Calculus & Analytical Geometry
Code: MT1003

Text Book

CALCULUS

METRIC VERSION | 9E

Single Variable Calculus

Early Transcendentals

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Reference Book

THOMAS'

FIFTEENTH EDITION

CALCULUS

EARLY TRANSCENDENTALS

1 Functions

Functions are a tool for describing the real world in mathematical terms.

Functions

Functions arise whenever one quantity depends on another. Consider the following four situations.

A. The area A of a circle depends on the radius r of the circle. The rule that connects r and A is given by the equation $A = \pi r^2$. With each positive number r there is associated one value of A, and we say that A is a *function* of r.

B. The human population of the world P depends on the time t. Table 1 gives estimates of the world population P at time t, for certain years. For instance,

$$P \approx 2{,}560{,}000{,}000 \qquad \text{when } t = 1950$$

For each value of the time t there is a corresponding value of P, and we say that P is a function of t.

C. The cost C of mailing an envelope depends on its weight w. Although there is no simple formula that connects w and C, the post office has a rule for determining C when w is known.

Function

A function f from a set D to a set Y is a rule that assigns a unique value y in Y to each x in D.

The set D is called the domain of f.

The set Y is called the codomain of f.

The range of f is the set of all y values as x varies throughout the domain.

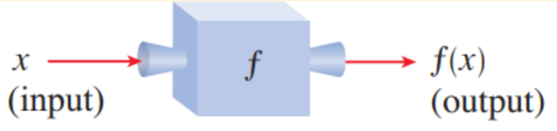
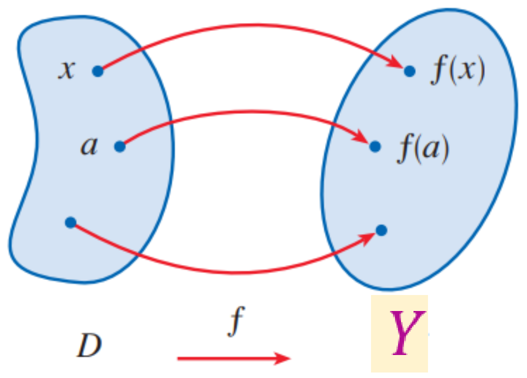


FIGURE
Machine diagram for a function f



Since the y value that we are getting depends upon the input x value that's why we say that x is independent variable and y is dependent variable and f is a function of x and write as $y = f(x)$.

If for a number α in D we have β in Y then we write $f(\alpha) = \beta$ and say that β is the value of f at α.

Representations of Functions

There are four possible ways to represent a function:

- verbally (by a description in words)
- numerically (by a table of values)
- visually (by a graph)
- algebraically (by an explicit formula)

Let's represent one function in all these ways...

Verbally

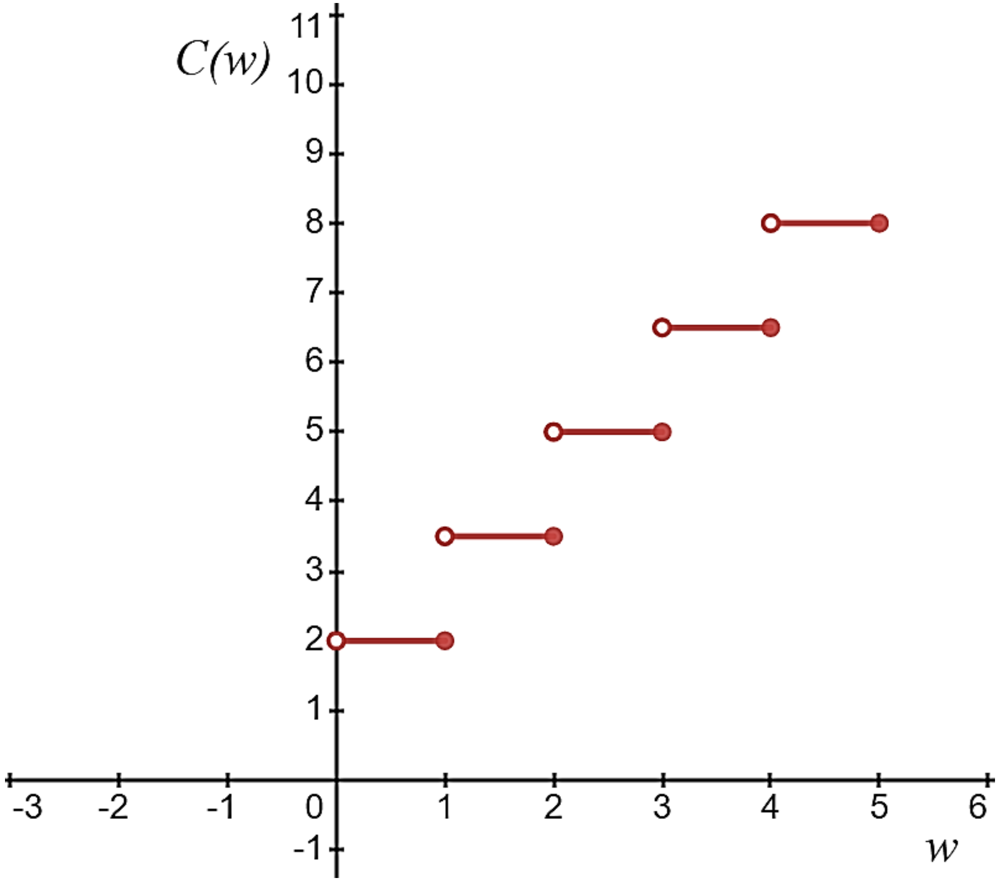
Let $C(w)$ be the cost of mailing a large envelope with weight w. The cost is \$2 for up to 1kg, plus \$1.5 for each additional kg (or less) up to 5 kg.

Numerically

Weight in Kg w	Cost in dollers $C(w)$
$0 < w \leq 1$	2
$1 < w \leq 2$	3.5
$2 < w \leq 3$	5
$3 < w \leq 4$	6.5
$4 < w \leq 5$	8

Visually Graph of a Function

The graph of a function f consists of all points (x, y) in the coordinate plane such that $y = f(x)$ and x is in the domain of f.



Algebraically

$$C(w) = \begin{cases} 2 & \text{if} \quad 0 < w \leq 1 \\ 3.5 & \text{if} \quad 1 < w \leq 2 \\ 5 & \text{if} \quad 2 < w \leq 3 \\ 6.5 & \text{if} \quad 3 < w \leq 4 \\ 8 & \text{if} \quad 4 < w \leq 5 \end{cases}$$

Python Code for Cost Function

```python
a=input('Please enter the weight of your parcel: ')
w=float(a)
if w>0 and w<=1:
        print('Please pay $2')
elif w>1 and w<=2:
        print('Please pay $3.5')
elif w>2 and w<=3:
        print('Please pay $5')
elif w>3 and w<=4:
        print('Please pay $6.5')
elif w>4 and w<=5:
        print('Please pay $8')
```

```
Please enter the weight of your parcel:  2.5

Please pay $5
```

Domain Convention

If a function is given by a formula and the domain is not stated explicitly, the convention is that the domain is the set of all numbers for which the formula makes sense and defines a real number.

EXAMPLE Find the domain of each function.

(a) $f(x) = \sqrt{x + 2}$ (b) $g(x) = \dfrac{1}{x^2 - x}$

SOLUTION

(a) Because the square root of a negative number is not defined (as a real number), the domain of f consists of all values of x such that $x + 2 \geqslant 0$. This is equivalent to $x \geqslant -2$, so the domain is the interval $[-2, \infty)$.

(b) Since

$$g(x) = \frac{1}{x^2 - x} = \frac{1}{x(x - 1)}$$

and division by 0 is not allowed, we see that $g(x)$ is not defined when $x = 0$ or $x = 1$. So the domain of g is

$$\{x \mid x \neq 0, x \neq 1\}$$

which could also be written in interval notation as

$$(-\infty, 0) \cup (0, 1) \cup (1, \infty)$$

EXAMPLE Verify the natural domains and associated ranges of some simple functions. The domains in each case are the values of x for which the formula makes sense.

Function	Domain (x)	Range (y)
$y = x^2$	$(-\infty, \infty)$	$[0, \infty)$
$y = 1/x$	$(-\infty, 0) \cup (0, \infty)$	$(-\infty, 0) \cup (0, \infty)$
$y = \sqrt{x}$	$[0, \infty)$	$[0, \infty)$
$y = \sqrt{4 - x}$	$(-\infty, 4]$	$[0, \infty)$
$y = \sqrt{1 - x^2}$	$[-1, 1]$	$[0, 1]$

▸ Example Find the domain and range of

$$\text{(a)}\ \ f(x) = 2 + \sqrt{x-1} \qquad \text{(b)}\ \ f(x) = (x+1)/(x-1)$$

Solution (a). Since no domain is stated explicitly, the domain of f is its natural domain, $[1, +\infty)$. As x varies over the interval $[1, +\infty)$, the value of $\sqrt{x-1}$ varies over the interval $[0, +\infty)$, so the value of $f(x) = 2 + \sqrt{x-1}$ varies over the interval $[2, +\infty)$, which is the range of f. The domain and range are highlighted in green on the x- and y-axes in Figure 0.1.15.

Solution (b). The given function f is defined for all real x, except $x = 1$, so the natural domain of f is

$$\{x : x \neq 1\} = (-\infty, 1) \cup (1, +\infty)$$

To determine the range it will be convenient to introduce a dependent variable

$$y = \frac{x+1}{x-1}$$

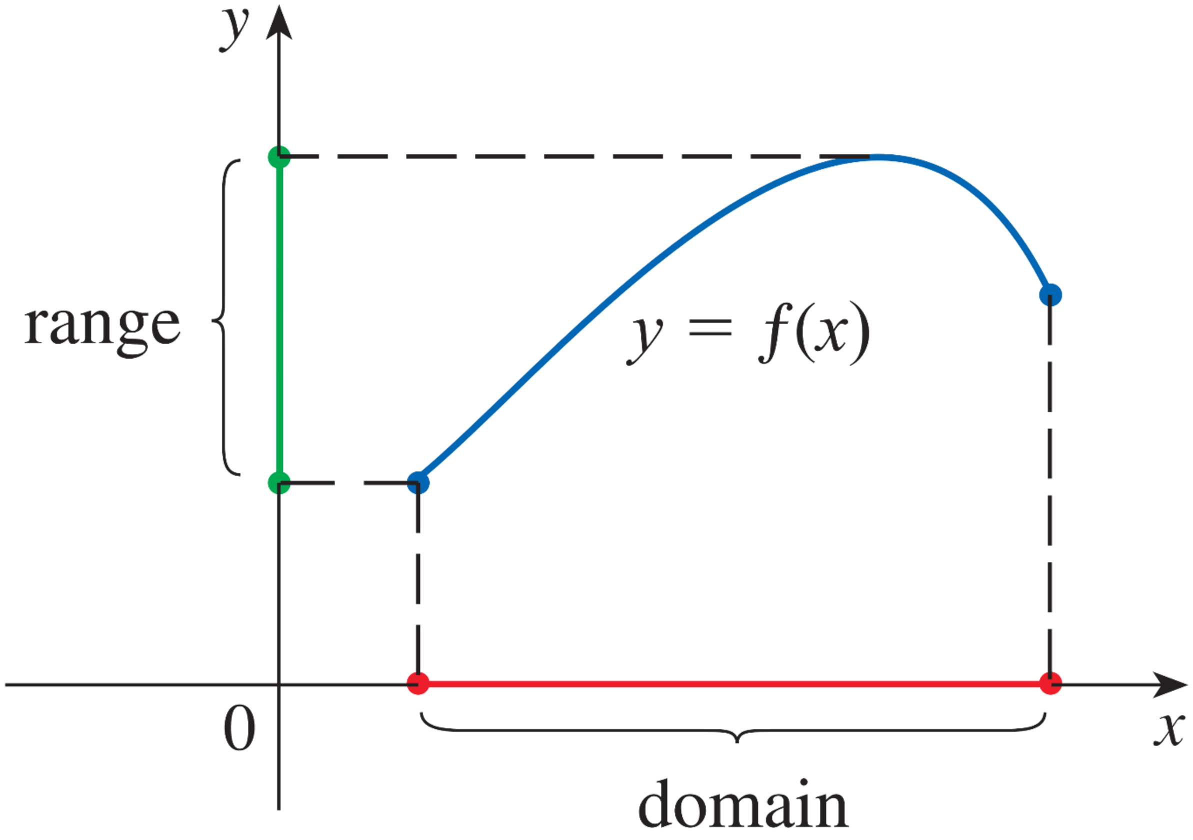
$$\begin{aligned} (x-1)y &= x+1 \\ xy - y &= x+1 \\ xy - x &= y+1 \\ x(y-1) &= y+1 \\ x &= \frac{y+1}{y-1} \end{aligned}$$

It is now evident from the right side of this equation that $y = 1$ is not in the range; otherwise we would have a division by zero. No other values of y are excluded by this equation, so the range of the function f is $\{y : y \neq 1\} = (-\infty, 1) \cup (1, +\infty)$, which agrees with the result obtained graphically. ◄

Finding Domain and Range from the graph

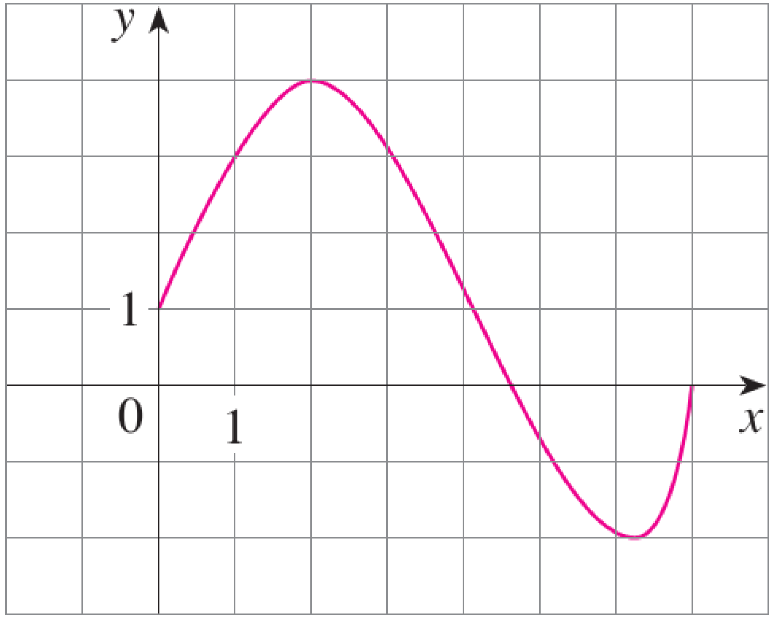
If the graph of a function is given then it is very easy to find its

domain and ran



Problem

The graph of a function f is shown in Figure



(a) Find the values of $f(1)$ and $f(5)$.

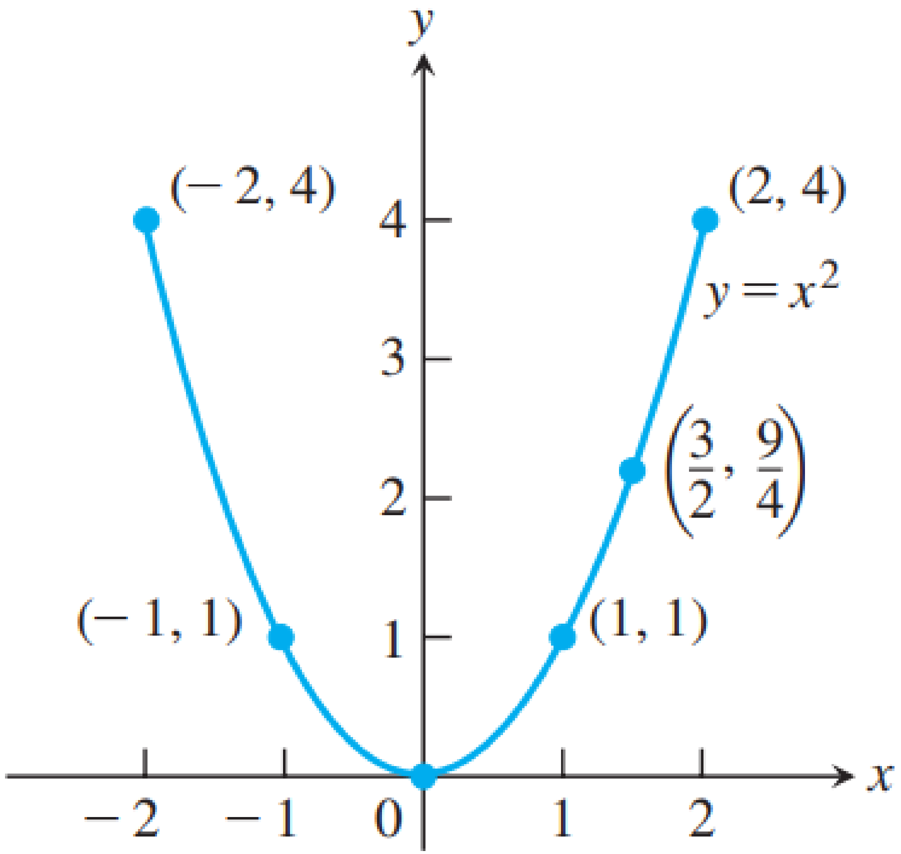
(b) What are the domain and range of f?

Practice Problem

Find the domain and range and sketch the graph of the function $h(x) = \sqrt{4 - x^2}$.

EXAMPLE Graph the function $y = x^2$ over the interval $[-2, 2]$.

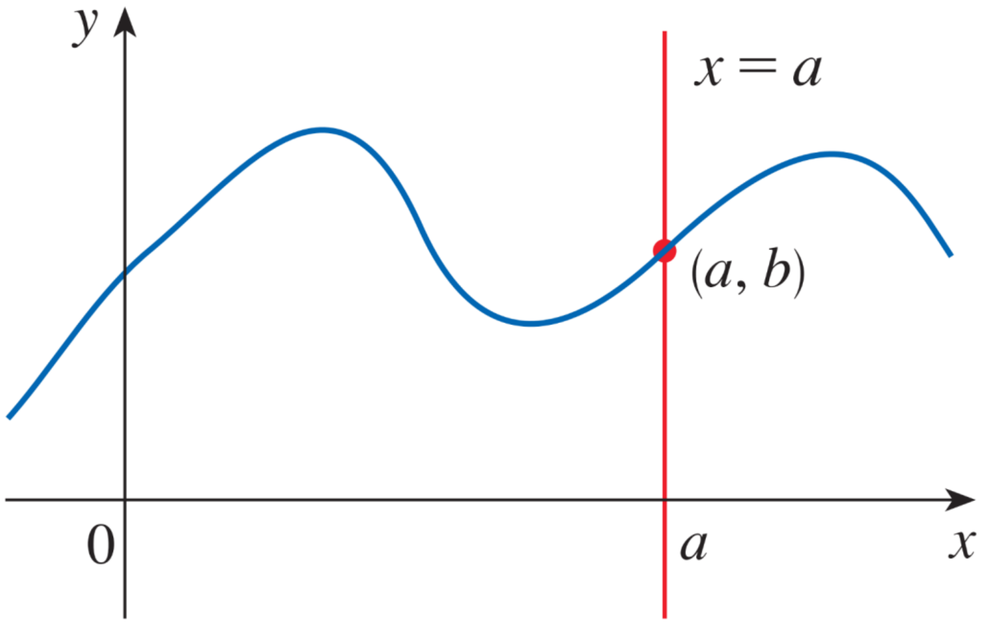
x	$y = x^2$
-2	4
-1	1
0	0
1	1
$\frac{3}{2}$	$\frac{9}{4}$
2	4

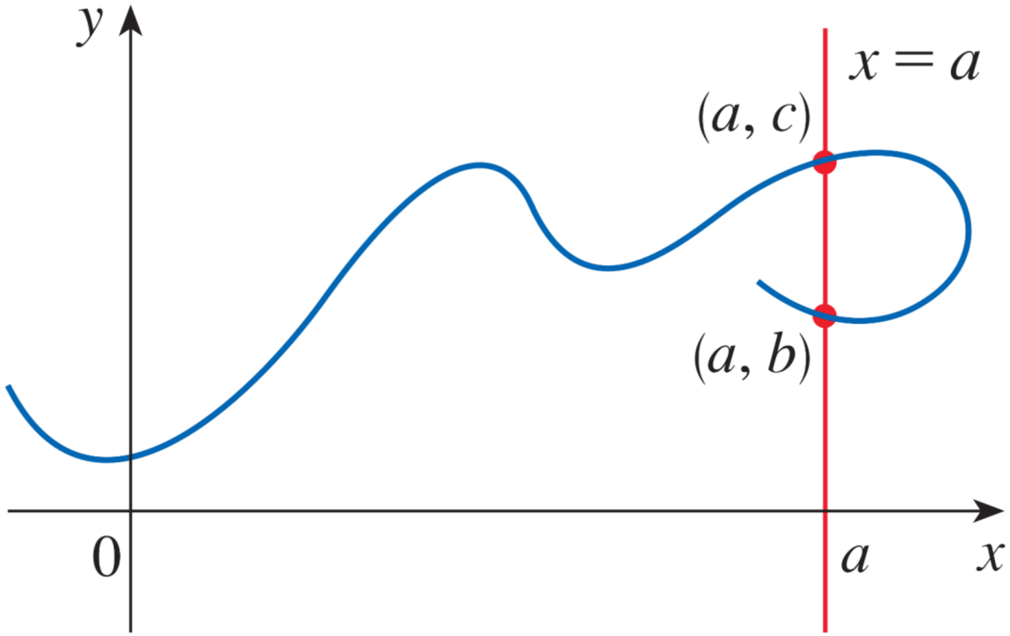


Vertical Line Test for checking whether a curve is the graph of a function

The graph of a function is a curve in the xy-plane. But the question arises: Which curves in the xy-plane are graphs of functions? This is answered by the following test.

The Vertical Line Test A curve in the xy-plane is the graph of a function of x if and only if no vertical line intersects the curve more than once.





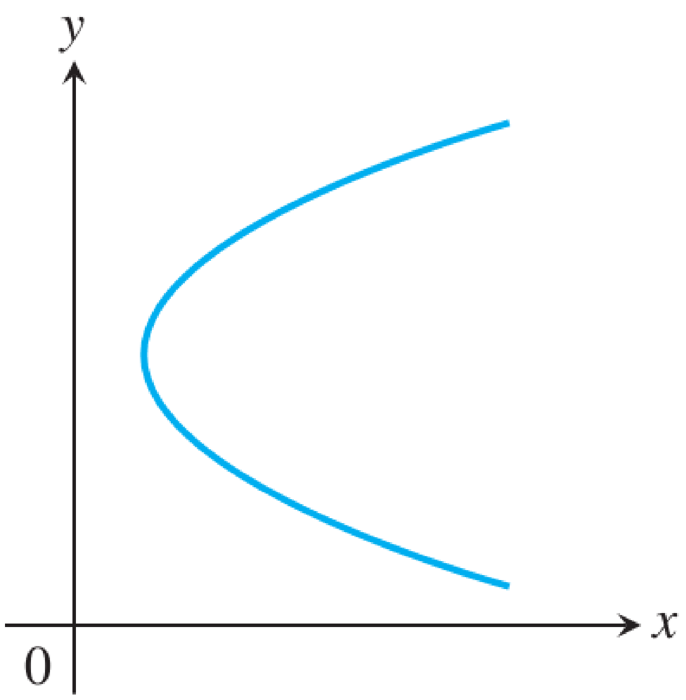
This curve represents a function

This curve doesn't represent a function.

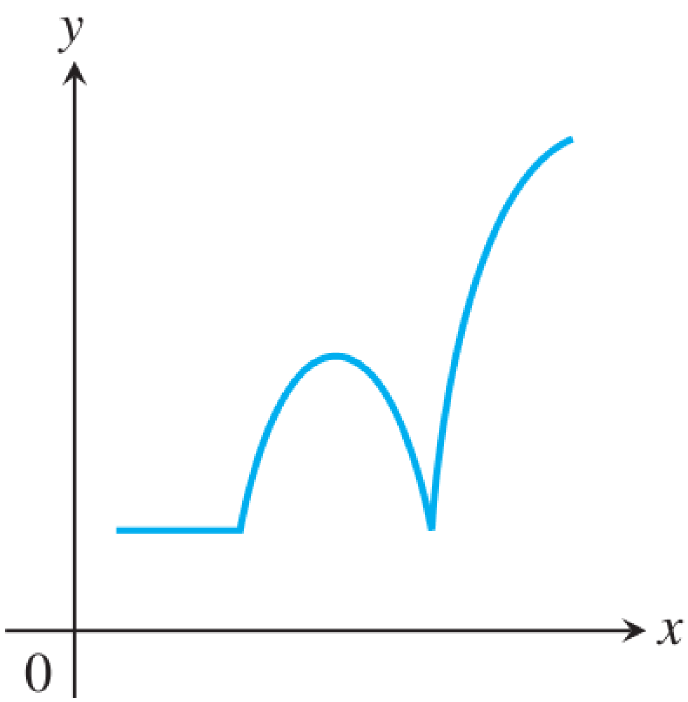
Practice Problem

Which of the graphs are graphs of functions of x, and which are not? Give reasons for your answers.

a.



b.



Piecewise-Defined Functions

Sometimes a function is described in pieces by using different formulas on different parts of its domain. One example is the **absolute value function**

$$|x| = \begin{cases} x, & x \geq 0 \quad \text{First formula} \\ -x, & x < 0 \quad \text{Second formula} \end{cases}$$

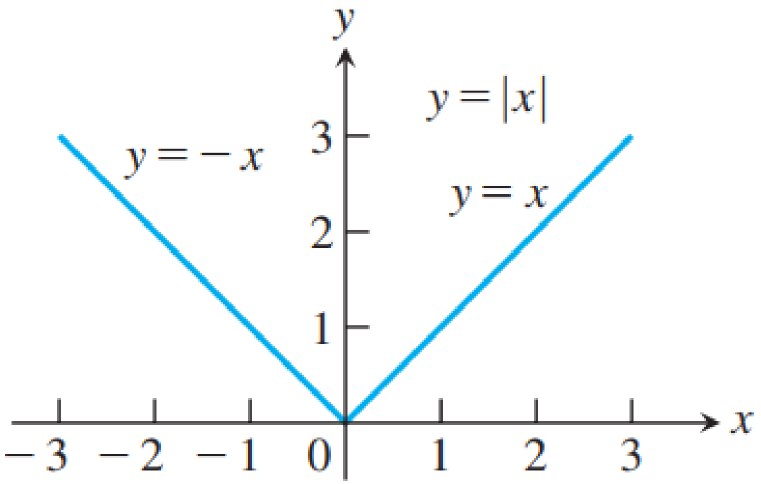


FIGURE The absolute value function has domain $(-\infty, \infty)$ and range $[0, \infty)$.

EXAMPLE A function f is defined by

$$f(x) = \begin{cases} 1 - x & \text{if } x \leqslant -1 \\ x^2 & \text{if } x > -1 \end{cases}$$

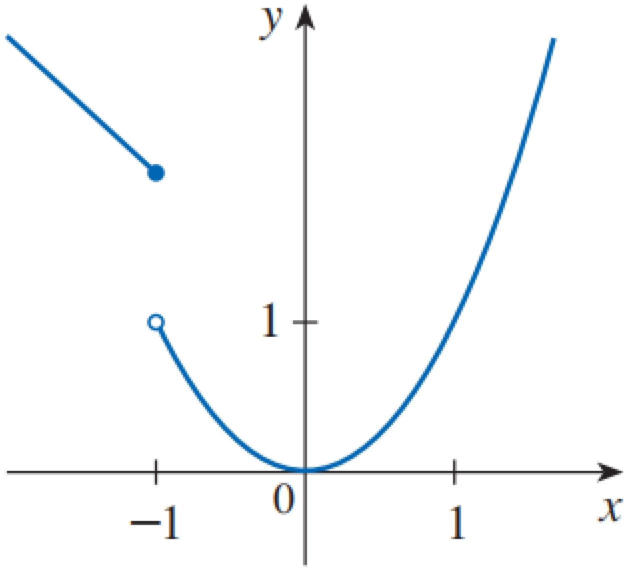
Evaluate $f(-2)$, $f(-1)$, and $f(0)$ and sketch the graph.

SOLUTION Remember that a function is a rule.

Since $-2 \leqslant -1$, we have $f(-2) = 1 - (-2) = 3$.

Since $-1 \leqslant -1$, we have $f(-1) = 1 - (-1) = 2$.

Since $0 > -1$, we have $f(0) = 0^2 = 0$.

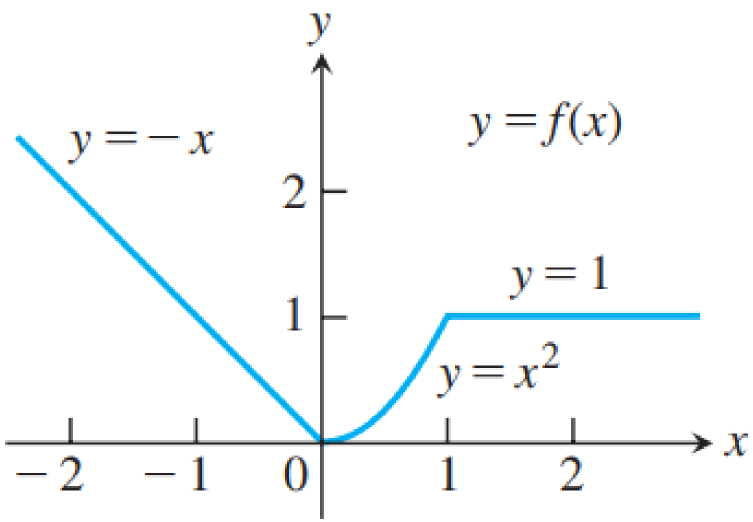


FIGURE

EXAMPLE The function

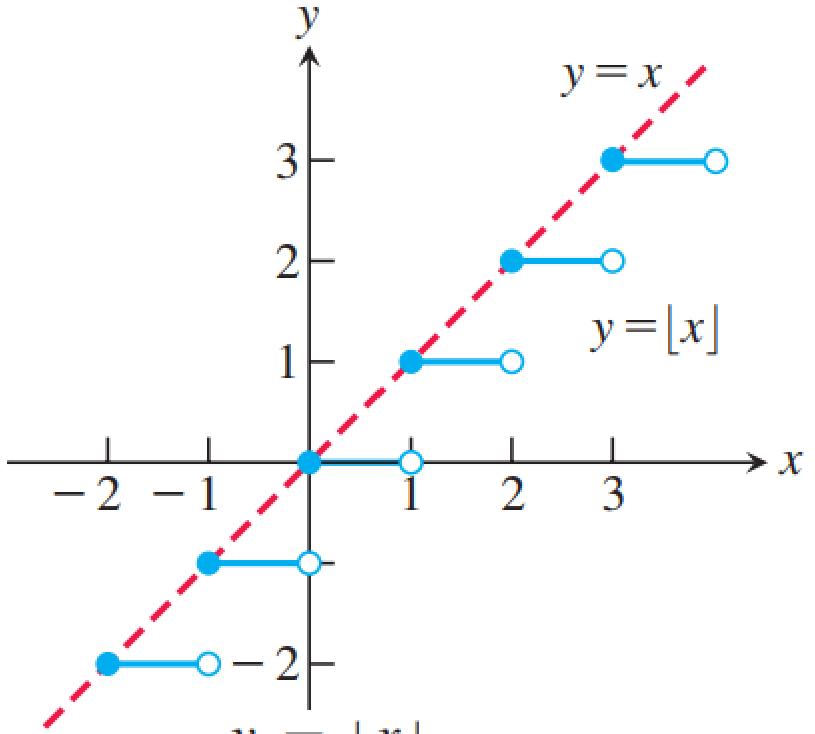
$$f(x) = \begin{cases} -x, & x < 0 \quad \text{First formula} \\ x^2, & 0 \leq x \leq 1 \quad \text{Second formula} \\ 1, & x > 1 \quad \text{Third formula} \end{cases}$$

is defined on the entire real line but has values given by different formulas, depending on the position of x. The values of f are given by $y = -x$ when $x < 0$, $y = x^2$ when $0 \leq x \leq 1$, and $y = 1$ when $x > 1$. The function, however, is *just one function* whose domain is the entire set of real numbers



EXAMPLE The function whose value at any number x is the *greatest integer less than or equal to* x is called the **greatest integer function** or the **integer floor function**. It is denoted $\lfloor x \rfloor$. Figure 1.10 shows the graph. Observe that

$$\lfloor 2.4 \rfloor = 2, \quad \lfloor 1.9 \rfloor = 1, \quad \lfloor 0 \rfloor = 0, \quad \lfloor -1.2 \rfloor = -2,$$
$$\lfloor 2 \rfloor = 2, \quad \lfloor 0.2 \rfloor = 0, \quad \lfloor -0.3 \rfloor = -1, \quad \lfloor -2 \rfloor = -2.$$

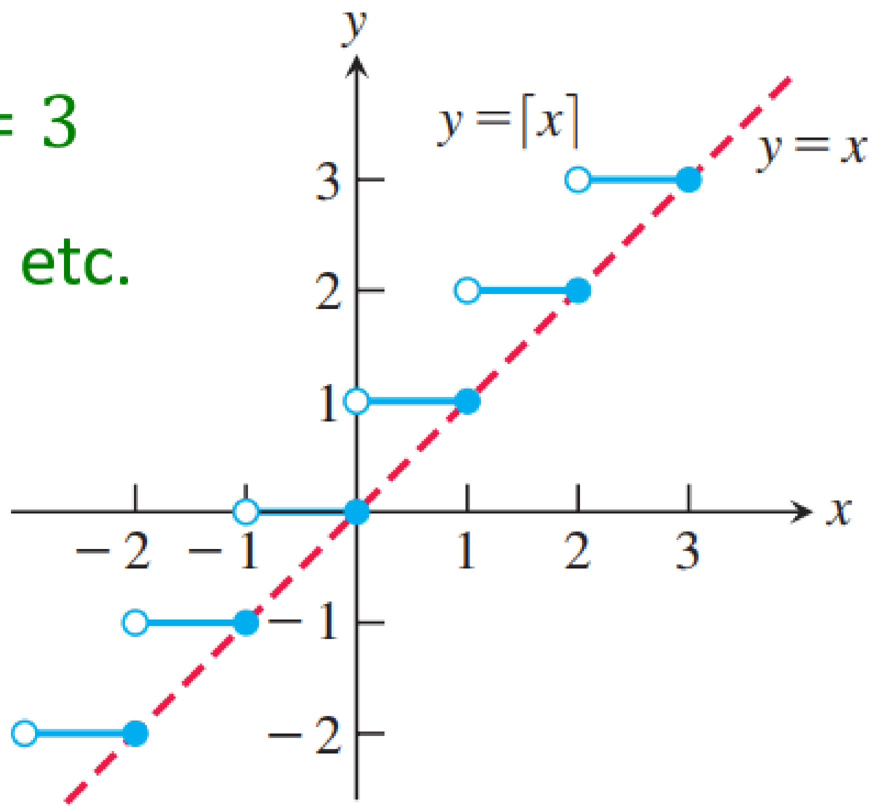


$y = \lfloor x \rfloor$

EXAMPLE The function whose value at any number x is the *smallest integer greater than or equal to* x is called the **least integer function** or the **integer ceiling function**. It is denoted $\lceil x \rceil$. Figure 1.11 shows the graph. For positive values of x, this function might represent, for example, the cost of parking x hours in a parking lot that charges \$1 for each hour or part of an hour.

$\lceil 1 \rceil = 1,$ $\quad \lceil 1.1 \rceil = 2,$ $\quad \lceil 2.8 \rceil = 3$

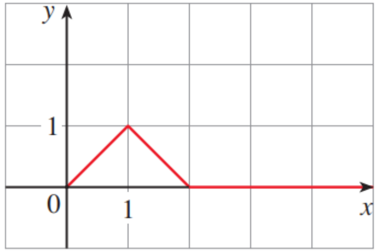
$\lceil -0.1 \rceil = 0,$ $\quad \lceil -2.9 \rceil = -2$ $\quad$ etc.



EXAMPLE Find a formula for the function f graphed in Figure .

SOLUTION The line through $(0, 0)$ and $(1, 1)$ has slope $m = 1$ and y-intercept $b = 0$, so its equation is $y = x$. Thus, for the part of the graph of f that joins $(0, 0)$ to $(1, 1)$, we have

$$f(x) = x \qquad \text{if } 0 \leqslant x \leqslant 1$$



FIGURE

The point-slope form of the equation of a line is $y - y_1 = m(x - x_1)$.

The line through $(1, 1)$ and $(2, 0)$ has slope $m = -1$, so its point-slope form is

$$y - 0 = (-1)(x - 2) \qquad \text{or} \qquad y = 2 - x$$

So we have
$$f(x) = 2 - x \qquad \text{if } 1 < x \leqslant 2$$

We also see that the graph of f coincides with the x-axis for $x > 2$. Putting this information together, we have the following three-piece formula for f:

$$f(x) = \begin{cases} x & \text{if } 0 \leqslant x \leqslant 1 \\ 2 - x & \text{if } 1 < x \leqslant 2 \\ 0 & \text{if } x > 2 \end{cases}$$

■

EXAMPLE If $f(x) = 2x^2 - 5x + 1$ and $h \neq 0$, evaluate $\dfrac{f(a+h) - f(a)}{h}$.

SOLUTION We first evaluate $f(a+h)$ by replacing x by $a+h$ in the expression for $f(x)$:

$$\begin{aligned} f(a+h) &= 2(a+h)^2 - 5(a+h) + 1 \\ &= 2(a^2 + 2ah + h^2) - 5(a+h) + 1 \\ &= 2a^2 + 4ah + 2h^2 - 5a - 5h + 1 \end{aligned}$$

Then we substitute into the given expression and simplify:

$$\begin{aligned} \frac{f(a+h) - f(a)}{h} &= \frac{(2a^2 + 4ah + 2h^2 - 5a - 5h + 1) - (2a^2 - 5a + 1)}{h} \\ &= \frac{2a^2 + 4ah + 2h^2 - 5a - 5h + 1 - 2a^2 + 5a - 1}{h} \\ &= \frac{4ah + 2h^2 - 5h}{h} = 4a + 2h - 5 \end{aligned}$$

■

Even Functions and Odd Functions: Symmetry

The graphs of *even* and *odd* functions have special symmetry properties.

DEFINITIONS A function $y = f(x)$ is an

even function of x if $f(-x) = f(x)$,

odd function of x if $f(-x) = -f(x)$,

for every x in the function's domain.

The graph of an even function is **symmetric about the *y*-axis.**

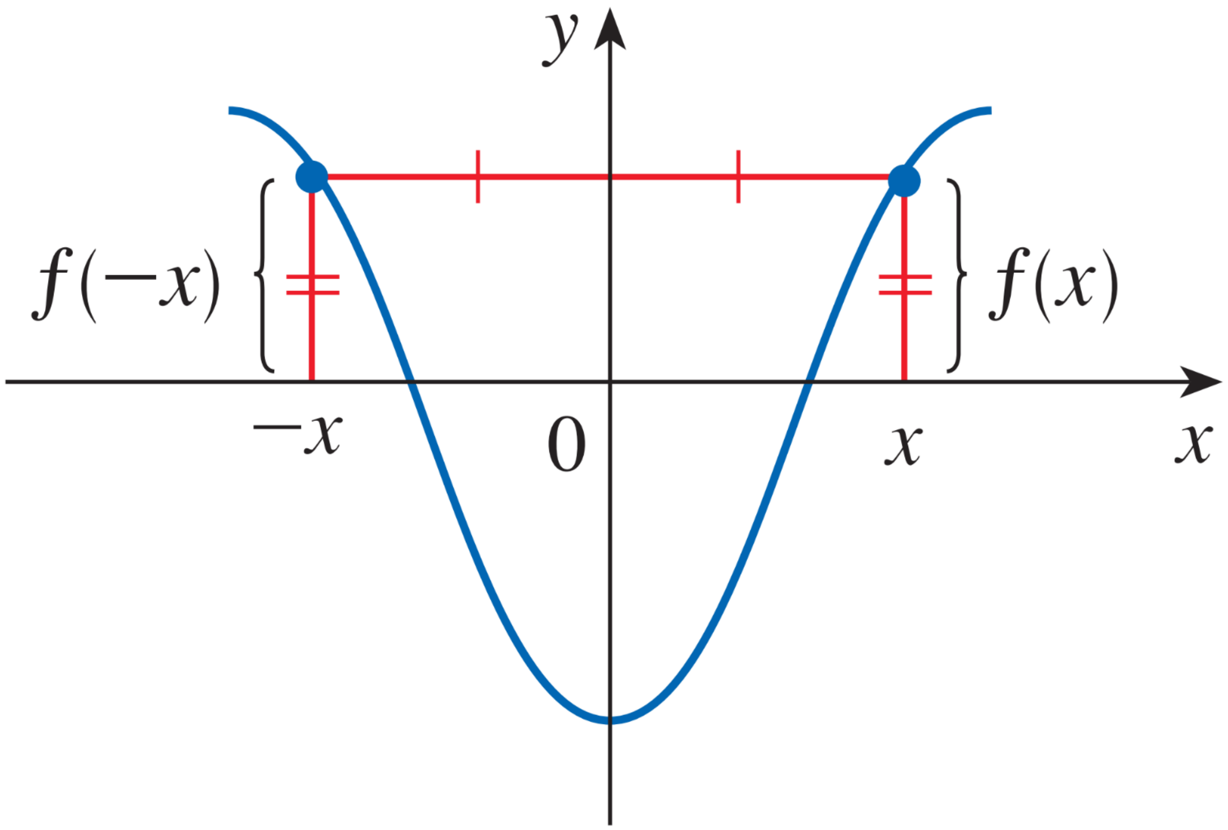


FIGURE
An even function

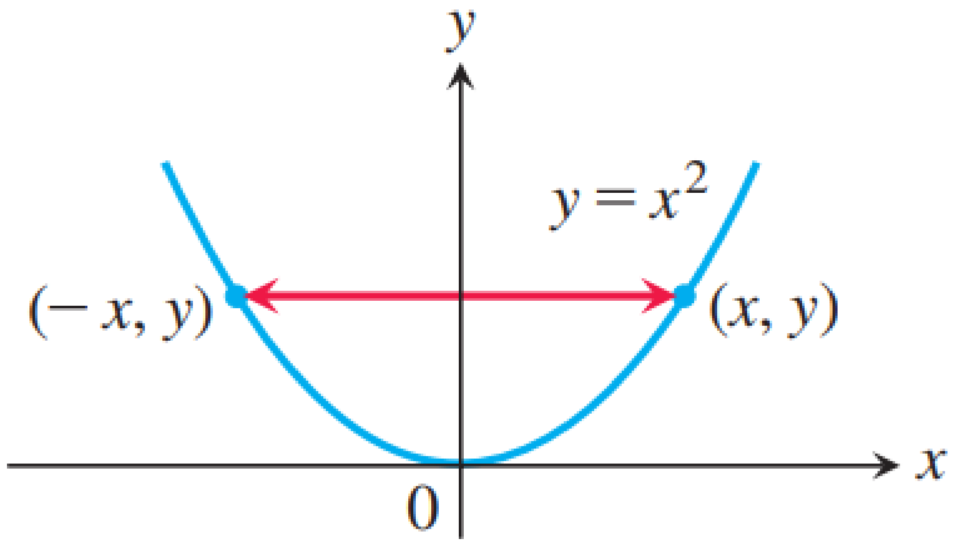


FIGURE
An even function

The graph of an odd function is **symmetric about the origin**.

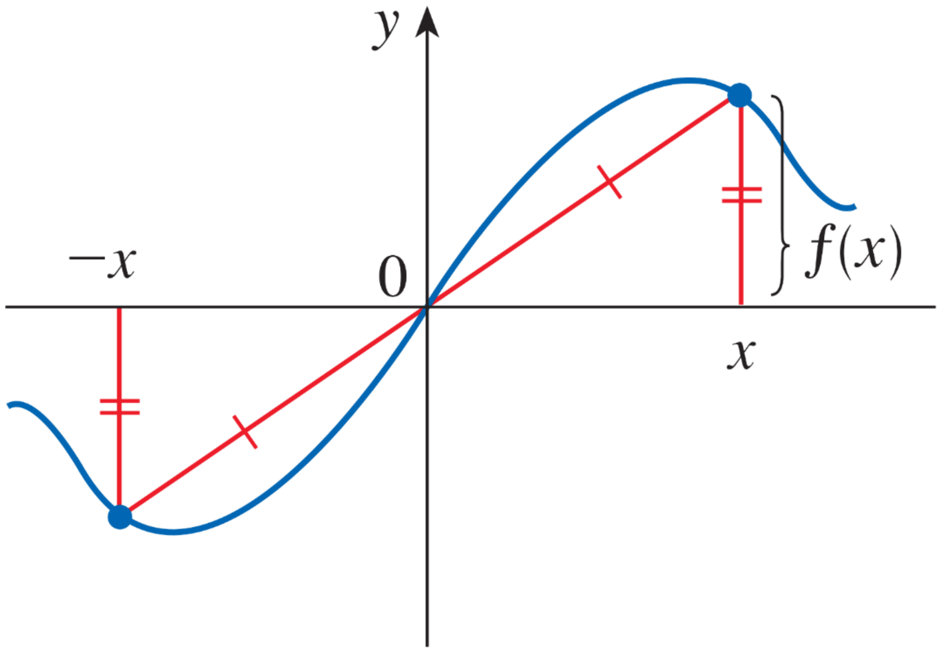


FIGURE
An odd function

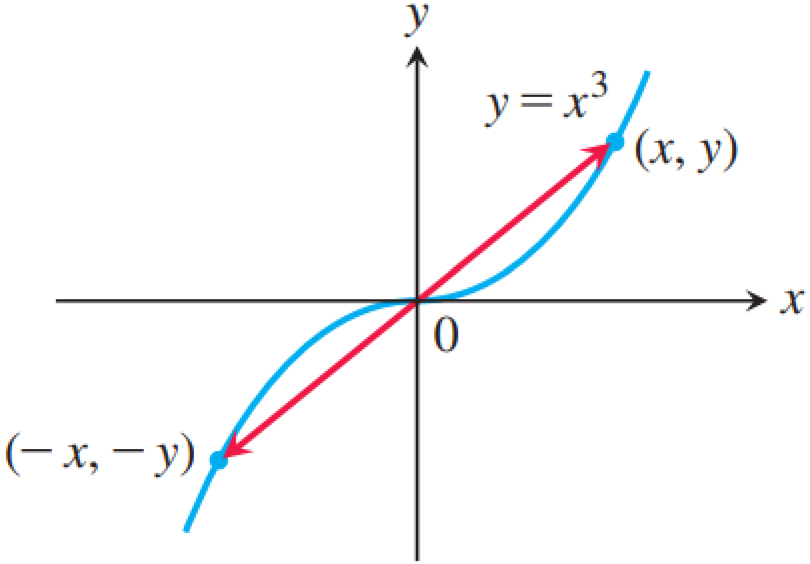


FIGURE
An odd function

EXAMPLE Determine whether each of the following functions is even, odd, or neither even nor odd.

(a) $f(x) = x^5 + x$ (b) $g(x) = 1 - x^4$ (c) $h(x) = 2x - x^2$

SOLUTION

(a)
$$\begin{aligned} f(-x) &= (-x)^5 + (-x) = (-1)^5 x^5 + (-x) \\ &= -x^5 - x = -(x^5 + x) \\ &= -f(x) \end{aligned}$$

Therefore f is an odd function.

(b)
$$g(-x) = 1 - (-x)^4 = 1 - x^4 = g(x)$$

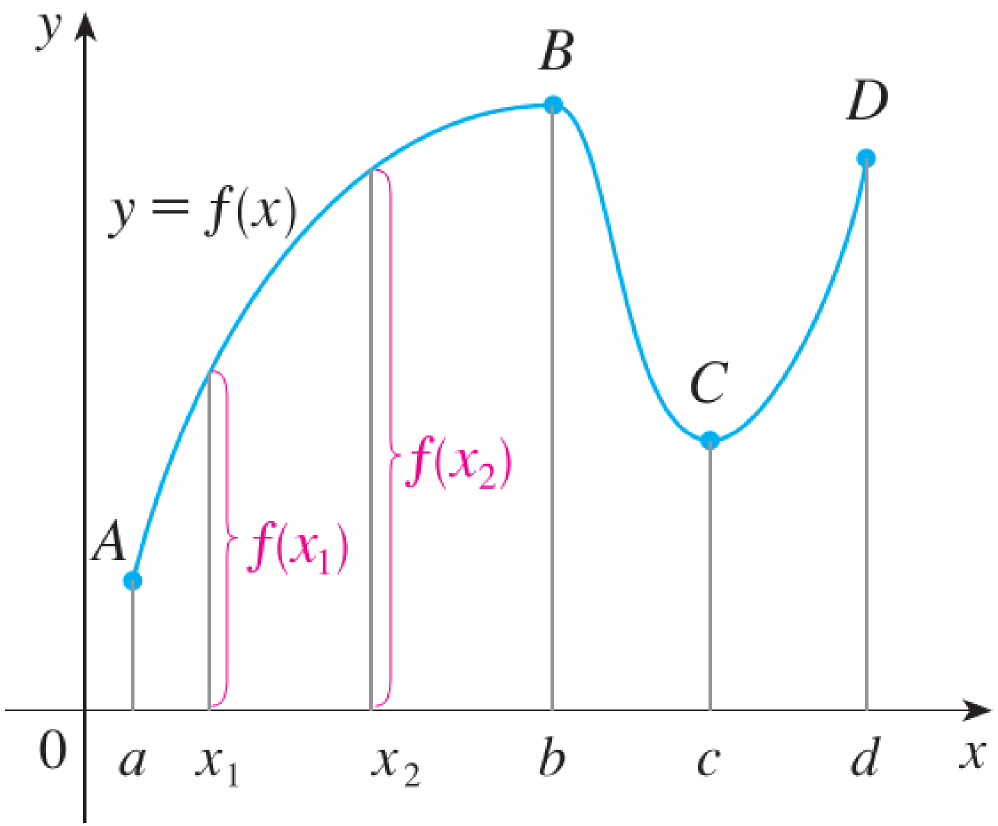
So g is even.

(c)
$$h(-x) = 2(-x) - (-x)^2 = -2x - x^2$$

Since $h(-x) \neq h(x)$ and $h(-x) \neq -h(x)$, we conclude that h is neither even nor odd.

Increasing and Decreasing Functions

The graph shown rises from A to B, falls from B to C, and rises again from C to D. The function f is said to be increasing on the interval [a, b], decreasing on [b, c], and increasing again on [c, d].



Increasing and Decreasing Functions

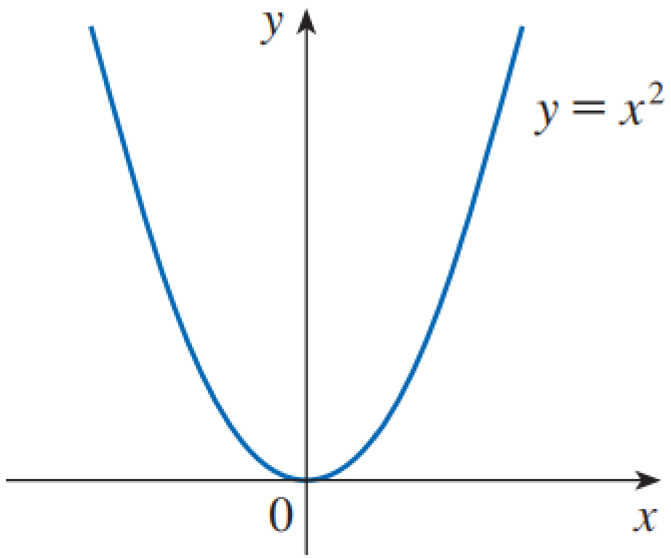
If the graph of a function climbs or rises as you move from left to right, we say that the function is *increasing*. If the graph descends or falls as you move from left to right, the function is *decreasing*.

DEFINITIONS Let f be a function defined on an interval I and let x_1 and x_2 be two distinct points in I.

1. If $f(x_2) > f(x_1)$ whenever $x_1 < x_2$, then f is said to be **increasing** on I.
2. If $f(x_2) < f(x_1)$ whenever $x_1 < x_2$, then f is said to be **decreasing** on I.

Example

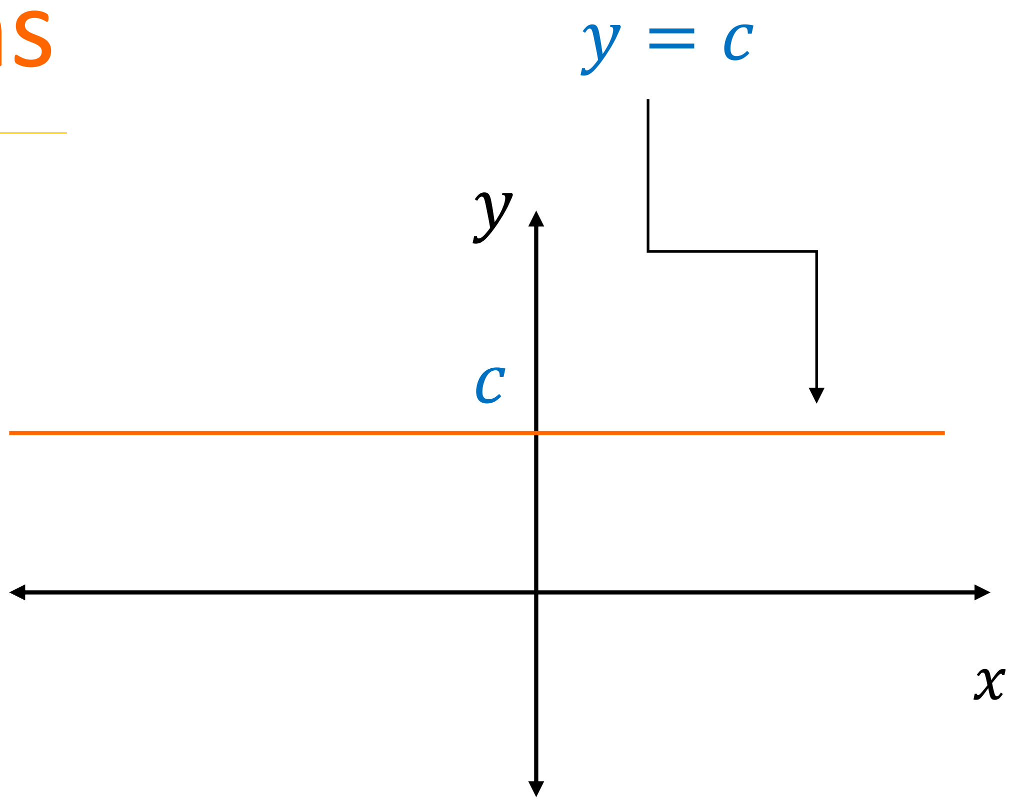
The function $f(x) = x^2$ is decreasing on the interval $(-\infty, 0]$ and increasing on the interval $[0, \infty)$.



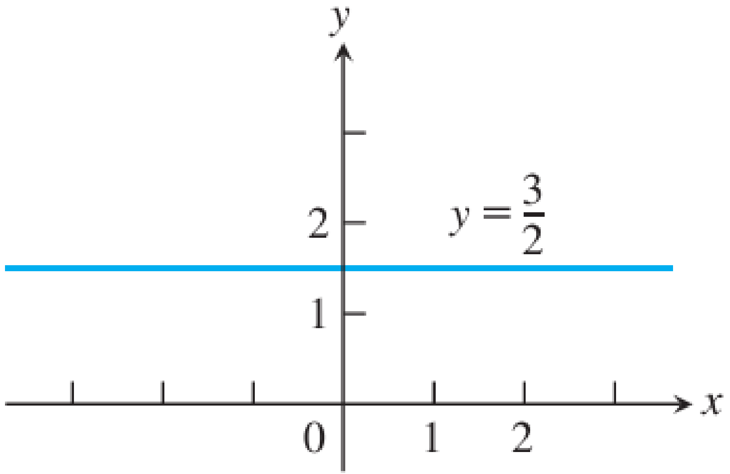
Graph of some basic Functions

Constant Function

$$f(x) = c$$



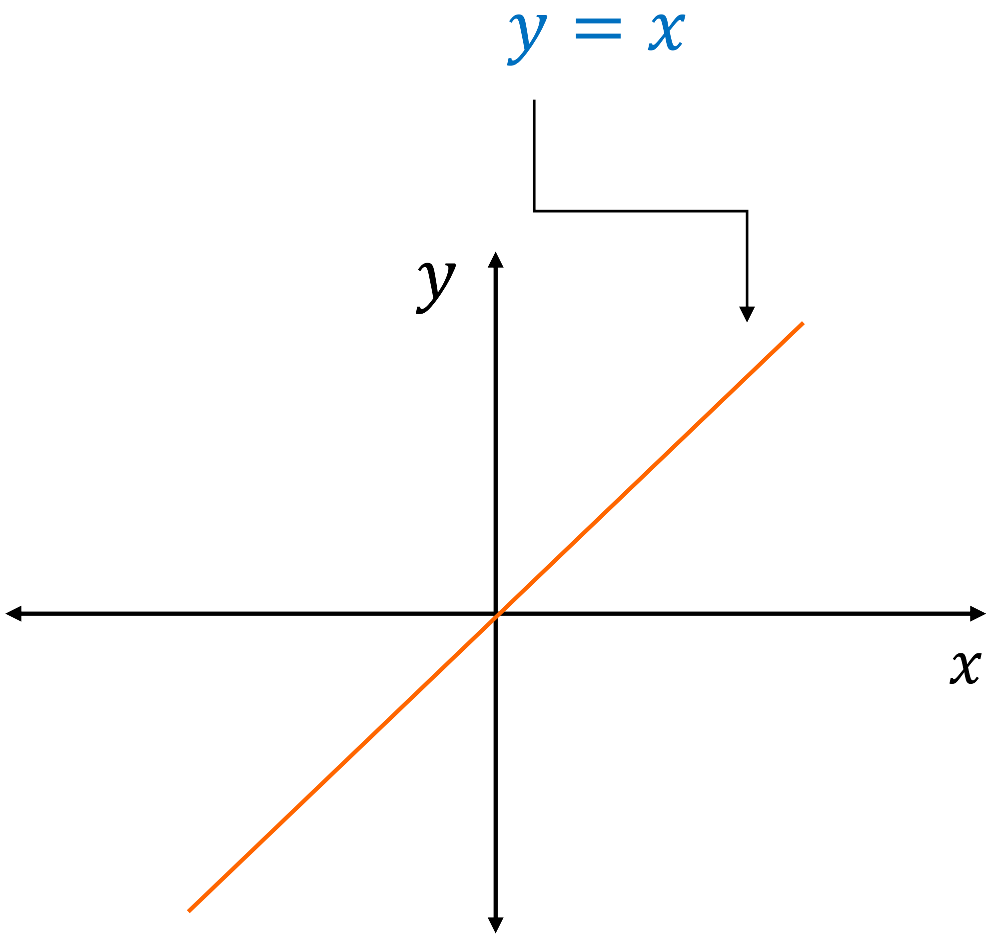
Example



The graph of a constant function is a horizontal line.

Identity Function

$$f(x) = x$$

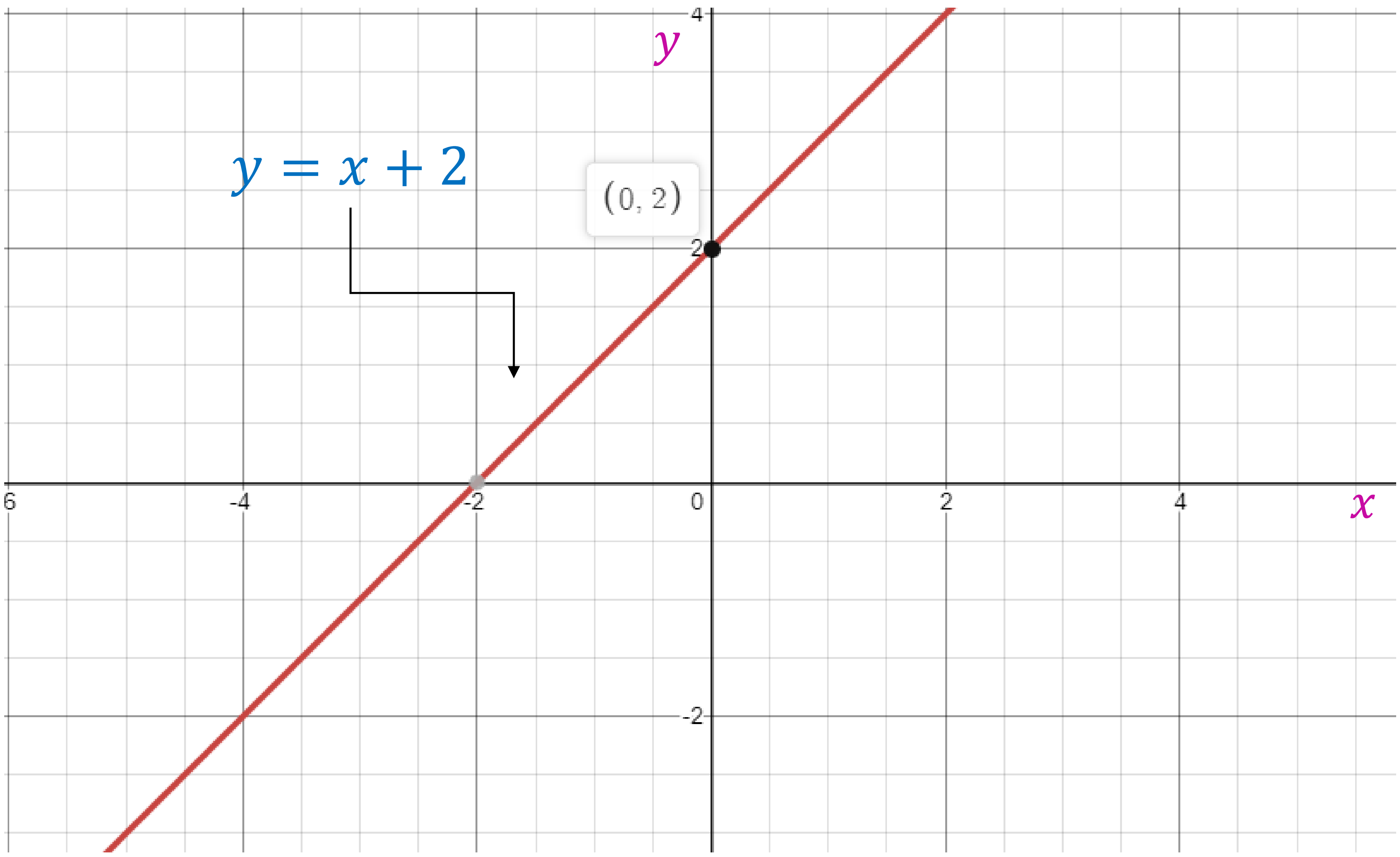


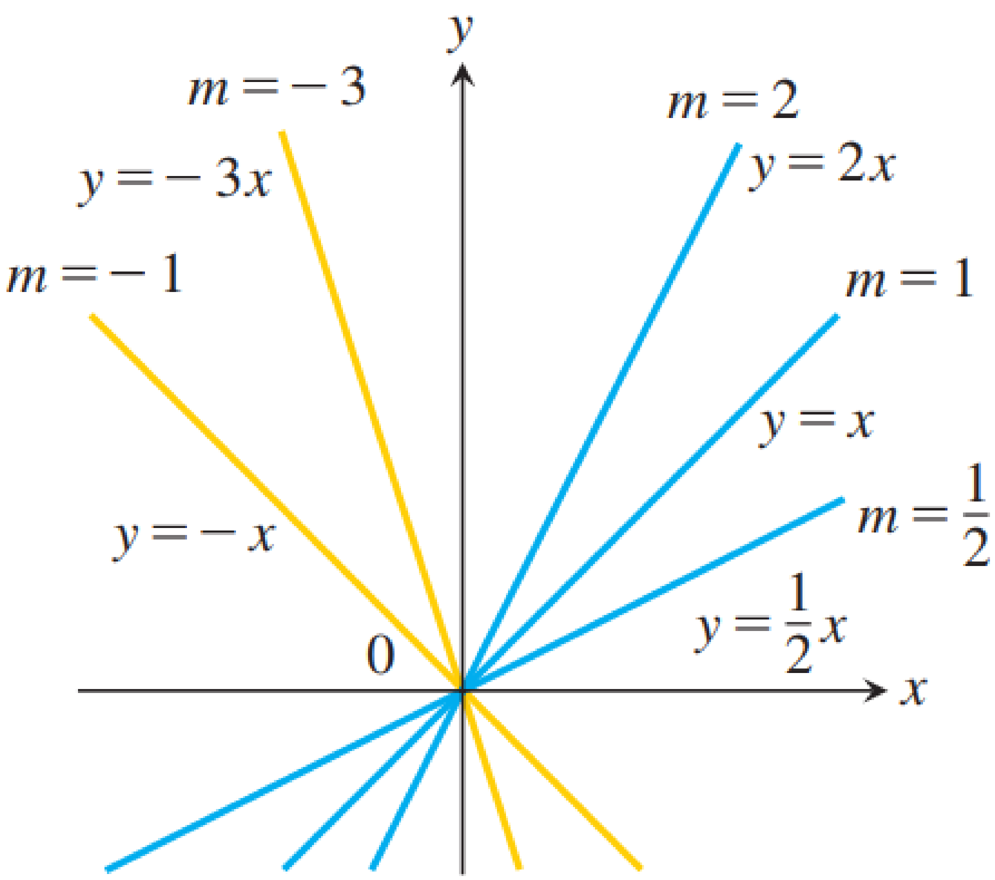
Linear Function

$$f(x) = mx + c$$

The graph of a linear function is a non vertical line. The number m is the slope of the line and c is the y-intercept.

Example: $y = x + 2$ is an equation of line that has slope 1 and y-intercept 2.


$y = x + 2$
$(0, 2)$
y
x
6
-4
-2
0
2
4
4
2
-2



(a)

Practice Problem

Climate change : Recent studies indicate that the average surface temperature of the earth has been rising steadily. Some scientists have modeled the temperature by the linear function $T = 0.02t + 8.50$, where T is temperature in °C and t represents years since 1900.

(a) What do the slope and T-intercept represent?

(b) Use the equation to predict the average global surface temperature in 2100.

Problem

Demand function for CDs : After studying sales for several months, the owner of a large CD retail outlet knows that the number of new CDs sold in a day (called the demand) decreases as the retail price increases. Specifically, his data indicate that at a price of $\$14$ per CD an average of 400 CDs are sold per day, while at a price of $\$17$ per CD an average of 250 CDs are sold per day. Assume that the demand d is a linear function of the price p.

a. Find and graph the demand function $d = f(p) = mp + b$.

b. According to this model, how many CDs (on average) are sold at a price of $\$20$?

SOLUTION

a. Two points on the graph of the demand function are given: $(p, d) = (14, 400)$ and $(17, 250)$. Therefore, the slope of the demand line is

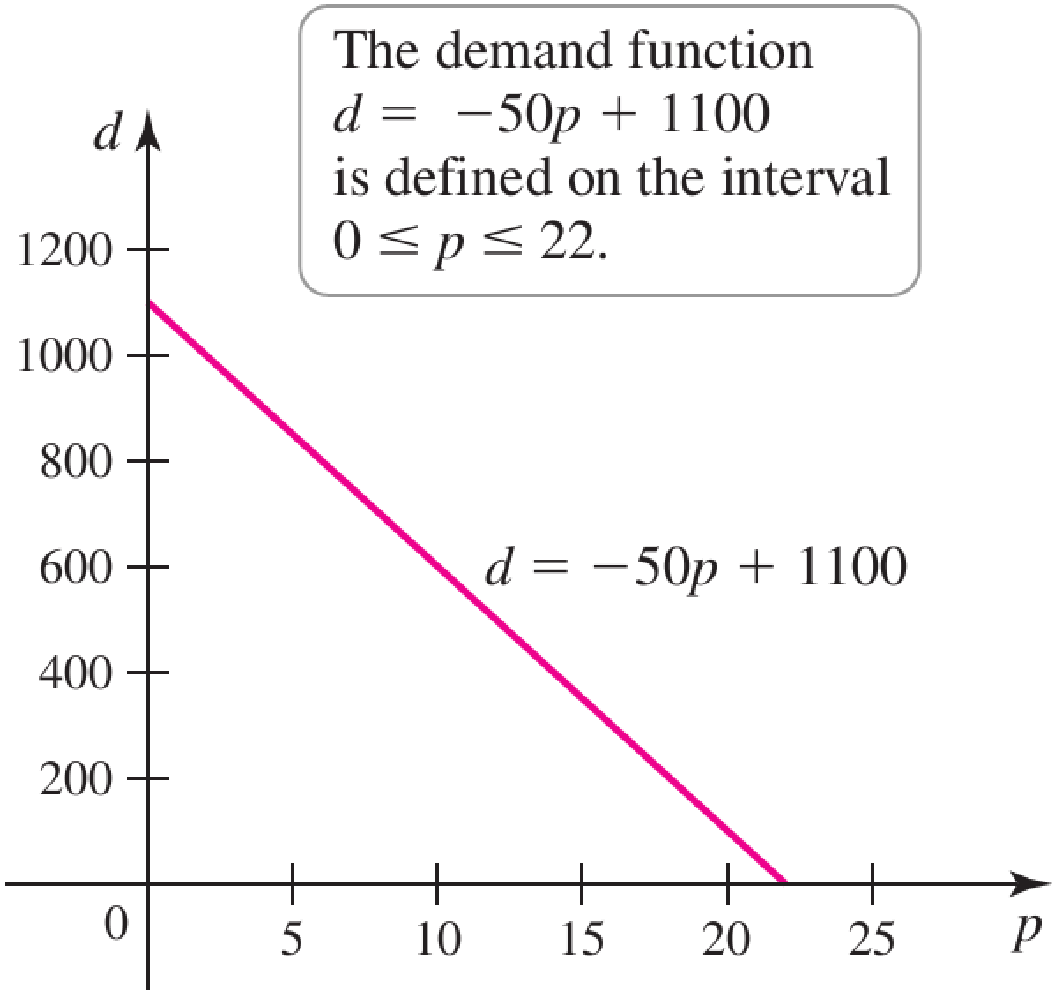
$$m = \frac{400 - 250}{14 - 17} = -50 \text{ CDs per dollar.}$$

It follows that the equation of the linear demand function is

$$d - 250 = -50(p - 17).$$

Expressing d as a function of p, we have $d = f(p) = -50p + 1100$

b. Using the demand function with a price of \$20, the average number of CDs that could be sold per day is $f(20) = 100$.



Practice Problem

The period T of a pendulum is measured for pendulums of several different lengths L. Based on the following data, does T appear to be a linear function of L?

L (cm)	20	30	40	50
T (s)	0.9	1.1	1.27	1.42

Practice Problem

A city's population was $30{,}700$ in the year 2010 and is growing by 850 people a year.

(a) Give a formula for the city's population , P , as a function of the number of years , t , since 2010.

(b) What is the population predicted to be in 2021?

(c) When is the population expected to reach $45{,}000$?

Polynomials

A function P is called a polynomial if

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + a_2 x^2 + a_1 x + a_0$$

where n is a nonnegative integer and the numbers $a_0, a_1, a_2, \ldots a_n$ are constants called the coefficients of the polynomial. If the leading coefficient $a_n \neq 0$, then the degree of the polynomial is n.

Examples:

$P(x) = 2x^6 + 5x^5 + 7x^3 - 8x^2 + x - 90$ is a polynomial of degree 6.

$P(x) = -8x^4 + x + 1$ is a polynomial of degree 4.

$f(x) = x^{\frac{1}{2}} + x + 1$ is not a polynomial because the power of first term is not a positive integer.

A polynomial of degree 1 is of the form $P(x) = a_1 x + a_0$ and so is a linear function.

A polynomial of degree 2 is of the form $P(x) = a_2 x^2 + a_1 x + a_0$ and is called a quadratic function.

Its graph is always a parabola obtained by shifting the parabola $y = ax^2$.

The parabola opens upward if $a_2 > 0$ and downward if $a_2 < 0$.

Power Functions

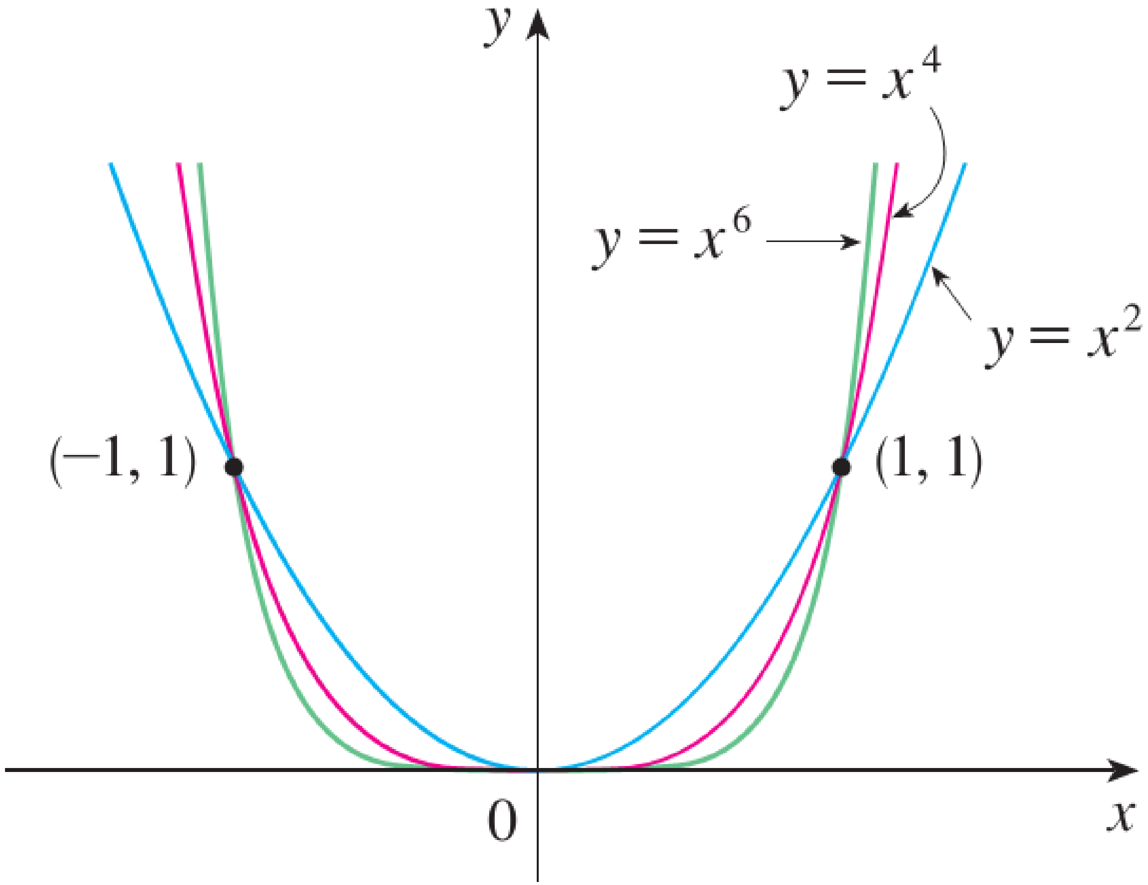
A function of the form $f(x) = x^a$, where a is a constant, is called a power function.

Examples $f(x) = x^2$, $g(x) = x^5$, etc.

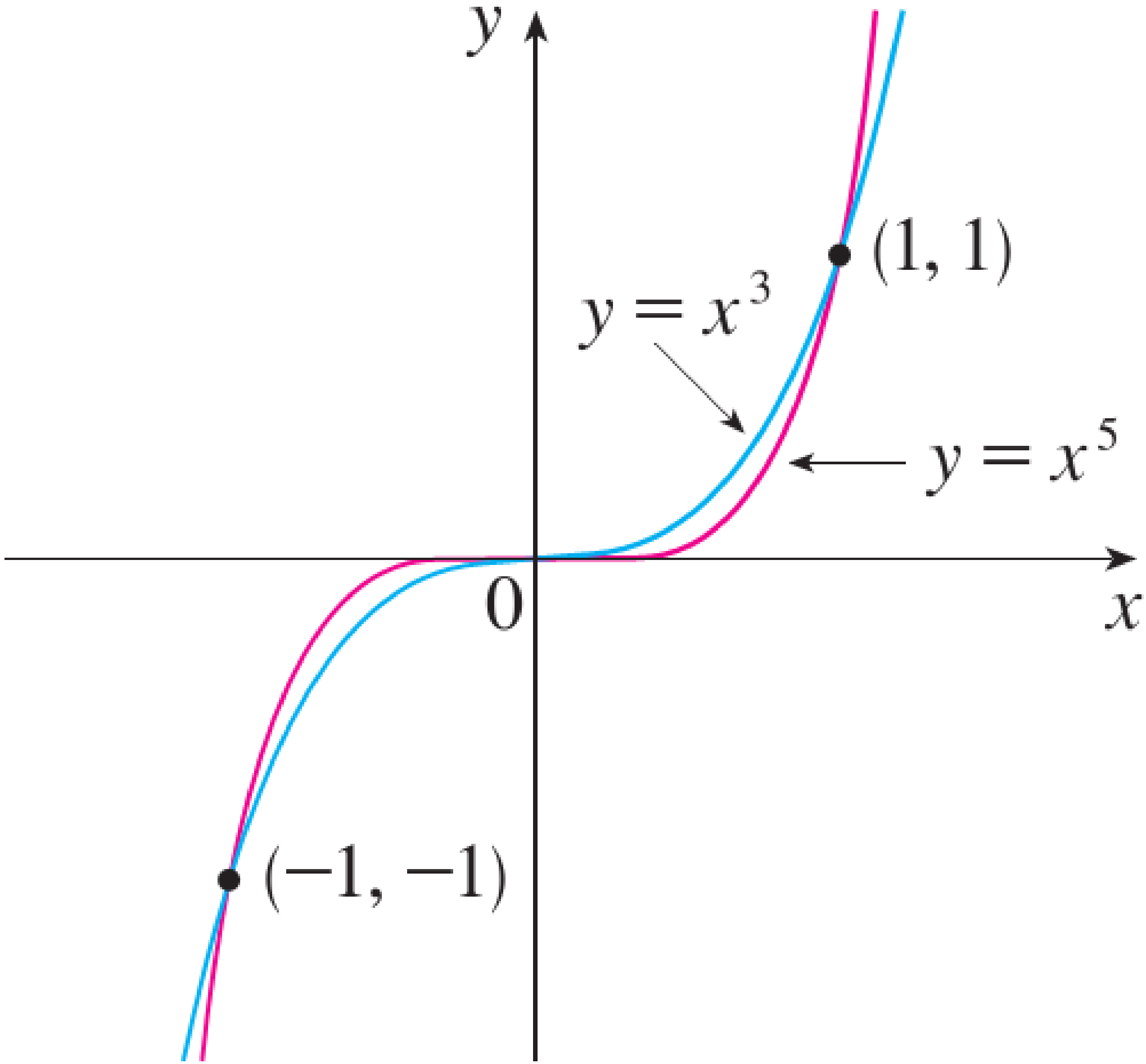
(i) $a = n$, where n is a positive integer

- The general shape of the graph of $f(x) = x^n$ depends on whether n is even or odd.
- If n is even, then $f(x) = x^n$ is an even function and its graph is similar to the parabola $y = x^2$.

- If n is odd, then $f(x) = x^n$ is an odd function and its graph is similar to that of $y = x^3$. As n increases, the graph of $y = x^n$ becomes flatter near 0 and steeper when $|x| > 1$. (If x is small, then x^2 is smaller, x^3 is more smaller than x^2 and so on.)



y
$y = x^4$
$y = x^6$
$y = x^2$
(−1, 1)
(1, 1)
0
x



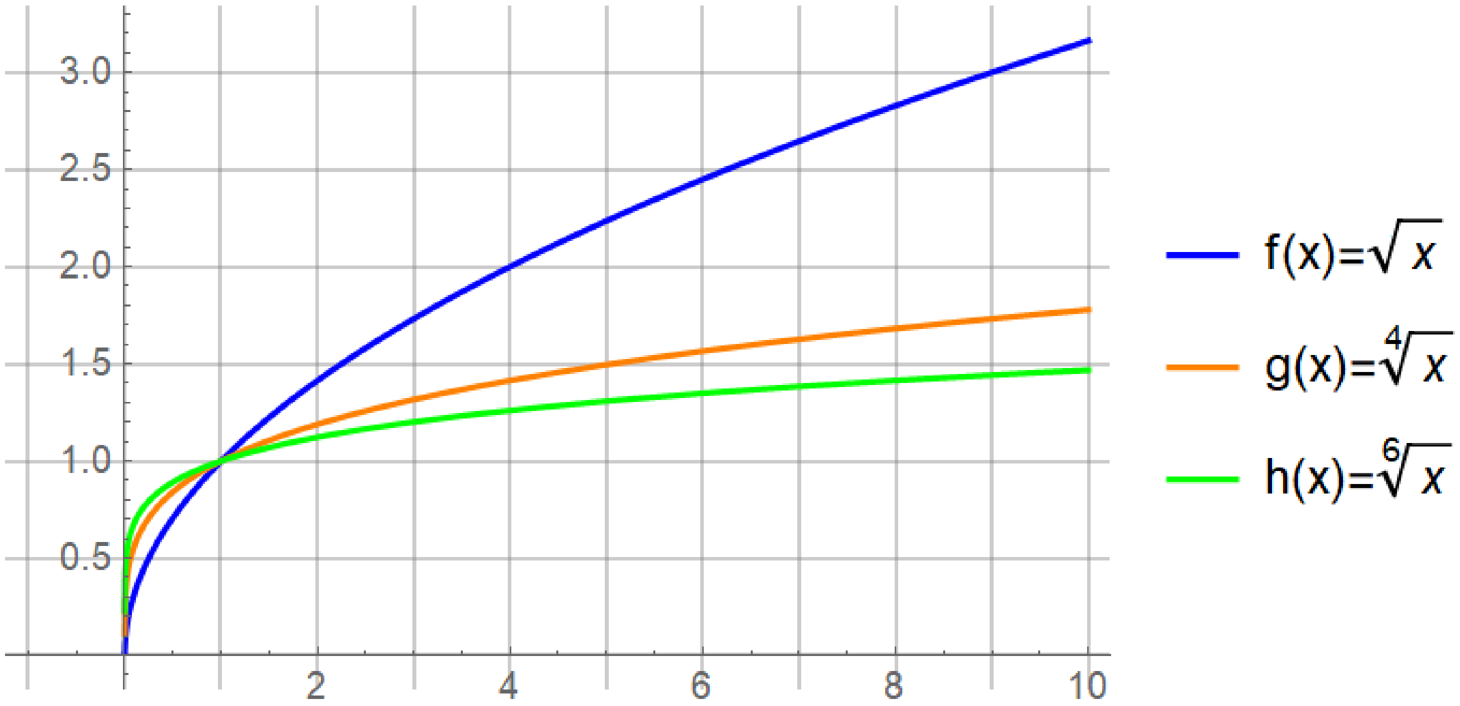
y
(1, 1)
$y = x^3$
$y = x^5$
0
x
(−1, −1)

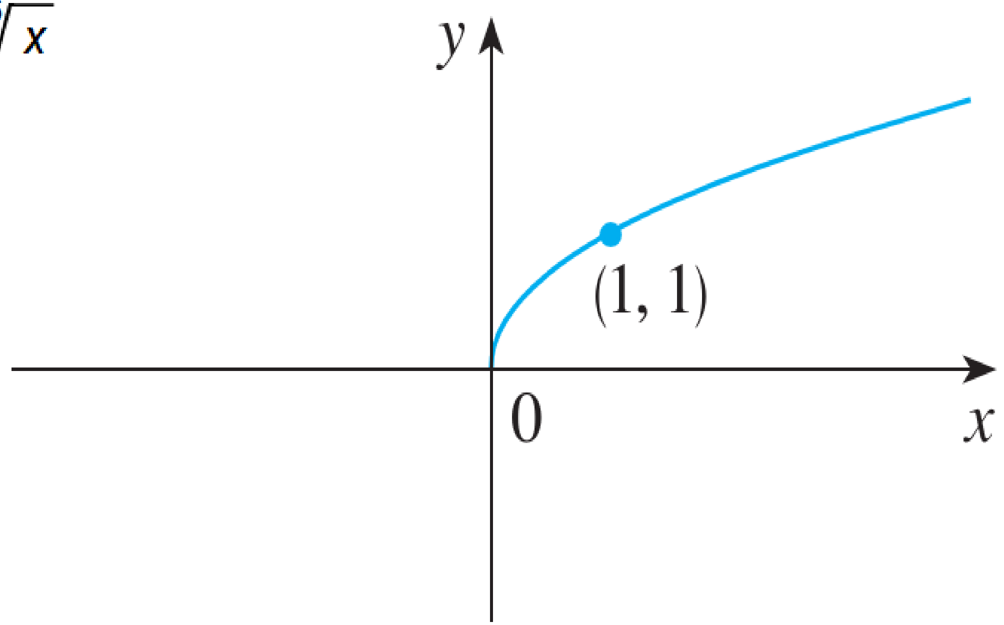
(ii) $a = \frac{1}{n}$, where n is a positive integer

- The function $f(x) = x^{\frac{1}{n}} = \sqrt[n]{x}$ is a root function.

- For $n = 2$ it is the square root function $f(x) = \sqrt{x}$ whose domain is $[0, \infty)$ and whose graph is the upper half of the parabola $x = y^2$.

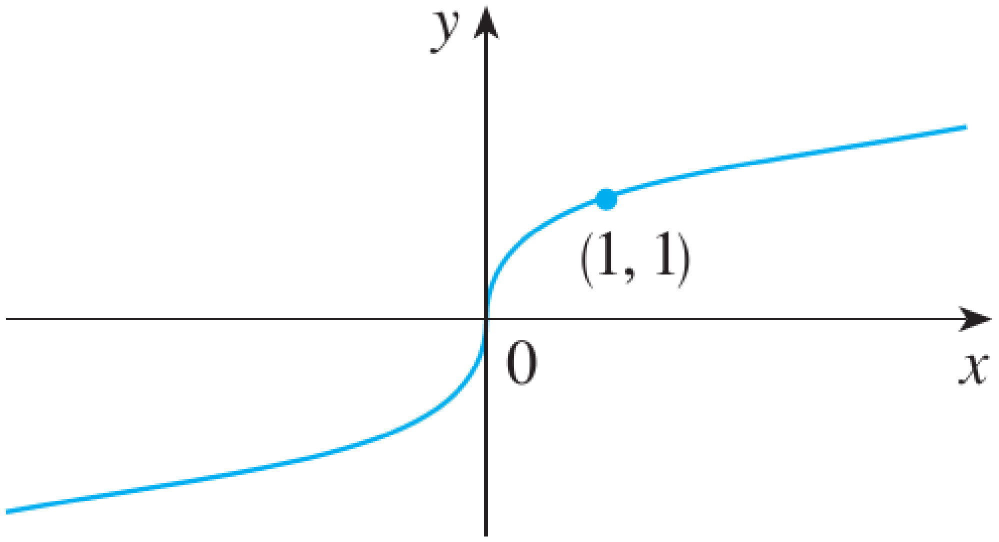
- For other even values of n, the graph of $y = \sqrt[n]{x}$ is similar to that of $y = \sqrt{x}$.



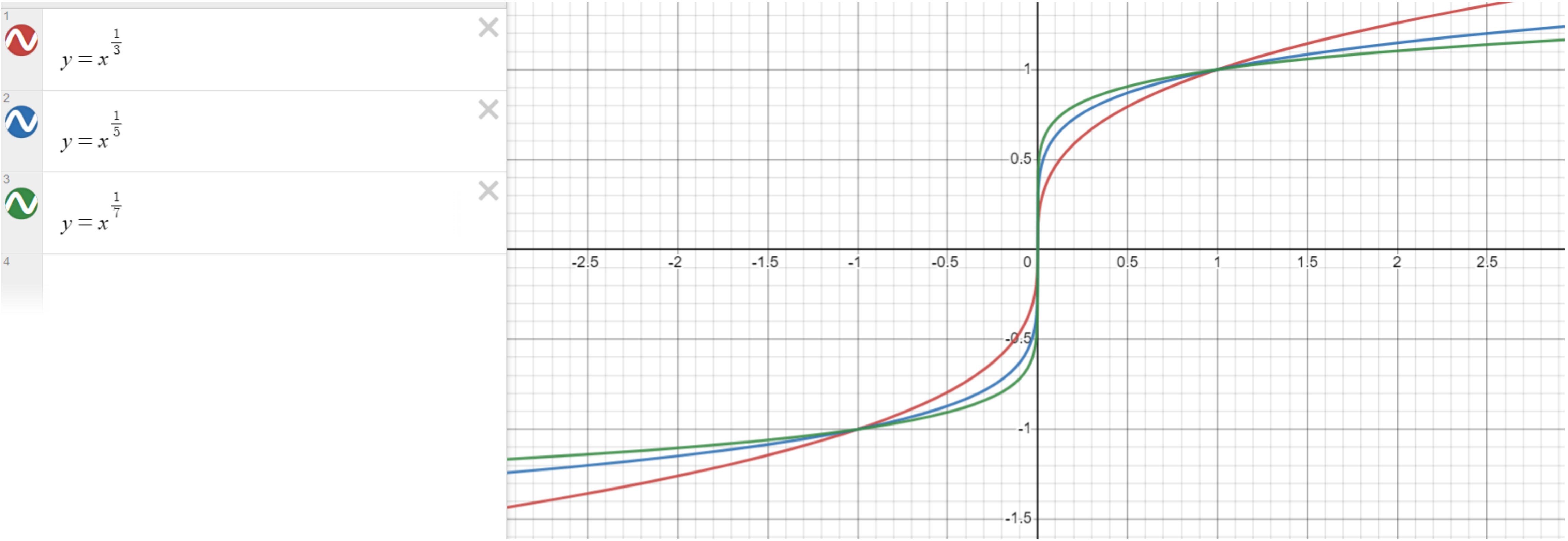


$$f(x) = \sqrt{x}$$

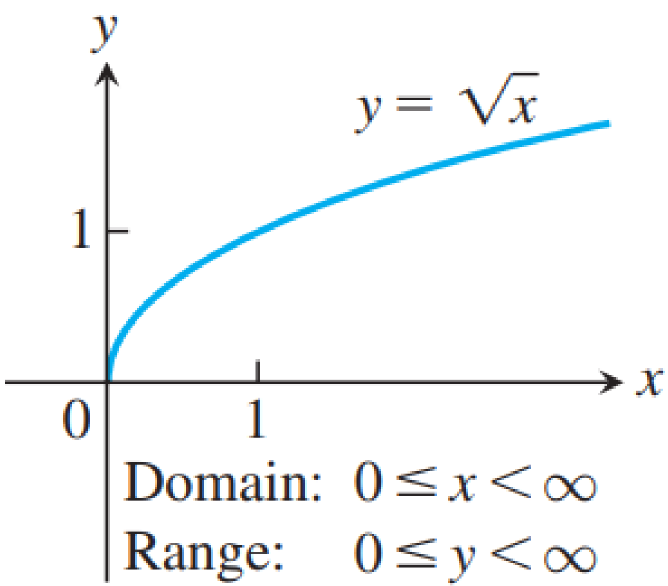
For $n = 3$ we have the cube root function $f(x) = \sqrt[3]{x}$ whose domain is R (recall that every real number has a cube root) and whose graph is shown in Figure.

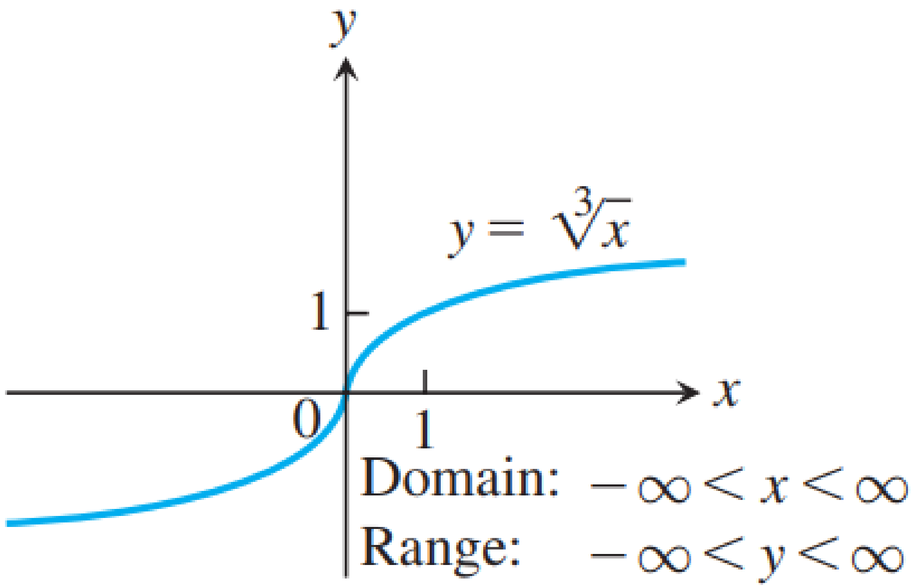


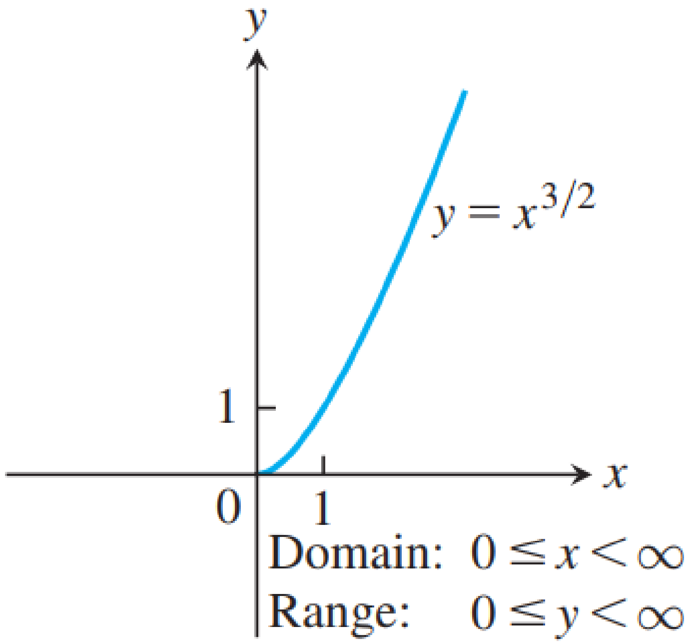
$$f(x) = \sqrt[3]{x}$$

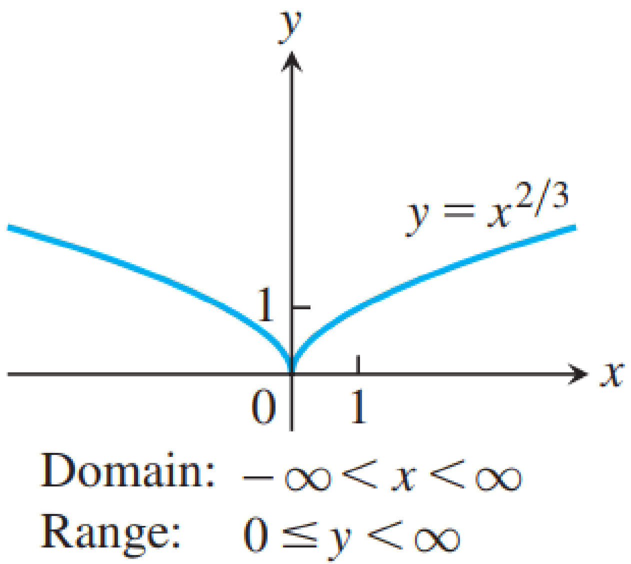


$y = x^{\frac{1}{3}}$
$y = x^{\frac{1}{5}}$
$y = x^{\frac{1}{7}}$
-2.5
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2
2.5
-1.5
-0.5


$y = \sqrt{x}$
y
x
1
0
1
Domain: $0 \leq x < \infty$
Range: $0 \leq y < \infty$

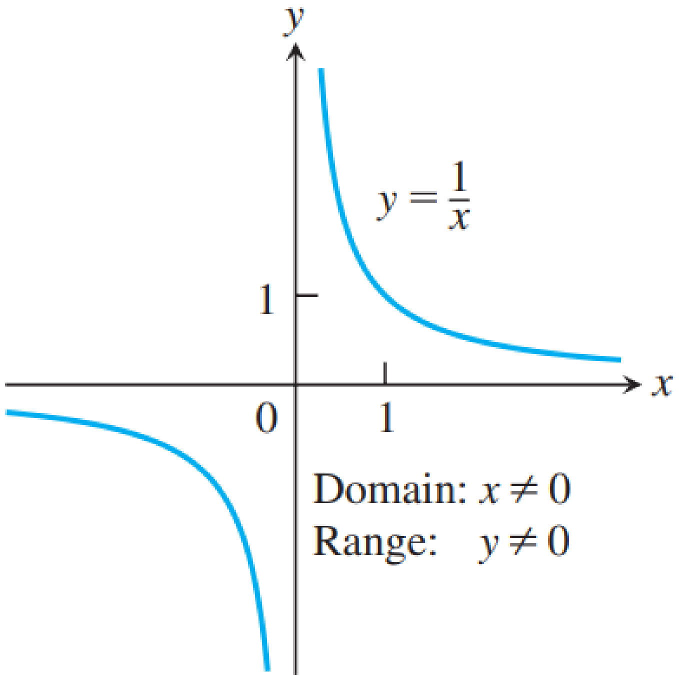

$y = \sqrt[3]{x}$
y
x
1
0
1
Domain: $-\infty < x < \infty$
Range: $-\infty < y < \infty$


$y = x^{3/2}$
y
x
1
0
1
Domain: $0 \leq x < \infty$
Range: $0 \leq y < \infty$

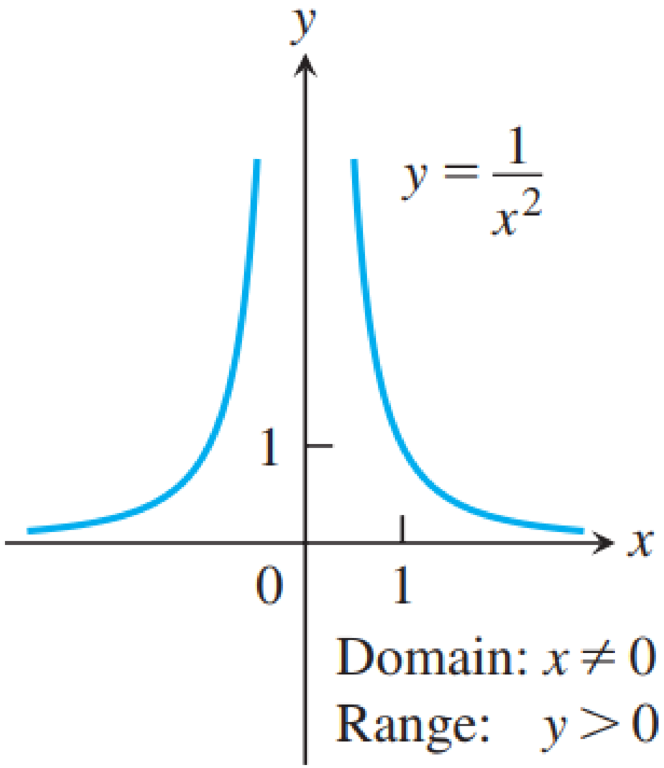

$y = x^{2/3}$
y
x
1
0
1
Domain: $-\infty < x < \infty$
Range: $0 \leq y < \infty$

(ii) $a = -1$

The graph of the reciprocal function $f(x) = x^{-1} = \frac{1}{x}$ is shown in the figure.



(a)



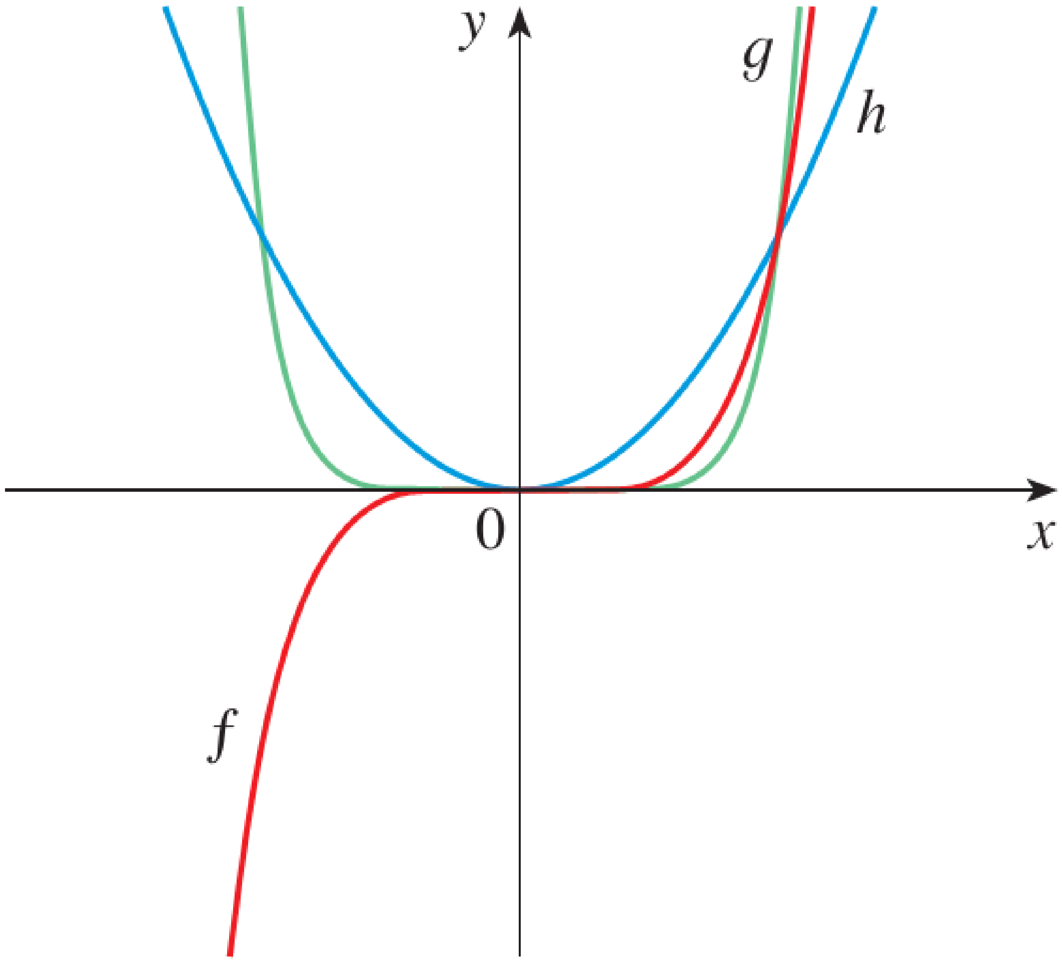
(b)

Practice Problem

Match each equation with its graph.

(Don't use a computer or graphing calculator.)

(a) $y = x^2$ (b) $y = x^5$ (c) $y = x^8$



Rational Functions

A **rational function** f is a ratio of two polynomials:

$$f(x) = \frac{P(x)}{Q(x)}$$

where P and Q are polynomials.

The domain consists of all values of x such that $Q(x) \neq 0$.

Example

$$f(x) = \frac{2x^4 - x^2 + 1}{x^2 - 4}$$

is a rational function with domain $\{x \mid x \neq \pm 2\}$.

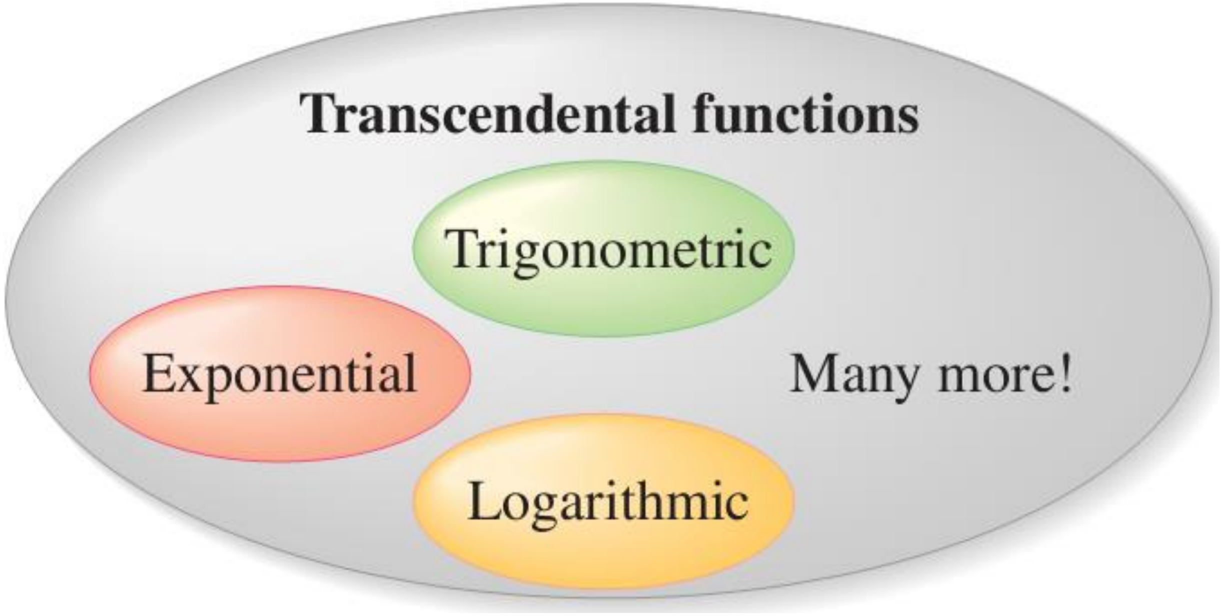
Algebraic Function

A function f is called an **algebraic function** if it can be constructed using algebraic operations (such as addition, subtraction, multiplication, division, and taking roots) starting with polynomials. Any rational function is automatically an algebraic function. Here are two more examples:

$$f(x) = \sqrt{x^2 + 1} \qquad\qquad g(x) = \frac{x^4 - 16x^2}{x + \sqrt{x}} + (x - 2)\sqrt[3]{x + 1}$$

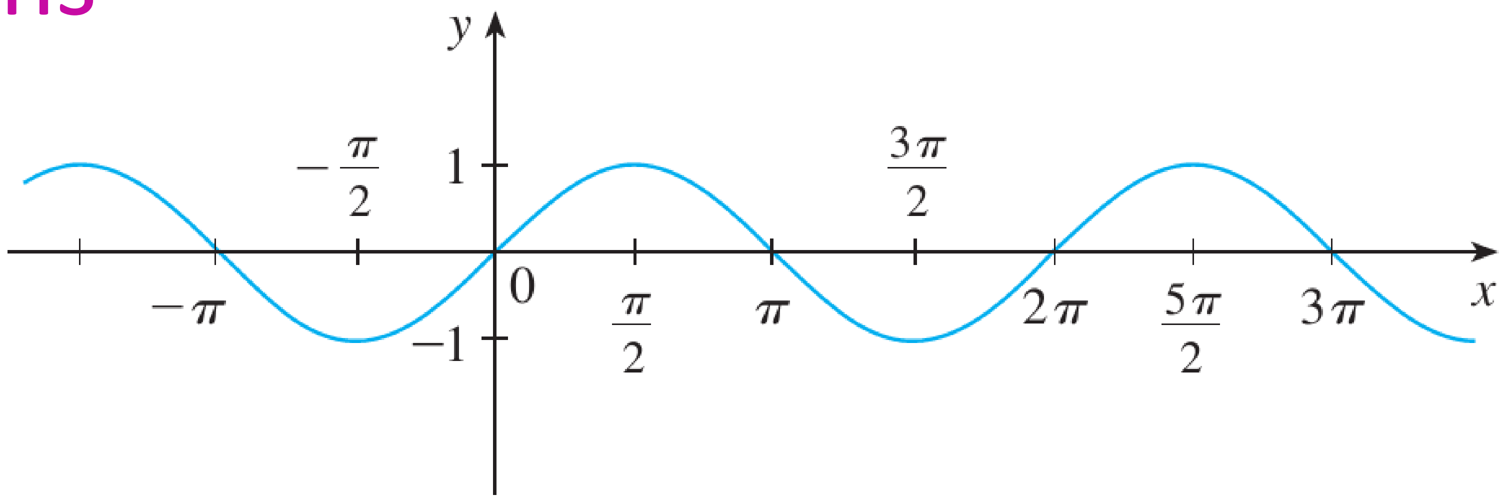
Transcendental Functions

These are functions that are not algebraic. They include the trigonometric, inverse trigonometric, exponential, and logarithmic functions, and many other functions as well.

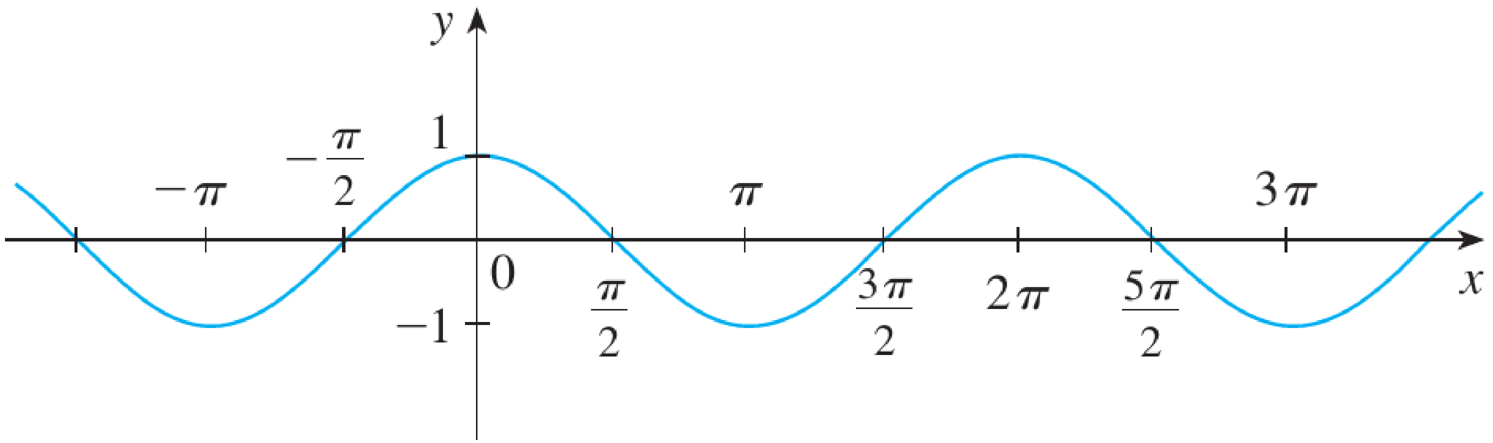


Trigonometric Functions

$f(x) = \sin x$



$g(x) = \cos x$



Homework

Sketch the graph of all trigonometric functions.

Notice that for both the sine and cosine functions the domain is $(-\infty, \infty)$ and the range is the closed interval $[-1, 1]$. Thus, for all values of x, we have

$$\boxed{-1 \leqslant \sin x \leqslant 1 \qquad -1 \leqslant \cos x \leqslant 1}$$

or, in terms of absolute values,

$$|\sin x| \leqslant 1 \qquad |\cos x| \leqslant 1$$

An important property of the sine and cosine functions is that they are periodic functions and have period 2π. This means that, for all values of x,

$$\boxed{\sin(x + 2\pi) = \sin x \qquad \cos(x + 2\pi) = \cos x}$$

EXAMPLE Find the domain of the function $f(x) = \dfrac{1}{1 - 2\cos x}$.

SOLUTION This function is defined for all values of x except for those that make the denominator 0. But

$$1 - 2\cos x = 0 \iff \cos x = \frac{1}{2} \iff x = \frac{\pi}{3} + 2n\pi \quad \text{or} \quad x = \frac{5\pi}{3} + 2n\pi$$

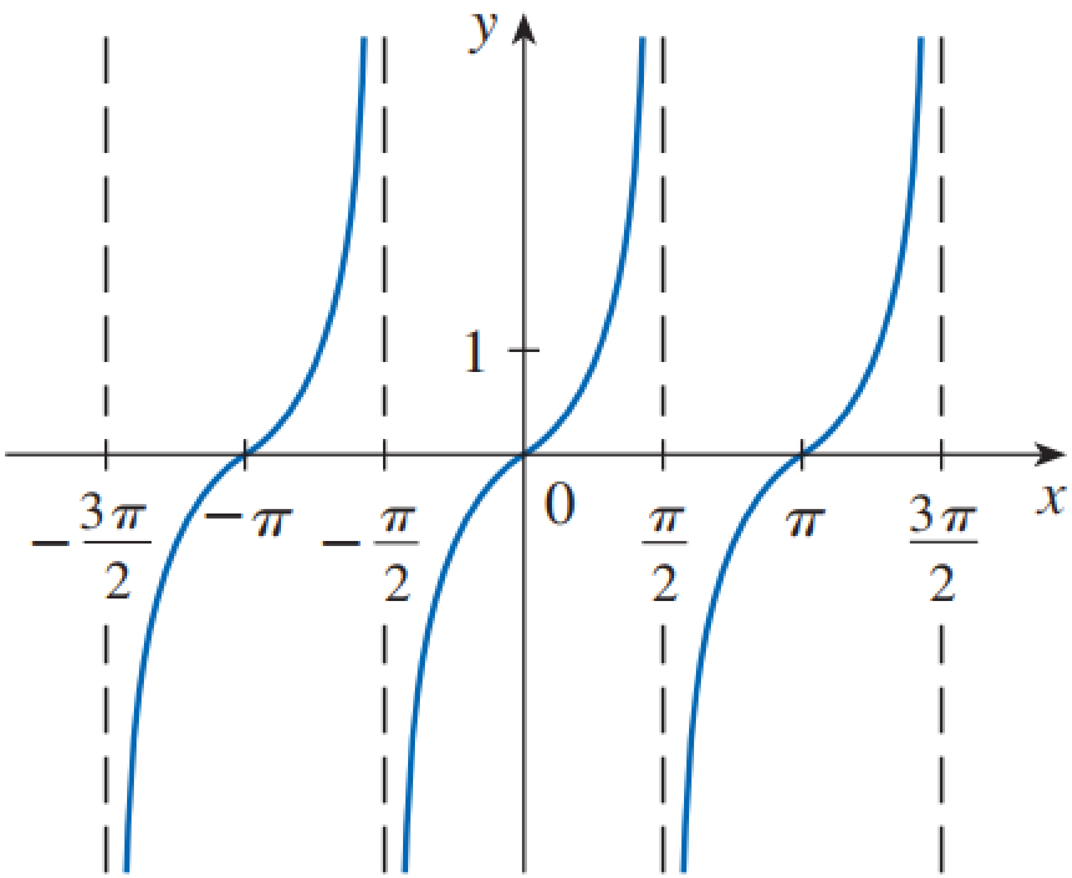
where n is any integer (because the cosine function has period 2π). So the domain of f is the set of all real numbers except for the ones noted above. ■

The tangent function is related to the sine and cosine functions by the equation

$$\tan x = \frac{\sin x}{\cos x}$$

and its graph is shown in Figure . It is undefined whenever $\cos x = 0$, that is, when $x = \pm\pi/2, \pm3\pi/2, \ldots$. Its range is $(-\infty, \infty)$. Notice that the tangent function has period π:

$$\tan(x + \pi) = \tan x \qquad \text{for all } x$$



FIGURE

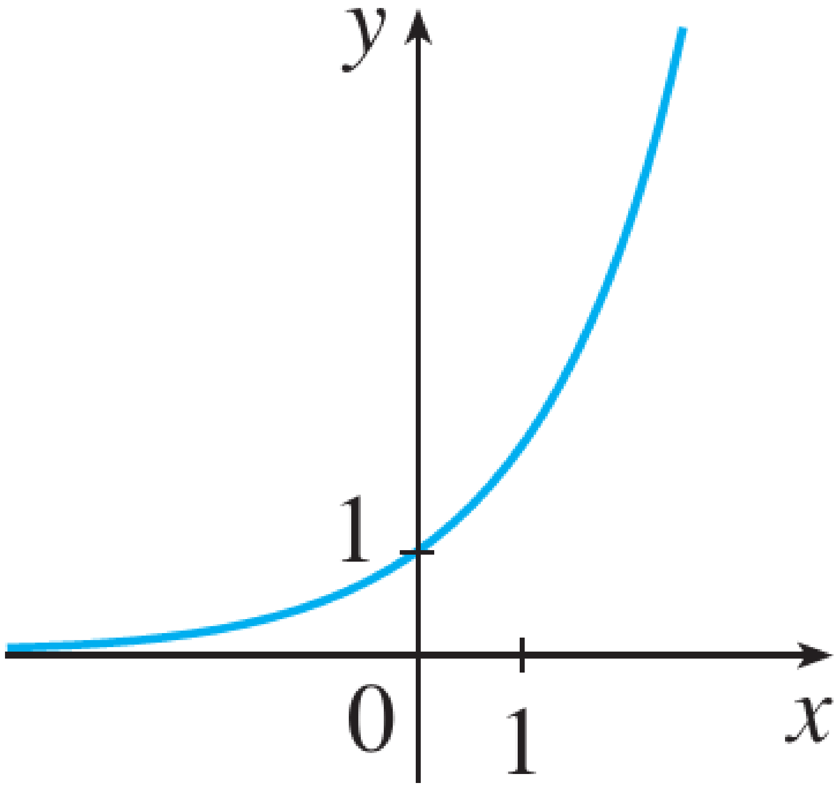
$y = \tan x$

Exponential Functions

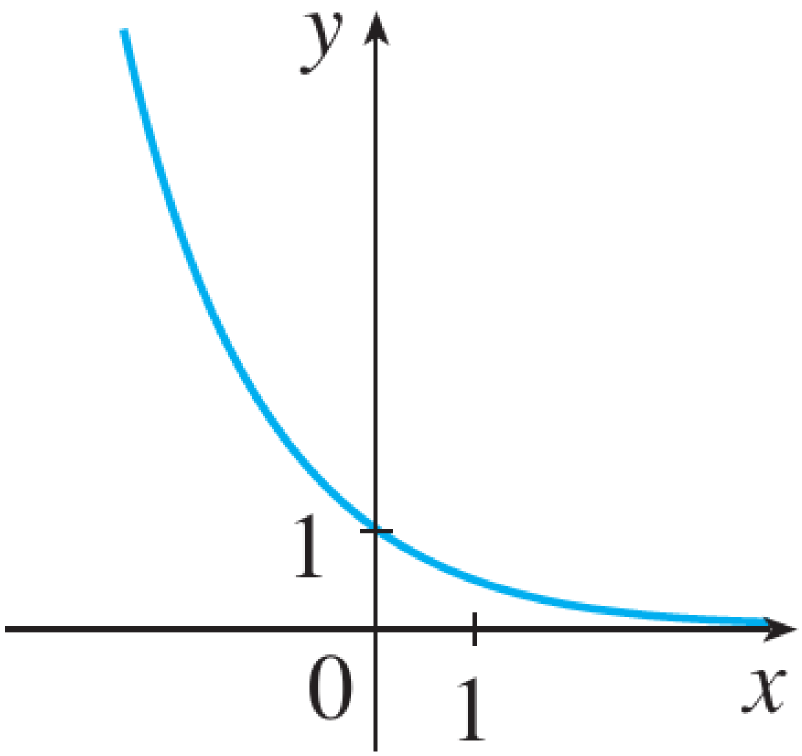
The **exponential functions** are the functions of the form $f(x) = b^x$, where the base b is a positive constant.

The graphs of $y = 2^x$ and $y = (0.5)^x$ are shown in Figure **on the next slide**.

In both cases the domain is $(-\infty, \infty)$ and the range is $(0, \infty)$.

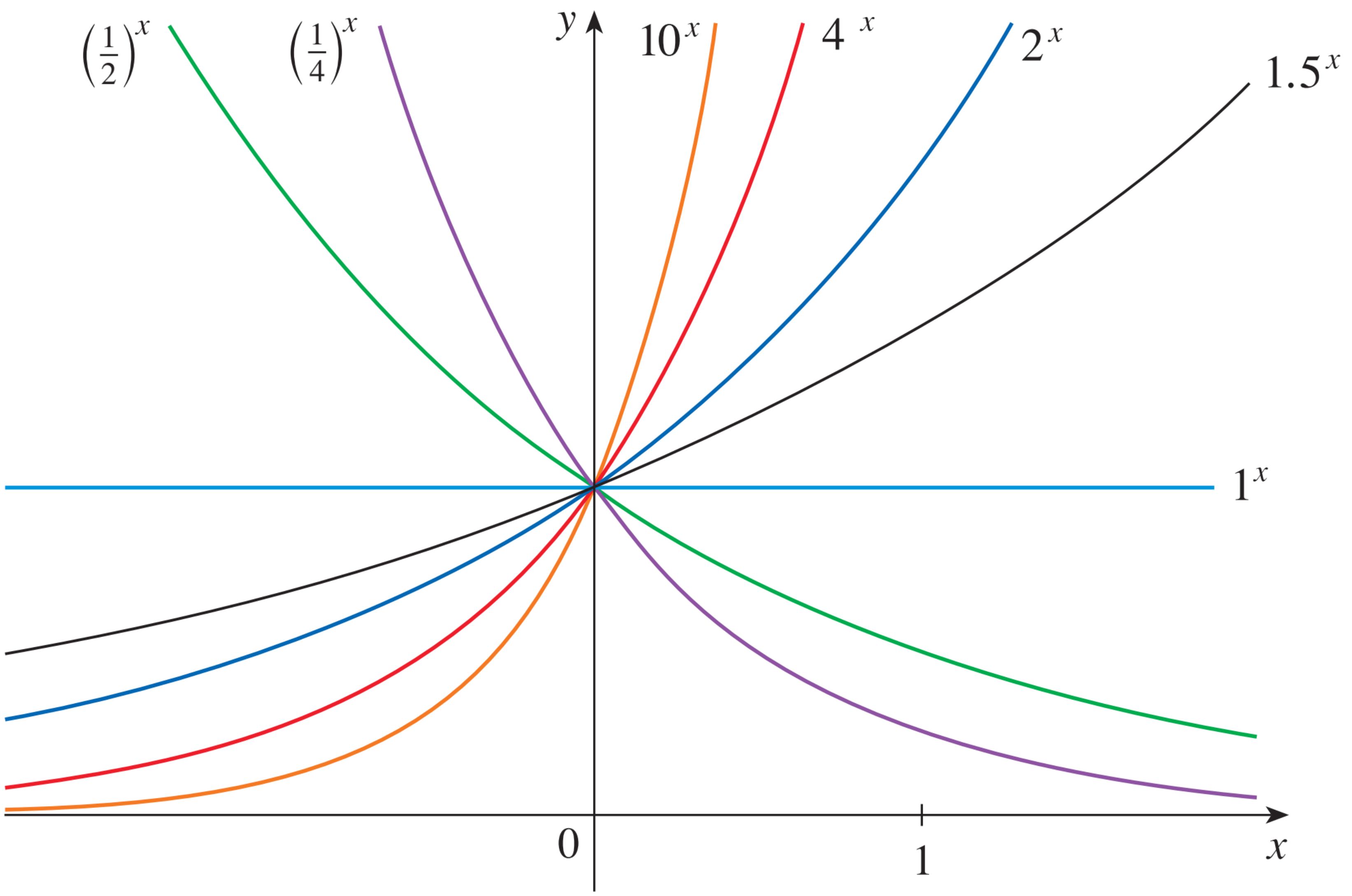


$y = 2^x$



$y = (0.5)^x$

Graph of some Exponential Functions

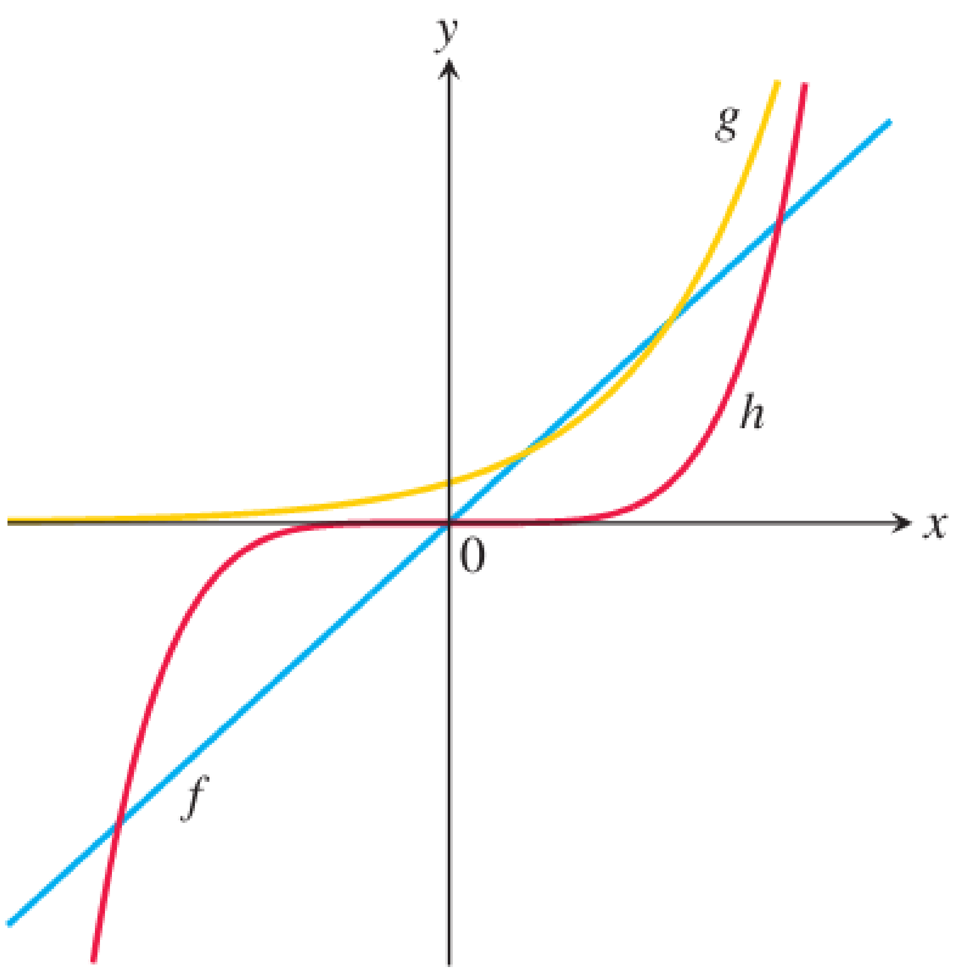


Practice Problem

Match each equation with its graph.

(Don’t use a computer or graphing calculator.)

a. $y = 5x$ **b.** $y = 5^x$ **c.** $y = x^5$



Practice Problem

Wingspan and weight : The weight W (in pounds) of a bird (that can fly) has been related to the wingspan L (in inches) of the bird by the power function $L = 30.6\, W^{0.3952}$.

(i) An eagle has a wingspan of about 90 inches. Use the model to estimate the weight of the eagle.

(ii) An ostrich weighs about 300 pounds. Use the model to estimate what the wingspan of an ostrich should be in order for it to fly.

(iii) The wingspan of an ostrich is about 72 inches. Use your answer to part (b) to explain why ostriches can't fly.

Hints :

$$L = 30.6W^{.3952}$$

$$90 = 30.6w^{.3952}$$

$$\frac{90}{30.6} = w^{.3952}$$

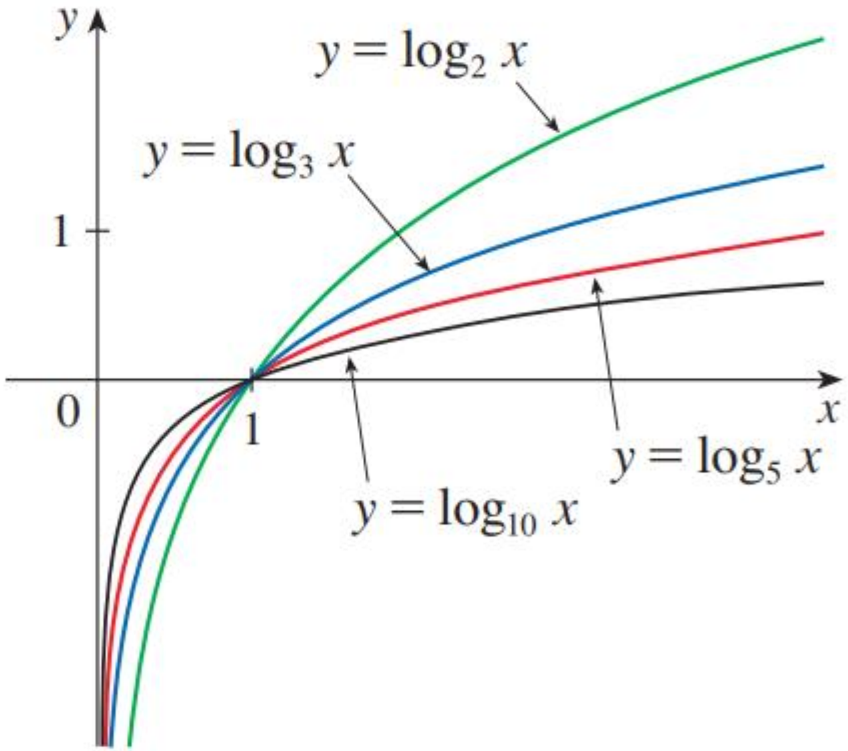
$$w = \left(\frac{90}{30.6}\right)^{\frac{1}{.3952}} \approx 15.3$$

$$L = 30.6(300)^{.3952}$$

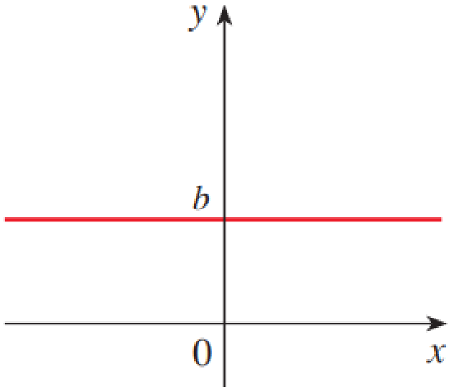
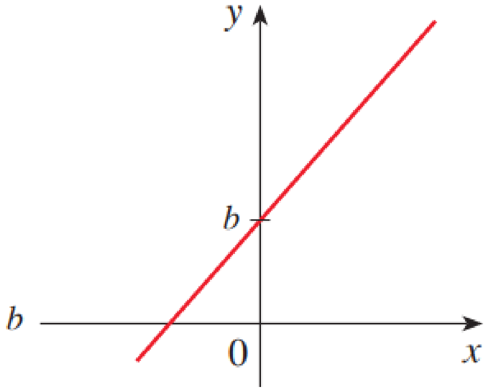
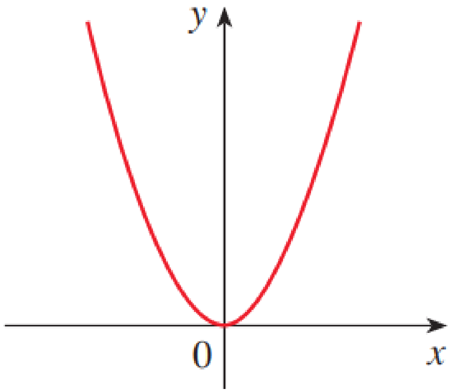
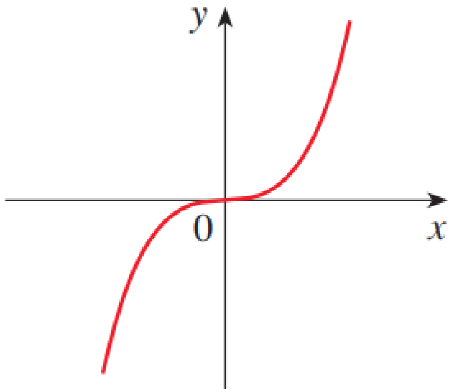
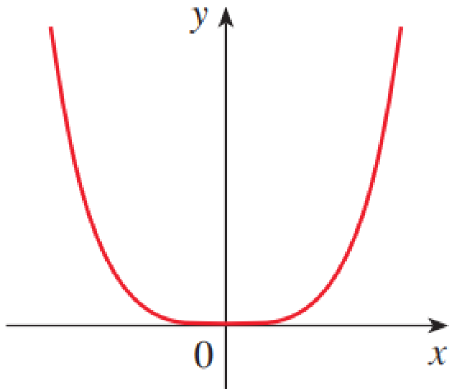
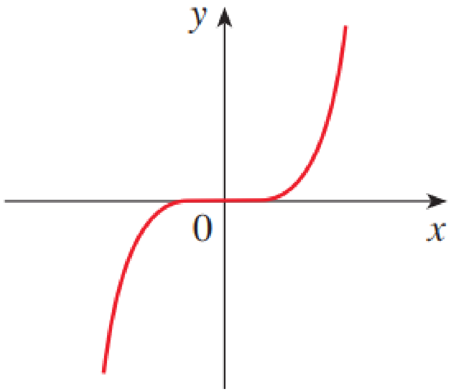
$$\approx 291.5$$

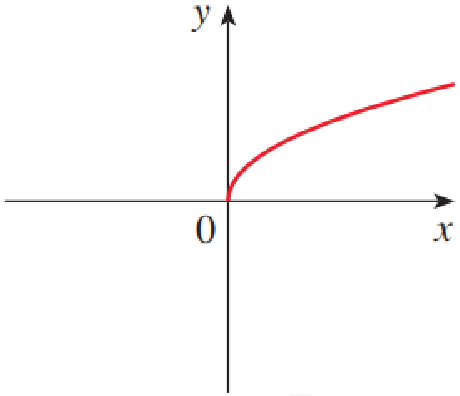
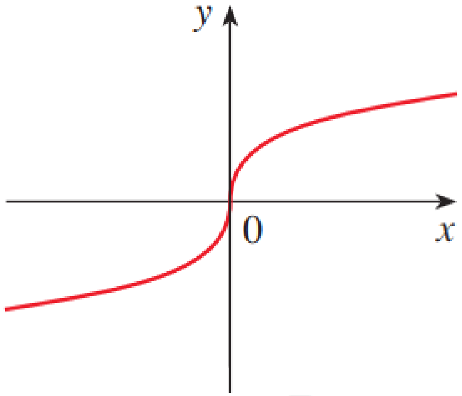
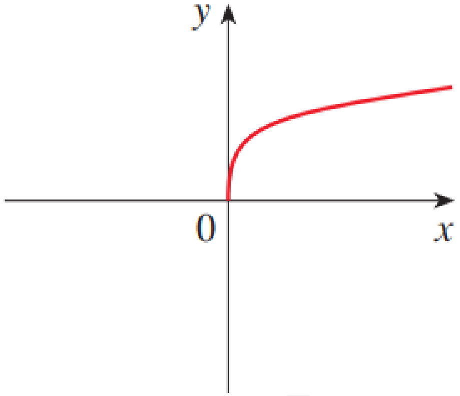
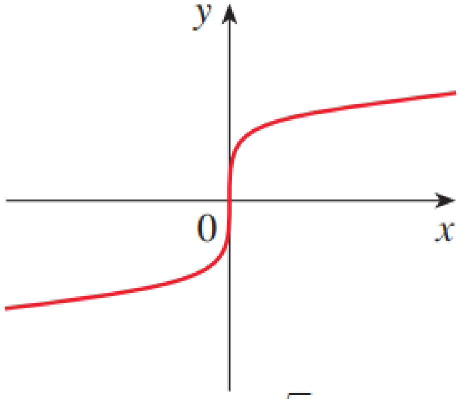
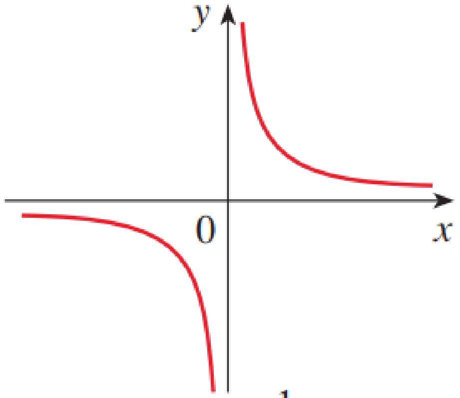
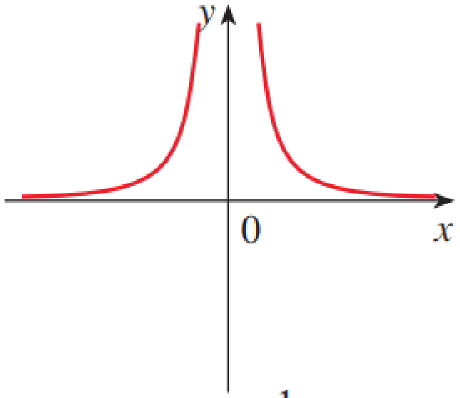
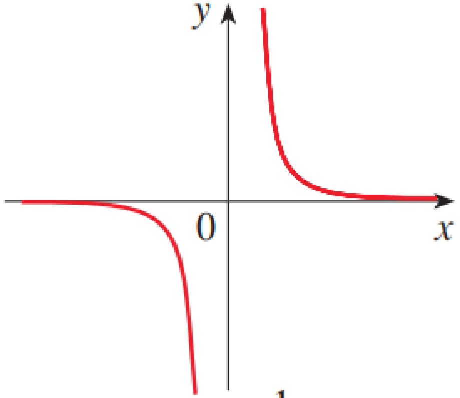
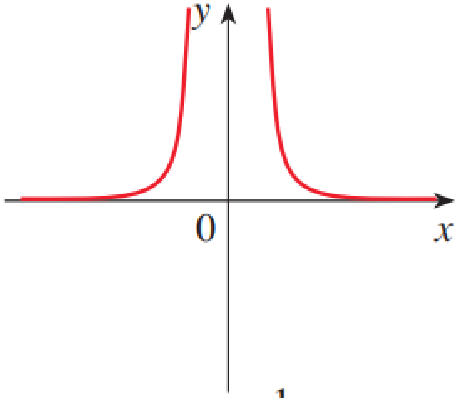
Logarithmic Functions

The **logarithmic functions** $f(x) = \log_b x$, where the base b is a positive constant, are the inverse functions of the exponential functions.



In each case the domain is $(0, \infty)$ and the range is $(-\infty, \infty)$.

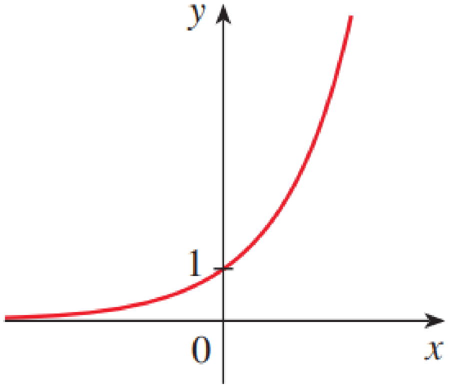
Linear Functions $f(x) = mx + b$	 $f(x) = b$  $f(x) = mx + b$
Power Functions $f(x) = x^n$	 $f(x) = x^2$  $f(x) = x^3$  $f(x) = x^4$  $f(x) = x^5$

Root Functions $f(x) = \sqrt[n]{x}$	 $f(x) = \sqrt{x}$	 $f(x) = \sqrt[3]{x}$	 $f(x) = \sqrt[4]{x}$	 $f(x) = \sqrt[5]{x}$
Reciprocal Functions $f(x) = \dfrac{1}{x^n}$	 $f(x) = \dfrac{1}{x}$	 $f(x) = \dfrac{1}{x^2}$	 $f(x) = \dfrac{1}{x^3}$	 $f(x) = \dfrac{1}{x^4}$

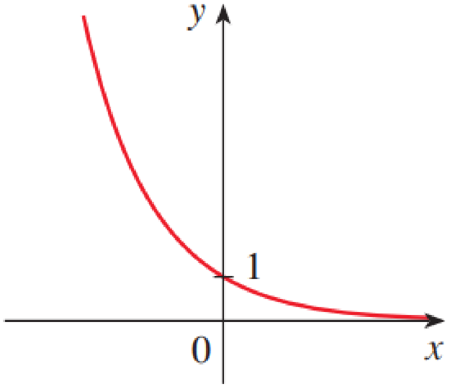
Exponential and Logarithmic Functions

$f(x) = b^x$

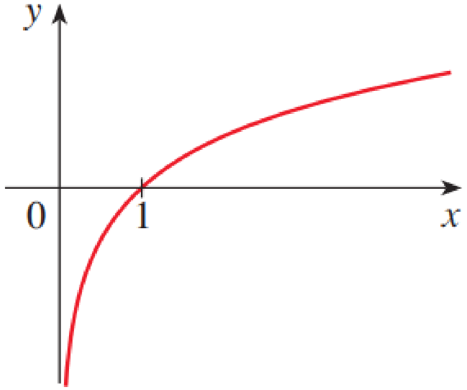
$f(x) = \log_b x$



$f(x) = b^x \;\; (b > 1)$



$f(x) = b^x \;\; (b < 1)$



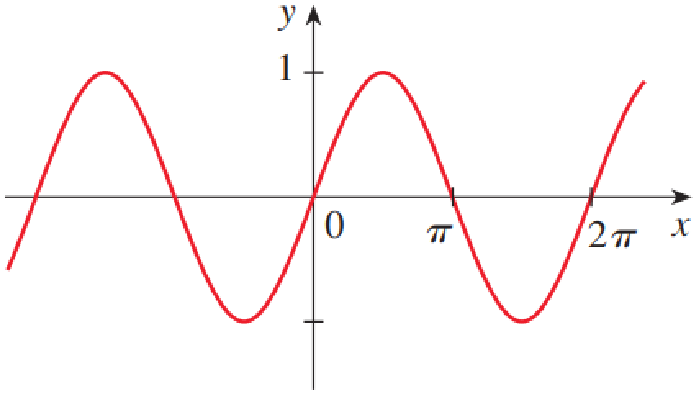
$f(x) = \log_b x \;\; (b > 1)$

Trigonometric Functions

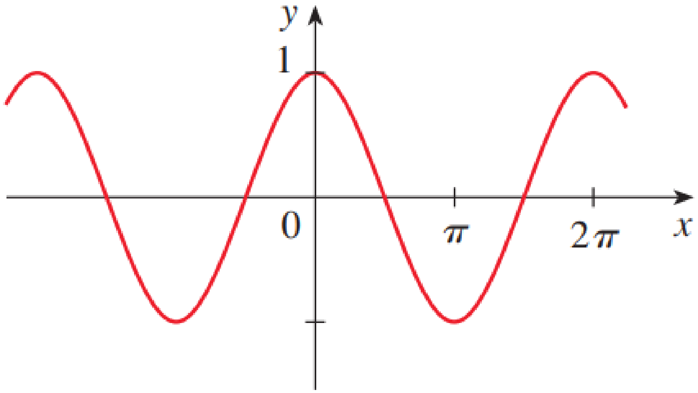
$f(x) = \sin x$

$f(x) = \cos x$

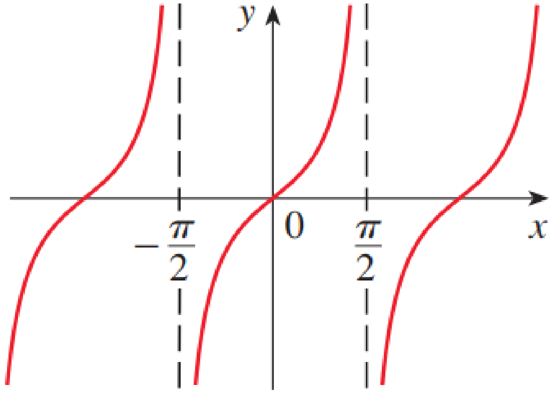
$f(x) = \tan x$



$f(x) = \sin x$



$f(x) = \cos x$



$f(x) = \tan x$

New Functions from Old Functions

Translation

Vertical and Horizontal Shifts Suppose $c > 0$. To obtain the graph of

$y = f(x) + c$, shift the graph of $y = f(x)$ a distance c units upward

$y = f(x) - c$, shift the graph of $y = f(x)$ a distance c units downward

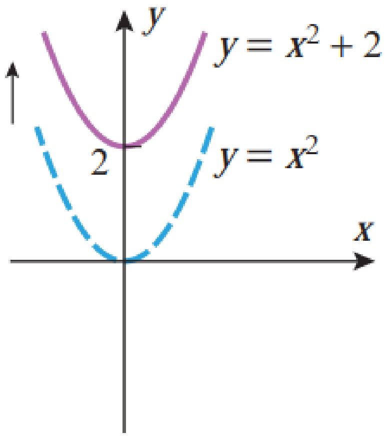
$y = f(x - c)$, shift the graph of $y = f(x)$ a distance c units to the right

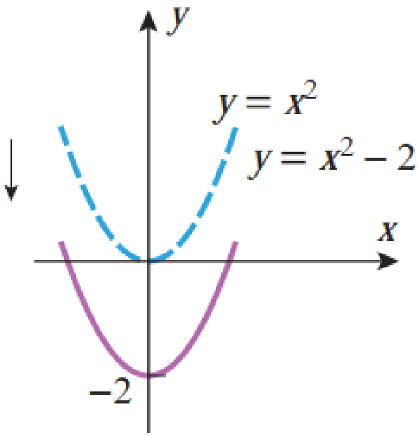
$y = f(x + c)$, shift the graph of $y = f(x)$ a distance c units to the left

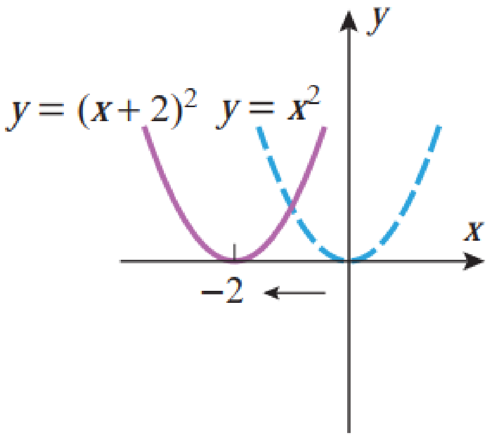
TRANSLATION PRINCIPLES

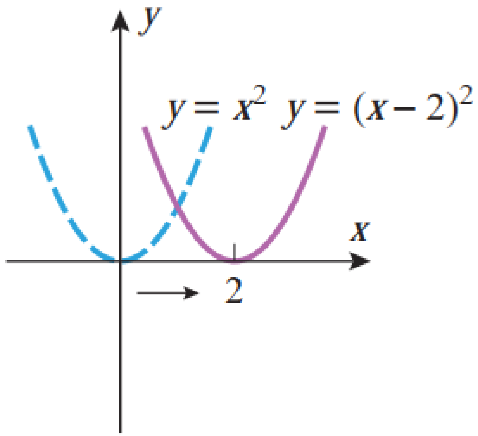
OPERATION ON $y = f(x)$	Add a positive constant c to $f(x)$	Subtract a positive constant c from $f(x)$	Add a positive constant c to x	Subtract a positive constant c from x
NEW EQUATION	$y = f(x) + c$	$y = f(x) - c$	$y = f(x + c)$	$y = f(x - c)$
GEOMETRIC EFFECT	Translates the graph of $y = f(x)$ up c units	Translates the graph of $y = f(x)$ down c units	Translates the graph of $y = f(x)$ left c units	Translates the graph of $y = f(x)$ right c units

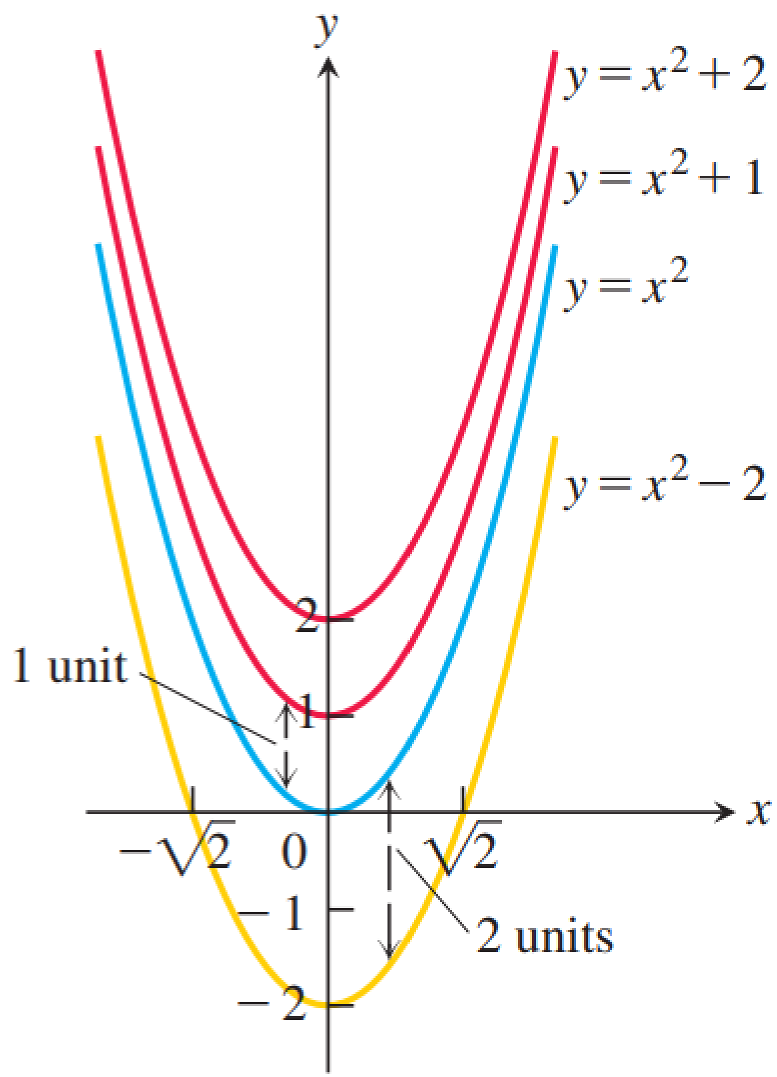
EXAMPLE

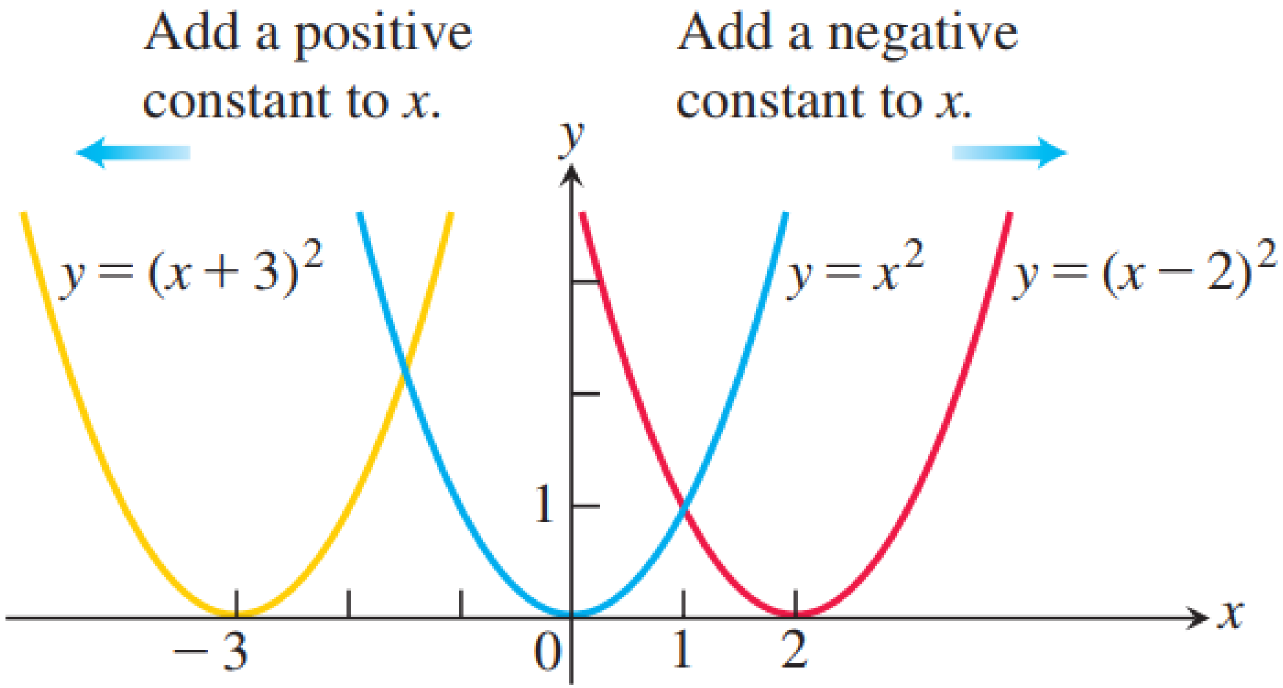




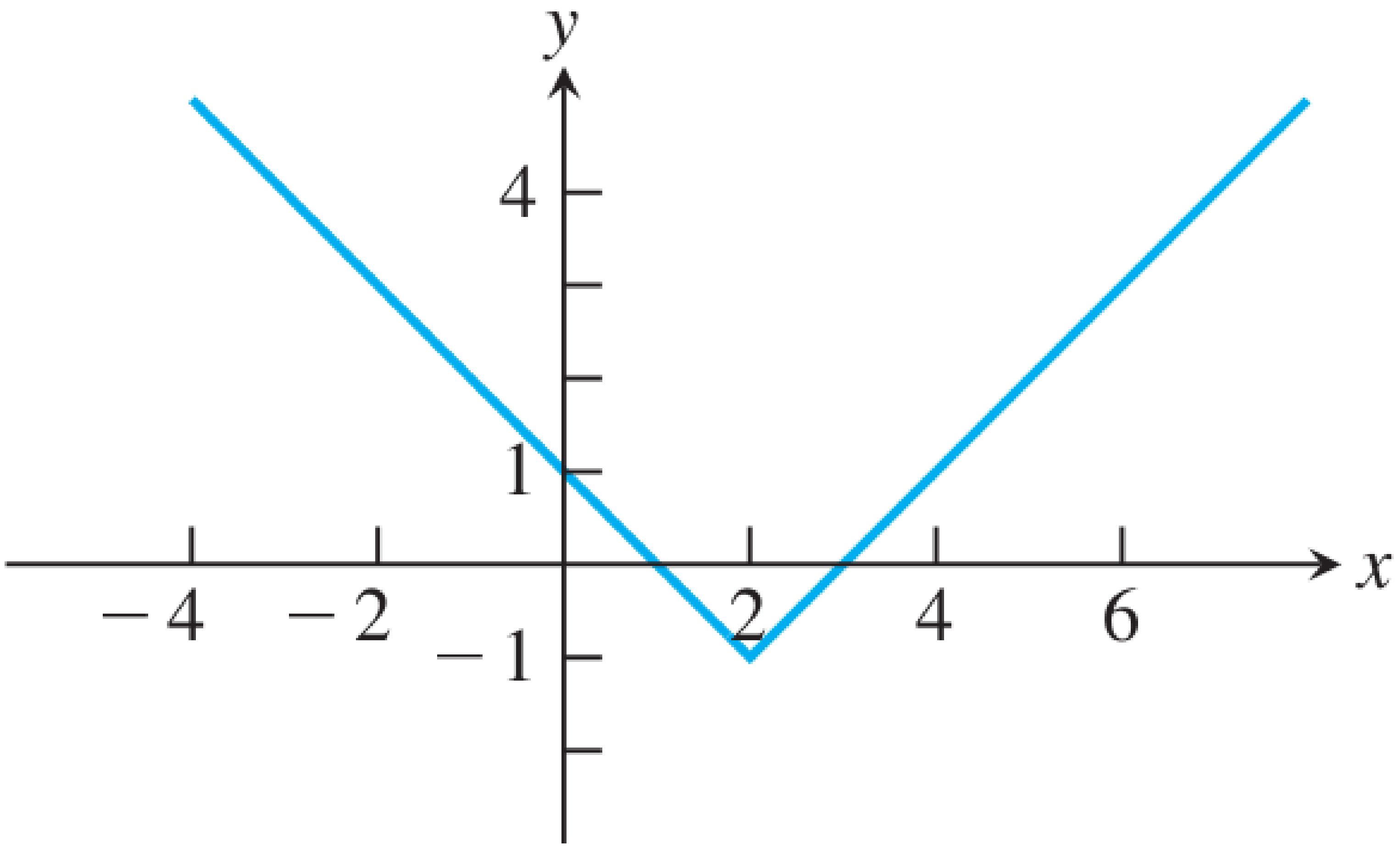





y
$y = x^2 + 2$
$y = x^2 + 1$
$y = x^2$
$y = x^2 - 2$
1 unit
2
1
$-\sqrt{2}$
0
$\sqrt{2}$
x
-1
2 units
-2


Add a positive constant to x.
Add a negative constant to x.
y
$y = (x + 3)^2$
$y = x^2$
$y = (x - 2)^2$
1
-3
0
1
2
x

Problem: Find a formula for the function graphed.

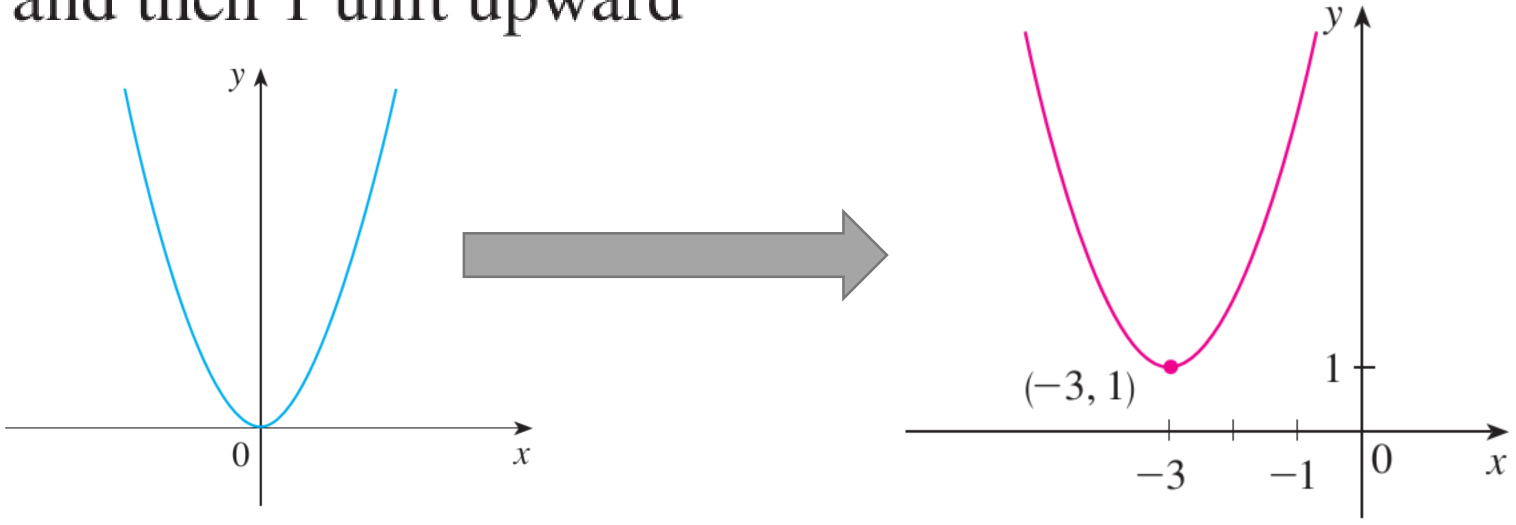


Problem Sketch the graph of the function $f(x) = x^2 + 6x + 10$.

SOLUTION Completing the square, we write the equation of the graph as

$$y = x^2 + 6x + 10 = (x + 3)^2 + 1$$

This means we obtain the desired graph by starting with the parabola $y = x^2$ and shifting 3 units to the left and then 1 unit upward



(a) $y = x^2$

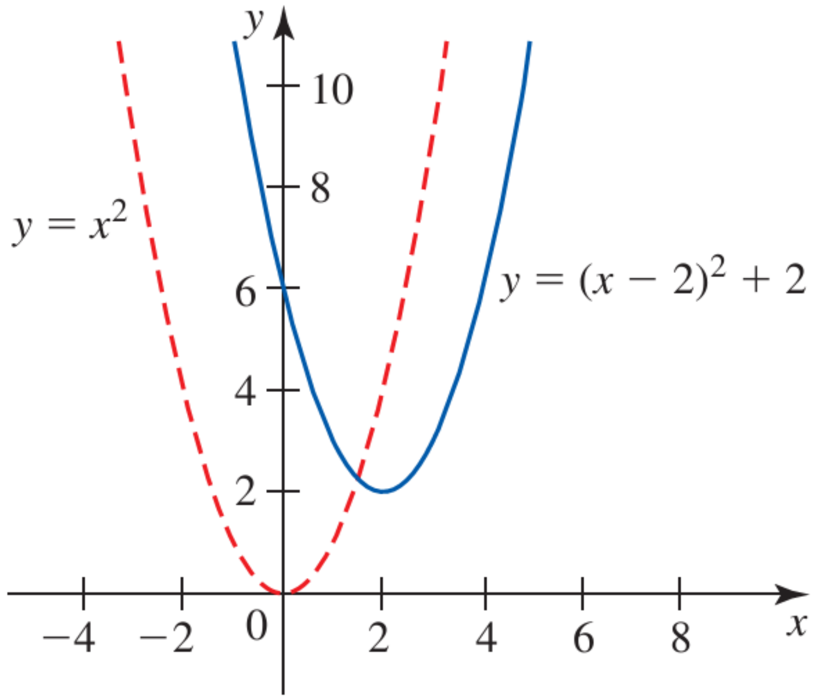
(b) $y = (x + 3)^2 + 1$

Problem Sketch the graph of the function f defined by $f(x) = x^2 - 4x + 6$.

Solution By completing the square, we can rewrite the given equation in the form

$$\begin{aligned} y &= [x^2 - 4x + (-2)^2] + 6 - (-2)^2 \\ &= (x - 2)^2 + 2 \end{aligned}$$

We see that the required graph can be obtained from the graph of $y = x^2$ by shifting it 2 units to the right and 2 units upward

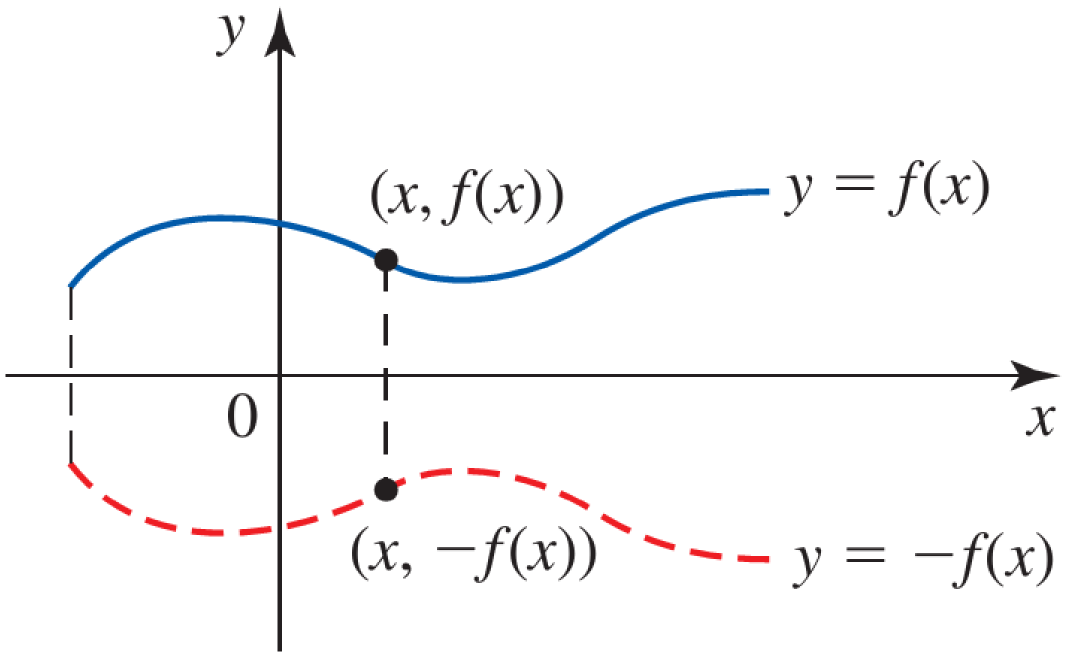


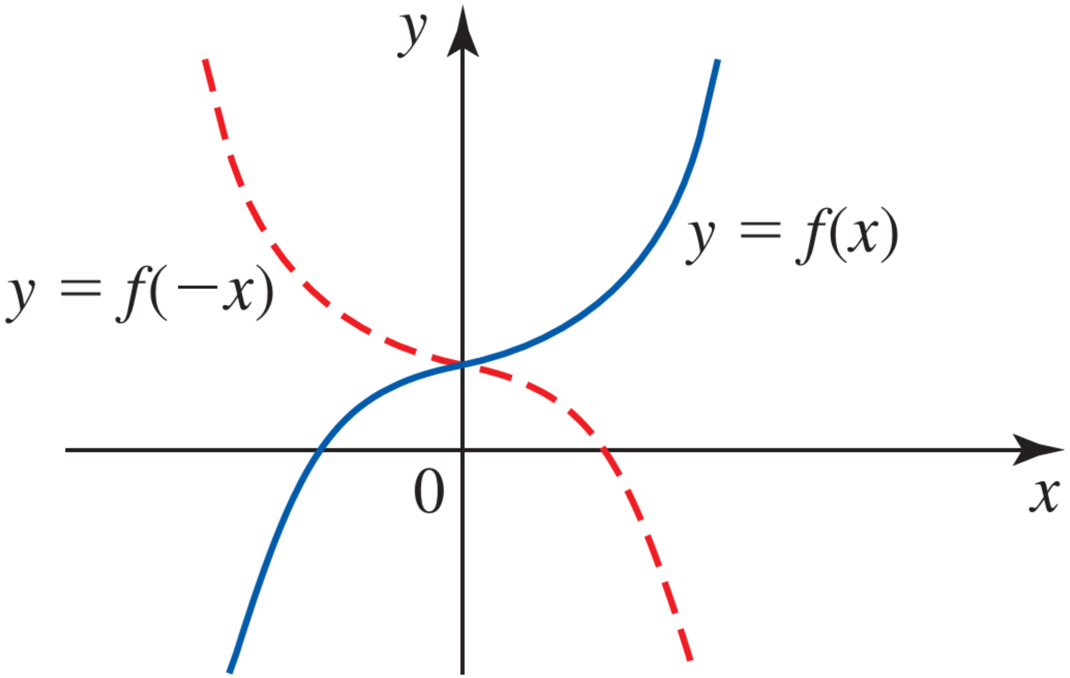
Reflection

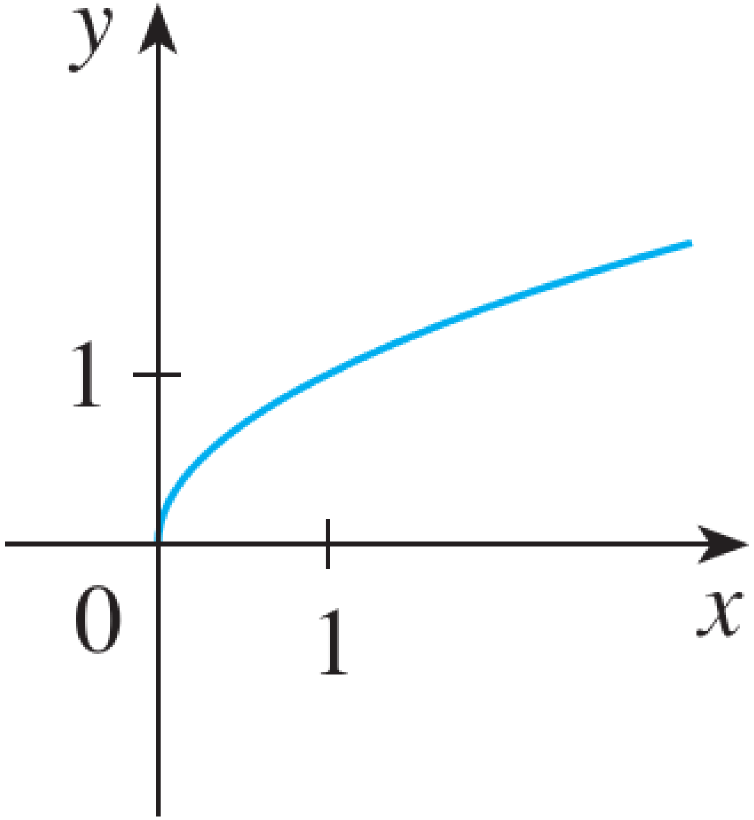
To obtain the graph of

(i) $y = -f(x)$, reflect the graph of $y = f(x)$ about the x-axis.

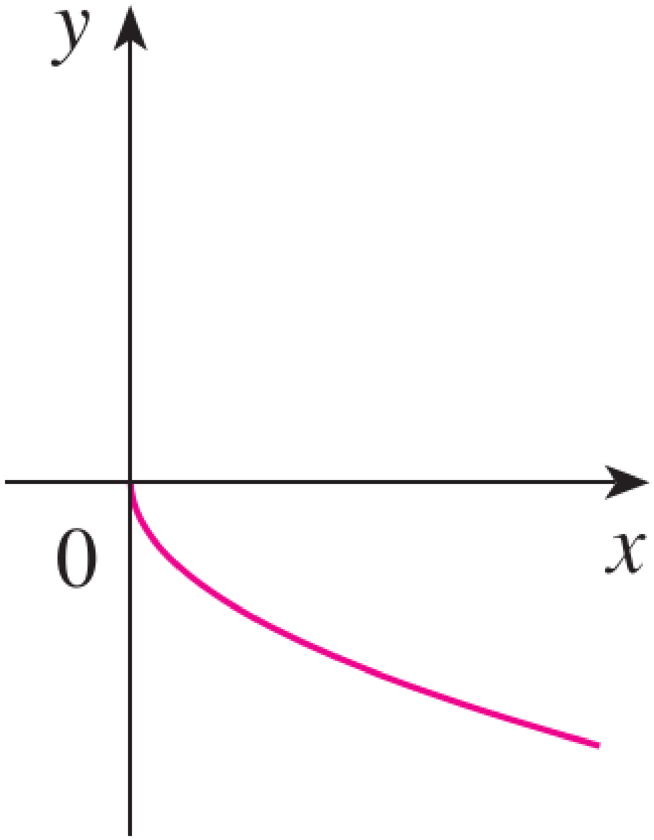
(ii) $y = f(-x)$, reflect the graph of $y = f(x)$ about the y-axis.



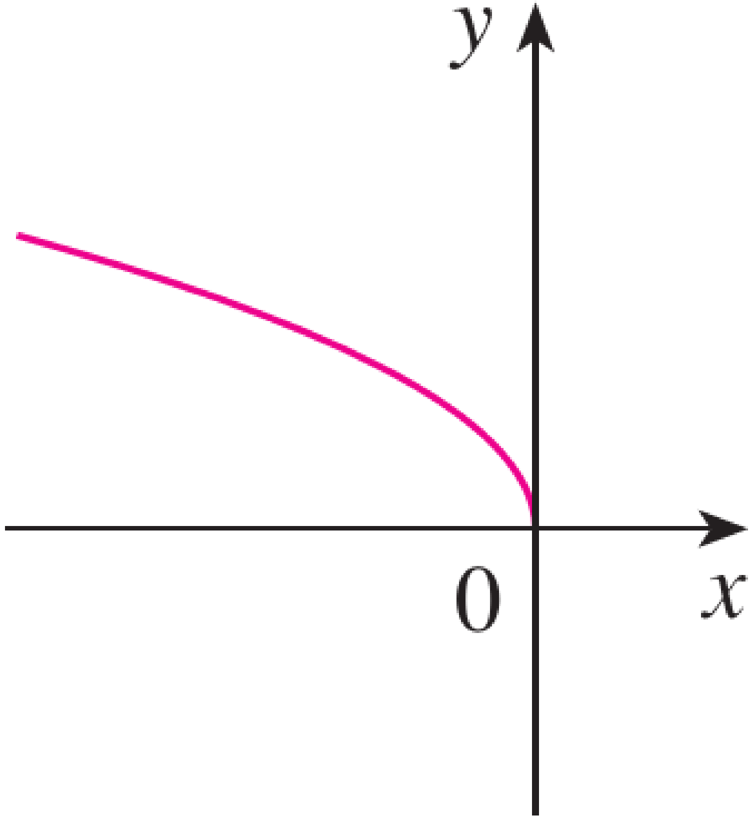




$y = \sqrt{x}$



$y = -\sqrt{x}$



$y = \sqrt{-x}$

Stretching & Shrinking

Suppose $c > 1$. To obtain the graph of

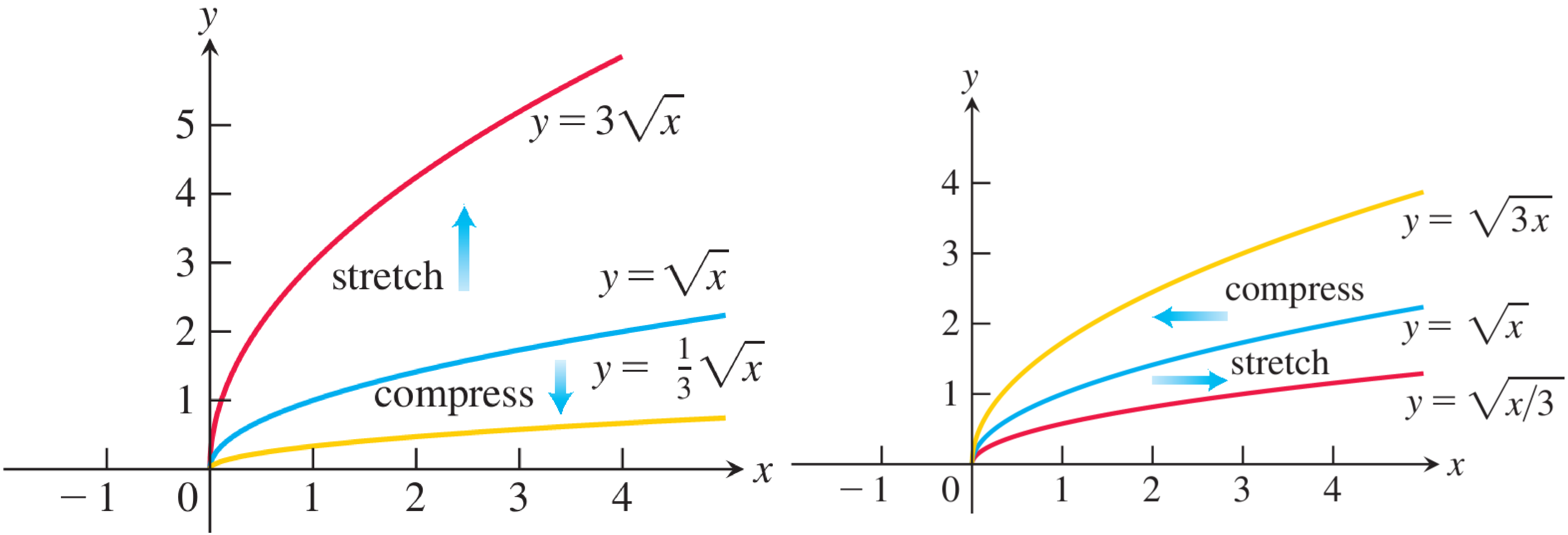
(i) $y = cf(x)$, stretch the graph of $y = f(x)$ vertically by a factor of c.

(ii) $y = \left(\frac{1}{c}\right) f(x)$, shrink the graph of $y = f(x)$ vertically by a factor of c.

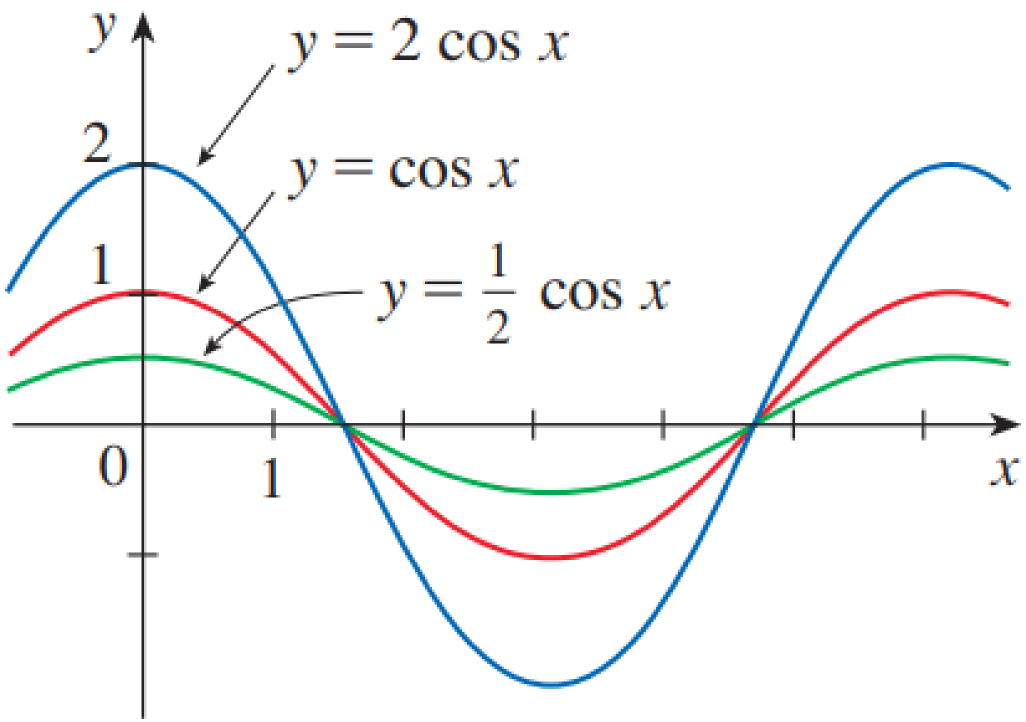
(iii) $y = f(cx)$, shrink the graph of $y = f(x)$ horizontally by a factor of c.

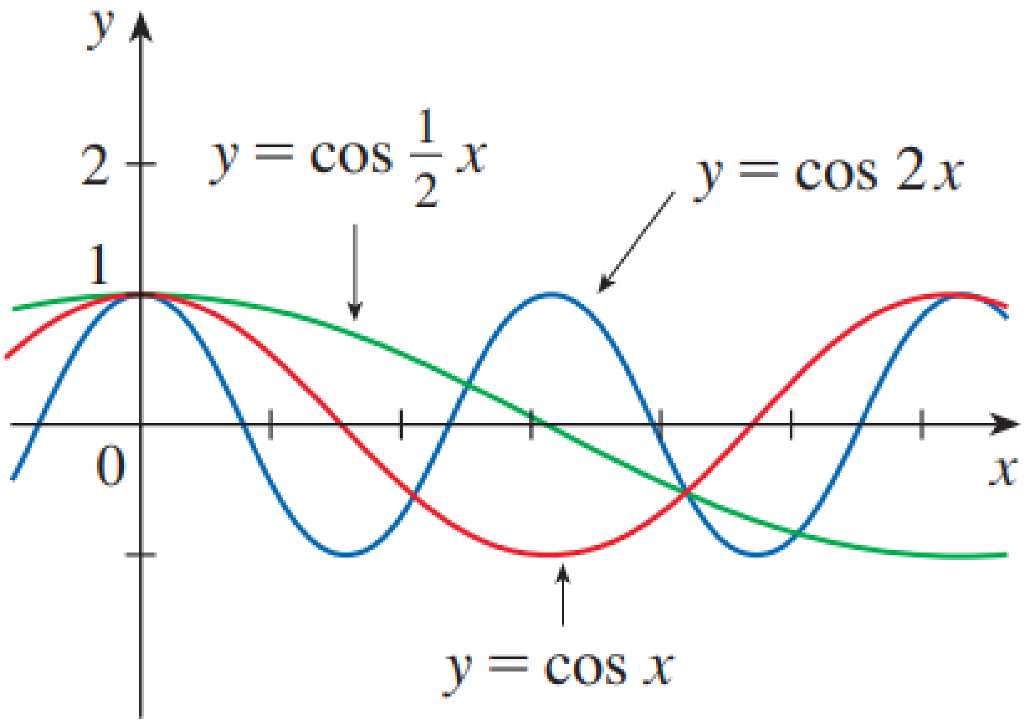
(iv) $y = f\left(\frac{x}{c}\right)$, stretch the graph of $y = f(x)$ horizontally by a factor of c.

Example: We can get the graph of the function $y = 3\sqrt{x}$ by vertically stretching the graph of $y = \sqrt{x}$ by a factor of 3.



Vertically stretching and shrinking of the graph $y = \sqrt{x}$ by a factor of 3.


y
y = 2 cos x
2
y = cos x
1
y = 1/2 cos x
0
1
x


y
2
y = cos 1/2 x
y = cos 2x
1
0
x
y = cos x

Practice Problems

Graph each function, not by plotting points, but by starting with the graph of one of the standard function and apply an appropriate transformation:

(a) $y = |2x| + 1$

(b) $y = 5\sin x$

(c) $y = 2x^2 + 4x + 2$

(d) $y = \cos(3x)$

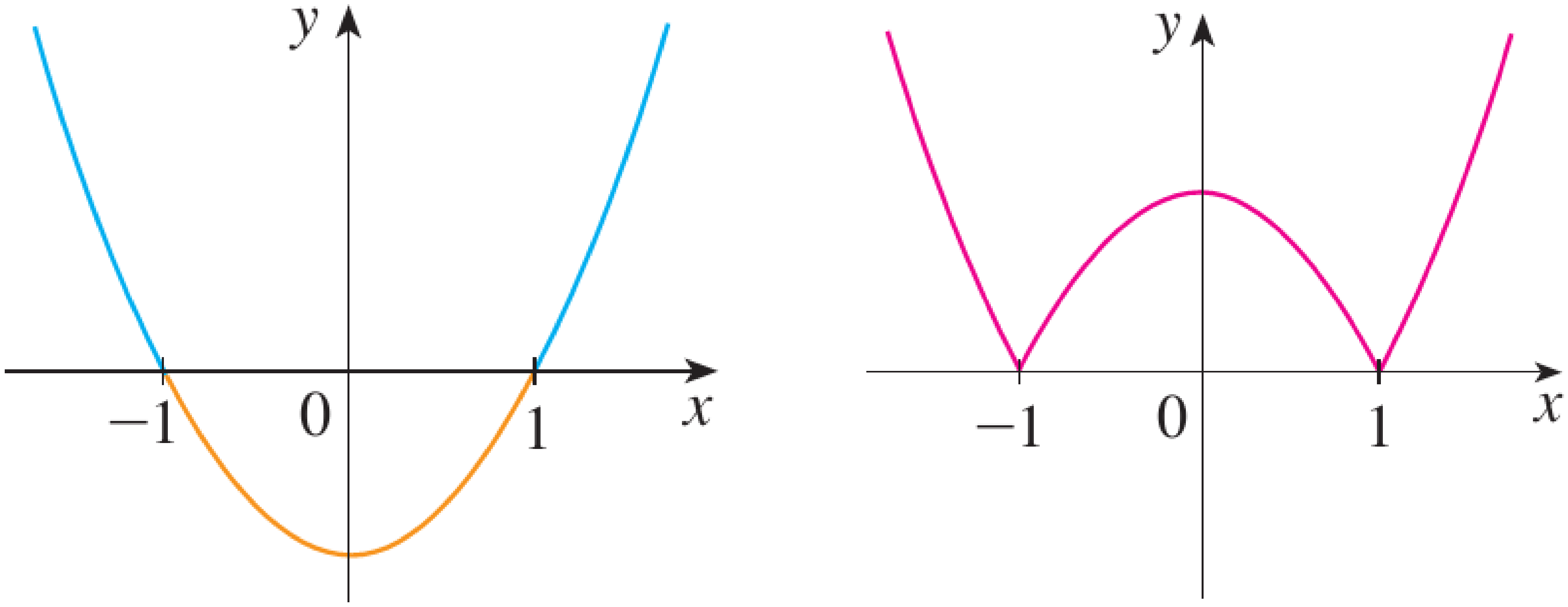
Problem

Sketch the graph of the function $y = |x^2 - 1|$.

SOLUTION We first graph the parabola $y = x^2 - 1$ by shifting the parabola $y = x^2$ downward 1 unit.

We see that the graph lies below the x-axis when $-1 < x < 1$, so we reflect that part of the graph about the x-axis to obtain the graph of $y = |x^2 - 1|$

Graph is on next slide…



(a) $y = x^2 - 1$ (b) $y = |x^2 - 1|$

Practice Problems

Graph each function, not by plotting points, but by starting with the graph of one of the standard function and apply an appropriate transformation:

(i) $f(x) = -\sqrt{x-1} + 2$

(ii) $g(x) = x^2 + 2x + 1$

(iii) $h(x) = \left|-x^2 + 5\right|$

(iv) $p(x) = |x+1| - 5$

(v) $q(x) = \lceil x \rceil - 1$

(vi) $r(x) = \dfrac{1}{1-x}$

(vii) $s(x) = \lfloor x \rfloor + 1$

(viii) $t(x) = \dfrac{1}{x^2} + 1$

(ix) $u(x) = |\cos x|$

(x) $v(x) = -x^2 + 1$

(xi) $t(x) = \ln(x+1)$

(xii) $w(x) = e^x - 1$

Combination of Functions

DEFINITION

Let f and g be functions with domains A and B, respectively. Then their sum $f + g$, difference $f - g$, product fg, and quotient f/g are defined as follows:

$$(f + g)(x) = f(x) + g(x) \quad \text{with domain } A \cap B \tag{1a}$$

$$(f - g)(x) = f(x) - g(x) \quad \text{with domain } A \cap B \tag{1b}$$

$$(fg)(x) = f(x)g(x) \quad \text{with domain } A \cap B \tag{1c}$$

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} \quad \text{with domain } \{x \mid x \in A \cap B \text{ and } g(x) \neq 0\} \tag{1d}$$

Problem

Let f and g be functions defined by $f(x) = \sqrt{x}$ and $g(x) = \sqrt{3 - x}$. Find the domain and the rule for each of the functions $f + g$, $f - g$, fg, and f/g.

Solution

The domain of f is $[0, \infty)$, and the domain of g is $(-\infty, 3]$. Therefore, the domain of $f + g$, $f - g$, and fg is

$$[0, \infty) \cap (-\infty, 3] = [0, 3]$$

The rules for these functions are

$$(f+g)(x) = f(x) + g(x) = \sqrt{x} + \sqrt{3-x}$$

$$(f-g)(x) = f(x) - g(x) = \sqrt{x} - \sqrt{3-x}$$

and

$$(fg)(x) = f(x)g(x) = \sqrt{x}\sqrt{3-x} = \sqrt{3x-x^2}$$

For the domain of f/g we must exclude the value of x

for which $g(x) = \sqrt{3-x} = 0$ or $x = 3$.

Therefore, f/g is defined by

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{\sqrt{x}}{\sqrt{3 - x}} = \sqrt{\frac{x}{3 - x}}$$

with domain $[0, 3)$.

Function	Formula	Domain
$f+g$	$(f+g)(x)=\sqrt{x}+\sqrt{1-x}$	$[0,1]=D(f)\cap D(g)$
$f-g$	$(f-g)(x)=\sqrt{x}-\sqrt{1-x}$	$[0,1]$
$g-f$	$(g-f)(x)=\sqrt{1-x}-\sqrt{x}$	$[0,1]$
$f\cdot g$	$(f\cdot g)(x)=f(x)g(x)=\sqrt{x(1-x)}$	$[0,1]$
f/g	$\frac{f}{g}(x)=\frac{f(x)}{g(x)}=\sqrt{\frac{x}{1-x}}$	$[0,1)\,(x=1\text{ excluded})$
g/f	$\frac{g}{f}(x)=\frac{g(x)}{f(x)}=\sqrt{\frac{1-x}{x}}$	$(0,1]\,(x=0\text{ excluded})$

Composing Functions

Composition is another method for combining functions. In this operation the output from one function becomes the input to a second function.

Definition Given two functions f and g, the **composite function** $f \circ g$ (also called the **composition** of f and g) is defined by

$$(f \circ g)(x) = f(g(x))$$

The domain of $f \circ g$ is the set of all x in the domain of g such that $g(x)$ is in the domain of f. In other words, $(f \circ g)(x)$ is defined whenever both $g(x)$ and $f(g(x))$ are defined.

The functions $f \circ g$ and $g \circ f$ are usually quite different.

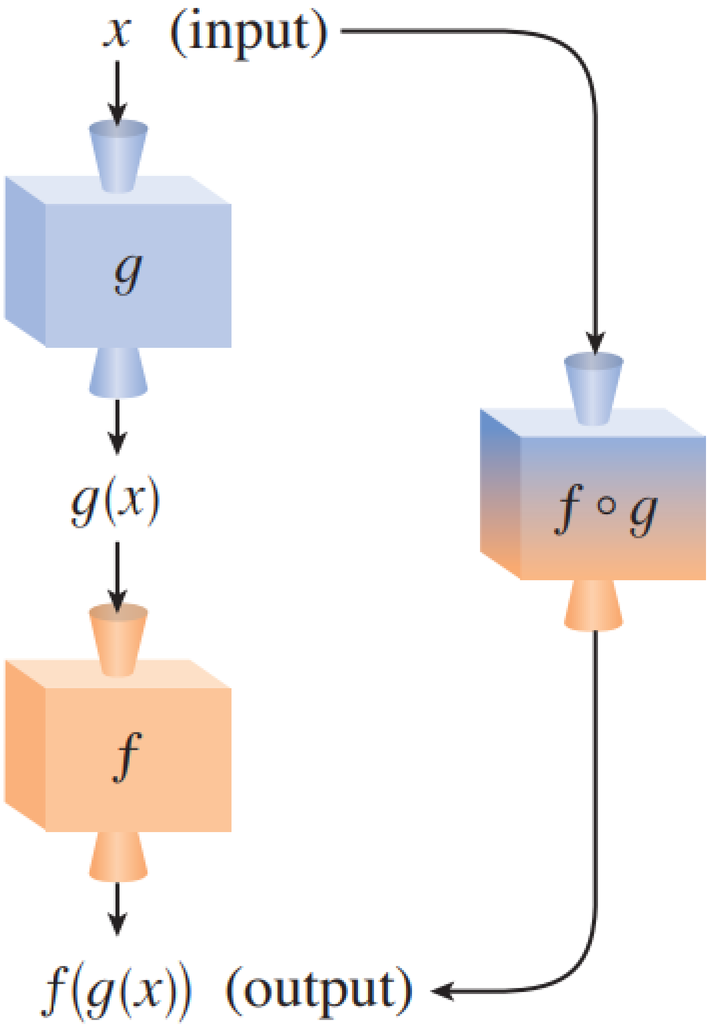


FIGURE
The $f \circ g$ machine is composed of the g machine (first) and then the f machine.

Problem

For $f(x) = x^2 + 1$ and $g(x) = \sqrt{x - 2}$, find the compositions $f \circ g$ and $g \circ f$ and identify the domain of each.

Solution

First, we have

$$(f \circ g)(x) = f(g(x)) = f(\sqrt{x-2})$$
$$= (\sqrt{x-2})^2 + 1 = x - 2 + 1 = x - 1.$$

It's tempting to write that the domain of $f \circ g$ is the entire real line, but look more carefully. Note that for x to be in the domain of g, we must have $x \geq 2$. The domain of f is the whole real line, so this places no further restrictions on the domain of $f \circ g$. Even though the final expression $x - 1$ is defined for all x, the domain of $(f \circ g)$ is $\{x \mid x \geq 2\}$.

For the second composition,

$$\begin{aligned}(g \circ f)(x) &= g(f(x)) = g(x^2 + 1) \\ &= \sqrt{(x^2 + 1) - 2} = \sqrt{x^2 - 1}.\end{aligned}$$

The resulting square root requires $x^2 - 1 \geq 0$ or $|x| \geq 1$. Since the "inside" function f is defined for all x, the domain of $g \circ f$ is $\{x \in \mathbb{R} \mid |x| \geq 1\}$, which we write in interval notation as $(-\infty, -1] \cup [1, \infty)$.

EXAMPLE If $f(x) = \sqrt{x}$ and $g(x) = x + 1$, find

(a) $(f \circ g)(x)$ **(b)** $(g \circ f)(x)$ **(c)** $(f \circ f)(x)$ **(d)** $(g \circ g)(x)$.

Solution

Composition	**Domain**
(a) $(f \circ g)(x) = f(g(x)) = \sqrt{g(x)} = \sqrt{x+1}$	$[-1, \infty)$
(b) $(g \circ f)(x) = g(f(x)) = f(x) + 1 = \sqrt{x} + 1$	$[0, \infty)$
(c) $(f \circ f)(x) = f(f(x)) = \sqrt{f(x)} = \sqrt{\sqrt{x}} = x^{1/4}$	$[0, \infty)$
(d) $(g \circ g)(x) = g(g(x)) = g(x) + 1 = (x + 1) + 1 = x + 2$	$(-\infty, \infty)$

To see why the domain of $f \circ g$ is $[-1, \infty)$, notice that $g(x) = x + 1$ is defined for all real x but $g(x)$ belongs to the domain of f only if $x + 1 \geq 0$, that is to say, when $x \geq -1$. ■

Notice that if $f(x) = x^2$ and $g(x) = \sqrt{x}$, then $(f \circ g)(x) = (\sqrt{x})^2 = x$. However, the domain of $f \circ g$ is $[0, \infty)$, not $(-\infty, \infty)$, since $\sqrt{x}$ requires $x \geq 0$.

EXAMPLE Find $f \circ g \circ h$ if $f(x) = x/(x + 1)$, $g(x) = x^{10}$, and $h(x) = x + 3$.

SOLUTION

$$\begin{aligned}(f \circ g \circ h)(x) &= f(g(h(x))) = f(g(x + 3)) \\ &= f((x + 3)^{10}) = \frac{(x + 3)^{10}}{(x + 3)^{10} + 1}\end{aligned}$$

■

So far we have used composition to build complicated functions from simpler ones. But in calculus it is often useful to be able to *decompose* a complicated function into simpler ones, as in the following example.

EXAMPLE Given $F(x) = \cos^2(x + 9)$, find functions f, g, and h such that $F = f \circ g \circ h$.

SOLUTION Since $F(x) = [\cos(x + 9)]^2$, the formula for F says: first add 9, then take the cosine of the result, and finally square. So we let

$$h(x) = x + 9 \qquad g(x) = \cos x \qquad f(x) = x^2$$

Then
$$\begin{aligned}(f \circ g \circ h)(x) &= f(g(h(x))) = f(g(x + 9)) = f(\cos(x + 9)) \\ &= [\cos(x + 9)]^2 = F(x)\end{aligned}$$

■

Practice Problems 1

Evaluating Composite Functions Use the graphs of f and g to evaluate each expression. If the result is undefined, explain why.

(a) $(f \circ g)(3)$

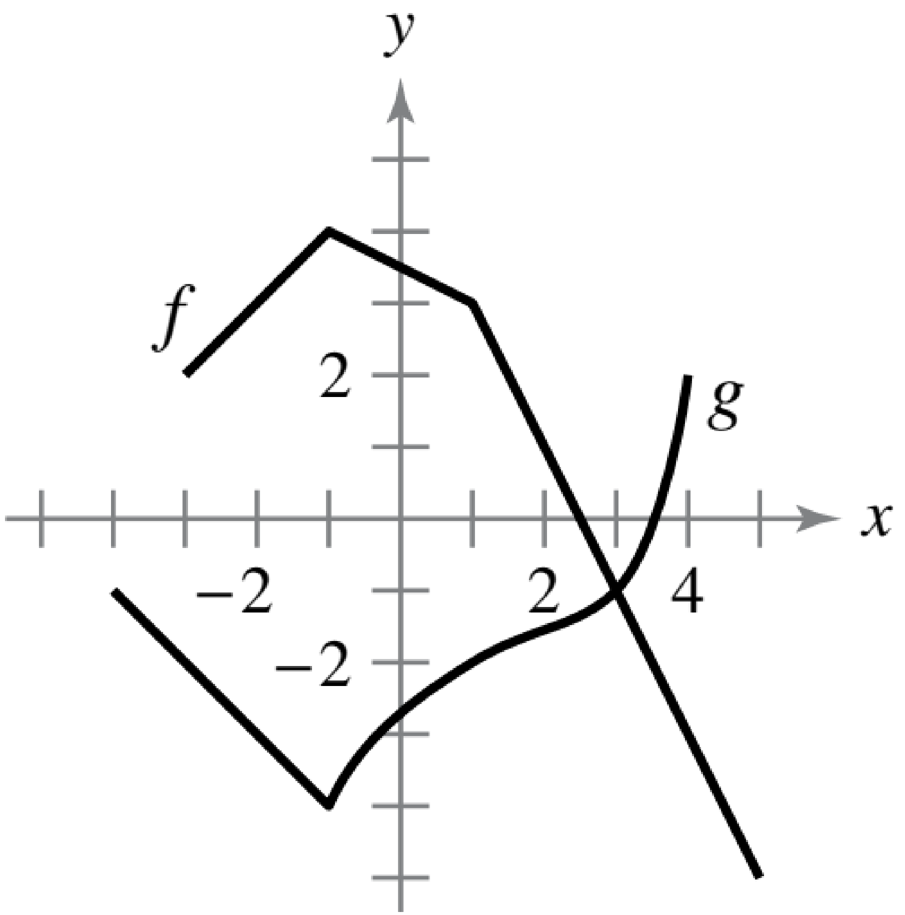
(b) $g(f(2))$

(c) $g(f(5))$

(d) $(f \circ g)(-3)$

(e) $(g \circ f)(-1)$

(f) $f(g(-1))$



Practice Problem 2

State the domain of the following functions.

(i) $f(x) = \frac{e^x + \ln(x)}{\sqrt{x+1}}$

(ii) $g(x) = \sqrt{|x|}$

(iii) $h(x) = \tan^{-1}(x^2 + x + 1)$

(iv) $R(x) = \tan^{-1}(x^2 + x + 1)$

(v) $L(x) = \sin^{-1}(x^2 + 2x)$

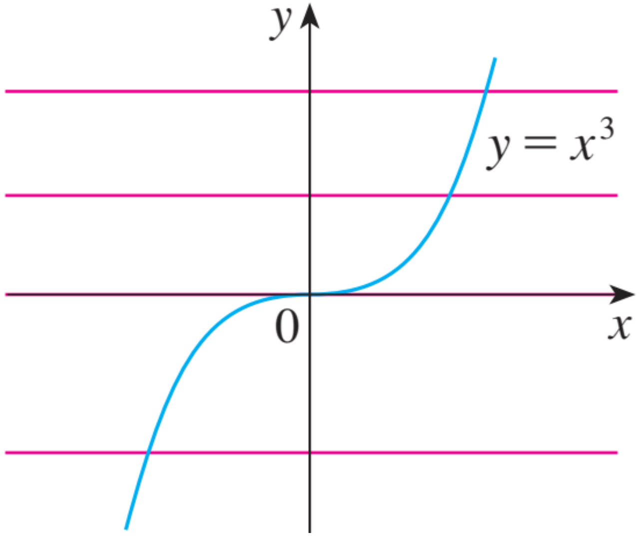
(vi) $s(x) = \ln(-(x + 1))$

(vii) $r(x) = \cos^{-1}\left(\frac{1}{x}\right)$

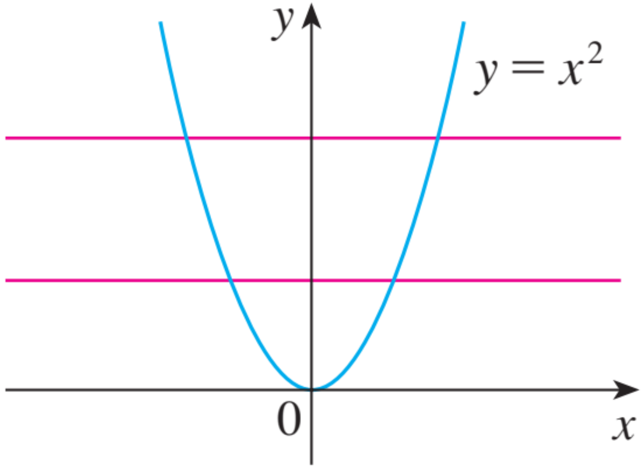
Inverse Functions

DEFINITION A function $f(x)$ is **one-to-one** on a domain D if $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$ in D.

Horizontal Line Test A function is one-to-one if and only if no horizontal line intersects its graph more than once.



$f(x) = x^3$ is one-to-one.



$g(x) = x^2$ is not one-to-one.

One-to-one functions are important because they are precisely the functions that possess inverse functions according to the following definition.

Definition Let f be a one-to-one function with domain A and range B. Then its **inverse function** f^{-1} has domain B and range A and is defined by

$$f^{-1}(y) = x \iff f(x) = y$$

for any y in B.

This definition says that if f maps x into y, then f^{-1} maps y back into x.

CAUTION Do not mistake the -1 in f^{-1} for an exponent. Thus

$$f^{-1}(x) \quad \text{does } \textit{not} \text{ mean} \quad \frac{1}{f(x)}$$

$$f^{-1}(f(x)) = x \qquad \text{for every } x \text{ in } A$$

$$f(f^{-1}(x)) = x \qquad \text{for every } x \text{ in } B$$

5 How to Find the Inverse Function of a One-to-One Function f

STEP 1 Write $y = f(x)$.

STEP 2 Solve this equation for x in terms of y (if possible).

STEP 3 To express f^{-1} as a function of x, interchange x and y. The resulting equation is $y = f^{-1}(x)$.

EXAMPLE Find the inverse function of $f(x) = x^3 + 2$.

SOLUTION According to (5) we first write

$$y = x^3 + 2$$

Then we solve this equation for x:

$$x^3 = y - 2$$
$$x = \sqrt[3]{y - 2}$$

Finally, we interchange x and y:

$$y = \sqrt[3]{x - 2}$$

Therefore the inverse function is $f^{-1}(x) = \sqrt[3]{x - 2}$.

EXAMPLE Find the inverse of $y = \frac{1}{2}x + 1$, expressed as a function of x.

Solution

1. *Solve for x in terms of y:*

$$\begin{aligned} y &= \frac{1}{2}x + 1 \\ 2y &= x + 2 \\ x &= 2y - 2. \end{aligned}$$

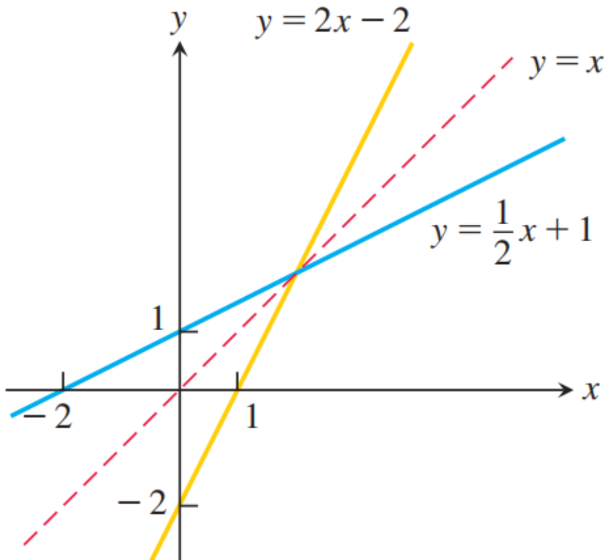
The graph satisfies the horizontal line test, so the function is one-to-one (Fig. 1.59).

2. *Interchange x and y:* $y = 2x - 2$.

Expresses the function in the usual form, where y is the dependent variable.

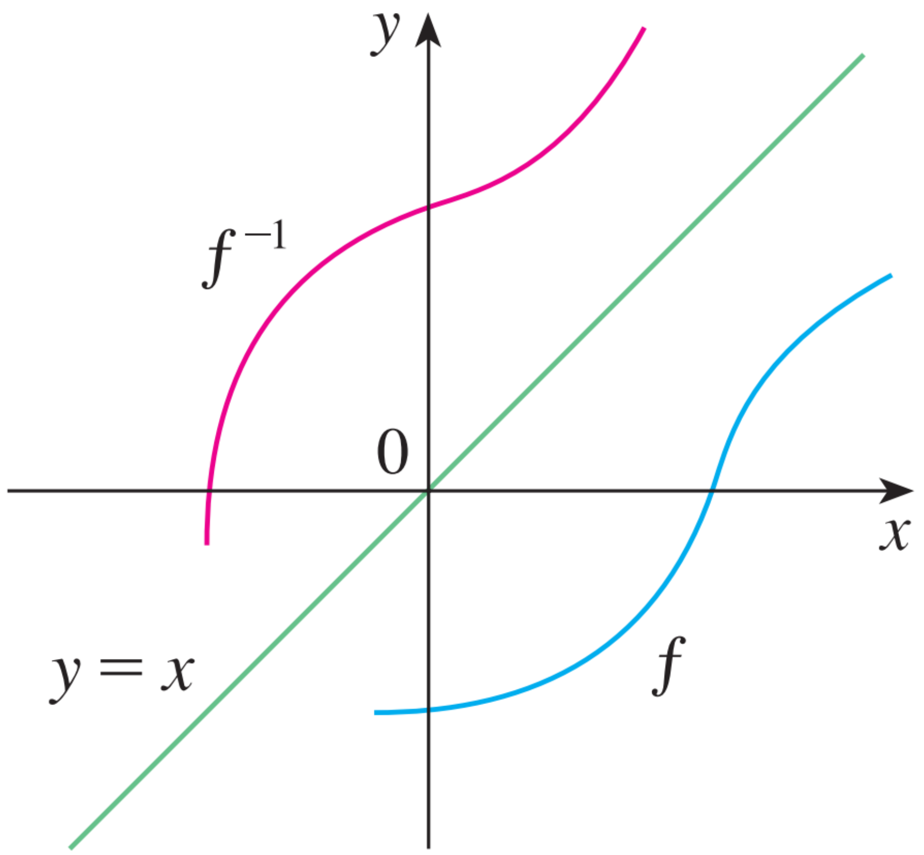
$$f^{-1}(f(x)) = 2\left(\frac{1}{2}x + 1\right) - 2 = x + 2 - 2 = x$$

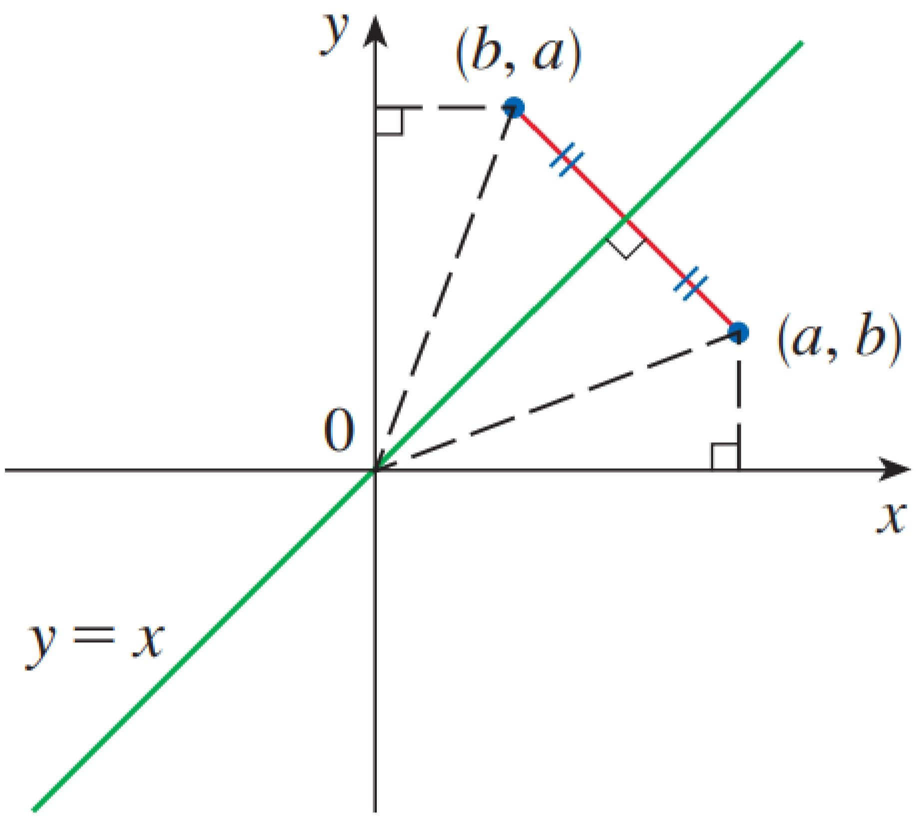
$$f(f^{-1}(x)) = \frac{1}{2}(2x - 2) + 1 = x - 1 + 1 = x.$$

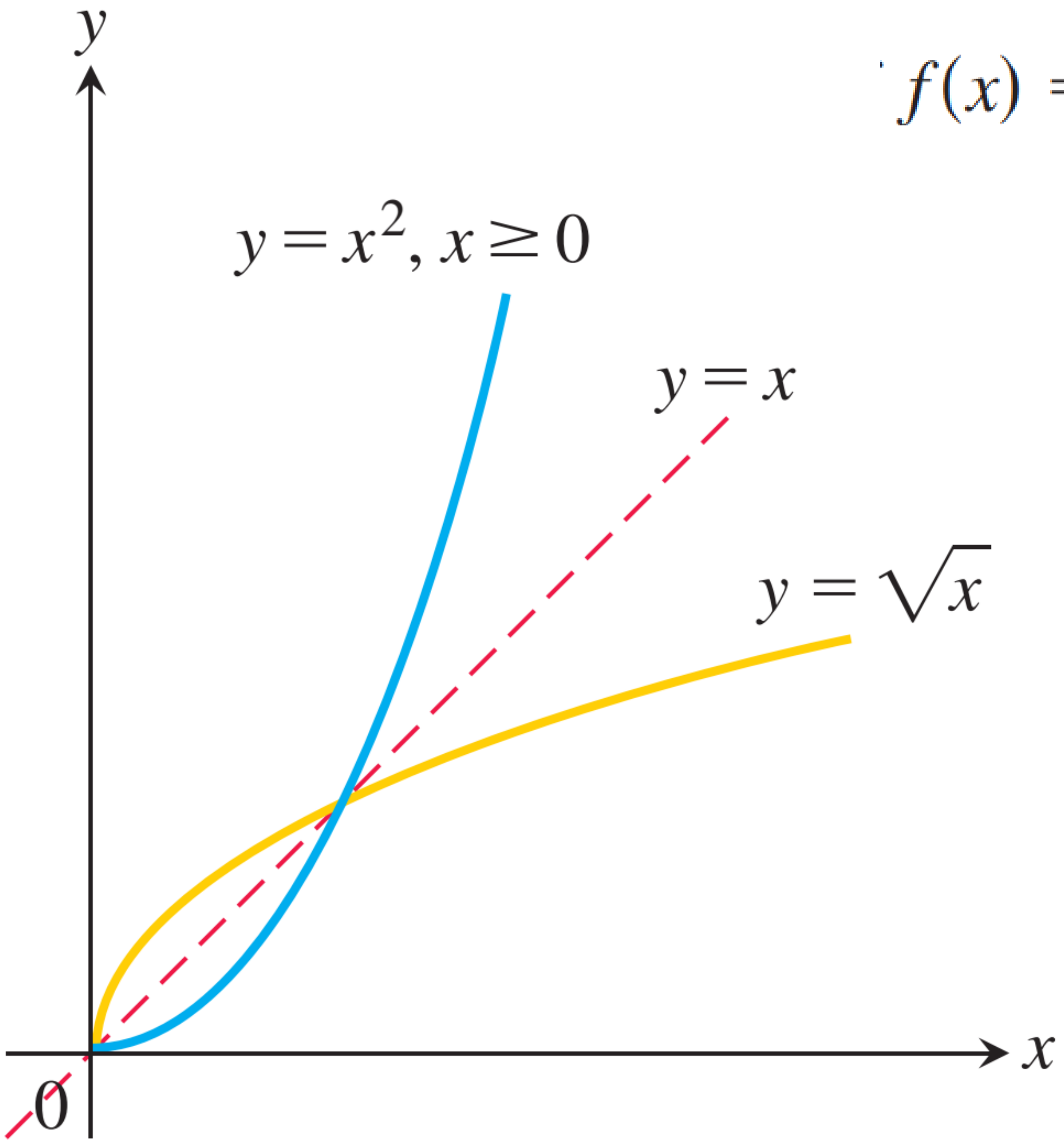


Graph of f^{-1} from the graph of f

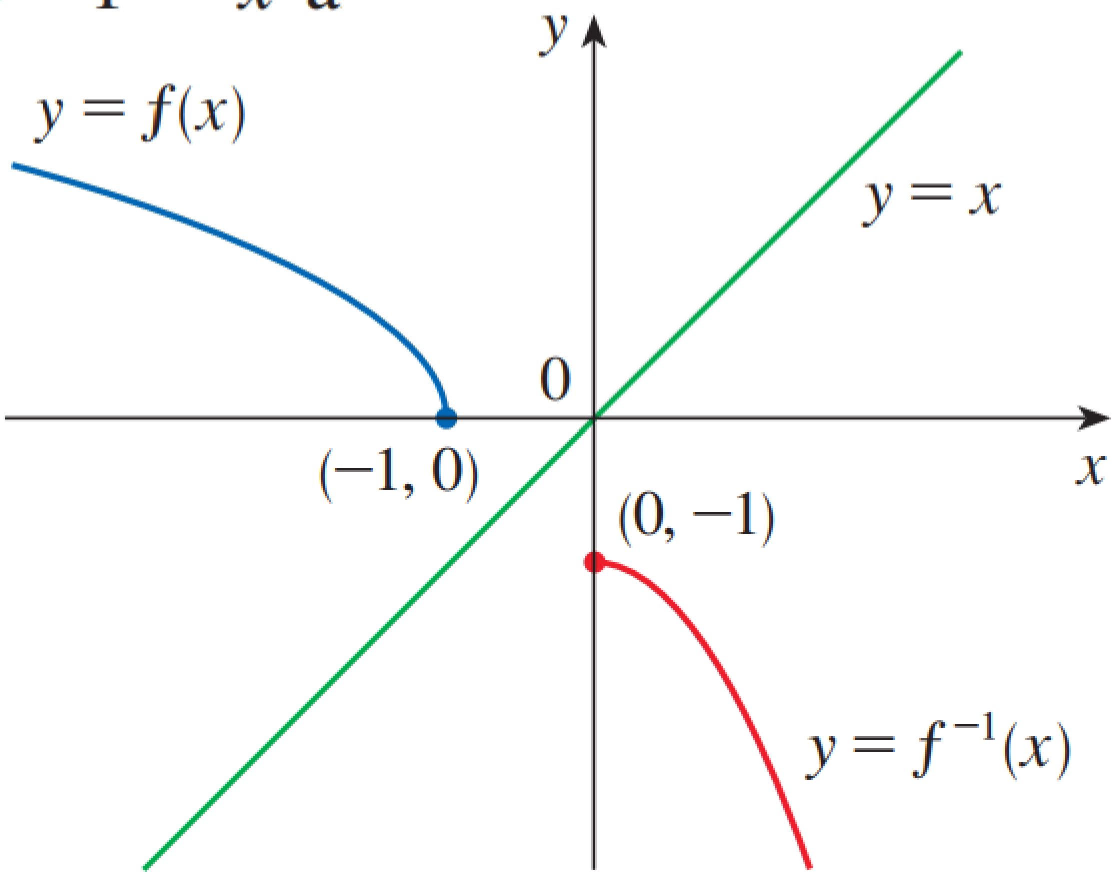
The graph of f^{-1} is obtained by reflecting the graph of f about the line $y = x$.





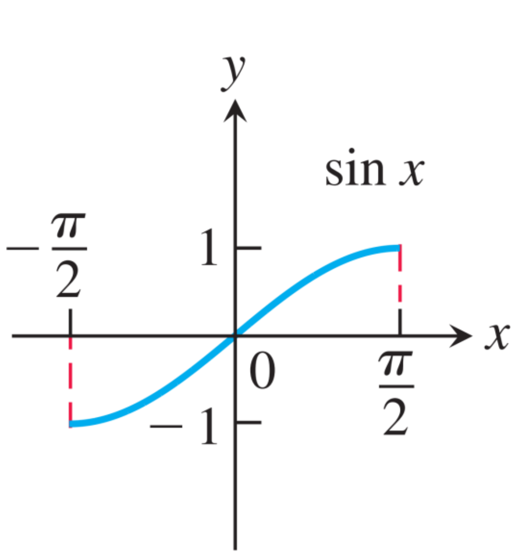


$f(x) = \sqrt{-1 - x}$ a

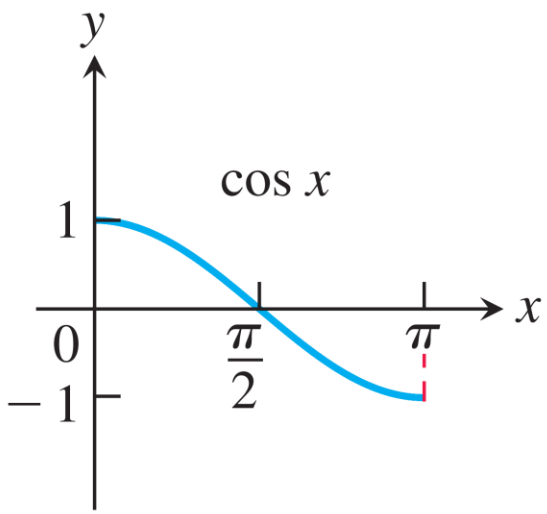


$f^{-1}(x) = -x^2 - 1, x \geqslant 0.$

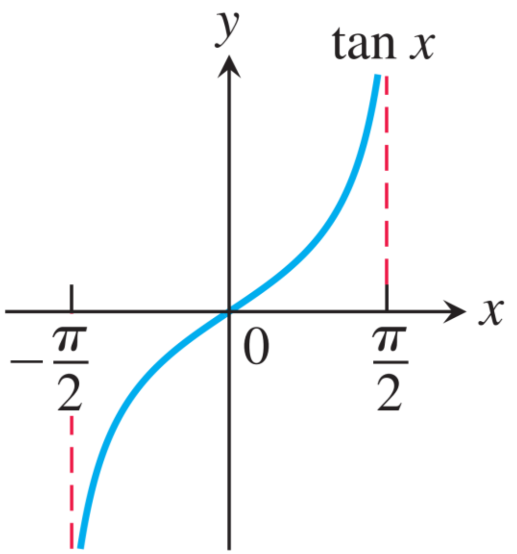
Domain restrictions that make the trigonometric functions one-to-one



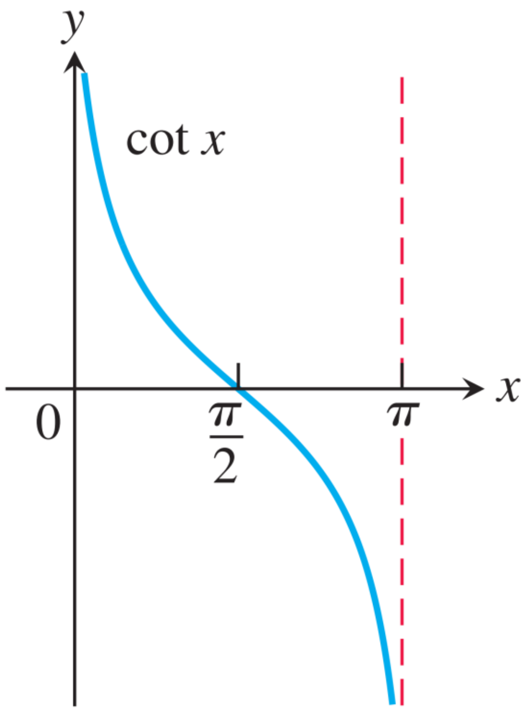
$y = \sin x$
Domain: $[-\pi/2, \pi/2]$
Range: $[-1, 1]$



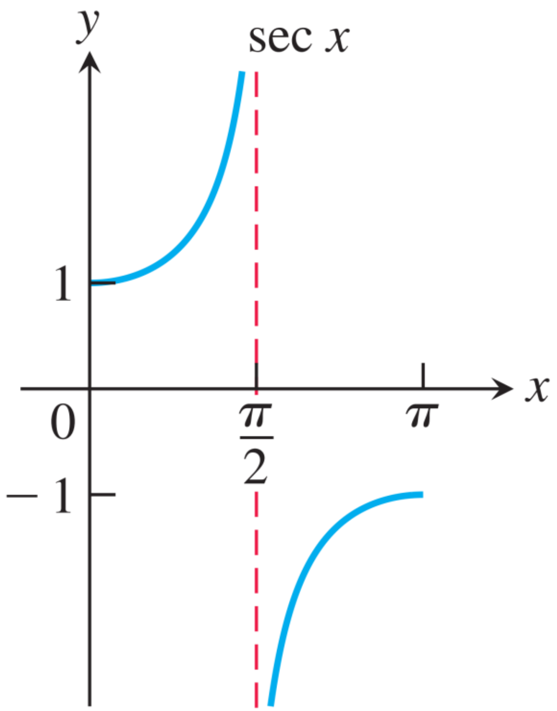
$y = \cos x$
Domain: $[0, \pi]$
Range: $[-1, 1]$



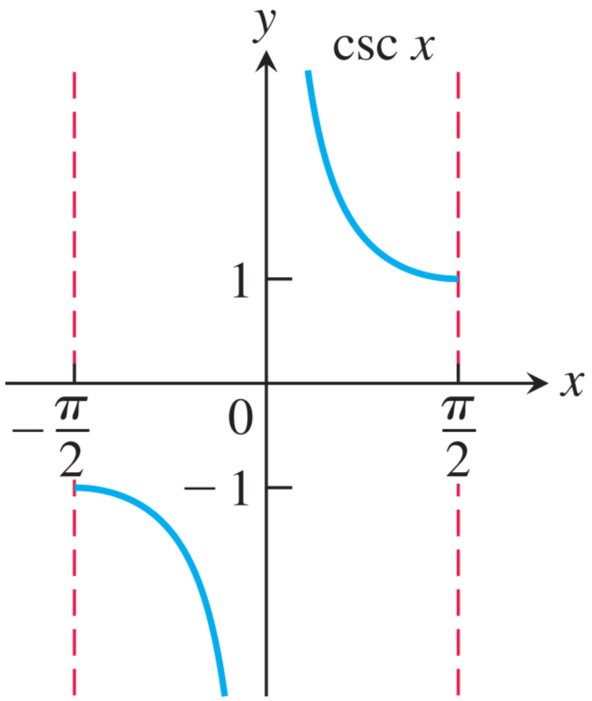
$y = \tan x$
Domain: $(-\pi/2, \pi/2)$
Range: $(-\infty, \infty)$



$y = \cot x$
Domain: $(0, \pi)$
Range: $(-\infty, \infty)$



$y = \sec x$
Domain: $[0, \pi/2) \cup (\pi/2, \pi]$
Range: $(-\infty, -1] \cup [1, \infty)$



$y = \csc x$
Domain: $[-\pi/2, 0) \cup (0, \pi/2]$
Range: $(-\infty, -1] \cup [1, \infty)$

Logarithmic Functions

DEFINITION The **logarithm function with base** a, written $y = \log_a x$, is the inverse of the base a exponential function $y = a^x (a > 0, a \neq 1)$.

$$\log_e x \quad \text{is written as} \quad \ln x.$$
$$\log_{10} x \quad \text{is written as} \quad \log x.$$

The function $y = \ln x$ is called the **natural logarithm function**, and $y = \log x$ is often called the **common logarithm function**.

Laws of Logarithms If x and y are positive numbers, then

1. $\log_b(xy) = \log_b x + \log_b y$
2. $\log_b\left(\dfrac{x}{y}\right) = \log_b x - \log_b y$
3. $\log_b(x^r) = r\log_b x \qquad$ (where r is any real number)

EXAMPLE Use the laws of logarithms to evaluate $\log_2 80 - \log_2 5$.

SOLUTION Using Law 2, we have

$$\log_2 80 - \log_2 5 = \log_2\left(\frac{80}{5}\right) = \log_2 16 = 4$$

because $2^4 = 16$. ■

Natural Logarithms

$$\log_e x = \ln x$$

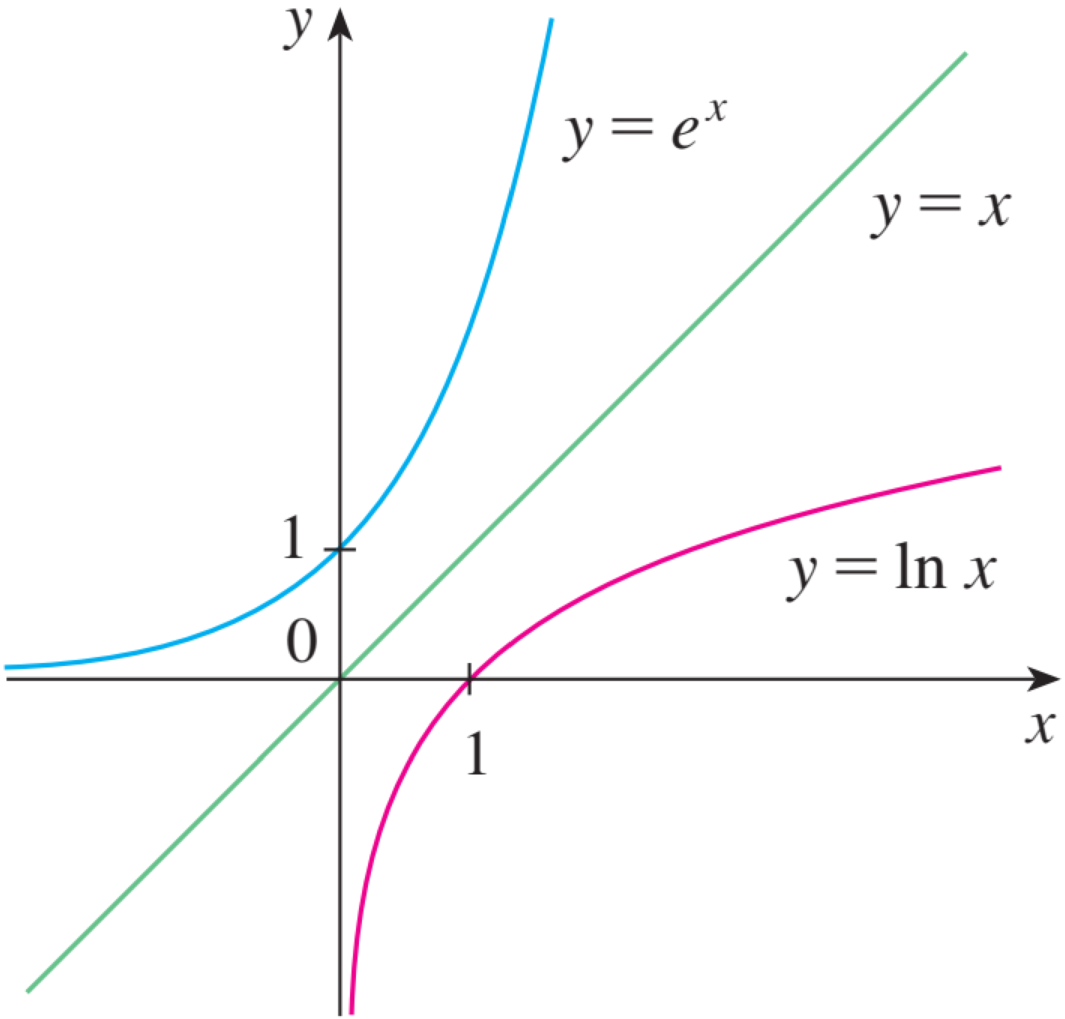
$$\ln x = y \iff e^y = x$$

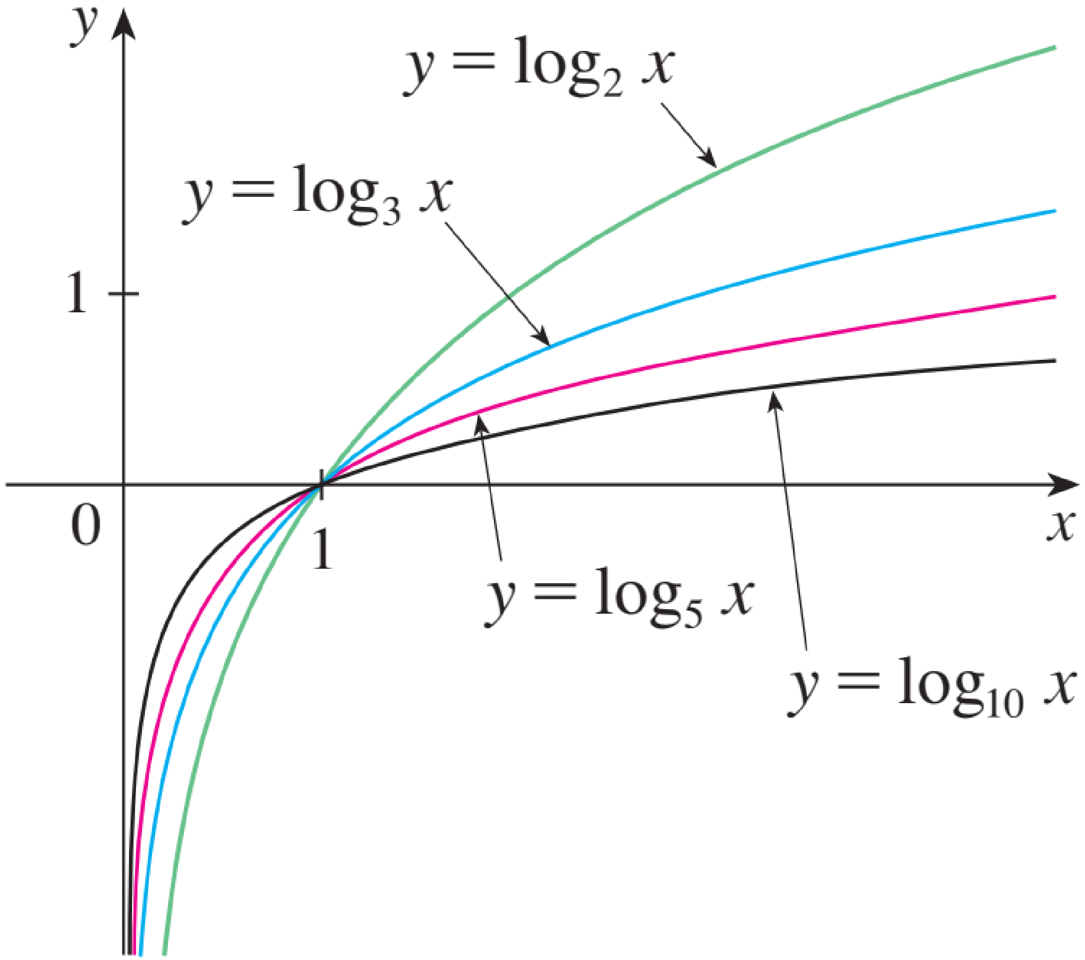
$$\ln(e^x) = x \qquad x \in \mathbb{R}$$
$$e^{\ln x} = x \qquad x > 0$$

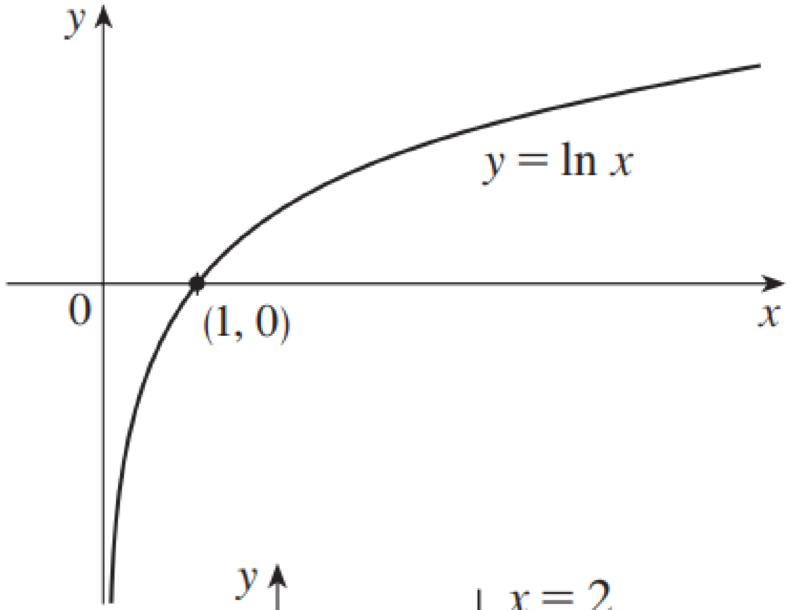
EXAMPLE Use the laws of logarithms to expand $\ln \dfrac{x^2\sqrt{x^2+2}}{3x+1}$.

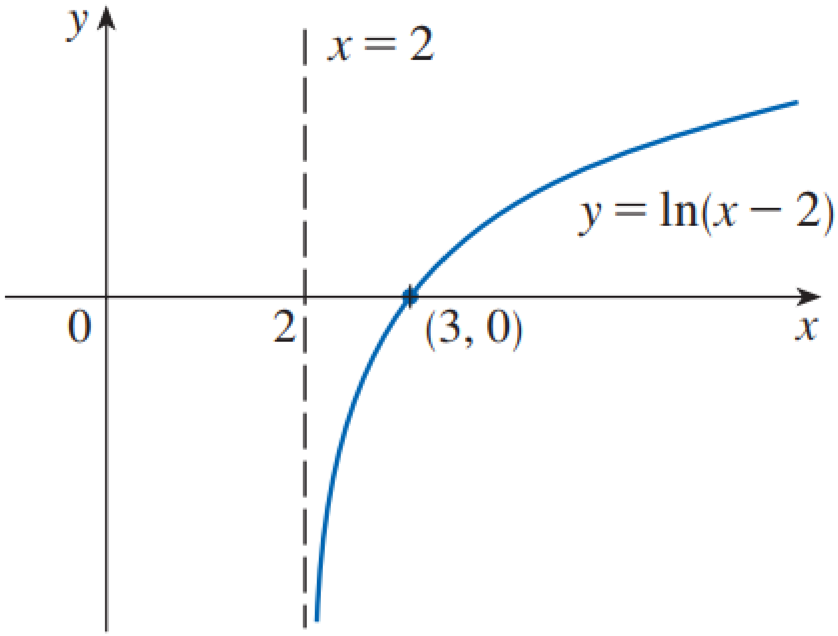
SOLUTION Using Laws 1, 2, and 3 of logarithms, we have

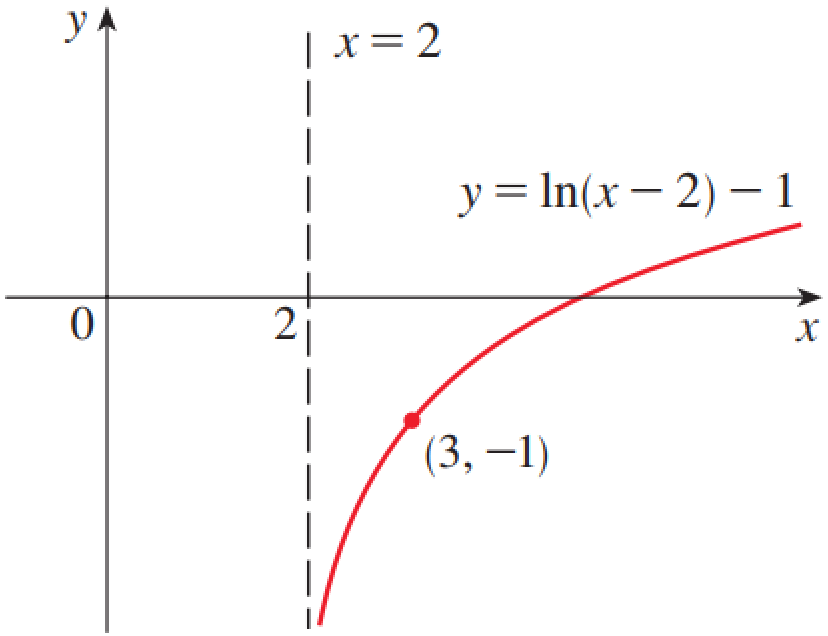
$$\begin{aligned}\ln \frac{x^2\sqrt{x^2+2}}{3x+1} &= \ln x^2 + \ln\sqrt{x^2+2} - \ln(3x+1)\\ &= 2\ln x + \tfrac{1}{2}\ln(x^2+2) - \ln(3x+1)\end{aligned}$$


y
$y = e^x$
$y = x$
$y = \ln x$
1
0
1
x


y
$y = \log_2 x$
$y = \log_3 x$
1
0
1
x
$y = \log_5 x$
$y = \log_{10} x$


y
$y = \ln x$
0
(1, 0)
x


y
$x = 2$
$y = \ln(x - 2)$
0
2
(3, 0)
x


y
$x = 2$
$y = \ln(x - 2) - 1$
0
2
x
(3, −1)