An Operational Measure of Liquidity

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What is liquidity? Kenneth Boulding says that "liquidity is a *quality* of assets which... is not a very clear or easily measurable concept" (1955, p. 310). According to John Maynard Keynes:

> There is, clearly, no absolute standard of "liquidity" but merely a scale of liquidity—a varying premium of which account has to be taken... in estimating the comparative attractions of holding different forms of wealth. The conception of what contributes to "liquidity" is a partly vague one, changing from time to time and depending on social practice and institutions. [1936, p. 240]

Similarly, Helen Makower and Jacob Marschak observe that

> ..."liquidity" has so often been used to cover all properties of money indiscriminately that it seems better not to use it for any of the separate properties of money. We thus resign ourselves to giving up "liquidity" as a measurable concept: it is, like the price level, a bundle of measurable properties. [1938, p. 284]

However, they also note that the term liquidity suggests "the fact that money is easily transformable (on the market) into other assets and is thus an effective instrument for manoeuvring" (p. 284). Closely related is the notion of liquidity due to Jack Hirshleifer who said that liquidity is "an asset's capability over time of being realized in the form of funds available for immediate consumption or reinvestment—proximately in the form of money" (1968, p. 1).[1] The notion of liquidity presented here most closely resembles Hirshleifer's.

The purpose of this paper is to present a precise definition of liquidity in terms of its most important characteristic—the time until an asset is exchanged for money. We then show that this definition is compatible with several other useful notions of liquidity.

Whereas academic economists do not possess a definition of liquidity as a measurable concept (though they do mention an assortment of its attributes), other workers in the area casually respond that liquidity is the length of time it takes to sell an asset (i.e., convert into cash); thus cash is considered the most liquid asset, while stocks listed on the NYSE are viewed as more liquid than collectibles, precious metals, jewels, real estate, and capital goods. The problem with this view of liquidity is the lack of precision and casual reference to "the" length of time it takes to convert the asset into cash.

This length of time is a function of a number of factors including frequency of offers (i.e., difficulty in locating a buyer), impediments to the transfer of legal title (viz, the time it takes to verify legal ownership as in a title or patent search and the right to dispose of the asset as in a leasehold interest, dealership, or letter stock), the costs associated with holding the asset, and, most importantly, the price at which you (the owner) are willing to sell. If your minimal price is too dear, then it might never be sold. On the other hand, if the price is exceedingly low (and legal niceties such as proof of ownership are readily established), then the asset might be sold in a very short period of time.

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[1] His reference to the time dimension is particularly relevant in regard to a premature sale (see Section II, Part C).

Any thoughtful response to clarify the meaning of liquidity must incorporate the idea that the price demanded be "reasonable." The approach suggested here incorporates this idea as it consists in embedding the sale of the asset in a search environment, discerning a sales policy that maximizes the expected discounted value of the net proceeds associated with the sale, and defining the asset's liquidity to be the expected time until the asset is sold when following the optimal policy.[2]

Clearly the concepts of liquidity and money are intimately connected. As defined here an asset's liquidity is the optimal expected time to transform the asset into money. A distinguishing characteristic of money is its role as a medium of exchange.[3] From this perspective, money is desirable because of the ease with which it can be exchanged for other commodities. If we rank commodities by their liquidity, our definition is equivalent to money being the most liquid asset. An exchange of commodity i for commodity j is accomplished most swiftly by first trading i for money and then trading money for j.[4] The expected time to go from i to money corresponds to our measure of i's liquidity. The expected time to go from i to j measures the liquidity of the (i, j) transaction. The crucial point is that in going from commodity i to money, the individual follows an optimal selling policy, and in going from money to j, the individual pursues an optimal buying policy. The approach taken here is novel in that rational behavior under uncertainty, as exhibited by adherence to optimal stopping rules, is *the* defining characteristic of liquidity. This perspective illuminates both the demand for money,[5] and portfolio analysis.[6]

The environment in which the sale of the asset occurs is presented in Section I; there we define the expected time of sale as our measure of an asset's liquidity. The compatibility of this measure with other notions of liquidity is demonstrated in Section II. In particular, we show that our definition is compatible with Keynes (Part A) and that liquidity increases with (i) the market interest rate (Part B), (ii) the thickness of a market with brisk trading (Theorem 1), and (iii) the predictability of offers (Theorem 2). Furthermore, this last result is consonant with the concept of an efficient market. One expects that any change in a parameter which leads to an increase in liquidity also would lead to a decrease in the discount associated with a quick sale. Theorem 3 (Part E) fulfills this expectation for the search cost c.

In the third and final section we proffer a simple search model with a future golden investment opportunity. Theorem 5 reveals that the choice of a more liquid initial investment enhances the investor's ability to profit from the arrival of the golden opportunity: as per our definition, liquidity provides flexibility.

[2] Hirshleifer was the first author to explicitly note the importance of uncertainty and search in determining an asset's liquidity. He forcefully observed: "It is immediately evident that uncertainty is of the essence here"; and "limitations of information may prevent buyers and sellers from finding one another, at least without incurring the costs and uncertainties of a search process" (1968, pp. 1–2).

[3] See Boulding (pp. 310–11) for a lucid discussion of liquidity and the role of money. A deep and influential analysis of the role of money in economic theory is contained in the work of Robert Clower (1977).

[4] Armen Alchian (1977) suggests that the transactions costs in trading i for j will be minimized via trading i for money and money for j with the first trade being effected by a specialist in commodity i and the second trade by a specialist in commodity j. An expanded discussion of this point that includes the importance of search and information is presented in Karl Brunner and Allan Meltzer (1971).

[5] Milton Friedman's demand for money function (1957) contains a variable u that represents uncertainty, among other things. However, uncertainty is a minor actor in his theory of money. Instead of being the tail, Friedman's u variable is now the dog (see p. 9). In a heuristic formulation, Jack Carr and Michael Darby (1981) have based a short-run money demand function upon the effects of money supply shocks on money holdings given reservation prices (presumably the result of optimal search behavior) set by asset sellers and buyers.

[6] Most portfolio analyses (for example, James Tobin, 1958) assume that the appropriate measure of risk is that associated with immediate sale of the assets held in the portfolio. But immediate sale may not be optimal.

I. The Setting

Search is the fundamental feature of the arena in which the sale of the asset is to take place; the setting is very much akin to the standard job search or house selling model. The search environment is characterized by four objects: c_i, T_i, X_i, and β. First, there are the costs of owning/operating the asset as well as the cost of attempting to sell the asset. In the discrete time framework, the net operating and search cost for period i is denoted c_i.

Second, one offer arrives at each time in the set $\{S_i: i=1,2,\dots\}$ of arrival times. The random arrival times S_i satisfy

$$S_i = \sum_{j=1}^{i} T_j,$$

where the integer-valued random variables $T_i \geq 0$ need not be either independent or identically distributed.

The ith price offered is a nonnegative random variable X_i.[7] In the standard search paradigm (see Section II), the X_i are independent, identically distributed, and independent of $\{T_i\}$. None of these three assumptions is invoked here. As evidenced in equation (1) below, our formulation can be structured so that either it does not permit the seller to accept any offer other than the one most recently tendered so recall of past offers is not allowed, or it does permit the seller to accept any of the tendered offers so recall is allowed.

Finally, all expenditures and receipts are discounted[8] at the rate β so that the present value of a dollar received in period i is β^i. The seller seeks to maximize the expected discounted value of his net receipts. More formally, the discounted net receipts $R(\tau)$ associated with a stopping time τ is given by

$$\text{(1a)} \quad R(\tau) = \beta^\tau Y_{N(\tau)} - \sum_{i=1}^{\tau} \beta^i c_i$$

$$\text{(1b)} \quad Y_i = \begin{cases} X_i & \text{if no recall} \\ \max(X_1, \dots, X_i) & \text{if recall allowed,} \end{cases}$$

where $N(\tau) = \max\{n: S_n \leq \tau\}$ is the random number of offers that the seller observes when employing the decision rule τ and the random variable $Y_{N(\tau)}$ is the size of the accepted offer. Consequently, the seller chooses a stopping rule τ^* in the set T of all stopping rules (we do not require $P(\tau < \infty) = 1$) such that[9]

$$\text{(2)} \quad ER(\tau^*) = \max\{ER(\tau): \tau \in \mathrm{T}\}.$$

The manifest value of the asset is $V^* \equiv ER(\tau^*)$, and the length of time it takes to realize the asset's value and to convert the asset into cash is the random variable τ^*. (In making this statement we are implicitly assuming that there is no lag between the time an acceptable offer is made and the time that the seller is paid.) We propose to use $E\tau^*$ as the measure of an asset's liquidity with an increase in $E\tau^*$ corresponding to a decrease in liquidity.

The key point is that for any given asset (with its concomitant cost function c_i, arrival times $\{S_i\}$, and offers $\{X_i\}$), there is an optimal policy τ^* which determines the value $ER(\tau^*)$ of the asset. The asset's liquidity is determined by τ^*.[10]

[7]Though it could be accounted for, the impact of inflation upon asset prices is not considered in this analysis.

[8]If the time to sale is relatively short, then discounting will have little impact—though an individual with special short-run opportunities or critical consumption needs could have a high time value of money.

[9]We unabashedly assume the existence of an optimal rule. The assumptions of the standard search paradigm (see the discussion in Section II) ensure the existence of an optimal rule.

[10]Simplicity led us to elect the mean of τ^* as our measure of liquidity. Another increasing function of the distribution of τ^* could have been selected. In particular, the comparative statics results found in Section II, Parts A–D, remain unchanged by any such selection.

(continued)

II. Compatibility with Other Notions of Liquidity

Our measure of liquidity is not only internally consistent but also compatible with a good deal of what economists have said. In regard to its consistency we note that $E\tau^* = 0$ for money so that money is perfectly liquid: it is the most liquid asset.

Second, an illiquid asset is one that can't be sold, or rather one with $E\tau^* = \infty$. This can occur when there are informational asymmetries or structural constraints that induce the potential buyers to undervalue the asset; that is, its worth to the current owner exceeds its assessed or actual worth to any potential buyer. Informational asymmetries arise in the context of a business in which there are many cash transactions and the company's books are not a reliable guide to revenues. Structural constraints such as tax considerations in which only some assets "may be burdened by transaction duties" (Hirshleifer, 1972, p. 137) provide another example of impaired marketability which can render an asset totally illiquid.[11]

To analyze how this might come about, suppose that c_i, the net search and operating cost per period, is $c < 0$ for all i. In addition, suppose that no buyer is willing to offer more than $-c\beta/(1-\beta)$. The policy τ^* of never accepting an offer at or below $-c\beta/(1-\beta)$ yields the owner an expected discounted value of $-c\Sigma_{i=1}^{\infty}\beta^i = -c\beta/(1-\beta)$ so τ^* is indeed optimal. Moreover, $\tau^* \equiv \infty$ so $E\tau^* = \infty$, and the asset is illiquid.

The "standard" search paradigm is utilized extensively in the ensuing analysis. It entails (i) a constant search cost so $c_i \equiv c$, (ii) one offer tendered each and every period so $T_i \equiv 1$, (iii) independent offers X_i drawn from the same known probability distribution F, and (iv) recall of past offers. With these assumptions the existence of an optimal rule τ^* is guaranteed, and τ^* has the following representation: $\tau^* = \min\{n\colon X_n \geq \xi\}$, where ξ is referred to as the reservation price. Thus, the seller accepts the first offer greater than or equal to his reservation price ξ; consequently, τ^* is a geometric random variable with parameter $P(X_1 \geq \xi)$, the probability that a given offer is successful in effecting the asset's sale. Furthermore, it is clear upon reflection and easy to demonstrate that $\xi = V^*$. (See our 1976a, b papers for a full discussion of the standard search model and its variants.)

A. *Compatibility with Keynes*

According to Keynes, one asset is more liquid than another if it is "more certainly realisable at short notice without loss" (1930, p. 67). In the context of the standard search paradigm, Keynes' definition is equivalent to ours if we interpret "at short notice," "more certainly realisable," and "without loss" to mean "in one period," "has a higher probability p of being sold in one period," and "in accord with the optimal policy." To see this, recall that the asset is sold if and only if the offer price is ξ or larger, and merely observe that $p = P(X_i \geq \xi)$ is related to $E\tau^*$ via $E\tau^* = 1/p$ so that liquidity increases with p.

B. *Liquidity and Impatience*

Continuing with the standard search paradigm, recall that the reservation price ξ_β is a function of the discount factor β. As per equation (13) of our earlier paper (1976a), the asset's reservation price satisfies

$$c = H(\xi) - \xi(1-\beta)/\beta, \tag{3}$$

where $H(x) = \int_x^\infty (y-x)\,dF(y)$. Differentiating the first-order condition (3) with respect to β yields

$$d\xi/d\beta = \xi/\beta^2[1 - F(\xi) + (1-\beta)/\beta] > 0. \tag{4}$$

Hence, ξ_β is a strictly increasing function of β. Consequently, an increase in β leads to an

This invariance is due to the first-order stochastic dominance of the time τ^* until the sale of the asset induced by the increase in the reservation price ξ. But any analysis in the vein of Theorem 3 would thereby be rendered much more complicated.

[11]A structural characteristic leading to this situation arises when the asset is a business and the current owner's managerial talents in running this business substantially exceed his talents (and implicit wages) in any other employment.

increase in $E\tau^*$ as $E\tau^* = 1/P(X_1 \geq \xi_\beta)$ with ξ_β strictly increasing in β. That is, an increase in the market interest rate or in the asset holder's time preference (a lower value of β) leads to an increase in the asset's liquidity. This demonstrates two facts. First, because more impatience (as might arise from increased consumption needs that only can be satisfied via the expenditure of wealth in the form of money) leads to a more liquid asset, impatience and liquidity preference are commensurate in that they vary directly. As expected, an increase in liquidity preference leads to an increase in liquidity itself. Second, because an asset's liquidity depends upon the discount factor, it is a property of the asset holder as well as an intrinsic property of the asset itself. Nevertheless, to the extent that dispersion of the offer price distribution, rate of receipt of offers, and relative costs of soliciting offers are common across all sellers, the ranking of assets in terms of liquidity will be similar across individuals, regardless of their degree of impatience. With this view we return to the notion that liquidity is determined by characteristics of the asset; characteristics of the seller have virtually no impact.

C. *Liquidity and Thickness of the Market*

When there are many transactions per day of a homogeneous asset such as wheat or long-term Treasury bonds, the market for the asset is thick. On the other hand, the more idiosyncratic the asset, as is the case if it is one-of-a-kind (a work of art or a castle), or has a limited set of uses (a germ-free, refrigerated warehouse or a special purpose lathe), the thinner the market becomes. The number of transactions in a market is a function of several factors, including the frequency of offers received by any particular asset. Accordingly, the thickness of the market for an asset is said to increase with the frequency of offers. Theorem 1 below establishes the direct, though weaker than anticipated, connection between our measure of liquidity and the thickness of the market.

THEOREM 1: *An increase in the frequency of offers causes the expected time of sale to decrease (and liquidity to increase) if either the interest rate is near zero or the frequency of offers is very high.*

PROOF:

To begin the analysis, suppose offers arrive according to a Poisson process with rate λ, and let α be the continuous-time interest rate. Then β_λ, the one-period (a period is the time interval until the next offer) discount factor, satisfies $\beta_\lambda = \lambda/(\alpha+\lambda)$. Define T_λ and τ_λ^* to be the time of sale and the number of offers received until the sale, respectively, when offers arrive at rate λ. As $1/\lambda$ is the expected time between offers, we have $ET_\lambda = E\tau_\lambda^*/\lambda$.

Differentiating with respect to λ and utilizing (4) and $E\tau_\lambda^* = 1/[1-F(\xi_\lambda)]$ yields

$$\begin{aligned} dET_\lambda/d\lambda &= -E\tau_\lambda^*/\lambda^2 + \lambda^{-1}\, dE\tau_\lambda^*/d\lambda \\ &= -\left[\lambda^2(1-F(\xi_\lambda))\right]^{-1} \\ &\quad + f(\xi_\lambda)[1-F(\xi_\lambda)]^{-2}\lambda^{-1} \\ &\qquad \times (d\xi_\lambda/d\beta_\lambda)\cdot(d\beta_\lambda/d\lambda) \\ &= -\left[\lambda^2(1-F(\xi_\lambda))\right]^{-1} \\ &\quad + \alpha\xi_\lambda f(\xi_\lambda)\left\{\lambda^3[1-F(\xi_\lambda)]^2\right. \\ &\qquad \left.\times[1-F(\xi_\lambda)+\alpha/\lambda]\right\}^{-1}. \end{aligned}$$

Noting that both the reservation price ξ and the expected time $[1-F(\xi)]^{-1}$ until an acceptable offer is received are bounded for α near zero and λ large, the above expression for $dET_\lambda/d\lambda$ reveals that its sign is negative when α is near zero and also when λ is large.

The expected time of sale is the product of the expected time between offers and the expected number of offers received until the asset is sold. A decrease in the expected time $1/\lambda$ between offers, the first term in the product, causes the discount factor β_λ to increase. This leads to an increase in the reservation price and hence to an increase in the number of offers received, the second term in the product. Theorem 1 asserts that the net effect is negative if discounting has little impact (because the time to sale is short or the interest rate is small). In a practical sense, Theorem 1 implies that liquidity, as defined here, increases with thickness for all of the familiar highly organized markets (such as the NYSE) characterized by brisk trading; Theorem 1 is uninformative as regards thin markets.

D. *Liquidity and Predictability*

In Jacob Marschak's view, the word liquidity "denotes a bundle of two measurable properties and is therefore itself not measurable" (1938, p. 323). The two properties he refers to are "plasticity," that is, the ease "of manoeuvring into and out of various yields after the asset has been acquired," and "the low variability of its price." A version of this view of liquidity might provide the following definition: an asset is liquid if it *can be sold quickly* at a *predictable price*. By this definition, liquidity is a two-dimensional attribute.

Consider commodities such as wheat and long-term Treasury bonds. The market for both assets is nearly perfect in that the attempt to sell even as much as one million dollars worth of these assets will have only a minute effect upon "the market price." Moreover, there is a ready (and highly organized) market for both assets with a multitude of transactions taking place each weekday. The transaction can be effected in a matter of minutes. Consequently, it is indisputable that these assets can be sold quickly. On this dimension they would be seen to be near-money.[12]

Recently, however, interest rates have been highly volatile; fluctuations of as much as 9 percent in a single day (recall Federal Reserve Chairman Volcker's announcement of October 6, 1979) have occurred. And the wheat market has a long history of volatility. Thus, neither of these assets rates high on the dimension of "predictable price."

Predictability, we maintain, is an expression of concern with adverse events or downside-risk, that is, safety. As such it ignores and fails to account for the occurrence of favorable events or upside-risk. Our measure of liquidity implicitly utilizes both the adverse and the favorable events by requiring that the asset be sold at its "fair market price" where the price is derived from the seller's optimization (see equations (1) and (2)).

To see the relation between predictability and our measure of liquidity, let $W_i = X_i - \mu$ so $EW_i = 0$ and parameterize predictability by the following representation of the offers: $X_i = \mu + \varepsilon W_i$.

Naturally, a decrease in ε is interpreted as an increase in predictability. An increase in ε is a mean-preserving increase in risk of the sort that might properly be labeled a dilation. We shall limit our investigation to dilations because other mean-preserving increases in risk are less regular in that the concomitant change in liquidity they induce can be either an increase or a decrease.

The seller's problem is to choose a stopping rule τ_ε in the set T of all stopping rules to maximize

$$\begin{aligned} E&\left[\beta^\tau(\mu+\varepsilon W_\tau)-c(\beta+\ldots+\beta^\tau)\right] \\ &= -\left(c\beta/(1-\beta)\right)+E\beta^\tau \\ &\qquad\times\left[\mu+\left(c\beta/(1-\beta)\right)+\varepsilon W_\tau\right] \\ &= -\left(c\beta/(1-\beta)\right)+\varepsilon E\beta^\tau \\ &\qquad\times\left[\left(\mu+\left(c\beta/(1-\beta)\right)\right)/\varepsilon+W_\tau\right]. \end{aligned}$$

Equivalently, the seller seeks to maximize

$$E\beta^\tau(\mu_\varepsilon+W_\tau), \tag{5}$$

where $\mu_\varepsilon = (\mu + (c\beta/(1-\beta)))/\varepsilon$. When there is total predictability, that is, when $\varepsilon = 0$,

[12] One might say that such an asset is perfectly marketable. Not only is the owner capable of effecting a quick sale, there is nothing to be gained (on average) from waiting for a better price; i.e., a quick sale can be effected at the market price. This raises the question of whether the concept we have provided measures liquidity or marketability. In our view, this question is largely semantic.

$\tau_\varepsilon \equiv 1$ if $\mu > -c\beta/(1-\beta)$; otherwise, $\tau_\varepsilon = \infty$. In view of this fact we shall assume that $\mu > -c\beta/(1-\beta)$ so that $\mu_\varepsilon > 0$ and μ_ε decreases as ε increases.

It is our intention to show that an increase in the mean μ_ε induces an earlier sale, and, concomitantly, an increase in liquidity.

The stopping problem expressed in (5) is the discounted version of the standard job search problem with a search cost of zero. When $EX_1 = \mu$, the solution is a reservation price ξ_μ such that the seller accepts the offer if and only if it equals or exceeds ξ_μ. As demonstrated in our earlier paper (1976a, p. 164), ξ_μ is the unique solution to

$$0 = H_\mu(x) - rx, \tag{6}$$

where $\beta = 1/(1+r)$ and $H_\mu(x) = H(x-\mu)$.[13]

The decreasing nature of H (recall $H'(x) = -(1-F(x))$) yields the following two facts.

LEMMA 1: *If $\delta^+ > \delta$, then $\xi_{\delta+} > \xi_\delta$.*

PROOF:

From the definition of H_δ we have

$$\begin{aligned} H_{\delta+}(\xi_\delta) - r\xi_\delta &= H(\xi_\delta - \delta^+) - r\xi_\delta \\ &= H(\xi_\delta - \delta) - r\xi_\delta + H(\xi_\delta - \delta^+) \\ &\quad - H(\xi_\delta - \delta) \\ &= H(\xi_\delta - \delta^+) - H(\xi_\delta - \delta) > 0. \end{aligned}$$

LEMMA 2: *If $\delta^+ > \delta$, then $\xi_{\delta+} - \xi_\delta < \delta^+ - \delta$.*

PROOF:

By the definition of H_δ,

$$\begin{aligned} H_{\delta+}(\xi_\delta + \delta^+ - \delta) - r(\xi_\delta + \delta^+ - \delta) &= H(\xi_\delta - \delta) - r\xi_\delta - r(\delta^+ - \delta) \\ &= -r(\delta^+ - \delta) < 0, \end{aligned}$$

so $\xi_\delta + \delta^+ - \delta > \xi_{\delta+}$.

THEOREM 2: *The asset's liquidity is an increasing function of its predictability; that is, $E\tau_\varepsilon$ is an increasing function of ε.*[14]

PROOF:

Fix $\varepsilon^+ > \varepsilon$ so that $\delta^+ \equiv \mu_{\varepsilon+} < \mu_\varepsilon \equiv \delta$. Applying Lemma 2 (with the roles of δ^+ and δ reversed there), we obtain

$$\begin{aligned} p^+ &\equiv P(W_1 + \delta^+ \geq \xi_{\delta+}) \\ &= P(W_1 \geq \xi_{\delta+} - \delta^+) < P(W_1 \geq \xi_\delta - \delta) \equiv p \end{aligned}$$

so that $E\tau_{\varepsilon+} = 1/p^+ > 1/p = E\tau_\varepsilon$.

As the connection between the predictability of the asset's price and the thickness of the asset's market, though presumably direct, is tenuous, we dispense with further comment on this connection. The connection between Theorem 2 and the notion of efficient markets does merit discussion. Theorem 2 states that liquidity decreases with the asset's risk so, absent further specification of the source of risk, it might appear that Theorem 2 contradicts the concept of an efficient market. In a fully informed market there is nothing to be gained by waiting to sell a risky asset, whereas Theorem 2 implies an optimal waiting time which is strictly positive and strictly increasing with the asset's risk. In our analysis the source of uncertainty emanates from the random selection of the particular agent seeking to purchase the asset; in particular, the agents value the asset differentially. Accordingly, the arrival of a

[13] If $P(W_i \leq y) = F(y)$ and $F_\mu(y) = P(W_i + \mu \leq y) = F(y-\mu)$, then

$$\begin{aligned} H_\mu(x) &= \int_x^\infty (y-x)\,dF_\mu(y) = \int_x^\infty (y-x)\,dF(y-\mu) \\ &= \int_{x-\mu}^\infty (z-(x-\mu))\,dF(z) = H(x-\mu). \end{aligned}$$

[14] The reverse result holds if we assume $\mu < -c\beta/(1-\beta)$.

low offer correctly has no impact upon the owner's opinion (an opinion shared by the market) of the asset's value, for the offer does not constitute new information. On the other hand, the source of variation in an efficient markets setting is the arrival of new information. As new information arrives, say in the form of a low offer, each agent, including the asset owner, simultaneously revalues the asset. In short, risk is the embodiment of heterogeneous preferences in one analysis and the arrival of new, commonly shared information in the other. In view of this discussion, it is clear that financial assets traded in a thick, efficient market will be exceedingly liquid.

E. *Liquidity and the Discount Attending Premature Sale*

For some, liquidity corresponds to the following idea of discount.[15,16] Suppose an asset has a value v, but the likely price at which it can be sold in a "quick sale" is only $(1-d/100)v$. Then the discount d associated with the quick sale measures the asset's liquidity—the higher the value of d, the less liquid the asset.

We can incorporate this idea in our search setting. To do so, interpret a quick sale as a constraint that the conversion to cash takes place within a fixed (and perhaps short) amount of time t.[17] Then only policies in the set $T_t = \{\tau \in \mathrm{T}: \tau \leq t\}$ are permitted. Hence, the seller seeks a stopping rule τ_t in the set T_t such that

$$(7) \quad V_t \equiv ER(\tau_t) = \max\{ER(\tau): \tau \in T_t\}.$$

The corresponding discount is $100(1-V_t/V^*)$ which we label $d(t)$.

Makower and Marschak describe their concept of saleability "as the relationship between the selling price and the time which the seller must wait in order to get it" (p. 280). Continuing in this vein, they state that "the influence of time on the selling price is due to the seller's finding more buyers." With these ideas, their deterministic "price-time schedule" is very much akin to the function V_t and the waiting for offers in the search environment is not very different from their idea of waiting in order to find more buyers.

This formalization of liquidity is not necessarily the same as the one proposed earlier, for it can easily happen that the discount $d_1(t)$ for asset 1 is less than the discount $d_2(t)$ for asset 2, yet $E\tau_1^* > E\tau_2^*$. More generally, $d_1(t)-d_2(t)$ can change sign as t increases.

While we readily acknowledge that there are instances in which our proposed measure $E\tau^*$ of liquidity is not commensurate with the notion of liquidity embedded in the discount $100(1-V_t/V^*)$, we expect that these two measures will agree frequently. In fact, just as an increase in the cost c of search leads to an increase in liquidity as measured by the expected time to sale, we demonstrate in Theorem 3 that an increase in c also causes the discount $100(1-V_n/V^*)$ to decrease for all horizon lengths n. Even though these two measures are not mathematically equivalent, this result suggests they are compatible in a practical sense.

THEOREM 3: *In the context of the standard search paradigm with recall, an increase in the*

[15] Hirshleifer asserts that "Illiquid assets... are those characterized by a relatively large discount for 'premature' realization" (1972, p. 137).

[16] Roland McKean uses liquidity "to mean merely 'moneyness'," and asserts that "Usually, an asset's liquidity is described to include the probabilities of getting various fractions of the going price plus the time period necessary to liquidate the asset" (1949, p. 509). These "fractions" correspond to our idea of discount. Like Marschak, McKean believes that an operational definition of liquidity is not possible:

> Since these components cannot be measured, there is little to be gained by breaking the notion down. Perhaps it is sufficient to say that the more nearly we regard an asset as substitutable for money, or the more it partakes of the same attractions possessed by money-holdings, the more liquidity the asset has. [p. 509–10]

[17] Alternatively, suppose that the optimal stopping rule is preempted by an event requiring immediate disposal of the asset. This event can be interpreted as either a once-in-a-lifetime investment opportunity or a catastrophe. The time at which preemption occurs is a random variable and can be included in the formulation of the stopping rule problem.

cost of search causes both $E\tau^*$ and $(1 - V_n/V^*)$ to decrease, $n = 1,2,\ldots$, where V_n, defined in (7), *is the value of the asset when it must be sold within n periods.*

PROOF:

Differentiating the first-order condition (3) with respect to c yields

$$(8) \qquad \xi' = -\beta/[1-\beta F(\xi)] < 0.$$

Hence, an increase in c causes ξ, and in turn $E\tau^* = 1/P(X_1 \geq \xi)$, to decrease.

It is clear upon reflection that ξ represents not only the asset's reservation price but also its value; that is, $\xi = V^*$. Consequently, in order to demonstrate that the discount decreases with c it suffices to show

$$(9) \qquad d[V_n/\xi]/dc \geq 0, \qquad n = 1,2,\ldots.$$

To begin the analysis it behooves us to notice that

$$(10a) \qquad V_1 = \beta(\mu - c)$$

$$(10b) \qquad V_{n+1} = -\beta c + \beta F(\xi) V_n + \beta \int_{\xi}^{\infty} x\, dF(x), \qquad n \geq 1$$

where $\mu = EX_1$ and we have used the fact (see our 1976a paper, p. 170) that the reservation price when n periods remain is ξ, $n = 1,2,\ldots$. (This fact provides an enormous simplification in the analysis vis-à-vis the case of no recall.) From (8) and (10a) we obtain

$$(11) \quad (d/dc)[V_1/\xi] = \left\{-\xi + \frac{\beta(\mu - c)}{1 - \beta F(\xi)}\right\}\beta/\xi^2,$$

whereas manipulation of (3) produces

$$(12) \quad \mu - c = (\xi/\beta) - \int_0^{\xi}(\xi - x)\, dF(x).$$

Inserting (12) into (11) generates

$$(13) \quad \xi^2(1-\beta F(\xi))(d/dc)[V_1/\xi] = \beta^2 \int_0^{\xi} x\, dF(x).$$

The nonnegativity of $d[V_1/\xi]/dc$ is palpable from its representation in (13).

To simplify the rather complex expressions in $d[V_n/\xi]/dc$, we shall write F and f in place of $F(\xi)$ and $f(\xi)$ and $D \equiv \xi F(\xi) - \int_0^{\xi} x\, dF(x)$. From (10) and (12) we have

$$\begin{aligned}(14) \quad (V_2 - V_1)/\beta &= \beta F(\mu - c) - \mu + \int_{\xi}^{\infty} x\, dF(x) \\ &= \beta F\left[\xi/\beta - \int_0^{\xi}(\xi - x)\, dF(x)\right] - \int_0^{\xi} x\, DF(x) \\ &= (1 - \beta F) D,\end{aligned}$$

whereas iterating the first differences obtained via (10) leads to

$$\begin{aligned}(15) \quad V_{n+1} - V_n &= \beta F(V_n - V_{n-1}) = \ldots \\ &= (\beta F)^{n-1}(V_2 - V_1), \quad n \geq 1.\end{aligned}$$

From (14) we easily realize

$$(16) \quad (d/dc)(V_2 - V_1) = \beta F(1 - \beta F) + \beta D - \beta^2 D f \xi'.$$

Employing (8), (14), and (16) in conjunction with (15) yields

$$\begin{aligned}(17) \quad &\xi^2 (d/dc)[(V_{n+1} - V_n)/\xi] \\ &= (\beta F)^{n-1}\{\xi\beta F(1-\beta F) + \xi\beta D + \beta^2 D\} \\ &\quad + \beta^2 D \xi (\beta F)^{-1} f \xi' \\ &\times \{(\beta F)^{n-1}[n(1 - \beta F) - 1]\}, \quad n = 1,2,\ldots\end{aligned}$$

As differentiation is a linear operator and $V_{n+1} = \Sigma_{i=1}^{n}(V_{i+1} - V_i) + V_1$, equations (16) and (17) enable us to conclude that $[\gamma \equiv \beta F]$

$$(18)\quad \xi^2(1-\beta F)(d/dc)[V_{n+1}/\xi]$$
$$= (1-\gamma^n)\left[\xi\beta F(1-\beta F)+\xi\beta D+\beta^2 D\right]$$
$$+\beta^2\int_0^{\xi} x\,dF(x) - \beta^3 D\xi(\beta F)^{-1}$$
$$\times f\left\{(1-\gamma)\sum_{i=1}^{n} i\gamma^{i-1} - \sum_{i=1}^{n}\gamma^{i-1}\right\}.$$

Because the term in braces equals $-n\gamma^n < 0$ and $D \geq 0$, all of the terms on the right-hand side of (17) are nonnegative.

The value of Theorem 3 resides in its demonstration of the compatibility of the two measures rather than in the conclusion that an increase in the cost of search leads to an increase in liquidity.[18] In fact, this conclusion is somewhat counterintuitive. We offer two distinct arguments to diminish the disturbing aspects of this counterintuitive result.

First, there need be no connection between costly offers and infrequent offers. For instance, if the asset earns a large net rent and there is an out-of-pocket expense associated with obtaining an offer, then an increase in the frequency of offers could change the sign of the search cost from negative to positive.

Second, when properly viewed, this result raises nary an eyebrow in a labor market context. Theorem 3 asserts that the expected duration of unemployment is shorter for workers with high search costs: their reservation wage is lower; hence they more readily accept offers of employment. If worker B has a higher search cost than A, his (expected) period of unemployment is shorter. If A can signal his desirability more easily than B, then we anticipate that A will have a lower search cost and, therefore, a longer duration of unemployment. In the same vein, suppose C has the same search cost as A but C is a less able worker. In particular, suppose each offer received by C is δ less than the corresponding offer received by A. As shown in Theorem 4, the expected duration of unemployment is shorter for A.[19] Clearly, a long period of unemployment is not synonymous with an inferior employee: workers with short periods of unemployment may be the ones with good job prospects (as per Theorem 4) or impaired ability to signal their worth (as per Theorem 3). Similarly, unless all other aspects are identical, the less liquid asset need not be inferior.

THEOREM 4: *Let ξ and ξ_δ denote the reservation wage in the standard search paradigm when the offer distributions F and F_δ satisfy $F_\delta(t) = P(X+\delta \leq t) = P(X \leq t-\delta) = F(t-\delta)$ with $\delta > 0$; thus, each offer in the second problem is δ larger than each corresponding offer in the original problem. If $\beta < 1$, then $\xi_\delta < \xi + \delta$ and τ^* is (stochastically) larger than τ_δ^*. If $\beta = 1$, then $\xi_\delta = \xi + \delta$ and τ^* has the same distribution as τ_δ^*.*

PROOF:

Footnote 13 reveals that $H_\delta(t) = H(t-\delta)$. Suppose $\beta < 1$ and $\xi_\delta \geq \xi + \delta$. Substituting into the first-order condition (3) yields

$$H(\xi) - \frac{1-\beta}{\beta}\xi = c = H(\xi_\delta - \delta)$$
$$-((1-\beta)/\beta)\xi_\delta$$
$$< H(\xi) - ((1-\beta)/\beta)\xi,$$

[18] To ameliorate this counterintuitive result we might use the expected discounted cost of search in place of the expected time of sale as our operational measure of liquidity. However, as shown in ch. 3 of our forthcoming book, this operational measure also can increase with increases in c.

[19] The explanation of this phenomenon is implicit in the proof of Theorem 4: an upward shift in the mean of the offer distribution causes the opportunity cost of search to increase. On the other hand, if F_δ, the offer distribution for A, arises from a multiplicative shift with $\delta > 1$ (so $F_\delta(t) = F(t/\delta)$), then the impact of this shift is easily seen to be equivalent to a decrease in the search cost which induces a longer duration of unemployment. Thus, the direction of change in the duration of unemployment due to an increase in the worker's ability, reflected in his offer distribution, is sensitive to the form in which this increased ability is embodied.

as $H(\cdot)$ is a decreasing function and $\xi_\delta \geq \xi + \delta > \xi$ by assumption. This contradiction reveals that $\xi_\delta < \xi + \delta$. Consequently, $P(X_\delta \geq \xi_\delta) = P(X + \delta \geq \xi_\delta) = P(X \geq \xi_\delta - \delta) \geq P(X \geq \xi)$ so that τ^* is stochastically larger than τ_δ^*.

III. Liquidity as Flexibility[20]

An investment of funds today obviously reduces the range of options open to the agent tomorrow. This flexibility aspect of liquidity is implicit in Section I. Both the search paradigm, in general, and the liquidity search model, in particular, are based on the opportunity cost doctrine—the cost of holding one asset is the return that could be achieved by investing in the next best asset. Most of search theory employs models in which these opportunity costs are constant. However, if the agent's future opportunities differ from his current opportunities, he may eschew commitments that yield an inflexible or illiquid portfolio. In terms of our definition, he may avoid investments with large values of $E\tau^*$.

The key feature of the simple search model we propose is the existence of a single golden investment opportunity that becomes available at some future date. This feature of the investment environment causes the constant opportunity costs to vanish. By specifying the functional form of the offer distribution F, we are able to demonstrate the investor's preference for a more liquid/flexible current investment.

At time 0 the investor's endowment is ξ, all in the form of cash. A set of assets parameterized by $\lambda > 0$ is available for purchase at cost ξ, where asset λ has the associated offer distribution F_λ and search cost c_λ. For simplicity in presentation, assume that each asset's reservation price ξ_λ satisfies $c_\lambda = H_\lambda(\xi_\lambda)$ rather than (3), the discounted form of the first-order condition. The value ξ_λ of asset λ equals its purchase cost ξ, and each asset generates a constant flow of income at the rate $\alpha\xi$, where $\alpha > 0$ is the continuous-time interest rate (i.e., $e^{-\alpha}$ plays the role of β). Assume that the investor purchases exactly one asset at time 0 and consumes its income stream as it flows in. Thus, after selecting an asset, say λ_0, for purchase at time 0, the investor's endowment is λ_0 at each point in time.

At *one* point T in time in the future, the investor will be presented with the opportunity to invest in the golden asset. He will have enough time to solicit exactly one offer for the asset he owns in order to generate cash to purchase the golden asset. For the purposes of our analysis it does not matter if T is a random variable with known distribution, deterministic and specified in advance, or uncertain in the sense of Frank Knight (1921). What is important is that there be time to generate but one offer. (This will be the case if T is a geometric random variable.)

The golden asset is divisible and has constant returns to scale. Each dollar invested in the golden asset generates a constant flow of income at the rate r per unit time. As implied by its name, the income flow associated with an investment of ξ dollars in the golden asset exceeds $\alpha\xi$; that is, $r > \alpha$.

Let $G(\lambda)$ be the expected gain to search at time T when asset λ was purchased at time 0. If the observed value x of the offer X_λ were to be invested in the golden asset, the resulting cash flow would be rx. This investment is made only if $rx > \alpha\xi$; otherwise, the investor foregoes investment in the golden asset and retains asset λ. Hence, $\max\{rX_\lambda/\alpha; \xi\} - \xi_\lambda$ is the return to search, and $G(\lambda)$, the expected gain, is given by

$$
\begin{aligned}
(19)\quad G(\lambda) &= E\max\{rX_\lambda/\alpha; \xi\} - c_\lambda - \xi \\
&= \frac{r}{\alpha}E\max\{X_\lambda; \alpha\xi/r\} - c_\lambda - \xi \\
&= \frac{r}{\alpha}\left\{\frac{\alpha\xi}{r}F_\lambda\left(\frac{\alpha\xi}{r}\right) + \int_{\alpha\xi/r}^{\infty} x\,dF_\lambda(x)\right\} - c_\lambda - \xi \\
&= \frac{r}{\alpha}\left\{\frac{\alpha\xi}{r} + \int_{\alpha\xi/r}^{\infty}\left(x - \frac{\alpha\xi}{r}\right)dF_\lambda(x)\right\} - c_\lambda - \xi \\
&= (r/\alpha)H_\lambda(\alpha\xi/r) - c_\lambda,
\end{aligned}
$$

[20] The discussion in this section is in the spirit of Albert Hart (1942), John Hicks (1974), and, especially, Robert Jones and Joseph Ostroy (1984).

where H_λ is the usual H function associated with the offer distribution F_λ.

Recalling that $r > \alpha$ and H_λ is a nonincreasing function, we observe that

$$G(\lambda) > H_\lambda(\alpha\xi/r) - c_\lambda \geq H_\lambda(\xi) - c_\lambda = 0,$$

so search is profitable for each asset λ.

Most of us believe that liquidity is sought to provide flexibility—be it to meet special consumption exigencies or special (golden) investment opportunities. In order to test the validity of this conventional wisdom in the context of our simple search model with a golden opportunity, we shall assume further that the offer distributions F_λ are exponential: $F_\lambda(x) = 1 - e^{-\lambda x}$. Consequently, $H_\lambda(x) = e^{-\lambda x}/\lambda$ and $c_\lambda = e^{-\lambda\xi}/\lambda$. With exponential offers we have

$$E\tau_\lambda^* = [1 - F_\lambda(\xi_\lambda)]^{-1} = 1/e^{-\lambda\xi}$$

so $E\tau_\lambda^*$ increases with λ: liquidity decreases as λ increases. From (19) and F_λ exponential we obtain

$$G(\lambda) = \frac{r}{\alpha}\frac{1}{\lambda}e^{-\lambda\xi\alpha/r} - e^{-\lambda\xi}/\lambda$$

and then, because $r/\alpha > 1$,

$$\lambda^2\frac{dG(\lambda)}{d\lambda} = \frac{r}{\alpha}\left\{-\frac{\lambda\xi\alpha}{r}e^{-\lambda\xi\alpha/r} - e^{-\lambda\xi\alpha/r}\right\}$$
$$-\{-\lambda\xi e^{-\lambda\xi} - e^{-\lambda\xi}\} = (\lambda\xi + 1)e^{-\lambda\xi}$$
$$-(\lambda\xi + (r/\alpha))e^{-\lambda\xi\alpha/r} < 0.$$

Thus, the expected gain to search also decreases with λ. More formally, we have established the following theorem.

THEOREM 5: *In the simple search model (with timeless search), a set of initially available assets with exponential offer distributions, and a subsequently available golden asset, the risk-neutral investor improves his expected return by selecting a more liquid asset for his initial investment.*

As conjectured, the choice of a more liquid initial investment does indeed enhance and facilitate the investor's ability to profit from the arrival of the golden opportunity. We view this result as providing an endorsement of John Hicks' remark that "by holding the imperfectly liquid asset the holder has narrowed the band of opportunities which may be open to him..." (p. 43–44). By choosing a less liquid asset, the investor has more nearly "locked himself in." In particular, note that the probability $(1 - \exp\{-\lambda\xi\alpha/r\})$ of not investing in the golden asset—being locked in—increases as the asset's liquidity decreases.

Although ours is not an equilibrium analysis and the extent to which this result is robust remains to be investigated, our analysis of the simple search model with a golden asset is tantalizingly suggestive of a broad range of macroeconomic phenomena that might successfully be treated with our approach to liquidity and liquidity preference. Though it may be somewhat grandiose, we conclude by paraphrasing Robert Jones and Joseph Ostroy's (p. 26) remarks concerning the profession's long but spotty treatment of flexibility: the difficulty of providing a definition of liquidity in such a way as to have universal application and the difficulty of obtaining formal results without model-specific qualifications may account for the very limited role accorded to liquidity in contemporary theory. However, the connection between liquidity and (risky) investment decisions is too compelling to be ignored.

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