

Details

The dimension of an irreducible representation of the Lie algebra G , characterized by its highest weight, is obtained from the Weyl dimension formula.

This formula involves the product over all the positive roots of the inner products between them and the chosen weight, shifted by the Weyl vector.

The collection of inner products is encoded by labels attached to the vertices of the periodic quiver of roots.

For simply - laced Lie algebras (ADE cases) this quiver is obtained from the adjacency matrix of the Dynkin diagram and from the matrices that describe the fusion algebra of $SU(2)$ at an appropriate level. For non simply - laced cases the calculation is similar but it involves specific scaling factors for the Coxeter orbits of short and long roots.

The syntax `DimensionIrrepLie["G", list]` uses the above algorithm, for all choices of the Lie algebra G .

The second argument, `list`, is the list of components of the highest weight of the chosen irreducible representation in the basis of fundamental weights.

If the first argument is dropped, $G=A_r$ is assumed. However, if the first argument is absent (so G is assumed to be $A_r=su(r+1)$) the program does not use the general algorithm described before to determine the dimension; rather it uses the following explicit expression (which actually comes from using the previous algorithm), where r denotes the rank and where the list of components is denoted (x_j) :

$$\frac{\prod_{p=1}^r \left(\prod_{s=1}^{1-p+r} \left(p + \sum_{j=s}^{-1+p+s} x_j \right) \right)}{\text{BarnesG}[2 + r]}$$

Standard notations for classical groups and their Lie algebras:

$A_r = su(r+1)$, the Lie algebras of the special unitary groups.

$D_r = so(2r)$, the Lie algebras of the special orthogonal groups (even case).

$B_r = so(2r+1)$, the Lie algebras of the special orthogonal groups (odd case).

$C_r = sp(r)=usp(2r)$, the Lie algebras of the (compact) symplectic groups.

Exceptional simple Lie groups: E_6, E_7, E_8, G_2, F_4 .

Ordering of fundamental weights, in the list of components, is determined by the ordering of vertices of the associated Dynkin diagram, or by the list of simple roots.

This order is standard (from left to right) for A_r . The two special simple roots of D_r are the last two.

The special simple root in E_6, E_7 , or E_8 , is the last one.

Non simply-laced cases: For B_r , the last root to the right is the unique short simple root. For G_2 , the short simple root is also to the right. For C_r , the short roots (all simple roots but one) are to the left. For F_4 there should be no ambiguity. The ordering can also be inferred from the dimensions of fundamental representations (see the Examples section).

The syntax `DimensionIrrepLie["G", list, q]` calculating the q -dimension uses the same algorithm as in the classical case, but the inner products associated with the vertices of the periodic quiver of roots are calculated in terms of q -numbers. The result is expressed in terms of variables `qnum[n,q]` which have no value but represent the q -numbers n , defined as $(q^n - q^{-n})/(q - q^{-1})$.

The q -dimension can be obtained in terms of q by substituting (using Rule) `qnum[n,q]` to the previous expression. See Examples.

The formal variable q can be taken real, or specialised to a complex root of unity $\text{Exp}\left[\frac{i\pi}{\kappa}\right]$, but in the latter case the variable κ should be large enough for the chosen irrep to exist.

The classical situation (usual dimensions) is recovered when q goes to 1.

In[]:=

Possible Issues

Study of quantum dimensions: `DimensionIrrepLie["G", ruplet, q]` treats the third argument as the literal symbol q .

If your session already defines a value for q (e.g. $q = 5$), that value will propagate into the output (you'll see `qnum[\dots , 5]`).

To avoid unintended substitutions, ensure that q is unassigned before calling the function (`Clear[q];`; alternatively, call the function in a localized environment (`Block[{ q }, DimensionIrrepLie["G", ruplet, q]]`).

In the case of quantum dimensions at roots of unity, the value of $kappa$ in $q = \text{Exp}(I \text{ Pi} / kappa)$ should be large enough for the chosen representation to exist (integrability condition).

In the case of quantum dimensions at roots of unity, the simplification of the generated trigonometric expressions can be slow.