PHYS 526 (Quantum Field Theory I) Notes

Rio Weil

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Introduction:

This is a set of lecture notes taken from UBC's PHYS 526 (Graduate Quantum Field Theory I) course, taught by Dr. Gordon Semenoff. The course covers many-particle systems, second quantization, degenerate Fermi and Bose gases, The action principle and Noether's theorem, Non-relativistic space-time symmetries, relativistic field theories, real scalar quantum field theory, Emergent relativistic symemtry, Dirac field theory, photons, functional methods, and perturbative quantum electrodynamics. If any errors are found in the notes, feel free to email me at ryoheiweil@phas.ubc.ca.

Contents

1 Motivation and Many Particles

1.1 Why QFT?

1. Natural way to study QM systems with large number of DOFs

2. To reconcile special relativity and quantum mechanics ("QM + SR = QFT"). No way to do "regular" QM in a relativistic setting. Mainly because $E = mc^2$, so energy can convert into mass; e.g. highly energetic collisions in the LHC which produce a large number of particles. You need a framework which can account for a large type and an arbitrary number of quantum-mechanical particles.

3. If you take point quantum mechanics and replace the NRQM Hamiltonian (with non-relativistic momentum) with the relativistic version of $H = \sqrt{p^2c^2 + m^2c^4} = mc^2 + \frac{p^2}{2m} + \cdots$, you find that the particle disperses (much like in the non-relativistic case) but it spreads in such a way that it violates causality; i.e. it can disperse outside of the light cone. There is no way to repair this in single-particle point quantum mechanics. QFT fixes this beautifully. It introduces an antiparticle, and says that the acausal process is actually a superposition of two processes, one with the particle and one with the antiparticle, and one "tune" the superposition so there is a destructive interference of the acausal behavior.

1.2 A one-particle QM system

Let's review a one-particle system. It is described by a wavefunction $\psi(\mathbf{x}, t)$, which satisfies the Schrodinger equation:

$$i\hbar \frac{\partial}{\partial t} \psi(\mathbf{x}, t) = \left(-\frac{\hbar^2 \mathbf{\nabla}^2}{2m} + V(\mathbf{x}) \right) \psi(\mathbf{x}, t). \tag{1.1}$$

In the above equation, $H = -\frac{\hbar^2 \mathbf{\nabla}^2}{2m} + V(\mathbf{x})$ is the Hamiltonian (the so called "energy"), where the first term is the kinetic energy and the second term is the position-dependent potential energy (e.g. due to gravitational interaction, electronic interactions). Note we assume that the potential is velocity-independent to simplify things. We take the momentum to be:

$$\mathbf{p} := -i\hbar \mathbf{\nabla} \tag{1.2}$$

so the kinetic energy is of course:

$$\frac{\mathbf{p}}{2m} = -\frac{\hbar^2 \mathbf{\nabla}^2}{2m}. \tag{1.3}$$

The nabla operator is defined as $\mathbf{\nabla} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right)$, and so $\mathbf{\nabla}^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$. Note that the Schrodinger equation is linear (and its validity can be confirmed in experiment, though we take it as an axiom in NRQM). To the wavefunction we can associate a probability amplitude:

$$|\psi(\mathbf{x}, t)|^2 d^3x = \text{ Probability of finding particle in volume } d^3x \text{ at position } \mathbf{x}, \text{ time } t. \tag{1.4}$$

And since we must find the particle somewhere, we have the normalization condition:

$$\int |\psi(\mathbf{x}, t)|^2 d^3x = 1. \tag{1.5}$$

We have not yet specified where the particle is allowed to be. If the particle is confined to some region (e.g. a box) then we require the enforcement of boundary conditions on the wavefunction (e.g. $\psi(x, 0) = \psi(x, L) = 0$ for a infinite square well). For our purposes, we will take $\mathbf{x} \in \mathbb{R}^3$ (no confinement), with the boundary condition specified by the normalization condition (but sometimes we even relax this, e.g. with plane waves, where we might specify the BC as the existence of the Fourier transform).

1.3 A many-particle QM system

We now move to a many-particle quantum mechanical system. How does the Schrodinger equation look in this case? For an identical N-particle system, a natural generalization is:

$$i\hbar\frac{\partial}{\partial t}\psi(\mathbf{x}_1,\mathbf{x}_2,\ldots,\mathbf{x}_N,t) = \left(\sum_{i=1}^{N}\frac{-\hbar^2\boldsymbol{\nabla}_i^2}{2m} + V(\mathbf{x}_1,\mathbf{x}_2,\ldots,\mathbf{x}_N)\right)\psi(\mathbf{x}_1,\mathbf{x}_2,\ldots,\mathbf{x}_N,t). \tag{1.6}$$

But why is this a natural generalization? Suppose Alice and Bob have a particle each, and are studying the particles in two far-apart labs. They both analyze their experiment using a one-particle Schrodinger equation (Alice should not have to take into account Bob's particle on the other side of the world, and vise versa! Physics should be local). Perhaps they are doing similar experiments, and start exchanging emails, and want to describe the system as a composite. The natural way to create a composite system would be to multiply the wavefunction of particle one by the wavefunction of particle two:

$$\psi(\mathbf{x}_1,\mathbf{x}_2,t) = \psi_1(\mathbf{x}_1,t)\psi_2(\mathbf{x}_2,t) \tag{1.7}$$

this follows naturally from the probabilistic interpretation of the wavefunctions (when we compose two probability distributions, we take their product, not their sum). We could then show that the product of the two wavefunctions satisfies the composite SE Eq. (1.6) if they satisfy their individual one-particle Schrodinger equations, i.e.:

$$i\hbar\frac{\partial}{\partial t}\psi_1(\mathbf{x}_1,t)\psi_2(\mathbf{x}_2,t) = \left(-\frac{-\hbar^2\boldsymbol{\nabla}^2}{2m} - \frac{\hbar^2\boldsymbol{\nabla}_2^2}{2m} + V(\mathbf{x}_1) + V(\mathbf{x}_2)\right)\psi_1(\mathbf{x}_1,t)\psi_2(\mathbf{x}_2,t). \tag{1.8}$$

However, if we introduce an interaction between the two particles (e.g. a Coloumb interaction), taking the composite wavefunction as the product no longer becomes valid; however Eq. (1.6) still holds.

We now return to the assumption that the particles are identical. Of course this means that $m_1 = m_2 = \ldots m_N$, but this has the more interesting implication that V is symmetric in its arguments, i.e.

$$V(\mathbf{x}_{P(1)},\mathbf{x}_{P(2)},\ldots,\mathbf{x}_{P(N)}) = V(\mathbf{x}_1,\mathbf{x}_2,\ldots,\mathbf{x}_N) \tag{1.9}$$

for any permutation $(P(1),\ldots P(N))$ of $(1,\ldots N)$. There are $N!$ permutation of N objects. This is what it means for the particles to be identical, as they feel interactions in a way such that the interaction is left unchanged by swapping any of the particles.

2 Many Particles Continued, Second Quantization

2.1 Bosons and Fermions

Recall the many-particle Schrodinger Equation:

$$i\hbar\frac{\partial}{\partial t}\psi(\mathbf{x}_1,\ldots,\mathbf{x}_N,t) = \left(\sum_{i=1}^{N}\frac{-\hbar^2\boldsymbol{\nabla}_i^2}{2m} + V(\mathbf{x}_1,\ldots,\mathbf{x}_N)\right)\psi(\mathbf{x}_1,\ldots,\mathbf{x}_N,t). \tag{2.1}$$

This is the fundamental mathematical problem we are to solve when doing QM. There are depressingly few examples which are exactly solvable (almost none), such as single-particle potentials (where potentials such as the harmonic oscillator, hydrogen atom, infinite square well are exactly solvable). If we include two-particle potentials and $N \geq 3$, things are very difficult. There are a few low-dimensional examples solved with sophisticated techniques, and a couple solutions for $N = 2$, but that's about it. We've seen a few of these in prior QM courses. It may be depressing that we are talking about equations we can't solve, but we don't have to write down solutions; there are at least existence theorems for solutions (requiring boundary conditions/initial values; the initial value determines it uniquely and deterministically at later times).

We have gone out of our way to make the particles identical here; all particles have the same mass m, and the potential is a symmetric function of its arguments (it is unchanged under permutation of indices on the coordinates). This gives this equation a very high degree of symmetry[1]. This tells us that if we manage to find a solution, we obtain another solution by permuting the labels:

$$\psi(\mathbf{x}_1,\ldots,\mathbf{x}_N,t) \to \psi(\mathbf{x}_{P(1)},\ldots,\mathbf{x}_{P(N)},t) \tag{2.2}$$

so one solution gives us $N!$ solutions, and then by the principle of superposition we actually obtain an infinite number. But to be different quantum states, they should be linearly independent as vectors, i.e.:

$$c_1\psi + c_2\psi_P = 0 \tag{2.3}$$

can only be solved by $c_1 = c_2 = 0$. If they are linearly independent, then $\psi_P = -\frac{c_1}{c_2}\psi$ (or $e^{i\varphi}\psi$ if the states are normalized). So which is it? At this point, mathematics doesn't help us, but mother nature does come to the rescue and chooses one of these; nature says that they always have to be linearly dependent; so there is only one state[2]. After you find one solution, you have *not* found $N!$ solutions, but just the one. Given this, we can consider a particular interchange where we swap of the two labels. If we do it twice, we should come back to the same state:

$$\psi_P = -\frac{c_1}{c_2}\psi = \left(-\frac{c_1}{c_2}\right)^2\psi_P \tag{2.4}$$

so this tells us that $(-c_1/c_2)^2 = 1$, i.e. $-c_1/c_2$ is either 1 or -1. If we do an interchange of labels, we will have two cases:

$$\psi(\mathbf{x}_{P(1)},\ldots,\mathbf{x}_{P(N)},t) = \psi(\mathbf{x}_1,\ldots,\mathbf{x}_N,t) \tag{2.5}$$

where no change happens for any permutation; such particles are known as **bosons**. Or, we can have:

$$\psi(\mathbf{x}_{P(1)},\ldots,\mathbf{x}_{P(N)},t) = (-1)^{\deg(P)}\psi(\mathbf{x}_1,\ldots,\mathbf{x}_N,t) \tag{2.6}$$

where $\deg(P)$ is the number of neighbours necessary to interchange to put the labels back in order (this is easily seen to be defined mod 2). Such particles are known as **fermions**[3].

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[1]Symmetry will be a central focus in future lectures.

[2]Perhaps the only time in life where nature picks the simplest path...

[3]These two are the only possibilities in three dimensions; in lower dimensions particles known as *anyons* are also possible (and useful for fault-tolerant topological quantum computing, as it turns out!), but these are outside the scope of the course. In relativistic physics, there are theorems that tell you particles are always bosons/fermions, but these theorems always have caveats; so in some sense this uniqueness of particle types comes down to what has been observed (and certainly this is the case for NRQM).

Note that this affects the counting of states quite severely. For indistinguishable particles, we have only one state (rather than a multiplicity of states) when we find a solution. Bosons are said to follow Bose-Einstein statistics, while Fermions follow Fermi-Dirac[4] statistics.

2.2 Particles with Spin

How do we account for the spin of particles? If we were just writing the SE for one particle with spin, we would write:

$$i\hbar \frac{\partial}{\partial t} \psi_\sigma(\mathbf{x}, t) = \left(-\frac{\hbar^2}{2m} \mathbf{\nabla}^2 + V(\mathbf{x}) \right) \psi_\sigma(\mathbf{x}, t) \tag{2.7}$$

where σ is a discrete index that runs over the possible spin polarizations of the particle:

$$\sigma = -J, -J+1, \ldots, J-1, J. \tag{2.8}$$

Of course the potential could have a dependence on the spin (e.g. spin-orbit coupling, spin-spin coupling):

$$\sum_{\tau=-J}^{\tau} V_\sigma^\tau(\mathbf{x}) \psi_\tau(\mathbf{x}, t). \tag{2.9}$$

Though we do not consider such interactions here, they are of immense importance in nuclear and AMO physics. If we have multiple particles with spin, our wavefunction can now be written as:

$$\psi_{\sigma_1, \ldots, \sigma_N}(\mathbf{x}_1, \ldots, \mathbf{x}_N, t). \tag{2.10}$$

If we want to symmetrize or anti-symmetrize, we permute both the position and the spin labels:

$$\psi_{\sigma_1, \ldots, \sigma_N}(\mathbf{x}_1, \ldots, \mathbf{x}_N, t) = \pm \psi_{\sigma_{P(1)}, \ldots, \sigma_{P(N)}}(\mathbf{x}_{P(1)}, \ldots, \mathbf{x}_{P(N)}, t). \tag{2.11}$$

2.3 The Potential

Another comment is on the multi-particle potential energy function. If the particles do not interact with one another, we have:

$$V(\mathbf{x}_1, \ldots, \mathbf{x}_N) = \sum_{i=1}^{N} V(\mathbf{x}_i) \tag{2.12}$$

But we could also have the sum of two-body potentials:

$$V(\mathbf{x}_1, \ldots, \mathbf{x}_N) = \sum_{i=1}^{N} V(\mathbf{x}_i) + \sum_{i<j} V(\mathbf{x}_i, \mathbf{x}_j) \tag{2.13}$$

But if we study nuclear or condensed matter physics, we also have higher-body interactions:

$$V(\mathbf{x}_1, \ldots, \mathbf{x}_N) = \sum_{i=1}^{N} V(\mathbf{x}_i) + \sum_{i<j} V(\mathbf{x}_i, \mathbf{x}_j) + \sum_{i<j<k} V(\mathbf{x}_i, \mathbf{x}_j, \mathbf{x}_k) + \ldots. \tag{2.14}$$

Now we might think; how might we separate/determine these in a unique way? The experimentalist answer is to put particles in one, or two, or three (or more) at a time to determine the N-particle forces one at a time. For the purposes of our course, we will generally limit ourselves to studying up to two-body interactions. Three-body interactions and higher rarely do things for us (exception: nucleons in the nucleus).

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[4]Named after their founders; though really it was Bose that wrote to Einstein first, and the F-D was really due to Pauli...

2.4 Second Quantization

We've written down a problem we will never solve; we have done this in order to rewrite the problem. This rewrite is required for a few reasons; first, we may have an open quantum system, so the number of particles is *not* fixed (we can get around it in the picture we have painted above, e.g. by using the average number of particles, but it isn't ideal). Another point to make is that for a finite number of particles and an infinite volume, we get zero density; we would prefer to describe things with finite (instead of zero) density. We could fix this in the current picture by putting space into a finite box, but again this is another band-aid we require. Perhaps the biggest reason for the rewrite is for mathematical elegance. We now leave physics behind for a few moments to construct a useful formalism.

We keep the spin, and write down an object $\psi_\sigma(\mathbf{x})$. Note that ψ here is *not* a wavefunction here. It is instead an operator. Operators operate on states, the most trivial of which is given by the empty vacuum $|0\rangle$. This is the state of our many-particle system with no particles in it. We will assume that $\psi_\sigma(\mathbf{x})$ has the property that:

$$\psi_\sigma(\mathbf{x})|0\rangle = 0. \tag{2.15}$$

And we will also assume that $|0\rangle$ is normalized:

$$\langle 0|0\rangle = 1. \tag{2.16}$$

Note that $|0\rangle$ is not the zero vector of our vector space (as the above normalization condition should make clear). We will also assume the existence (and uniqueness) of the dual space, which contains the bra $\langle 0|$ which satisfies:

$$\langle 0|\psi^{\dagger\sigma}(\mathbf{x}) = 0. \tag{2.17}$$

We then need only one more step; the commutation relations that these operators obey. We will assume that:

$$[\psi_\sigma(\mathbf{x}), \psi_\rho(\mathbf{y})] = 0, \quad \forall \mathbf{x}, \mathbf{y}, \sigma, \rho \tag{2.18}$$

Taking the Hermitian adjoint of the above, we obtain:

$$[\psi^{\dagger\sigma}(\mathbf{x}), \psi^{\dagger\rho}(\mathbf{y})] = 0, \quad \forall \mathbf{x}, \mathbf{y}, \sigma, \rho. \tag{2.19}$$

We need something that doesn't commute for these to be operators (rather than numbers); we have a relation reminiscent of the annhilation/creation operators of the quantum harmonic oscillator:

$$[\psi_\sigma(\mathbf{x}), \psi^{\dagger\rho}(\mathbf{y})] = \delta_\rho^\sigma \delta^3(\mathbf{x} - \mathbf{y}). \tag{2.20}$$

When we do relativistic physics, the contra and covariant coordinates (up/down labels) will become important; for now they are just labels without too much significance (just helps us to keep track). We can now consider a family of states:

$$|0\rangle, \psi^{\dagger\sigma}(\mathbf{x})|0\rangle, \psi^{\dagger\sigma}(\mathbf{x})\psi^{\dagger\rho}(\mathbf{y})|0\rangle, \ldots \tag{2.21}$$

we can think of these as the basis vectors of our vector space, given the name the *Fock space*. A general vector in Fock space is given by a linear combination of the above basis vectors. Among these vectors, we can find the matrix elements of ψ_σ, ψ^σ and so on. We can now write down the density operator:

$$\rho(\mathbf{x}) = \psi^{\dagger\sigma}(\mathbf{x})\psi_\sigma(\mathbf{x}). \tag{2.22}$$

When we pair an covariant/contravariant quantity (up/down label), there is an implied sum (Einstein summation convention); we have omitted a $\sum_{\sigma=-J}^{J}$ in the above expression. Now, if we operate the density operator on the vacuum state, we have:

$$\rho(\mathbf{x})|0\rangle = \psi^{\dagger\sigma}(\mathbf{x})\psi_\sigma(\mathbf{x})|0\rangle = 0. \tag{2.23}$$

We can use the commutation algebra to look at an arbitrary basis vector and the action of $\rho(\mathbf{x})$ on it:

$$\rho(\mathbf{x})\psi^{\dagger\sigma_1}(\mathbf{x}_1)\ldots\psi^{\dagger\sigma_N}(\mathbf{x}_N)|0\rangle = \sum_{i=1}^{N}\delta(\mathbf{x}-\mathbf{x}_i)\psi^{\dagger\sigma_1}(\mathbf{x}_1)\ldots\psi^{\dagger\sigma_N}(\mathbf{x}_N)|0\rangle \tag{2.24}$$

In other words, the basis states are eigenstates of $\rho(\mathbf{x})$ with eigenvalues $\sum_{i=1}^{N}\delta(\mathbf{x}-\mathbf{x}_i)$.

Note that the basis states are not really physical quantum mechanical states; the particles are at fixed positions and at fixed spins. Another point to make is that since the ψ^σs commute with each other, it is a completely symmetric state, i.e. a state of bosons. It's not possible to say which bosons are sitting at which position, and has bose-einstein statistics built in. Nature has been very kind to us; if nature did not have such statistics, we would not be able to use such construction.

Next time, we will discuss what happens when the ψ_σs are fermions rather than bosons. We will also look for how we formulate the Schrodinger equation in this second quantization language.

A remark on the Fock space; it is a continuously infinite basis. Not clear that this is a separable Hilbert space, which is where we do QM in. One of the tenets is that the basis for such a space is countable (of which what we have is not). We could improve this by replacing the construction of $\psi^{\dagger\sigma}(\mathbf{x})$s with $\int d^x f_i(\mathbf{x})\psi^{\dagger\sigma}(\mathbf{x})$ where f_i are square integrable functions. But this is not actually discrete. You can use a Cantor diagonalization argument to show that there will be an uncountable number of states. To get a Hilbert space, you need to restrict yourself to states with a finite total number of particles. Then consider Cauchy sequences of such basis states, and consider all of the basis states plus limits of such Cauchy sequences, giving a Hilbert space. But this is a subtetly that we will not really consider for the remainder of the course.

3 Second Quantization Continued

3.1 Second Quantization in the Schrodinger Picture

Last time, we were in the middle of a mathematical construction. We established a Hilbert space[5], with creation/annihilation operators with commutation relations:

$$[\psi_\sigma(\mathbf{x}), \psi^{\dagger\rho}(\mathbf{y})] = \delta^\rho_\sigma \delta^3(\mathbf{x} - \mathbf{y}). \tag{3.1}$$

and

$$[\psi_\sigma(\mathbf{x}), \psi_\rho(\mathbf{y})] = [\psi^{\dagger\sigma}(\mathbf{x}), \psi^{\dagger\rho}(\mathbf{y})] = 0. \tag{3.2}$$

and we could obtain basis states by operating (mutually commuting)[6] creation operators on the vacuum state:

$$\psi^{\dagger\sigma_1}(\mathbf{x}_1)\dots\psi^{\dagger\sigma_N}(\mathbf{x}_N)|0\rangle. \tag{3.3}$$

where the vacuum state is normalized:

$$\langle 0|0\rangle = 1 \tag{3.4}$$

and is sent to zero by the anhilation operators:

$$\psi_\sigma(\mathbf{x})|0\rangle = 0, \langle 0|\psi^{\dagger\sigma}(\mathbf{x}) = 0. \tag{3.5}$$

Now, consider a superposition:

$$\int d^3x_1\dots d^3x_N \psi_{\sigma_1\dots\sigma_N}(\mathbf{x}_1,\dots,\mathbf{x}_N,t)\psi^{\dagger\sigma_1}(\mathbf{x}_1)\dots\psi^{\dagger\sigma_N}(\mathbf{x}_N)|0\rangle. \tag{3.6}$$

where one can think of $\psi_{\sigma_1\dots\sigma_N}(\mathbf{x}_1,\dots,\mathbf{x}_N,t)$ as the coefficients of the sum (though of course the labels vary continuously, so we have integrals instead) - we will see shortly that this is the wavefunction of the system. Of course we also implicitly sum over the spin labels according to the Einstein summation convention. We will worry about normalization later on. Note that we still have not done anything physical here, so let us now do that; we want to set up our many-particle Schrodinger Equation in this second quantization language. Let us give the integral in Eq. (3.6) a name; let's call it $|\Psi(t)\rangle$. We want to find some sort of equation which tells us how this object evolves in time. This should be given by:

$$i\hbar\frac{\partial}{\partial t}|\Psi(t)\rangle = H|\Psi(t)\rangle. \tag{3.7}$$

such that the above equation is equivalent to the old Schrodinger Equation (1.6); but note that it is written much more economically. In order to do so, we need to specify the Hamiltonian. In order to do so, we use the fact that the potential term in the SE can be decomposed into one, two, three, and in general N body interaction terms. We obtain:

$$
\begin{aligned}
H = &\int d^3x \frac{\hbar^2}{2m}\boldsymbol{\nabla}\psi^{\dagger\sigma}(\mathbf{x})\cdot\boldsymbol{\nabla}\psi_\sigma(\mathbf{x}) + \int d^3x V(\mathbf{x})\psi^{\dagger\sigma}(\mathbf{x})\psi_\sigma(\mathbf{x}) \\
&+ \frac{1}{2!}\int d^3x d^3y V(\mathbf{x},\mathbf{y})\psi^{\dagger\sigma}(\mathbf{x})\psi^{\dagger\rho}(\mathbf{y})\psi_\rho(\mathbf{y})\psi_\sigma(\mathbf{x}) \\
&+ \frac{1}{3!}\int d^3x d^3y d^3z V(\mathbf{x},\mathbf{y},\mathbf{z})\psi^{\dagger\sigma}(\mathbf{x})\psi^{\dagger\rho}(\mathbf{y})\psi^{\dagger\tau}(\mathbf{z})\psi_\tau(\mathbf{z})\psi_\rho(\mathbf{y})\psi_\sigma(\mathbf{x}) \\
&+ \dots
\end{aligned}
\tag{3.8}
$$

Clearly if we did not have a decomposition of the potential into N-body terms, this decomposition would not work out. Note that in the above Hamiltonian, we have assumed that the potential interaction doesn't

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[5]Small mathematical detail we will not worry about; quantum states belong to a projective space, rather than a true vector space.
[6]If the particles were somehow distinguishable, this entire construction would fail; this commutativity would not make sense.

care about the spin (but if it did, we would make the Vs into a matrix, which depends on the spin; see the textbook for a more general formula). The terms in the above sum get successively complex (three body interactions are already a nightmare[7]), but for most physical scenarios we only have up to two-body interactions.

We look back at Eq. (3.7); on the LHS the time derivative affects only the $\psi_{\sigma_1 \ldots \sigma_N}(\mathbf{x}_1, \ldots, \mathbf{x}_N, t)$. On the RHS we get a mixture of terms from H acting on $|\Psi(t)\rangle$. We could then collect terms to see how the various terms act on $\psi_{\sigma_1 \ldots \sigma_N}(\mathbf{x}_1, \ldots, \mathbf{x}_N, t)$; doing so, we would completely reproduce the old N-body Schrodinger equation in (1.6), with the only caveat that we have not specified the particle number. To this end, we consider the number operator:

$$\mathcal{N} = \int d^3 x \psi^{\dagger \sigma}(\mathbf{x}) \psi_\sigma(\mathbf{x}) \tag{3.9}$$

Which when acted on an arbitrary state $|\Psi(t)\rangle$ counts the particle number. So we could supplement Eq. (3.7) with Eq. (3.9), and pairing this with knowledge of what $\psi^{\dagger \sigma}(\mathbf{x})$, $\psi_\sigma(\mathbf{x})$ are from the commutation relations, we have a completely equivalent formulation of quantum mechanics. Note that we have really done nothing here, just rewrote the same thing in a different language.

We have established an example of a non-relativistic quantum field theory. There is a further step we can take to write this down, however. Note that all of our work above was in the Schrodinger picture, where the states are time-dependent and the operators are time-independent. We will find it useful to recast this in the Heisenberg picture[8], where the states are time-independent and the operators are time-dependent. Let us begin this reconstruction now.

3.2   Review of the Heisenberg Picture

We can write the time-dependent quantum state that solves Eq. (3.7) as:

$$|\Psi(t)\rangle = e^{-\frac{i}{\hbar} H t} |\psi(0)\rangle. \tag{3.10}$$

Where $|0\rangle$ is the initial state of the system at time zero. Now, the expectation value of an operator $\mathcal{O}$ in the Schrodinger picture can be written as:

$$\langle \mathcal{O} \rangle (t) = \langle \Psi(t) | \mathcal{O} | \Psi(t) \rangle. \tag{3.11}$$

Now, substituting in Eq. (3.10) to the above, we have:

$$\langle \mathcal{O} \rangle (t) = \langle \Psi(0) | e^{\frac{i}{\hbar} H t} \mathcal{O} e^{-\frac{i}{\hbar} H t} | \psi(0) \rangle \tag{3.12}$$

So now we can define the time dependent operator:

$$\mathcal{O}(t) = e^{\frac{i}{\hbar} H t} \mathcal{O} e^{-\frac{i}{\hbar} H t} \tag{3.13}$$

and hence the expectation value can be written in the Heisenberg picture as:

$$\langle \mathcal{O} \rangle (t) = \langle \Psi(0) | \mathcal{O}(t) | \psi(0) \rangle. \tag{3.14}$$

And the time-dependence of the operators are described by the Heisenberg equation of motion:

$$\frac{\partial}{\partial t} \mathcal{O}(t) = \frac{i}{\hbar} [H, \mathcal{O}(t)]. \tag{3.15}$$

For most QM problems, this is a much harder picture to solve problems in; however this is the way we will proceed, as things will look much more like a QFT in this formalism.

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[7]They are of interest in nuclear physics, but we won't generally concern ourselves with them here
[8]Though note that this rewrite would not be doing anything physically

3.3 Second Quantization in the Heisenberg Picture

In the Heisenberg picture, we can study the time evolution of the annhilation/creation operators:

$$i\hbar \frac{\partial}{\partial t}\psi_\sigma(\mathbf{x},t) = [\psi_\sigma(\mathbf{x},t),H]. \tag{3.16}$$

Note that H here is time independent, but ψ_σ has acquired a time-dependence in the Heisenberg picture. Now taking the commutation relations for ψ_σ from our previous construction, we find:

$$[\psi_{\sigma_1}(\mathbf{x}_1,t),\psi_{\sigma_2}(\mathbf{x}_2,t)] = 0 \tag{3.17}$$

in the case where the time ts are the same. It is the same story with the $\psi^{\dagger\sigma}$s:

$$[\psi^{\dagger\sigma_1}(\mathbf{x}_1,t),\psi^{\dagger\sigma_2}(\mathbf{x}_2,t)] = 0 \tag{3.18}$$

and the final commutation relation:

$$[\psi_{\sigma_1}(\mathbf{x}_1,t),\psi^{\dagger\sigma_2}(\mathbf{x}_2,t)] = \delta^{\sigma_2}_{\sigma_1}\delta^3(\mathbf{x}_1 - \mathbf{x}_2). \tag{3.19}$$

In order to remind us that the times should be the same when we look at the above operators, we call these the *equal-time commutation relations*. They give us enough algebra to figure out what (3.16). So let us do just that.

$$i\hbar \frac{\partial}{\partial t}\psi_\sigma(\mathbf{x},t) = -\frac{\hbar^2\mathbf{\nabla}^2}{2m}\psi_\sigma(\mathbf{x},t) + V(\mathbf{x})\psi_\sigma(\mathbf{x},t)$$
$$+ \int d^3y V(\mathbf{x},\mathbf{y})\psi^{\dagger\rho}(\mathbf{y})\psi_\rho(\mathbf{y})\psi_\sigma(\mathbf{x},t) + \dots \tag{3.20}$$

This is a lot simpler, and it *looks* like $\psi_\sigma(\mathbf{x},t)$ satisfies a Schrodinger equation up to the second term (even though it is an operator and not a wavefunction). However then it becomes nonlinear as we add more terms (recall that the SE is linear). It does appear to satisfy the propogation of waves, and we will call it a *field equation*. So, in the Heisenberg picture, we obtain an equivalent formulation of quantum mechanics by imposing the equal-time commutation relations and the field equation, as well as the number operator constraint:

$$\mathcal{N}|\Psi\rangle = N|\Psi\rangle. \tag{3.21}$$

So this is an example of a full-blown quantum field theory, which we could turn around and use field-theoretic methods to attack. It is very easy to generalize this to the case where we have an infinite number of particles. It is also easy to generalize this to a system where the number of particles is not fixed (where we get rid of the constraint imposed by Eq. (3.21)).

3.4 From Bosons to Fermions

Everything we said here concerns bosons (from the construction of the operators to the Fock space states). For fermions, we want totally antisymmetric states. In order to do this, we simply replace the commutation relations of the annhilation/creation operators with anticommutation relations. By this one simple change, we obtain the fermionic field theory rather than the bosonic field theory.

Also note; if we have multiple types of particles, we have multiple field equations, with cross terms in the equations of motion; but we will not be too concerned about this scenario here.

A fun note: the number of fermions is either even or odd. There is no dynamical process which can change the parity (fermion superselection rule; important for (e.g.) condensed matter in the study of topological insulators). The flip of the sign of the state should not change anything about the universe (irrelevancy of the global phase).

Next time, we will try to solve a simple many-particle system in the Heisenberg picture. We will see that it is not as abstract as it currently looks!

4 Weakly Interacting Particles

4.1 Review of QFT construction

To briefly recap, we have taken the N-particle SE:

$$i\hbar\frac{\partial}{\partial t}\psi_{\sigma_1\ldots\sigma_N}(\mathbf{x}_1,\ldots,\mathbf{x}_N,t) = \left(\sum_{i=1}^{N}-\frac{\hbar^2\mathbf{\nabla}_i^2}{2m}+\sum_{i<j}\lambda\delta(\mathbf{x}_i-\mathbf{x}_j)\right)\psi_{\sigma_1\ldots\sigma_N}(\mathbf{x}_1,\ldots,\mathbf{x}_N,t) \tag{4.1}$$

(here with a short-range repulsive interaction) with the normalization condition:

$$\int d^3x_1\ldots d^3x_N\psi^{\dagger\sigma_1\ldots\sigma_N}(\mathbf{x}_1,\ldots\mathbf{x}_N,t)\psi_{\sigma_1\ldots\sigma_N}(\mathbf{x}_1,\ldots\mathbf{x}_N,t) = 1. \tag{4.2}$$

and we have shown that it is completely equivalent to the field equation:

$$\left(i\hbar\frac{\partial}{\partial t}+\frac{\hbar^2\mathbf{\nabla}^2}{2m}\right)\psi_\sigma(\mathbf{x},t) = \lambda\psi^{\dagger\rho}(\mathbf{x},t)\psi_\rho(\mathbf{x},t)\psi_\sigma(\mathbf{x},t) \tag{4.3}$$

where the ψ_σ are field operators with equal-time commutation relations:

$$[\psi_\sigma(\mathbf{x},t),\psi^{\dagger\rho}(\mathbf{y},t)] = \delta_\sigma^\rho\delta^3(\mathbf{x}-\mathbf{y}). \tag{4.4}$$

(the other commutation relations are zero). Actually, the two are almost equivalent; we also have to put in the number operator:

$$\mathcal{N} = \int d^3\psi^{\dagger\sigma}(\mathbf{x},t)\psi_\sigma(\mathbf{x},t) \tag{4.5}$$

which specifies the number of particles in the system; note that $\mathcal{N}$ commutes with H and so is time-independent (and we can take the t in the above expression to be whatever we like; useful when we want to apply the equal-time commutation relations). We will in the near future find a more clever way to show that this is time-independent.

The field equation and the operators already give us a QFT; the number operator is auxilary and only necessary for a complete equivalence to the old formulation. We can also bring the information about the Hamiltonian along with us (useful as the eigenvalues of the Hamiltonian provide valuable information):

$$H = \int d^3x\frac{\hbar^2}{2m}\mathbf{\nabla}\psi^{\dagger\sigma}(\mathbf{x},t)\cdot\mathbf{\nabla}\psi_\sigma(\mathbf{x},t) + \frac{\lambda}{2}\psi^{\dagger\sigma}(\mathbf{x},t)\psi^{\dagger\rho}(\mathbf{x},t)\psi_\rho(\mathbf{x},t)\psi_\sigma(\mathbf{x},t). \tag{4.6}$$

But again we will find a more sophisticated way to derive these later (from the field equations and commutation relations) using the fact that quantities are conserved.

Even for this very simple interaction potential, it is impossible to solve this analytically (for 1-D some work may be possible). So what do we do? Perhaps we can solve an approximation to it; we look at the limit in which the interaction is small/weak. We should have to define what "small" means, but let us avoid that for the moment.

In statistical mechanics, we never say the particles are never completley non-interacting, as the particles need to transfer energy in order to reach thermal equilibrium. We assume the interaction is there, but quite weak. But similar to that case, at a first pass we can just throw away the interaction and solve the non-interacting scenario.

4.2 Weakly (Non) Interacting Particles (Bosons)

In the limit of no interaction ($\lambda \to 0$) we have:

$$\left(i\hbar \frac{\partial}{\partial t} + \frac{\hbar^2 \boldsymbol{\nabla}^2}{2m} \right) \psi_\sigma(\mathbf{x}, t) = 0. \tag{4.7}$$

Now we have a linear rather than a nonlinear PDE to solve. Not only this, but the one above is very easy to solve; there is no x or t dependence in the above, so we can solve it simply by a Fourier transform. Plugging in a plane wave, we have:

$$\left(i\hbar \frac{\partial}{\partial t} + \frac{\hbar^2 \boldsymbol{\nabla}^2}{2m} \right) e^{i\mathbf{k}\cdot\mathbf{x} - \frac{i}{\hbar}\frac{\hbar k^2}{2m}t} = 0 \tag{4.8}$$

where the above equation is easily verified by the observations that $\frac{\partial}{\partial t} e^{i\omega t} = i\omega e^{i\omega t}$ and $\boldsymbol{\nabla} e^{i\mathbf{k}\cdot\mathbf{x}} = i\mathbf{k} e^{i\mathbf{k}\cdot\mathbf{x}}$. However note we have really found a continuum of solutions, as the above works for any value of $\mathbf{k}$. At this point we need to convince ourselves that we have found the complete set of solutions, but of course the set of plane waves are complete so we have just that. A most general solution is written as a linear combination:

$$\psi_\sigma(\mathbf{x}, t) = \int \frac{d^3k}{(2\pi)^{3/2}} e^{i\mathbf{k}\cdot\mathbf{x} - i\frac{\hbar k^2}{2m}t} a_\sigma(\mathbf{k}) \tag{4.9}$$

where the $a_\sigma(\mathbf{k})$ can be thought of the coefficients of the expansion. $\psi^{\dagger\sigma}(\mathbf{x}, t)$ can then easily be found to be:

$$\psi^{\dagger\sigma}(\mathbf{x}, t) = \int \frac{d^3k}{(2\pi)^{3/2}} e^{-i\mathbf{k}\cdot\mathbf{x} + i\frac{\hbar k^2}{2m}t} a^{\dagger\sigma}(\mathbf{k}). \tag{4.10}$$

Using the commutation relations for the field operators, we can then find the commutation relations for the $a_\sigma/a^{\dagger\sigma}$ to be:

$$[a_\sigma(\mathbf{k}), a_\rho(\mathbf{l})] = [a^{\dagger\sigma}(\mathbf{k}, a^{\dagger\rho}(\mathbf{l})] = 0 \tag{4.11}$$

$$[a_\sigma(\mathbf{k}), a^{\dagger\rho}(\mathbf{l})] = \delta_\sigma^\rho \delta^3(\mathbf{k} - \mathbf{l}). \tag{4.12}$$

which is extremely similar in form to the commutation algebra of the ψs. We can then construct the vector space that the as act on. Letting $|0\rangle$ be the empty vacuum state, we have:

$$a_\sigma(\mathbf{k})|0\rangle = 0, \forall \mathbf{k}, \sigma \tag{4.13}$$

and the vector space has a basis composed of vectors of the form:

$$a^{\dagger\sigma_1}(\mathbf{k}_1) \ldots a^{\dagger\sigma_N}(\mathbf{k}_N)|0\rangle. \tag{4.14}$$

The dual statement of the above is:

$$\langle 0|a^{\dagger\sigma}(\mathbf{k}) = 0, \forall \mathbf{k}, \sigma \tag{4.15}$$

We can now calculate matrix elements using the commutation algebra:

$$\langle 0|a_{\sigma_1}(\mathbf{k}_1) \ldots a_{\sigma_m}(\mathbf{k}_m) a^{\dagger\rho_1}(\mathbf{l}_1) \ldots a^{\dagger\rho_m}(\mathbf{l}_n)|0\rangle = \delta_{mn} \sum_P \delta(\mathbf{k} - \mathbf{l}_{P(1)})\delta_{\sigma_1}^{\rho_{\sigma(1)}} \ldots \delta(\mathbf{k}_n - \mathbf{l}_{P(m)})\delta_{\sigma_n}^{\rho_{P(m)}} \tag{4.16}$$

which is messy, but really comes from the fact that the matrix element is symmetric in its arguments of $\mathbf{k}_i, \mathbf{l}_j$. Sometimes due to this symmetry we enforce a $\frac{1}{N!}$ to normalize for all permutations (which can be useful for some applications). This concludes the boson story, but what about fermions?

4.3 Weakly (Non) Interacting Particles (Fermions)

For bosons, we initially enforced symmetry of the wavefunction in the arguments. For fermions, we enforce antisymmetry instead. All commutators become anticommutators; and the structure of the above argument holds basically exactly the same with the commutation relations for ψ replaced with anti-commutation relations. When we get to the as, we also replace the commutators with anticommutators.

When we compute the matrix elements, we get negative signs from the anticommutation relations, so:

$$\langle 0|a_{\sigma_1}(\mathbf{k}_1)\ldots a_{\sigma_m}(\mathbf{k}_m)a^{\dagger\rho_1}(\mathbf{l}_1)\ldots a^{\dagger\rho_m}(\mathbf{l}_n)|0\rangle = \delta_{mn}\sum_P (-1)^{\deg(P)}\delta(\mathbf{k}-\mathbf{l}_{P(1)})\delta^{\rho_{\sigma(1)}}_{\sigma_1}\ldots\delta(\mathbf{k}_n-\mathbf{l}_{P(m)})\delta^{\rho_{P(m)}}_{\sigma_n}$$

(4.17)

There's not much of a difference so far; but we will find the many-particle states are profoundly different for bosons and fermions.

4.4 Understanding our solution to the theory

We now ask: in what sense have we "solved" this theory? To start, we can take our plane wave solution and plug it into the number and Hamiltonian operators. Doing so, we obtain:

$$\mathcal{N} = \int d^3k\, a^{\dagger\sigma}(\mathbf{k})a_\sigma(\mathbf{k}) \tag{4.18}$$

$$H = \int d^3k\, \frac{\hbar^2\mathbf{k}^2}{2m}a^{\dagger\sigma}(\mathbf{k})a_\sigma(\mathbf{k}). \tag{4.19}$$

We find that this closely resembles the harmonic oscillator, and also that $\mathcal{N}$ and H are explicitly time-independent. Also, if we take our states (Eq. (4.14)), we see that they are indeed eigenstates of these operators (can be determined through the commutation algebra, or by interpreting these as harmonic oscillator eigenstates[9]):

$$\mathcal{N}a^{\dagger\sigma_1}(\mathbf{k}_1)\ldots a^{\dagger\sigma_N}(\mathbf{k}_N)|0\rangle = Na^{\dagger\sigma_1}(\mathbf{k}_1)\ldots a^{\dagger\sigma_N}(\mathbf{k}_N)|0\rangle \tag{4.20}$$

$$Ha^{\dagger\sigma_1}(\mathbf{k}_1)\ldots a^{\dagger\sigma_N}(\mathbf{k}_N)|0\rangle = \left(\sum_{i=1}^N \frac{\hbar^2\mathbf{k}_i^2}{2m}\right)\left(a^{\dagger\sigma_1}(\mathbf{k}_1)\ldots a^{\dagger\sigma_N}(\mathbf{k}_N)|0\rangle\right). \tag{4.21}$$

Note that this discussion applies equally as well to bosons and fermions.

4.5 Degenerate Fermi Gas - Vacuum State

We have found a complete solution in the non-interacting limit, from which we can draw out some information. If we think about a high-energy state where bosons/fermions can be treated the same, can we derive familiar results (e.g. high T limit for ideal gas/ideal gas law)? As a teaser for next time: we will start to study fermionic systems, which are slightly easier to understand. Consider a state with an infinite number of fermionic particles (as we want a finite density, if we have infinite space then we need infinite particles; in CM you may instead consider a finite number of particles in a box with boundary conditions). We will like to look at the low-energy states of such a system. If I had only one particle, its lowest energy state would be $\mathbf{k} = \mathbf{0}$, the state with a constant wavefunction. If I had two fermions, the first can have energy zero, but the second one cannot; this is because if we had two fermions in the same state, then $\{a^{\dagger\sigma}(\mathbf{k})a^{\dagger\sigma}(\mathbf{k})\} = (2a^{\dagger\sigma}(\mathbf{k}))^2 = 0$ (although we could have two different spin states); so $a^{\dagger\sigma}(\mathbf{k}))^2|0\rangle = 0$ is the zero vector and hence not a real quantum state. This is of course the famous *Pauli exclusion principle*.

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[9]The quote that "space is just a bunch of harmonic oscillators" is salient here.

The next $\mathbf{k}$ above zero is slightly ill-defined in the infinite-space limit (as we have a continuum)[10] but let us imagine it. The N-particle ground state would be:

$$|\mathcal{O}\rangle = \prod_{\mathbf{k}<k_F,\sigma} \left(a^{\dagger\sigma}(\mathbf{k})\right)|0\rangle. \tag{4.22}$$

Note that the above is really mathematical nonsense due to the continuity of $\mathbf{k}$. We call $|\mathcal{O}\rangle$ the vacuum (different from the empty vacuum $|0\rangle$). Note that the vacuum state we will stick with for the rest of the course, while the empty vacuum we will abandon when we get to relativistic field theory; it is not accessible in that limit. Let us define the vacuum state algebraically instead of the nonsense definition we have above (though we can heuristically understand the below algebraic constraints based on the nonsense equation we have above):

$$a^{\dagger\sigma}(\mathbf{k})|\mathcal{O}\rangle = 0 \text{ if } |\mathbf{k}| \leq k_F \tag{4.23}$$

$$a_\sigma(\mathbf{k})|\mathcal{O}\rangle = 0 \text{ if } |\mathbf{k}| > k_F. \tag{4.24}$$

$$\langle\mathcal{O}|\mathcal{O}\rangle = 0. \tag{4.25}$$

Note that the $\leq$ does not really matter in the above equation; the Fermi surface where $|\mathbf{k}| = k_F$ is a set of zero measure. k_F here is known as the Fermi wavenumber and from this we can construct $\hbar k_F$ the Fermi momentum, and $\epsilon_F = \frac{\hbar^2 k_F^2}{2m}$ the Fermi energy.

This construction is how we will deal with fermions next day!

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[10]Here it might be nice to be a CM physicist instead...

5 Degenerate Fermi Gas

5.1 Review of the Weakly Interacting Fermion Construction

Last time, we started looking at the Fermi gas; it is what you get when you throw away the interactions between fermions (if you retain the interactions, you have something known as a Fermi liquid). Recall that we quantized the field:

$$\psi_\sigma(\mathbf{x}, t) = \int \frac{d^3k}{(2\pi)^{3/2}} e^{i\mathbf{k}\cdot\mathbf{x} - i\frac{\hbar \mathbf{k}^2}{2m}t} \alpha_\sigma(\mathbf{k}). \tag{5.1}$$

We recall that we showed how no two fermions could occupy the same state. So building up the ground state of an N-particle fermion state, we start with the lowest energy state and place one (actually two, accounting for spin) particle per energy level. Since $\epsilon \propto \mathbf{k}^2$, all states up to a spherical surface known as the *Fermi sphere* become filled.

   We tried to construct this state (the vacuum), but it resulted in some mathematical ugliness (with an infinite continuous product of creation operators up to some $\mathbf{k}$... mathematical nonsense). But we got around this by talking about the state directly and postulating its properties. We say that:

$$\begin{aligned} \alpha_\sigma^\dagger(\mathbf{k})|\mathcal{O}\rangle &= 0 \quad |\mathbf{k}| \leq k_F \\ \alpha_\sigma(\mathbf{k})|\mathcal{O}\rangle &= 0 \quad |\mathbf{k}| > k_F. \end{aligned} \tag{5.2}$$

though whether the states on the Fermi sphere are filled or empty will not be of particular importance to us.

5.2 Particles and Holes

Let us now rename some things:

$$\begin{aligned} \alpha_\sigma(\mathbf{k}) &= a_\sigma(\mathbf{k}) \quad \text{if } |\mathbf{k}| > k_F \\ \alpha_\sigma(\mathbf{k}) &= b_\sigma^\dagger(-\mathbf{k}) \quad \text{if } |\mathbf{k}| < k_F \end{aligned} \tag{5.3}$$

so it looks like we have two kinds of creation/annhilation operators. They are in a sense the same, but we label them like this to make the defining formulas for $|\mathcal{O}\rangle$ simpler, as now we can write them as:

$$\begin{aligned} a_\sigma(\mathbf{k})|\mathcal{O}\rangle &= 0 \quad |\mathbf{k}| > k_F \\ b_\sigma^\dagger(\mathbf{k})|\mathcal{O}\rangle &= 0 \quad |\mathbf{k}| < k_F. \end{aligned} \tag{5.4}$$

And we can write an excited state as:

$$(a^{\dagger\sigma_1}(\mathbf{k}_1)\dots b_{\rho_1}^\dagger(\mathbf{l}_1))|\mathcal{O}\rangle \tag{5.5}$$

where the as create fermions, and the bs create holes (annihilating what was already there). We can therefore write the quantized field operator as:

$$\psi_\sigma(\mathbf{x}, t) = \int_{|\mathbf{k}|>k_F} \frac{d^3k}{(2\pi)^{3/2}} e^{i\mathbf{k}\cdot\mathbf{x} - i\frac{\hbar \mathbf{k}^2}{2m}t} a_\sigma(\mathbf{k}) + \int_{|\mathbf{k}|<k_F} \frac{d^3k}{(2\pi)^{3/2}} e^{-i\mathbf{k}\cdot\mathbf{x} - i\frac{\hbar \mathbf{k}^2}{2m}t} b_\sigma^\dagger(\mathbf{k}) \tag{5.6}$$

where we have done a change of variables $\mathbf{k} \to -\mathbf{k}$ in the second integral. So, ψ has two parts; it either annhilates a particle, or creates a hole. The only funny part of the expression is that the phases have opposite signs, but the energy part does not. It looks like the holes somehow gives us back negative energy. There is something we can do to repair this; let us redefine the energy. As it is currently, we have $\epsilon = \frac{\hbar^2\mathbf{k}^2}{2m}$. What if instead we add a constant[11]:

$$\frac{\hbar^2\mathbf{k}^2}{2m} \to \frac{\hbar^2\mathbf{k}^2}{2m} - \frac{\hbar^2 k_F^2}{2m} = \frac{\hbar^2\mathbf{k}^2}{2m} - \epsilon_F \tag{5.7}$$

---

[11]Which we are told all the time in QM we are allowed to do; note that really this is a lie, but let us not worry about it.

if we do this and plug this into the energy, we get something nicer, as both particles and holes will have positive energy relative to the vacuum state (we return to this statement shortly). The net effect on the Hamiltonian is the substitution:

$$H \to H - \epsilon_F \mathcal{N}. \tag{5.8}$$

Now, if we want to calculate the number operator, we have:

$$\mathcal{N} = \int_{|\mathbf{k}|>k_F} d^3k a^{\dagger\sigma}(\mathbf{k})a_\sigma(\mathbf{k}) - \int_{|\mathbf{k}|<k_F} d^3k b^\dagger_\sigma(\mathbf{k})b^\sigma(\mathbf{k}) + \rho V \tag{5.9}$$

The negative sign on the b integral comes from the fact that we interchange the order to get the correct order of operators, but then we get a negative sign from the anticommutation, and we add ρV (with ρ the density and V the volume... an infinite constant. It is counting all of the particles in the Fermi sea/inside the Fermi sphere). If we now write down the Hamiltonian, we have:

$$H - \epsilon_F \mathcal{N} = \int_{|\mathbf{k}|>k_F} d^3k \frac{\hbar^2(\mathbf{k}^2 - k_F^2)}{2m} a^{\dagger\sigma}(\mathbf{k})a_\sigma(\mathbf{k}) - \int_{|\mathbf{k}|<k_F} \frac{\hbar^2(k_F^2 - \mathbf{k}^2)}{2m} b^\dagger_\sigma(\mathbf{k})b^\sigma(\mathbf{k}) + \phi V \tag{5.10}$$

where the change in the sign of the energy is from the anticommutation of the bs, and we also add the energy (an infinite constant representing the energy of all particles within the Fermi sphere) to compensate for this anticommutation. Where:

$$\Phi = \langle H - \epsilon_F N \rangle = \phi V \tag{5.11}$$

is the Grand canonical free energy. Note if we open our system, the fermions were go out of the system until none are left (they tend to want to escape to minimize the free energy); in order for this to not happen, there is a chemical potential μ which can be seen as a sort of "gate voltage" of the fermions. It is an adjustable parameter we can tweak to coax the fermions back in. So then, determining the density has to do with extremizing the free energy.

Having do this, we notice that the energy of our excitations are all positive! So our shifting of the Hamiltonian was a good choice.

5.3 Equations of State and Physical Parameters

How do we figure out the parameters in the above expression (e.g. the density)? Let's return to the particle number in our old formalism:

$$\mathcal{N} = \int d^3k \alpha^{\dagger\sigma}(\mathbf{k})\alpha_\sigma(\mathbf{k}) \tag{5.12}$$

From this, let's calculate the expectation value for the vacuum state[12]:

$$\rho V = \langle \mathcal{O}|\mathcal{N}|\mathcal{O}\rangle = \int_{|\mathbf{k}|<k_F} (2J+1)\delta^3(\mathbf{0}) \tag{5.13}$$

where we have noted that the integral vanishes when $|\mathbf{k}| > k_F$ due to how the vacuum is defined, and we have replaced the αs with a terrible anticommutator. Here $\delta^3(\mathbf{k})$ is defined as:

$$\delta^3(\mathbf{k}) = \int \frac{d^3x}{(2\pi)^3} e^{i\mathbf{k}\cdot\mathbf{x}} \tag{5.14}$$

so then:

$$\delta^3(\mathbf{0}) = \int \frac{d^3x}{(2\pi)^3} = \frac{V}{(2\pi)^3} \tag{5.15}$$

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[12]Note we will be very mathematically relaxed in our manipulations here... e.g. interchanging orders of matrix elements. It may be possible to justify this with working in a finite system and then taking limits, but we get the same answer so let us be cavalier about it.

so the above expectation value becomes:

$$\rho V = \int_{|\mathbf{k}|<k_F} (2J+1)\frac{V}{(2\pi)^3} \tag{5.16}$$

and if we now cancel out the volume, we remove the infinite quantities from both sides:

$$\rho = \int_{|\mathbf{k}|<k_F} \frac{2J+1}{(2\pi)^3} = \frac{(2J+1)}{(2\pi)^3}\frac{4\pi}{3}k_F^3 = \frac{(2J+1)}{6\pi^2}k_F^3 \tag{5.17}$$

where $\frac{4\pi}{3}k_F^3$ is the volume contained in the Fermi-sphere of radius k_F. Another calculation we can do is to calculate the internal energy; this is just the expectation value of our original Hamiltonian, which works out in a very similar way:

$$U = \langle \mathcal{O}|H|\mathcal{O}\rangle = V\frac{(2J+1)}{(2\pi)^3}\int_{|\mathbf{k}|<k_F} d^3k\frac{\hbar^2\mathbf{k}^2}{2m}. \tag{5.18}$$

We go to polar coordinates to calculate the above:

$$U = V\frac{(2J+1)}{(2\pi)^3}\frac{\hbar^2}{2m}\int_0^{k_F} k^2 dk \int_0^\pi \sin(\theta)d\theta \int_0^{2\pi} d\varphi k^2 = V\frac{(2J+1)}{(2\pi)^3}\frac{\hbar^2}{2m}\left(\frac{k_F^5}{5}\right)(4\pi) \tag{5.19}$$

So we conclude:

$$U = V\frac{(2J+1)}{(2\pi)^2}\frac{\hbar^2}{m}\frac{k_F^5}{5} = uV \tag{5.20}$$

where u is the energy density. So, we can now use these two results to solve for k_F and then eliminate it from our expressions. This gives us equations of state, which contain valuable (nontrivial!) physical information. We find from Eq. (5.17):

$$k_F = \left[\frac{6\pi^2\rho}{(2J+1)}\right]^{1/3} \tag{5.21}$$

and so plugging into Eq. (5.20) we have:

$$u = \frac{\hbar^2}{2m}\frac{(2J+1)}{10\pi^2}\left[\frac{6\pi^2\rho}{(2J+1)}\right]^{5/3} \tag{5.22}$$

so $u \propto \rho^{5/3}$. We can then find various other physical quantities by using thermodynamic relations, e.g. the pressure as a function of energy density (note we have assumed that everything is at constant zero temperature here):

$$P = -\left.\frac{\partial U}{\partial V}\right|_N = \frac{2}{3}u \tag{5.23}$$

We can also find the chemical potential:

$$\mu = \left.\frac{\partial U}{\partial N}\right|_V = \epsilon_F \tag{5.24}$$

where in the above calculation we would write $U = uV$, and then write $\rho = N/V$. So we find at zero temperature that indeed the chemical potential is the Fermi energy (of course if one turns on interactions, or the temperature, then this will no longer hold!) What's more, if we look at Φ, we have:

$$\Phi = \langle H - \mu\mathcal{N}\rangle = \phi V \implies \phi = -P \tag{5.25}$$

5.4   A Simple Example of a Quantum Phase Transition

Note that there is a more detailed discussion of the above in the notes, and we have covered the main points here. But let us discuss one thing that is not in the notes. We recall the interplay between the fermions wanting to get away and the chemical potential to coax them back in. We now consider a simple model of a *quantum phase transition*[13]. We assume that we can tune the chemical potential; as we tune, the density varies. If $\mu = 0$, the density is zero. If we make it negative, the density stays at zero. So, as we increase μ, the density becomes nonzero as we cross $\mu = 0$; signalling a phase transition in the system.
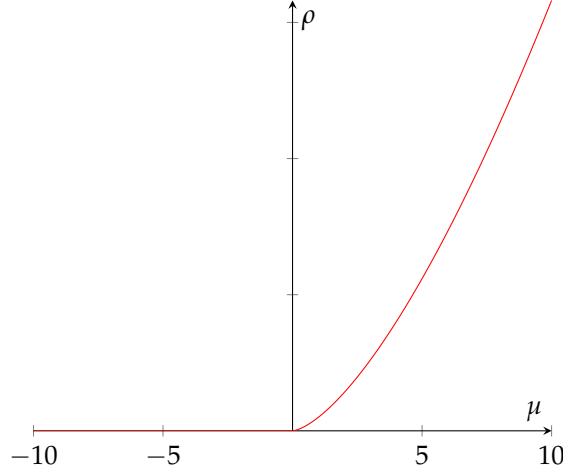
Figure 5.1: Plot of ρ vs. μ (arbitrary units). ρ is zero for $\mu \leq 0$ and $\rho \propto \mu^{3/2}$ for $\mu > 0$, signalling a phase transition in the system.

The fact that $\rho \propto \mu^{3/2}$ is obtainable by realizing that $k_F \propto \rho^{1/3}$ (Eq. (5.21)) and $\mu = \epsilon_F \propto k_F^2$ (Eqs. (5.24) and the definition of Fermi energy). Since $\varphi \propto P \propto u$ (Eqs. (5.25) and (5.23)), and $u \propto \rho^{5/3}$ (Eq. (5.17)), we find that the free energy φ is related to the chemical potential by $\varphi \propto \mu^{5/2}$.

Since we have to take three derivatives of φ w.r.t. μ before we get a discontinuity (at $\mu = 0$), by the Ehrenfest classification of phase transitions, this is a third order phase transition. This discussion can also be taken as something that highlights the difference between Fermi gas and Fermi liquid behaviour; interactions change the properties of the system significantly.

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[13]One of the main models discussed in Sachdev's book by this title!

6 Degenerate Bose Gas

6.1 Setting up the Boson Case

Last time we discussed the Fermi gas, though we did not introduce any interactions. This is a starting point of doing condensed matter physics, nuclear physics (where the nucleus could be thought of as a small Fermi gas; though of course the interactions are strong in this case), neutron stars etc. When we add interactions, we get a Fermi liquid, where calculations become intractable; though we can apply computational methods in some cases. In a way, the strong interaction has been solved via a computer calculation; this is impressive as the models have been shown to work! Today, we instead discuss the degenerate Bose gas. We cover it as the second topic as it is more complex; due to Bose-Einstein condensation. We're stuck at zero temperature, and a gas of quantum-mechanical bosons at $T = 0$ (indeed, below some critical temperature T_c) forms a BE. If we're looking at particles without charge, we're looking at some kind of *superfluid*. The noninteracting case is so ugly that we don't study it; we add a small interaction to make things more sensible, so the use of "fluid" is really correct, here.

We study a zero-temperature state of a box of bosons. If we ignore all interactions, the energy takes the form:

$$\epsilon = \frac{\hbar^2 \mathbf{k}^2}{2m}. \tag{6.1}$$

Here, more than one particle can occupy the lowest energy state, so we might take our quantized field with the creation operator, and do something that looks nonsensical:

$$(\alpha^\dagger(0))^N |0\rangle \tag{6.2}$$

Of course this looks crazy, as these particles with $\mathbf{k} = \mathbf{0}$ have infinite wavelength...so perhaps we should do something else, but what else is there? We could say that there is confinement within a box (of box L with hard boundary conditions), but then the wavefunction would follow boundary conditions (in each direction).
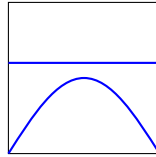
Figure 6.1: Two possible views as the problem. We can constrain the bosons to a box of side length L with hard boundaries, constraining the ground state wavefunction to look like $\sqrt{\frac{2}{L}} \sin(\frac{2\pi x}{L})$. Alteratively, we can assume our system is a finite patch out of infinite space, in which case the ground state wavefunctions are plane waves with infinite wavelength, i.e. constant in space.

In fact, the wavefunction and the energy would be quite sensitive to the boundary conditions. This is not something that we really want[14]. We could do the opposite and remove the walls, and assume that our system is a finite patch within infinite space. Then a piece of the system should look fairly generic. But what happens then? The wavefunction would be constant over all space if $\mathbf{k} = \mathbf{0}$, so the probability of finding it anywhere in space is equal. So, our state is not something with a constant particle number. So at the outset, we can take this point of view that the particle number is not fixed; an approach that particle physicists and field theorists love. One could instead take a closed system and study it through this method, but there is not a consensus as to whether the two approaches are actually equivalent. But we will take the field theorists approach, here.

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[14]as theorists in a QFT class, anyway; maybe not if you work at SBQMI down the street.

Motivated by this argument, we consider the state:

$$|\mathcal{O}\rangle = \sum_{N=0}^{\infty} c_N \left(\alpha^\dagger(\mathbf{0}) \right)^N |0\rangle. \tag{6.3}$$

This is still not quite sensible; perhaps we can take something like:

$$\alpha_f^\dagger = \int d^3x f(\mathbf{x}) \alpha^{dag}(\mathbf{k}) \tag{6.4}$$

for f square integrable if we wanted to be a tad more rigorous, but let us not worry ourselves with this very much. How do we characterize such states? One thing we can notice is:

$$\langle \mathcal{O} | \psi(\mathbf{x}, t) | \mathcal{O} \rangle \neq 0. \tag{6.5}$$

The field operator annihilates a boson, but it will still have a nonzero overlap with the original vacuum state, so the overall expectation value will be nonzero (see HW2). Note that this is okay for bosons, but we would never see this for fermions. This is hugely degenerate if the bosons are non-interacting.

6.2 Coherent States

Let us talk for a little about coherent states, which have the above expectation value property. At $t = 0$, we consider the field-operator commutation relations:

$$[\psi(\mathbf{x}), \psi^\dagger(\mathbf{y})] = \delta^3(\mathbf{x} - \mathbf{y}) \tag{6.6}$$

and:

$$[\psi^\dagger(\mathbf{x}), \psi^\dagger(\mathbf{y})] = [\psi(\mathbf{x}), \psi(\mathbf{y})] = 0. \tag{6.7}$$

If we act the annhilation operator on the empty vacuum, we have:

$$\psi(\mathbf{x})|0\rangle = 0. \tag{6.8}$$

We do not concern ourselves with the spin. Now, we consider the state:

$$|\eta\rangle = e^{\int d^3x \left(\eta^*(\mathbf{x})\psi(\mathbf{x}) - \psi^\dagger(\mathbf{x})\eta(\mathbf{x}) \right)} |0\rangle \tag{6.9}$$

where the operator in the exponential is anti-hermitian and so the overall operator is unitary. If we write the normal ordered version of the above, we obtain:

$$|\eta\rangle = e^{-\int d^3x \eta^*(\mathbf{x})\eta(\mathbf{x})} e^{-\int d^3x \eta(\mathbf{x})\psi^\dagger(\mathbf{x})} |0\rangle \tag{6.10}$$

where we have got rid of the ψ term by considering that this annhilates the vacuum state. We can write the action of the annihilation operator on the coherent state as:

$$\psi(\mathbf{x})|\eta\rangle = \eta(\mathbf{x})|\eta\rangle. \tag{6.11}$$

So that's a cool property! It's also a unitary transform of the vacuum state, so it is normalized:

$$\langle \eta | \eta \rangle = 1. \tag{6.12}$$

So this is an example of the state where:

$$\langle \psi | \psi(\mathbf{x}) | \eta \rangle = \eta(\mathbf{x}). \tag{6.13}$$

This is an example of a "good" coherent state (as opposed to a bad one) and very often in physics we used bad ones. What could go wrong? For example, the integral $\int d^3x \eta^*(\mathbf{x})\eta(\mathbf{x})$ could diverge if $\eta(\mathbf{x})$ is a constant. This often happens, actually (and we will proceed to work with this right now). The bad coherent state is ubiquotous; any interaction state of charges particles produces a "coherent" state of soft photons, but this is a bad coherent state (it has no overlap with any state which has a finite number of photons). Every QED interaction produces an infinite number of photons, which fly away undetected, but seem to be always there. So, we drop "true/good" coherent states for now, and work with bad ones; which will still give us sensible results.

6.3 Landau's Argument for Superfluidity

Note that in the textbook that there is a section which reviews Landau's famous argument about the quasiparticle spectrum and critical velocity of a superfluid. It uses Galilean symmetry, and so we don't cover it here (and the treatment of the book is so refined, that it is probably better to read it there). There is however the idea of a superfluid flowing through a pipe, without resistance, and so even if we introduce some interactions (between the particles, and with the pipe), it will flow through the pipe without any resistance. This is to some extent seen in the lab, and there is seen that if the superfluid goes too fast then the superfluid starts to feel resistance. Landau's argument goes as follows. For the energy to dissipate, there should be some excitations in the fluid. So, there should be some viscosity that creates a travelling wave/ripple (sometimes called quasiparticles) in the fluid. Let us say this is wavelike (everything is quantum mechanical here), with a wavenumber $\mathbf{k}$ and frequency $\omega(\mathbf{k})$.
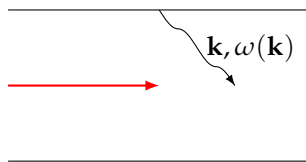
Figure 6.2: In Landau's argument for superfluids, we consider a superfluid travelling in a pipe, and wavelike excitations/quasiparticles with wavenumber $\mathbf{k}$ and frequency $\omega(\mathbf{k})$.

He then argues that this should be energetically favourable if the velocity is larger than a critical velocity v_c:

$$v_c = \min_{\mathbf{k}} \frac{\omega(\mathbf{k})}{|\mathbf{k}|}. \tag{6.14}$$

The argument is beautiful in how it invokes Galilean relativity/symmetry (do read about it in your own time!) Note the stark difference from the free particle case; as then we would have $\omega(\mathbf{k}) \propto \mathbf{k}^2$ so $v_c = 0$, i.e. free particles are not superfluidic. What does work is if we have a sound wave, as then $\omega(\mathbf{k}) = v_s|\mathbf{k}|$; here the critical velocity is the speed of sound.

So, the important goal for us; can we find this using our model? What do the elementary excitations look like?

A small aside; this argument doesn't really agree with experiments. v_c as measured is smaller than what we would expect from Landau's argument. The explanation is that the dispersion relation $\omega(\mathbf{k})$ as measured in experiment takes the form of the roton curve as seen in the below figure. Then v_c is not the slope of the purple curve (as would be predicted theoretically) but instead is the slope of the red curve (less than the theoretically predicted value).

6.4 Introducing a Small Interaction to our QFT

We introduce a weak repulsive interaction between the bosons that supresses the Bose-Einstein condensation. They will want to be far apart and run away from our system; so we add a chemical potential to balance this and draw them back in (so we end up with a finite density at the end). Let us write down a Hamiltonian[15] for this:

$$H = \int d^3x \left(\frac{\hbar^2}{2m} \boldsymbol{\nabla}\psi^\dagger(\mathbf{x}) \cdot \boldsymbol{\nabla}\psi(\mathbf{x}) - \mu\psi^\dagger(\mathbf{x})\psi(\mathbf{x}) + \frac{\lambda}{2}\psi^\dagger(\mathbf{x})\psi^\dagger(\mathbf{x})\psi_\mathbf{x}\psi(\mathbf{x}) \right) \tag{6.15}$$

where the last term is a ball-bearing potential, with λ "small" (and positive; else the spectrum will not be bounded from below, as the energy from all the particles sitting on top of each other will be negative

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[15]A remark: model building is often much easier than the actual toying with the (e.g. field) equations...
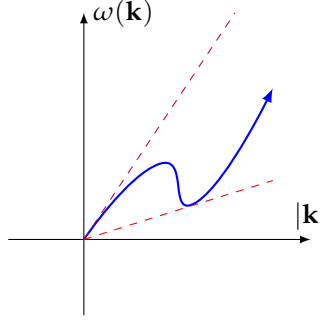
Figure 6.3: Plot of the experimental dispersion relation for superfluids (roton dispersion relation). From this we can see the reason for the experimental disagreement of the Landau argument for determining the critical velocity v_c of superfluids, as the local minimum in the roton curve shifts the critical velocity to be the slope of the red curve, rather than the Landau theoretical prediction of the purple curve.

infinity). We also enforce the commutation relations:

$$[\psi(\mathbf{x},t), \psi^\dagger(\mathbf{y},t)] = \delta^3(\mathbf{x}-\mathbf{y}), \quad [\psi^\dagger(\mathbf{x},t), \psi^\dagger(\mathbf{y},t)] = [\psi(\mathbf{x},t), \psi(\mathbf{y},t)] = 0 \tag{6.16}$$

and also introduce the vacuum state $|\mathcal{O}\rangle$, which is the lowest-eigenvalue eigenstate of the (bounded-from-below) Hamiltonian. It is possible that the expectation value is nonvanishing, so let us anticipate this and introduce a function $\eta(\mathbf{x})$:

$$\langle\mathcal{O}|\psi(\mathbf{x},t)|\mathcal{O}\rangle = \eta(\mathbf{x}) \neq 0. \tag{6.17}$$

How do we treat this possibility in a systematic way? One way would be to write:

$$\psi(\mathbf{x},t) = \eta(\mathbf{x},t) + \tilde\psi(\mathbf{x},t) \tag{6.18}$$

where η is the classical part and $\tilde\psi$ is the quantum/non-classical part, i.e. which follows:

$$\langle\mathcal{O}|\tilde\psi(\mathbf{x},t)|\mathcal{O}\rangle = 0. \tag{6.19}$$

So, there will be a classical part of the Hamiltonian where we forget about the $\tilde\psi$s, and in this term we would like ηs which minimize the Hamiltonian. Then we can solve a classical field equation, check the extremum of the functional etc. We can argue that the η that minimizes the Hamiltonian goes as:

$$\eta \sim \frac{1}{\sqrt{\lambda}}. \tag{6.20}$$

At very weak coupling, this classical piece is emphasized. We then might ask what the size of the quantum part is. Since ψ needs to obey equal-time commutation relations, and η drops out of these (as it is classical and drops out of the commutation relations), then since $\tilde\psi$ follows the commutation relations (where λ shows up nowhere), this tells us that $\tilde\psi$ is of order unity:

$$\tilde\psi \sim \lambda^0 = 1. \tag{6.21}$$

If we make λ small, we can then ignore anything but the classical η term. We can consider an asymptotic expansion of η (asymptotic as the first term is singular) and $\tilde\psi$, each of which have corrections which can be solved order by order. This is an arbitrarily good approach so long as λ is arbitrary small and nonzero. Let us take it (next class!)

Another thing we might be interested in is the "smoothest" possible states, i.e. states that have some symmetry. A smooth surface has translation symmetry. We expect that the low-energy states of our theory

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here may be smooth. Suppose η was a constant; then the most important part of our field is smooth. This is like going back to our condensate at the start of our lecture, where our wavefunctions were constant throughout all space. We then simply search for a constant which minimizes the Hamiltonian. The kinetic energy terms are zero if η is a constant, and we can minimize the other terms to find:

$$\eta = \sqrt{\frac{\mu}{\lambda}} \sim \frac{1}{\sqrt{\lambda}}. \tag{6.22}$$

From our theory we can calculate the density:

$$\rho \cong \frac{\mu}{\lambda} \tag{6.23}$$

and the (grand canonical free) energy:

$$\Phi = -\frac{\mu^2}{2\lambda} + \ldots = -\frac{\lambda}{2}\rho^2 + \ldots \tag{6.24}$$

which is also equal to the negative of the pressure:

$$P = -\Phi = \frac{\lambda}{2}\rho^2. \tag{6.25}$$

So already this theory gives us a lot to work with, but we have yet to show that it is in fact a superfluid; this is something we will calculate and verify next class, by figuring out what $\omega(\mathbf{k})$ is for our theory.

7 Degenerate Bose Gas Continued

7.1 A Review of the Boson Gas Hamiltonian

Recall the Hamiltonian we were working with in studying the Bose gas/liquid:

$$H = \int d^3x \left(\frac{\hbar^2}{2m} \boldsymbol{\nabla}\psi^\dagger \cdot \boldsymbol{\nabla}\psi - \mu\psi^\dagger\psi + \frac{\lambda}{2}\psi^\dagger\psi^\dagger\psi\psi \right) \tag{7.1}$$

where:

$$\psi = \eta + \tilde{\psi} \tag{7.2}$$

with η is the classical part and $\tilde{\psi}$ the quantum part. We can interpret this as:

$$\langle \mathcal{O}|\tilde{\psi}|\mathcal{O}\rangle = 0. \tag{7.3}$$

We noted that the classical part η was very important in the weak-coupling limit, as $\eta \sim \frac{1}{\sqrt{\lambda}}$. Meanwhile, $\tilde{\psi}$ obeys commutation relations for which no λ shows up, so $\tilde{\psi} \sim \lambda^0$. However, we expect that there are an infinite series of correction, so really:

$$\psi = \eta + \tilde{\psi} + \delta\eta + \lambda\delta\tilde{\psi} \tag{7.4}$$

Where $\delta\eta \sim \sqrt{\lambda}$. For small λ (though this is not entirely a trivial statement; λ has dimensions, so what it is small compared to?), it is meaningful to analyze the leading terms (i.e. a classical Hamiltonian). We can plug in η to where the ψs are in the Hamiltonian, and we get something that looks like a potential:

$$V(\eta) = \mathcal{V}(-\mu|\eta|^2 + \frac{\lambda}{2}|\eta|^4) \tag{7.5}$$

apologies for the confusing notation; V on the left is the potential, $\mathcal{V}$ on the right is a volume. Minimizing V (the potential) with respect to η, we find:

$$\eta = \sqrt{\frac{\mu}{\lambda}}. \tag{7.6}$$

There is something funny with this identification; $V(\eta)$ depends only on the norm of η, but in the expression for η we have chosen it to be real. From a minimization perspective:

$$\eta = \sqrt{\frac{\mu}{\lambda}}e^{i\theta} \tag{7.7}$$

are valid minima for all $\theta \in \mathbb{R}$. So the minimum is not unique. However, we can proceed by just choosing an angle of our choice, and it will not matter. Why is this the case?

7.2 A Brief Foray Into Spontaneous Symmetry Breaking

Looking at the Hamiltonian, we notice that there is a symmetry; namely, the Hamiltonian is unchanged by the introduction of some phase $\psi \mapsto \psi e^{i\theta}$. So, this tells us that what we choose for the phase of η should not matter.

   The idea is that the potential is completely symmetric under rotation, but the solution is not. This is something known as spontaneous symmetry breaking. Classical analogy: which way does chalk fall when one lets go of it from the top position. A priori, there is a symmetry; there is no preference to how it falls. But when we actually do the experiment, it falls somewhere as we perturb it somehow when releasing (breaking the symmetry) or an air molecule bounces off of it and causes it to fall. Quantum mechanically, the chalk is described by the rigid rotator Hamiltonian:

$$H = \frac{I}{2}\dot{\theta}^2 \tag{7.8}$$
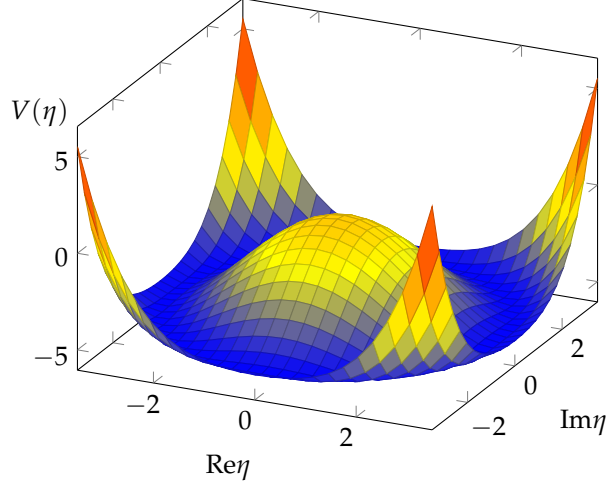
Figure 7.1: 3D plot of the potential V as a function of η. We take $\mathcal{V} = 1, \mu = 1, \lambda = 1/10$. The blue ring corresponds to $\eta = \sqrt{\frac{\mu}{\lambda}} e^{i\theta}$ which minimizes the potential, which is radially symmetric.

If I look at the eigenfunctions, we have:

$$\psi(\theta) = e^{in\theta} \tag{7.9}$$

The ground state is a superposition of all of these angles, which averages out to zero. But this is clearly not the right state; it is in some orientation and it is frozen there. The idea is that the chalk pointing at some angle is actually an excited state; we have something like:

$$\psi(\theta) = \sum_n e^{in(\theta - \theta_c)} \tag{7.10}$$

and it relaxes towards the ground state given sufficient time; however this time depends on the moment of inertia, which here it is very large. If the chalk was small enough, then we would see it undergo this relaxation; since it is macroscopic, the relaxation time is very large.

Why this chalk discussion? This is to say that why the quantum mechanical description of the chalk may be correct, it is really not particularly reasonable; it makes more sense to treat this classically. So, returning back to the idea of spontaneous symmetry breaking. This is a ubiqitous phenomenon (in condensed matter, particle physics, etc.). For example, consider ferromagnetism. The true ground state of a ferromagnetic is some superposition, but if we look at it, it will not decay to the ground state during the lifetime of the universe. So, we can just think about this classically.

Having now chosen a solution, we can't see the symmetry of the whole potential anymore; the fluctuations (as obtained by calculating the higher order terms) cannot "see" the whole energy landscape (one can think that the "moment of inertia" is so large that it cannot explore the entire ring of minima).

A takeaway from all of this; spontaneous symmetry-breaking is a phenomena we often see when classical effects dominate over quantum ones.

7.3 Solving the System

So, we've solved for the leading term of the expansion of ψ! Since it's dominant, we could just calculate the internal energy, or the grand canonical free energy from what we have so far. We can find the GC free energy to be:

$$\phi = -\frac{\lambda}{2}\rho^2 = -P. \tag{7.11}$$

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We can find corrections to this by calculating the contributions from the higher order terms (these corrections should be small, if our asymptotic expansion is actually reasonable). In particular, here this expansion works at low density.

Next, we can plug $\psi = \eta + \tilde{\psi}$ into the field equation (see HW2) or the Hamiltonian (which we do now), and see what we get out. We find:

$$H' = -\frac{\lambda}{2}\rho^2 V + \int d^3x \left(\frac{\hbar^2}{2m} \boldsymbol{\nabla}\tilde{\psi}^\dagger \cdot \boldsymbol{\nabla}\tilde{\psi} + \mu\tilde{\psi}^\dagger\tilde{\psi} + \frac{\mu}{2}\tilde{\psi}\tilde{\psi} + \frac{\mu}{2}\tilde{\psi}^\dagger\tilde{\psi}^\dagger \right) + \mathcal{O}(\lambda) \tag{7.12}$$

Now the Hamiltonian does not look normal in the $\tilde{\psi}$ and $\tilde{\psi}^\dagger$s anymore; this is due to the broken symmetry. We can also plug in our expansion into the commutation relations:

$$[\tilde{\psi}(\mathbf{x},t), \tilde{\psi}^\dagger(\mathbf{y},t)] = \delta^3(\mathbf{x} - \mathbf{y}) \tag{7.13}$$

Let us comment on the $\mathcal{O}(\lambda)$ corrections; we first have that:

$$-\frac{\lambda}{2}\rho^2 V = -\frac{\mu^2}{2\lambda}V \tag{7.14}$$

So this is indeed the leading term for small λ. The integral term is of order $\lambda^0 = 1$. Then the corrections are of order λ, so we can control these by taking small λ.

For large derivatives, the kinetic energy terms dominate, but for smaller derivatives the other terms in the integral become important.

By plugging things back in, we get a linear field equation for $\tilde{\psi}$ (The Bogolibov-de-Gennes equation); see HW2. But let us attack this Hamiltonian directly:

$$\psi(\mathbf{x}, 0) = \int \frac{d^3k}{(2\pi)^{3/2}} e^{i\mathbf{k}\cdot\mathbf{x}} \alpha(\mathbf{k}) \tag{7.15}$$

this is really just a change of basis in function space; the whole integral is like a unitary matrix that rotates the basis. Let us go to the αs; why? Because the Hamiltonian is translation invariant, we would suspect that the momentum would be an important quantity, and $\mathbf{k}$ is closely related to momentum, so it is good to analyze. The Hamiltonian reads:

$$H = -\frac{\mu^2}{2\lambda}V + \int d^3k \left(\frac{\hbar^2 k^2}{2m}\alpha^\dagger(\mathbf{k})\alpha(\mathbf{k}) + \mu\alpha^\dagger(\mathbf{k})\alpha(\mathbf{k}) + \frac{\mu}{2}\alpha(-\mathbf{k})\alpha(\mathbf{k}) + \frac{\mu}{2}\alpha^\dagger(\mathbf{k})\alpha^\dagger(-\mathbf{k}) \right) \tag{7.16}$$

From going to k-space, we have a separate Hamiltonian for each value of $\mathbf{k}$, and the first term ($\alpha^\dagger(\mathbf{k})\alpha(\mathbf{k})$) is diagonalized. However, it is still not clear what the eigenvalues of the elementary excitations are. We need to diagonalize this Hamiltonian; let us try this. First we note that the Hamiltonian is of quadratic form, so we should be able to diagonalize it. We let:

$$\begin{aligned}
\alpha(\mathbf{k}) &= \cosh\theta(\mathbf{k})a(\mathbf{k}) - \sinh\theta(\mathbf{k})a^\dagger(-\mathbf{k}) \\
\alpha^\dagger(\mathbf{k}) &= \cosh\theta(\mathbf{k})a^\dagger(\mathbf{k}) - \sinh\theta(\mathbf{k})a(-\mathbf{k})
\end{aligned} \tag{7.17}$$

Wait, where do the hyperbolic functions come from? We know that the αs follow the commutation relations:

$$[\alpha(\mathbf{k}), \alpha^\dagger(\mathbf{l})] = \delta^3(\mathbf{k} - \mathbf{l}) \tag{7.18}$$

so we want the as to do the same:

$$[a(\mathbf{k}), a^\dagger(\mathbf{l})] = \delta^3(\mathbf{k} - \mathbf{l}) \tag{7.19}$$

I don't have a lot of freedom in the coefficients if I want the commutation relations to hold; the hyperbolic functions come out of this.

Eqs. (7.17) is known as a Bogoliubov transformation. We can plug this into the Hamiltonian and get an even more complicated expression with $\sin \theta(\mathbf{k})$s and as, and we can adjust $\theta(\mathbf{k})$ until we get something proportional to $a^\dagger(\mathbf{k})a(\mathbf{k})$. When we do so, we end up with:

$$H = -\frac{\mu^2}{2\lambda}V + \int d^3k E(\mathbf{k})a^\dagger(\mathbf{k})a(\mathbf{k}) + E_0 \tag{7.20}$$

where the constant at the end comes from the corrections from interchanging the order of operators (as a consistency check: this constant should be small compared to the leading order constant). It is a kind of zero-point energy. In order for the above to hold, we find the constraint that:

$$\tanh 2\theta(\mathbf{k}) = \frac{\mu}{\frac{\hbar^2 \mathbf{k}^2}{2m} + \mu} \tag{7.21}$$

and $E(\mathbf{k})$ follows:

$$E(\mathbf{k}) = \sqrt{\left(\frac{\hbar^2 \mathbf{k}^2}{2m} + \mu\right)^2 - \mu^2} \tag{7.22}$$

so we obtain a dispersion relation for $\mathbf{k}$! The cool thing is we can now use this dispersion relation combined with the Landau result to get a testable prediction of the critical velocity for superfluids. For small $\mathbf{k}$, the above reduces to:

$$E(\mathbf{k}) \approx \sqrt{\frac{\hbar^2}{2m}\mu}|\mathbf{k}| \tag{7.23}$$

So the Landau criterion would tell us that:

$$v_c = \sqrt{\frac{\hbar^2}{2m}\mu}. \tag{7.24}$$

This turns out to be wrong most of the time, as superfluids are in general not weakly interacting. But, when an experimental group created a weakly-interacting Bose-Einstein condensate, they were able to indeed verify that the above expression holds!

This concludes our discussion; HW2 attacks the same problem from the perspective of the equations of motion.

7.4   A Small Teaser for Next Lecture

We've developed our QFT, rewrote many-particle QM as a QFT, and studied some simple (but useful!) examples (such as the Fermi gas[16] and superfluids). Now, we go into developing the ideas of QFT a little further. One might wonder why we need something else; this is because this "something else" gives us great power and a nicer way to study field theories when we have to do perturbation theory. It may be an unexpected step, but we will temporarily regress from doing QFT to doing classical field theory. We will consider the *action principle*. We will consider some classical field $\psi_\sigma(\mathbf{x}, t)$ (bosons only; anti-commuting fermions would not be pure functions, but would instead be operators belonging to an algebra...). We will then ask if we can derive our field equation from a variational principle and an action. Of course the answer is yes (else we wouldn't bring it up), and we can even write down the Lagrangian density and the action which will yield our classical field equation. The action is:

$$S = \int dt d^3x \mathcal{L} \tag{7.25}$$

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[16]Quite a useful "first-order" approximation for metals.

where the Lagrangian density is:

$$\mathcal{L} = i\hbar\psi^{\dagger\sigma}\frac{\partial}{\partial t}\psi_\sigma - \frac{\hbar^2}{2m}\boldsymbol{\nabla}\psi^{\dagger\sigma}\cdot\boldsymbol{\nabla}\psi_\sigma + \mu\psi^{\dagger\sigma}\psi_\sigma - \frac{\lambda}{2}\left(\psi^{\dagger\sigma}\psi_\sigma\right)^2. \tag{7.26}$$

However, we have lost something: our quantum field equation had a particular ordering of the ψ and $\psi^\dagger$s in the interaction part. We have to recover this somehow. This could be a small or large problem; for us it is small because we know how it has to look like ahead of time. But, something like the ordering problem in treating Einstein's gravity is not resolved. In general, this is something one needs to solve case-by-case.

What then, do we gain? The Lagrangian encodes the commutator as well as the field equation. One can encode the first term as:

$$\left(i\hbar\psi^{\dagger\sigma}\right)\left(\frac{\partial}{\partial t}\psi_\sigma\right) = P\dot{Q} \tag{7.27}$$

and note the connection with the Hamiltonian:

$$L = P\dot{Q} - H(P,Q). \tag{7.28}$$

One can consider the Poisson bracket:

$$\{\psi_\sigma(\mathbf{x},t), i\hbar\psi^{\dagger\rho}(\mathbf{y},t)\} = \delta_\sigma^\rho\delta^3(\mathbf{x}-\mathbf{y}) \tag{7.29}$$

and when quantizing one adds an $i\hbar$ and replaces the Poisson bracket with a commutator. So there is that benefit of less writing. We also gain the analysis of symmetries and conserved quantities that we know from classical field theory, which we can lift to the QFT (for example, conservation of energy, momentum).

This is not just a construct; we will learn how to use functional integrals, and the (classical) action plays a very large role there (we integrate over it). It will stay with us for a long time.

8 The Action Principle

We revert a little bit back into classical field theory. Technically, the field equation and the commutation relations we have constructed define our QFT, and give us all the information we need. So changing the formalism needs some justification; that justification is that classical field theory has a lot of structure, we can lift back into QFT to learn more about QFT. This logic is fairly obscure in most textbooks, but we will explore it a bit deeper. We will find the classical to quantum transition to be relatively straightforwards. For generic quantum theories, this is not the case, but quantum field theories tend to be simple as far as mechanical things go. This is not just because we like simple things, but because the internal consistency of quantum field theories doesn't let them be extremely complicated.

8.1 The Action & Lagrangians

The action is defined as the following integral over spacetime:

$$S = \int d^3x dt \mathcal{L} = S[\psi, \psi^\dagger] \tag{8.1}$$

where $\mathcal{L}$ is the Lagrangian density. It is a functional; it maps functions $\psi, \psi^\dagger$ onto a number S. We consider the Lagrangian density:

$$\mathcal{L} = \frac{i\hbar}{2} \psi^{\dagger\sigma} \frac{\partial \psi_\sigma}{\partial t} - \frac{i\hbar}{2} \frac{\partial \psi^{\dagger\sigma}}{\partial t} \psi_\sigma - \frac{\hbar^2}{2m} \boldsymbol{\nabla} \psi^{\dagger\sigma} \cdot \boldsymbol{\nabla} \psi_\sigma + \mu \psi^{\dagger\sigma} \psi_\sigma - \frac{\lambda}{2} (\psi^{\dagger\sigma} \psi_\sigma)^2. \tag{8.2}$$

this is the Lagrangian that would recover our non-relativistic quantum field theory (we just conjured this up, with some guidelines; namely that it reproduces the field equation that we want. The field equation is the primitive here, and the Lagrangian we reverse engineer[17]). From this Lagrangian we can not only recover the field equation but also the commutator.

Note that for free field theories, we will in general only consider Lagrangians that are quadratic in the fields; this allows for the equations of motion to be linear. For interacting field theories we may have higher order terms.

Now, if we consider the relation of the Lagrangian to the Hamiltonian:

$$\mathcal{L} = P\dot{Q} - H(P,Q) = i\hbar \psi^{\dagger\sigma} \frac{\partial \psi_\sigma}{\partial t} - \dots \tag{8.3}$$

We can identify Q with $\psi_\sigma(\mathbf{x},t)$ and P with $i\hbar\psi^{\dagger\rho}(\mathbf{x},t)$. The Poisson bracket then gives us:

$$\{\psi_\sigma(\mathbf{x},t), i\hbar\psi^{\dagger\rho}(\mathbf{x},t)\} = \delta^\rho_\sigma \delta^3(\mathbf{x}-\mathbf{y}) \tag{8.4}$$

So if we then go from classical to quantum by changing the Poisson brackets to commutators and introducing a factor if $i\hbar$, we have:

$$[\psi_\sigma(\mathbf{x},t), i\hbar\psi^{\dagger\rho}(\mathbf{x},t)] = i\hbar\delta^\rho_\sigma \delta^3(\mathbf{x}-\mathbf{y}) \implies [\psi_\sigma(\mathbf{x},t), \psi^{\dagger\rho}(\mathbf{x},t)] = \delta^\rho_\sigma \delta^3(\mathbf{x}-\mathbf{y}). \tag{8.5}$$

Next, how do we get the field equation? The action principle states that the action functional, considered as a mapping of classical fields obeying the appropriate boundary conditions to the real numbers, is stationary when it is evaluated on the field configurations which obey the classical equations of motion, that is, the classical field equation. So, the Lagrangian encodes the field equation, and we can suss it out through some calculus of variations/functional calculus.

---

[17]But note that once we start to familiarize ourselves with Lagrangians, we will find it easier to write down a Lagrangian and derive the field equations etc. from there. It is a slightly shorter presentation of all of the data encoded in the field equation and commutation relations.

8.2 Deriving the Euler-Lagrange Equations

We write down a formal mathematical criteria for what this means. $\psi, \psi^\dagger$ are a "stationary point" (of course one remembers these are actually functions) when:

$$S[\psi + \delta\psi, \psi^\dagger + \delta\psi^\dagger] = S[\psi, \psi^\dagger] + O((\delta\psi)^2, (\delta\psi^\dagger)^2, \delta\psi\delta\psi^\dagger) = S[\psi, \psi^\dagger] + \delta S \tag{8.6}$$

One might ask what a "nearby point" in function space actually is (i.e. what does $\delta\psi$ mean?); we avoid this discussion as our applications tend to be simple.

We can now define the variation of the action as follows (by considering the variation of the Lagrangian, treating it as a function of $\psi_\sigma, \psi^{\dagger\sigma}$ and its spatial/time derivatives):

$$\delta S = \int dt d^3 x \delta\mathcal{L}$$

$$= \int dt d^3 x \left[\delta\psi_\sigma(\mathbf{x}, t) \frac{\partial\mathcal{L}}{\partial\psi_\sigma(\mathbf{x}, t)} + \delta\dot{\psi}_\sigma(\mathbf{x}, t) \frac{\partial\mathcal{L}}{\partial\dot{\psi}_\sigma(\mathbf{x}, t)} + \delta\boldsymbol{\nabla}\psi_\sigma(\mathbf{x}, t) \frac{\partial\mathcal{L}}{\partial\boldsymbol{\nabla}\psi_\sigma(\mathbf{x}, t)} + (\psi_\sigma \iff \psi^{\dagger\sigma}) \right] \tag{8.7}$$

where $(\psi_\sigma \iff \psi^{\dagger\sigma})$ denotes we have all the same terms, replacing the ψs with their Hermitian conjugate. We can then consider that:

$$\delta(\boldsymbol{\nabla}\psi_\sigma(\mathbf{x}, t)) = \boldsymbol{\nabla}(\delta\psi_\sigma(\mathbf{x}, t)) \tag{8.8}$$

i.e. the variation of the derivative is the derivative of the variation, and the same with the time derivatives:

$$\delta(\frac{\partial}{\partial t}\psi_\sigma(\mathbf{x}, t)) = \frac{\partial}{\partial t}(\delta\psi_\sigma(\mathbf{x}, t)) \tag{8.9}$$

We can then rewrite the variation of the action as:

$$\delta S = \int dt d^3 x \left[\delta\psi_\sigma(\mathbf{x}, t) \left(\frac{\partial\mathcal{L}}{\partial\psi_\sigma} - \frac{\partial}{\partial t}\frac{\partial\mathcal{L}}{\partial\dot{\psi}_\sigma} - \boldsymbol{\nabla}\cdot\frac{\partial\mathcal{L}}{\partial\boldsymbol{\nabla}\psi_\sigma} \right) + \frac{\partial}{\partial t}\left(\delta\psi_\sigma\frac{\partial\mathcal{L}}{\partial\dot{\psi}_\sigma} \right) + \boldsymbol{\nabla}\cdot\left(\delta\psi_\sigma\frac{\partial\mathcal{L}}{\partial\boldsymbol{\nabla}\psi_\sigma} \right) + (\psi_\sigma \iff \psi^{\dagger\sigma}) \right] \tag{8.10}$$

We look at the second and third terms in the above integral. It is a four-divergence, so Gauss's theorem allows us to rewrite its spacetime volume integral as a surface integral at the boundaries of space and time. While we don't concern ourselves with the details of Dirchlet vs. Neumann boundary conditions, we assume that there is some BC which makes the surface terms vanish:

$$\frac{\partial}{\partial t}\left(\delta\psi_\sigma\frac{\partial\mathcal{L}}{\partial\dot{\psi}_\sigma} \right) = \boldsymbol{\nabla}\cdot\left(\delta\psi_\sigma\frac{\partial\mathcal{L}}{\partial\dot{\psi}_\sigma} \right) = 0 \tag{8.11}$$

So, if we set $\delta S = 0$, this tells us that the first quantity in brackets must vanish if $\delta\psi_\sigma(\mathbf{x}, t)$ has an arbitrary profile; we thus get the Euler-Lagrange equations:

$$\frac{\partial\mathcal{L}}{\partial\psi_\sigma(\mathbf{x}, t)} - \frac{\partial}{\partial t}\frac{\partial\mathcal{L}}{\partial\dot{\psi}_\sigma(\mathbf{x}, t)} - \boldsymbol{\nabla}\cdot\frac{\partial\mathcal{L}}{\partial\boldsymbol{\nabla}\psi_\sigma(\mathbf{x}, t)} = 0$$

$$\frac{\partial\mathcal{L}}{\partial\psi^{\dagger\sigma}(\mathbf{x}, t)} - \frac{\partial}{\partial t}\frac{\partial\mathcal{L}}{\partial\dot{\psi}^{\dagger\sigma}(\mathbf{x}, t)} - \boldsymbol{\nabla}\cdot\frac{\partial\mathcal{L}}{\partial\boldsymbol{\nabla}\psi^{\dagger\sigma}(\mathbf{x}, t)} = 0 \tag{8.12}$$

In "normal" classical mechanics, we don't have the third term, but in our case we do have a spatial dependence. A note: one can see that this already lifts nicely to relativistic physics. Another note: potentials can cause some complication, here.

We can get the field equations from the EL-equations, and we have already derived the commutator. There is still the issue of operator ordering. In this non-relativistic theory, it is the only place that we see it. But in general this is something one must keep in mind.

Other than recovering what we previously had, we actually get more! This is in the context of classical field theory, and it has to do with symmetries.

8.3   Symmetry and Noether's Theorem

The kinds of symmetry we are interested in are where there is some infinitesimal transformation (i.e. continuous symmetries). For example a rotation, or translation (an example of one without an infinitesimal transformation would be parity).

We take our field variable, and we transform it to some other field variable:

$$
\psi_\sigma(\mathbf{x}, t) \rightarrow \tilde{\psi}_\sigma(\mathbf{x}, t) = \psi_\sigma(\mathbf{x}, t) + \delta\psi_\sigma(\mathbf{x}, t)
$$
$$
\psi^{\dagger\sigma}(\mathbf{x}, t) \rightarrow \tilde{\psi}^{\dagger\sigma}(\mathbf{x}, t) = \psi^{\dagger\sigma}(\mathbf{x}, t) + \delta\psi^{\dagger\sigma}(\mathbf{x}, t)
\tag{8.13}
$$

Where we can take the transformation to be linear as we consider infinitesimal transformations (higher orders negligeble). We can now consider how the Lagrangian transforms under this. We say that this is a *symmetry* (this is our definition) if $\delta\mathcal{L}$ can be organized (algebraically; without looking at the equations of motion) in the following way:

$$
\delta\mathcal{L} = \frac{\partial}{\partial t} R(\mathbf{x}, t) + \boldsymbol{\nabla} \cdot \mathbf{J}(\mathbf{x}, t)
\tag{8.14}
$$

If things drop off at infinity, then this is a way of saying that the action doesn't change. Some textbooks require the Lagrangian to be invariant; we do not enforce this constraint here.

Now, having defined a symmetry, we can invoke the equations of motion and see what more we can learn. If we assume that the equation of motion is obeyed, then we have that:

$$
\delta\mathcal{L} = \frac{\partial}{\partial t}\left(\delta\psi_\sigma \frac{\partial\mathcal{L}}{\partial\dot{\psi}_\sigma}\right) + \boldsymbol{\nabla} \cdot \left(\delta\psi_\sigma \frac{\partial\mathcal{L}}{\partial\boldsymbol{\nabla}\psi_\sigma}\right) + (\psi_\sigma \iff \psi^{\dagger\sigma})
\tag{8.15}
$$

So we now have two equations for $\delta\mathcal{L}$; their difference should then be zero:

$$
0 = \frac{\partial}{\partial t}\left[\delta\psi_\sigma(\vec{x}, t)\frac{\partial\mathcal{L}}{\partial\dot{\psi}_\sigma(\mathbf{x}, t)} + \delta\psi^{\dagger\sigma}(\vec{x}, t)\frac{\partial\mathcal{L}}{\partial\dot{\psi}^{\dagger\sigma}(\mathbf{x}, t)} - R(\mathbf{x}, t)\right]
$$
$$
+ \boldsymbol{\nabla} \cdot \left[\delta\psi_\sigma(\vec{x}, t)\frac{\partial\mathcal{L}}{\partial\boldsymbol{\nabla}\psi_\sigma(\mathbf{x}, t)} + \delta\psi^{\dagger\sigma}(\vec{x}, t)\frac{\partial\mathcal{L}}{\partial\boldsymbol{\nabla}\psi^{\dagger\sigma}(\mathbf{x}, t)} - \mathbf{J}(\mathbf{x}, t)\right]
\tag{8.16}
$$
$$
\implies \frac{\partial}{\partial t}\mathcal{R}(\mathbf{x}, t) + \boldsymbol{\nabla} \cdot \boldsymbol{\mathcal{J}}(\mathbf{x}, t) = 0
$$

in other words, we have a conservation law! The symmetry implies a conservation law, and a systematic way of finding the conserved quantity; this is the power of the Lagrangian. This is *Noether's Theorem*. $\mathcal{R}$ is the Noether charge density and $\boldsymbol{\mathcal{J}}$ is the Noether current density. If we now integrate, we find:

$$
\frac{\partial}{\partial t}\int_\Omega d^3x\, \mathcal{R}(\mathbf{x}, t) = -\oiint_{\partial\Omega} d\mathbf{n} \cdot \boldsymbol{\mathcal{J}}(\mathbf{x}, t)
\tag{8.17}
$$

In otehr words the change in the Noether charge is just the Noether current that flows across the boundary. We can now lift this to quantum field theories, where we will find great use of these results. Next time we will explore various examples of symmetries, and see what Noether's theorem has to say about their associated conserved quantities.

9 Symmetries & Noether's Theorem

Last time we reverted from quantum to classical field theory. We did this to cast our field theory in terms of a Lagrangian and equations of motion derived from them; this had the benefit of allowing us to use Noether's theorem, which tells us about conserved quantities and how to determine what they are. We can then lift these equations of motion back into our QFT, showing that operators in our QFT are also conserved. There are two problems in the way; the first is operator ordering issues, and the second is that products of operators at the same point requires some mathematical definition, which can be add odds with the conservation law! This can actually lead to interesting physics though; in particle physics these are known as anomalies, e.g. in pion decay. The miracle is that this fooling around with operators leads to effects that are experimentally observable. Abstract and seemingly ad-hoc mathematics have physical impact!

But we won't go there (at least for now); we will first study some examples of symmetries and Noether's Theorem. We need an example theory before we find symmetries of said theory; let's write down our favourite one:

$$\mathcal{L} = \frac{i\hbar}{2}\psi^\dagger \frac{\partial \psi}{\partial t} - \frac{i\hbar}{2}\frac{\partial \psi^\dagger}{\partial t}\psi - \frac{\hbar^2}{2m}\boldsymbol{\nabla}\psi^\dagger \cdot \boldsymbol{\nabla}\psi - \frac{\lambda}{2}(\psi^\dagger\psi)^2 \tag{9.1}$$

where we have omitted the **x**s and spin indices for brevity. The symmetrized time derivative part ensures that it is real. We also don't include chemical potential for now. But note it is useful for operator ordering issues; changing the order of terms generates extra constants, and the chemical potential can be used to absorb these ambiguities.

9.1 Phase Symmetry

One transformation that leaves $\mathcal{L}$ invariant is to multiply ψ by some phase (we can see this as each ψ is paired with $\psi^\dagger$). So we can write down an infinitesmal version of this transformation:

$$\delta\psi = i\theta\psi$$
$$\delta\psi^\dagger = -i\theta\psi^\dagger \tag{9.2}$$

we can test that this is indeed a symmetry by seeing if we can write the Lagrangian as a total derivative under it; but it is even easier than that here; just plugging it in, to linear order we see that:

$$\delta\mathcal{L} = 0 \tag{9.3}$$

i.e. the Lagrangian density doesn't change at all. We call this *phase symmetry*. Going back to our expression from last class of the Noether current and Noether charge density, we have:

$$\mathcal{R}(\mathbf{x},t) = \delta\psi\frac{\partial \mathcal{L}}{\partial \dot\psi} + \delta\psi^\dagger\frac{\partial \mathcal{L}}{\partial \dot\psi} = \hbar\theta\psi^\dagger\psi = 0 \tag{9.4}$$

We see what we would have called the number operator in the QFT popping out on the RHS; in other words, the particle number is a conserved Noether charge of some kind. We can also write:

$$\mathcal{J} = \delta\psi\frac{\partial \mathcal{L}}{\partial \boldsymbol{\nabla}\psi} + \delta\psi^\dagger\frac{\partial \mathcal{L}}{\partial(\boldsymbol{\nabla}\psi^\dagger)} = 0. \tag{9.5}$$

so we also get information about the flux of the particle number. We have the following expression for the particle current:

$$\mathcal{J} = i\theta\frac{\hbar^2}{2m}(\boldsymbol{\nabla}\psi^\dagger)\psi - \frac{i\theta\hbar^2}{2m}(\psi^\dagger\boldsymbol{\nabla}\psi) \tag{9.6}$$

If this follows a continuity equation, then so will the above multiplied by a constant; so we learn:

$$\frac{\partial}{\partial t}(\psi^\dagger \psi) + \boldsymbol{\nabla} \cdot \left(\frac{i\hbar}{2m} \boldsymbol{\nabla} \psi^\dagger \psi - \frac{i\hbar}{2m} \psi^\dagger \boldsymbol{\nabla} \psi \right) = 0 \tag{9.7}$$

We can define the density:

$$\rho(\mathbf{x}, t) = \psi^{\dagger\sigma}(\mathbf{x}, t)\psi_\sigma(\mathbf{x}, t) \tag{9.8}$$

and the current:

$$\mathbf{J}(\mathbf{x}, t) = -\frac{i\hbar}{2m} \left(\psi^{\dagger\sigma}(\mathbf{x}, t)\boldsymbol{\nabla}\psi_\sigma(\mathbf{x}, t) - \boldsymbol{\nabla}\psi^{\dagger\sigma}(\mathbf{x}, t)\psi_\sigma(\mathbf{x}, t) \right) \tag{9.9}$$

So this is our first example of a conservation law (note: this follows from Noether's Theorem, which we proved last time!); phase symmetry leads to the conservation of particle number. Note that the above example is operator ordering ambiguous, so it should lift nicely to QFT; it tells us if our theory has a phase symmetry, then:

$$\frac{\partial}{\partial t}\mathcal{N} = \frac{\partial}{\partial t}\int d^3x \rho(\mathbf{x}, t) = 0 \tag{9.10}$$

We knew this already for specific examples, but now we have a more general argument; we know it to be true for any theory with a phase symmetry. Now, what else can we do?

9.2 Translational Symmetry and the Stress Tensor

We would expect that our theory is translation invariant in both time and in space, as there are now explicit ts or xs anywhere. These should lead to conservation of energy and momentum, respectively. A time translation would be:

$$\psi(\mathbf{x}, t) \to \psi(\mathbf{x}, t + \epsilon) \approx \psi(\mathbf{x}, t) + \epsilon\frac{\partial}{\partial t}\psi(\mathbf{x}, t) \tag{9.11}$$

and a space translation would be:

$$\psi(\mathbf{x}, t) \to \psi(\mathbf{x} + \boldsymbol{\epsilon}, t) \approx \psi(\mathbf{x}, t) + \boldsymbol{\epsilon}\boldsymbol{\nabla}\psi(\mathbf{x}, t) \tag{9.12}$$

So we could then write:

$$\delta\psi = (\epsilon\frac{\partial}{\partial t} + \boldsymbol{\epsilon} \cdot \boldsymbol{\nabla})\psi$$
$$\delta\psi^\dagger = (\epsilon\frac{\partial}{\partial t} + \boldsymbol{\epsilon} \cdot \boldsymbol{\nabla})\psi^\dagger \tag{9.13}$$

since $\mathcal{L}$ does not depend on t or $\mathbf{x}$, if we make the above infinitesimal transformations, the change in the Lagrangian would look like:

$$\delta\mathcal{L} = (\epsilon\frac{\partial}{\partial t} + \boldsymbol{\epsilon} \cdot \boldsymbol{\nabla})\mathcal{L} = \frac{\partial}{\partial t}(\epsilon\mathbf{L}) + \boldsymbol{\nabla} \cdot (\boldsymbol{\epsilon}\mathcal{L}) \tag{9.14}$$

so it satisfies a symmetry criterion, and therefore there is a conserved charge; in fact there are four, one for each component of ϵ (like a conserved four-vector)! Then there are 3 conserved currents for each of these. The best way to write this is as a 4x4 matrix; let's set this up.

$$\mathcal{R}(\mathbf{x}, t) = \left(\epsilon\frac{\partial}{\partial t} + \boldsymbol{\epsilon} \cdot \boldsymbol{\nabla} \right) \psi\frac{\partial\mathcal{L}}{\partial\dot{\psi}} + \left(\epsilon\frac{\partial}{\partial t} + \boldsymbol{\epsilon} \cdot \boldsymbol{\nabla} \right) \psi^\dagger\frac{\partial\mathcal{L}}{\partial(\boldsymbol{\nabla}\psi^\dagger)} - (\epsilon\mathcal{L}) \tag{9.15}$$

and there are one of these for each of the four ϵs. The current densities are analogous:

$$\mathcal{J}(\mathbf{x}, t) = \left(\epsilon\frac{\partial}{\partial t} + \boldsymbol{\epsilon} \cdot \boldsymbol{\nabla} \right) \psi\frac{\partial\mathcal{L}}{\partial(\boldsymbol{\nabla}\psi)} + \left(\epsilon\frac{\partial}{\partial t} + \boldsymbol{\epsilon} \cdot \boldsymbol{\nabla} \right) \psi^\dagger\frac{\partial\mathcal{L}}{\partial(\boldsymbol{\nabla}\psi^\dagger)} - \epsilon\mathcal{L} \tag{9.16}$$

we now write this as a matrix, known as the energy-momentum (stress) tensor:

$$\begin{pmatrix} \mathbb{T}^{tt} & \mathbb{T}^{ta} \\ \mathbb{T}^{bt} & \mathbb{T}^{ba} \end{pmatrix} \tag{9.17}$$

where the first row is the time row, and the latter three are the space rows (and analogously for the columns). a, b range over xyz. The conservation laws for each of these quantities looks like:

$$\frac{\partial}{\partial t} \mathbb{T}^{tt} + \frac{\partial}{\partial (x^b)} \mathbb{T}^{bt} = 0 \tag{9.18}$$

$$\frac{\partial}{\partial t} \mathbb{T}^{ta} + \frac{\partial}{\partial (x^b)} \mathbb{T}^{ba} = 0. \tag{9.19}$$

where the derivatives act on the first indices, and the second label points to the symmetry we are talking about. We can then read off the components of the stress tensor:

$$\mathbb{T}^{tt} = \frac{\partial \psi}{\partial t} \frac{\partial \mathcal{L}}{\partial \dot{\psi}} + \frac{\partial \psi^\dagger}{\partial (\dot{\psi}^\dagger)} \frac{\partial \mathcal{L}}{\partial (\dot{\psi}^\dagger)} - \mathcal{L} = \mathcal{H} = \frac{\hbar^2}{2m} \boldsymbol{\nabla} \psi^\dagger \cdot \boldsymbol{\nabla} \psi + \frac{\lambda}{2} (\psi^\dagger \psi)^2 \tag{9.20}$$

we recognize this as the Hamiltonian density. So what is conserved here is the integral of this over space, i.e. the total energy! The other components are:

$$\mathbb{T}^{ta} = \frac{\partial \psi}{\partial (x^a)} \frac{\partial \mathcal{L}}{\partial \dot{\psi}} + \frac{\partial \psi^\dagger}{\partial (x^a)} \frac{\partial \mathcal{L}}{\partial (\dot{\psi}^\dagger)} \tag{9.21}$$

note we've stripped off the ϵ as it is just an overall factor. We also don't include the $-\mathcal{L}$ term in the above. This we can recognize as the momentum density, whose integral is the spatial momentum; which is conserved. You can get the three space components by varying $a = x, y, z$, and each of these are conserved.

Note that writing the stress tensor does not do anything except save us some writing. In conclusion, we have the conserved quantities:

$$U = \int d^3x \, \mathbb{T}^{tt}(\mathbf{x}, t) \tag{9.22}$$

$$P^a = \int d^3x \, \mathbb{T}^{ta}(\mathbf{x}, t) \tag{9.23}$$

Does this lift to the quantum field theory? The momentum is easy because it turns out to be quadratic. Just looking at $\mathbb{T}^{ta}$, we can see that it only depends on the first terms of the Hamiltonian; it doesn't care about the interactions. Any interaction that has space translation invariance will have the same expression for the momentum density and the moementum itself. Moreover, it is the same as the number operator case where there is no operator ordering ambiguity.

However, there is an ambiguity for the $\mathbb{T}^{tt}$/energy term. This is more or less an ad-hoc procedure where we guess ordering until it works. Luckily we don't encounter this issue very much, and in this case we already know what the ordering is. So we know how to deal with it already.

9.3 Galilean Relativity

A teaser for next day: translation symmetry in space and time give us a tensor. There are some more symmetries, though; in a sense symmetries of a generic theory (symmetries that lots of theories share), which go in the direction of relativity. Even in non-relativistic physics, there are concepts of relativity. One example is Galilean symmetry/relativity (experiments in a moving car should yield the same results as that done in a stationary lab). On a primitive level, we can go back to Newtonian mechanics:

$$m\ddot{\mathbf{x}} = \mathbf{0} \tag{9.24}$$

and this equation has a Galilean symmetry. If we replace $\mathbf{x}(t)$ with $\tilde{\mathbf{x}}(t) = \mathbf{x}(t) + \mathbf{v}t$, we should still agree that Newton's second law holds. And they do, because the second derivative kills the $\mathbf{v}t$ term. How does one lift this to quantum mechanics? Well, we have a Schrodinger equation:

$$\left(i\hbar \frac{\partial}{\partial t} + \frac{\hbar^2 \mathbf{\nabla}^2}{2m} \right) \psi(\mathbf{x}, t) = 0 \tag{9.25}$$

How do we do a Galilean transformation here? Inside the wavefunction, let's try changing our coordinate; $\psi(\mathbf{x}, t) \to \psi(\mathbf{x} + \mathbf{v}t, t)$. But then this doesn't obey the original SE... we have to modify it:

$$\left(i\hbar \frac{\partial}{\partial t} - i\mathbf{v} \cdot \mathbf{\nabla} + \frac{\hbar^2 \mathbf{\nabla}^2}{2m} \right) \psi(\mathbf{x} + \mathbf{v}t, t) = 0. \tag{9.26}$$

This looks strange (we would want the new wavefunction to obey the same equation, after all). So what if we introduce a phase factor, i.e. consider:

$$\psi(\mathbf{x}, t) \to e^{i \frac{m\mathbf{v} \cdot \mathbf{x}}{\hbar}} \psi(\mathbf{x} + \mathbf{v}t, t) \tag{9.27}$$

But then we have:

$$\left(i\hbar \frac{\partial}{\partial t} + \frac{\hbar^2 \mathbf{\nabla}^2}{2m} + \frac{m\mathbf{v}^2}{2} \right) e^{i \frac{m\mathbf{v} \cdot \mathbf{x}}{\hbar}} \psi(\mathbf{x} + \mathbf{v}t, t) = 0 \tag{9.28}$$

(look in the notes for the correct formula...) so still our transformed wavefunction doesn't obey the SE. But it does look a bit closer, at the very least. So we have another redefinition:

$$\left(i\hbar \frac{\partial}{\partial t} + \frac{\hbar^2 \mathbf{\nabla}^2}{2m} \right) e^{i \frac{m\mathbf{v} \cdot \mathbf{x}}{\hbar} - i \frac{m\mathbf{v}^2}{2} t} \psi(\mathbf{x} + \mathbf{v}t, t) = 0. \tag{9.29}$$

There isn't a great intuition for why this looks so complicated... but in any case, this is it, and we can lift this to our quantum field equation (as part of our field equation looks exactly like the SE), and then for the interaction term there are no explicit $\mathbf{x}$s or ts and the whole theory is phase invariant, so these things cancel. Next day we will write down the infinitesimal transformations for this case, show that $\delta \mathcal{L}$ obeys the required relation, and write down a conservation law (but the quantity that is conserved is not so clear in this case; it will be derived quantity).

10   Galilean Symmetry

Last time we ended up the middle of a discussion of Galilean symmetry. We observed there that if $\psi(\mathbf{x}, t)$ was a solution of the Schrodinger equation, we could write down another solution:

$$e^{-\frac{i}{\hbar}\frac{mv^2}{2}t - i\frac{m}{\hbar}\mathbf{v}\cdot\mathbf{x}}\psi(\mathbf{x} + \mathbf{v}t, t). \tag{10.1}$$

This is a Galilean boost. Galilean symmetry is present in many classical mechanics systems. Landau and Lifschiz mention it in their textbook, where they argue that Galilean symmetry is the reason for why which $T = \frac{1}{2}mv^2$ (rather than any other power of v; see https://physics.stackexchange.com/questions/535/why-does-kinetic-energy-increase-quadratically-not-linearly-with-speed/14752#14752 for a very nice argument that uses Galilean symmetry to show this). This seems to be a symmetry of our equation of motion; let us now test if it is a symmetry of our full theory. We showed that it was a symmetry for the free theory; and it should also be a symmetry for our interacting theory of the type we have been discussing. If we either have hyper-local interactions or interactions that only depend on relative distance, they should have this symmetry.

10.1   Verifying the Symmetry

To test this on the level of Lagrangians, we first write down the infinitesimal version of the transformation. We let the velocity be infinitesimal and taylor expand to first order, and then strip off $\mathbf{v}$:

$$\delta\psi = t\boldsymbol{\nabla}^a\psi(\mathbf{x}, t) - i\frac{m}{\hbar}x^a\psi(\mathbf{x}, t) \tag{10.2}$$

And the $\psi^\dagger$ version of this is just replacing everything with its complex conjugate:

$$\delta\psi^\dagger = t\boldsymbol{\nabla}^a\psi^\dagger(\mathbf{x}, t) + i\frac{m}{2}x^a\psi^\dagger(\mathbf{x}, t) \tag{10.3}$$

To confirm that this is a symmetry, we calculate the first variation of the Lagrangian density, and use algebra to see if we can express it as in Eq. (8.14). In fact we can:

$$\delta\mathcal{L} = \boldsymbol{\nabla}^a(t\mathcal{L}(\mathbf{x}, t)) \tag{10.4}$$

Let us go through this calculation for the free-field Lagrangian, looking at it term-by-term.

$$
\begin{aligned}
\delta(i\hbar\psi^\dagger\dot{\psi}) &= i\hbar\delta\psi^\dagger\dot{\psi} + i\hbar\psi^\dagger\frac{\partial}{\partial t}(\delta\psi) \\
&= i\hbar(t\boldsymbol{\nabla}^a\psi^\dagger + i\frac{m}{2x}x^a\psi^\dagger)\dot{\psi} + i\hbar\psi^\dagger(t\boldsymbol{\nabla}^a - i\frac{m}{2}x^a)\psi + i\hbar\psi^\dagger\boldsymbol{\nabla}^a\psi \\
&= i\hbar t\boldsymbol{\nabla}^a(\psi^\dagger\dot{\psi}) + i\hbar\psi^\dagger\boldsymbol{\nabla}^a\psi \\
&\quad + \boldsymbol{\nabla}^a(ti\hbar\psi^\dagger\dot{\psi}) + i\hbar\psi^\dagger\boldsymbol{\nabla}^a\psi
\end{aligned}
\tag{10.5}
$$

We have two terms; but the second term will be cancelled out by the next term in the Lagrangian density. It comes from the fact that $\delta\psi$ and $\delta\psi^\dagger$ have gradients $\boldsymbol{\nabla}^a$ acting on them and these do not commute with the x^as. So cancelling the extra bits, we end up with:

$$\delta\mathcal{L} = \boldsymbol{\nabla}^a(t\mathcal{L}(\mathbf{x}, t)) \tag{10.6}$$

We could also argue that the extra terms cancel out from the phase invariance. This demonstrates that this indeed is a symmetry from our technical definition. We could have expected this of course; we have a time dependent translation $\mathbf{x} \to \mathbf{x} + \mathbf{x}t$, and we had a phase factor out front to cancel out everything.

10.2 Noether's Theorem and Galilean Symmetry

The Noether charge density is given by:

$$B^a = t\mathbb{T}^{ta} + mx^a\rho \tag{10.7}$$

i.e. time times the noether charge density for spatial translation (momentum/the stress tensor!) plus the Noether charge for phase (density, where $\rho = \psi^\dagger\psi$). So this is the charge density conserved under Galilean boosts, and there are three of them corresponding to three directions. We also have a current density:

$$\mathcal{B}^{ab} = t\mathbb{T}^{ba} + mx^a J^b \tag{10.8}$$

again, these are just components of the stress tensor plus the particle current. We don't actually derive much new here. We recall that translation symmetry tells us:

$$\frac{\partial}{\partial t}\mathbb{T}^{ta} + \nabla_b\mathbb{T}^{ba} = 0 \tag{10.9}$$

and phase symmetry tells us:

$$\frac{\partial}{\partial t}\rho + \nabla_b J^b = 0 \tag{10.10}$$

And the above two together almost imply:

$$\frac{\partial}{\partial t}B^a + \nabla_b\mathcal{B}^{ab} = 0 \tag{10.11}$$

But not quite, because the $\frac{\partial}{\partial t}$ hits the t and the ∇ hits the x. We do have:

$$\mathbb{T}^{ta} + mJ^a = 0. \tag{10.12}$$

so this tells us that the momentum density is the mass times the particle current density; a very intuitive result, but one required by Galilean invariance. This is at the level of densities. What does it mean on a deeper sense? We can look at the conservation laws for Galilean symmetry. The following is a constant in time:

$$t\int d^3x\,\mathbb{T}^{ta} + \int d^3x\,mx^a\rho = \text{Constant} \tag{10.13}$$

where we have integrated the Noether charge density. We can think of the second term as the average position of the center of mass. Or, we can divide the entire equation by it and rewrite it as follows:

$$\int d^3x\,x^a\rho(\mathbf{x}, t) = -\frac{1}{m}\left[\int d^3x\,\mathbb{T}^{ta}(\mathbf{x}, t)\right]t + \langle x^a(0)\rangle \tag{10.14}$$

i.e. the average position of the COM is the initial position plus the average velocity times time... it simply means that the COM of the system moves at a constant speed. We already expected this, but now we have a concrete criteria to test for Galilean invariance!

10.3 Benefits of Lifting Galilean Symmetry to QFT

Can we learn something more practical from this? Let's look at something really cool; let's see what these symmetries tell us when we lift them to QFT. We won't go through the technicalities of this process (if it doesn't work, the problem tends to be really hard, or unsolvable anyway). We expect Galilean symmetry to survive this lifting[18] Let us see what the benefits of this symmetry are.

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[18]Of course we are assuming that the system is non-relativistic; relativistic corrections are not Galilean invariant. In the relativistic case we would look at Lorentz invariance, instead.

Perhaps we want to calculate a correlation function like the following:

$$W(\mathbf{x_1}, t_1, \mathbf{x_2}, t_2) = \langle \mathcal{O} | \psi(\mathbf{x_1}, t_1) \psi^\dagger(\mathbf{x_2}, t_2) | \mathcal{O} \rangle. \tag{10.15}$$

Let's assume we have Galilean symmetry at the quantum level. Then presumably the ground state is Galilean invariant; even more primitively, it is translation invariant. Since it's an eigenstate of the Hamiltonian, time translation just gives rise to a phase which can be removed by shifting the Hamiltonian by a constant, anyway. Explicitly, we have the equation:

$$|\mathcal{O}\rangle = e^{-\frac{i}{\hbar} H t} |\mathcal{O}\rangle \tag{10.16}$$

Space translation should look very similar, but instead of the Hamiltonian, we would put the momentum operator and space in the imaginary exponential:

$$|\mathcal{O}\rangle = e^{-\frac{i}{\hbar} \mathbf{P} \cdot \hat{a}} |\mathcal{O}\rangle \tag{10.17}$$

A homogenous system should have these symmetries. Such translations should be generated by unitary operations, for which the ground state is invariant under. For Galilean symmetry, the situation is more complicated, but we may think that there is similarly a unitary operator associated with a Galilean boost. Let us assume there is one without knowing its form; we know it should have the action:

$$|\mathcal{O}\rangle = U|\mathcal{O}\rangle \tag{10.18}$$

$$U^\dagger \psi(\mathbf{x}, t) U = e^{-\frac{i}{\hbar} \frac{mv^2}{2} t - i \frac{m}{\hbar} \mathbf{v} \cdot \mathbf{x}} \psi(\mathbf{x} + \mathbf{v} t, t) \tag{10.19}$$

in a system with this symmetry, U with these properties should exist. What do they tell us about the correlation functions? If we generate the time translation, and plug in the translated object into the correlation function (i.e. put in):

$$\langle \mathcal{O} | U^\dagger \psi(\mathbf{x_1}, t_1) U U^\dagger \psi^\dagger(\mathbf{x_2}, t_2) U | \mathcal{O} \rangle. \tag{10.20}$$

where U is the time translation, we learn that the correlation *is only a function of* $t_1 - t_2$. This is because the e^{iH} in the middle is only a function of $t_1 - t_2$ and the e^{iH} at the end doesn't depend on anything. The only dependence is on the difference! The exact same argument follows for positions, where one can prove the correlation only depends on $\mathbf{x_1} - \mathbf{x_2}$. In other words, symmetry has reduced the number of things that W can depend on from 8 to 4.

Rotation symmetry is not discussed yet, but it is in HW3! But it is really the same story. We will find that the ground state is rotation invariant, and so we can generate a rotation, and so we learn that W can only be a function of a rotationally invariant combination of $\mathbf{x_1}, \mathbf{x_2}$, i.e. W only depends on $(\mathbf{x_1} - \mathbf{x_2})^2, t_2 - t_1$ (only on the magnitude of the vector!!) So there are only two variables that the correlation functions can depend on! And we haven't even used Galilean symmetry yet! If we do apply it, we find[19]:

$$W((\mathbf{x_1} - \mathbf{x_2})^2, t_1 - t_2) = e^{\frac{im}{\hbar} \frac{(\mathbf{x_1} - \mathbf{x_2})^2}{(2(t_1 - t_2))}} f(t_1 - t_2) \tag{10.21}$$

so symmetry has reduced our problem appreciably. We can go no further with symmetry; for the free field theory f is a constant. Interactions seem to only modify this by multiplying by some function of time. We should feel powerful at this point; for Galilean symmetry we only need to find some f that depends on time. In conclusion: symmetry simplifies a lot! We can write objects such as correlation functions to depend on only one variable, for example.

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[19]Note; there might be some singular behaviour if the positions/times are equal; some more work might be necessary in this case

10.4 Scale Invariance

The symmetries discussed so far have been in some sense "generic" in that they apply to many systems; but there are a few more interesting symmetries that occur only in very special systems. This perhaps makes them less interesting, but they have been studied in depth over the last 20 years or so, especially in systems such as cold atoms. These are symmetries relating to changing scales; scale transformations, and something related known as conformal transformations (also known as Special Schrodinger transformations). We will discuss them briefly here as they are likely to come up when one studies CM physics, cold atom physics etc. . .

If we put the interaction to zero and look at the free field equation, we have an equation with a scale symmetry:

$$\left(i\hbar \frac{\partial}{\partial t} + \frac{\hbar^2 \nabla^2}{2m} \right) \psi(\mathbf{x}, t) = 0 \tag{10.22}$$

At first this might seem strange, as there appear to be dimensionful quantities like $\hbar, m$ in the above. But nevertheless, if $\psi(\mathbf{x}, t)$ is a solution, then so is

$$\Lambda^{d/2} \psi(\Lambda x, \Lambda^2 t) \tag{10.23}$$

where the Λ is the scaling factor, and the $\Lambda^{d/2}$ appears so that the transformed solution is still normalized. Going back to the example with the correlation functions, the scale invariance determines $f(t_1 - t_2) = (t_1 - t_2)^d$ times some overall constant and so up to a constant we have determined the correlation function completely! The two-point correlation functions have been completely solved for us. One might wonder; if we had interactions here, could we use this to solve interacting theories? The answer is yes, and this is what makes this very interesting. What makes this difficult is that if we add interactions, adding linear terms as operators in general messes up the scale symmetry; there are only few examples we can add. The Coulomb interaction is one that might seem scale-invariant (it doesn't seem to contain any dimensionful quantities at first) but when one tries to add it in, one finds that indeed scale invariance is violated. In the simple case we consider above, we are safe with no interactions. With interactions, if the coupling constant values are tuned very precisely (i.e. to "fixed points") then we do have scale invariance.

Note that since in the free field equation the equation is symmetric, we might expect the same of the action. Indeed, one can confirm this by looking at the infinitesimal transformation:

$$\delta \psi = \left(\mathbf{x} \cdot \nabla - 2t \frac{\partial}{\partial t} + \frac{d}{2} \right) \psi \tag{10.24}$$

and if we construct the Noether charge and current density, because the above terms are like translations in space in time, algebraically what comes out is something that contains the stress tensor (again).

A philosophical question: Why is lifting classical field theory to QFT so successful? Because for many situations we have weak coupling (e.g. E&M, gravity, weak interactions) and so the structure is preserved when we take the classical behaviour to quantum (and of course this is why the classical approaches to these topics are quite accurate, and we study it in undergrad).

11 Scale and Conformal Symmetry

11.1 Non-Relativstic Conformal Field Theory

We've been studying the "space-time" symmetries of our non-relativistic QFT. Though space-time is a bit of a stretch, as the non-relativistic theories don't really have a grounding in geometry (though literature does exist on "non-relativistic" general relativity, casting Galilean transformations as isometries of non-relativistic spacetime. But this is a bit contrived...) In the real world, non-relativistic symmetries are just symmetries for things that are moving slowly (we should be able to derive them from SR by taking the non-relativistic limit)! This is a more useful point of view; in the world we live in, almost everything we would analyze with the kind of field theory we have been discussing is made of particles which have a relativistic foundation behind them. With that said, let us wrap up this discussion with a short introduction to non-relativistic CFT.

   This isn't a particularly new subject; it's origin dates back to the 1970s. That said it has only been developed quite recently in the context of cold atoms, and critical behaviour (in the sense of a phase transition). The connection is that (in a rough sense); when something undergoes a phase transition (e.g. a paramagnet becoming ferromagnetic) there are fluctuations on all length scales; such that if we change the resolution at which we view the magnet, it does not look much different at different scales. This tends to characterize a phase transition; you get something roughly scale-invaraint. We started about the mathematical machinery behind scale invariance last day. If we have a theory that is symmetric under scale transformation, it looks the same under different scales!

   We've written down a free field theory (so we actually have an example of a theory for which this holds) that has the property of scale invariance:

$$\left(i\hbar \frac{\partial}{\partial t} + \frac{\hbar^2 \boldsymbol{\nabla}^2}{2m} \right) \psi(\mathbf{x}, t) = 0 \tag{11.1}$$

so (for example) the degenerate Fermi gas we discussed has scale symmetry. A real Fermi gas may have interactions that break this scale invariance, but still perhaps in certain regimes it may look the same (e.g. a piece of copper looks the same $1\text{m}, 10\text{m}, 100\text{m}$ away; it is scale invariant at large distances/in the infrared, even though this may break at smaller scales). We identified the infinitesimal transformation:

$$\delta \psi = \left(\mathbf{x} \cdot \boldsymbol{\nabla} + 2t \frac{\mathrm{d}}{\mathrm{d}t} + \frac{d}{2} \right) \psi(\mathbf{x}, t) \tag{11.2}$$

and under this transformation, the Lagrangian density of this theory transforms (in what is known as the *scale transformation*) as:

$$\delta \mathcal{L} = \boldsymbol{\nabla} \cdot (\mathbf{x} \mathcal{L}) + \frac{\mathrm{d}}{\mathrm{d}t} (2t\mathcal{L}) \tag{11.3}$$

so it can be written in terms of total derivatives; it is therefore a symmetry! We can now turn the crank and write down the Noether charges and currents. But before that, we note that the above is accompanied often with a *conformal transformation* (also known as a Special Schrodinger transformation/symmetry), which reads:

$$\delta \psi = \left(t^2 \frac{\mathrm{d}}{\mathrm{d}t} + t\mathbf{x} \cdot \boldsymbol{\nabla} - i \frac{m}{\hbar} \frac{\mathbf{x}^2}{2} + \frac{d}{2} t \right) \psi \tag{11.4}$$

here it isn't really clear where this comes from; the origin will be clearer in the relativistic case. The free field Lagrangian density transforms as:

$$\delta \mathcal{L} = \frac{\partial}{\partial t} (t\mathcal{L}) + \boldsymbol{\nabla} \cdot (t\mathbf{x} \mathcal{L}) \tag{11.5}$$

so again this is a symmetry. We will see why conformal symmetry often comes along in a theory with scale symmetry (and we will see some similar things in the assignment, where the symmetry of the stress tensor

under the spatial indices implies rotational invariance. There is also a connection with translation/number invariance implying Galilean symmetry etc.)

11.2 Noether's Theorem for Scale Symmetry

We can write down the Noether charge as:

$$D = \delta\psi\frac{\partial\mathcal{L}}{\partial\dot\psi} + \delta\psi^\dagger\frac{\partial\mathcal{L}}{\partial(\dot\psi^\dagger)} - 2t\mathcal{L} \tag{11.6}$$

and the current as:

$$\boldsymbol{\mathcal{D}} = \delta\psi\frac{\partial\mathcal{L}}{\partial(\boldsymbol{\nabla}\psi)} + \delta\psi^\dagger\frac{\partial\mathcal{L}}{\partial(\boldsymbol{\nabla}\psi^\dagger)} - \mathbf{x}\mathcal{L} \tag{11.7}$$

A handy tip if we work with fermions to deal with the potential signs: If we have a Lagrangian, and we want to compute $\delta\psi\frac{\partial\mathcal{L}}{\partial\dot\psi}$; then it's second order in the ψs so it commutes with everything, until we get to ψ and we replace it. We can now go look up the Lagrangian and see what we get:

$$D = t2\mathbb{T}^{tt} + x_b\mathbb{T}^{tb} \tag{11.8}$$

The ath Noether charge is:

$$\boldsymbol{\mathcal{D}}^a = 2t\mathbb{T}^{at} + x_b\mathbb{T}^{ab} - \frac{d}{2}\frac{\hbar^2}{2m}\boldsymbol{\nabla}^a(\psi^\dagger\psi) \tag{11.9}$$

where we note the last $d/2$ piece gives us a bit of a mess (unlike the rest of the terms which come from the space/time transformations); it's like a phase transformation, but it's missing an i; so its really something else that we have not seen yet. This is indeed conserved:

$$\frac{\partial}{\partial t}D + \boldsymbol{\nabla}_a\boldsymbol{\mathcal{D}}^a = 0 \tag{11.10}$$

by Noether's theorem. If we assume translation invariance:

$$\partial_t\mathbb{T}^{tt} + \boldsymbol{\nabla}_a\mathbb{T}^{at} = 0$$
$$\partial_t\mathbb{T}^{tb} + \boldsymbol{\nabla}_a\mathbb{T}^{ab} = 0 \tag{11.11}$$

we then obtain:

$$2\mathbb{T}^{tt} + \sum_a\mathbb{T}^{aa} - \frac{d}{t}\frac{\hbar^2}{2m}\boldsymbol{\nabla}^2\rho = 0. \tag{11.12}$$

where $\rho = \psi^\dagger\psi$. This almost looks like the trace of the stress tensor (up to the junk at the end, and the factor of 2 in the front), so this is *almost* saying that scale invariance implies a stress tensor.

11.3 Improving the Stress Tensor

The stress tensor has an ambiguity; in fact, any Noether current does. Way at the beginning when we test for a symmetry, we try to see if $\delta\mathcal{L}$ can be written as a total derivative using algebra. This is an ambiguous process if there exist quantities in the variation which can be added to the two derivative terms and cancel. This ambiguity then carries through to the whole calculation, to the point where there may be additional things you can add to the current... so then which current is which? There isn't a great answer for non-relativistic physics; in the relativistic case, we have gravity and so we can pick the current that couples to gravity. In NR we have no right to do this, but we can pick a different stress tensor that is still consistent with the physics. The improvement proceeds as follows. We add something to our previous stress tensor. If the previous stress tensor was symmetric it should be symmetric, and it should

obey the same conservation laws, and it should be written as some derivative. The candidate that satisfies all three of these qualifications are:

$$\tilde{\mathbb{T}}^{ab} = \mathbb{T}^{ab} - \frac{d}{d-1}\left(\delta^{ab}\boldsymbol{\nabla}^2 - \boldsymbol{\nabla}^a\boldsymbol{\nabla}^b\right)(\psi^\dagger\psi) \tag{11.13}$$

note that the time-time and time-space components to not change. With the above modification, note that we still have conservation, if $\mathbb{T}^{ab}$ is symmetric then so is $\tilde{\mathbb{T}}^{ab}$ etc. So indeed we have an ambiguity. Why is this improved? For this improved stress tensor, scale invariance implies:

$$2\tilde{\mathbb{T}}^{tt} + \sum_a \tilde{\mathbb{T}}^{aa} = 0 \tag{11.14}$$

We consider a homogenous/isotropic system, and take the expectation value of the above. The expectation value of the first term is the energy density, and the diagonal components of the stress tensor are pressures, which are all the same as we assume isotropy. We therefore find:

$$2U - dP = 0. \tag{11.15}$$

In other words:

$$U = \frac{d}{2}P \tag{11.16}$$

which is a prediction of scale invariance! We obtain an equation of state of a scale invariant theory. We computed this above for the degenerate Fermi gas; we can now go back and check that this indeed holds for $d = 3$! Also, note that Eq. (11.14) can now be used as a criterion for scale invariance. Also also; note that if we add interaction, all bets are off!

Note: the SE may look like a scale invariant problem, but the SE always has a bound state with a binding energy; how can we have a binding energy for a theory with no dimensionful paramaters? The problem is we have a singular potential (e.g. a dirac delta) so we need some extra boundary conditions for the behaviour of the wavefunction at these points. This turns out to be equivalent to specifying the binding energy, but quantifying this violates scale invaraince. Finding scale invariant theories is hard. Though, one can usually tune parameters/coupling constants to get there; unfortunately this usually appears in places where calculations are intractable; either coupling constants are too large to use perturbation theory, or we set them all to zero and we have a trivial/Gaussian fixed point). Other fixed points have been shown to exist, e.g. the unitarity point where a state just gains some binding symmetry, or the point where phase symmetry gets broken etc.

11.4   Noether's Theorem for "Conformal" Symmetry

There's certainly less intuititon for the conformal symmetry... it's not really a conformal symmetry in that conformal symmetries preserve angles. The transformation in Eq. (11.4) doesn't really satisfy this. But, it is of interest, so we can write down the Noether current density (and let's assume we've done the whole song and dance of improving the stress tensor so as to get rid of the junk terms...) - actually we did not have time to do this. But let's see how we can get conformal symmetry:

Let's assume we have translation symmetry in time and space as in Eq. (11.11) and also have phase symmetry:

$$\frac{\partial\rho}{\partial t} + \boldsymbol{\nabla}\cdot\mathbf{J} = 0 \tag{11.17}$$

and Galilean symmetry:

$$\mathbb{T}^{ta} = -mJ^a \tag{11.18}$$

and scale invariance:

$$2\tilde{\mathbb{T}}^{tt} + \sum_a \tilde{\mathbb{T}}^{aa} = 0 \tag{11.19}$$

These symmetries imply the conformal symmetry (intriguingly, we don't need rotational symmetry!).

Note that the effects of these symmetries constrain various (2/3) point correlation functions E.g. deuteron has a low binding energy relative to the masses, almost a critical system with conformal symmetry, one can do calculations of matrix elements and compare this to experimental scattering data etc.

Next class: So far, there haven't been unintuitive leaps to what we have done. Everything has been methodical; just doing mathematics (doing nothing)! We've just been playing with things; there haven't been challenges to ou; physical intuition. To go to the next level, we need to understand special relativity to some level, and we will construct these relativistic systems. We will do this by couching SR as a type of symmetry (a true symmetry of spacetime). We can then build field theories that have these symmetries. We have the technical tools at our disposal, but we do require a bit of a leap. What is different? What we've learnt about QM is still QM, we've just placed it in another context. But for example take the pi-meson generator at TRIUMF. To describe this, we go from a state with a few nuclei to a state with a whole bunch of pions. We need a QM theory where the number of particles is not fixed. QFT is a natural place for this. $E = mc^2$ tells you that if you have the right energies and quantum numbers, you can create particles. The FT description is ideal for this; we can write down dynamical processes that do these things. This is why QFT is often called the natural marriage of QM and special relativity. But there is another deep reason, namely causality. What happens is if we ask "what if we did single particle QM?" with position $\mathbf{x}$, momentum $\mathbf{p}$, with commutation relations $[x_i, p_j] = i\hbar\delta_{ij}$, and the relativistic Hamiltonian:

$$H = \sqrt{\mathbf{p}^2 c^2 + m^2 c^4}. \tag{11.20}$$

Things go very wrong! When we calculate wavepacket spreads, there is a nonzero probability that the particle could have travelled superliminally (because the wavepacket is a superposition of a huge range of possible momenta). We can show that sometimes we will detect it outside of the light cone. We violate some very fundamental principles. Causality is not a philosophical principle, but an experimental fact that we do not see particles that go faster than the speed of light[20]. So we would discard this theory because it doesn't describe nature (that said if we do find tachyons, maybe we dig this back up).

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[20]Throwback to the experiment that detected "superliminal neutrinos" because of a loose cabel

12 Relativistic Quantum Mechanics, Space-Time Coordinates

12.1 The Naive setup

Let us discuss for a few minutes a subject that "doesn't exist" - we will see why through the discussion! Think of a quantum-mechanical particle with position $\mathbf{X}$, momentum $\mathbf{P}$, such that these follow the canonical commutation relations $[X_i, P_j] = i\hbar\delta_{ij}$. We will give this particle a relativistic expression for the energy:

$$H = \sqrt{P^2c^2 + m^2c^4} \tag{12.1}$$

We can now ask about this system. If we were naively trying to combine QM with SR, this would be the first kind of thing we would try. In this kind of system, we can discuss eigenstates of position and momentum (although of course these are not eigenstates in the true sense; they are not normalized):

$$H|p\rangle = \sqrt{p^2c^2 + m^2c^4}|p\rangle \tag{12.2}$$

the question of how this eigenstate evolves in time is a simple one; its a stationary state of the system, so only evolves according to some phase:

$$|p(t)\rangle = e^{-\frac{i}{\hbar}Ht}|p\rangle = e^{-\frac{i}{\hbar}\sqrt{p^2c^2+m^2c^4}t}|p\rangle. \tag{12.3}$$

Of course nothing interesting happens to eigenstates of the Hamiltonian with time. The more interesting question is what would happen to an eigenstate of position? We can characterize this as the matrix element $brax_f e^{-\frac{i}{\hbar}Ht}|x_i\rangle$ which is the probability (transition) amplitude that a particle starts at position x_i and some time later is found at position x_f. From the above, we can calculate this as:

$$\langle x_f|e^{-\frac{i}{\hbar}Ht}|x_i\rangle = \int d^3p \langle x_f|p\rangle e^{-\frac{i}{\hbar}\sqrt{p^2c^2+m^2c^4}t}\langle p|x_i\rangle \tag{12.4}$$

Where we have inserted the resolution of the identity $\int d^3p |p\rangle\langle p|$. The inner products of momentum and position we are familiar as:

$$\langle p|x\rangle = \frac{e^{-i\frac{\mathbf{p}}{\hbar}\cdot\mathbf{x}}}{(2\pi\hbar)^{3/2}} \tag{12.5}$$

So then:

$$\langle x_f|e^{-\frac{i}{\hbar}Ht}|x_i\rangle = \int \frac{d^3p}{(2\pi\hbar)^3} e^{i\frac{\mathbf{p}}{\hbar}\cdot(\mathbf{x}_f-\mathbf{x}_i)-\frac{i}{\hbar}\sqrt{p^2c^2+m^2c^4}t} \tag{12.6}$$

but this final expression we derive with the standard (naive) assumptions has problems.

12.2 Problem 1 - Lack of Lorentz Invariance

For one, it is not Lorentz invariant. Why should an probability distribution change if we put the experiment on a train? Or if we put ourselves on a train and interpret the experiment? It transforms something like a charge density; the time component of a four-vector. This is because we have setup $\psi^\dagger\psi$ to transform like the time-component of a four-vector, even though we want to normalize the wavefunction and this normalization should hold between all frames.

We could fix this up to be Lorentz invariant via an easy fix up; we insert a factor of the energy downstairs:

$$\langle x_f|e^{-\frac{i}{\hbar}Ht}|x_i\rangle = \int \frac{d^3p}{(2\pi\hbar)^3} \frac{mc^2}{\sqrt{pc^2+m^2c^4}} e^{i\frac{\mathbf{p}}{\hbar}\cdot(\mathbf{x}_f-\mathbf{x}_i)-\frac{i}{\hbar}\sqrt{p^2c^2+m^2c^4}t} \tag{12.7}$$

there is nothing in our theory that tells us why we do this, but it does work (and it will be something we get from QFT). When $pc^2 \ll m^2c^4$ (i.e. in the non-relativistic limit) the factor is one so we recover

the NRQM result. Another note: inserting this factor has significant effects, where we have difficulty localizing the particle (this carries through to quantum field theory - we cannot localize a particle to a position smaller than its wavelength. But in NRQM we can do this; the wavefunction of a particle can be as sharply peaked as we need it to be).

12.3 Problem 2 - Causality Violations
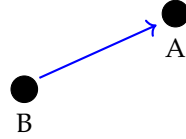
The next problem is harder to patch up.

Figure 12.1: Consider an experiment where Bob releases a particle and Alice is some distance away. Under naive relativistic QM, Alice has a non-zero chance of measuring the particle to be at her position, even if she is so far away such that light would not be able to reach her in that time from when Bob released it.

When we do the calculation for the probability of finding the particle some arbitrarily far distance away, we find that we have a non-zero probability regardless of the distance. To see this, let us return to the above transition probability. We can evaluate the integral approximately using the edge-of-the-wedge theorem. We note that $\langle x_f | e^{-\frac{i}{\hbar} H t} | x_i \rangle$ is analytic in the lower half of the complex time plane; this comes from the fact that that there the $e^{-\frac{i}{\hbar} \sqrt{p^2 c^2 + m^2 c^4} t}$ leads to an exponential supression of the integral. As time comes to the real axis, it becomes much less nice (it becomes something like a distribution). We can define the value on the real axis as the limit taken from the analytic lower half time plane. How does this tell us anything? If the theory were causal, for small t we would have to find that the integral is zero. But then we have some arc in the complex plane for which the function is zero. But if this region is in the region where the function is analytic, then by analytic continuation we could show that it would be zero everywhere.

So the last step to show is that it is indeed analytic in this region. In the lower-t plane, we can use Cauchy's integral formula to write:

$$f(t_0) = \oint \frac{dt}{2\pi i} \frac{f(t)}{t - t_0} \tag{12.8}$$

where $f(t) = \langle x_f | e^{-\frac{i}{\hbar} H t} | x_i \rangle$ is our integral of interest. The value of the integral does not depend on the contour taken; let us deform the contour so that part of the contour lies on the real axis. We can then push the t_0 up until it is on the real axis; at this point the formula still holds, and so $f(t_0)$ is an analytic function on the real axis as well. Therefore by analytic continuation, the integral is zero everywhere. Contradiction!

We therefore conclude that our theory violates causality. This would be ok if the universe actually behaved like this, but as far as we know we have never observed this; physics appears to be causal. So, this theory doesn't reflect reality, and we should throw it away.
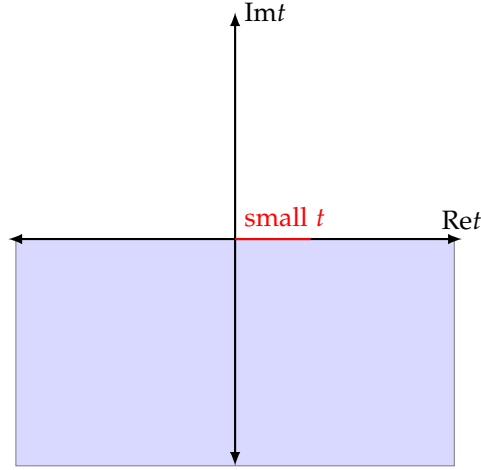
Figure 12.2: View of the complex t-plane. The integral for $\langle x_f | e^{-\frac{i}{\hbar}Ht} | x_i \rangle$ (Eq. (12.7)) is analytic in the lower-half plane due to the exponential supression in the integral. For the theory to be causal, the integral must vanish for small real t (in red). However, if this is indeed the case that the integral vanishes for an arc in $\mathbb{C}$ and is analytic on the arc, we may use analytic continuation (taylor expanding the integral along the arc where it vanishes) to conclude that the integral vanishes throughout the entire complex plane; a contradiction.
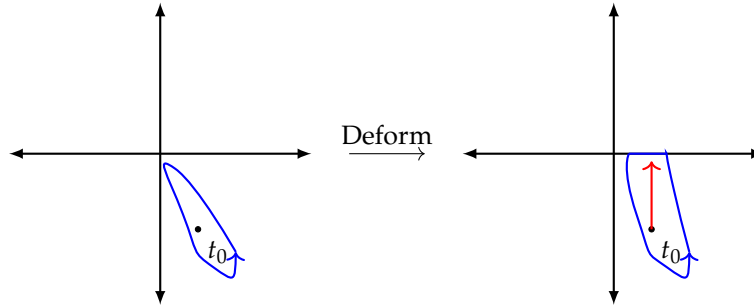
Figure 12.3: In the region where it is analytic (i.e. the lower-half plane), we can take $\langle x_f | e^{-\frac{i}{\hbar}Ht} | x_i \rangle$ at some t_0 to be defined by an arbitrary contour integral in the lower-half plane by Cauchy's Integral Formula. However, we can then deform the contour such that that part of the contour lies on the small real t region where the integral must vanish for the theory to be causal. We can then push up t_0 (take a limit) until it lies on the real axis in the small t region. But then Cauchy's integral formula still holds there, so in fact the integral is analytic on the real axis, and in particular is analytic and vanishing in the small t region; it therefore vanishes everywhere by the previous analytic continuation observation (giving us the desired contradiction).

12.4 The QFT fix-up

How does QFT fix this? First, we get the factor of $\frac{mc^2}{\sqrt{pc^2+m^2c^4}}$ to preserve lorentz invariance. But the other thing we do is we make use of the negative energy branch of the energy spectrum:

$$E = -\sqrt{p^2c^2 + m^2c^4}. \tag{12.9}$$

When we calculate the same process in QFT, something gets added! We have two amplitudes:

$$\langle x_f|e^{-\frac{i}{\hbar}Ht}|x_i\rangle = \int \frac{d^3p}{(2\pi\hbar)^3}\left[\frac{mc^2}{\sqrt{pc^2+m^2c^4}}e^{i\frac{\mathbf{p}}{\hbar}\cdot(\mathbf{x}_f-\mathbf{x}_i)-\frac{i}{\hbar}\sqrt{p^2c^2+m^2c^4}t} - \frac{mc^2}{\sqrt{pc^2+m^2c^4}}e^{i\frac{\mathbf{p}}{\hbar}\cdot(\mathbf{x}_f-\mathbf{x}_i)+\frac{i}{\hbar}\sqrt{p^2c^2+m^2c^4}t}\right] \tag{12.10}$$

i.e. we add a term that looks like time is running backwards. How do we interepret this? The first term/process has the interpretation of Bob having a particle and Alice detects it. The second term/process is as follows. When trying to detect a particle, Alice produces a particle-antiparticle pair, the antiparticle which goes the other direction and annhilates Bob's particle. We find in QFT that outside of the light cone that these two processes perfectly destructively interfere, and so causality is preserved!

An interesting note: if we repeated this analysis with a massless particle, we would find that the probability only has support on the lightcone (As we would expect; massless particles should travel on the lightcone!)

Another note: the addition of this second term destroys our analyticity argument as we now have one term that is analytic in the lower-t plane and the other is analytic in the upper-t plane. So we do not get the contradiction that the integral vanishes everywhere, as we did before.

Does this fixup fix everything? It seems to. They seem to be Lorentz invariant, and causal. This is why QFT gains the title of the natural marriage of quantum mechanics and special relativity (and string theory is known as the natural marriage of QM and GR - each are the only currently known solution to unifying these theories). Although this cannot be said of string theory yet, QFT at least seems to pass some very stringent experimental tests.

12.5 Special Relativity and Minkowski Spacetime

Once we go to a relativistic setting, it is natural to discuss space-time transformations (which are still translations, rotations and the like) which are symmetries of the space-time in which the QFT lives. We then try to write down the QFT such that these symmetries are not violated. We already know how to deal with this a little bit; we know Noether's theorem, and how to turn the crank and find conservation laws. We will therefore proceed to approach symmetries from a different point of view; as arising from the geometries of spacetime.

The space-time we consider is Minkowski spacetime; as much as we love curved spacetime, we won't consider it in this course here. It is described by coordinates:

$$\mathbf{x} = (x^1, x^2, x^3) \equiv x^a, \quad a = 1, 2, 3 \tag{12.11}$$

which describes three-dimensional infinite Euclidean space. The indices on the xs now have a position that means something. It is always up on a coordinate (or at least for the next while - x_a is meaningless, for now). We will add to this another coordinate for time to make this a space-time (but we will give it units of distance to make it symmetric):

$$x^0 \equiv ct \tag{12.12}$$

where x^0 ranges over the real line. Combining these, we obtain a four-vector:

$$x^\mu = (x^0, x^1, x^2, x^3), \quad \mu = 0, 1, 2, 3. \tag{12.13}$$

We will generally use latin letters for regular vectors, and greek letters for four-vectors (though sometimes we leave it off entirely when the object is clear from context). The cartesian coordinates are not the only coordinates possible, but they are quite convenient; symmetries are quite apparent here.

12.6 Coordinates on a space-time

If we have a general spacetime, we could think about a coordinate system; all this would be is a dictionary for the various points in the spacetime, with points x^μ, y^μ etc. There is of course a bunch of mathematical structure behind this, e.g. continuity/smoothness, this is done formally by locally mapping points (homeomorphism) to $\mathbb{R}^n$. We won't need this formality, but we will takeaway the idea of a coordinate transformation. Suppose we have a different coordinate system, with points $\tilde{x}^\mu, \tilde{y}^\mu$ etc. We could then compare dictionaries and figure out how to translate in between them. We could then define a transformation in between them: $\tilde{x}^\mu(x^\mu)$. This is a mathematical way of writing down a transformation, and the transformation inherits the "smoothness" of the coordinates (it is differentiable, invertible...). Formally, the invertiblity criterion is:

$$\det(\frac{\partial}{\partial x^\mu}\tilde{x}^\nu) \neq 0. \tag{12.14}$$

12.7 Scalar Fields

We consider a scalar field $\phi(x^\mu)$; this is a physical entity that takes a value at each point in spacetime (for example; the temperature in a room). If someone else comes along with a different set of coordinates, we should agree with the value of the scalar field; so we should agree that:

$$\phi(x) = \tilde{\phi}(\tilde{x}) \tag{12.15}$$

so a scalar field is defined in this way; with how it undergoes coordinate transformations.

We will discuss next time, but we can consider that displacement transforms like:

$$dx^\nu = \frac{\partial x^\mu}{\partial \tilde{x}^\nu} d\tilde{x}^\nu \tag{12.16}$$

So perhaps a vector field transforms in the same way:

$$A^\mu(x) = \frac{\partial x^\mu}{\partial \tilde{x}^\nu} d\tilde{A}^\nu(x) \tag{12.17}$$

13 Fields, Metrics, and Space-Time Symmetry

13.1 Review - Coordinates, Transformations, Scalar Fields

Last time, we were considering a coordinate system for our spacetime (which has requirements such as smoothness that we take as a given); we have a four-vector x^μ which represents a point in the spacetime. We then considered a different coordinate system $\tilde{x}^\mu$ and a change-of-coordinates transformation $x^\mu \to \tilde{x}^\mu(x)$ (four smooth functions in four variables). This function should be invertible, with $\tilde{x}^\mu \to x^\mu(\tilde{x})$ the inverse.

We can then discuss things that live in our spacetime, such as the scalar field. The term scalar says something about this; namely it tells us how we should translate values of the field under different coordinate transformations. We have some scalar field $\phi(x)$, and in some different set of coordinates we may have $\tilde{\phi}(\tilde{x})$ but at the same point the data should match, so:

$$\phi(x) = \tilde{\phi}(\tilde{x}) \tag{13.1}$$

13.2 Vector Fields

This is a straightforward generalization of a vector field; at each point in space we associate a vector with magnitude and direction. If temperature was an example of a scalar field, then wind would be a good example of a vector field; the wind has some magnitude and direction at each point in space. But now something to consider; when we transform between coordinates, we not only have to translate magnitudes, but also the direction of the wind. We say a (contravariant) vector field should transform like a differential:

$$dx^\mu = \frac{\partial x^\mu}{\partial \tilde{x}^\mu} d\tilde{x}^\nu \tag{13.2}$$

so:

$$A^\mu(x) = \frac{\partial x^\mu}{\partial \tilde{x}^\nu} \tilde{A}^\nu(\tilde{x}) \tag{13.3}$$

note the up and down indices here are quite important. x always has an up index. The index on the derivative is a bit interesting. We consider a transformation of the derivative:

$$\frac{\partial}{\partial x^\nu} = \frac{\partial \tilde{x}^\nu}{\partial x^\mu} \frac{\partial}{\partial \tilde{x}^\nu} \tag{13.4}$$

and we see that this is upside-down compared to the vector field transformation; so something different is happening here. We could find a vector field that transforms under this "upside-down" rule, and we would distinguish this by putting a down index on it (this would be a covariant) vector field:

$$A_\mu(x) = \frac{\partial \tilde{x}^\nu}{\partial x^\mu} \tilde{A}_\nu(\tilde{x}) \tag{13.5}$$

Note: we remind the reader that we work with Einstein summation convention, and so when we have two indices paired they are summed over. Since the partial derivative transforms covariantly, it is natural to think of it as an object as a down index:

$$\partial_\mu = \frac{\partial}{\partial x^\mu}, \quad \tilde{\partial}_\mu = \frac{\partial}{\partial \tilde{x}^\mu} \tag{13.6}$$

13.3 Tensor Fields

Tensor fields are a generalization of vector fields; they are things with more than one index. One example is the inertia tensor with two indices takes the form of a 3x3 matrix in 3-dimensional space. The relativistic description of an EM field is another. We can lift our transformations of vectors into tensors in the most obvious way:

$$\mathbb{T}^{\mu_1 \cdots \mu_m}{}_{\nu_1 \dots n}(x) = \frac{\partial x^{\mu_1}}{\partial \tilde{x}^{\rho_1}} \cdots \frac{\partial x^{\mu_m}}{\partial \tilde{x}^{\rho_m}} \frac{\partial \tilde{x}^{\sigma_1}}{\partial x^{\nu_1}} \cdots \frac{\partial \tilde{x}^{\sigma_n}}{\partial x^{\nu_n}} \tilde{\mathbb{T}}^{\rho_1 \cdots \rho_m}{}_{\sigma_1 \dots \sigma_n}(\tilde{x}) \tag{13.7}$$

13.4 Proper Time and Spacetime Metrics

At this point, we know nothing about the spacetime beyond the fact that I know I can index it and measure things. We need more than that; we will need to embed objects into this spacetime somehow. We will need some rules for doing so, and rules thar we can all agree on.

Let's take a pet snail[21] and put him at x^μ in our spacetime. Now let's say he moves to $x^\mu + dx^\mu$. We then record the time that elapses on the wristwatch. This we would call the proper time.
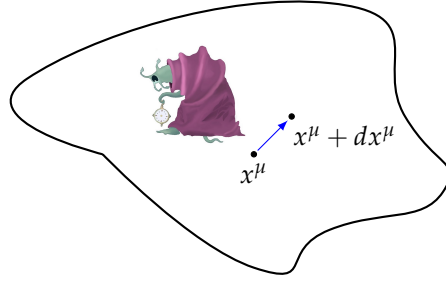

Figure 13.1: We take our pet snail (actually pictured is a slug, but close enough) and put him at coordinate x^μ in our spacetime. We then define the proper time as the time that elapses on his watch when he moves from x^μ to $x^\mu + dx^\mu$.

This should vary linearly with how far he travells. We should also have a perscription for the path taken to be somehow "sensible"; say $x^\mu + t dx^\mu$ for $0 \leq t \leq 1$. We want some way of figuring out what this proper time is. If it's linear in x^μ, but this has an index and proper time is just a number. So it would be useful to square the propert time; this would be bi-linear in x^μ, perhaps with an expression:

$$d\tau^2 = dx^\mu g_{\mu\nu}(x) dx^\nu \tag{13.8}$$

this defines a matrix $g_{\mu\nu}$ of some kind. It turns out to be a fairly special one; why? If we both watch the snail and record the proper time in the same way, we should agree on that proper time for every possible path he should take. We can therefore map out this matrix. Even if we restrict ourselves to timelike paths, there is enough to map out this tensor. The physics tells us that the times must agree, and so:

$$d\tau^2 = dx^\mu g_{\mu\nu}(x) dx^\nu = d\tilde{x}^\mu \tilde{g}_{\mu\nu}(\tilde{x}) d\tilde{x}^\nu \tag{13.9}$$

This tells us that g transforms as a tensor field! And so:

$$g_{\mu\nu}(x) = \frac{\partial \tilde{x}^\rho}{\partial x^\mu} \frac{\partial \tilde{x}^\sigma}{\partial x^\nu} \tilde{g}_{\rho\sigma}(\tilde{x}). \tag{13.10}$$

We give this tensor field a name of being a *metric*, or *metric tensor*. This encodes a lot of information about our spacetime, as it in a sense measures distances. We will not go into the direction of curvature and general relativity here, but this is certainly a fascinating direction.

13.5 Minkowski Spacetime

We are only example in a specific example of spacetime - Minkowski spacetime. We define it as a spacetime where there exists a coordinate system such that:

$$g_{\mu\nu}(x) = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{13.11}$$

---

[21] Let's assume that snails exist...

the poor snail notices that there are limits on his motion coming from the minus sign; he cannot for example go backwards in time. Note we have to correct the formula in Eq. (13.9) we had for $d\tau^2$ due to this minus sign:

$$d\tau^2 = -dx^\mu g_{\mu\nu}(x)dx^\nu = -d\tilde{x}^\mu \tilde{g}_{\mu\nu}(\tilde{x})d\tilde{x}^\nu \tag{13.12}$$

Note that this is known as the east-coast convention; one could alternatively take:

$$g_{\mu\nu}(x) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{13.13}$$

and this would be the west-coast convention. Despite our location at UBC, we adopt the east-coast convention.

Some background; the WC convention was developed by high-energy physicists, while the EC convention was developed by relativists. The former has the benefit of energies being positive, the latter has the benefit of distances being positive. Both are correct and often one takes the convention most convenient for the setting (depending on whether you discuss particle energies, or geometry).

13.6 Infinitesimal Coordinate Transformations

We consider infinitesimal coordinate transformations:

$$\tilde{x}^\mu = x^\mu + f^\mu(x) \tag{13.14}$$

We can think of f^μ as small but otherwise arbitrary. It also looks like a vector field, but is not quite one (it coordinate transforms in a funny way). Scalar fields transform under infinitesimal transformations as:

$$\phi(x) = \tilde{\phi}(\tilde{x}) = \phi(x) + \delta\phi(x) + f^\mu(x)\partial_\mu\phi(x) \tag{13.15}$$

where we have Taylor expanded in f for the last term. As usual we consider only up to linear order. Cancelling terms, we obtain:

$$\delta\varphi(x) = -f^\mu(x)\partial_\mu\varphi(x) \tag{13.16}$$

We can then abstract this to various other fields. One example is the contravariant vector field:

$$A^\mu(x) = \frac{\partial x^\mu}{\partial \tilde{x}^\nu}\tilde{A}^\nu(\tilde{x}) \tag{13.17}$$

Note the beauty of infinitesimal transformations in that they are very easy to invert:

$$x^\mu = \tilde{x}^\mu - f^\mu(\tilde{x}) \tag{13.18}$$

and so

$$\frac{\partial x^\mu}{\partial \tilde{x}^\nu} = \delta^\mu_\nu - \partial_\nu f^\mu \tag{13.19}$$

therefore:

$$A^\mu(x) = A^\mu(x) - \partial_\nu f^\mu(x)A^\nu(x) + \delta A^\mu(x) + f^\nu(x)\partial_\nu A^\mu(x) \tag{13.20}$$

From here the zeroth order bit cancells out, and so:

$$\delta A^\mu(x) = -f^\nu \partial_\nu A^\mu(x) + \partial_\nu f^\mu(x)A^\nu(x) \tag{13.21}$$

where the first term comes from the transformation of the coordinate and the second term comes from the transformation of the frame. The scalar field has the first part (and in fact every field will have it)! The extra terms are the derivatives of the transformation function with the indices matched up. Doing the same for covariant fields:

$$\delta A_\mu(x) = -f^\nu(x)\partial_\nu A_\mu(x) - \partial_\mu f^\nu(x)A_\nu(x) \tag{13.22}$$

And as a particular example, the metric transforms as:

$$\delta g_{\mu\nu}(x) = -f^\lambda(x)\partial_\lambda g_{\mu\nu}(x) - \partial_\mu f^\lambda(x)g_{\lambda\nu}(x) - \partial_\nu f^\lambda(x)g_{\mu\lambda}(x) \tag{13.23}$$

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13.7 The definition of space-time symmetry

Having discussed metrics, what else can one say about a space? One thing that would be interesting is symmetry. Maybe we can define symmetry of spacetime as follows; symmetry is a change of coordinates such that the rules for calculating proper time remain exactly the same. We need the metric to calculate the proper time; so if we calculate the same proper time under a coordinate transformation, the metric better be left identical. Putting this together, we can say that *A symmetry of space-time is a coordinate transformation such that* $\delta g_{\mu\nu}(x) = 0$. So to discover symmetries, we can dream up a large class of fs, and see for what class we have $\delta g_{\mu\nu}(x) = 0$. This wouldn't be a useful definition of symmetry was plentiful, and in general it is quite rare to find them. Precisely, the maximum number of spacetime symmetries turns out to be $d(d+1)/2$[22]. This is obtained by counting the solutions to the differential equation obtained by setting $\delta g_{\mu\nu}(x) = 0$. An equivalent definition of symmetry is therefore: $\hat{f}^\mu(x)$ *is a symmetry if:*

$$\hat{f}^\lambda(x)\partial_\lambda g_{\mu\nu}(x) + \partial_\mu \hat{f}^\lambda(x)g_{\lambda\nu}(x) + \partial_\nu \hat{f}^\lambda(x)g_{\mu\lambda}(x) \tag{13.24}$$

i.e. $\hat{f}^\mu(x)$ *the Killing vector satisfies the Killing equation.* One might wonder if the form of the equation changes in different coordinate systems; this turns out to not be the case (and would require some differential geometry knowledge). This gives us a systematic way to discuss symmetries; something we will continue with next class.

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[22]The number of independent components of a $d \times d$ symmetric tensor - though typically there are less, there are only 3 maximally symmetric spaces, namely Minkowski, and the spaces of constant positive or constant negative curvature. For 2D surfaces, this means the only maximally symmetric surfaces are the plane, sphere, and the one that looks locally like a Pringle (I missed the formal name that Gordon said here). In 4D we have Minkowski, DeSitter and Anti DeSitter space.

14 Symmetries of Minkowski Space-Time, Conformal Transformations

14.1 Metric Tensors

We have been discussing the metric tensor field $g_{\mu\nu}(x)$. This is special because it is used to measure proper times. There is one small thing we have forgot to mention which we should now fill in. What is $g^{\mu\nu}(x)$? The way we define this is the inverse of the metric with down indices. Therefore:

$$g^{\mu\nu}(x)g_{\mu\lambda}(x) = \delta^{\mu}_{\lambda} \tag{14.1}$$

w here δ^{μ}_{λ} is the 4×4 matrix with ones on the diagonal and zeros everywhere else. We can study how this transforms (and the answer is: it will not!)

$$\delta^{\mu}_{\lambda} = \frac{\partial x^{\mu}}{\partial \tilde{x}^{\rho}}\frac{\partial \tilde{x}^{\sigma}}{\partial x^{\lambda}}\tilde{\delta}^{\sigma}_{\rho} \tag{14.2}$$

the above equation only holds if $\tilde{\delta}^{\sigma}_{\rho}$ is identically a unit matrix. So, it looks the same in every coordinate system. It is sometimes called an *invariant tensor* for this reason (of which there are fairly few).

Note that $g^{\mu\nu}$ is also a tensor (a contravariant one).

Note that we have here assumed that the metric is invertible. If it is not, then our proper time experiments would have some strange results indeed... the snail could travel and no time could pass on the wristwatch, for example. This doesn't seem very physical. So we say that this shouldn't be possible; the metric should be invertible everywhere (technical terminology: the metric should be non-degenerate).

14.2 Contracting Fields, Raising/Lowering Indices

We can consider the contraction of two vector fields:

$$A_{\mu}(x)A'^{\mu}(x) \tag{14.3}$$

this object transforms like a scalar field; this makes sense, as it should transform like something without any indices.

In the same vein, once we have this tensor g, we can use it to raise indices on vector (tensor) fields:

$$A^{\mu}(x) = g^{\mu\nu}(x)A_{\nu}(x) \tag{14.4}$$

and similarly:

$$A_{\mu}(x) = g_{\mu\nu}A^{\nu}(x) \tag{14.5}$$

A note: haven't discussed spinors and their transformations; this is because there isn't really a transformation law for them. This might be a bit disquieting that this simple structure we have been exploring does not cover all possible physical objects (e.g. electrons are described by spinors). This will eventually be something we need to consider.

14.3 Symmetries of Space-Time

A symmetry of space-time is a coordinate transformation:

$$x^{\mu} \rightarrow \tilde{x}^{\mu} = x^{\mu} + f^{\mu}(x) \tag{14.6}$$

that leaves the metric invariant:

$$\delta g_{\mu\nu}(x) = 0. \tag{14.7}$$

So literally, g looks the same in both coordinate systems. We can't tell if we are doing physics in one coordinate systems or the other.

An example of a coordinate transformation that *isn't* a symmetry: the transformation that takes us from cartesian to polar coordinates. This is *not* a symmetry transformation. The schrodinger equation looks different in this coordinate system!

An example of a coordinate transformation that *is* a symmetry: a transformation that rotates/changes the direction of the Cartesian axes. In this case the laws of physics look exactly the same.

We can write explicitly the Killing equation:

$$\hat{f}^\lambda(x)\partial_\lambda g_{\mu\nu}(x) + \partial_\mu \hat{f}^\lambda(x)g_{\lambda\nu}(x) + \partial_\nu \hat{f}^\lambda(x)g_{\mu\lambda}(x) = 0. \tag{14.8}$$

The solutions $\hat{f}^\lambda(x)$ of which are called Killing vectors.

14.4 Killing Equation and Vectors for Minkowski Space-Time

Let's look at the Killing vectors for Minkowski space. Recall that Minkowski space is defined by the fact that we can go to a coordinate system where the metric looks like:

$$g_{\mu\nu}(x) = \eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{14.9}$$

We can plug this into the equation and find the Killing equation for Minkowski space:

$$\partial_\mu f^\lambda \eta_{\lambda\mu} + \partial_\nu f^\lambda \eta_{\mu\lambda} = 0. \tag{14.10}$$

If we now define:

$$\hat{f}_\mu(x) = \eta_{\mu\nu}\hat{f}^\nu(x) \tag{14.11}$$

then the Killing equation reads:

$$\partial_\mu \hat{f}_\nu + \eta_\nu \hat{f}_\mu = 0. \tag{14.12}$$

and we can start searching for our Killing vectors. Note that the above is a homogenous equation, which helps us with our search (we can look for solutions of zeroth order and linear order). They are:

1. $\hat{f}_\mu = C_\mu$, constants (of which there are 4). These correspond to time and space translation symmetry.

2. $\hat{f}_\mu = \omega_{\mu\nu}x^\nu$. We observe that this is indeed a solution with $\omega_{\mu\nu}$ anti-symmetric, so $\omega_{\mu\nu} = -\omega_{\nu\mu}$. These are the Lorentz transformations (in the broader sense); 3 spatial rotations and 3 Lorentz boosts.

Higher order polynomials do not solve the equation; this is it! This is as many Killing vectors as a space-time can have (Minkowski space is maximally symmetric) - there are 10 total for $D = 4$.

14.5 Conformal Transformations

Under a conformal transformation, the rules change slightly (this is a slight extension of our concept of symmetry). This is important as there are some QFTs that have this kind of enhanced symmetry (not technically a symmetry of the space as we have defined it). Conformal transformations are coordinate transformations for which the metric transforms to something proportional to the metric:

$$\delta g_{\mu\nu}(x) = \Omega(x)g_{\mu\nu}(x). \tag{14.13}$$

This is the Conformal Killing equation. In Minkowski space, it looks like:

$$\partial_\mu \hat{f}_\nu(x) + \partial_\nu \hat{f}_\mu(x) - \frac{2}{D}\eta_{\mu\nu}\partial_\lambda \hat{f}^\lambda(x) = 0. \tag{14.14}$$

Where D is the dimension of the space. One can check that the Killing vectors for Minkowski space are divergence free, so the third term vanishes and (of course) we have that the original Killing equation is satisfied. How many additional solutions are there? Not too many. Again it is homogenous so we can try the same strategy of looking for polynomial solutions. We find the conformal Killing vectors:

1. $\hat{f}_\mu C_\mu$

2. $\hat{f}_\mu = \omega_{\mu\nu} x^\nu$ for ω antisymmetric.

3. $\hat{f}^\mu = \lambda x^\mu$. A scale transformation.

4. $\hat{f}^\mu = x^\mu x^\nu b_\nu - b^\mu \frac{x^\nu x_\nu}{2}$ where b is some constant four-vector (of which there are four possible choices).

In the above, we have x with a lower index. It is defined as:

$$x_\nu = \eta_{\mu\nu} x^\mu \tag{14.15}$$

once we are in Minkowski space and agree on using Cartesian coordinates, we can raise and lower the indices on x (and on anything) with η. We can think of this as an inner product/space-time invariant:

$$x^\nu x_\nu = \eta_{\mu\nu} x^\mu x^\nu = \mathbf{x}^2 - (x^0)^2 \tag{14.16}$$

and this may be familiar from a previous course in special relativity.

As discussed before, conformal invariance can give us some extra symmetries when studying some systems; e.g. the classical Ising model on its second order phase transition; this turns out to be a QFT with $\hbar = 1$! It's not a quantum system, yet it obeys the rules of a QFT with a conformal symmetry. The conformal symmetry and some of its constraints have been exploited to calculate the critical exponents of the Ising model to world-record accuracy, matching numerical simulations perfectly.

14.6 Finite Lorentz Transformations

So, we've discussed the infinitesimal transformations. Let's say a few words about the finite case. We consider the Lorentz transforms:

$$x^\mu \to \tilde{x}^\mu = x^\mu + \omega^\mu_\nu x^\nu \tag{14.17}$$

So we write this as:

$$\tilde{x}^\rho = \Lambda^\rho_\mu x^\mu \tag{14.18}$$

where:

$$\Lambda^\rho_\mu \approx \delta^\rho_\mu + \omega^\rho_\mu \tag{14.19}$$

We consider the corresponding transformation for the metric:

$$\eta_{\mu\nu} = \frac{\partial \tilde{x}^\rho}{\partial x^\mu} \frac{\partial \tilde{x}^\sigma}{\partial x^\mu} \eta_{\rho\sigma} = \Lambda^\rho_\mu \Lambda^\sigma_\nu \eta_{\rho\sigma} \tag{14.20}$$

Lorentz transformations are the set of all Λs which obey the above equation. We have been studying the infinitesimal ones. We know that:

$$(\det \Lambda)^2 = 1 \tag{14.21}$$

and so:

$$\det \Lambda = \pm 1 \tag{14.22}$$

The $\det \Lambda = 1$ transformations are known as proper. It can be shown that the Λs here form a group. If Λ_1, Λ_2 obey the equation, then so do Λ_1^{-1}, $\Lambda_1 \Lambda_2$ etc.

Some objects are invariant under Lorentz transformations; for example scalars, and by extension any contraction of vector fields $A^\mu(x) A_\mu(x)$.

15 Real Scalar Field Theory

We've been discussing special relativity; now we use it to build up our relativistic quantum field theory. When we discussed our non-relativistic QFT, we didn't discuss geometry all too much; but here it will be useful to inform us about the kinds of space-time symmetries that our theory should possess as well. Our world doesn't look fairly symmetric; that said physics has had quite a bit of success exploring theories with symmetries to explain our universe; and elementary particle physics does appear to possess these.

15.1 Setting up the Real Scalar Field Theory

Let's look at a simple (in fact the simplest) example of a relativistic QFT. This will be simple in the sense that we will not have to worry about many indices. We consider a real scalar field theory. Consider a scalar field $\phi(x)$, and we want to consider a dynamical system where ϕ is a degree of freedom, and is relativistic. We can start writing down the Lagrangian density guided by what we have learned in the last few lectures. Some are convention dependent; there could be derivative terms in ϕ, and these are usually written first. The normalization of the derivative term fixes the normalization of the field ϕ, as well. We want to organize the derivatives such that they have Lorentz symmetry. We thus obtain the term:

$$-\frac{1}{2}\partial_\mu\phi(x)\partial^\mu\phi(x) \tag{15.1}$$

We could have terms with more derivatives (so-called effective field theories); while they would not be excluded in classical field theories, quantum field theories do not like them very much; so we do not consider them. We also could have something quadratic in ϕ:

$$-\frac{m^2}{2}\phi^2(x) \tag{15.2}$$

We could in principle have a term linear in ϕ, but we can get rid of it assuming that the Lagrangian has a symmetry such that it is unchanged under replacing ϕ with $-\phi$.

There are other terms that could contain ϕ, such as:

$$-\frac{\lambda}{4!}\phi^4(x) \tag{15.3}$$

(of course all the odd terms have to vanish by the $-\phi \leftrightarrow \phi$ symmetry). Of course we could have in principle ϕ^6, ϕ^8 terms etc. but these are not allowed due to renormalizability conditions. So in principle we have written everything down that could be contained in our Lagrangian density:

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\phi(x)\partial^\mu\phi(x) - \frac{m^2}{2}\phi^2(x) - \frac{\lambda}{4!}\phi^4(x) \tag{15.4}$$

We have tried to organize the Lagrangian density such that it transforms like a scalar under Lorentz transformations. Let's begin by studying what this Lagrangian tells us. The Euler-Lagrange equation in relativistic form for a scalar field is very nice; it reads:

$$\partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi(x))} = \frac{\partial\mathcal{L}}{\partial\phi(x)} \tag{15.5}$$

We thus find the equation of motion:

$$(-\partial^2 + m^2)\phi(x) + \frac{\lambda}{3!}\phi^3(x) = 0. \tag{15.6}$$

For λ finite this is very hard to solve, so in general the approach is to treat this in perturbation theory with small λ. If we set $\lambda = 0$, we obtain the Klein-Gordon equation:

$$(-\partial^2 + m^2)\phi(x) = 0. \tag{15.7}$$

and this term in (15.6) is the Klein-Gordon term, with wave operator $-\partial^2 + m^2$. The $\frac{\lambda}{3!}\phi^3(x)$ non-linear term can be thought of the interaction term. This is in analogy to our previous non-relativistic QFT, where we had a free field equation and added on the interaction terms. Note there isn't really a place here for an operator ordering ambiguity; everything commutes with everything here.

15.2   Natural Units

We work in a system of units where $\hbar = 1, c = 1$. Note that if we had not made this choice, then there would be factors of $\hbar, c$ in the $-\frac{m^2}{2}\phi^2(x)$ term. We can of course figure out where these should go by studying the units, but with natural units we are saved the hassle.

15.3   Commutation Relations

Note that this is the kind of system where the Lagrangian has velocity squared times some function of position:

$$\mathcal{L} = \frac{1}{2}\frac{\partial\phi}{\partial t}\frac{\partial\phi}{\partial t} + \dots \tag{15.8}$$

so the canonical momentum is $P = \frac{\partial\phi}{\partial t}$, and the canonical position $X = \phi$. We then have the Poisson bracket:

$$\{\phi, \frac{\partial}{\partial t}\phi\} = \delta() \tag{15.9}$$

and lifting this to the QFT, we have the commutator relation:

$$[\phi(x), \frac{\partial}{\partial t}\phi] = i\hbar\delta() \tag{15.10}$$

Concretely, we have the equal-time commutation relations of:

$$[\phi(x), \frac{\partial}{\partial y^0}\phi(y)]\delta(x^0 - y^0) = i\delta(x - y) \tag{15.11}$$

Of course the positions commute with each other:

$$[\phi(x), \phi(y)]\delta(x^0 - y^0) = 0 \tag{15.12}$$

As do the momenta:

$$[\frac{\partial}{\partial x^0}\phi(x), \frac{\partial}{\partial y^0}\phi(y)]\delta(x^0 - y^0) = 0 \tag{15.13}$$

One might wonder if particle physicists are missing any theories by only considering Lagrangian ones; but this is a discussion for another day.

15.4   Symmetry

We recall the infinitesimal transformation of the scalar field:

$$\delta\phi(x) = -f^\mu(x)\partial_\mu\phi(x) \tag{15.14}$$

where if this is a space-time symmetry then f^μ is a Killing vector; otherwise it is a generic change of coordinates. Let us keep it general for now; we see from algebra that:

$$\delta\mathcal{L}(x) = -f^\mu(x)\partial_\mu\mathcal{L}(x) + (\partial_\mu f_\nu + \partial_\nu f_\mu)\frac{1}{2}\partial^\mu\phi\partial^\nu\phi \tag{15.15}$$

We need to show that this transforms like the derivative of something. So let us rearrange using the product rule:

$$\delta \mathcal{L}(x) = \partial_\mu(-f^\mu(x)\mathcal{L}(x)) + (\partial_\mu f_\nu + \partial_\nu f_\mu)\frac{1}{2}\left(\partial^\mu \phi \partial^\nu \phi + \eta^{\mu\nu}\mathcal{L}\right) \tag{15.16}$$

Now if f^μ is a Killing vector, then $\partial_\mu f_\nu + \partial_\nu f_\mu = 0$ from the Killing equation in Minkowski space! So $\delta \mathcal{L}$ can be written as a total derivative - it is a symmetry in the Noetherian sense if f^μ is a Killing vector/we have a space-time symmetry:

$$\delta \mathcal{L}(x) = -f^\mu(x)\partial_\mu \mathcal{L}(x) \tag{15.17}$$

Applying Noether's theorem, we can obtain the Noether charge density and current. By Noether's theorem:

$$\mathcal{J}^\mu = \delta\phi\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi} - \mathcal{R}^\mu = \hat{f}^\nu\partial_\nu\phi\partial^\nu\phi + \hat{f}^\mu\mathcal{L} \tag{15.18}$$

where $\mathcal{R}^\mu = (R, \mathbf{J})$ from our old formulation of Noether's theorem (not in relativistic notation). So by Noether's theorem, this is conserved:

$$\partial_\mu \mathcal{J}^\mu = 0. \tag{15.19}$$

15.5   A Faster Noether's Theorem

We came here with a variation of φ where f was arbitrary, obtaining:

$$\delta \mathcal{L}(x) = \partial_\mu(-f^\mu(x)\mathcal{L}(x)) + (\partial_\mu f_\nu + \partial_\nu f_\mu)\frac{1}{2}\left(\partial^\mu \phi \partial^\nu \phi + \eta^{\mu\nu}\mathcal{L}\right) \tag{15.20}$$

Now, what if we actually imposed the equation of motion (something we have been told not to do)? If we vary the Lagrangian density and impose the the equations of motion (from the EL equation), then the above variation in $\mathcal{L}(x)$ has to be written as a total derivative. How do we make th esecond term in the above expression a total derivative? By writing $\partial_\mu(\dots)$ and $\partial_\nu()$... but then this implies the $\partial_\mu\left(\partial^\mu \phi \partial^\nu \phi + \eta^{\mu\nu}\mathcal{L}\right) = 0$ which gives us the result in a quick way!

15.6   The Stress Tensor

Another connection; let us write the Noether current as a contraction of the Killing vector:

$$\mathcal{J}^\mu = \hat{f}^\nu\left(\partial_\nu\varphi\partial^\mu\varphi + \eta^\mu_\nu\mathcal{L}\right) \tag{15.21}$$

We may call the above expression in the brackets a stress tensor $\mathbb{T}^\mu_\nu(X)$ and from the conservation:

$$\partial_\mu\mathbb{T}^{\mu\nu}(x) = 0. \tag{15.22}$$

So, in our context here we have the Stress tensor definition:

$$\mathbb{T}^{\mu\nu}(x) = \partial_\nu\varphi\partial^\mu\varphi + \eta^\mu_\nu\mathcal{L} \tag{15.23}$$

So: the Nother current for a symmetry $\delta\varphi = -\hat{f}^\mu\partial_\mu\varphi$ is:

$$\mathcal{J}^\mu = \hat{f}_\nu(x)\mathbb{T}^{\mu\nu}(x). \tag{15.24}$$

and this conserved stress tensor summarizes the conservation laws of spacetime symmetries. There are no operator ordering ambiguities, and the theory is not constructed to have any other symmetries, so we are in some sense done. But we still learn things. The time-time component gives the energy density, the time-space component gives the momentum density, and the other components give us stresses/strains and so on. Recall we also had this in non-relativistic physics, and this is exactly the analog. The construction of things like angular momentum operators from $\mathbb{T}$ in the non-relativistic setting generalizes here, by considering the various space-time symmetries of the form:

$$\hat{f}_\nu(x) = C_\nu + \omega_{\nu\lambda}x^\lambda. \tag{15.25}$$

15.7 Non-Interacting Theory

In order to gain intuition, let us try to solve this theory in the limit where it is solvable, i.e. when we set the nonlinear term to $\lambda = 0$. In this case, the field equation looks as follows:

$$(-\partial^2 + m^2)\varphi(x) = 0 \tag{15.26}$$

This can be solved (as usual) by the plane wave ansatz:

$$\varphi(x) \sim e^{ik_\mu x^\mu} \tag{15.27}$$

this solves the equation if:

$$k^\mu k_\mu + m^2 = 0. \tag{15.28}$$

Which tells us the relativistic dispersion relation:

$$k^0 = \pm\sqrt{\mathbf{k}^2 + m^2} \tag{15.29}$$

We need both solutions. The plane waves are intrinsically complex, but of course φ is an intrinsically real scalar field, i.e. $\varphi = \varphi^*$ in classical field theory of $\varphi = \varphi^\dagger$ in quantum field theory. We need plane waves on both sides to superimpose them and cancel out the imaginary components:

$$\varphi(x) = \int \frac{d^3k}{\sqrt{(2\pi)^3 2\omega(k)}} \left(e^{i\mathbf{k}\cdot\mathbf{x} - i\omega t} a(\mathbf{k}) - e^{-i\mathbf{k}\cdot\mathbf{x} + i\omega t} a^\dagger(\mathbf{k}) \right) \tag{15.30}$$

where $\omega(k) = \sqrt{k^2 + m^2}$. One could check that the plane waves were complete, orthonormal (and so on) and that this is a solution to the QFT. In the QFT the $a, a^\dagger$s become operators with commutation relations:

$$[a(\mathbf{k}), a^\dagger(\mathbf{l})] = \delta^3(\mathbf{k} - \mathbf{l}) \tag{15.31}$$

which implies that:

$$[\varphi(x), \frac{\partial}{\partial y^0}\varphi(y)]\delta(x^0 - y^0) = \delta^3(\mathbf{x} - \mathbf{y}) \tag{15.32}$$

In fact if we look at the Hamiltonian:

$$H = \int d^3x \, \mathbb{T}^{00}(x) \tag{15.33}$$

and we plug in our solution and do some algebra, we find:

$$H = \int d^3k \, \omega(\mathbf{k}) a^\dagger(\mathbf{k}) a(\mathbf{k}) + C \tag{15.34}$$

i.e. a bunch of (non-interacting) harmonic oscillators plus a constant (which comes from the operator ordering), which many QFT courses tell you is harmless (as we are only interested in relative energies) but is not quite physically true... of course we have things like gravity which depend on the total energy density. But since we don't feel ourselves gravitating to some infinite energy density from the scalar field, then we should set it to zero to be consistent with physics:

$$H = \int d^3k \, \omega(\mathbf{k}) a^\dagger(\mathbf{k}) a(\mathbf{k}). \tag{15.35}$$

This in some sense is an ambiguity of the theory (and one of the mysteries of QFT)! It is an arbitrary constant, but gets a bit hairy when you think about it too much; e.g. the dark energy density in our theory

here would be infinite, but it is finite in our universe. We won't need it for anything because we won't be venturing off into quantum gravity, but it would be important there.

The momentum are:

$$P^a = \int d^3x \, \mathbb{T}^{0a}(x) = \int d^3k \, k^a a^\dagger(\mathbf{k}) a(\mathbf{k}) + C \tag{15.36}$$

here we might be able to argue that the constant is zero by an isotropy argument, but we run into troubles with frame dependence etc.... but let us not worry about it too much and just set it to zero.

Finally, let us construct quantum states for our theory. We have the vacuum $|0\rangle$, which is normalized:

$$\langle 0|0\rangle = 1 \tag{15.37}$$

and is annhilated by the annhilation operators:

$$a(\mathbf{k})|0\rangle = 0, \quad \langle 0|a^\dagger(\mathbf{k}) = 0 \quad \forall \mathbf{k} \tag{15.38}$$

We then have our basis:

$$\left\{ |0\rangle, a^\dagger(\mathbf{k})|0\rangle, \frac{a^\dagger(\mathbf{k}_1)a^\dagger(\mathbf{k}_2)|0\rangle}{\sqrt{2}}, \dots \right\} \tag{15.39}$$

and we now more or less have a complete solution of our theory. As soon as we turn on λ we have no solution, but can obtain a solution via perturbation theory, where we learn how to include corrections. The whole course here on in is learning how to include these corrections in a systematic way, and in fact this is all we know how to do (other than trying to solve the theory numerically). For this particular theory, things have been studied up to λ^8, with roughly 8! Feynman diagrams...

16 Correlation Functions, Real Scalar Field Theory II

16.1 A Review of Defining Field Theories

The discussion is getting excited! We are finally getting to a non-relativistic quantum field theory; albeit one that we are not, in general, able to solve. However, it is a fact that there are very few solvable field theories. They tend to have special properties, e.g. be in dimension 1 (can still be useful for studying some QM systems) but the systems for which QFT was developed for (i.e. Quantum Electrodynamics) is analytically unsolvable. So our procedure will be to remove the interactions, solve it exactly, and add back the interactions in perturbation theory. And this is basically the state of the art; doing free field theory and perturbation theory on top. And standard model/particle physics is basically based completely on this.

We also had a few different presentations of our theory. We began by writing down a Lagrangian that looks symmetric. Then if there is any doubt, we check that this is indeed the case. This then defines the field theory, as all of the data of the field theory is encoded in the Lagrangian (really the Lagrangian density, but the entire world calls it the Lagrangian). For our free real scalar field theory, we had:

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{m^2}{2}\phi^2 \tag{16.1}$$

We alternatively could take our starting point as the field equation and commutation relations:

$$(-\partial^2 + m^2)\phi = 0 \tag{16.2}$$

$$[\phi(x), \dot{\phi}(y)]\delta(x^0 - y^0) = i\delta^3(\mathbf{x} - \mathbf{y}) \tag{16.3}$$

but the Lagrangian method gives us a little more structure from Noether's theorem. We could be clever and guess the conservation laws using the field equation/commutation relation approach, but we would have to be quite clever. Today, we will want to discuss yet a third way of encoding the data of the field theory; and that is in correlation functions.

16.2 Correlation Function Definition of Field Theories

By a correlation function, we mean:
$$\langle 0|\phi(x_1)\dots\phi(x_n)|0\rangle \tag{16.4}$$

and another way to write down our QFT is to write down all possible correlation functions. At first this seems like a clumsy way to proceed; in the other methods we can write down the theory in a few lines. In this method we have an (infinite) set of correlation functions to write down. Fortunately, there will be a great simplifying principle.

This set of expectation values are given the name of Wightman functions (Wightman being a scientist who tried to axiomatize field theories - to some small level of success). We therefore write:

$$W(x_1\dots x_n) = \langle 0|\phi(x_1)\dots\phi(x_n)|0\rangle \tag{16.5}$$

Note that we will like to consider time-ordered correlation functions:

$$\langle 0|T\phi(x_1)\dots\phi(x_n)|0\rangle \tag{16.6}$$

where T is an operator that orders the $\phi(x_i)$s to go from earlier to later times. This is nice as it makes the correlator symmetric in the xs. This was realized by Dyson early on in modern field theory, who noticed that time-dependent perturbation theory in QFT simplifies significantly if correlation functions are written in this time-ordered form. For this reason they are called Dyson correlation functions, and we denote:

$$\Gamma(x_1\dots x_n) = \langle 0|T\phi(x_1)\dots\phi(x_n)|0\rangle \tag{16.7}$$

and these Γs tend to be the useful/tractable correlation functions that we will generally consider.

There are places where this formulation of field theory is useful, e.g. in Monte Carlo approaches to field theories.

16.3   Returning to our Real Scalar Relativistic Free Field Theory

Last time we found that our real scalar relativistic free field theory (with Lagrangian in Eq. (16.1) and field equation/commutation relations given in Eqs. (16.2), (16.3)) and we found the solutions to be:

$$\phi(x) = \int \frac{d^3k}{\sqrt{(2\pi)^3 2\omega(\mathbf{k})}} \left(e^{i\mathbf{k}\cdot\mathbf{x} - i\omega t} a(\mathbf{k}) - e^{-i\mathbf{k}\cdot\mathbf{x} + i\omega t} a^\dagger(\mathbf{k}) \right) \tag{16.8}$$

with commutation relations:

$$[a(\mathbf{k}), a^\dagger(\mathbf{l})] = \delta^3(\mathbf{k} - \mathbf{l}) \tag{16.9}$$

Note that there are other conventions for the $\frac{1}{\sqrt{2\omega(\mathbf{k})}}$; we have a real scalar field, so it should be invariant under Lorentz transformations. But the $\delta(\mathbf{k} - \mathbf{l})$ and the as do not. So we introduce the $\frac{1}{\sqrt{2\omega(\mathbf{k})}}$ in our free field theory solution to compensate for the weird transformations that the as obey. At this point, some textbooks avoid this problem by introducing a different normalization for the commutators:

$$[a(\mathbf{k}), a^\dagger(\mathbf{l})] = 2\omega(\mathbf{k})\delta^3(\mathbf{k} - \mathbf{l}) \tag{16.10}$$

and then the integration measure:

$$\int \frac{d^3k}{2\omega(\mathbf{k})} \tag{16.11}$$

would be Lorentz invariant. So there is a more invariant way to write all of these things. It is left as an exercise to the reader tha the above is a Lorentz invariant measure, and that $2\omega(\mathbf{k})\delta^3(\mathbf{k} - \mathbf{l})$ is Lorentz invariant.

Here we have an explicit solution to the theory! We have a vacuum state $|0\rangle$ where:

$$a(\mathbf{k})|0\rangle = 0 \quad \forall \mathbf{k} \tag{16.12}$$

we also have the dual statement:

$$\langle 0|a^\dagger(\mathbf{k}) = 0 \quad \forall \mathbf{k} \tag{16.13}$$

And then we can establish a basis:

$$\left\{ |0\rangle, a^\dagger(\mathbf{k})|0\rangle, \frac{a^\dagger(\mathbf{k}_1)a^\dagger(\mathbf{k}_2)|0\rangle}{\sqrt{2}}, \ldots \right\} \tag{16.14}$$

And there is of course a natural inner product we get in this space we get from contracting with the dual and using the commutation relations and so on. We then would investigate Cauchy sequences and all of their limits, and then the space of all Cauchy sequences + their limits gives us the Hilbert space we want. These details aren't needed really, but they are of course there. As we discussed previously, we can write down the energy-momentum tensor and from there obtain things such as the momentum and the Hamiltonian:

$$P^\mu = \int d^3x T^{0\mu}(x) \tag{16.15}$$

and so:

$$P^0 = H = \int d^3k\, \omega(\mathbf{k}) a^\dagger(\mathbf{k}) a(\mathbf{k}) \tag{16.16}$$

$$\mathbf{P} = \int d^3k\, \mathbf{k}\, a^\dagger(\mathbf{k}) a(\mathbf{k}) \tag{16.17}$$

16.4 Transformations of the Real Scalar Field

note that we can therefore find:

$$[P^\mu, \phi(x)] = i\partial^\mu \phi(x) \tag{16.18}$$

i.e. the momentum generates some space-time translation of the field! We could also write down a Noether charge corresponding to Lorentz invariance

$$M^{\mu\nu} = \int d^3x (x^\mu T^{0\nu} - x^\nu T^{0\mu}) \tag{16.19}$$

which looks a lot like the angular momentum operator we are familiar with. We could then calculate:

$$[M^{\mu\nu}, \phi(x)] = i(x^\mu p^\nu - x^\nu \partial^\mu)\phi(x) \tag{16.20}$$

(note that in the above computation we can set the time to be whatever we like; $t = 0$ may be convenient). So we have the infinitesimal Lorentz transformations. In more detail:

$$\Lambda^\mu_\nu \approx \delta^\mu_\nu + \omega^\mu_\nu \tag{16.21}$$

and to first order we have:

$$\delta\phi(x) = -f^\nu \partial_\nu \phi \tag{16.22}$$

the coordinates transform under the transformations as:

$$x^\mu = \tilde{x}^\mu = \Lambda^\mu_\nu x^\nu = x^\mu + \omega^\mu_\nu x^\nu. \tag{16.23}$$

and therefore:

$$\delta\phi(x) = \omega_{\mu\nu} x^\mu \partial^\nu \phi = \frac{1}{2}\omega_{\mu\nu}(x^\mu \partial^\nu - x^\nu \partial^\mu)\phi. \tag{16.24}$$

Note: It is a pretty hard job to get the finite transformations from the infinitesimal ones, as the former are more complicated. But all we really need to do is that it is indeed possible (e.g. via exponentiation).

Before we move on: We see that $M^{\mu\nu}$ is a Hermitian operator that produces an infinitesimal transformation. So exponentiating it gives us a unitary operator that implements the finite Lorentz transform $U(\Lambda)$. This has the action:

$$U(\Lambda)\phi(x)U^\dagger(\Lambda) = \phi(\Lambda^{-1}x) \tag{16.25}$$

where we have considered the coordinate transformation $\tilde{\phi}(\tilde{x}) = \phi(x)$ with $\tilde{x} = \Lambda x$, and so $\tilde{\phi}(\Lambda x) = \phi(x)$ and therefore $\tilde{\phi}(x) = \phi(\Lambda^{-1}x)$. So some transformation $U(\Lambda)$ certainly exists, but we will not worry too much about precisely its form. The translation operator is a bit simpler:

$$e^{iP_\mu a^\mu}\phi(x)e^{-iP_\mu a^\mu} = \phi(x + a) \tag{16.26}$$

where this identity could be realized by Taylor expanding the exponentials and using $[P^\mu, \phi(x)] = i\partial^\mu \phi(x)$.

Some comments before we move on; the transformation operators are infinite dimensional, while if we considered fields with spin this would have finite representations. Here we have a non-compact group, so they are not Hermitian. There is no finite-dimensional unitary representations of the Lorentz group (theorem). But $U(\Lambda)$ is a unitary operator, so it must be infinite dimensional.

Another comment; let us throw away the vacuum energy term in our Hamiltonian. Because then we have that the vacuum state satisfies:

$$P^\mu |0\rangle = 0 \tag{16.27}$$

and so:

$$e^{iP_\mu a^\mu}|0\rangle = |0\rangle \tag{16.28}$$

and further:

$$M^{\mu\nu}|0\rangle = 0 \tag{16.29}$$

$$U(\Lambda)|0\rangle = |0\rangle. \tag{16.30}$$

Physically, one can think of these relations as "moving, or transforming nothing gives us nothing".

These are important relations and they tell us something interesting.

16.5 Correlation Functions for Relativistic Scalar Field theory

One thing we can do is calculate correlation functions. For example:

$$W(x_1.x_2) = \langle 0|\phi(x_1)\phi(x_2)|0\rangle \tag{16.31}$$

we previously learned that this is only a function of $x_1 - x_2$, but let us see if we can recover this result here in a different way:

$$
\begin{aligned}
W(x_1, x_2) &= \langle 0|e^{iP_\mu x_1^\mu}\phi(0)e^{-iP_\mu(x_1^\mu - x_2)}\phi(0)e^{-iP_\mu x_2^\mu}|0\rangle \\
&= \langle 0|\phi(0)e^{-iP_\mu(x_1^\mu - x_2^\mu)}\phi(0)|0\rangle \\
&\sim W(x_1 - x_2) \\
&\sim W((x_1 - x_2)^2) = W((x_1 - x_2)^\mu \eta_{\mu\nu}(x_1 - x_2)^\nu)
\end{aligned}
\tag{16.32}
$$

(Sorry, I missed the explanation for why we can then conclude that it only depends on the square).

We could also insert a sum over a complete set of states $\mathbb{I} = \sum_n |n\rangle\langle n|$ inside the correlator to find:

$$W(x_1, x_2) = \sum_n |\langle n|\phi(0)|0\rangle|^2 e^{iP_\mu(x_2 - x_1)^\mu} \tag{16.33}$$

Note that:

$$e^{iP_\mu(x_2 - x_1)^\mu} = e^{-iP^0(x_2 - x_1)^0 + i\mathbf{p}\cdot(\mathbf{x}_1 - \mathbf{x}_2)} \tag{16.34}$$

We can note that as a function of $(x_2 - x_1)^0$, the function is analytic in the lower half plane. Here we learn that if we have the P_μ generator, even in an interacting field theory, if the vacuum is invariant we would obtain an analyticity property for correlation functions. This is in some sense a universal truth about relativistic scalar field theories. To see if what we are doing makes sense, we can always check this analyticity property.

As a teaser; this kind of analyticity can be used to prove a very radical theorem whose applications are not very well understood. That theorem is the Reeh-Schlieder Theorem. (there is a beautiful lecture by Ed Witten about this). This theorem says that if we take an open set Ω of spacetime, and take $x_i \in \Omega$, then the set of states $\phi(x_1)\dots\phi(x_n)|0\rangle$ we can generate are dense in the Hilbert space of our quantum field theory, i.e. this class of states allows us to approximate *any* state in our QFT arbitrarily well. If you give me 5 minutes and a patch of grass, I suddenly have all the states I need to make Jupiter! This seems like nonsense; but it is in fact true! We will also discuss more conventional uses of analyticity, later on...

Gordon wonders: Can Reeh-Schlieder be proven for a non-relativistic theory; e.g. that of the electron gas? Then we could prove it experimentally.

17 Real Scalar Field Theory III, Dirac Field Theory

Last time, we discussed a real scalar field theory, and demonstrated some consequences of translational and Lorentz symmetry (Poincare symmetries). The discussion was a bit abstract; let's now move to a more concrete calculation; we claimed things were analytic last class, but let's actually show this via an example.

17.1 Deriving the Free-Field Feynman Propagator

Recall the solutions:

$$\phi(x) = \int \frac{d^3k}{\sqrt{(2\pi)^3 2\omega(\mathbf{k})}} \left(e^{i\mathbf{k}\cdot\mathbf{x} - i\omega(\mathbf{k})x^0} a(\mathbf{k}) + e^{-i\mathbf{k}\cdot\mathbf{x} + i\omega(\mathbf{k})t} a^\dagger(\mathbf{k}) \right) \tag{17.1}$$

these were solutions to the field equation, and is a solution to the QFT if we enforce the equal-time commutation relations on $a, a^\dagger$.

Having an exact solution for free fields, we can look at some simple correlation functions and see what they look like. It shouldn't be difficult to convince yourself that after finding the two-point correlation functions for free fields, you can find arbitrary N-point correlation functions. Let us begin there; it is important enough to get its own name; the *Feynman propagator*. We denote it as:

$$\Delta(x, y) = \langle 0 | J\phi(x)\phi(y) | 0 \rangle \big|_{\lambda=0} \tag{17.2}$$

The two point function of the interacting theory is also interesting (and has its own name), and we will get to this soon. We can write the Feynman propagator as follows:

$$\Delta(x, y) = \theta(x^0 - y^0)\langle 0 | \phi(x)\phi(y) | 0 \rangle + \theta(y^0 - x^0)\langle 0 | \phi(y)\phi(x) | 0 \rangle \tag{17.3}$$

where the Heaviside step functions enforce the time ordering. We can take and plug in our expressions for ϕ and plug them in; using the creation/annhilation relations, this is sufficient to do this calculation. Doing so, we find:

$$\Delta(x, y) = \theta(x^0 - y^0) \int \frac{d^3k}{(2\pi)^3 2\omega(\mathbf{k})} e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y}) - i\omega(x^0-y^0)} + \theta(y^0 - x^0) \int \frac{d^3k}{(2\pi)^3 2\omega(\mathbf{k})} e^{-i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y}) + i\omega(x^0-y^0)} \tag{17.4}$$

notice how the second term is just the first term with x, y interchanged; the time ordering makes things symmetric as a function of the coordinates. Last time, we discussed the analyticity of Wightman functions; the first term is analytic in the lower half plane, and then has support in the the right half plane due to the Heaviside, so is analytic in the fourth quadrant. The second term is analytic in the upper half plane and has support in the left half plane due to the Heaviside, so is analytic in the second quadrant. This turns out to be a very useful observation; this is because this allows us to change what we mean by time by adding a complex number to it (analytically continuing where the time goes); in some sense putting $t \mapsto it$. This is known as a Wick rotation, and doing this we can make Minkowski space look like Euclidean space. This can be a useful starting point for our field theory; it is technically a wonderful thing! We get to work with a much nicer metric without the minus sign.

17.2 Another way to derive the Free-Field Feynman Propagator

We have derived the two-point function, but not as it appears in most textbooks, or the most useful form for us. We could do it the technical/long way or we can take a pedagogical shortcut. Recall the solution to our QFT solves the field equation:

$$(-\partial^2 + m^2)\phi(x) = 0 \tag{17.5}$$

So what happens if we take this operator and operate it on the time-ordered two-point function? We would find:

$$(-\partial^2 + m^2)\langle 0 | J\phi(x)\phi(y) | 0 \rangle = 0. \tag{17.6}$$
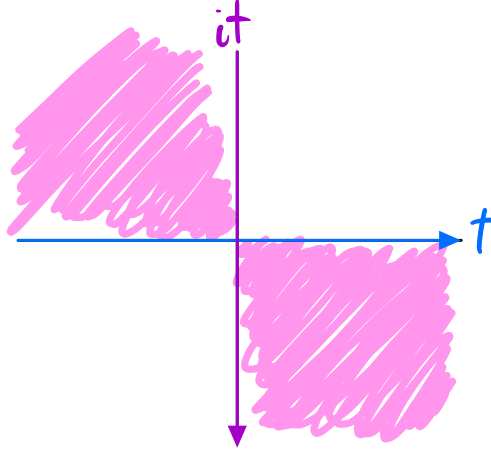
Figure 17.1: The first term in the above integral is analytic in the fourth quadrant and the second term in the above integral is analytic in the second quadrant. We can then take our contour integral for time which goes along the real axis, and rotate it so it lies on the imaginary axis (Wick rotation). Doing so, our Minkowski space looks Euclidean!

Can we just pull this in and get zero? No, not quite; ∂^2 contains time derivatives, and we have a time ordering of the fields inside the expectation value; we need to take this into account. Let's consider the time derivative:

$$\frac{\partial^2}{\partial(x^0)^2}\langle 0|J\phi(x)\phi(y)|0\rangle \tag{17.7}$$

and pulling this in and using the product rule and that the derivative of a step function is the dirac delta, as well as the equal time commutation relations:

$$
\begin{aligned}
\frac{\partial^2}{\partial(x^0)^2}\langle 0|J\phi(x)\phi(y)|0\rangle &= \frac{\partial}{\partial x^0}\left(\langle 0|J\frac{\partial}{\partial x^0}\phi(x)\phi(y)|0\rangle + \delta(x^0-y^0)\cancel{\langle 0|[\phi(x),\phi(y)]|0\rangle}\right)\\
&= \langle 0|J\frac{\partial^2\phi(x)}{\partial(x^0)^2}\phi(y)|0\rangle + \delta(x^0-y^0)\langle 0|[\frac{\partial}{\partial x^0}\phi(x),\phi(y)]|0\rangle = -i\delta(x-y)
\end{aligned}
\tag{17.8}
$$

We therefore learn:

$$(-\partial^2 + m^2)\Delta(x,y) = -i\delta(x-y) \tag{17.9}$$

so $\Delta(x,y)$ is a Green's function! Note that this would not be the case for interacting fields as the wave equation is modified. So, we can short circuit all of our work that we have been doing as we can just find a solution to the Green's function equation above. Naively, we can solve this using a Fourier transform so we find:

$$\Delta(x,y) = \int \frac{d^4k}{(2\pi)^4}e^{ik_\mu(x-y)^\mu}-\frac{i}{k^2+m^2} \tag{17.10}$$

where $k^2 = \mathbf{k}^2 - (k^0)^2$. Here the problem becomes apparent; we have a singularity in the above expression. We need to enforce boundary conditions. We take the singularity at k^0 and resolve it by adding a small imaginary part into the denominator, in such a way such that when we do the k^0 integral (which we could do via Cauchy's integral theorem, if we like):

$$\Delta(x,y) = \int \frac{d^4k}{(2\pi)^4}e^{ik_\mu(x-y)^\mu}-\frac{i}{k^2+m^2-i\epsilon} \tag{17.11}$$

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which fixes the ambiguity and gives us a time-ordered boundary condition. We leave it as an exercise (though it is detailed in the textbook) for how we can write this as something that has two poles, then use partial fractions to separate the two pole terms, and use Cauchy's integral formula to get to:

$$\Delta(x, y) = \theta(x^0 - y^0) \int \frac{d^3k}{(2\pi)^3 2\omega(\mathbf{k})} e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})-i\omega(x^0-y^0)} + \theta(y^0 - x^0) \int \frac{d^3k}{(2\pi)^3 2\omega(\mathbf{k})} e^{-i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})+i\omega(x^0-y^0)}$$

(17.12)

The discussion of analyticity follows exactly as we had before as we have the same expression. There is also a comment we can make about k^0. We have poles on the real axis originally, which we add a small $+i\epsilon$ to the denominator to shift them above/below it so we can integrate over the real axis. This enforces boundary conditions.

We can also consider doing an path integral whose interior does not contain either of the poles (pictured below). Adding it to the integral along the real axis, and taking the boundary to infinity, we get an integral just along the imaginary axis; something that looks Euclidean. This turns out to be extremely useful when we do calculations; we go from a slightly fishy Minkowski space integral to a Euclidean integral that is completely well-defined.
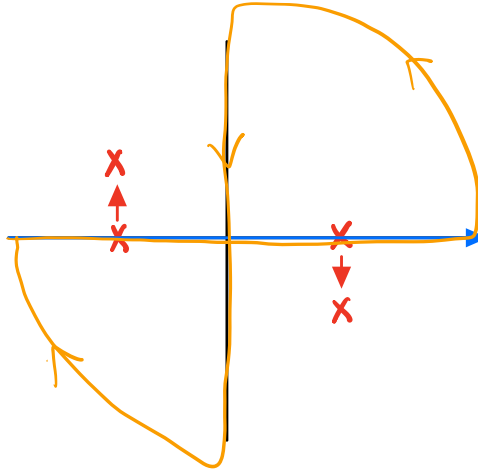

Figure 17.2: The poles (in red) which originally sat on the time axis have been shifted above/below the real axis via the inclusion of the $-i\epsilon$ in the denominatr of the propogator integral. This allows for a meaningful integral to be taken over the real axis. We can also consider a path integral whose interior contains none of the poles; adding this with the integral over the real axis, we obtain our integral over the imaginary time axis, wherein Minkowski space looks Euclidean.

A few other comments; recall in our discussion of wavepacket dispersion where we added two terms, one for particle and one for antiparticle. This is not the same as the addition of two terms we see here. We were seeing a different kind of Green's function, i.e. a retarded Green's functions (vs. the time-ordered Green's function we see here). This is the appropriate Green's function if we want the past to influence the future. This is a bit of an interesting subtle point; to get causality, we need to do a bit more work compared to what we have done here.

We can use some combinatorics to obtain all other correlation functions; so the free field scalar field theory is completely solved! We will put off studying the interacting field theory until we have a couple more free field theories under our belt. After this in the textbook, we have a chapter about emergent relativistic field theory. It will not be covered in lecture, but it is useful in studying relativistic field theories that go beyond particle physics (e.g. phonons, electrons in graphene). Field theories would be confined to a quite narrow range if we only look at the standard model; the condensed matter side of

things offers some interesting cases of study! If we get to the end of the course and have discussed everything else, perhaps we can come back to it.

17.3   Introduction to the Dirac Field

There is a basic scalar field amidst the elementary particles - the celebrated Higgs field. The next step up adds spin to the mix (the scalar field has no spin); in particular spin-1/2 as the simplest one. This is the *Dirac Field*, which describes fermions. Since it describes spin, we need another index which at least runs over two. But beyond this, we also need our causality arguments, which requires the existence of an antiparticle. The scalar field did not appear to have this; it did have a negative frequency branch, but this was not a different kind of particle, really. This could happen for dirac-looking field theories as well (though in this case this is not a dirac field, but instead we have majorana particles! which are their own antiparticles). A direct example; electrons and electron holes in the Fermi sea.

   The above discussion suggests that we need a 4-component wavefunction (2 spin states for the particle and antiparticle each). This would suggest the guess:

$$H =? \begin{pmatrix} \sqrt{p^2 + m^2} & 0 & 0 & 0 \\ 0 & \sqrt{p^2 + m^2} & 0 & 0 \\ 0 & 0 & -\sqrt{p^2 + m^2} & 0 \\ 0 & 0 & 0 & -\sqrt{p^2 + m^2} \end{pmatrix} \tag{17.13}$$

but this is ugly, and the spin is basically trivial here; and we don't have a way of knowing that the spin is trivial unless we know things about group theory and representations of the Lorentz group. We would expect that there would be not just two degenerate spin states, as we have here (think relativistic corrections to the hydrogen spectrum!) So the above formula is not correct. Dirac came up with a better idea; can we find some matrices with the correct Hermiticity properties such that the eigenvalues are the same as what we have written above, and it is linear in momentum?

$$H = i\boldsymbol{\alpha} \cdot \boldsymbol{\nabla} + \beta m \tag{17.14}$$

In writing this, Dirac discovered antiparticles; he was however extremely confused with this discovery (even after the experimental discovery of the proton...) It was confusing as it was brilliant. These matrices, in order to have the same spectra, have to be realized as a quaternion factorization of the original guess we had above. They obey the algebra:

$$\left\{ \alpha^a, \alpha^b \right\} = 2\delta^{ab}\mathbb{I}, \quad \beta^2 = \mathbb{I}, \quad \left\{ \alpha^a, \beta \right\} = 0. \tag{17.15}$$

and are Hermitian:

$$(\alpha^a)^\dagger = \alpha^a, \beta^\dagger = \beta. \tag{17.16}$$

These are the Dirac Matrices. To study time evolution, we take this Hamiltonian and put it into the Schrodinger equation:

$$i\frac{\partial}{\partial t}\psi = (i\boldsymbol{\alpha} \cdot \boldsymbol{\nabla} + \beta m)\psi \tag{17.17}$$

where all we have really done is substituted in the Schrodinger operator for the Dirac operator. If we were to derive this theory from the Lagrangian density, we would have something that looks very similar to the original Lagrangian density for our non-relativistic QFT:

$$\mathcal{L} = i\psi_\sigma^\dagger \frac{\partial}{\partial t}\psi_\sigma + \dots \tag{17.18}$$

with the equal-time anticommutation relations:

$$\{\psi_a(x), \psi_b^\dagger(y)\}\delta(x^0 - y^0) = \delta(x - y) \tag{17.19}$$

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$$\{\psi_a^\dagger(x), \psi_b^\dagger(y)\}\delta(x^0 - y^0) = \{\psi_a(x), \psi_b(y)\}\delta(x^0 - y^0) = 0. \tag{17.20}$$

At this point, there is no reason to expect that the states are fermions; one way to figure this out would be that if we had bosons, we would have an unstable theory as we would fill up the negative energy states as much as we want. We need the Pauli principle to stabilize the Fermi sea and ensure that the negative energy states are already filled. Another more formal way to see this; if we use the commutation relations, we would find that the Hamiltonian is not bounded from below. Note that the Poisson bracket actually becomes modified in the case where we have anti-commuting vs. commuting classical fields.

Next time we explore solutions to this theory. Before then; a question; is this theory actually relativistic? We set up the equation such that the spectrum is relativistic with $\sqrt{p^2 + m^2}$. The details we have yet to work out, the answer will (of course) be yes, however.

18 Dirac Field Theory II

18.1 The Dirac Equation

We have been considering the Dirac equation:

$$i\frac{\partial}{\partial t}\psi = \left(i\boldsymbol{\alpha} \cdot \boldsymbol{\nabla} + \beta m\right)\psi \tag{18.1}$$

where $\alpha^a, \alpha^b, \alpha^c, \beta$ are Hermitian matrices (else the above operator would not be Hermitian) that obey an algebra.

Why is this the right thing to do? For one, we can see that the above gives the correct dispersion relation (and the cool part - we can do this without even knowing what the matrices are! We only need to know the algebra of commutation relations we obey). How? We consider the square of the above operator, and use that the various terms anticommute. The square of the Hamiltonian has eigenvalues $p^2 + m^2$, so the Hamiltonian itself must have $\pm \left\{\sqrt{p^2 + m^2}\right\}$. Alright, then how many eigenvalues are there? We can observe that $\alpha^1\alpha^2\alpha^3$ anticommutes with everything, and so when we operate them on the eigenvector of the Hamiltonian, we get another eigenvector but with the opposite sign. Explicitly:

$$H\psi_\omega = \omega\psi_\omega \tag{18.2}$$

where:

$$\omega^2 = -\boldsymbol{\nabla}^2 + m^2 \tag{18.3}$$

and so:

$$H\alpha^1\alpha^2\alpha^3\psi_\omega = -\omega\alpha^1\alpha^2\alpha^3\psi_\omega \tag{18.4}$$

In addition, $\alpha^1\alpha^2\alpha^3$ is nonsingular as $(\alpha^1\alpha^2\alpha^3)^{-1} = (\alpha^1\alpha^2\alpha^3)^\dagger$. In fact we find that $\alpha^1\alpha^2\alpha^3$ has an equal number (2) of eigenvalues $+1$ and -1. So from this we find that there are an equal number of $+\sqrt{p^2 + m^2}$ and $-\sqrt{p^2 + m^2}$ eigenvalues for the Hamiltonian. This discussion is not a big deal, but is meant to show that playing with the algebra is quite profitable - this tells us that any representation of the algebra works. Someday, we may choose an explicit representation to solve the Dirac Equation, but for now let us keep things representation-free and consider space-time symmetry.

18.2 Space-Time symmetry for the Dirac Field Theory

Multiplying both sides of the Dirac equation by $i\beta$, we find:

$$i\left(i\beta\frac{\partial}{\partial t}\right)\psi = \left(i(i\beta\boldsymbol{\alpha}) \cdot \boldsymbol{\nabla} + im\right)\psi \tag{18.5}$$

this looks more Lorentz covariant than it did in the previous formula. To emphasize this even further, let us rename $i\beta = \gamma^0$ and $i\beta\alpha^a = \gamma^a$. Since $\beta^2 = 1$ we find that $(\gamma^0)^2 = -1$. We can then rewrite the above equation as:

$$\left(\gamma^\mu\partial_\mu + m\right)\psi = 0. \tag{18.6}$$

where:

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}. \tag{18.7}$$

So the γs here compactify the Dirac equation, and their commutation algebra in some sense encodes the algebra of the dirac matrices. We can even more consisely write the above using the Feynman slash notation $\slashed{\partial} = \gamma^\mu\partial_\mu$ to find:

$$(\slashed{\partial} + m)\psi = 0. \tag{18.8}$$

So far we have just played with notation; let us now study the symmetries. Recall that the symmetries of the fields we have discussed (scalar, vector, tensor) have symmetries defined by their coordinate transformation properties. So we should do this for the Dirac field. But this is something we have not done yet; spinors don't have a representation as a field. It turns out that the Dirac field transforms like a scalar field - more technically it behaves like a spinor under a sort of "changing of frames" (informally phrased here...) but under transformations behaves like a scalar. It it interesting that somehow nature does not take on the simplest possible geometrical features. Already with the very common field describing (e.g.) electrons, the discussion has already become a bit complicated.

So, we have the Dirac field as in Eq. (18.6), and we can study the space-time symmetries. If we just had the field equation, we would consider some infinitesimal transformation of the fields:

$$\psi \to \psi + \delta\psi \tag{18.9}$$

and this would be a symmetry if:

$$(\slashed{\partial} + m)\psi = 0 \implies (\slashed{\partial} + m)\delta\psi = 0. \tag{18.10}$$

Let's start with translation invariance.

18.2.1 Translation Invariance

Translation Invariance would correspond to:

$$\delta_\mu \psi(x) = -\partial_\mu \psi(x). \tag{18.11}$$

Translation of course being generated by a derivative. It is easy to show that this satisfies the symmetry criterion:

$$0 = \partial_\mu (0) = \partial_\mu \left((\slashed{\partial} + m)\psi\right) \implies (\slashed{\partial} + m)(\partial_\mu \psi) = 0 \tag{18.12}$$

So this is shown to be a symmetry. Of course this is not the most useful way, as to apply Noether's theorem to determine the conserved current we require a Lagrangian for the theory.

18.2.2 Lorentz Invariance

A lorentz transformation would correspond to:

$$\delta\psi = \omega_{\mu\nu}\left(x^\mu \partial^\nu - x^\nu \partial^\mu\right)\psi + S\psi \tag{18.13}$$

where we need a matrix S in order to have the correct transformation. We then have (using the product rule - the $\slashed{\partial}$ has derivatives!):

$$(\slashed{\partial} + m)\delta\psi = \omega_{\mu\nu}(\gamma^\mu \partial^\nu - \gamma^\nu \partial^\mu)\psi + \omega_{\mu\nu}\left(x^\mu \partial^\nu - x^\nu \partial^\mu\right)(\slashed{\partial} + m)\psi + [\gamma^\mu, S]\partial_\mu \psi + S(\slashed{\partial} + m)\psi \tag{18.14}$$

From the dirac equation the second term and fourth term vanish, so:

$$(\slashed{\partial} + m)\delta\psi = \omega_{\mu\nu}(\gamma^\mu \partial^\nu - \gamma^\nu \partial^\mu)\psi + [\gamma^\mu, S]\partial_\mu \psi \tag{18.15}$$

The guess for S is:

$$S = -\omega_{\mu\nu}\frac{1}{4}[\gamma^\mu, \gamma^\nu] \tag{18.16}$$

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we could get this by process of elimination, considering all of the possible Hermitian matrices in our basis. Let's look at the commutator of γ^μ with $\mathbb{S}$:

$$
\begin{aligned}
[\gamma^\mu, \mathbb{S}] &= -\frac{1}{4}\omega_{\rho\sigma}[\gamma^\mu, [\gamma^\rho, \gamma^\sigma]] \\
&= -\frac{1}{4}\omega_{\rho\sigma}\left(\gamma^\mu\gamma^\rho\gamma^\sigma - \gamma^\mu\gamma^\sigma\gamma^\rho - \gamma^\rho\gamma^\sigma\gamma^\mu + \gamma^\sigma\gamma^\rho\gamma^\mu\right) \\
&= -\frac{1}{4}\omega_{\rho\sigma}\left(\gamma^\mu\gamma^\rho\gamma^\sigma - \gamma^\mu\gamma^\sigma\gamma^\rho - \gamma^\rho\left(2\eta^{\sigma\mu} - \gamma^\mu\gamma^\sigma\right) + \gamma^\sigma\left(2\eta^{\rho\mu} - \gamma^\mu\gamma^\rho\right)\right) \\
&= -\frac{1}{4}\omega_{\rho\sigma}\left(\gamma^\mu\gamma^\rho\gamma^\sigma - \gamma^\mu\gamma^\sigma\gamma^\rho + \gamma^\rho\gamma^\mu\gamma^\sigma - \gamma^\rho\gamma^\mu\gamma^\rho - 2\gamma^\sigma\eta^{\sigma\mu} + 2\gamma^\sigma\eta^{\rho\mu}\right) \\
&= -\frac{1}{4}\omega_{\rho\sigma}(4)\left(\eta^{\mu\rho}\gamma^\sigma - \eta^{\mu\sigma}\gamma^\rho\right) \\
&= \omega_{\rho\sigma}(\gamma^\sigma\partial^\rho - \gamma^\rho\partial^\sigma)\psi
\end{aligned}
\tag{18.17}
$$

So we conclude:

$$
[\gamma^\mu, \mathbb{S}]\psi = -\omega_{\mu\nu}(\gamma^\mu\partial^\nu - \gamma^\nu\partial^\mu)\psi
\tag{18.18}
$$

and therefore:

$$
(\slashed{\partial} + m)\delta\psi = 0.
\tag{18.19}
$$

So we have identified a symmetry!

$$
\delta\psi = \left(\omega_{\mu\nu}\left(x^\mu\partial^\nu - x^\nu\partial^\mu\right)\psi - \omega_{\mu\nu}\frac{1}{4}[\gamma^\mu, \gamma^\nu]\right)\psi
\tag{18.20}
$$

However, again we don't have a systematic way of identifying the conserved Noether current. We need a Lagrangian density for that. Note that the space and time overlaps in the above (e.g. $[\gamma^0, \gamma^a]$) are Hermitian rather than anti-Hermitian. The boosts are not unitary. And there is a good reason for this; if they were, $\psi^\dagger\psi$ would be invariant. But this is density, which is not a Lorentz scalar (it is rather the time component of a current). Lorentz symmetries are not like the usual non-relativistic symmetries where $\psi^\dagger\psi$ is invariant, where a symmetry should not change the normalization of the wavefunction.

18.3   Lagrangian Density for the Noether current

So - we could try to find a Noether current. In order to do this, let's be pragmatic and find the Lagrangian density for the Dirac field theory (i.e. a Lagrangian density whose variation gives the Dirac equation).

We have the Dirac equation:

$$
(\gamma^\mu\partial_\mu + m)\psi(x) = 0.
\tag{18.21}
$$

so our Lagrangian density should look like:

$$
\begin{aligned}
\mathcal{L} &= -i\psi^\dagger\gamma^0\gamma^0\frac{\partial}{\partial t}\psi - \dots \\
&= -i(\psi^\dagger\gamma^0)(\slashed{\partial} + m)\psi \\
&= -i\bar{\psi}(\slashed{\partial} + m)\psi
\end{aligned}
\tag{18.22}
$$

where we have defined $\bar{\psi} = \psi^\dagger\gamma^0$. We strongly suspect this is Lorentz invariant (at least certainly its critical points are)... Applying Noether's theorem for translations is basically trivial:

$$
\delta\psi = -\partial_\mu\psi \implies \delta\mathcal{L} = -\partial_\mu\mathcal{L}
\tag{18.23}
$$

and we can use this to find the conserved Noether's current. The Lorentz transformations takes a little more work, but not too much from our work above:

$$\delta\psi = \left(\omega_{\mu\nu}\left(x^\mu\partial^\nu - x^\nu\partial^\mu\right)\psi - \omega_{\mu\nu}\frac{1}{4}[\gamma^\mu,\gamma^\nu]\right)\psi$$
$$\implies \delta\mathcal{L} = \omega_{\mu\nu}(x^\mu\partial^\nu - x^\nu\partial^\mu)\mathcal{L} = \partial^\mu(-2\omega_{\mu\nu}x^\nu\mathcal{L})$$

(18.24)

Our question for next time; are we able to write down a stress tensor that we could use for all of these symmetries?

19 Dirac Field Theory III

We've written down the Lagrangian density for the Dirac field theory:

$$\mathcal{L} = -i\bar{\psi}(\slashed{\partial} + m)\psi \tag{19.1}$$

where $\bar{\psi} = \psi^\dagger \gamma^0$. We are discussing fermions, so the above Lagrangian concerns anti-commuting classical fields. Last time we discussed the Lorentz invariance of the Dirac Field Theory; now we can use Noether's theorem to find the conserved Noether current.

Before that; a quick question; why do we want a $-i$ there? Because $(\gamma^0)^2 = -1$ and therefore with the $-i$ we get the desired $i\psi^\dagger \frac{\partial}{\partial t}\psi$ term when expanding. It isn't a huge problem if we get the normalization wrong, but we've chosen the canonical normalization here.

Additionally, in the textbook, the derivative has been symmetrized. In the above form, the derivative only acts on the right. In the textbook, there is a $\overset{\leftrightarrow}{\slashed{\partial}} = \overset{\rightarrow}{\slashed{\partial}} - \overset{\leftarrow}{\slashed{\partial}}$ in the Lagrangian due to the symmetrization.

Let's go ahead and analyze the symmetries of this theory. The most important one will not be the space-time translation symmetry, but rather the symmetry under phase transformations.

19.1 Phase Symmetry

We consider a variation of the fields:

$$\delta\psi = i\psi, \quad \delta\bar{\psi} = -i\bar{\psi} \tag{19.2}$$

It is easy to see that the Lagrangian is invariant under this phase transformation:

$$\mathcal{L} = -i\bar{\psi}(\slashed{\partial} + m)\psi \mapsto -i(-i\bar{\psi})(\slashed{\partial} + m)(i\psi) = -i\bar{\psi}(\slashed{\partial} + m)\psi = \mathcal{L} \tag{19.3}$$

and so $\delta\mathcal{L} = 0$. Noether's theorem tells us that there is a conserved current:

$$\mathcal{J}^\mu = \delta\psi \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)} + \delta\bar{\psi}\frac{\partial\mathcal{L}}{\partial(\partial_\mu\bar{\psi})} = \bar{\psi}\gamma^\mu\psi \tag{19.4}$$

And therefore:

$$\partial_\mu\mathcal{J}^\mu = \partial_\mu\left(\bar{\psi}\gamma^\mu\psi\right) = 0. \tag{19.5}$$

Let's give this an overall factor of minus one for notational consistency (so when we have $(\gamma^0)^2 = -1$ the minus sign cancells):

$$\mathcal{J}^\mu(x) = -\bar{\psi}\gamma^\mu\psi \tag{19.6}$$

and so much like the previous case, we have the conservation of particle number due to phase symmetry.

19.2 Spacetime Symmetry

One of these symmetries is just translations:

$$\delta_{(\mu)}\psi = -\partial_\mu\psi \tag{19.7}$$

Then:

$$\delta_{(\mu)}\mathcal{L} = -\partial_\mu\mathcal{L} \tag{19.8}$$

as there is no explicit coordinate dependence in the Lagrangian density. So, this is a symmetry and so we can write:

$$\mathcal{J}^\nu_{(\mu)}(x) = \delta_{(\mu)}\psi\frac{\partial\mathcal{L}}{\partial(\partial_\nu\psi)} + \delta_{(\mu)}\bar{\psi}\frac{\partial\mathcal{L}}{\partial(\partial_\nu\bar{\psi})} + \eta^\nu_\mu\mathcal{L}(x). \tag{19.9}$$

The second term is zero as there is no $\partial_\nu \bar{\psi}$ dependence in the Lagrangian density. The first term we can easily read off, so the above evauates to:

$$\mathcal{J}^\nu_{(\mu)}(x) = -i\bar{\psi}\gamma^\mu \partial_\mu \psi - i\eta^\mu_\nu \bar{\psi}(\not{\partial} + m)\psi. \tag{19.10}$$

Since we've now derived the conserved current (independent of the equation of motion), we are now free to rewrite it in a cleaner form using the EoM. Doing so, we can eliminate the last term. We can then write the above as:

$$\mathcal{J}^\nu_{(\mu)}(x) = \mathbb{T}^\nu_{0\mu}(x) \tag{19.11}$$

Then:

$$\partial_\nu \mathbb{T}^\nu_{0\mu} = i\bar{\psi}\not{\partial}\partial_\mu \psi + i\bar{\psi} \overleftarrow{\not{\partial}} \partial_\mu \psi \tag{19.12}$$

The EoM tells us:

$$(\not{\partial} + m)\psi = 0, \quad \psi^\dagger \left(\overleftarrow{\not{\partial}} + m \right) = 0. \tag{19.13}$$

and multiplying on the right by γ^0:

$$\bar{\psi} \left(- \overleftarrow{\not{\partial}} + m \right) = 0 \tag{19.14}$$

and hence:

$$\partial_\nu \mathbb{T}^\nu_{0\mu} = i\bar{\psi}\not{\partial}\partial_\mu \psi + i\bar{\psi} \overleftarrow{\not{\partial}} \partial_\mu \psi = 0. \tag{19.15}$$

It's worth pointing out that this stress tensor seems problematic off the bat; it's not symmetric. This turns out to be not a problem, though.

19.3 Lorentz Symmetry

The infinitesimal Lorentz transformation of the fields reads:L

$$\delta\psi = \omega_{\mu\nu} \left((x^\mu \partial^\nu - x^\nu \partial_\mu)\psi - \frac{1}{4}[\gamma^\mu, \gamma^\nu] \right) \psi \tag{19.16}$$

note that $[\gamma^\mu, \gamma^\nu]$ is often called the spin tensor.

We're going to take a bit of a simplifying shortcut to proceed. We can plug this into the Lagrangian density, and we will find that:

$$\delta\mathcal{L} = \omega_{\mu\nu}(x^\mu \partial^\nu - x^\nu \partial^\mu)\mathcal{L}(x) \tag{19.17}$$

which doesn't look like a symmetry, but by the antisymmetry of ω we can exchange the order of the derivative and hence the above becomes something like $\delta\mathcal{L} = \partial(\dots)$ and so is a symmetry. We find the conserved Noether current to be:

$$\mathcal{J}^\lambda(x) = (x^\nu \mathbb{T}^{\lambda\mu}_0 - x^\mu \mathbb{T}^{\lambda\nu}_0)\omega_{\mu\nu} - \frac{i}{4}\bar{\psi}[\gamma^\lambda, [\gamma^\mu, \gamma^\nu]]\psi\omega_{\mu\nu}. \tag{19.18}$$

We then have:

$$0 = \partial_\lambda \mathcal{J}^\lambda = (\mathbb{T}^{\nu\mu}_0 - \mathbb{T}^{\mu\nu}_0)\omega_{\mu\nu} + \omega_{\mu\nu}\partial_\lambda \left(\frac{i}{4}\bar{\psi}[\gamma^\lambda, [\gamma^\mu, \gamma^\nu]]\psi \right) \tag{19.19}$$

the above identity could be proven using algebra, but let's just rely on Noether's theorem to save us the Hassle. However, the above identity does give us something nice; it allows us to improve the stress tensor, symmetrizing it:

$$\mathbb{T}^{\mu\nu}(x) = \frac{i}{2}\bar{\psi} \left(\gamma^\mu \partial^\nu + \gamma^\nu \partial^\mu \right) \psi \tag{19.20}$$

If we contract the above with a Killing vector, we find:

$$\mathcal{J}_f^\lambda = \mathbb{T}^{\mu\nu}\hat{f}_\nu(x). \tag{19.21}$$

What is more, if the fermion is massless, the $\mathbb{T}^{\mu\nu}$ has vanishing trace! This is because when we take the trace, $\gamma^\mu\partial^\nu$ turns into a $\slashed{\partial}$. Therefore $\mathbb{T}^{\mu\nu}\hat{f}_\nu(x)$ is a conserved current even if $\hat{f}$ is not a Killing vector but a conformal Killing vector.

If we look at the Hamiltonian here, we get the (matrix-valued) Dirac Hamiltonian which we have to confirm is actually sensible. This will have negative and positive eigenvalues. If we chose bosons instead of fermions, it would not be bounded from below; as bosons will arbitrarily fill the negative energy states. We have the so called "Dirac sea", which is stabilized in the case of Fermions (as it is totally filled), like the case of the Fermi sea. Note that in constrast to the Fermi sea however, we have an infinitely deep sea! Another contrast; in the Fermi gas system we have a metal, while in the Dirac field system we have an insulator as there is an energy gap between the dirac sea (negative energy states) and the positive energy states.
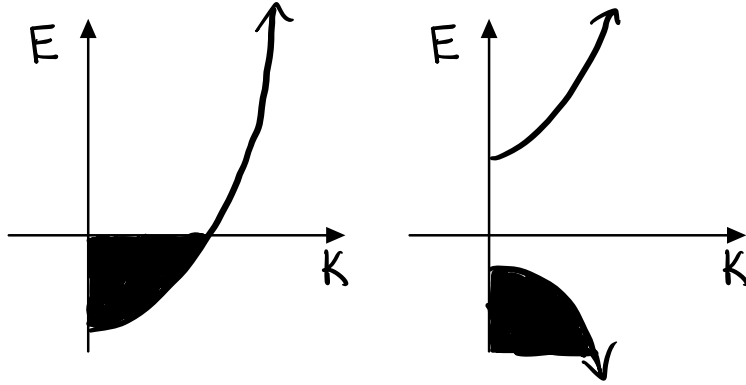

Figure 19.1: Sketch of the energies of a degenerate Fermi gas (left) and the Dirac field (right). The shaded region corresponds to the Fermi and Dirac "seas", respectively.

The infinitely deep quality of the Dirac sea has some interesting implications. It gives us a way to violate some symmetries, for example, e.g. the symmetry that shifts all levels by a unit. In the dirac sea case, if we moved up one unit we would create a particle in the positive energy region and if we moved down unit we would create a hole in the negative energy region.

19.4   Starting to Solve the Dirac Equation

So, we've pretty comprehensively studied the symmetries of the Dirac theory. The last thing left is to actually solve it. We consider the representation:

$$\gamma^0 = \begin{pmatrix} 0 & \mathbb{I} \\ -\mathbb{I} & 0 \end{pmatrix}, \quad \gamma^a = \begin{pmatrix} 0 & \sigma^a \\ \sigma^a & 0 \end{pmatrix} \tag{19.22}$$

where σ^a are the Pauli matrices:

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{19.23}$$

We then have that rotations in the *ab* plane (about the *c* axis) looks like:

$$\frac{i}{4}[\gamma^a, \gamma^b] = -\epsilon^{abc} \begin{pmatrix} \frac{\sigma^c}{2} & 0 \\ 0 & \frac{\sigma^c}{2} \end{pmatrix} \tag{19.24}$$

where the commutators are easily computed using the Pauli algebra. This is a nice representation as the spins are indeed actual spins/rotations, in a sense.

We consider a four-spinor plane wave ansatz:

$$\psi(x) = \begin{pmatrix} u \\ v \end{pmatrix} e^{i(\mathbf{k} \cdot \mathbf{x} - \omega(\mathbf{k})t)} \tag{19.25}$$

In the chosen matrix representation, the Dirac equation with the above ansatz becomes:

$$\begin{pmatrix} m & i\omega + i\sigma \cdot \mathbf{k} \\ -i\omega + i\sigma \cdot \mathbf{k} & m \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = 0. \tag{19.26}$$

and so if we solve the eigenvalues and eigenvectors of the above, we are done! We can see that the spin plays a very active role here due to the $\sigma \cdot \mathbf{k}$ (spin-momentum coupling). We can see that spin is not a good quantum number here as the spin rotation messes it up. What is a good quantum number is if we look for the eigenvalues of $\sigma \cdot \mathbf{k}$ - this is related to the spin along the direction of motion of the particle, or the *helicity*. We will discuss it more next time, where we will conclude our discussion of the Dirac field. We will then move onto photons!

20 Dirac Theory IV, Photon Field I

To finish up our discussion of the Dirac field, we discuss its solutions. We then move onto the photon field. One thing we notice as we get to more realistic fields is that they get more complicated. We started by looking at the scalar field, which corresponds to the Higgs field (and some other emergent fields). The more realistic field is the Dirac field, which describes electrons and other spin-1/2 massive particles. The quantization is only a tiny bit more complicated than it was with the scalar field. The next level up is gravity, or Yang-Mills, which have the same sort of complications, but more complicated.

20.1 Solving the Dirac Equation

We wish to solve the Dirac equation:

$$(\partial\!\!\!/ + m)\psi(x) = 0. \tag{20.1}$$

in other words, find the kernel of the $\partial\!\!\!/ + m$ operator. We notice there are no xs or ts in the expression, so it is automatic what we have to do; make a plane wave ansatz!

$$\psi(x) = \psi_k e^{ik^\mu x_\mu} = \begin{pmatrix} u(k) \\ v(k) \end{pmatrix} e^{i\mathbf{k}\cdot\mathbf{x} - i\omega(\mathbf{k})t} \tag{20.2}$$

This is part of it. The other part is we need to know what the γ^μ matrices are; it is easier to solve the equation if we choose an explicit representation. We choose the representation:

$$\gamma^0 = \begin{pmatrix} 0 & \mathbb{I} \\ -\mathbb{I} & 0 \end{pmatrix}, \quad \gamma^a = \begin{pmatrix} 0 & \sigma^a \\ \sigma^a & 0 \end{pmatrix} \tag{20.3}$$

With this ansatz and choice of representation, the dirac equation becomes:

$$\begin{pmatrix} m & -i\omega + i\sigma\cdot\mathbf{k} \\ i\omega + i\sigma\cdot\mathbf{k} & m \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = 0. \tag{20.4}$$

where if there is just a number, there is implicitly a 2x2 identity matrix. We have to solve the above; this is easy! The only complications are with the $\sigma\cdot\mathbf{k}$; we will to this end look for u, v that are eigenvectors of this:

$$\sigma\cdot\mathbf{k}v_\lambda = \lambda v_\lambda \tag{20.5}$$

And this will have the very nice effect of reducing the matrix we are diagonalizing into a matrix of only numbers. Solving for the eigenvalues is very easy; notice:

$$(\sigma\cdot\mathbf{k})^2 = k^a k^b \sigma^a \sigma^b = \frac{1}{2}k^a k^b \{\sigma^a, \sigma^b\} = \frac{1}{2}k^a k^b 2\delta_{ab} = \mathbf{k}^2 \tag{20.6}$$

and so:

$$\lambda^2 = \mathbf{k}^2 \implies \lambda = \pm|\mathbf{k}| \tag{20.7}$$

How do we figure out if they are plus or minus? Well of course $\text{Tr}(\sigma^a) = 0$ and so the eigenvalues must sum to zero, so there is one $+|\mathbf{k}|$ eigenvalue and one $-|\mathbf{k}|$ eigenvalues. So:

$$\sigma\cdot\mathbf{k}v_s = s|\mathbf{k}|v_s, \quad s = \begin{cases} +1 \\ -1 \end{cases} \tag{20.8}$$

So we can plug this back into our matrix equation; since the matrix mixes u, v we can write them with the same eigenvalue:

$$\begin{pmatrix} m & -i\omega + is|\mathbf{k}| \\ i\omega + is|\mathbf{k}| & m \end{pmatrix} \begin{pmatrix} u_s \\ v_s \end{pmatrix} = 0. \tag{20.9}$$

for this to have a solution, we have to have a non-empty kernel, i.e. the determinant must vanish, so writing down the determinant of the above:

$$m^2 - (\omega^2 - s^2\mathbf{k}^2) = 0 \tag{20.10}$$

with $s^2 = 1$, we easily find the frequency to be:

$$\omega = \pm\sqrt{\mathbf{k}^2 + m^2} \tag{20.11}$$

From this we can find a relation between the spinors u_s, v_s:

$$i(\omega + s|\mathbf{k}|)u_s + mv_s = 0 \tag{20.12}$$

which then we obtain:

$$v_s = -i\frac{\omega + s|\mathbf{k}|}{m}u_s \tag{20.13}$$

we may want to normalize the spinors:

$$|u_s|^2 + |v_s|^2 = 1 \tag{20.14}$$

and so:

$$|u_s|^2 + \left(\frac{\omega + s|\mathbf{k}|}{m}\right)^2|u_s|^2 = 1 \tag{20.15}$$

Therefore:

$$\frac{m^2 + \mathbf{k}^2 + \omega^2 + 2s\omega|\mathbf{k}|}{m^2}|u_s|^2 = 1 \tag{20.16}$$

And since the first two terms in the above add up to ω^2, we find:

$$\frac{2\omega(\omega + s|\mathbf{k}|)}{m}|u_s|^2 = 1 \tag{20.17}$$

and so:

$$u_s = \frac{m}{\sqrt{2\omega(\omega + s|\mathbf{k}|)}}\hat{u}_s \tag{20.18}$$

where is a spinor which satisfies:

$$\sigma \cdot \mathbf{k}\hat{u}_s = s|\mathbf{k}|\hat{u}_s \tag{20.19}$$

$$\hat{u}_s^\dagger\hat{u}_s = 1. \tag{20.20}$$

Let us review; we have found four solutions; we have the defining equation for $\hat{u}_s$ which gives us two solutions (one for $s = \pm 1$) and then we get two solutions each when plugging into the equation for u_s (as we have $\omega = \pm\sqrt{\mathbf{k}^2 + m^2}$) which determines v_s. This makes sense with our expectation that a four-by-four matrix equation should have four solutions!

So, we have the solutions to the Dirac equation:

$$\psi_{s\omega} = \begin{pmatrix} u_{s\omega} \\ v_{s\omega} \end{pmatrix} \tag{20.21}$$

with:

$$\psi_{s\omega}^\dagger\psi_{s\omega} = 1. \tag{20.22}$$

The most general solution would be a superposition (over $\mathbf{k}, \omega, s$), so:

$$\psi(x) = \int \frac{d^3k}{(2\pi)^{3/2}} \left[e^{i\mathbf{k}\cdot\mathbf{x} - i\sqrt{\mathbf{k}^2 + m}t} \sum_s \begin{pmatrix} u_{s\omega}(\mathbf{k}) \\ v_{s\omega}(\mathbf{k}) \end{pmatrix} a_s(\mathbf{k}) + e^{-i\mathbf{k}\cdot\mathbf{x} + i\sqrt{\mathbf{k}^2 + m^2}t} \sum_s \begin{pmatrix} u_{s-\omega}(\mathbf{k}) \\ v_{s-\omega}(\mathbf{k}) \end{pmatrix} b_s^\dagger(\mathbf{k}) \right] \tag{20.23}$$

note that we don't have all the junk we did in the scalar field case, as here our objects are more akin to non-relativistic fermions, which only had the plane wave normalization. s is known as the Helicity. It is not its spin projection along some fixed axis, but rather the projection of spin along its motion (which is some axis, but not a fixed one; it can change as it moves around). To get the equal time commutation realtions for the field ψ, we enforce:

$$\{a_s(\mathbf{k}), a^\dagger_{s'}(\mathbf{l})\} = \delta_{ss'}\delta^3(\mathbf{k}-\mathbf{l}) \tag{20.24}$$

$$\{b_s(\mathbf{k}), b^\dagger_{s'}(\mathbf{l})\} = \delta_{ss'}\delta^3(\mathbf{k}-\mathbf{l}) \tag{20.25}$$

(and all other combinations that we do note write are zero). which implies:

$$\{\psi_a(\mathbf{x}), \psi^\dagger_b(\mathbf{y})\}\delta(x^0 - y^0) = \delta_{ab}\delta^4(x-y) \tag{20.26}$$

so in some sense we have solved our quantum field theory! We can proceed to create our Hilbert (more accurately, Fock) space. We have our vacuum state $|\mathcal{O}\rangle$, which corresponds to the state with all the negative energy states (in the Dirac sea) filled and all of the positive energy ones empty. This is normalized:

$$\langle \mathcal{O}|\mathcal{O}\rangle = 1 \tag{20.27}$$

and such that it is annihilated by the particle/hole annihilation operators:

$$a_s(\mathbf{k})|\mathcal{O}\rangle = 0, \quad b_s(\mathbf{k})|\mathcal{O}\rangle = 0 \tag{20.28}$$

and we have the basis states:

$$|\mathcal{O}\rangle, a^\dagger_s(\mathbf{k})|\mathcal{O}\rangle, b^\dagger_s(\mathbf{k})|\mathcal{O}\rangle, a^\dagger_{s_1}(\mathbf{k}_1)a^\dagger_{s_2}(\mathbf{k}_2)|\mathcal{O}\rangle, \ldots \tag{20.29}$$

We can integrate the $\mathbb{T}^{tt}$ component of the stress tensor to get the Hamiltonian:

$$H = \int d^3\mathbb{T}^{00} = \sum_s \int d^3k \sqrt{k^2 + m^2}\left(a^\dagger_s(\mathbf{k})a_s(\mathbf{k}) + b^\dagger_s(\mathbf{k})b_s(\mathbf{k})\right) \tag{20.30}$$

and we can see these are just the number operators times the energies of the particles/holes (anti-particles) they are counting. This is a stable system. If we had made the mistake of considering this system as bosonic, what would have happened? We would replace the anti-commutators with commutators, and it would follow from our Lagrangian density. We could have gone down what turns out to be the wrong path, and everything would be fine up to the construction of the Fock space (we would have more space because no exclusion principle), but anyway. But something goes wrong when we look at the Hamiltonian. If we had bosons instead of fermions, we would have:

$$H = \int d^3\mathbb{T}^{00} = \sum_s \int d^3k \sqrt{k^2 + m^2}\left(a^\dagger_s(\mathbf{k})a_s(\mathbf{k}) - b^\dagger_s(\mathbf{k})b_s(\mathbf{k})\right) \tag{20.31}$$

this is terrible! The Hamiltonian is not bounded from below. By exciting the bs, we can get an arbitrarily low energy. This would be ok for a free field theory, but as soon as we try to apply it to something in the real world (which couples it to things), we would get transitions, which pushes the system to lower energy states, making it highly unstable. This construction only makes sense if the fields are fermions. It is not a stable theory if we have bosons. This is part of a bigger theorem known as the spin-statistics theorem (a complicated spinoff of our discussion about analyticity). This can be proved that a Lorentz invariant theory has a spin-statistics theorem. It has to do with the fact that if you have a half integer spin, the field is double valued; (spinors have $R_n(2\pi) = -1$, $R_n(4\pi) = 1$), and combining this with lorentz invariance yields the spin-statistics theorem.

By the way, if we calculate the number operators:

$$\mathcal{N} = \int d^3x \psi^\dagger(x)\psi(x) = \sum_s \int d^3k (a^\dagger_s(\mathbf{k})a_s(\mathbf{k}) - b^\dagger_s(\mathbf{k})b_s(\mathbf{k})) \tag{20.32}$$

This makes the number operators very suitable for describing electric charge; the as create electrons, bs create positrons.

20.2 The Photon Field and Maxwell's Equations

We now proceed to study the quantum field theory for the photons. This will be a weakly coupled bosonic theory. Recall from our initial discussion about bosonic fields that Bose fields act somewhat classically. We can proceed to get the Lagrangian from this, and we get something that we are intimately familiar with; classical electrodynamics! So, we can carry these equations of motion over and use them to study the theory of quantum electrodynamics. This might be suspect; how might we know that there aren't terms proportional to $\frac{1}{\hbar}$ etc.? In some sense we don't know, but being a QFT there are certain conditions that need to be mathematically satisfied. These constraints tell us that we don't need to correct Maxwell's equations.

This isn't to say that there aren't effective field theories, which (e.g.) can be used to study photon-photon scattering (a rare event) which looks like something beyond classical electrodynamics (as classical waves do not scatter off of each other) but this is understood as a quantum process, where the photons disintegrate into "virtual" electrons and positrons which scatter from each other and reannhilate etc.

In any case, we have a pretty good guess for how to start with talking about photons; just write down maxwell's equations! In nonrelativistic notation, they are written in any classical electrodynamics text-book. But they are much more consisely written in relativistic notation. We consider the electromagnetic tensor $\mathcal{F}^{\mu\nu}$ with $\mathcal{F}^{0a} \sim E^a$ and $\mathcal{F}^{ab} \sim \epsilon^{abc} B^c$. We can then forget about how Lorentz transforms work, and just use that $\mathcal{F}^{\mu\nu}$ transforms like a tensor! When we stick the electromagnetic fields into $F^{\mu\nu}$, the Maxwell equations are simple to write (we have two sets):

$$\partial_\mu \mathcal{F}^{\mu\nu}(x) = \mathcal{J}^\nu(x) \tag{20.33}$$

$$\partial_\mu \mathcal{F}_{\nu\lambda}(x) + \partial_\nu \mathcal{F}_{\lambda\mu}(x) + \partial_\lambda \mathcal{F}_{\mu\nu}(x) = 0. \tag{20.34}$$

What is normally done to proceed from here is to write down a solution to the above equation. This might be familiar as a Bianchi identity. The equation tells us that it is closed, and on simple spaces this tells us it is exact, and so a solution of this is written as:

$$\mathcal{F}_{\mu\nu}(x) = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{20.35}$$

i.e. in terms of a four-vector field A_μ. Given some topology and boundary considerations, this is a unique solution. After determining this, we can take A_μ to be the dynamical variable, and then use the first set of Maxwell equations to determine how it is governed. At the classical level, A_μ is not particularly important

21 Photon Field II

In the last lecture we concluded our discussion of the Dirac Field, and started our discussion of the Photon Field. Let us continue this now.

21.1 Relativistic Classical Electrodynamics - Getting the Field Equation

As we said last time, to get the degrees of freedom and the classical equations of motion/classical structure of the theory, we should be able to use Maxwell's theory. This statement comes with the assumption that the theory is weakly coupled, and when this happens the physics is dominated by classical behavior. If we assume that classical electrodynamics is this behavior, then Maxwell's equations provide us with the equations of motion. We wrote them in the relativistic form:

$$\partial_\mu \mathcal{F}^{\mu\nu} = \mathcal{J}^\nu \tag{21.1}$$

$$\partial_\mu \mathcal{F}_{\nu\lambda} + \partial_\nu \mathcal{F}_{\lambda\mu} + \partial_\lambda \mathcal{F}_{\mu\nu} = 0 \tag{21.2}$$

and we then said that the solution to the second equation could be cast in terms of a vector potential as:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{21.3}$$

Note that the first set of Maxwell's equations describes a kind of divergence, and the above equation a curl. We can plug this curl equation back into the divergence one to obtain equations of motion. We can also obtain the equations of motion from a Lagrangian density:

$$\mathcal{L} = -\frac{1}{4}\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu} + A_\mu \mathcal{J}^\mu \tag{21.4}$$

The Euler-Lagrange equations give us back the first set of Maxwell's equations:

$$\partial_\lambda \frac{\partial \mathcal{L}}{\partial(\partial_\lambda A_\nu)} - \frac{\partial \mathcal{L}}{\partial A_\nu} = 0. \tag{21.5}$$

computing the derivatives:

$$-\partial_\lambda F^{\lambda\nu} + \mathcal{J}^\nu = 0 \implies \partial_\lambda F^{\lambda\nu} = \mathcal{J}^\nu \tag{21.6}$$

We therefore have obtained the equation of motion, or the field equation; it is nothing more than Maxwell's equations. The other Maxwell's equation is solved automatically as $\mathcal{F}$ depends on A.

So we obtain the field equation; the other piece of structure we need is the commutators.

21.2 Commutator Relations and Covariant Quantization

In order to obtain these, we should study the time dependent terms in the Lagrangian density. These are something like:

$$\mathcal{L} = \frac{1}{2}\left(\frac{\partial}{\partial t}\mathbf{A}(x)\right)^2 - \frac{\partial}{\partial t}\mathbf{A}(x) \cdot \boldsymbol{\nabla} A_0(x) + \ldots \tag{21.7}$$

We see immediatley that $\frac{\partial}{\partial(x^0)}A^0$ does not appear. This means that we are working in a constrained system. A great reference for this is a paper by Dirac in the Canadian Journal of Mathematics. However, we don't solve it in this way. Solving it like a constraint system, it is not clear that the system is Lorentz convariant (though the process may be more logically clear). In our process, we try to keep things as Lorentz covariant as long as possible, as this helps us a lot when we consider calculations. So, we proceed via the method of *covariant quantization*. Gordon used to not like this, and physically oriented colleagues like Bill Unruh also dislikes it. But it is very handy in quantization of bosonic string theory, so those doing

HEP theory may become used to it. It does give the same answer, so it seems alright, though it does have some strange aspects.

We note that our theory is currently Gauge invariant, as if we take:

$$A_\mu \to A_\mu + \partial_\mu \Lambda \tag{21.8}$$

then $\mathcal{F}_{\mu\nu}$ remains unchanged. Further, the other term in the Lagrangian goes as:

$$A^\mu \mathcal{J}^\mu \to A^\mu \mathcal{J}^\mu + \partial_\mu \Lambda \mathcal{J}^\mu = A_\mu \mathcal{J}^\mu + \partial_\mu (\Lambda \mathcal{J}^\mu) \tag{21.9}$$

where the last equality follows if $\partial_\mu \mathcal{J}^\mu = 0$ (which is true if the Maxwell equations hold). This qualifies this as a symmetry of the system, but it's not just any symmetry as Λ is a fairly arbitrary function. Really it should not be treated as a symmetry, but rather a signal that A^μ has too many degrees of freedom in it. We could have perhaps expected this; A^μ is a four field, but a photon only has two polarizations; so somehow we should expect that there is some redundancy. We can make use of this redundancy, however, as we can always gauge transform:

$$A_\mu \to \tilde{A}_\mu = A_\mu + \partial_\mu \Lambda \tag{21.10}$$

such that $\partial^\mu \tilde{A}_\mu = 0$ (fixing the Gauge). This allows us to cancel out a term in the Maxwell's equations:

$$\partial^2 A^\nu - \cancel{\partial^\nu \partial^\mu A_\mu} = J^\nu \tag{21.11}$$

which looks more familiar. The Lagrangian density looks like:

$$\tilde{\mathcal{L}} = -\frac{1}{2} \partial_\mu A^\nu \partial^\mu A_\nu + A_\nu \mathcal{J}^\nu \tag{21.12}$$

so we can take our theory to be defined by the above equation plus the condition that $\partial_\mu A^\mu = 0$. Now, if we look at the time-dependent terms here, we have:

$$\mathcal{L} = \left(\frac{\partial \mathbf{A}}{\partial x^0}\right)^2 - \left(\frac{\partial A^0}{\partial x^0}\right)^2 + \dots \tag{21.13}$$

We also see that:

$$\partial^2 A_\mu = \mathcal{J}_\mu \tag{21.14}$$

and:

$$\partial^2 \left(\partial^\mu A_\mu\right) = 0 \tag{21.15}$$

where we use that $\partial_\nu \mathcal{J}^\nu = 0$. Note that if this condition holds at the initial time, the condition $\partial_\mu A^\mu = 0$ remains to hold.

(Some comment about a term that will plague us and lead to complications, which I missed)

We can look now at the canonical position and momenta, consider their Poisson bracket, and promote them to commutation relations:

$$[A_\mu(x), \frac{\partial}{\partial y^0} A_\nu(y)]\delta(x^0 - y^0) = i\eta_{\mu\nu}\delta^4(x - y) \tag{21.16}$$

21.3 The Zero-Current Case

Let us for the moment set $\mathcal{J}^\mu = 0$ so we can solve something. We then have:

$$\partial^2 A^\mu = 0 \tag{21.17}$$

but we also recall that $\partial_\mu A^\mu = 0$ that we should impose somehow. The strategy we should take to impose it is to take the commutation relations and the field equations, solve them and make a Fock space for

their solutions, and in the Fock space, impose the Gauge fixing condition as a physical state condition (restricting ourselves to a subset of the Fock space where the physical space condition is obeyed).

As we have done many times, we solve the field equation as a superposition of plane waves:

$$A_\mu(x) = \int \frac{d^3k}{\sqrt{(2\pi)^2 2|\mathbf{k}|}} \left[e^{i\mathbf{k}\cdot\mathbf{x} - i|\mathbf{k}|t} a_\mu(\mathbf{k}) + e^{-i\mathbf{k}\cdot\mathbf{x} + i|\mathbf{k}|t} a_\mu^\dagger(\mathbf{k}) \right] \tag{21.18}$$

Note that in the classical case, the electromagnetic fields are real; in the quantum case, this translates to A_μ being Hermitian. To this end we add the negative frequency states in the above equation. In order for the equal-time commutation relations to hold, we require that:

$$[a_\mu(\mathbf{k}), a_\nu^\dagger(\mathbf{k})] = \eta_{\mu\nu}\delta^3(\mathbf{k} - \mathbf{l}) \tag{21.19}$$

This looks like a solution to the first state of things! We can in fact define a vacuum state $|0\rangle$ which is annihilated by all of the a_μs:

$$a_\mu(\mathbf{k})|0\rangle = 0, \quad \langle 0|a_\mu^\dagger(\mathbf{k}) = 0 \tag{21.20}$$

and is normalized:

$$\langle 0|0\rangle = 1 \tag{21.21}$$

and we can use this to construct a basis for our Fock space:

$$\left\{ |0\rangle, a_\mu^\dagger|0\rangle, a_{\mu_1}^\dagger(\mathbf{k}_1) a_{\mu_2}^\dagger(\mathbf{k}_2)|0\rangle, \ldots \right\} \tag{21.22}$$

if we then look at what the Hamiltonian is for this space, we find that the Hamiltonian is diagonal in this basis. This solves the first part of the theory; everything is fixed about the space of states.

Now, consider the norm of the one-photon states:

$$\left| \int d^3k \mathcal{J}^\mu(\mathbf{k}) a_\mu^\dagger(\mathbf{k})|0\rangle \right|^2 = \int d^3k \xi^\mu(\mathbf{k}) \xi_\mu^*(\mathbf{k}) \tag{21.23}$$

However we run into a problem; the RHS is not positive! It seems like with our construction we run into space with non-positive norm, which is nonsensical. We are therefore interested in some physical subspace of the entire space, where the constraint $\partial_\mu A^\mu(x) = 0$ is realized. One thing we could try is to say that:

$$\partial_\mu A^\mu(x)|\text{phys}\rangle = 0. \tag{21.24}$$

for all physical states $|\text{phys}\rangle$. This seems like the obvious thing, but like many places in field theory, the obvious thing does not work as we find that there are no physical states at all, as the above operator has no kernel. We can try something a bit weaker; take the positive frequency components:

$$\partial_\mu A^\mu(x)^{(+)}|\text{phys}\rangle = 0. \tag{21.25}$$

We find that this does imply:

$$\langle \text{phys}|\partial_\mu A^\mu(x)|\text{phys}\rangle = \langle \text{phys}|(\partial_\mu A^\mu(x)^{(+)} + \partial_\mu A^\mu(x)^{(-)})|\text{phys}\rangle = 0. \tag{21.26}$$

We now ask if our generic one-photon case:

$$\int d^3q J^\mu(\mathbf{q}) a_\mu^\dagger(\mathbf{q})|0\rangle \tag{21.27}$$

is physical. So we operate $k_\mu a^\mu(k)$ on it and see when it vanishes. This will give the result that the state is a physical state if it obeys:

$$k_\mu \xi^\mu(k) = 0. \tag{21.28}$$

This determines:

$$\xi^0(k) = \frac{1}{|\mathbf{k}|}\mathbf{k} \cdot \boldsymbol{\xi}(k) \tag{21.29}$$

and the norm of our field becomes:

$$\left|\int \xi^\mu a_\mu^\dagger |0\rangle\right|^2 = \int d^3k \xi^\mu(k)\xi_\mu^*(k) = \int d^3k \xi^i(k)\left(\delta_{ij} - \frac{k_i k_j}{\mathbf{k}^2}\right)\xi^i(k)^* \tag{21.30}$$

where we have written the norm entirely in terms of the spatial pieces, using that the time component is no longer independent.

Now we ask if we have fixed our problem? We are close. The eigenvalues of $\left(\delta_{ij} - \frac{k_i k_j}{\mathbf{k}^2}\right)$ are $1, 1, 0$. So there are no negative norm states anymore (good)! but there are still zero norm states which is quite strange. Note that we looked at one photon states here, but the textbook goes through the easy generalization for multi-photon states:

$$\left|\int d^3k_1 \ldots d^3k_n \eta^{\mu_1 \ldots \mu_n}(k_1 \ldots k_n)a_{\mu_1}^\dagger(k_1) \ldots a_{\mu_n}^\dagger(k_n)|0\rangle\right|^2 = n!\int d^3k_1 \ldots d^3k_n \xi^{\mu_1 \ldots \mu_n}(k_1 \ldots k_n)\xi_{\mu_1 \ldots \mu_n}^*(k_1 \ldots k_n) \tag{21.31}$$

But the physical state condition is that $k_{\mu_i}\eta^{\mu_1 \ldots \mu_n}(k_1 \ldots k_n) = 0$ and so we get:

$$= n!\int d^3k_1 \ldots d^3k_n \xi^{i_1 \ldots i_n}(k_1 \ldots k_n)T_{i_1 j_1 \ldots i_n j_n}\xi^{j_1 \ldots j_n}(k_1 \ldots k_n)^* \tag{21.32}$$

this is now non-negative, but it can still be zero as the Ts do have zero eigenvalues.

In summary, there are physical states with zero norm. This is actually not good enough as states have positive norm in a normal quantum mechanical system. In a normal quantum mechanical system, we would say a state with zero norm is just the zero vector. We have to therefore get rid of these somehow.

We consider an equivalence relation on physical states $|\text{phys}\rangle \sim |\text{phys}'\rangle$ if:

$$\left\||\text{phys}\rangle - |\text{phys}\rangle'\right\|^2 = 0 \tag{21.33}$$

This gets rid of the zero norm states, because if $\left\||\text{phys}\rangle\right\|^2 = 0$, then:

$$\left\||\text{phys}\rangle - \mathbf{0}\right\|^2 = 0 \tag{21.34}$$

and so the physical state is equivalent to the zero vector in the physical space. The equivalence class of zero norm vectors is just the trivial state equivalent to zero. Other vectors outside of this equivalence class will have nonzero norm. This makes sense as an equivalence relation partitions a set into disjoint equivalence classes. These classes become our physical states. We still have to argue that this equivalence relation is sensible, however. We will show this next time - however what does sensible mean? In real QM, this means that if i take the expectation value of any operator with two equivalent states, they take the same answer. If this is not the case, then the equivalence relation is just nonsense. So at most we have to show that this is reproduced.

22 Photon Field III, Functional Methods I

Last time, we studied the physical states of the photon field. We took these as the states for which:

$$\partial_\mu A^\mu(x)^{(+)}|\text{phys}\rangle = 0. \tag{22.1}$$

From this we obtained:

$$\int dq_1 \dots dq_n \xi^{\mu_1 \dots \mu_n}(q_1 \dots q_n) a^\dagger_{\mu_1}(q_1) \dots a^\dagger_{\mu_n}(q_n)|0\rangle \tag{22.2}$$

The physical state condition is that $q_1 \xi^{\mu_1 \dots \mu_n}(q_1 \dots q_n) = 0$. After imposing the condition, we find:

$$\langle \xi|\xi\rangle = n! \int dq_1 \dots dq_n \xi^{i_1 \dots i_n}(q_1 \dots q_n) T_{i_1 j_1} \dots T_{i_n j_n} \xi^{j_1 \dots j_n}(q_1 \dots q_n) \tag{22.3}$$

Where:

$$T_{ij} = \delta_{ij} - q_1 q_j / \mathbf{q}^2 \tag{22.4}$$

which have eigenvalues $1, 1, 0$. So, the physical state condition has removed the negative norm states (it chooses the subset of states where the norm is non-negative). We still need to remove the zero norm states, however.

22.1 An aside - Massive photons

Note: If we have a massive photon, to the Lagrangian density we would add a photon mass term:

$$\mathcal{L} = -\frac{1}{4}\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu} - \frac{m^2}{2}A_\mu A^\mu + \dots \tag{22.5}$$

in this theory, we obtain the equation of motion:

$$-\partial_\mu \mathcal{F}^{\mu\nu} + m^2 A^\nu = 0 \tag{22.6}$$

Taking a four-divergence of this, we would get a different dispersion relation, and the T_{ij}s would differ, in particular:

$$T_{ij} = \delta_{ij} - \frac{q_i q_j}{q^2 + m^2} \tag{22.7}$$

Note that this object does *not* have zero eigenvalues, however; it has eigenvalues $1, 1, \frac{m^2}{q^2+m^2}$. Really this is describing a spin-1 photon with three polarizations. For massive photons we are already done just by imposing the physical state condition, as the zero eigenvalue goes away (but note in the $m \to 0$ limit we find that the eigenvalue is 0).

Although experimental bounds on the photon mass are tiny, tiny small ($10^{-39}m_e$), it doesn't rule out a zero mass entirely. There are some famous field theory textbooks (and in some technical computations) where they take m to be finite all the way through to avoid the complicated argument we will proceed to do.

22.2 Correcting the Zero-Norm States

We still need to cure the zero norm condition. To this end we enforce an equivalence relation:

$$|\text{phys}\rangle \sim |\text{phys}'\rangle \tag{22.8}$$

If

$$\left\||\text{phys}\rangle - |\text{phys}'\rangle\right\|^2 = 0. \tag{22.9}$$

this immediately rules out the zero norm states, as if $|\text{phys}\rangle$ is zero norm, then:

$$\langle\text{phys}|\text{phys}\rangle = 0 \tag{22.10}$$

and so:

$$\big\| |\text{phys}\rangle - \mathbf{0} \big\|^2 = \big\| |\text{phys}\rangle \big\|^2 = \langle\text{phys}|\text{phys}\rangle = 0. \tag{22.11}$$

So, every zero norm state is equivalent to the zero vector.

However, we still have to show the following. If $|\text{phys}\rangle \sim |\text{phys}'\rangle$, then:

$$\langle\text{phys}|O|\text{phys}\rangle = \langle\text{phys}'|O|\text{phys}'\rangle. \tag{22.12}$$

for an observable O. Further, O must be a gauge invariant operator, so:

$$O(a_\mu(k), a_\mu^\dagger(k)) = O(a_\mu + k_\mu\xi(k), a_\mu^\dagger + k_\mu\xi^*(k)) \tag{22.13}$$

Ok, let's try to make an argument for this. Consider one particle states $a_\mu^\dagger(k)|0\rangle$. We decompose this state; this is a bit of a complicated story (recall polarization in undergraduate quantum and electrodynamics)... but to this end let us consider projections e_s^μ. They are orthogonal to k_μ (the direction of propogation) and so:

$$e_s^\mu k_\mu = 0 \tag{22.14}$$

but the projectors have zero time component so $e_s^0 = 0$. This tells us that the spatial component of the vectors are orthogonal:

$$e_s^a k^a = 0 \tag{22.15}$$

There is a two-dimensional subspace orthgonal to k^a, which gives us two projectors. For example if $\mathbf{k} = (0,0,k)$ then $e_1^\mu = (0,1,0,0)$ and $e_2^\mu = (0,0,1,0)$.

So we get two physical polarization states of the photon:

$$e_s^\mu a_\mu^\dagger(k)|0\rangle. \tag{22.16}$$

for $s = 1,2$. We can also consider k^μ (which is orthogonal to the two polarization part by construction). For the final vector we can consider $e_L^\mu = \delta_0^\mu = (1,0,0,0)$ So we have the two additional states:

$$k^\mu a_\mu^\dagger(k)|0\rangle \tag{22.17}$$

$$e_L^\mu a_\mu^\dagger(k)|0\rangle = \delta_0^\mu a_\mu^\dagger(k)|0\rangle = a_0^\dagger(k)|0\rangle \tag{22.18}$$

The physical state condition is that:

$$k_\nu a^\nu(x)|\text{phys}\rangle = 0. \tag{22.19}$$

We then find that $e_s^\mu a_\mu^\dagger(k)|0\rangle, k^\mu a_\mu^\dagger(k)|0\rangle$ pass the condition, and $a_0^\dagger(k)|0\rangle$ fails; so we just keep the first two.

We now consider that since $k_\nu a^\nu(x)|\text{phys}\rangle = 0$, then:

$$\Big\| k^\mu a_\mu^\dagger(k)|\text{phys}\rangle \Big\|^2 = 0. \tag{22.20}$$

Let's consider then the equivalence class of $e_s^\mu a_\mu(k)|0\rangle$. The equivalence class will be:

$$e_s^\mu a_\mu(k)|0\rangle + k^\mu a_\mu^\dagger(k)|\text{phys}\rangle \tag{22.21}$$

where $|\text{phys}\rangle$ ranges over all physical states. This is a huge equivalence class! Now, if we have an operator $O(a, a^\dagger)$, we can see that it will have the same expectation value for members of the same equivalence class. So $\langle O\rangle$ will be representative independent, so long as:

$$[O(a, a^\dagger), k^\mu a_\mu^\dagger(k)] = 0 \tag{22.22}$$

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and

$$[O(a, a^\dagger), k^\mu a_\mu(k)] = 0 \tag{22.23}$$

This concludes the discussion of the theory of the photon for now; hopefully we have a good sense of the gymnastics required to have sensible photon states. Already things are complicated, without a straightforwards quantization. Although perhaps this was not the clearest presentation, the result is exactly the same as if we used constraints instead; and at least this formalism is covariant!

22.3 A First Look at Quantum Electrodynamics

We have our theory of the photon field, and then we add electrons and positrons by adding the dirac field; we then couple them together:

$$\mathcal{L} = -\frac{1}{4}\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu} - i\bar{\psi}(\slashed{\partial} + m)\psi - e\bar{\psi}\gamma^\mu\psi A_\mu(x) \tag{22.24}$$

where we have coupled the vector potential to the number current (times the charge). The thing we need for the coupling is that the current is conserved.

People like to emphasize that the above is gauge invariant; recall that the Gauge transformations read:

$$A_\mu \to A_\mu + \partial_\mu \Lambda \tag{22.25}$$

The new theory above is not gauge invariant as written, but we can make it Gauge invariant by transforming the dirac field as:

$$\psi \to e^{ie\Lambda}\psi, \quad \bar{\psi} \to \bar{\psi}e^{-ie\Lambda} \tag{22.26}$$

so if we combine the free photon gauge invariance condition with the above phase transformation conditions, we obtain a gauge invariance for the coupled theory.

Note that if we already had renormalizability in mind as a condition we require, this is already the most general model we could write down.

We can now proceed to discuss the equations of motion of the theory, commutation relations etc. We will not really do this yet, because we will take another detour and finish recasting our QFT in terms of correlation functions. We will discuss perturbation theory at length - however you may recall from QM that ordinary PT is quite difficult and gets messy quite quickly. So, we will want to do the most elegant formulation.

22.4 Intro to Functional Methods

We will build up our toolbox for calculating correlation functions. We are interested in calculating time-ordered correlation functions, e.g.:

$$\langle 0|TA_{\mu_1}(x_1)\dots A_{\mu_n}(x_n)\psi_{a_1}(y_1)\dots\psi_{a_k}(y_k)\bar{\psi}_{b_1}(z_1)\dots\bar{\psi}_{b_k}(z_k)|0\rangle \tag{22.27}$$

Note that we have the same number of dirac fields k in the correlation function. We had phase symmetry in the classical field theory which we lifted to the quantum field theory. This means that our correlation functions should also have phase symmetry. This implies that correlation functions vanish unless they have the same number of ψ and $\bar{\psi}$. Note that we have enforced a gauge symmetry here yet (and the above is not gauge invariant), so we do slightly wrong things, and at some point we will enforce conditions in order to ensure that we only discuss gauge invariant objects.

As a teaser for next class; we will study functional calculus, where we study functionals. Functionals are mappings from functions to numbers, $F[f]$. They do not have to be real numbers (they can be complex) and similarly f can be real or complex. The interesting question is then how can we take derivatives/integrals of functionals w.r.t. functions. E.g. how do $\frac{\delta}{\delta f(x)}F[f]$ behave? We can jump straight to the gun with taking path integrals, but in this course we will develop some functional calculus first,

reformulate field theory in terms of functional differential equations, and then look at the solutions of these.

A little more of a teaser of how we can consider derivatives. If we expand in a complete set of square integrable functions:

$$f(x) = \sum_n c_n f_n(x) \tag{22.28}$$

and plug it into the functional:

$$F[f] = F[c_n] \tag{22.29}$$

the functional becomes a function of an infinite number of a coefficients. We can then define:

$$\frac{\delta F}{\delta f(x)} = \sum_n \frac{\partial F}{\partial c_n} f_n(x) \tag{22.30}$$

this we look at next time and figure out some properties; we can then use these derived rules instead of referring to the definition next day.

23 Functional Methods

A functional $F[\varphi]$ takes in a function $\varphi(x)$ and outputs a number (a function of a function). Today we will study how to do calculus with functionals.

23.1 Functional Derivative

Is it possible to take a derivative of a functional? We might think to do something like $F[\varphi + \delta\varphi] - F[\varphi]$ and divide by the variation in φ... we can therefore define:

$$F[\varphi + \delta\varphi] - F[\varphi] = \int dx \delta\varphi(x) \frac{\delta F}{\delta\varphi(x)} + \ldots \tag{23.1}$$

However we may want to be a little more concrete about this. We therefore consider a complete set of functions $\{f_1(x), f_2(x), \ldots f_n(x), \ldots\}$ of which some examples that come to mind are plane waves and the eigenfunctions of the quantum harmonic oscillator. If we have a complete set of functions, we can expand out φ in these functions:

$$\varphi(x) = \sum_n c_n f_n(x) \tag{23.2}$$

Let us also assume that our set of functions are orthonormal:

$$\int dx f_n(x) f_m(x) = \delta_{mn} \tag{23.3}$$

and the completeness condition gives:

$$\sum_n f_n(x) f_n(y) = \delta(x - y) \tag{23.4}$$

So, the coefficients in our derivative are obtained in the straightforward way using the orthogonality relation:

$$c_n = \int dx \varphi(x) f_n(x) \tag{23.5}$$

And so in principle if we know the c_ns we can reconstruct φ, and if we know φ we can calculate the c_ns. IF we now consider plugging in φ into the functional F in this expanded form, we have that F has now become a function of all of the c_ns:

$$F[\varphi] \to F[c_1, c_2, \ldots, c_n, \ldots] \tag{23.6}$$

We could then define the functional derivative as:

$$\frac{\delta F}{\delta\varphi(x)} = \sum_n f_n \frac{\partial}{\partial c_n} F[c_1, c_2, \ldots] \tag{23.7}$$

This is not a formula because there is an infinite number of c_ns (in general) and thus we have an infinite series, but we can get by only manipulating ordinary derivatives, something we should be familiar from first-year calculus. With this definition we can derive various formulas for the functional derivative:

$$\frac{\delta}{\delta\varphi(x)}(F_1 F_2) = \frac{\delta F_1}{\delta\varphi(x)} F_2 + F_1 \frac{\delta F_2}{\delta\varphi(x)} \tag{23.8}$$

$$\frac{\delta}{\delta\varphi(x)} g(F[\varphi]) = \frac{\partial g}{\partial F} \frac{\delta F}{\delta\varphi(x)} \tag{23.9}$$

I.e. analogs to the familiar product rule and chain rule formulas. We also obtain (treating $\varphi(y)$ as a functional):

$$\frac{\delta}{\delta\varphi(x)} \varphi(y) = \delta(x - y) \tag{23.10}$$

These rules are generally all we need to know about a functional derivative, and we will rarely have to refer back to the full definition. For example if we consider:

$$K[\varphi] = \int dx_1 \ldots dx_n K(x_1, \ldots x_n) \varphi(x_1) \ldots \varphi(x_n) \tag{23.11}$$

then:

$$\frac{\delta K[\varphi]}{\varphi(x)} = n \int dx_1 \ldots dx_{n-1} K(x, x_1, \ldots, x_{n-1}) \varphi(x_1) \ldots \varphi(x_{n-1}) \tag{23.12}$$

Another thing we might think to do is the functional derivative of an exponential:

$$\frac{\delta}{\delta \varphi} e^{i \int dx' J(x') \varphi(x')} = i J(x) e^{i \int dx' J(x') \varphi(x')} \tag{23.13}$$

Or take the functional derivative of a Gaussian functional:

$$\frac{\delta}{\delta \varphi(x)} e^{-\frac{1}{2} \int dx' dy \varphi(x') \Delta(x', y) \varphi(y)} = - \int dy \Delta(x, y) \varphi(y) e^{-\frac{1}{2} \int dx' dy \varphi(x') \Delta(x', y) \varphi(y)} \tag{23.14}$$

and we can even consider taking the second derivative:

$$\frac{\delta}{\delta \varphi(x)} \frac{\delta}{\delta \varphi(y)} = -\Delta(x, y) e^{-\frac{1}{2} \int \varphi \Delta \varphi} + \int dz_1 dz_2 \Delta(x, z_1) \Delta(x, z_2) \varphi(z_1) \varphi(z_2) e^{-\frac{1}{2} \int \varphi \Delta \varphi} \tag{23.15}$$

23.2 Functional Integral

While functional differentiation is relatively easy, functional integrals are more difficult. How do we make sense of something like $\int d\varphi(x) F[\varphi]$? We really are integrating over an infinite-dimensional space, so things get a bit tricky; not all functions will be integrable here. We do the same expansion of $\varphi(x) = \sum_n c_n f_n(x)$. There is a question of boundary conditions here; perhaps we want φs that go to zero exponentially quickly at infinity, and so we might pick a basis (for the f_ns) that has the same property, for example the harmonic oscillator basis.

Having chosen a basis[23], we can define our functional integral as:

$$\int d\varphi(x) F[\varphi] \equiv \int dc_1 dc_2 \ldots F[c_1, c_2, \ldots] \tag{23.16}$$

so we have an infinite number of integrals over real variables. This is still hard, but sensible. In general there are very few cases in which this can actually be solved, and if it was possible to solve all functional integrals all of QFT would be resolved.

23.3 Functional Gaussian Integral

As an example - we will need to know the functional integral of the Gaussian $e^{-\frac{1}{2} \int dx dy \varphi(x) \Delta(x, y) \varphi(y)}$. But we cannot just integrate any Gaussian; the Δ appearing in it should have a positive definite kernel (positive eigenvalues). We want to find:

$$I = \int [d\varphi(x)] e^{-\frac{1}{2} \int dx dy \varphi(x) \Delta(x, y) \varphi(y)} = \int dc_1 dc_2 \ldots e^{-\frac{1}{2} \sum_{mn} \Delta_{mn} c_n} \tag{23.17}$$

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[23]Note that the actual value of the integral is not basis-dependent, despite the seemingly basis-dependent definition we have written down; up to a possible complication with boundary conditions, the transformation between various bases will be some unitary transformation, and hence the functional integral will be equivalent up to the absolute value of a determinant of a unitary transformation - which is just $|e^{i\varphi}| = 1$.

Where $\Delta_{mn} = \int dxdy f_m(x)\Delta(x,y)f_n(y)$. Δ_{mn} here is real and symmetric. The integral is still quite ugly because of the infinite number of c_ns - we would know how to do the finite case. We do the physicist's hack and stop the sum at some integer K. Then Δ is a $K \times K$ (finite!) real symmetric matrix. Recalling our linear algebra course, we know that such a matrix can be diagonalized by a unitary transformation:

$$\Delta_{mn} = \mathcal{O}_{mp}\Delta_p\mathcal{O}^\dagger_{pn} \tag{23.18}$$

where the $\mathcal{O}$s are unitary (I guess we say orthogonal here because they are real):

$$\mathcal{O}\mathcal{O}^\dagger = \mathcal{O}^\dagger\mathcal{O} = \mathbb{I} \tag{23.19}$$

So then we write the sum as:

$$-\frac{1}{2}\sum_{mnp} c_m\mathcal{O}_{mp}\Delta_p\mathcal{O}^\dagger_{pn}c_n \tag{23.20}$$

We can then change our variables of integration $c \to \mathcal{O}c$. When we do so, the integration measure changes by the Jacobian (absolute value); but here the Jacobian is orthogonal and hence $\det\mathcal{O} = \pm 1$ and so the absolute value is just one. So, the integral becomes:

$$I = \int dc_1 dc_2 \ldots e^{-\frac{1}{2}\sum_p \Delta_p c_p^2} \tag{23.21}$$

we've done something that is valid for finite matrices... we will later push the upper limit to infinity to recover our original functional integral. Doing this integral is now easy! We just need to remember how to do a Gaussian integral:

$$I = \prod_p \int dc\, e^{-\frac{1}{2}\Delta_p c^2} \tag{23.22}$$

To compute this, we can recall the trick of going into polar coordinates:

$$\int dc\, e^{-\frac{1}{2}\Delta c^2} = \sqrt{\int dc_1 dc_2 e^{-\frac{1}{2}\Delta \mathbf{c}^2}} = \sqrt{\int_0^{2\pi} d\varphi \int_0^\infty c\, dc\, e^{-\frac{1}{2}\Delta c^2}} = \sqrt{\frac{2\pi}{\Delta}} \tag{23.23}$$

Therefore we conclude:

$$I = \prod_{p=1}^K \sqrt{\frac{2\pi}{\Delta_p}} \tag{23.24}$$

we now recall the intiial comment that Δ should have positive definite kernel. The reason for this is because we require the eigenvalues to be positive in the product above. If Δ_p is negative we get imaginary terms, and if $\Delta_p = 0$ then the product diverges. We can write I as:

$$I = \frac{1}{\sqrt{\det(\Delta/2\pi)}} \tag{23.25}$$

where we use that the determinant of a real matrix is just the product of the eigenvalues.

There are not many Δs for which this integral makes sense, if we make it go to infinity. Most of the time how we make it sensible is to restore the cutoff in some way. We will not really encounter this integral in this course, though we will see it in stat mech next term. What we will be interested in is doing these integrals with correlation fucntions, and then dividing by the integrals without the insertions, e.g:

$$I = \frac{\int [d\varphi(x)]e^{-\frac{1}{2}\int dxdy\varphi(x)\Delta(x,y)\varphi(y)}\varphi(z_1)\varphi(z_2)\ldots\varphi(z_l)}{\int [d\varphi(x)]e^{-\frac{1}{2}\int dxdy\varphi(x)\Delta(x,y)\varphi(y)}} \tag{23.26}$$

here the determinants cancel so we do not have to worry about it very much. We will want to find this integral, and it is not very hard but there is a fast way to do it, by equating it to:

$$I = \frac{\delta}{\delta J(z_1)} \cdots \frac{\delta}{J(z_l)} \left[\frac{\int [d\varphi(x)] e^{-\frac{1}{2}\int \varphi \Delta \varphi + \int dy J(y)\varphi(y)}}{\int [d\varphi(x)] e^{-\frac{1}{2}\varphi \Delta \varphi}} \right]_{J=0} \tag{23.27}$$

i.e. the functional derivatives of a generating functional. Using the symmetry of the integration domain and the symmetry of the integration measure in function space, we can do a change of variable $\varphi \rightarrow \varphi + \Delta^{-1}J$; then we find the Js and φs decouple into two quadratic terms:

$$I = \frac{\delta}{\delta J(z_1)} \cdots \frac{\delta}{J(z_l)} \left[\frac{\int [d\varphi] e^{-\frac{1}{2}\varphi \Delta \varphi} e^{\frac{1}{2}\int J\Delta J}}{\int [d\varphi] e^{-\frac{1}{2}\int \varphi \Delta \varphi}} \right]_{J=0} \tag{23.28}$$

Then the integrals in φ cancel, and we are just left with the J part; we end up with the simple answer of:

$$I = \sum_{\text{pairings}} \prod_{\langle i,j \rangle \text{pairs}} \Delta^{-1}(z_i, s_j) \tag{23.29}$$

Next time we will apply these techniques to our real simple scalar field theory. We will find (at least for the free field theory) that this will be quite a nice description.

24 Correlation Functions and the Generating Functional

24.1 Reviewing the Free Scalar Field Theory

Last time, we learned how to do functional derivatives and functional integrals. In this lecture we study how to analyze a quantum field theory using functional methods. We will take an example of the simplest possible quantum field theory, that is of the real scalar field $\phi(x)$.

We start with the free field theory - we will discuss interacting fields later. We have the action given by:

$$S[\phi] = \int d^4x \mathcal{L}(x) \tag{24.1}$$

and the Lagrangian:

$$\mathcal{L}(x) = -\frac{1}{2}\partial_\mu \phi(x)\partial^\mu \phi(x) - \frac{m}{2}\phi^2(x) \tag{24.2}$$

Equivalently, we can describe the theory with the field equation (the Klein-Gordon equation):

$$(-\partial^2 + m^2)\phi(x) = 0. \tag{24.3}$$

and the equal time-commutation relations of:

$$[\phi(x), \frac{\partial}{\partial y^0}\phi(y)]\delta(x^0 - y^0) = i\delta^4(x - y) \tag{24.4}$$

We can solve this via the superposition of plane waves:

$$\phi(x) = \int \frac{d^3k}{\sqrt{(2\pi)2k^0}}\left(e^{ik_\mu x^\mu}a(\mathbf{k}) + e^{-ikx}a^\dagger(\mathbf{k})\right) \tag{24.5}$$

and for ϕ to have the correct commutation relations, $a, a^\dagger$ satisfy:

$$[a(\mathbf{k}), a^\dagger(\mathbf{l})] = \delta^3(\mathbf{k} - \mathbf{l}) \tag{24.6}$$

and note that $k^0 = \sqrt{\mathbf{k}^2 + m^2}$. We can then define our basis of states starting with the vacuum state which satisfies:

$$\langle \mathcal{O}|\mathcal{O}\rangle = 1, a(\mathbf{k})|\mathcal{O}\rangle = 0, \langle \mathcal{O}|a^\dagger(\mathbf{k}) = 0 \tag{24.7}$$

and (in the case of bosons) have multi-particle states:

$$|\mathbf{k}_1, \ldots \mathbf{k}_n\rangle = \frac{1}{\sqrt{n!}}a^\dagger(\mathbf{k}_1)\ldots a^\dagger(\mathbf{k}_n)|\mathcal{O}\rangle \tag{24.8}$$

where the $\mathbf{k}_1 \ldots \mathbf{k}_n$ can be interchanged with each other as the $a^\dagger$s commute. The energy is just the sum of the energies of each particle:

$$P^0 = \sum_{j=1}^{n}\sqrt{\mathbf{k}_j^2 + m} \tag{24.9}$$

and the momentum the sum of the momenta:

$$\mathbf{P} = \sum_{j=1}^{n}\mathbf{P}_n \tag{24.10}$$

This is free-field theory; it is nice and solveable. We want to now cast it into a new form (we have discussed two in detail already, that being the field equation + commutation relations, and the Lagrangian). Namely, correlation functions!

24.2 Correlation Function Recasting

We consider n-point functions:

$$\langle\mathcal{O}|T\phi(x_1)\ldots\phi(x_n)|\mathcal{O}\rangle \tag{24.11}$$

where T is the time-ordering operator:

$$T\phi(x_1)\phi(x_2) = \theta(x_1^0 - x_2^0)\phi(x_1)\phi(x_2) + \theta(x_2^0 - x_1^0)\phi(x_2)\phi(x_1) \tag{24.12}$$

where θ is the Heaviside step function:

$$\theta(x) = \begin{cases} 1 & x > 0 \\ 0 & x \leq 0 \end{cases} \tag{24.13}$$

Note that $T\phi(x_1)\phi(x_2)$ is symmetric in $x_1 \leftrightarrow x_2$! We can see this immediately in the 2-point function here, and it is easy to see that it generalizes to n-point functions, as the time-ordering will always reorganize the order of the ϕs regardless of how we might write the x_is.

It seems a bit strange to go from a compact definition of the field theory with the Lagrangian to an infinite number of correlation functions. It indeed would be totally silly if there was not a way to compactly write down the set of correlation functions. To this end, we will use functional techniques.

24.3 The Generating Functional

To begin, we search for a generating functional $Z[J]$. The way it is defined is if we take our n-point function, and multiply them by these functions J, integrate over the points, then sum to infinity and define the leading term as one. I.e.:

$$Z[J] = 1 + \sum_{n=1}^{\infty} \frac{i^n}{n!} \int d^4x_1 \ldots d^4x_n J(x_1) \ldots J(x_n)\langle\mathcal{O}|T\phi(x_1)\ldots\phi(x_n)|\mathcal{O}\rangle \tag{24.14}$$

We can see that this contains all of our correlation functions. In order to get them out, we take functional derivatives and take J to zero:

$$\langle\mathcal{O}|T\phi(x_1)\ldots\phi(x_n)|\mathcal{O}\rangle = \frac{1}{i}\frac{\delta}{\delta J(x_1)} \ldots \frac{1}{i}\frac{\delta}{\delta J(x_n)}Z[J]\Big|_{J=0} \tag{24.15}$$

So if we can study this nice compact form. Note that there is an equivalent way of writing this as:

$$Z[J] = 1 + \sum_{n=1}^{\infty} \frac{1}{n}\langle\mathcal{O}|T(i\int d^4x J(x)\phi(x))^n|\mathcal{O}\rangle \tag{24.16}$$

where we have correctly assumed the commutativity of the time-ordering and the integral. We can then use the definition of the exponential function and write:

$$Z[J] = \langle\mathcal{O}|Je^{i\int d^4x J(x)\phi(x)}|\mathcal{O}\rangle. \tag{24.17}$$

We want to understand $Z[J]$ a little bit better.

24.4 Finding a Functional Differential Equation

In order to do so, we look for a functional differential equation it solves, and then solve that differential equation. To begin, we look at the first derivative of $Z[J]$:

$$\frac{1}{i}\frac{\delta Z[J]}{\delta J} = \sum_{n=0}^{\infty} \frac{i^n}{n!} \int dx^1 \ldots dx_n J(x_1) \ldots J(x_n)\langle\mathcal{O}|T\phi(x)\phi(x_1)\ldots\phi(x_n)|\mathcal{O}\rangle \tag{24.18}$$

What I do next is to operate the wave operator (Klein-Gordon operator) on the above object. We are cavalier, and move the operator through all of the integrals. We might think that this results in the RHS vanishing, as $(-\partial^2 + m^2)\phi = 0$. But this is not quite correct. This is because we have a time-derivative in the wave operator, and the time-ordering operator makes the product of fields have a discontinuity in the time. Thus we should get delta functions in time when we look at the time derivatives. Let's work it out:

$$(-\partial^2 + m^2)\frac{1}{i}\frac{\delta Z[J]}{\delta J(x)}$$

$$= \sum_{n=0}^{\infty} \frac{i^n}{n!} \int dx_1 \dots dx_n J(x_1) \dots J(x_n)(-\partial^2 + m^2)\langle\mathcal{O}|T\phi(x)\phi(x_1)\dots\phi(x_n)|\mathcal{O}\rangle$$

$$= \sum_{n=0}^{\infty} \frac{i^n}{n!} \int dx_1 \dots dx_n J(x_1) \dots J(x_n)\left(\left(\frac{\partial}{\partial x^0}\right)^2\langle\mathcal{O}|T\phi(x)\phi(x_1)\dots\phi(x_n)|\mathcal{O}\rangle - \langle\mathcal{O}|T\left(\frac{\partial}{\partial x^0}\right)^2\phi(x)\phi(x_1)\dots\phi(x_n)|\mathcal{O}\rangle\right)$$

$$= \sum_{n=0}^{\infty} \frac{i^n}{n!} \int dx_1 \dots dx_n J(x_1) \dots J(x_n)\frac{\partial}{\partial x^0}\langle\mathcal{O}|\delta(x^0 - x_1^0)[\phi(x),\phi(x_1)]\dots|\mathcal{O}\rangle + \dots$$

$$= \sum_{n=0}^{\infty} \frac{i^n}{n!} \int dx_1 \dots dx_n J(x_1) \dots J(x_n)\sum_{i=1}^{m} -i\delta^4(x - x_i)\langle\mathcal{O}|T\phi(x_1)\dots\phi(x_{i-1})\phi(x_{i+1})\dots\phi(x_n)|\mathcal{O}\rangle$$

$$\text{(24.19)}$$

where we note that how the discontinuity due to the time ordering results in step functions, which when we take the time derivatives results in a dirac delta. Note the first time derivative commutes with the ordering, but the second time derivative gives us nonzero terms. The final expression has a sum over all of the ϕs, with the $\phi(x)$ dropping out and the $\phi(x_i)$ dropping out to be a dirac delta in each term. Using this, we can compare this back to the first term. One delta function per term liberates a J, and the rest sums up to the generating functional again. Hence, the slightly messy but absolutely correct discussion can be summarized in the result:

$$(-\partial^2 + m^2)\frac{1}{i}\frac{\delta}{\delta J(x)}Z[J] = J(x)Z[J] \tag{24.20}$$

which is a functional differential equation for $Z[J]$. Now, if we use:

$$\frac{1}{Z[J]}\frac{\delta}{\delta J(x)}Z[J] = \frac{\delta}{\delta J(x)}\ln(Z[J]) \tag{24.21}$$

so dividing the functional differential equation by $Z[J]$ we have:

$$(-\partial^2 + m^2)\frac{1}{i}\frac{\delta}{\delta J(x)}\ln Z[J] = J(x) \tag{24.22}$$

24.5 Solving the Functional Differential Equation

We need to solve this equation; we do this by finding a Green function. It satisfies the equation:

$$(-\partial^2 + m^2)g(x,y) = \delta(x - y) \tag{24.23}$$

They are notoriously ambiguous; for example we can add any solution the KG equation and we would still have a solution. Let assume we have fixed the ambiguity somehow, and have found a Green function. Then we can turn the derivative equation into an integral equation:

$$\frac{\delta}{\delta J(x)}\ln Z[J] = \int ig(x,y)J(y)d^4y \tag{24.24}$$

Note that an integrability condition of the above is that $g(x,y)$ is symmetric, else the RHS would not be a derivative of anything. Now, we can guess the functional anti-derivative of $\ln Z[J]$ to be:

$$\ln Z[J] = \frac{i}{2} \int dx dy J(x) g(x,y) J(y) + C \tag{24.25}$$

We can fix the constant by the boundary condition $Z[0] = 1$, so $C = 0$. Then, we can take the exponential of both sides to find:

$$Z[J] = e^{\frac{i}{2} \int d^4 x d^4 y J(x) g(x,y) J(y)} \tag{24.26}$$

so if we know the Green function, we have obtained a beautifully compact representation of the correlation functions! So, let's find it. What we do know is that two functional derivatives give us the two-point function:

$$\langle \mathcal{O} | T\phi(x_1)\phi(x_2) | \mathcal{O} \rangle = -\frac{\delta^2}{\delta J(x_1) J(x_2)} Z[J] \bigg|_{J=0} = -ig(x_1, x_2) \tag{24.27}$$

i.e. the two-point functions are Green functions! From this we obtain the specific Green function that we want; it is specifically the two-point function! To confirm this, let's operate the wave operator on it:

$$(-\partial^2 + m^2)\langle \mathcal{O} | T\phi(x_1)\phi(x_2) | \mathcal{O} \rangle = \Delta(x_1, x_2) = -ig(x,y) \tag{24.28}$$

We can use the time derivative and the product rule (keeping in mind the stepside functions that come from the time ordering) to find that indeed:

$$(-\partial^2 + m^2)\langle \mathcal{O} | T\phi(x_1)\phi(x_2) | \mathcal{O} \rangle = -i\delta(x_1 - x_2) \tag{24.29}$$

so this is a Green's function. This is known as the *Feynman propogator*. Note that in specifying it as the two-point function we have fixed a boundary condition that resolves the ambiguity in the Green function. We can get an explicit form of $\Delta(x_1, x_2)$ (down to some integrals which can't be evaluated fully - they evaluate to some modified Bessel functions) by plugging in the solution to the field theory:

$$\Delta(x_1, x_2) = \Theta(x_1^0 - x_2^0) \int \frac{d^3 k}{(2\pi)^3 2k^0} e^{ik(x_1 - x_2)} + \Theta(x_2^0 - x_1^0) \int \frac{d^3 k}{(2\pi)^3 2k^0} e^{-ik(x_1 - x_2)} \tag{24.30}$$

Then:

$$-ig(x_1, x_2) = -i \int \frac{d^4 k}{(2\pi)^4} \frac{e^{ik_\mu (x_1 - x_2)^\mu}}{k_\nu k^\nu + m^2 - i\epsilon} \tag{24.31}$$

Operating $-\partial^2 + m^2$ on both sides, we can see that we get out $-\delta(x_1 - x_2)$ as required. The boundary condition is fixed by the ϵ we put in the denominator. Specifically, note the sign here. Its role is to specify what happens when $k_\nu k^\nu = -m^2$. The ϵ moves the real zeroes of k to those with an imaginary part.

Now, if we do the k^0 integral using Cauchy's integral formula (noting the poles in the complex plane off the real axis), one can show that the integral reproduces the above solution in Eq. (24.30). We don't go through it here; it is a technical exercise in complex analysis.

When we find this, we are convinced that we know what Δ is. Given this concrete expression, and given that it is proportional to the Green function, we can come back to the generating functional and put in Δ:

$$\Delta(x,y) = \langle \mathcal{O} | T\phi(x)\phi(y) | \mathcal{O} \rangle = -i \int \frac{d^4 k}{(2\pi)^4} e^{ik(x-y)} \frac{1}{k^2 + m^2 - i\epsilon} \tag{24.32}$$

and this compact expression contains all information about the time-ordered correlation functions in our QFT! Just take a bunch of functional derivatives of it.

Note that in the free field theory case we could solve for it exactly, and found it to be a simple form - however it will still be a useful construct for when we discuss interacting fields, starting next lecture.

25 Functional Integrals for the Scalar Field

Last time, we discussed how to find the generating functional for the correlation functions of a free scalar theory. Recall that a free scalar field theory obeys the free field equation:

$$(-\partial^2 + m^2)\phi(x) = 0 \tag{25.1}$$

and the generating functional is given by:

$$Z[J] = \langle \mathcal{O}|Te^{i\int dyJ(y)\phi(y)}|\mathcal{O}\rangle \tag{25.2}$$

We then found the explicit expression:

$$Z[J] = e^{-\frac{1}{2}\int dxdy J(x)\Delta(x,y)J(y)} \tag{25.3}$$

where:

$$\Delta(x,y) = \langle \mathcal{O}|T\phi(x)\phi(y)|\mathcal{O}\rangle = -i\int \frac{d^4k}{(2\pi)^4}e^{ik(x-y)}\frac{1}{k^2 + \mu^2 - i\epsilon} \tag{25.4}$$

note where not shown, $k^2 = k_\mu k^\mu$ and $kx = k_\mu x^\mu$ etc. Mote that it is easy to show that Δ is a Green function:

$$(-\partial^2 + m^2)\Delta(x,y) = -i\int \frac{d^4k}{(2\pi)^4}\frac{e^{ik(x-y)}}{k^2 + m^2 - i\epsilon}(k^2 + m^2) = -i\delta^4(x-y) \tag{25.5}$$

This is something very special to us; it is the Feynman propogator.

25.1 Rewriting the Generating Functional

What we will now do is take this beautiful compact formula, and undo it just a little bit. This is not to understand the free field theory better (we can't really do better than what we have done already) but rather to give something which we can more easily generalize to interacting quantum field theory. Let us write the generating functional as the ratio of Gaussian integrals of classical fields:

$$Z[J] = \frac{\int [d\phi]e^{i\int d^4x\left(-\frac{1}{2}\partial_\mu\phi(x)\partial^\mu\phi - \frac{m^2}{2}\phi^2(x) + J(x)\phi + i\frac{\epsilon}{2}\phi^2(x)\right)}}{\int [d\phi]e^{i\int d^4x\left(-\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{m^2}{2}\phi^2 + i\frac{\epsilon}{2}\phi^2(x)\right)}} \tag{25.6}$$

When we do the Gaussian integral, we get a Gaussian in J with an inverse of the quadratic form $\phi(x)(-\partial^2 + m^2)\delta(x-y)\phi(y)$, which is just the Green function, so:

$$Z[J] = e^{\frac{i}{2}\int dxdy J(x)g(x,y)J(y)} \tag{25.7}$$

Now, we discussed that $\Delta(x,y) = -ig(x,y)$, so the result of doing this functional integral is exactly:

$$Z[J] = e^{-\frac{1}{2}\int dxdy J(x)\Delta(x,y)J(y)}. \tag{25.8}$$

How does the functional integral know that the Green function is the Feynman propogrator? Really, we should be integrateing with a little damping factor; thats the $\frac{i\epsilon}{2}\phi^2(x)$ in the numerator. It makes the integral more convergent. That gives us an $i\epsilon$ that tells us what the Green function should be; it should be:

$$g(x,y) = \frac{1}{-\partial^2 + m^2 - i\epsilon} \tag{25.9}$$

which gives the correct expression under the fourier transform. Note that the $i\epsilon$s in the integral are almost always implicit; they are rarely written. In this form, we can note that the integrand is just a classical action. Note that before this $\phi(x)$ was an operator/quantum field, but in this integration formula it is a classical function.

25.2 Wick's Theorem

Another way to write the functional integral is to write it as:

$$\langle \mathcal{O}|T\phi(x_1)\dots\phi(x_n)|\mathcal{O}\rangle = \frac{\int [d\phi]e^{iS[\phi]}\phi(x_1)\dots\phi(x_n)}{\int [d\phi]e^{iS[\phi]}} \tag{25.10}$$

Where the action is:

$$S[\phi] = \int d^4x \left(-\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{m^2}{2}\phi^2 \right) \tag{25.11}$$

this integral we can do easily. The way to do it is to come back to it as its representation as a generating function. Doing so, these integrals become a sum of products of two point functions:

$$\langle \mathcal{O}|T\phi(x_1)\dots\phi(x_n)|\mathcal{O}\rangle = \sum_{\text{pairings}}\prod_{ij\text{ pairs}}\Delta(x_i, x_j) \tag{25.12}$$

Note that this is only possible when n is even; when n is odd we cannot do this pairing. But actually by symmetry considerations the integral vanishes when ϕ is odd, so this case is trivial anyway. This expression of the correlation function as the sum of products of two-point functions is known as Wick's theorem - which will be a very useful one indeed.

As an example, Wick's theorem says:

$$\frac{\int [d\phi]e^{iS}\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)}{\int [d\phi]e^{iS}} = \Delta(x_1, x_2)\Delta(x_3, x_4) + \Delta(x_1, x_3)\Delta(x_2, x_4) + \Delta(x_1, x_4)\Delta(x_2, x_3) \tag{25.13}$$

So, this concludes our discussion of the free field. To find correlation functions, we just take our classical action and throw in the fields for which we wish to compute the correlation function for.

25.3 Interacting Scalar Field

We consider the Interacting Scalar Field, with Lagrangian:

$$\mathcal{L}(X) = -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{m^2}{2}\phi^2 - \frac{\lambda}{4!}\phi^4 \tag{25.14}$$

Note that we do not add the ϕ^3 term given the desired invariance under $\phi \to -\phi$. We obtain the field equation:

$$(-\partial^2 + m^2)\phi(x) = -\frac{\lambda}{3!}\phi^3(x) \tag{25.15}$$

which is a nonlinear equation which we do not know how to solve. The commutation relations remain unchanged:

$$[\phi(x), \frac{\partial}{\partial y^0}\phi(y)]\delta(x^0 - y^0) = i\delta(x - y) \tag{25.16}$$

Now, let's consider the time-ordered correlation functions

$$\langle \mathcal{O}|T\phi(x_1)\dots\phi(x_n)|\mathcal{O}\rangle \tag{25.17}$$

Note that since the theory is not analytically solvable, we do not have a clear definition for $|\mathcal{O}\rangle$ as before. Instead, we assume that somewhere in the background here that there is some Hamiltonian for which $|\mathcal{O}\rangle$ is the ground state. There is more dicussion to be had about this, e.g. Lorentz invariance might constrain the energy to be zero, but let us move on for now. We take this vacuum and take the expectation of the

product of n interacting fields. To figure out what these correlation functions are, we might again look for a generating functional:

$$Z[J] = \langle \mathcal{O}|Je^{i\int d^4xJ(x)\phi(x)}|\mathcal{O}\rangle \tag{25.18}$$

How do we look for the generating functional? In the free field theory we looked for a functional differential equation. We can try to find the same here. Let us operate the KG wave operator on the time-ordered correlation function; unlike the free field case, we get $(-\partial^2 + m^2)\phi = -\frac{\lambda}{3!}\phi$ whenever the KG operator acts on terms of that form. So:

$$(-\partial^2 + m^2)\langle\mathcal{O}|T\phi(x_1)\dots\phi(x_n)|\mathcal{O}\rangle = -\frac{\lambda}{3!}\langle\mathcal{O}|T\phi^3(x_1)\phi(x_2)\dots\phi(x_j)|\mathcal{O}\rangle - i\sum_{l=1}^{n}\delta(x_1 - x_l)\langle\mathcal{O}|J\phi(x_2)\dots\phi(x_{l-1})\phi(x_{l+1})\dots\phi($$

$$\tag{25.19}$$

Multiplying by $\frac{i^{n-1}}{(n-1)!}\int dx_2\dots dx_nJ(x_2)\dots J(x_n)$, we obtain:

$$(-\partial^2 + m^2)\frac{1}{i}\frac{\delta}{\delta J(x)}Z = J(x)Z - \frac{\lambda}{3!}\left(\frac{1}{i}\frac{\delta}{\delta J(x)}\right)^3 Z[J] \tag{25.20}$$

Unlike what we had before which was a first order functional differential equation (and hence solvable), we now have a third order functional differential equation which we cannot solve analytically. We can however construct a formal solution which will be useful. Consider:

$$Z[J] = e^{-i\int dy\frac{\lambda}{4!}\left(\frac{1}{i}\frac{\delta}{\delta J(y)}\right)^4}e^{-\frac{1}{2}\int dxdyJ(x)\Delta(x,y)J(y)} \tag{25.21}$$

and it is possible to show that this satisfies the above differential equation.If we take $\lambda \to 0$ and compute $\Delta(x,y)$, we find that the Green function that goes into the above is just $i\delta^4(x - y)$ (i times the Feynman propogator) as we had before. This is a formal solution as we don't have a closed form expression for this fourth order functional derivative in the exponential. It is however an excellent starting point for doing perturbation theory, as we can taylor expand in powers of the coupling constant λ. Then, by computing functional derivatives (easy), we can compute the generating functional order by order in λ. We will come back to this formula once in a while, because it is a rather nice and compact expression. However, there is another useful expression, and that will be to rewrite the RHS as a functional integral:

$$Z[J] = \frac{\int[d\phi]e^{iS[\phi]+i\int dxJ(x)\phi(x)}}{\int[d\phi]e^{iS[\phi]}} \tag{25.22}$$

with:

$$S[\phi] = \int d^4x\left(-\frac{1}{2}\partial_\mu\phi(x)\partial^\mu\phi(x) - \frac{m^2 - i\epsilon}{2}\phi^2(x) - \frac{\lambda}{4!}\phi^4(x)\right) \tag{25.23}$$

So then:

$$\langle\mathcal{O}|T\phi(x_1)\dots\phi(x_n)|\mathcal{O}\rangle = \frac{\int[d\phi]e^{iS[\phi]}\phi(x_1)\dots\phi(x_n)}{\int[d\phi]e^{iS[\phi]}} \tag{25.24}$$

For the generic field theory, this cannot be solved. We can however expand things out in a Taylor series:

$$\langle\mathcal{O}|T\phi(x_1)\dots\phi(x_n)|\mathcal{O}\rangle = \frac{\sum_{k=0}^{\infty}\frac{i^k}{k!}\left(\frac{-\lambda}{4!}\right)^k\int d\phi e^{iS_0[\phi]}\left(\int d^4x\phi^4(x)\right)^k\phi(x_1)\dots\phi(x_n)}{\sum_{k=0}^{\infty}\frac{i^k}{k!}\left(\frac{-\lambda}{4!}\right)^k\int d\phi e^{iS_0[\phi]}\left(\int d^4k\phi^4(x)\right)^k} \tag{25.25}$$

where:

$$S_0[\phi] = \int d^4y\left(-\frac{1}{2}\partial_\mu\phi(xy\partial^\mu\phi(y) - \frac{m^2 - i\epsilon}{2}\phi^2(y)\right) \tag{25.26}$$

Since we just have the free-field action here, we can use Wick's theorem to compute the n-point functions as the sum of products of two-point functions.

We will continue to pursue the route of perturbation theory to study these non-analytically solvable theories. Stay tuned!

TODO - perhaps fill in some notes on the lecture I missed?

26 Perturbation Theory

We've been discussing scalar field theory for a while now, and this is because we want to use it as an example for how to use perturbation theory to study systems that are not analytically solvable.

We wish to calculate $\langle \mathcal{O}|T\phi(x_1)\ldots\phi(x_n)|\mathcal{O}\rangle$ as a series in λ. These correlation functions tell us a lot about the theory. For example, the spectral theorem (which is a technical point which we may come back to later) tells us that we can use the two-point functions to deduce something about the energy and momentum of the field theory. From other correlation functions beyond two-point functions, we can use them to calculate scattering matrix element. Scattering (S) matrix elements describe a scattering event, where fields interact (go in and come and out in a reorganized form). The matrix elements give quantum amplitudes for reorganization - somewhat analogous to transition amplitudes that you would have computed in your quantum mechanics course.

We have already started this process - we have already found the $\lambda = 0$ contribution, in the form of Wick's theorem. As a reminder, Wick's theorem told us that for the free field theory:

$$\langle \mathcal{O}|T\phi(x_1)\ldots\phi(x_n)|\mathcal{O}\rangle = \sum_{\text{pairings}} \prod_{\text{pairs}} \Delta(x_i, x_j) \tag{26.1}$$

So for the interacting field theory:

$$\langle \mathcal{O}|T\phi(x_1)\ldots\phi(x_n)|\mathcal{O}\rangle = \sum_{\text{pairings}} \prod_{\text{pairs}} \Delta(x_i, x_j) + O(\lambda) \tag{26.2}$$

and we now want to probe the $O(\lambda)$ part. This is in some sense where the interesting things occur - if we neglect it, then the particles do not interact so the scattering matrix is highly uninteresting indeed (but of course this makes sense, as the $\lambda = 0$ limit is the free/non-interacting QFT).

26.1 Counterterms

We consider the Lagrangian:

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{m^2}{2}\phi^2 - \frac{\lambda}{4!}\phi^4 \tag{26.3}$$

what we are doing is modelling - note in this field theory we really only have three parameters (m, λ, and the scale of ϕ). So, if we model something with this field theory, we calculate three things and match them to experiment, and see if they agree!

If we add any higher order terms, our theory becomes un-normalizable - well, actually, we have to add a bunch more parameters at every order, i.e. becomes a theory with infinite number of parameters.

Note that we are allowed to re-scale ϕ in this theory - this is where the term of renormalization may have first came from (now it means something else). We can rewrite the above with some rescaling of the theory:

$$\mathcal{L} = \frac{Z}{2}\partial_\mu\phi\partial^\mu\phi - \frac{m^2}{2}Z_m\phi^2 - \frac{\lambda}{4!}Z_\lambda\phi^4 \tag{26.4}$$

where now we can call m the actual mass, λ the actual strength of the interaction, and the Zs are now the tunable parameters. Now we do perturbation theory in λ. For this theory, it has been done to order ~ 5 or so. But the complexity increases steeply with the order, so PT is not a trivial procedure here.

To the end, suppose that the renormalization constants can be expanded our in a power series in λ (the physical/actual coupling constant):

$$Z = 1 + z^{(1)}\lambda + z^{(2)}\lambda^2 + \ldots = 1 + z \tag{26.5}$$

$$Z_m = 1 + z_m^{(1)}\lambda + z_m^{(2)}\lambda^2 + \ldots = 1 + z_m \tag{26.6}$$

$$Z_\lambda = 1 + z_\lambda^{(1)}\lambda + z_\lambda^{(2)} + \lambda^2 = 1 + z_\lambda \tag{26.7}$$

So we can write the full Lagrangian as the free field Lagrangian plus the interaction terms:

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_I \tag{26.8}$$

where:

$$\mathcal{L}_0 = -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{m^2}{2}\phi^2 \tag{26.9}$$

$$\mathcal{L}_I = -\frac{z}{2}\partial_\mu\phi\partial^\mu\phi - \frac{m^2}{2}z_m\phi^2 - \frac{\lambda Z_\lambda}{4!}\phi^4 \tag{26.10}$$

Now we want to compute something perturbatively.

26.2   Perturbative Calculation

Let's compute the two-point function:

$$D(x,y) = \langle 0|T\phi(x)\phi(y)|0\rangle = \Delta(x,y) + O(\lambda) \tag{26.11}$$

i.e. the free field-propogator plus corrections of order λ and higher. Recall that:

$$\Delta(x,y) = \int \frac{d^4k}{(2\pi)^4} e^{ik_\mu(x-y)^\mu} \frac{-i}{k^2 + m^2 - i\epsilon} \tag{26.12}$$

We recall the functional integral representation of $D(x,y)$:

$$D(x,y) = \frac{\int d\phi(x)e^{i\int d^4x\mathcal{L}_0 + i\int d^4x\mathcal{L}_I(x)}\phi(x)\phi(y)}{\int d\phi(x)e^{i\int d^4x\mathcal{L}_0 + i\int d^4x\mathcal{L}_I}} \tag{26.13}$$

now, let us take the RHS of the above equation and Taylor expand in λ. To zeroth order, we set $\mathcal{L}_I = 0$ and we just have the free-field theory two-point function $\Delta(x,y)$. Going to first order, we keep everything to first order in λ:

$$D(x,y) = \Delta(x,y) + \frac{\int d\phi(x)e^{i\int L_0(x)}\phi(x)\phi(y)\left(-i\int d^4z\left(\frac{\lambda}{4!}\phi^4(z) + \frac{\lambda m}{2}z_m^{(1)}\phi^2(z) + \frac{\lambda z^{(1)}}{2}\partial_\mu\phi(z)\partial^\mu\phi(z)\right)\right)}{\int d\phi e^{i\int \mathcal{L}_0}}$$

$$- \frac{\int d\phi e^{i\int L_0}\phi(x)\phi(y)\int d\phi e^{i\mathcal{L}_0}\left(-i\int dz\frac{\lambda\phi^4}{4!} + \frac{mz^{(1)}m\lambda}{2}\phi^2 + \frac{z^{(1)}}{2}\lambda(\partial\phi)^2\right)}{(\int d\phi e^{i\int \mathcal{L}_0})^2}$$

$$\tag{26.14}$$

The second term comes from the expansion of the numerator, the third from the expansion of the denominator.

We can now use Wick's theorem to calculate the terms, as we are just left with Gaussian integrals! In particular, let us draw some Feynman diagrams to determine the contributions - this is just a way of counting up the pairings.

$$\phi^4(z)$$

$$\phi(x)$$

$$\phi(y)$$
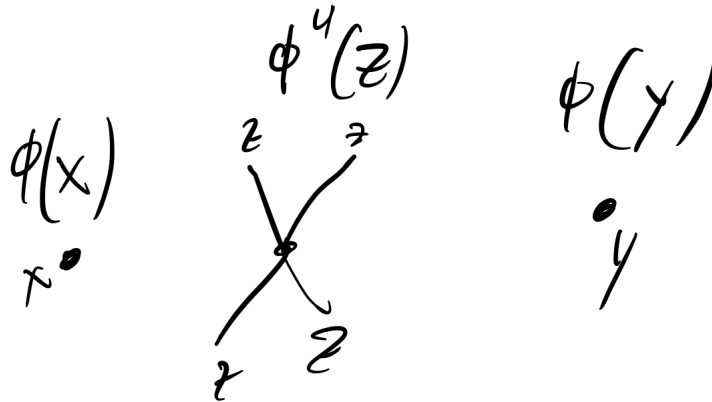
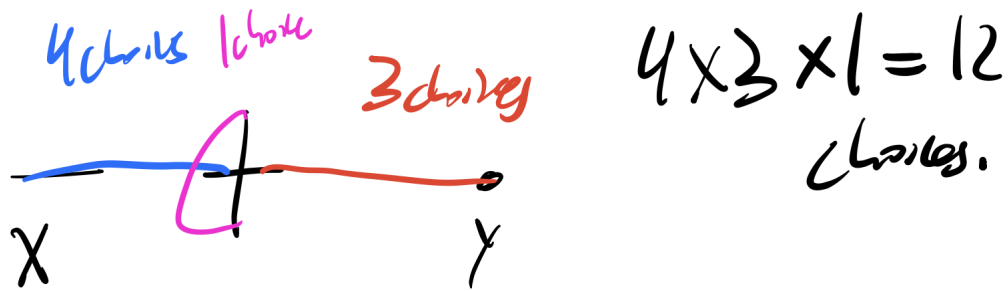First, lets think about the $\phi(x)\phi(y)\phi^4(z)$ term.

The simplest possible pairing would be pairing $\phi(x)$ and $\phi(y)$, and then the $\phi^4(z)$ to itself. There are 3 possible ways to pair the legs of the $\phi^4(z)$ term. We then obtain from Wick's theorem:

$$-i\frac{3\lambda}{4!}\int dz\Delta^2(z,z)\Delta(x,y). \tag{26.15}$$

z   3 ways!
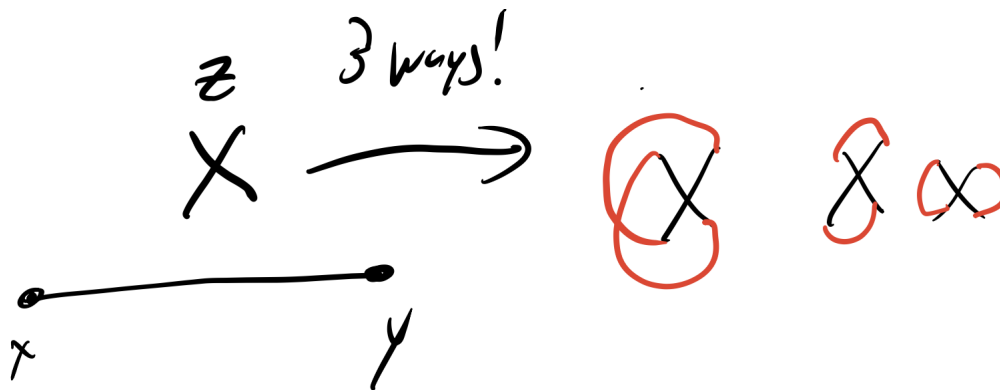
All of the other possibilities have the $\phi(x)$ and $\phi(y)$ pairing with the $\phi^4(z)$. There are four legs to choose from for x, then three remaining legs to choose from for y, and then the remaining two legs pair with itself, so we have $4 \cdot 3 \cdot 1 = 12$ choices and so:

$$-i\frac{4\cdot 3}{4!}\lambda \int dz\Delta(z,z)\Delta(x,z)\Delta(z,y) \tag{26.16}$$

4 choices 1 choice   3 choices $4 \times 3 \times 1 = 12$
                                         choices.

That's it for the pairings! Then we worry about the counterterms, i.e. the terms:

$$\phi(x)\phi(y)\left(\frac{\lambda m}{2}z_m^{(1)}\phi^2(z) + \frac{\lambda z^{(1)}}{2}\partial_\mu\phi(z)\partial^\mu\phi(z)\right)$$ (26.17)

For the first thing we have just a line, one pairing between $\phi(z), \phi(z)$ so:

$$-\frac{i\lambda}{2}\int dz(-z^{(1)}\partial^2\Delta(z,z) - m^2 z_m^{(1)}\Delta(z,z))$$ (26.18)
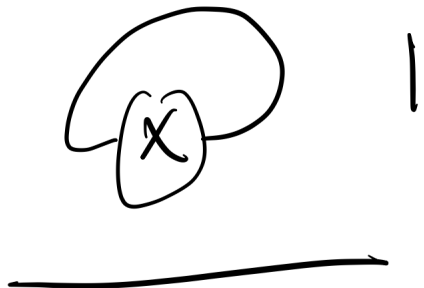
For the next, we have two pairings between $\phi(z), \phi(x)$ and $\phi(z), \phi(y)$ and so:

$$-i\lambda\int dz\Delta(x,z)(-z^{(1)}\partial^2 + m^2 z_m^{(1)})\Delta(z,y)$$ (26.19)

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$2 \, w \eta)$
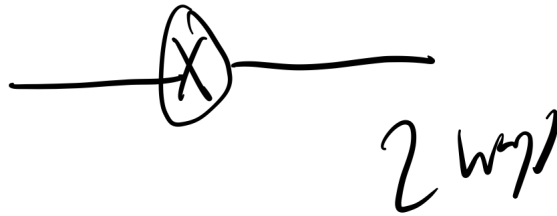
Now, let's worry about the third term in our expansion of $D(x,y)$. What will happen is when we look at the pairings, it will be the sum of:
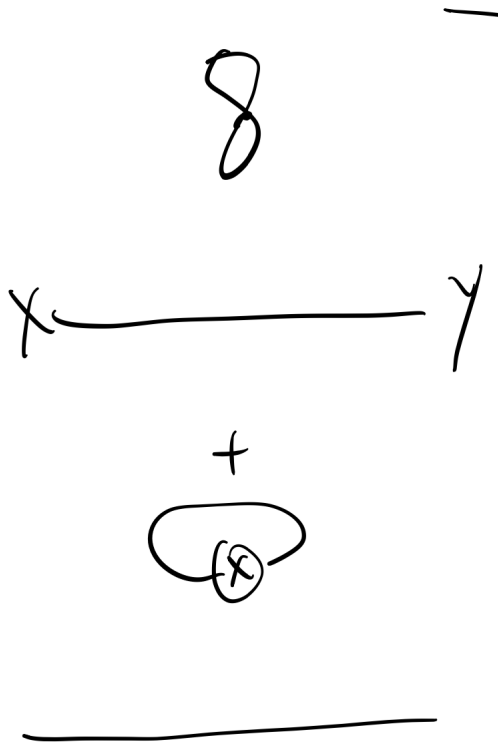
And this will actually cancel out the terms in the first term! We can prove that this always happens, using functional integral techniques. This is known as Goldstone's theorem. Anyone who calculates Feynman diagrams should really take their hats off to him because it saves us a lot of time!

Why those two? Each of the two that are removed have a subdiagram that is connected to external legs. This is known as a vacuum bubble. Goldstone's theorem tells us we may ignore such vacuum bubble terms.

Now the punchline:

$$D(x,y) = \Delta(x,y) - i\frac{\lambda}{2}\int d^4z\Delta(x,z)\Delta(z,z)\Delta(zy) - i\lambda \int d^4z\Delta(x,z)(-z^{(1)}\partial^2 + m^2z_m^{(1)})\Delta(z,y) + O(\lambda^2)$$

(26.20)

So our final result is quite short! At this point, we could choose parameters such that we are as close to free field theory as we could get. We could choose the $m^2 z_m^{(1)}$ to cancel out $\Delta(z, z)$ and the $z^{(1)}$ to be zero as we have no other derivative terms. And this would be a wise choice - this would keep m^2 to the actual mass, and give a nice normalization for ϕ. At this point we will stop.

A comment - this is not a convergent series (this can be seen because the series has a radius of convergence of zero, as it diverges for when $\lambda < 0$); it is an asymptotic expansion. We can go up to a certain order and stop when it starts to blowup. For QED its about 50 orders in α, for QCD its only a few orders (note the coupling constant is $\sim \frac{1}{3}$ so strong interactions are hard!)

Next time we make this more sophisticated.

27 Feynman Diagrams

27.1 Feynman Rules

Last time, we did a perturbative calculation of something. We did it with a straightforward Taylor expansion (in the coupling constant) of the functional integral expression for the two-point function, and then used our ability to do Gaussian functional integrals to get the result. But we never actually did a functional integral! We already did that previously (Wick's Theorem) so we just quoted the result of that - the only thing left to do was to do the combinatorics of the interactions. In doing so, we were able to work out these combinatorics in a straightforwards way using Feynman diagrams. As you imagine, all of this can be automated somehow (and in fact there are packages that draw all Feynman diagram that contribute to some order to some correlation functions), and this is a usually nice thing to use. It's easy to forget if you do it by hand... (anecdote about a group who worked on Yang-Mills for many years, but had forgotten just one diagram, and got scooped by a group in Russia). There's quite a bit to the science of Feynman diagrams, and in this course we just scratch the surface. We already used them last time, and now let's use them some more. One thing we might have noticed from last time is there are rules for how to begin and how to do the computation. These we can formalize and turn into a list of *Feynman Rules* - we don't copy them here, but look in the textbook for a full list! For simple computations, you will find that it is straightforwards, but the computation quickly begins to increase at higher order in perturbation theory.

27.2 Simplification 1 - Momentum Space

Recall our (first-order) perturbative computation of the two-point function:

$$D(x,y) = \Delta(x,y) - \frac{i\lambda}{2} \int dz \Delta(x,y)\Delta(z,z)\Delta(z,y) - i\lambda \int dz \Delta(x,z)(-z^{(1)}\partial^2 + z_m^{(1)}m^2)\Delta(z,y) + \dots \quad (27.1)$$

Now using that $D(x,y) = D(x-y)$ and so $\Delta(x,y) = \Delta(x-y)$ (from our Lorentz invariant symmetry analysis of two-point correlation functions from earlier), we find that the above expression turns into convolutions. This expression therefore simplifies drastically (into just a product) if we take a Fourier transform and look at momentum space. We therefore can simplify by going into momentum space:

$$D(x,y) = \int \frac{d^4k}{(2\pi)^4} e^{ik^\mu(x-y)_\mu} D(k^2) \quad (27.2)$$

for the free-field propogator, we know exactly what it is:

$$\Delta(x,y) = \int \frac{d^4k}{(2\pi)^4} e^{ik^\mu(x-y)_\mu} \frac{-i}{k^2 + m^2 - i\epsilon} \quad (27.3)$$

So if we substitute this into (27.1), we find:

$$D(k^2) = \frac{-i}{k^2 + m^2 - i\epsilon} + \frac{-i}{k^2 + m^2 - i\epsilon} \left(-\frac{i\lambda}{2} \int \frac{d^4p}{(2\pi)^4} \frac{-i}{p^2 + m^2 - i\epsilon} \right) \frac{-i}{k^2 + m^2 - i\epsilon}$$
$$+ \frac{-i}{k^2 + m^2 - i\epsilon} \left(-i\lambda(z^{(1)}k^2 + z_m^{(1)}m^2) \right) \frac{-i}{k^2 + m^2 - i\epsilon} + \dots \quad (27.4)$$

so to get a closed form expression, the only thing we have to do is calculate the integral appearing in the above expression. Unfortunately, the integral is divergent. However, if we just integrate over p^0 its convergent, and we have the same integral as we looked at a couple lectures ago (where we calculated a first-order energy shift, and then could absorb this shift into the mass, which was formally infinite...). It

is a very similar story to what we had before, but more systematic. One such way this is so is that the integral is Lorentz covariant.

One last thing is the counterterms sitting there; we can choose $z_m^{(1)}$ to cancel the integral and $z^{(1)} = 0$ as there is no other k^2 dependence. This would be an ok thing to do, but there is an arbitrariness in choosing the counterterms, known as scheme dependence. If the integral had k dependence, we would have to be more carful...

The second thing to notice is the form of this expression. Drawing them diagramatically, we have:

If we continued to higher orders, we first have repetitions of what came before (but now in pairs):

But also new terms:

Which is all the terms to order λ^2 - this already seems like a lot! Is there any way to systematically simplify or distinguish? Yes, you can topologically characterize the diagrams . It would be useful if

it allowed us to forget about some terms. In fact we can. We can categorize reducible and irreducible diagrams. Irreducible - can't snip somewhere and retain something non-trivial. So, the reducible ones are the repeated (paired) diagrams here, and the irreducible ones are the new ones. The question is then; is it possible to just focus on the irreducible diagrams and obtain all diagrams from there?

27.3 Simplification 2 - Compute Connected, Irreducible Diagrams

Yes. We consider an expansion as follows:

Where the black dots correspond to irreducible diagrams. Ones that are repetitions of what we have already calculated are obtained by joining irreducible (sometimes called one-particle irreducible) diagrams together. This cuts down on the multiplicity significantly. Note that when we put things together, the combinatorial factors organize themselves such that this is just a simple geometric series. Gordon doesn't know an easy way to prove this; every QFT textbook simply states it. The textbook gives a proof sketch, but it is complicated.

Consider the computation of a four-point function. Some diagrams that contribute here are disconnected two-point diagrams. These contribute a lot of multiplicity and it would be nice to only have to focus on the connected ones. However, we need a well-defined way to put them back together to get the full result. It turns out there is a rather clever way. We've chosen a field theory with a symmetry such that every diagram needs two legs coming out of it.

We already knew it before this discussion, but also note a **Simplification 0** of sorts - **Goldstone's Theorem**!

27.4 Diagram Combinatorics - Connected Correlation Functions

We wouldn't be able to discuss this meaningfully without knowing the combinatorics involved. In fact there are functional methods involved here. Let us discuss the case of connected correlation functions.

Given the generating functional $Z[J]$, we then have $W[J] = \ln Z[J]$ is the generating functional for connected correlation functions. So in other words, the connected correlation function is:

$$\Gamma_C(x_1, \ldots x_n) = \frac{1}{i}\frac{\delta}{\delta J(x_1)} \cdots \frac{1}{i}\frac{\delta}{\delta J(x_n)} W[J]\Big|_{J=0} \tag{27.5}$$

We thus obtain a formula for connected correlation functions in terms of ordinary ones. For two-point correlation functions this formula is almost trivial:

$$\Gamma_C(x_1, x_2) = D(x_1, x_2) - \langle \phi(x_1) \rangle \langle \phi(x_2) \rangle \tag{27.6}$$

but in the field theory we work with I believe the expectation values vanish. For the four-point function, we get something a little more complicated:

$$\Gamma_C(x_1, x_2, x_3, x_4) = \Gamma(x_1, x_2, x_3, x_4) - \Gamma(x_1, x_2)\Gamma(x_3, x_4) - \Gamma(X_1, x_3)\Gamma(x_2, x_4) - \Gamma(x_1, x_4)\Gamma(x_2, x_3) \tag{27.7}$$

where Γ without the C denotes the time ordered n-point correlation function. We may want to invert this, as we want the time-ordered correlation functions in terms of the connected ones, rather than the other way around.

Note - see the proof in the textbook for more details - we see that the expansion of a connected correlators in Feynman diagrams only use connected Feynman diagrams. We can do the same thing for the irreducible correlation functions (connected connected and irreducible correlation functions). This is a bit interesting, though we will not have a lot of time for it here. It will involve the Legendre transform, and it will come up often in a statistical mechanics class.

27.5   Generating Functional for Irreducible Correlation Functions

We will just talk about correlation functions for a moment, not Feynman diagrams. We will then define what we mean by irreducible in this context, and then it can be shown that such correlation functions can be written as a sum of irreducible Feynman diagrams. What we do is we return to $W[J]$, the connected generating functional. Let us define the one-point function as:

$$\phi(x) = \frac{1}{i}\frac{\delta}{\delta J(x)} W[J] \tag{27.8}$$

which differs a lot from the connected one-point function as we do not set $J = 0$. Then, take W and do a Legendre transform:

$$\Gamma[\phi] = W[J] - i \int J(x)\phi(x) \tag{27.9}$$

What we must do is take the equation for ϕ and solve it for J - this is terrible, so we solve it perturbatively. We then plug it in to get Γ as a functional of ϕ. And then, using the same terrible notation, the irreducible n-point function is obtained via functional derivatives of the classical field:

$$\Gamma_I(x_1, \ldots x_n) = \frac{1}{i}\frac{\delta}{\delta\phi(x_1)} \cdots \frac{1}{i}\frac{\delta}{\delta\phi(x_n)} \Gamma[\phi]\Big|_{\phi=?} \tag{27.10}$$

and then we set $J = 0$, which seems hard as we've lost track of J. We can see that taking the functional derivative of $\Gamma[\phi]$:

$$\frac{1}{i}\frac{\delta\Gamma[\phi]}{\delta\phi(x)} = -J(x) \tag{27.11}$$

so to put $J = 0$, we find a ϕ which is a soloution to the above equation, so Γ is at an extremum as a functional of ϕ:

$$\Gamma_I(x_1, \ldots x_n) = \frac{1}{i}\frac{\delta}{\delta\phi(x_1)} \cdots \frac{1}{i}\frac{\delta}{\delta\phi(x_n)} \Gamma[\phi]\Big|_{\phi=\bar{\phi}} \tag{27.12}$$

where:

$$\frac{1}{i}\frac{\delta\Gamma}{\delta\phi}\bigg|_{\phi=\bar{\phi}} = 0. \tag{27.13}$$

Usually $\bar{\phi} = 0$, but it doesn't have to be, e.g. when we have spontaneous symmetry breaking, where we have multiple zeroes and have to choose based on stability criteria, e.g. taking the second derivative and seeing if one has a sensible two-point function. We won't worry too much about this. What we really need is just a systematic realtionship between Γ_I and the Γ connected, Γ unindexed... for example:

$$\frac{1}{i^2}\frac{\delta^2\Gamma}{\delta\phi(x_1)\delta\phi(x_2)}\bigg|_{\phi=\bar{\phi}} = D^{-1}(x,y) \tag{27.14}$$

and then we can calculate higher ones, and the relationship between them and the ordinary correlation functions is one that removes all the pieces. Then, one can actually prove that the sum of irreducible diagrams produces an irreducible correlation functions.

This is a long story, and probably at the edge/beyond the edge of the curriculum of the course. The moral is that we can calculate irreducible diagrams, and then somewhere dig up an algorithm for reconstructing the full n-point function by these mechanism. Almost nobody does this in detail because you learn it via looking at a few simple correlation functions (e.g. two point is very simple). But its good to know something systematic is out there.

28

28.1 Finishing the 2-Point Function Calculation

We discussed quite a bit in a quasi-heuristic way about different types of correlation functions - generic, connected, irreducible. These are things that are not particularly clearly explained in most textbooks out there. I have tried to explain it in the textbook, if the lecture discussion was a bit unweildy/difficult to follow.

For now we only require the beginning of the argument. The class of all Feynman diagrams is the sum of all irreducible Feynman diagrams (diagramatically pictured below):
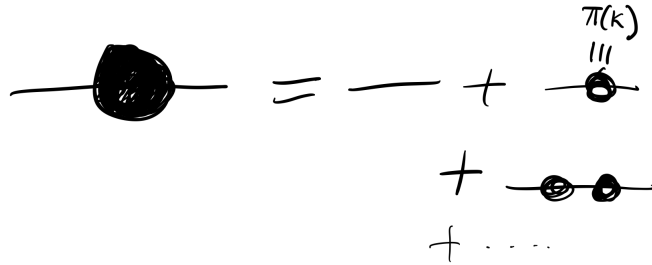
In momentum space, we had:

$$D(k) = \Delta(k) + \Delta(k)\Pi(k)\Delta(k) + \Delta(k)\Pi(k)\Delta(k)\Pi(k)\Delta(k) + \dots \tag{28.1}$$

The only question left - are we sure of the values of the coefficient in the above expansion to be one? How do we figure that out? We could explore it by testing the diagrams, or systematically via generating functions.

Since the above is a geometric series, let us sum it up!

$$
\begin{aligned}
D(k) &= \Delta(k) \left(1 + \Pi(k)\Delta(k) + (\Pi(k)\Delta(k))^2 + \dots \right) \\
&= \frac{\Delta(k)}{1 - \Pi(k)\Delta(k)} \\
&= \frac{\frac{-i}{k^2 + m^2 - i\epsilon}}{1 - \Pi(k)\frac{-i}{k^2 + m^2 - i\epsilon}} \\
&= \frac{-i}{k^2 + m^2 + i\Pi(k) - i\epsilon}
\end{aligned}
\tag{28.2}
$$

This appears like a translation of the mass term, and it is therefore called the self-energy.

Our Feynman diagram calculation looks like:

The extra factor of i in the diagram with the loop term will make the correction imaginary, hence $i\Pi(k)$ will be real.

28.2   4-Point Function

We consider:

$$\Gamma(x_1, \ldots x_n) = \langle \mathcal{O}|T\phi(x_1)\ldots\phi(x_n)|\mathcal{O}\rangle \tag{28.3}$$

and we also have Γ_C connected, Γ_I irreducible. We will study the irreducible one as there is a lot of redundant information in the full four-point function. Let us however consider the combinatorics that allows us to cast the full four-point function in terms of the irreducible ones. We first have:

$$\Gamma(x_1, \ldots, x_n) = \int \frac{d^4 p_1}{(2\pi)^4} \cdots \frac{d^4 p_n}{(2\pi)^4} e^{i(p_1 x_1 + \ldots p_n x_n)} \Gamma(p_1, \ldots, p_n)(2\pi)^4 \delta(p_1 + \ldots p_n) \tag{28.4}$$

Because of translation invariance, the Γ on the LHS is equal to itself translated by some constant in space. So on the LHS we should have the conservation of momentum; hence the δ function appearing. Now consider writing a four-point Γ as a bunch of connected terms:

$$\begin{aligned}
\Gamma(p_1, p_2, p_3, p_4) &= \Gamma(p_1, p_2)\Gamma(p_3, p_4) + \Gamma(p_1, p_3)\Gamma(p_2, p_4) + \Gamma_C(p_1, p_2, p_3, p_4) \\
&= \Gamma_C(p_1, p_2)\Gamma_C(p_3, p_4) + \Gamma_C(p_1, p_3)\Gamma_C(p_2, p_4) + \Gamma_C(p_1, p_2, p_3, p_4)
\end{aligned} \tag{28.5}$$

where we note that $\Gamma(p_1, p_2) = \Gamma_C(p_1, p_2)$ for two-point functions - actually only specifically when $p_2 = -p_1$. So really, $\Gamma(p_1, -p_1) = \Gamma_C(p_1, -p_1) = D(p)$. So, if we know the two-point functions and the connected four-point function, we can plug them into the above. The irreducible ones are now a short reach away.

The connected 4-point functions are ones with 4 legs sticking out. We make it reducible by adding something to one of the legs. If we factor our those things, what is left in the middle is irreducible.

irreducible   reducible
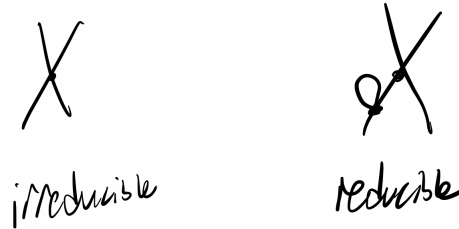
This tells me that:

$$\Gamma_C(p_1, p_2, p_3, p_4) = D(p_1)D(p_2)D(p_3)D(p_4)\Gamma(p_1, p_2, p_3, p_4) \tag{28.6}$$

So if I compute two-point functions and the irreducible four-point function to some order, I get the connected four-point function, which I can then get the full four-point function from. This is all easy algebra - no integrals!

Tadpole Diagram - here is a diagram we considered before:

But this is reducible; so the lines present are not actually real lines.

Ok, let's now calculate the irreducible 4-point function, as from it we can reconstruct what we already know. Let us start drawing some Feynman diagrams. If I have zero vertices, what I have is neither irreducible, nor connected.

vertex example
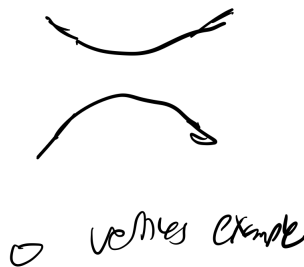
So our irreducible 4-point function begins at some power of λ. For λ^1, we have that the only way to get something irreducible is for the external legs to all connect with the vertex:
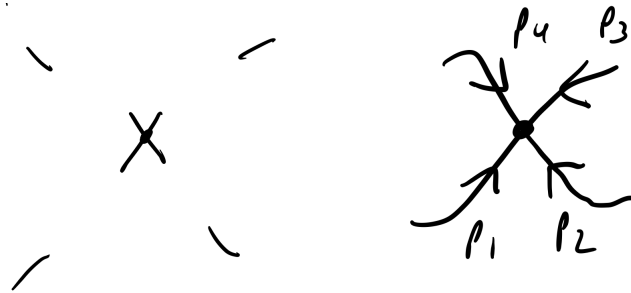

But there are 4 choices for the first leg, 3 for the second, 2 for the third, and one for the fourth, so we have a weight $\frac{-i}{4!}\lambda 4 \cdot 3 \cdot 2 \cdot 1 = -i\lambda$. So:

$$\Gamma_I(p_1, p_2, p_3, p_4) = -i\lambda \tag{28.7}$$

so this captures the interaction in the sense that it just produces the coupling constant! It is independent of the ps - as it is a pointlike interaction (the only thing being important is that the net momenta is zero).

Now, let's consider the next order - this should give the quantum corrections to the interaction (the first order result is really just what classical field theory would have told us). We go to λ^2. Our overall factor is $\frac{1}{2!}\left(\frac{-i\lambda}{4!}\right)^2$. We are only interested in the irreducible diagrams. The first line has to go to a vertex (and not another external leg), and we have 8 choices for where it could go. Now, there's a couple of choices. It can't go to one of the external legs, but it can go to either the first or the second vertex - there will be two classes. Let's consider the class where it goes to the first vertex. Next, I have to attach the remaining two legs. If I attach the remaining external legs to the first vertex, then the diagram is reducible; this is because then the remaining leg from the first vertex will have to connect to the other part of the diagram, and this can be cut to form two diagrams. So, there are four choices (on the second vertex) for the third leg, and three choices for the fourth leg. Now, there are two ways to pair up the four remaining legs from the vertices in a way that connects the two diagrams. So, the counting gives $\frac{1}{2!}\left(\frac{-i\lambda}{4!}\right)^2 \cdot 8 \cdot 3 \cdot 4 \cdot 3 \cdot 2.$
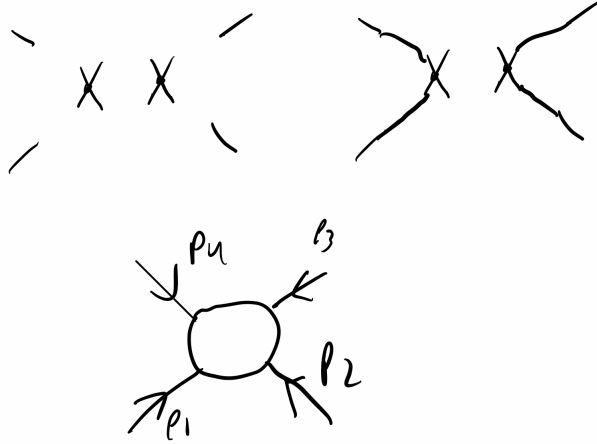
the other classes of possible diagrams look like:
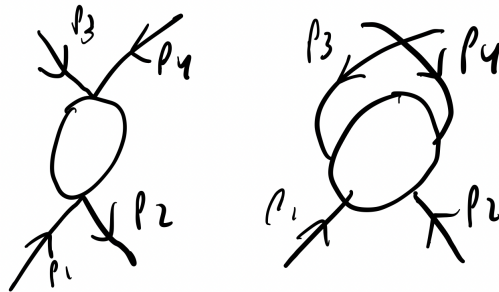
So we have three classes of diagrams, each with the same multiplicity, so overall the weight is $\frac{\lambda^2}{2}$. Now we have to go back to the analytic expressions, and recall what to put in. In momentum space this is very doable. Just put things into the diagram, and integrate over the remaining free momenta - of which there should be one per closed loop in the diagram (e.g. one momentum q for the one-loop diagrams we consider here).
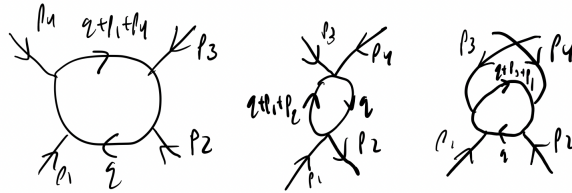
So our expression is:

$$\Gamma_I(p_1, p_2, p_3, p_4) = -i\lambda + \frac{\lambda^2}{2} \int \frac{d^4 q}{(2\pi)^4} \Delta(q)\Delta(q + p_1 + p_4)$$

$$+ \frac{\lambda^2}{2} \int \frac{d^4 q}{(2\pi)^4} \Delta(q)\Delta(q + p_1 + p_2) + \frac{\lambda^2}{2} \int \frac{d^4 q}{(2\pi)^4} \Delta(q)\Delta(q + p_1 + p_3) + O(\lambda^3)$$

(28.8)

so we must consider the integral:

$$I(p) = \frac{\lambda^2}{2} \int \frac{d^4 q}{(2\pi)^4} \frac{-i}{q^2 + m^2 - i\epsilon} \frac{-i}{(q + p)^2 + m^2 - i\epsilon}$$

(28.9)

but this diverges (logarithmically). So we must do something to make this sensible - regularization.

28.3 Dimensional Regularization

This has a long history and there is all sorts of tricks one can do, but they all amount to modifying this integral somehow such that the integral becomes by finite. We could for example replace Δ by some other function, and then we remove this modification later on. We could also put just an upper cutoff onto q. This is legitimate, but a little bit clumsy, as the translation invariance of the q of the integral is messed up by the cutoff momenta (though for logarithmic divergences, it doesn't mess it up in a very terrible way). We do the easiest fix up here, though logically the least transparent. We assume that we are not really in 4-dimensions, but in some other spacetime dimension, calculate the integral with that dimension by some parameter, and then analytically continue the result to four-dimension. This is known as *dimensional regularization*. Historically, it was important as it is a regularization that does not violate very many symmetries. In the literature, this is basically the only method used.

Note that we generally wish to expand in a dimensionless constant (e.g. λ). We therefore add a constant with dimensions of mass to compensate for the modified dimensionality of the integral:

$$I(p) = \frac{\lambda^2}{2}\mu^{4-2w} \int \frac{dq^{2w}}{(2\pi)^{2w}} \frac{-i}{q^2 + m^2 - i\epsilon} \frac{-i}{(q+p)^2 + m^2 - i\epsilon} \tag{28.10}$$

Even without have to introduce a explicit cutoff, we have had to introduce an explicit scale here. Now, we can assume w sufficiently small such that the integral will converge.

We will now simplify the formula using Feynman parameters:

$$\frac{1}{AB} = \int_0^1 d\alpha \frac{1}{(\alpha A + (1-\alpha)B)^2} \tag{28.11}$$

In the textbook there is a more general formula of the above. Why would I do this (introduce more integrals!?) - this is because this will allow us to combine the denominators. Our integral becomes:

$$I(p) = -\frac{\lambda^2 \mu^{4-2w}}{2} \int_0^1 d\alpha \int \frac{d^{2w}q}{(2\pi)^{2w}} \frac{1}{(q^2 + p^2\alpha(1-\alpha) + m^2 - i\epsilon)^2} \tag{28.12}$$

Note that we have translated q so as to get rid of a cross-term in the denominator. I could evaluate this via Cauchy's integral, but there would be some symmetry violations. Instead, we do a Rick rotation. We write the integral of q_0 which is along the real axis (blue, below) as a sum along the imaginary axis (red) plus a "figure-eight" type contour (purple).
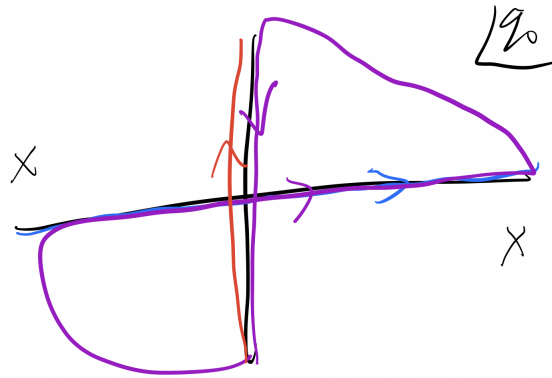
We change variables $q_0 \to iq_0, p_0 \to ip_0$:

$$I(p) = -\frac{i\lambda^2 \mu^{4-2w}}{2} \int_0^1 d\alpha \int \frac{d^{2w}q}{(2\pi^2)} \frac{1}{(q^2 + p^2\alpha(1-\alpha)m^2)^2} \tag{28.13}$$

We discuss how to carry this out next day, but we give the result today:

$$I(p) = -\frac{i\lambda^2 \mu^{4-2w}}{2} \int_0^1 d\alpha \frac{\Gamma(2-w)}{(4\pi)^w} \frac{1}{(p^2\alpha(1-\alpha)+m^2)^{2-w}} \tag{28.14}$$

The last integral is an unelightening hypergeometric type deal, so we leave it as an integral. Note that we have now I as a function of the dimensionality w. There, the Gamma-function diverges: $\Gamma(2-w) = \frac{1}{2-w} + \gamma + ...$ So this is now where the divergence is hiding. We still have to do something about it - but we now have it in a form where it is understandable. The next step will be to do an asymptotic expansion in 4-dimensions, and isolate the divergent and non-divergent part. This would give us the answer.

We've just done a one-loop integral. This kind of integral we will see again a few times, and the technique for computing it is more or less the same in all cases.

29 Renormalization

29.1 Review of the One-Loop Integral Calculation

Last time, we studied the computation of the one loop integral $\Gamma_I(k_1, k_2, k_3, k_4)$:
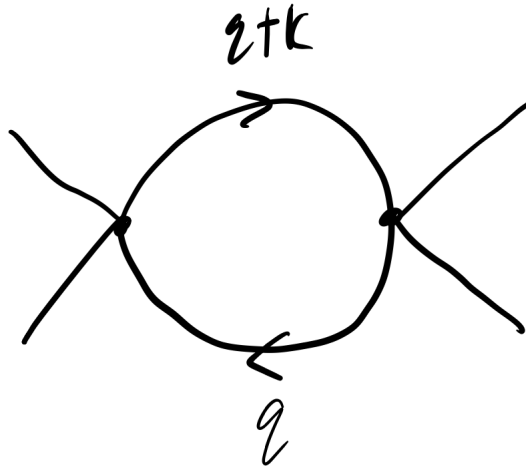
These integrals are generally tabulated in the literature, but it is instructive to go through it - they form the backbone of QFT calculations. Let us review it briefly. It is order two in the coupling, so we have a factor of $(-i\lambda)^2$. We then did the combinatorics, to find that we had a factor of $\frac{1}{2}$. The sophisticated Feynman diagrammers would tell you that this two is just the order of the discrete symmetries in this diagram. Going to the irreducible diagram we have to the general diagrams (Which can be reducible), all we do is attatch two-point functions on the external legs, e.g.
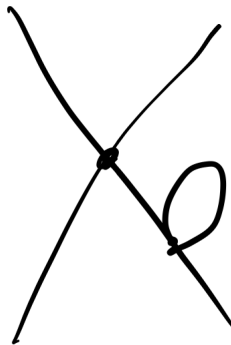
and this yields the correct combinatorics, and the reason why we know this formally is because we can go back to the generating functional, take 4 functional derivatives, do the combinatorics, and see that it indeed works out. We had an integral over the loop momentum (recall - one integral per closed loop in the diagram):

$$\frac{(-i\lambda)^2}{2} \int \frac{d^4q}{(2\pi)^4} \frac{-i}{q^2 + m^2 - i\epsilon} \frac{-i}{(q+k)^2 + m^2 - i\epsilon} \tag{29.1}$$

we then took on the task of actually doing this integral. We started by doing regularization, specifically via dimensional regularization:

$$\frac{\lambda^2}{2}\mu^{4-2w}\int\frac{d^{2w}q}{(2\pi)^{2w}}\frac{1}{q^2+m^2-i\epsilon}\frac{1}{(q+k)^2+m^2-i\epsilon} \tag{29.2}$$

where we introduce μ to keep the dimension of the integral consistent. We could then use Wick rotation/Feynman parameters; Wick rotation replaces $q_0 \to iq_0$ and $k_0 \to ik_0$ which makes the metric positive and Euclidean, and the integral becomes over real variables! We then introduce Feynman parameters:

$$\frac{1}{AB}=\int_0^1 d\alpha\frac{1}{(\alpha A+(1-\alpha)B)^2} \tag{29.3}$$

so then our integral becomes:

$$\frac{i\lambda^2}{2}\int_0^1 d\alpha\mu^{4-2w}\int\frac{dq^{2w}}{(2\pi)^{2w}}\frac{1}{(q^2+\alpha(1-\alpha)k^2+m^2)^2} \tag{29.4}$$

which is a nice way of phrasing it as it preserves the symmetry (here, Euclidean rotational symmetry, after doing the Wick rotation).

29.2 Computing the Dimensionally Regularized Integral

How do we do integrals that look like:

$$\int\frac{d^{2w}q}{(2\pi)^{2w}}\frac{1}{(q^2+m^2)^2} \tag{29.5}$$

If we recall the Gamma function:

$$\Gamma(s)=\int_0^\infty dx x^{s-1}e^x=\frac{1}{\alpha^s}\int_0^\infty dx x^{s-1}e^{x/\alpha} \tag{29.6}$$

so then:

$$
\begin{aligned}
\int\frac{d^{2w}q}{(2\pi)^w}\frac{1}{(q^2+m^2)^2} &= \frac{1}{\Gamma(2)}\int_0^\infty dx x\int e^{-x(q^2+m^2)}\frac{d^{2w}q}{(2\pi)^{2w}}\\
&= \frac{1}{\Gamma(2)}\int_0^\infty dx x e^{-xm^2}\left(\int\frac{d^2q}{(2\pi)^2}e^{-q^2x}\right)^w\\
&= \frac{1}{\Gamma(2)}\int_0^\infty dx x e^{-xm^2}\frac{1}{(4\pi)^w}\frac{q}{x^w}\\
&= \frac{\Gamma(2-w)}{(4\pi)^w\Gamma(2)}\frac{1}{(m^2)^{2-w}}
\end{aligned} \tag{29.7}
$$

and there's our result! There is a slightly more general procedure for this in the textbook, but this is basically it. Going back to our one-loop integral, we therefore conclude:

$$\frac{i\lambda^2}{2}\int_0^1 d\alpha\frac{\Gamma(2-w)}{(4\pi)^w\Gamma(2)}\left(\frac{\mu^2}{M^2}\right)^{2-w} \tag{29.8}$$

where M^2 is shorthand. This integral can be evaluated, but it gives some hypergeometric function which is not particularly enlightening, so we leave it as is. The only place where this diverges is for every even dimension $w \geq 4$. What we want to do is an asymptotic expansion in 4 dimensions; to this end:

$$\Gamma(2-w)=\frac{1}{2-w}+\gamma+O(2-w) \tag{29.9}$$

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where γ is the Euler-Mascheroni constant, and $O(2 - w) \to 0$ as $w \to 2$. Therefore returning to our integral:

$$i\frac{\lambda^2}{2} \int_0^1 d\alpha \frac{1}{(4\pi)^2} \left(\frac{1}{2-w} - \gamma\right) \left(1 + (2-w)\ln\frac{4\pi\mu^2}{M^2}\right)$$

$$= \frac{i\lambda^2}{32\pi^2}\frac{1}{2-w} + \frac{i\lambda^2}{32\pi^2} \int_0^1 d\alpha \ln(\frac{4\pi\mu^2 e^{-\gamma}}{\alpha(1-\alpha)k^2 + m^2})$$

(29.10)

We get a singularity coming from kinematics when the reaction can produce real particles as well as virutal particles. There are handwavey explanations for this; virtual particles have no choice but to recombine, but real particles have the option to fly away and become real particles. So we have competing processes of real/virtual particles, and then the unitarity of time evolution tells us that there is an imaginary part of the diagram, which corresponds to the cut singularity of the log...

If we stay in the Euclidean regime (rather than going into Minkowski) where $k^2 > 0$, we do not have to worry about this. Som there's our result for this particular Feynman diagram! There were more Feynman diagrams then this one that contributed to the two-point function, which had to do with how the legs were attatched:
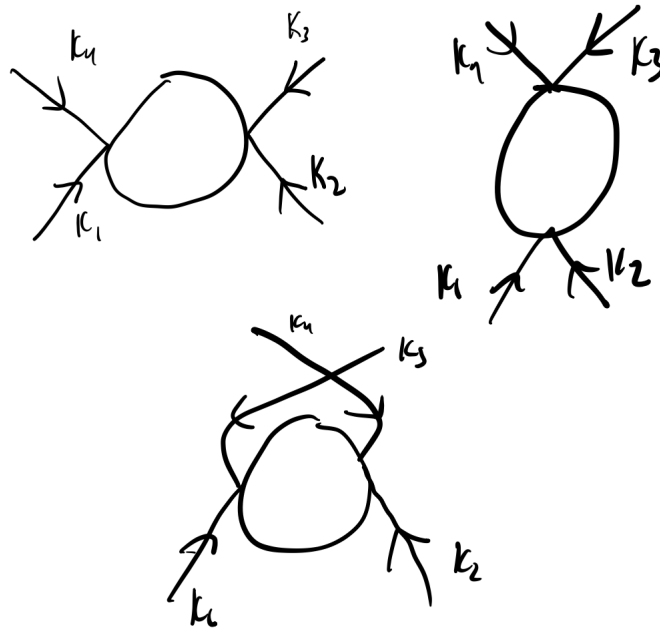
Summing over these is imposing bose symmetry, i.e. it should not matter how we permute the momenta. So then summing this with the lower order $(-i\lambda)$ term of:
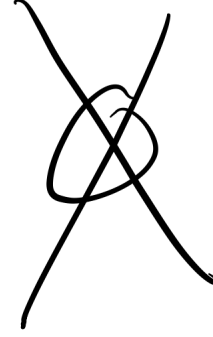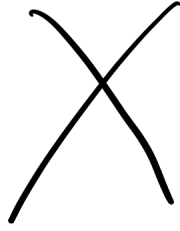
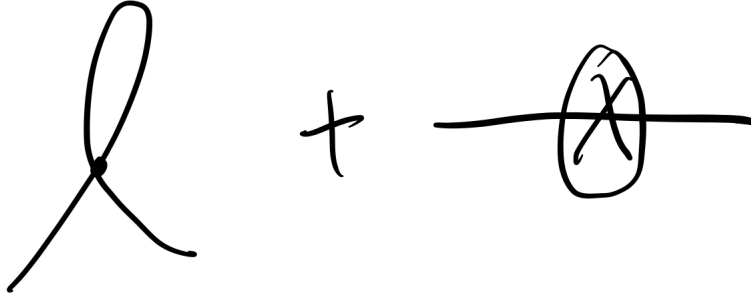as well as the counterterm of $-i\lambda^2 z_\lambda^{(1)}$, we are done!

So in conclusion:

$$\Gamma_I(k_1, k_2, k_3, k_4) = -i\lambda - i\lambda^2 z_\lambda^{(1)} + \frac{3i\lambda^2}{32\pi^2}\frac{1}{2-w}$$
$$+ \frac{i\lambda^2}{32\pi^2}\int_0^1 d\alpha \left(\ln\frac{4\pi\mu^2 e^{-\gamma}}{\alpha(1-\alpha)(k_1+k_4)^2 + m^2} + \ln\frac{4\pi\mu^2 e^{-\gamma}}{\alpha(1-\alpha)(k_1+k_2)^2 + m^2} + \ln\frac{4\pi\mu^2 e^{-\gamma}}{\alpha(1-\alpha)(k_1+k_3)^2 + m^2} \right)$$
$$+ O(\lambda^3)$$

$$(29.11)$$

Let us to the same order compute the two-point function, and then deal with the counterterm- its role will be to cancel out the singularity.

29.3   Computing the Two-Point function

Recall:

which corresponds to:

$$\Pi(k^2) = -\frac{i\lambda}{2}\int\frac{d^4q}{(2\pi)^4}\frac{-i}{q^2+m^2-i\epsilon} - i\lambda\left(z^{(1)}k^2 + z_m^{(1)}m^2\right) \qquad (29.12)$$

We need to integrate this; its a bit easier than the previous integral. Dimensionally regularize, put $q_0 \to iq_0$, and use the dimensional regularization formula. We then obtain:

$$\Pi(k^2) = -\frac{i\lambda}{2}\frac{\Gamma(1-w)}{(4\pi)^w\Gamma(1)}m^2\left(\frac{m^2}{\mu^2}\right)^{w-2} - i\lambda\left(z^{(1)}k^2 + z_m^{(1)}m^2\right) \qquad (29.13)$$

Again one can expand this asymptotically around $w = 2$. Note that $\Gamma(1-w)$ has a pole there. This yields:

$$\Pi(k^2) = \frac{i\lambda}{32\pi^2}m^2\frac{1}{1-w} - \frac{i\lambda_m}{32\pi^2}\ln\frac{m^2}{4\pi\mu^2 e^{1+\gamma}} - i\lambda\left(z^{(1)}k^2 + z_m^{(1)}m^2\right) \tag{29.14}$$

We now should fit the counterterms to observable physics - i.e. to cancel out the infinities, as two-point functions (observables) should not be infinite! Then there are the finite parts of the counterterms, which are determined by *schemes*. There are three popular ones; one is to choose the scheme so that the two-point function still has a singularity at $k^2 = -m^2$. The idea there is to choose the zs such that the singularity persists. It is easy to do here; the first term is k independent, so we can choose $z^{(1)} = 0$ and we can then select $z_m^{(1)}$ to cancel out the remaining things, so we reduce things to the propogator. This is what is known as "on-shell subtraction", as it places the singularity at the physical mass, preserving the analytic structure.

We need a criterion for $\Gamma_1(k_1, k_2, k_3, k_4)$ as well. We could simply say that for the four particles at rest (the same momentum/k_is), we say that that should be the physical coupling we call λ, and then choose $z_\lambda^{(1)}$ to cancel everything else when we plug in those momenta into the rest of the expression. We then get a finite result, no UV divergence, and a prediction - this in fact predicts the cross-section of the scattering of two scalar particles.

There are two other common schemes - one is *minimal subtraction*, where you choose the zs to cancel out the singularities. In this case the m appearing is not the actual mass of the particle, it is corrected by other terms. λ in this case is also not the physical coupling constant - both the physical mass and coupling constant are functions of m, λ. The nice thing about this scheme is that the counterterms do not depend on the mass, and μ as well. They only depend on the coupling constant. There is some techncial advantage in this, which we will see in the next lecture.

Finally, there is a $\overline{MS}$ (MS-bar) scheme. This cancels a singularity, as well as the $4\pi e^{-\gamma}$ (which again is just a constant term). There are other schemes, which we will learn a little about next time.

So, we have renormalized some of the important correlation functions here! We will also see what this is good for - i.e. constructing a scattering matrix. We can also construct the full correlation function from the irreducible correlation function, but this we will find will be unnecessary for calculating the scattering matrix. There's a little bit of magic involved. Before this, we will finish the perturbative calculation, by looking at the renormalization group. This is necessary because in the massless limit, the arguments of the logarithms scale as $\frac{1}{k^2}$ - but in PT we expand in small things, and depending on k this might not be the case. The PT turns out to have kinematical limitations, being only accurate when $k^2 \sim \mu^2$ and the logarithms are "small". We can fix this by making use of the fact that the μ^2 is arbitrary - the original theory did not depend on this. We can make use of this fact by scaling μ along with k.

30 Renormalization Group and Scattering for Scalar Field Theory

Recall - in PT we have counterterms to cancel out the infinities (because real data shouldn't have infinities). There are a few different renormalization scheme, and this is something that comes up for sure in QFT II where one studies strong interactions, QCD, Non-abelian gauge theories etc. As we've learned there is not really scheme-dependence on the final results, only scheme-dependence of the intermediate steps.

There are still a few things left to clean up from scalar field theory. One is the renormalization group, and the other is scattering theory/S-matrix.

30.1 Renormalization Group

We come to this from the least intuitive direction, that being renormalized quantum field theory. The condensed matter theorists have a much better approach to this, where one analyzes a second order phase transition of a classical model. On that side this is much clearer what one is doing. Here, it looks more like a mathematical trick, but unfortunately that's just how it looks from this perspective.

We have a parameter μ^2 that appeared when we were doing renormalization. This was just something we stuck in for dimensional regularization. If we did some kind of other regularization, there would be some other parameter - likely some kind of mass or momentum cutoff which we take to be large.

In 4-d, the coupling constant is dimensionless. But if we slide below 4 dimensions, it doesn't make sense to say something is large or small because dimensional quantities have to be small in relation to something. So, here we separated out μ as the dimensional part of the coupling constant to deal with this dimensionality. However, it does not go away when we come back to 4-dimensions, so it is there as another parameter in the theory. In some ways it is a parameter, but we are allowed to take advantage of the fact that it is completely arbitrary.

Let us go back to our theory before we introduced things like counterterms - calculating a correlation function, we have:

$$Z^{n/2}\Gamma(k_1,\ldots k_n) \tag{30.1}$$

which is just the correlation function in the renormalized theory times the factor $Z^{n/2}$. Note - the mass always stays a mass in any dimension of spacetime, so it is only λ that grows a dimensional part in different spacetime dimensions. But looking at the above, nothing depends on μ at this level. So, we take advantage of this by writing:

$$\mu\frac{\partial}{\partial\mu}\left(z^{n/2}\Gamma(k_1\ldots k_n)\right) = 0 \tag{30.2}$$

but this equation is not sensible because Z becomes infinite, so let us introduce an extra factor:

$$z^{-n/2}\mu\frac{\partial}{\partial\mu}\left(z^{n/2}\Gamma(k_1\ldots k_n)\right) = 0 \tag{30.3}$$

what happened to the μ-dependence? The bare quantities $m^2 z_m, \lambda\mu^{4-2w}z_\lambda$ do not depend on μ. They become μ-dependent in our subtraction scheme because m, λ do depend on it. So, we can rewrite our above equation in terms of renormalized quantities:

$$\left(\mu\frac{\partial}{\partial\mu} + \alpha m\frac{\partial}{\partial m} + \beta\frac{\partial}{\partial\lambda} + \frac{n}{2}\gamma\right)\Gamma(k_1,\ldots k_n) = 0 \tag{30.4}$$

this is identical to the above equation, taking into account the μ dependencies and expanding. α, β, γ are related to the derivatives of the renormalized parameters:

$$\alpha = \frac{1}{m}\mu\frac{\partial}{\partial\mu}m \tag{30.5}$$

$$\beta = \mu \frac{\partial}{\partial \mu} \lambda \tag{30.6}$$

$$\gamma = \frac{1}{z} \mu \frac{\partial}{\partial \mu} z \tag{30.7}$$

Note it's not really derivatives of μ, but rather derivatives by $\log \mu$. This is a renormalization group equation that tells us how the rest of the things in Γ change as we change μ. This turns out to be a difficult set of equations as there are a lot of interdependencies on parameters. It can however be made a bit easier. This is by using a subtraction scheme where the μ-dependence of things is simpler. This is likely why minimal subtraction was invented - there we only subtract the absolutely singular parts of the Feynman diagrams, so in an equation like $m^2 z_m, \lambda \mu^{4-2w} z_\lambda$, we only have dependencies on the coupling constant, so the RHS of α, β, γ only depend on the coupling constant, then Eq. (30.4) looks like a flow equation for the coupling constant.

30.2 Dimensional Analysis

Another thing we can do is dimensional analysis, as the momentum appearing in Γ are dimensional quantities. Doing so:

$$\Gamma(sk_1, sk_2, sk_3, sk_4) = s^{n-4} \Gamma(k_1, k_2, k_3, k_4, m/s, \mu/s, \lambda) \tag{30.8}$$

for example with the two point function $\Delta(k) = \frac{1}{k^2 + m^2}$ so:

$$\Delta(sk, m) = \frac{1}{s^2 k^2 + m^2} = \frac{1}{s^2} \frac{1}{k^2 + m^2/s^2} = \frac{1}{s^2} \Gamma(k^2 + m^2/s^2) = \frac{1}{s^2} \Delta(k, m/s) \tag{30.9}$$

Now if we operate on the correlation function by $s \frac{\mathrm{d}}{\mathrm{d}s}$:

$$s \frac{\mathrm{d}}{\mathrm{d}s} \Gamma(sk_1, sk_2, sk_3, sk_4) = (n - 4 - m \frac{\partial}{\partial m} - \mu \frac{\partial}{\partial \mu}) \Gamma \tag{30.10}$$

So between our two equations let us eliminate $\mu \frac{\partial}{\partial \mu}$, and obtain:

$$s \frac{\mathrm{d}}{\mathrm{d}s} \Gamma(sk_1, \ldots, sk_n, m, \lambda, \mu) = \left(n - 4 - \frac{n}{2} \gamma - (\alpha - 1) m \frac{\partial}{\partial m} + \beta \frac{\partial}{\partial \lambda} \right) \Gamma(k_1, \ldots, k_n, m/s, \lambda, \mu/s) \tag{30.11}$$

30.3 The Minimal Subtraction Scheme

We use this scheme because in this scheme, we have $\alpha(\lambda), \beta(\lambda), \gamma(\lambda)$ (some comment due to because we supress the singularities in this scheme), i.e. the functions only depend on the coupling constant. We then have a flow equation. We can integrate:

$$\mu \frac{\partial}{\partial \mu} \lambda = \beta(\lambda) \tag{30.12}$$

for the β function. For φ^4 theory, we find:

$$\beta(\lambda) = \frac{3\lambda^2}{16\pi^2} + O(\lambda^3) \tag{30.13}$$

if we replace μ in the above equation by a dimensionful parameter s:

$$s \frac{\partial}{\partial s} \lambda(s) = \beta(\lambda(s)) \tag{30.14}$$

Then the other equations become:

$$s \frac{\mathrm{d}}{\mathrm{d}s} m(s) = \alpha(\lambda(s)) m(s) \tag{30.15}$$

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$$s \frac{\mathrm{d}}{\mathrm{d}s} z(s) = \gamma(\lambda(s)) \tag{30.16}$$

We then obtain an equation of (30.11), by solving one equation, and plugging in the result to the next. The equation is:

$$\Gamma(sk_1, \ldots sk_n, m, \lambda, \mu) = s^{n-4} z(s) \Gamma(k_1, \ldots k_n, \frac{m(s)}{s}, \lambda(s), \mu) \tag{30.17}$$

30.4 Fixed points, β and coupling constants

Note the starting point solution to Eq. (30.12) is:

$$\int_{\lambda_0}^{\lambda} \frac{d\lambda'}{\beta(\lambda')} = \ln(s/s_0) \tag{30.18}$$

of course this is a perturbative solution, so λ must be kept small. We could study this equation a little bit, without even plugging in the β function. This tells us how the coupling constant goes as we scale things. We obtain a way to compare our n-point functions at different scales. One thing we could ask - we could as what happens when the s goes to zero. Well, m gets very big and λ scales as its dependence, etc. This is called the *renormalization group flow* and it can tell us something about renormalization group flow in a certain limit.

If the theory was massless, it is even easier. If we put the mass to zero, then all that happens is a coupling flow (in addition to a factor in front). If the mass were not there, then only the coupling flows. This has a very important application. IF we put the mass to zero and calculate the (irreducible) four point function (this is something we have done in the last lectures):

$$\Gamma_I(k_1, k_2, k_3, k_4) = -i\lambda + i\frac{\lambda^2}{32\pi^2} \left(\ln(\frac{4(k_1 + k_2)^2}{\mu}) + \ldots (k_1 + k_3) + \ldots (k_1 + k_4) \right) + O(\lambda^3) \tag{30.19}$$

If this is to be a converging series, the successive terms should be small. If we scale k by s, then we have a logarithm of the scale factor s - in fact it gets very large if we either make the scale large or small. This ruins the perturbation theory everywhere except where k is of the order μ. So, this tells us about the scale of which perturbation theory is good. Of course, logarithms are slowly growing so the scale must be much larger than μ to ruin the PT, but in principle it can be. All this work, and we get the rather depressing result that there is only some kinematical regime where things are ok, and in fact if our particles are massless then this regime does not include small momenta! This is very depressing - the small momenta regime is the one that is actually accessible via scattering experiments.

If we take this our base calculation, we can increase the scale on the ks not by scaling the ks on the RHS, but by letting the λ flow. This improves PT in an interesting way. What we are doing is anticipating is that there are higher powers in the perturbation theory. At every power of λ, there is a logarithm to some power of the momentum transfer divided by μ. We don't know this, but the RG actually tells us that it has to be so, because when we solve the flow equation we get all of these, and in fact it sums them up for us. It improves PT because if we put s on the ks on the LHS, then we put ss on the λs on the RHS.

If we plug in the known β function for the $\lambda\varphi^4$ scalar field theory, we obtain:

$$\frac{16\pi^2}{3} \int_{\lambda(1)}^{\lambda(s)} \frac{d\lambda'}{\lambda'^2} = \ln(s) \tag{30.20}$$

This is an equation we can solve in real time - simply doing the integral:

$$\frac{16\pi^2}{3} \left(-\frac{1}{\lambda(s)} + \frac{1}{\lambda(1)} \right) = \ln(s) \tag{30.21}$$

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Now solving this for $\lambda(s)$:

$$\frac{1}{\lambda(s)} = \frac{1}{\lambda(1)} - \frac{3}{16\pi^2}\ln(s) \qquad (30.22)$$

$$\lambda(s) = \frac{\lambda(1)}{1 - \frac{3}{16\pi^2}\lambda(1)\ln s} \qquad (30.23)$$

so now the news is half-good, because if s is small, then $\ln s$ is negative and so the denominator and positive and larger than 1, so $\lambda(s)$ is smaller than $\lambda(1)$! So in the small-s regime, PT is even more convergent than we are expecting. And as $s \to 0$, the coupling just goes to zero (infrared freedom), so in the small momentum regime things work out - things are accurate in this regime, and we are able to analyze low momentum transfer processes in this theory.

The bad news is on the other side - if $s > 1$ then the logarithm is positive and so the denominator is < 1 and so $\lambda(s)$ actually grows with s.
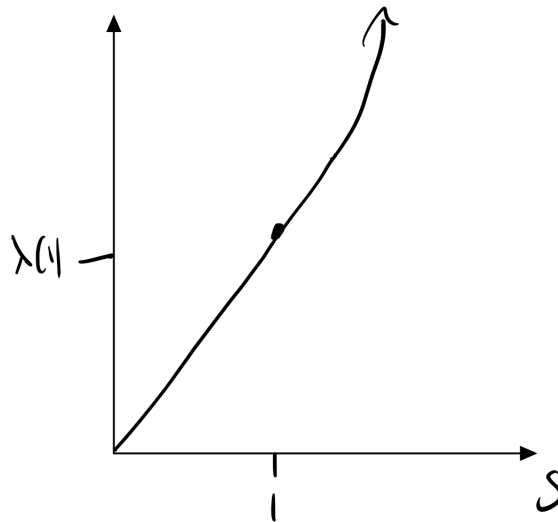
There is a bit of a dimensional transmutation going on here. If we put $m = 0$, then there is no dimensionful parameters in the field theory, but then one has appeared, and we do not know what it is, because the only information we have is our original theory with no dimensionality at all. So, we need to do experiments to figure it out. When TRIUMF says that the QCD is 200MeV is some statement about the... (couldn't follow)

There is a disaster here to note, where there is a crossover from the denominator being positive to negative. This is the Landau pole, and beyond this momentum the theory isn't sensible.

There is in an analysis in the notes of what happens depending on what the beta function looks like. As s diverges, $\beta(\lambda)$ has to be such that $\int_{\lambda_0}^{\lambda} \frac{d\lambda'}{\beta(\lambda')}$ diverges, which gives us a maximal value of s (couldn't follow). Note this is not fully resolved, and there is the possibility for a maximal scale, or that one does not exist depending on the dependency of β on λ. If there is a maximum scale then this theory is incomplete.
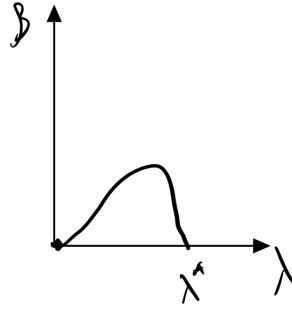
Other possibilities - β could have a zero at some fixed point $\lambda = \lambda^*$. If the dimension is 4 this cannot occur, if the dimension is less than 4 then the β function has an interesting shape:

$$\beta(\lambda) = \lambda(2\omega - 4) + \frac{3\lambda^2}{16\pi^2} \tag{30.24}$$

this is the β function in a double expansion, one of the parameters being λ and the other $2\omega - 4$ (and keeping terms to leading order). This β function looks as follows:

Note at high energies we have a flow to zero. This is because the coupling constant has a dimension, so the ratio of the coupling constant to the energy scale is dimensionless, so as the energy scale gets very big, the dimensionless scale gets small (Flows to field theory). On the opposite side, it flows to strong coupling at small momenta, but the nonlinearities in the theory conspire to stop the flow.
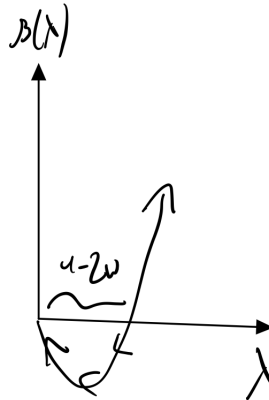
In dimensions less than 4, the theory gets stuck in one regime, and we are therefore able to obtain the entire theory in perturbation theory. ϵ-expansion for the Ising model in 2ω dimensions is very successful because the fixed point is extremely isolated - known as the Wilson-Fisher fixed point.

So, that's the renormalization group - there is a lot more to say but we leave that to a future course.

30.5   Scattering and the S-Matrix

So far we have studied the calculation of correlation functions. At some point we have to ask what they are good for. So, let's discuss that. Of course, the condensed matter theorists know that the two-point functions are useful for linear response theory (conductivity of metals etc.) Higher point correlation functions are less useful in CM. But anyway, what about in relativistic field theory? The main thing the correlation functions are useful in this context are in scattering theory.

So let's think about a scattering matrix. What do we do? We make a beam of particles (more or less free particles - far enough apart to not interact). So the energies are just the sum of the energies, the momenta the sum of the momenta etc. We can then model them as free particles.

In principle the scattering of two particles could create more, but let us work in the low-energy elastic scattering regime. Diagramatically, this may look as follows:
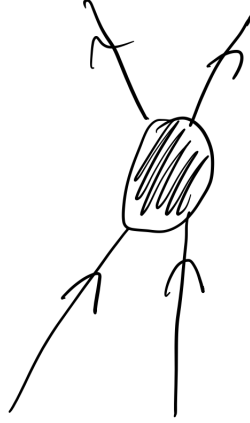
When we detect the particles, they would be far from the other particles, and only interacting with the detectors.

If the initial state of the particles is free incoming particles, we can model them with free fields:

$$(-\partial^2 + m^2)\phi_{in}(x) = 0 \tag{30.25}$$

then something happens which is certainly not free field theory, and we get an outgoing state. We give them enough time to interact, fly out etc. And then we get:

$$(-\partial^2 + m^2)\phi_{out}(x) = 0 \tag{30.26}$$

They might look a bit different than the incoming particles (of course), but they are the same kinds of particles by assumption (e.g. same mass). Of course ϕ_{in} could be an operator which we know everything about (we have studied free field theory to death)... ϕ_{out} is free, but also a bit more subtly as there is now an overlap of wavepackets.

So, we have an incoming state:

$$\frac{1}{\sqrt{2!}} a_{in}^\dagger(\mathbf{k}_1) a_{in}^\dagger(\mathbf{k}_2)|0\rangle \tag{30.27}$$

how do we describe the stuff that happened to the free particles? They come out, they are free particles again, so vectors in the same hilbert space, but it should be a mixture of what the incoming states could have been:

$$|\text{out}\rangle = \sum |\text{in}\rangle S \tag{30.28}$$

where S are the coefficients - the amplitude of finding the free particle states in the final state. This is the *S-matrix*. I can take it to be some operator acting on the incoming state, so let us write the above as:

$$|\text{out}\rangle = S^\dagger |\text{in}\rangle \tag{30.29}$$

There is a beautiful formula for this:

$$S = \ :e^{\int d^4x \phi_{in}(x)(-\partial^2 + m^2)\frac{\delta}{\delta J(x)}} : Z[J]\Big|_{J=0} \tag{30.30}$$

where : denotes normal ordering. The fact that S is an operator comes from the fact that ϕ_{in} is an operator on the Fock space. Normal ordering means we put all the as to the right and the $a^\dagger$s to the left.

There is a derivation of this that is not very complicated but a bit long. See textbook. To use the formula, one requires the on-shell subtraction scheme. This means that the two-point function should have a pole and residue as if it were a free-field two point function:

$$D(k^2) = \frac{-i}{k^2 + m^2 - i} + \text{finite as } k^2 \to -m^2 \tag{30.31}$$

If the counterterms are not chosen in this way, the m would/wouldn't be the actual mass (couldn't follow).

What this does for us is $\frac{\delta}{\delta J}$ inserts a ϕ into the expectation value... we take an n-point function generated by the $\frac{\delta}{\delta J}$s... we project:

$$\int d^4x \frac{e^{ikx}}{\sqrt{(2\pi)^3 2\sqrt{k^2 + m^2}}} (-\partial^2 + m^2)\langle 0|T\phi(x_1)\dots\phi(x_n)|0\rangle \tag{30.32}$$

I didn't follow anything that was said about what this represents. But for some reason, we need the n-point functions with the external legs amputated. This is not particularly irreducible or connected, it just has an amputation some way or another. For the φ^4 theory this is irreducible.

Next day we'll go through QED, and redo some perturbative things with small additional things that happen in the more complex QED setting.