the moment of inertia tensor **I** of a rigid body:

- ☐ $\lambda_1 + \lambda_2 + \lambda_3 = 0$ ✗
- ☐ $I_{ij} = I_{ji}$
- ☐ $I_{ii} \geq 0$
- ☐ $I_{ij} \geq 0$
- ☐ $\lambda_3 \leq \lambda_1 + \lambda_2$

*Solution.* The first is false (no reason it should be!) The second is true as the moment of inertia tensor is symmetric by construction. The third is true by definition of the diagonal elements. The fourth is false. The last one is tricky:

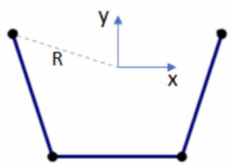
$$\lambda_1 + \lambda_2 = \rho \int (x^2 + y^2) dV + 2\rho \int z^2 dV \geq \rho \int x^2 + y^2 dV = \lambda_3$$

□

**Question 4** — 1 pts

Four equal masses are occupying four of the five corners of a pentagon and are connected by massless struts. The origin is at the centre of the pentagon. What are the **centre-of-mass** (CM) coordinates?

$\vec{R} = \frac{1}{M} \sum$




- ○ X=0, Y=-R/2
- ○ None of these
- ○ X=0, Y=-R/4
- ○ X=0, Y=0
- ○ X=0, Y=-R/5

*Solution.* From the definition $\mathbf{R} = \frac{1}{M} \sum_i \mathbf{r}_i m_i$, we see that the x component is immediately zero by symmetry. We could calculate the y component the pedestrian way by adding up all the contributions, or we can use a neat trick. If we complete the pentagon, then we have a center of mass at zero. Hence, we can imagine calculating the center of mass of the pentagon plus an "anti-mass" at the top vertex of the pentagon, which gives the easy result of $Y = -\frac{R}{4}$. □

**Question 8** 1 pts

A bead of mass mass m is threaded on a frictionless rod that lies in the **horizontal** plane and is forced to spin with **constant angular velocity** $\omega$ about an axis vertical through the midpoint of the rod. What is the Hamiltonian of the bead?

- ○ $H = \frac{p^2}{2m}$
- ○ $H = \frac{p^2}{2m} + \frac{m}{2}\omega^2 x^2$
- ○ $H = \frac{m}{2}(\dot{x}^2 + x^2\omega^2)$
- ○ $H = \frac{p^2}{2m} + \frac{m}{2}\omega^2 x^2 + mgx$
- ○ $H = \frac{p^2}{2m} - \frac{m}{2}\omega^2 x^2$

*Solution.* This question is somewhat deceptive; its tempting to use $\mathcal{H} = T + U$, but this **does not hold** here, as the coordinate transformation is not natural. Hence, we have to return to the definition $\mathcal{H} = \sum_i p_i \dot{q}_i - \mathcal{L}$. Doing so, we realize the answer is:

$$\mathcal{H} = \frac{p^2}{2m} - \frac{m}{2}\omega^2 x^2$$

Explicitly working it out step by step, we have that:

$$\mathcal{L} = T - U = T = \frac{m}{2}(\dot{x}^{@} + x^2\dot{\theta}^2)$$

There is no potential as we just rotate in a plane. Here we have that $\dot{\theta}^2 = \omega^2$, and in fact there is an explicit time dependence so the coordinate transformation is not natural. Hence:

$$\mathcal{L} = \frac{m}{2}(\dot{x}^2 + x^2\omega^2)$$

And deriving $p$ we have that:

$$p = \frac{\partial \mathcal{L}}{\partial \dot{x}} = m\dot{x}$$

And furthermore:

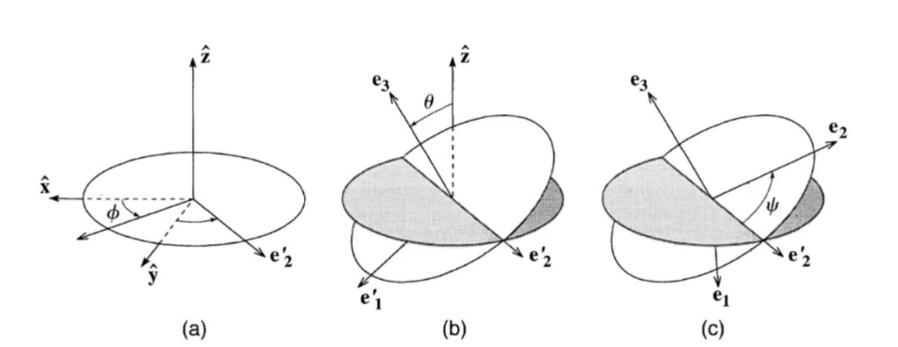
$$dotx = \frac{p}{m}$$

Then, we have that the Hamiltonian is given by:

$$\mathcal{H} = p\frac{p}{m} - \mathcal{L} = \frac{p^2}{m} - \frac{1}{2}\frac{p^2}{m} - \frac{m}{2}x^2\omega^2 = \frac{m}{2}(\dot{x}^2 + x^2\omega^2)$$

□

### 34.1.2 Euler Angles




$$\mathbf{R}_1(\phi) = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
$$\mathbf{R}_2(\theta) = \begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix}$$
$$\mathbf{R}_3(\psi) = \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

This is the convention for successive rotations to obtain the orientation of a rigid body. The first step is a rotation by $\phi$ about the $\hat{\mathbf{z}}$ axis. The second step is a rotation by angle $\theta$ about the new $\hat{\mathbf{e}}_2'$ axis. Finally, we rotate by $\psi$ about the $\hat{\mathbf{e}}_3$ axis.

### 34.1.3 Symmetric Top

A classic application of the Euler angles is the symmetric top, where we want the angular momentum in the body frame. Using the rotations, we derived $\boldsymbol{\omega}$ in terms of the Euler angles and in terms of the unit vectors in the body frame:

$$\vec{\omega} = (-\dot{\phi}\sin\theta\cos\psi + \dot{\theta}\sin\psi)\hat{\mathbf{e}}_1 + (\dot{\phi}\sin\theta\sin\psi + \dot{\theta}\cos\psi)\hat{\mathbf{e}}_2 + (\dot{\psi} + \dot{\phi}\cos\theta)\hat{\mathbf{e}}_3$$

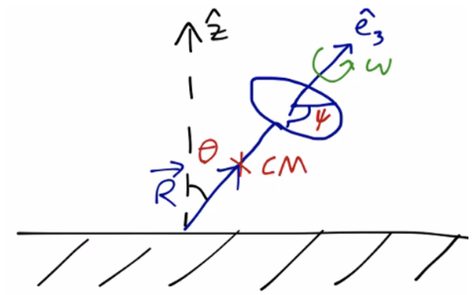
Using this, we found the kinetic energy:

$$T = \frac{1}{2}\mathbf{L}\cdot\boldsymbol{\omega} = \frac{1}{2}\left(\lambda_1\omega_1^2 + \lambda_2\omega_2^2 + \lambda_3\omega_3^2\right)$$

Of course, the kinetic energy is the same whether calculated in the body frame or lab frame; the product of vectors is invariant of frame, though the vectors themselves may change. We then assumed $\lambda_1 = \lambda_2$ (symmetric top) and from this we derived the Lagrangian:

$$\mathcal{L} = \frac{1}{2}\lambda_1\left(\dot{\phi}^2\sin^2\theta + \dot{\theta}^2\right) + \frac{1}{2}\lambda_3(\dot{\psi} + \dot{\phi}\cos\theta)^2 - MgR\cos\theta$$

#### 34.1.4 What if the tip is free to slide?




If this is the case, then we modify the Lagrangian, using the fact that we can always decouple the translational and rotational motions of the rigid body:

$$\mathcal{L} = \frac{1}{2}M\left(\dot{X}^2 + \dot{Y}^2 + \dot{Z}^2\right) + \frac{1}{2}\lambda_1^{\mathrm{cm}}\left(\dot{\phi}^2\sin^2\theta + \dot{\theta}^2\right) + \frac{1}{2}\lambda_3^{\mathrm{cm}}(\dot{\psi} + \dot{\phi}\cos\theta)^2 - MgR\cos\theta$$

Which we see is just the kinetic energy of the COM $\frac{1}{2}M\left(\dot{X}^2 + \dot{Y}^2 + \dot{Z}^2\right)$ plus the rotational terms we determined earlier. If we have steady precession, then $\theta =$ Const and hence $\dot{z} = -R\sin\theta\dot{\theta} = 0$ (as $z = R\cos\theta$). One subtlety is we have to relate the center of mass moments to the moments when we had the tip fixed. Using the parallel axis theorem, we have that:

$$\lambda_1^{tip} = \lambda_1^{COM} + MR^2$$

$$\lambda_3^{tip} = \lambda_3^{COM}$$

So we can relate these terms very quickly. We recall the formula for the precession frequency:

$$\Omega = \frac{\lambda_3\omega_3}{\lambda_1\cos\theta}$$

In the case of free precession. In the two situations where the tip is fixed vs free, we consider:

$$\Omega^{tip} = \frac{\lambda_3^{tip}\omega_3}{\lambda_1^{tip}\cos\theta} < \Omega^{COM}$$

So it actually precesses slower if we let the tip freely move!
Next day, we will look at canonical transformations.

# 35 Lecture 35

## 35.1 Lecture Notes - Course Review II

### 35.1.1 Examples of Canonical Transformations

Only transformations in which new coordinates obey Hamilton's equations of motion are canonical. Suppose we have our generalized position $q$ that maps to a new variable $Q = p$ and our generalized momentum $p$ that maps to a new variable $P = -q$. The original coordinates satisfy Hamilton's equations, so:

$$\dot{q} = \frac{\partial\mathcal{H}}{\partial p}, \quad \dot{p} = -\frac{\partial\mathcal{H}}{\partial q}$$

We therefore have that:

$$\dot{Q} = \dot{p} = -\frac{\partial\mathcal{H}}{\partial q} = -\frac{\partial\mathcal{H}}{\partial(-P)} = \frac{\partial\mathcal{H}}{\partial P}$$

So the first equation of motion is satisfied. Playing the same game with $\dot{P}$, we have:

$$\dot{P} = -\dot{q} = -\frac{\partial \mathcal{H}}{\partial p} = -\frac{\partial \mathcal{H}}{\partial Q}$$

So this also works out. Checking the (fundamental) Poisson brackets, we have that:

$$[Q, Q] = [p, p] = 0$$

$$[P, P] = [-q, -q] = [q, q] = 0$$

$$[Q, P] = [p, -q] = -[p, q] = -(-[q, p]) = [q, p] = 1$$

So we have checked that this is a canonical transformation in a completely equivalent way. Recall that we can come up with generating functions $F$ which link the old and new variables. One example (in the HW) was:

$$F_1(q, Q) = qe^Q$$

Now we can calculate the momenta:

$$p_i = \frac{\partial F_1}{\partial q_i}, \quad P_i = -\frac{\partial F_i}{\partial Q_i}$$

A second example is:

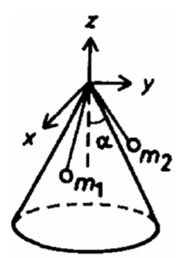
$$F_2(p, Q) = -(e^Q - 1)^2 \tan p$$

Where:

$$q_i = -\frac{\partial F_2}{\partial p_i}, \quad P_i = -\frac{\partial F_2}{\partial Q_i}$$

The significance of all of this is that there is a subclass of transformations, where due to the Jacobian determinant being equal to 1, the metric of phase space is preserved (this relates to a fundamental symmetry).

### 35.1.2 Practice Problem




Two equal masses are connected by a string of length $l$ that runs through the tip of a cone. One mass is free to move inside, the other moves without friction on the surface.

(a) Set up suitable generalized coordinates.

*Solution.* It is most natural to use spherical. For $m_1$ we use coordinates $(r, \theta, \phi)$. For $m_2$ we have coordinates $(l - r, \pi - \alpha, \beta)$. We have 6 variables, 4 independent, and two constrains. We put origin at the top of the cone. Take $r, \theta, \phi, \beta$ as our generalized coordinates. $\square$

(b) Find the Lagrangian and the equations of motion. Are there cylical coordinates?

*Solution.* Velocities are given by:

$$m_1 : (\dot{r}, r\dot{\theta}, r\dot{\phi}\sin\theta)$$

$$m_2 : (-\dot{r}, 0, (l-r)\dot{\beta}\sin(\pi-\alpha))$$

$$\mathcal{L} = \frac{m}{2}\left[2\dot{r}^2 + r^2\dot{\theta}^2 + r^2\dot{\phi}^2\sin^2\theta + (l-r)\dot{\beta}^2\sin^2(\pi-\alpha)\right] - mgr\cos\theta + mg(l-r)\cos\alpha$$

To find the equations of motion, we use the Euler Lagrange equations. We notice that the Lagrangian has no dependence on $\phi$ or $\beta$ and hence these coordinates are cyclic. □

(c) Find the Hamiltonian.

*Solution.* For the cyclic coordinates, we have:

$$p_\phi = mr^2 \dot{phi} \sin^2\theta = \text{Const.}$$

$$p_\beta = m(l-r)^2\dot{\beta}\sin^2(\pi-\alpha) = \text{Const.}$$

For the $r$ equations, we have:

$$2\ddot{r} - r(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta) + (l-r)\dot{\beta}^2\sin\alpha + g(\cos\theta + \cos\alpha) = 0$$

For the $\theta$ equation we have:

$$r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\dot{\phi}^2 =$$

□

(d) What is the angular velocity of the particle on the outside if it moves in a circular orbit?

*Solution.* We already calcualted $p_\phi$ and $p_\theta$ which were constant. Calculating the other two, we have:

$$p_r = \frac{\partial\mathcal{L}}{\partial\dot{r}} = 2m\dot{r}$$

$$p_\theta = \frac{\partial\mathcal{L}}{\partial\dot{\theta}} = mr^2\dot{\theta}$$

The Hamiltonian is given by:

$$\mathcal{H} = p_r\dot{r} + p_\theta\dot{\theta} + p_\phi\dot{\phi} + p_\beta\dot{\beta} - \mathcal{L}$$

Hence:

$$\mathcal{H} = \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} + \frac{p_\phi^2}{2mr\sin^2\theta} + \frac{p_\beta^2}{2m(l-r)^2\sin^2\alpha} + mgr\cos\theta - mg(L-r)\cos\alpha$$

Here we see that the Hamiltonian is just $\mathcal{H} = T + U$ as the transformation is indeed natural. Moving onto solving the question, in a circular orbit $r = \text{Const.}$, so we can therefore solve for:

$$\dot{\beta} = \frac{p_\beta^2}{2m(l-r)^2\sin^2\alpha}$$

□

# 36 Formula Sheet

This section lists most of the important formulas used throughout the year, albeit without explanation; it will likely be most helpful as a quick reference when doing homework problems of when studying. They are written in the order in which material was covered in the course.

## 36.1 The Variational Principle and Lagrangian Mechanics

Newton's Law

$$\sum_{i=1}^{n} \mathbf{F}_i(t) = m\ddot{\mathbf{r}}(t) \tag{1}$$

Generalized Coordinates:

$$q_i = q_i(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_n) \tag{2}$$

Action & Hamilton's Principle:

$$S[q_1, \ldots, q_n] = S[q_1(t), \ldots, q_n(t)] = \int_{t_1}^{t_2} \mathcal{L}(q_1, \dot{q}_1, \ldots, q_n, \dot{q}_n, t)dt \text{ is stationary for the true path} \tag{3}$$

The Lagrangian:

$$\mathcal{L} = T - U \tag{4}$$

Euler-Lagrange equations:

$$\frac{\partial f}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial f}{\partial y'} = 0 \text{ Where } f \text{ is the function being minimized} \tag{5}$$

Lagrange equation of motion:

$$\frac{\partial \mathcal{L}}{\partial q} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{q}} \tag{6}$$

Generalized Force:

$$\frac{\mathrm{d}\mathcal{L}}{\mathrm{d}q} \tag{7}$$

Generalized Momenta:

$$\frac{\mathrm{d}\mathcal{L}}{\mathrm{d}\dot{q}} \tag{8}$$

Center of Mass Coordinates:

$$\mathbf{R} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1 + m_2}, \quad \mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2 \tag{9}$$

Holonomic constraints:

$$f(\mathbf{r}_i, t) = 0 \tag{10}$$

Lagrange equation of motion with non-conservative force correction:

$$\frac{\partial \mathcal{L}}{\partial q} + F_{noncons} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{q}} \tag{11}$$

Lorentz Force Law:

$$m\ddot{\mathbf{r}} = q(\mathbf{E} + \dot{\mathbf{r}} \times \mathbf{B}) \tag{12}$$

Lagrangian for charged particle in EM field:

$$\mathcal{L} = T - U' = T - (qV - q\dot{\mathbf{r}} \cdot \mathbf{A}) \tag{13}$$

Generalized momentum for charged particle in EM field:

$$\mathbf{p} = m\dot{\mathbf{r}} + q\mathbf{A} \tag{14}$$

Relativistic spatial momentum:

$$\mathbf{p} = m_0\gamma\mathbf{v} \tag{15}$$

Noether's Theorem:

$$\frac{\mathrm{d}}{\mathrm{d}t} \sum_{j=1}^{n} \left( \frac{\partial \mathcal{L}}{\partial \dot{\tilde{q}}_j} \frac{\partial \tilde{q}_j}{\partial \alpha} \right) \Bigg|_{\alpha=0} = \frac{\mathrm{d}}{\mathrm{d}t} I(q_1, \dot{q}_1, \dots t) = 0 \tag{16}$$

Hamiltonian:

$$\mathcal{H} = \sum_j p_j \dot{q}_j - \mathcal{L} \tag{17}$$

Lagrange Multipliers:

$$\frac{\partial \mathcal{L}}{\partial q_i} + \sum_{k=1}^{m} \lambda_k \frac{\partial f_k}{\partial q_i} = \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \text{ for a system with } m \text{ holonomic constraints} \tag{18}$$

### 36.2 Coupled Oscillators

General solution to simple harmonic oscillator:

$$x(t) = A\cos(\omega_0 t - \delta), \quad \omega_0 = \sqrt{\frac{k}{m}} \tag{19}$$

Damped Oscillator ODE:

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0, \quad \beta = \frac{b}{2m} \tag{20}$$

Overdamped Solution:

$$x(t) = C_1 \exp(r_1 t) + C_2 \exp(r_2 t) = C_1 \exp\left(\left(-\beta + \sqrt{\beta^2 - \omega_0^2}\right) t\right) + C_2 \exp\left(\left(-\beta - \sqrt{\beta^2 - \omega_0^2}\right) t\right) \tag{21}$$

Critically Damped solution:

$$x(t) = C_1 \exp(-\beta t) + C_2 t \exp(-\beta t) \tag{22}$$

Underdamped Solution:

$$x(t) = A \exp(-\beta t) \cos\left(\left(\sqrt{\omega_0^2 - \beta^2}\right) t - \delta\right) \tag{23}$$

Damped Driven Oscillator ODE:

$$\ddot{z} + 2\beta\dot{z} + \omega_0^2 z = f_0 \exp(i\omega t) \tag{24}$$

Amplitude of solution $C\exp(i\omega t)$:

$$C = \frac{f_0}{-\omega^2 + 2i\beta\omega + \omega_0^2} = A\exp(i\delta) \tag{25}$$

Amplitude squared and phase:

$$A^2 = CC^* = \frac{f_0^2}{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}, \quad \delta = \arctan\left(\frac{2\beta\omega}{\omega_0^2 - \omega^2}\right) \tag{26}$$

General solution to $\ddot{x} + 2\beta\dot{x} + \omega_0^2 x + f_0 \cos(\omega t)$:

$$x(t) = A\cos(\omega t - \delta) + C_1 \exp(r_1 t) + C_2 \exp(r_2 t) = x_{periodic}(t) + x_{trans}(t) \tag{27}$$

Q factor:

$$Q = \frac{\omega_0}{2\beta} \tag{28}$$

General Coupled Oscillator Matrix equation:

$$\mathbb{M}\ddot{\mathbf{x}} = -\mathbb{K}\mathbf{x} \tag{29}$$

Eigenfrequency characteristic equation:

$$\det\Big(\mathbb{K} - \omega^2\mathbb{M}\Big) = 0 \tag{30}$$

Eigenmodes:

$$(\mathbb{K} - \omega\mathbb{M})\begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \mathbf{0} \tag{31}$$

Normal coordinates:

$$\mathbf{q}(t) = \sum_{i=1}^{n} \mathbf{a}_i \xi_i(t), \quad \text{such that } \ddot{\xi}_i + \omega_i^2 \xi_i = 0 \tag{32}$$

## 36.3 Mechanics in Non-Inertial Frames

Newton's law for linearly accelerating frame:

$$m\ddot{\mathbf{r}} = \mathbf{F} - m\mathbf{A} \tag{33}$$

Newton's Law with tidal force:

$$m\ddot{\mathbf{r}} = \mathbf{F}_g + \mathbf{F}_{tidal} = m\mathbf{g} - GMm\left(\frac{\hat{\mathbf{d}}}{d^2} - \frac{\hat{\mathbf{d}}_0}{d_0^2}\right) \tag{34}$$

Velocity in rotating frames

$$\mathbf{v} = \dot{\mathbf{r}} = \boldsymbol{\omega} \times \mathbf{r} \tag{35}$$

Addition of angular velocities:

$$\boldsymbol{\omega}_{31} = \boldsymbol{\omega}_{32} + \boldsymbol{\omega}_{21} \tag{36}$$

Time derivatives in non-inertial frames:

$$\left(\frac{\mathrm{d}\mathbf{Q}}{\mathrm{d}t}\right)_{S_0} = \left(\frac{\mathrm{d}\mathbf{Q}}{\mathrm{d}t}\right)_{S} + \boldsymbol{\Omega} \times \mathbf{Q} \tag{37}$$

Newton's Law in Rotating Frame:

$$m\ddot{\mathbf{r}} = \mathbf{F} + \mathbf{F}_{cor} + \mathbf{F}_{cent} + \mathbf{F}_{euler} = \mathbf{F} + 2m\dot{\mathbf{r}} \times \boldsymbol{\Omega} + m(\boldsymbol{\Omega} \times \mathbf{r}) \times \boldsymbol{\Omega} + m\mathbf{r} \times \dot{\boldsymbol{\Omega}} \tag{38}$$

## 36.4 Rigid Body Mechanics

Position of COM:

$$\mathbf{R} = \frac{1}{M}\sum_{\alpha} m_\alpha \mathbf{r}_\alpha \tag{39}$$

Momentum of COM:

$$\mathbf{P} = M\dot{\mathbf{R}} = \sum_{\alpha} m_\alpha \dot{\mathbf{r}}_\alpha \tag{40}$$

External force:

$$\mathbf{F}_{ext} = M\ddot{\mathbf{R}} \tag{41}$$

Angular momentum of spinning rigid bodies:

$$\mathbf{L} = \sum_{\alpha}(\mathbf{R} \times m_\alpha \dot{\mathbf{R}}) + \sum_{\alpha}(\mathbf{r}'_\alpha + m_\alpha \dot{\mathbf{r}}') = \mathbf{L}_{orbital} + \mathbf{L}_{spin} \tag{42}$$

Potential energy:

$$U = U_{ext} + U_{int} = U_{ext} + \sum_{i<j} U_{ij}(r_{ij}) \tag{43}$$

Kinetic energy:

$$T = \frac{1}{2}\sum_{\alpha} m_\alpha \dot{\mathbf{r}}_2^{\prime\alpha} \tag{44}$$

Angular momentum for rotation about z axis:

$$\mathbf{L} = \begin{bmatrix} L_x \\ L_y \\ L_z \end{bmatrix} = \sum_{\alpha}(\mathbf{r}_\alpha \times m_\alpha \mathbf{v}_\alpha) = \sum_{\alpha}(\mathbf{r}_\alpha \times m_\alpha(\omega\hat{\mathbf{z}} \times \mathbf{r}_\alpha)) = \sum_{\alpha} m_\alpha \omega \begin{bmatrix} -z_\alpha x_\alpha \\ -z_\alpha y_\alpha \\ x_\alpha^2 + y_\alpha^2 \end{bmatrix} = \begin{bmatrix} I_x\omega \\ I_y\omega \\ I_z\omega \end{bmatrix} \tag{45}$$

Inertia Tensor:

$$\mathbf{L} = \mathbb{I}\boldsymbol{\omega} = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix} \boldsymbol{\omega} \tag{46}$$

Inertia Tensor entries (discrete):

$$\mathbb{I} = \sum_{\alpha} m_\alpha \begin{bmatrix} (y_\alpha^2 + z_\alpha^2) & -x_\alpha y_\alpha & -x_\alpha z_\alpha \\ -y_\alpha x_\alpha & (z_\alpha^2 + x_\alpha^2) & -y_\alpha z_\alpha \\ -z_\alpha x_\alpha & -z_\alpha y_\alpha & (x_\alpha^2 + y_\alpha^2) \end{bmatrix}$$

Inertia Tensor entries (continuous)

$$\mathbb{I} = \int dV \rho(x,y,z) \begin{bmatrix} y^2 + z^2 & -xy & -xz \\ -xy & x^2 + z^2 & -yz \\ -xz & -yz & x^2 + y^2 \end{bmatrix}$$

Inertia Tensor entries (index notation):

$$I_{ij} = \int dV \rho(x,y,z) \left(\mathbf{r}^2 \delta_{ij} - r_i r_j\right) \tag{47}$$

Parallel Axis Theorem:

$$J_{ij} = I_{ij} + M(a^2\delta_{ij} - a_i a_j) \tag{48}$$

Principle Axis:

$$\mathbf{L} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} = \begin{bmatrix} \lambda_1\omega_1 \\ \lambda_2\omega_2 \\ \lambda_3\omega_3 \end{bmatrix} \tag{49}$$

Torque:

$$\boldsymbol{\Gamma} = \dot{\mathbf{L}} \tag{50}$$

Euler's Equations:

$$\begin{aligned} \Gamma_1 &= \lambda_1\dot{\omega}_1 - (\lambda_2 - \lambda_3)\omega_2\omega_3 \\ \Gamma_2 &= \lambda_2\dot{\omega}_2 - (\lambda_3 - \lambda_1)\omega_1\omega_3 \\ \Gamma_3 &= \lambda_3\dot{\omega}_3 - (\lambda_1 - \lambda_2)\omega_1\omega_2 \end{aligned} \tag{51}$$

Rotation matrices:

$$R_x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix}, \quad R_y = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}, \quad R_z = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{52}$$

Euler Angles:

1. First, rotate around z-axis by $\phi$.
2. Next, rotate around the new y-axis by $\theta$.
3. Then, rotate around the new z-axis by $\psi$.

Angular velocity vector with Euler Angles:

$$\boldsymbol{\omega} = \dot{\phi}\hat{\mathbf{z}} + \dot{\theta}\hat{\mathbf{e}}_2' + \dot{\psi}\hat{\mathbf{e}}_3 \tag{53}$$

Angular velocity vector solely in terms of rotated basis vectors

$$\boldsymbol{\omega} = (\dot{\theta}\sin\psi - \dot{\phi}\sin\theta\cos\psi)\hat{\mathbf{e}}_1 + (\dot{\theta}\cos\psi + \dot{\phi}\sin\theta\sin\psi)\hat{\mathbf{e}}_2 + (\dot{\phi}\cos\theta + \dot{\psi})\hat{\mathbf{e}}_3 \tag{54}$$

Kinetic energy for symmetric rigid body with $\lambda_1 = \lambda_2$:

$$T = \frac{1}{2}\left(\lambda_1\omega_1^2 + \lambda_2\omega_2^2 + \lambda_3\omega_3^2\right) = \frac{\lambda_1}{2}(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta) + \frac{\lambda_3}{2}(\dot{\psi} + \dot{\phi}\cos\theta)^2 \tag{55}$$

Lagrangian of spinning top:

$$\mathcal{L} = \frac{1}{2}\left[\lambda_3\left(\dot{\psi} + \dot{\phi}\cos\theta\right)^2 + \lambda_1\left(\dot{\phi}^2\sin^2\theta + \dot{\theta}^2\right)\right] - mgR\cos\theta \tag{56}$$

Conserved $L_3$:

$$L_3 = p_\psi = \frac{\partial\mathcal{L}}{\partial\dot{\psi}} = \lambda_3(\dot{\psi} + \dot{\phi}\cos\theta) = \lambda_3\omega_3 \tag{57}$$

Conserved $L_z$:

$$L_z = p_\phi = \frac{\partial\mathcal{L}}{\partial\dot{\phi}} = \lambda_1\dot{\phi}\sin^2\theta + \lambda_3(\dot{\phi}\cos\theta + \dot{\psi})\cos\theta \tag{58}$$

Total energy:

$$E = \frac{\lambda_1}{2}\dot{\theta}^2 + U_{eff}(\theta) \tag{59}$$

Effective Potential:

$$U_{eff}(\theta) = \frac{(p_\phi - p_\psi\cos\theta)^2}{2\lambda_1\sin^2\theta} + \frac{p_\psi^2}{2\lambda_3} + mgR\cos\theta \tag{60}$$

Oscillatons about the minimum/nutation:

$$\dot{\phi} = \frac{p_\phi - p_\psi\cos\theta}{\lambda_1\sin^2\theta} \tag{61}$$

## 36.5 Hamiltonian Mechanics

Hamiltonian Definition (again):

$$\mathcal{H} = \sum_i p_i\dot{q}_i - \mathcal{L} = \sum_i \frac{\partial\mathcal{L}}{\partial\dot{q}_i}\dot{q}_i - \mathcal{L} \tag{62}$$

Hamilton's equations of motion:

$$\frac{\partial\mathcal{H}}{\partial p_i} = \dot{q}_i, \frac{\partial\mathcal{H}}{\partial q_i} = -\dot{p}_i \tag{63}$$

Canonical transformation to $\mathcal{H} \mapsto K$, $q_i \mapsto Q_i$, $p_i \mapsto P_i$ must satisfy:

$$\frac{\partial K}{\partial P_i} = \dot{Q}_i, \frac{\partial K}{\partial Q_i} = -\dot{P}_i \tag{64}$$

Poisson Bracket:

$$[F, H] = \sum_j \frac{\partial F}{\partial q_j}\frac{\partial H}{\partial p_j} - \sum_j \frac{\partial F}{\partial p_j}\frac{\partial H}{\partial q_j} \tag{65}$$

Poisson bracket properties:

(a) Anti-symmetry $[F, G] = -[G, H]$, from which we obtain $[F, F] = 0$.

(b) Bilinearity $[aF + bG, H] = a[F, H] + b[G, H]$ and $[H, aF + bG] = a[H, F] + b[H, G]$

(c) Leibniz' Rule $[FG, H] = [F, H]G + F[G, H]$

(d) Jacobi Identity $[F, [G, H]] + [G, [H, F]] + [H, [F, G]] = 0$

Canonical relations:

$$[q_i, q_j] = 0, \quad [p_i, p_j] = 0, \quad [q_i, p_j] = \delta_{ij} \tag{66}$$

Canonical transformation to $\mathcal{H} \mapsto K$, $q_i \mapsto Q_i$, $p_i \mapsto P_i$ must satisfy:

$$[Q_i, Q_j] = 0, \quad [P_i, P_j] = 0, \quad [Q_i, P_j] = \delta_{ij} \tag{67}$$

Jacobian determinant:

$$\left| \frac{\partial(x, y)}{\partial u, v} \right| = \det \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix} \tag{68}$$

A transformation is canonical if:

$$\left| \frac{\partial(Q, P)}{\partial(q, p)} \right| = 1 \tag{69}$$

Liouville's Theorem:

$$\frac{\mathrm{d}V}{\mathrm{d}t} = 0 \text{ where } V \text{ is the phase space volume} \tag{70}$$

Canonical quantization:

$$[F, G]_P \mapsto \frac{1}{i\hbar}[F, G] \tag{71}$$

Heisenberg equation of motion:

$$\frac{\mathrm{d}O}{\mathrm{d}t} = \frac{1}{i\hbar}[O, H] + \frac{\partial O}{\partial t} \tag{72}$$

## 36.6 Scattering Theory

Number of Scattered Particles

$$N_{sc} = N_{inc}P(\text{hit}) = N_{inc}n_{target}\sigma L \tag{73}$$

Scattering Area of Hard Spheres:

$$\pi(R_1 + R_2)^2 \tag{74}$$

Mean Free path of air molecules:

$$\lambda = \frac{1}{n\sigma} \tag{75}$$

Solid Angle:

$$\Delta\Omega = \frac{A}{r^2} \tag{76}$$

Infinitesimal Solid angle:

$$d\Omega = \sin\theta d\theta d\phi \tag{77}$$

Differential cross section $\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}$ and how it relates to scattered particle number:

$$N_{sc}(\text{into } \mathrm{d}\Omega) = N_{inc}n_{target}\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}(\theta, \phi)\right)d\Omega \tag{78}$$

Differential Cross section:

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{b}{\sin\theta}\left|\frac{\mathrm{d}b}{\mathrm{d}\theta}\right| \tag{79}$$

Expression for $\Delta\phi$ (for solving for diff. cross section):

$$\Delta\phi(r) = \pi - \theta = 2b\int_{r_{min}}^{\infty} \frac{dr}{r^2\sqrt{1 - \frac{2U(r)}{mv_\infty^2} - \frac{b^2}{r^2}}} \tag{80}$$

Hard Sphere Differential Cross Section:

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{R^2}{4} \tag{81}$$

Coloumb Potential Differential Cross Section:

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{k^2 q_1^2 q_2^2}{16E^2}\frac{1}{\sin^4\left(\frac{\theta}{2}\right)} \tag{82}$$

Relating lab differential cross section to COM differential cross section:

$$\left(\frac{\partial\sigma}{\partial\Omega}\right)_{lab} = \left(\frac{\partial\sigma}{\partial\Omega}\right)_{COM}\left|\frac{\mathrm{d}(\cos\theta_{COM})}{\mathrm{dcos}\,\theta_{lab}}\right| \tag{83}$$

Where:

$$\left|\frac{\mathrm{d}(\cos\theta_{COM})}{\mathrm{dcos}\,\theta_{lab}}\right| = \frac{(1 + 2\lambda\cos\theta_{COM} + \lambda^2)^{3/2}}{|1 + \lambda\cos\theta_{COM}|} \tag{84}$$

## 36.7 Continuum Mechanics

1-D Wave equation:

$$\frac{\partial^2 u(x,t)}{\partial t^2} = c^2\,\frac{\partial^2 u(x,t)}{\partial x^2} \tag{85}$$

3-D generalization:

$$\frac{\partial^2 p}{\partial t^2} = c^2\nabla^2 p \tag{86}$$

Volume Forces:

$$\mathbf{F} = k\mathbf{A}dV \tag{87}$$

Stress and Strain:

- Stress = Force / Area = Pressure (Fluid)
- Stress = Tension / Area (Wire)
- Stress = Shear Force / Area (Shear)
- Strain = dV/V (Fluid)
- Strain = dl/l (Wire)
- Strain = dy/dx (Shear)

Young's Modulus:

$$Y = \frac{lk}{A} \tag{88}$$

Bulk Modulus:

$$dp = -B\frac{dV}{V} \tag{89}$$

Shear Modulus:

$$\frac{F}{A} = G\frac{dy}{dx} \tag{90}$$

Stress Tensor:

$$\mathbf{F}(d\mathbf{A}) = \sigma d\mathbf{A} \tag{91}$$

Displacement Tensor:

$$\mathbb{D} = \begin{bmatrix} \frac{\partial u_1}{\partial r_1} & \frac{\partial u_1}{\partial r_2} & \frac{\partial u_1}{\partial r_3} \\ \frac{\partial u_2}{\partial r_1} & \frac{\partial u_2}{\partial r_2} & \frac{\partial u_2}{\partial r_3} \\ \frac{\partial u_3}{\partial r_1} & \frac{\partial u_3}{\partial r_2} & \frac{\partial u_3}{\partial r_3} \end{bmatrix} \tag{92}$$

Strain Tensor:

$$\epsilon = \frac{1}{2}\left(\mathbb{D} + \mathbb{D}^T\right) \tag{93}$$

Hooke's Law for Isotropic Solids:

$$\sigma = 2\mu\epsilon + \lambda \operatorname{Tr}(\epsilon)\mathbb{I}, \quad \mu = G, \lambda = B - \frac{2}{3}\mu \tag{94}$$

## 36.8 Chaos Theory

Driven Damped Oscillator:

$$\ddot{\phi} + 2\beta\dot{\phi} + \omega_0 \sin\phi = \gamma\omega_0^2 \cos(\omega t) \tag{95}$$

Universal Feigenbaum Number:

$$\delta = \lim_{n\to\infty} \frac{\gamma_{n-1} - \gamma_{n-2}}{\gamma_n - \gamma_{n-1}} = 4.6692016\ldots \tag{96}$$

Logistic Map:

$$x_{t+1} = rx_t(1 - x_t) \tag{97}$$