Kinematics in Polar Coordinates:

$$\mathbf{v} = \dot{r}\hat{r} + r\dot{\phi}\hat{\phi}$$
$$\mathbf{a} = (\ddot{r} - r\dot{\phi}^2)\hat{r} + (r\ddot{\phi} + 2\dot{r}\dot{\phi})\hat{\phi} \tag{1}$$

Lagrange Multipliers (for a system with m holonomic constraints):

$$\frac{\partial \mathcal{L}}{\partial q_i} + \sum_{k=1}^{m} \lambda_k \frac{\partial f_k}{\partial q_i} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} \tag{2}$$

General solution to simple harmonic oscillator:

$$x(t) = A\cos(\omega_0 t - \delta), \quad \omega_0 = \sqrt{\frac{k}{m}} \tag{3}$$

Damped Oscillator ODE:

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0, \quad \beta = \frac{b}{2m} \tag{4}$$

Overdamped Solution:

$$\begin{aligned} x(t) &= C_1 \exp(r_1 t) + C_2 \exp(r_2 t) \\ &= C_1 \exp\left(\left(-\beta + \sqrt{\beta^2 - \omega_0^2}\right)t\right) + \\ &\quad C_2 \exp\left(\left(-\beta - \sqrt{\beta^2 - \omega_0^2}\right)t\right) \end{aligned} \tag{5}$$

Critically Damped solution:

$$x(t) = C_1 \exp(-\beta t) + C_2 t \exp(-\beta t) \tag{6}$$

Underdamped Solution:

$$x(t) = A\exp(-\beta t)\cos\left(\left(\sqrt{\omega_0^2 - \beta^2}\right)t - \delta\right) \tag{7}$$

Damped Driven Oscillator ODE:

$$\ddot{z} + 2\beta\dot{z} + \omega_0^2 z = f_0 \exp(i\omega t) \tag{8}$$

Amplitude of solution $C\exp(i\omega t)$:

$$C = \frac{f_0}{-\omega^2 + 2i\beta\omega + \omega_0^2} = A\exp(i\delta) \tag{9}$$

Amplitude squared and phase:

$$A^2 = CC^* = \frac{f_0^2}{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}$$
$$\delta = \arctan\left(\frac{2\beta\omega}{\omega_0^2 - \omega^2}\right) \tag{10}$$

General solution to $\ddot{x} + 2\beta\dot{x} + \omega_0^2 x + f_0\cos(\omega t)$:

$$\begin{aligned} x(t) &= A\cos(\omega t - \delta) + C_1 \exp(r_1 t) + C_2 \exp(r_2 t) \\ &= x_{periodic}(t) + x_{trans}(t) \end{aligned} \tag{11}$$

Q factor:

$$Q = \frac{\omega_0}{2\beta} \tag{12}$$

General Coupled Oscillator Matrix equation:

$$\mathbb{M}\ddot{\mathbf{x}} = -\mathbb{K}\mathbf{x} \tag{13}$$

Trial Solution:

$$\mathbf{z} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \mathbf{a}\exp(i\omega t) \tag{14}$$

Eigenfrequency characteristic equation:

$$\det\left(\mathbb{K} - \omega^2\mathbb{M}\right) = 0 \tag{15}$$

Eigenmodes:

$$(\mathbb{K} - \omega\mathbb{M})\begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \mathbf{0} \tag{16}$$

Normal coordinates:

$$\mathbf{q}(t) = \sum_{i=1}^{n} \mathbf{a}_i \xi_i(t), \quad \text{such that } \ddot{\xi}_i + \omega_i^2 \xi_i = 0 \tag{17}$$

Newton's law for linearly accelerating frame:

$$m\ddot{\mathbf{r}} = \mathbf{F} - m\mathbf{A} \tag{18}$$

Velocity in rotating frames

$$\mathbf{v} = \dot{\mathbf{r}} = \boldsymbol{\omega} \times \mathbf{r} \tag{19}$$

Addition of angular velocities:

$$\boldsymbol{\omega}_{31} = \boldsymbol{\omega}_{32} + \boldsymbol{\omega}_{21} \tag{20}$$

Time derivatives in non-inertial frames:

$$\left(\frac{\mathrm{d}\mathbf{Q}}{\mathrm{d}t}\right)_{S_0} = \left(\frac{\mathrm{d}\mathbf{Q}}{\mathrm{d}t}\right)_{S} + \boldsymbol{\Omega} \times \mathbf{Q} \tag{21}$$

Newton's Law in Rotating Frame:

$$\begin{aligned} m\ddot{\mathbf{r}} &= \mathbf{F} + \mathbf{F}_{cor} + \mathbf{F}_{cent} + \mathbf{F}_{euler} \\ &= \mathbf{F} + 2m\dot{\mathbf{r}} \times \boldsymbol{\Omega} + m(\boldsymbol{\Omega} \times \mathbf{r}) \times \boldsymbol{\Omega} + m\mathbf{r} \times \dot{\boldsymbol{\Omega}} \end{aligned} \tag{22}$$

Position of COM:

$$\mathbf{R} = \frac{1}{M}\sum_\alpha m_\alpha \mathbf{r}_\alpha \tag{23}$$

Momentum of COM:

$$\mathbf{P} = M\dot{\mathbf{R}} = \sum_\alpha m_\alpha \dot{\mathbf{r}}_\alpha \tag{24}$$

External force:

$$\mathbf{F}_{ext} = M\ddot{\mathbf{R}} \tag{25}$$

Angular momentum of spinning rigid bodies:

$$\begin{aligned} \mathbf{L} &= \sum_\alpha (\mathbf{R} \times m_\alpha \dot{\mathbf{R}}) + \sum_\alpha (\mathbf{r}'_\alpha + m_\alpha \dot{\mathbf{r}}') \\ &= \mathbf{L}_{orbital} + \mathbf{L}_{spin} \end{aligned} \tag{26}$$

Potential energy:

$$U = U_{ext} + U_{int} = U_{ext} + \sum_{i<j} U_{ij}(r_{ij}) \tag{27}$$

Kinetic energy:

$$T = \frac{1}{2}\sum_\alpha m_\alpha \dot{\mathbf{r}}'^{\alpha}_{2} \tag{28}$$

Angular momentum for rotation about z axis:

$$\begin{aligned} \mathbf{L} &= \begin{bmatrix} L_x \\ L_y \\ L_z \end{bmatrix} \\ &= \sum_\alpha (\mathbf{r}_\alpha \times m_\alpha \mathbf{v}_\alpha) \\ &= \sum_\alpha (\mathbf{r}_\alpha \times m_\alpha (\omega\hat{\mathbf{z}} \times \mathbf{r}_\alpha)) \\ &= \sum_\alpha m_\alpha \omega \begin{bmatrix} -z_\alpha x_\alpha \\ -z_\alpha y_\alpha \\ x_\alpha^2 + y_\alpha^2 \end{bmatrix} \\ &= \begin{bmatrix} I_x\omega \\ I_y\omega \\ I_z\omega \end{bmatrix} \end{aligned} \tag{29}$$

Inertia Tensor:

$$\mathbf{L} = \mathbb{I}\boldsymbol{\omega} = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix}\boldsymbol{\omega} \tag{30}$$

Inertia Tensor entries (discrete):

$$\mathbb{I} = \sum_\alpha m_\alpha \begin{bmatrix} (y_\alpha^2 + z_\alpha^2) & -x_\alpha y_\alpha & -x_\alpha z_\alpha \\ -y_\alpha x_\alpha & (z_\alpha^2 + x_\alpha^2) & -y_\alpha z_\alpha \\ -z_\alpha x_\alpha & -z_\alpha y_\alpha & (x_\alpha^2 + y_\alpha^2) \end{bmatrix}$$

Inertia Tensor entries (continuous)

$$\mathbb{I} = \int dV\rho(x,y,z)\begin{bmatrix} y^2 + z^2 & -xy & -xz \\ -xy & x^2 + z^2 & -yz \\ -xz & -yz & x^2 + y^2 \end{bmatrix}$$

Inertia Tensor entries (index notation):

$$I_{ij} = \int dV\rho(x,y,z)\left(\mathbf{r}^2\delta_{ij} - r_i r_j\right) \tag{31}$$

Parallel Axis Theorem ($\mathbf{a}$ is vector from center):

$$J_{ij} = I_{ij} + M(a^2\delta_{ij} - a_i a_j) \tag{32}$$

Principle Axis:

$$\mathbf{L} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}\begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} = \begin{bmatrix} \lambda_1\omega_1 \\ \lambda_2\omega_2 \\ \lambda_3\omega_3 \end{bmatrix} \tag{33}$$

Torque:

$$\boldsymbol{\Gamma} = \dot{\mathbf{L}} \tag{34}$$

Euler's Equations:

$$\begin{aligned} \Gamma_1 &= \lambda_1\dot{\omega}_1 - (\lambda_2 - \lambda_3)\omega_2\omega_3 \\ \Gamma_2 &= \lambda_2\dot{\omega}_2 - (\lambda_3 - \lambda_1)\omega_1\omega_3 \\ \Gamma_3 &= \lambda_3\dot{\omega}_3 - (\lambda_1 - \lambda_2)\omega_1\omega_2 \end{aligned} \tag{35}$$

Taylor Expansion of $f(x)$ About a:

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \mathcal{O}(x^3) \tag{36}$$

Small-Angle Approximation About Equilibrium Point:

$$\begin{aligned} \cos(\theta_0 + \epsilon) &\approx \cos\theta_0 - \epsilon\sin\theta_0 \\ \sin(\theta_0 + \epsilon) &\approx \sin\theta_0 + \epsilon\cos\theta_0 \end{aligned} \tag{37}$$

Miscellaneous Trig. Identities:

$$\sin(a+b) = \sin(a)\cos(b) + \cos(a)\sin(b) \tag{38}$$
$$\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b) \tag{39}$$
$$\sin(a-b) = \sin(a)\cos(b) - \cos(a)\sin(b) \tag{40}$$
$$\cos(a-b) = \cos(a)\cos(b) + \sin(a)\sin(b) \tag{41}$$