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Liquidity Premia and Transaction Costs

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ABSTRACT

Standard literature concludes that transaction costs only have a *second-order* effect on liquidity premia. We show that this conclusion depends crucially on the assumption of a constant investment opportunity set. In a regime-switching model in which the investment opportunity set varies over time, we explicitly characterize the optimal consumption and investment strategy. In contrast to the standard literature, we find that transaction costs can have a *first-order* effect on liquidity premia. However, with reasonably calibrated parameters, the presence of transaction costs still cannot fully explain the equity premium puzzle.

TRANSACTION COSTS ARE PREVALENT in almost all financial markets. Extensive research has been conducted on the optimal consumption and investment policy in the presence of transaction costs since the seminal work of Constantinides (1979, 1986). The presence of transaction costs significantly changes optimal consumption and investment strategies. For example, in the presence of transaction costs, continuous trade incurs infinite transaction costs, and thus even a small transaction cost can dramatically decrease the frequency of trade. However, most studies find that the utility loss due to the presence of transaction costs is small. In particular, Constantinides (1986) finds that the liquidity premium (i.e., the maximum expected return an investor is willing to exchange for zero transaction cost) is small relative to the transaction cost, even for a suboptimal trading strategy, and hence concludes that transaction costs only have a second-order effect for asset pricing. Indeed, in this framework it seems unlikely that transaction costs can have an important role in explaining the cross-sectional patterns of expected returns, the equity premium puzzle, or the small stock risk premia. This finding contrasts with many empirical studies that highlight the importance of transaction costs or related measures such as turnover in influencing the cross-sectional patterns of expected returns.[1]

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[1] See, for example, Eleswarapu (1997) and Brennan, Chordia, and Subrahmanyam (1998).

One of the common assumptions of the existing literature on optimal consumption and investment with transaction costs is that the investment opportunity set stays constant. For example, Constantinides (1986), Vayanos (1998), Liu and Loewenstein (2002), and Liu (2004) all assume that the expected stock return, the return volatility, and the liquidity (transaction costs) are constant throughout the investment horizon. Empirical research, however, documents a great deal of evidence that is inconsistent with the constant investment opportunity set hypothesis. For example, Schwert (1989) and Campbell and Hentschel (1992) conclude that the volatilities of stock returns vary substantially over time. Campbell (1991) and Lewellen (2004) find that expected returns on equities also change over time. In addition, the existence of large liquidity shocks (e.g., the 1987 crash, the 1998 Long Term Capital Management event) suggests that transaction costs can also vary over time.

Taking into account the stochastic nature of the investment opportunity set may qualitatively change the conclusion in the standard literature (e.g., Constantinides (1986)) that transaction costs only have a second-order effect. Intuitively, the presence of transaction costs reduces an investor's welfare through two mechanisms, namely, the transaction cost payment and the deviation from the optimal strategy in the no-transaction-cost case. If investment opportunity set parameters change over time, the optimal stock investment target in the no-transaction-cost case should also change over time. Thus, as market conditions change over time, compared to the constant-investment-opportunity-set case, an investor should rebalance more often to avoid being too far away from the target if the transaction cost rate is small. The relative impact of the transaction cost should therefore increase, mainly because of a greater transaction cost payment resulting from more frequent trading. In addition, if liquidity can vary dramatically over time, an investor should significantly change his trading strategy to minimize the loss from a bad liquidity shock, and thus the impact of transaction costs should also increase significantly, mainly due to the substantial deviation from the optimal strategy in the no-transaction-cost case.

To quantify this intuition, we build a model similar to that of Constantinides (1986) and Davis and Norman (1990), but with regime switching for fundamental parameters. Specifically, we consider the optimal consumption and investment problem for a small investor (i.e., one with no price impact) who derives constant relative risk aversion (CRRA) utility from intertemporal consumption and bequest.[2] The investor can invest in one risky asset and one risk-free asset. In contrast to most of the existing literature, we assume that the investment opportunity set is not constant and that there are two regimes ("Bull" and "bear") with different fundamental parameters such as expected return, volatility, and liquidity. One regime switches to the other regime at the first jump time of a market-independent but possibly regime-dependent Poisson process.[3]

[2] The bequest can also be interpreted as an exogenous need for cash.

[3] The investor we consider in this paper can be an institutional investor who does not have any price impact and who updates the estimates of fundamental parameters from time to time.

We explicitly characterize the solution to the investor's problem in a general setting.[4] Using parameter estimates from Ang and Bekaert (2002), our extensive numerical analysis demonstrates that in contrast to the standard conclusion that transaction costs only have a *second-order* effect, transaction costs can have a *first-order* effect if the investment opportunity set varies over time. Specifically, the liquidity premium to transaction cost (LPTC) ratio could be well above one. The consideration of a stochastic investment opportunity set makes this ratio typically more than 4 times and in many cases 10 times higher than what Constantinides finds, even for the optimal policies.[5] In addition, we find that the LPTC ratio increases as the Bull regime investment opportunity set becomes more and more attractive than the bear regime investment opportunity set. Intuitively, as the difference in, for example, expected returns increases, an investor invests more (less) in the risky asset in the Bull (bear) regime, revises the portfolio more dramatically, and thus incurs higher transaction costs. Our analysis therefore suggests that consideration of a time-varying investment opportunity set is an important factor in explaining the high volume of trade and the relation between transaction costs and the cross-sectional patterns of expected returns. However, our analysis also suggests that even when the investment opportunity set is time-varying, the magnitude of liquidity premia cannot be large enough to fully explain the equity premium puzzle (see Mehra and Prescott (1985)).

Unlike the no-transaction-cost case, smoothing of trading strategies across regimes is optimal in the presence of transaction costs. Without transaction costs, the optimal investment policy in one regime is independent of parameters in the other regime. In contrast, our analysis shows that in the presence of transaction costs, an investor optimally responds to changes in one regime by altering investment behavior in *both* regimes. For example, as the transaction cost in one regime increases (all else equal), the investor trades less in this regime and trades more in the other regime because the latter regime becomes relatively cheaper to trade in. This finding of cross-regime smoothing suggests that the presence of transaction costs can lead to patterns of optimal investment behavior that would seem suboptimal if only the current market conditions were considered.

As far as we know, in the literature with regime switching, this paper is the first to provide a verification theorem for a candidate solution, explicit bounds on the no-transaction regions, and the steady-state distribution function for the portfolio holding.

Our theoretical analysis suggests that extending the two-regime model to a multi-regime setting is straightforward but requires significantly more intensive computation. However, the qualitative results we obtain in the paper

[4] We also derive closed-form solutions up to some constants in special cases such as the no-intertemporal-consumption case. Results are not reported in this paper to save space, but are available from the authors.

[5] Recall that Constantinides (1986) uses a suboptimal consumption policy to emphasize how small the liquidity premium is.

should stay the same. For example, as long as the transaction cost is small relative to the changes in the optimal portfolio target across regimes, we expect an investor to optimally incur transaction costs when a regime switches. This fundamental intuition also applies to the case in which the investment opportunity set depends on a continuous state variable. Hence, jumps in the fundamental parameters in the financial market are not critical for our results.

Our paper is related to a large body of literature that addresses the optimal transaction policy for an investor facing transaction costs. Dumas and Luciano (1991) derive a closed-form solution for an investor with a long-term growth objective. Schroder (1995) uses a numerical method to solve for the optimal transaction policy in the presence of fixed costs. Vayanos (1998) derives the asset pricing impact of transaction costs in an overlapping generations framework. Leland (2000) examines a multiasset investment fund that is subject to transaction costs and capital gains taxes. He develops a relatively simple numerical procedure to compute the multidimensional no-transaction region. Lo, Mamaysky, and Wang (2004) study the effect of fixed transaction costs on asset prices and find that even small fixed costs can give rise to significant illiquidity discounts on asset prices. Shreve and Soner (1994) provide important theoretical results and an analysis of the liquidity premium. All these papers assume that the investment opportunity set is constant. Our analysis suggests that a stochastic investment opportunity set is an important consideration in generating both a higher volume of trade and a greater impact of transaction costs.

While some previous results characterize optimal policies in more general models (e.g., Koo (1992) and Loewenstein (2000)), they do not lead to transparent statements concerning liquidity premia and transaction frequency. Lynch and Balduzzi (2000) examine the impact of stock return predictability and transaction costs on portfolio rebalancing rules by discretizing both time and states to obtain a numerical approximation. However, due to some nonstandard modelling choices, it is difficult to interpret their results. For example, for some of their analysis they assume that consumption is financed by *costlessly* liquidating the stock and the riskless asset in proportions given by the pre-rebalancing portfolio weights. As a result, the post-consumption but pre-rebalancing portfolio weights are unchanged. When the investor rebalances these portfolio weights, a transaction cost proportional to post-consumption wealth is then incurred and paid by costlessly liquidating the stock and the riskless asset in proportions given by the post-rebalancing portfolio weights. Thus, an investor who does not want to change the dollar amount invested in the stock would first need to sell stock to finance consumption and then incur a transaction cost to buy back the stock. In contrast, in our model and the standard literature (e.g., Constantinides (1986)), the transaction cost payment and consumption withdrawals are financed using an optimal trading strategy. In addition, we provide a transparent and tractable model in which we can explicitly compute bounds on the transaction boundaries, liquidity premia, trading frequency, and expected lifetime transaction cost expenditures.

The rest of the paper is organized as follows. Section I presents the model with transaction costs and regime switching and provides characterizations of the solution. Numerical and graphical analysis is presented in Section II. Section III closes the paper. All of the proofs are in the Appendix.

I. Optimal Consumption and Investment

A. The Basic Model

Throughout this paper we assume a probability space $(\Omega, \mathcal{F}, P)$, where uncertainty and the filtration $\{\mathcal{F}_t\}$ are generated by a standard one-dimensional Brownian motion w and two independent Poisson processes representing the regime-switching risk and the mortality risk. We assume that all stochastic processes in this paper are adapted.

An investor can trade two assets, a money market account ("the bond") and a risky investment ("the stock"). There are two regimes, "Bull" (regime B) and "bear" (regime b). The fundamental parameters in the financial market may be regime dependent. We assume that regime i switches into regime j at the first jump time of an independent Poisson process with intensity λ_i, for $i, j \in \{B, b\}$. In regime i, the risk-free interest rate is r_i, and the investor can buy the stock at the ask $S_t^A = (1+\theta_i)S_t$ or sell the stock at the bid $S_t^B = (1-\alpha_i)S_t$, where $\theta_i \geq 0$ and $0 \leq \alpha_i \leq 1$ represent the proportional transaction cost rates and S_t satisfies

$$\frac{dS_t}{S_t} = \mu_i\, dt + \sigma_i\, dw_t, \tag{1}$$

where all parameters are positive constants and $\mu_i > r_i$.

In regime $i \in \{B, b\}$, when $\theta_i + \alpha_i > 0$ the above model gives rise to equations governing the evolution of the dollar amount invested in the bond, x_t, and the dollar amount invested in the stock, y_t:

$$dx_t = r_i x_t\, dt - (1+\theta_i)\, dI_t + (1-\alpha_i)\, dD_t - c_t\, dt, \tag{2}$$

$$dy_t = \mu_i\, y_t\, dt + \sigma_i\, y_t\, dw_t + dI_t - dD_t, \tag{3}$$

where c_t is the consumption rate and the processes D and I represent the cumulative dollar amount of sales and purchases of the stock, respectively. These processes are nondecreasing and right-continuous adapted processes with $D(0) = I(0) = 0$. Let x_0 and y_0 be the given initial positions in the bond and in the stock, respectively. Let $\Theta(x_0, y_0)$ denote the set of admissible trading strategies (c, D, I) such that (2) and (3) are satisfied, $c_t \geq 0$, $\int_0^t c_s\, ds < \infty$ for all t, and the investor is always solvent, that is,[6]

$$x_t + (1-\alpha_i)y_t \geq 0, \quad \forall t \geq 0 \text{ and } i \in \{B, b\}. \tag{4}$$

Similar to Constantinides (1986), we consider a CRRA investor who derives von Neumann–Morgenstern time additive utility from intertemporal consumption c with weight $1-k$ and bequest at death with weight k, with a time discount

[6] The assumption that $\mu_i > r_i$ implies that the investor never shorts the stock and thus that $y_t \geq 0$.

rate of ρ. For simplicity, we assume that death occurs at the first jump time of an independent Poisson process with intensity δ. Thus, after integrating out the mortality risk, the investor solves

$$\sup_{(c,D,I)\in\Theta(x_0,y_0)} E\left[\int_0^\infty e^{-(\rho+\delta)t}\left((1-k)\frac{c_t^{1-\gamma}}{1-\gamma}+k\delta\frac{(x_t+(1-\alpha_i)y_t)^{1-\gamma}}{1-\gamma}\right)dt\right]. \quad (5)$$

B. Optimal Policies with No Transaction Costs

In this section, we solve the optimal consumption and portfolio selection problem in the absence of transaction costs, that is, $\theta_i = \alpha_i = 0$, under the regime-switching model presented in the previous section. The results in this section serve as a benchmark for the subsequent analysis.

In this case, the cumulative purchases and sales of the stock can be of infinite variation. Let τ_i be the first jump time since the beginning of regime i. The investor's problem in regime $i \in \{B, b\}$ can be rewritten as

$$V_i(W) = \sup_{\{y_t:t\geq 0\}} E\left[\int_0^{\tau_i} e^{-(\rho+\delta)t}\left((1-k)\frac{c_t^{1-\gamma}}{1-\gamma}+k\delta\frac{W_t^{1-\gamma}}{1-\gamma}\right)dt + e^{-(\rho+\delta)\tau_i}V_j(W_{\tau_i})\right] \quad (6)$$

subject to

$$dW_t = r_i W_t\,dt + (\mu_i - r_i)y_t\,dt + \sigma_i y_t\,dw_t, \quad (7)$$

where $W_t \equiv x_t + y_t \geq 0$ and $V_j(W)$ is the value function in regime $j \neq$ i.

Under regularity conditions on V_i and V_j, the Hamilton–Jacobi–Bellman (HJB) equations take the form

$$\sup_{(c_i,y_i)}\left\{\frac{1}{2}\sigma_i^2 y_i^2 V_{iWW} + r_i W V_{iW} + (\mu_i - r_i)y_i V_{iW} - c_i V_{iW} \right.$$
$$\left. -(\rho+\delta+\lambda_i)V_i + \lambda_i V_j + (1-k)\frac{c_i^{1-\gamma}}{1-\gamma} + k\delta\frac{W^{1-\gamma}}{1-\gamma}\right\} = 0, \quad (8)$$

where $i, j \in \{B, b\}, i \neq j$. We conjecture

$$V_i(W) = M_i\frac{W^{1-\gamma}}{1-\gamma}, \text{ for a constant } M_i > 0, \quad i \in \{B, b\}. \quad (9)$$

By the first-order conditions, we have

$$c_i = \left(\frac{V_{iW}}{1-k}\right)^{-\frac{1}{\gamma}} \quad \text{and} \quad y_i = -\frac{(\mu_i - r_i)V_{iW}}{\sigma_i^2 V_{iWW}}.$$

Then, plugging (9) into (8), we have that M_i and M_j satisfy the system of equations

$$-(\eta_i + \lambda_i)M_i + \gamma(1-k)^{1/\gamma} M_i^{1-1/\gamma} + \lambda_i M_j + k\delta = 0, \tag{10}$$

where

$$\eta_i = \rho + \delta - (1-\gamma)\left(r_i + \frac{\kappa_i^2}{2\gamma}\right), \quad \kappa_i = \frac{\mu_i - r}{\sigma_i}, \quad i, j \in \{B, b\}, \quad i \neq j. \tag{11}$$

To ensure the existence of an optimal solution, we adopt the following assumption, similar to that of Merton (1971):

ASSUMPTION 1: $\eta_i > 0, i \in \{B, b\}$.

LEMMA 1: *Under Assumption 1, there is a unique solution* (M_B, M_b) *to (10).*

Proof: See the Appendix.

The following verification theorem shows that, indeed, our conjecture is correct.

THEOREM 1: *Under Assumption 1 for regime* $i \in \{B, b\}, V_i$ *defined in (9) is the value function defined in (6). In addition, the optimal consumption policy is* $c_i^* = (\frac{M_i}{1-k})^{-\frac{1}{\gamma}} W$ *and the optimal fraction of wealth invested in the stock is* $\pi_i^* = \frac{\mu_i - r_i}{\gamma\sigma_i^2}$.

Proof: This follows from an argument similar to (although simpler than) that presented in the proof of Theorem 2. See the Appendix. Q.E.D.

Theorem 1 implies that both the optimal consumption and the optimal dollar amount invested in the stock are constant fractions of the investor's wealth in each regime. Even though the investor smooths consumption across regimes (as reflected in the fact that M_i and M_j are jointly determined by equation (10)), the optimal investment policy is myopic in the sense that π_i^* only depends on the current regime parameters. This follows from the fact that the risk of regime switching is unhedgeable using the existing securities and that the investor can rebalance at regime-switching time without any transaction costs. As we show later, the presence of transaction costs makes the optimal trading strategy no longer myopic.

C. Optimal Policies with Transaction Costs

Now suppose there are transaction costs in both regimes, that is, $\theta_i + \alpha_i > 0, i \in \{B, b\}$. Using similar notation to that in the previous section, we can rewrite the investor's problem as

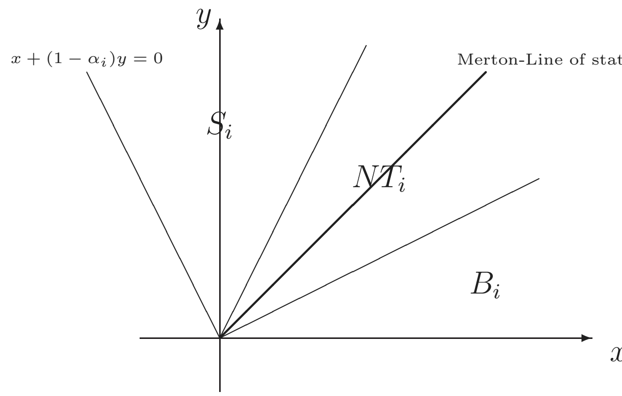

Figure 1. The solvency region splits into the buy region B_i, sell region S_i, and no-trade region NT_i.

$$v_i(x, y) = \sup_{(c,D,I)} E\left[\int_0^{\tau_i} e^{-(\rho+\delta)t}\left((1-k)\frac{c_t^{1-\gamma}}{1-\gamma} + k\delta\frac{(x_t + (1-\alpha_i)y_t)^{1-\gamma}}{1-\gamma}\right)dt + e^{-(\rho+\delta)\tau_i} v_j(x_{\tau_i}, y_{\tau_i})\right], \quad (12)$$

subject to (2) through (4), where $v_i(x, y)$ denotes the value function in regime i.

It can be easily verified that the value functions v_B and v_b are concave and homogeneous of degree $1-\gamma$ in (x, y) (see, for example, Constantinides (1979) and Fleming and Soner (1993), Lemma VIII.3.2). As in Davis and Norman (1990) and Liu and Loewenstein (2002), the solvency region $\mathcal{S}_i$ splits into three regions: A "no-trading" (NT_i) region, a "buy" (B_i) region, and a "sell" (S_i) region. The homogeneity of v_i implies that the transaction boundaries are straight lines in the (x, y) plane (see Figure 1). In addition, there exists an interval $[\underline{z}_i, \bar{z}_i]$ such that in regime i, the investor trades only the minimum amount to keep the bond to stock ratio,

$$z \equiv \frac{x}{y},$$

inside the interval. We depict this analysis in Figures 2 and 3. Figure 2 shows the solvency region when the two no-transaction regions (NT_B and NT_b) are separated. Intuitively, this case occurs when the difference between the two regimes is large (e.g., the expected return on the stock in the Bull regime is sufficiently greater than that in the bear regime), the regime transition probability is low, and the transaction cost is relatively small. Otherwise, the no-transaction regions can be overlapped, as shown in Figure 3.

The optimal transaction policy above implies that the HJB equation takes the following form:

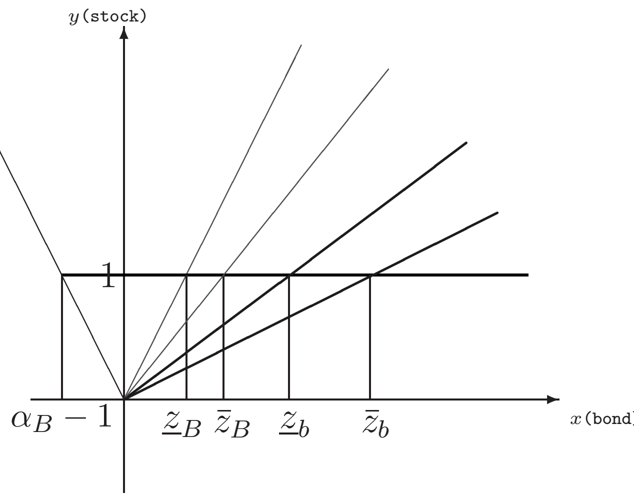

Figure 2. The solvency region with separated no-trade regions.

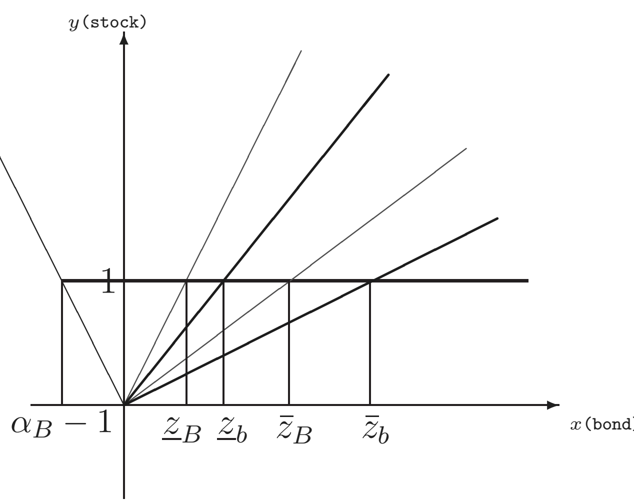

Figure 3. The solvency region with overlapped no-trade regions.

$$\frac{1}{2}\sigma_i^2 y^2 v_{iyy} + r_i x v_{ix} + \mu_i y v_{iy} + \frac{\gamma(1-k)^{1/\gamma}}{1-\gamma} v_{ix}^{1-1/\gamma}$$

$$-(\rho+\delta)v_i + k\delta\frac{(x+(1-\alpha_i)y)^{1-\gamma}}{1-\gamma} + \lambda_i(v_j - v_i) = 0, \tag{13}$$

for $j \neq i$ in NT_i. In the sell region, the investor transacts immediately to the sell boundary. Therefore,

$$v_i(x, y) = A_i \frac{(x + (1 - \alpha_i) y)^{1-\gamma}}{1 - \gamma}, \tag{14}$$

where A_i is a positive constant to be determined. Similarly, in the buy region, the investor transacts immediately to the buy boundary. Therefore,

$$v_i(x, y) = B_i \frac{(x + (1 + \theta_i) y)^{1-\gamma}}{1 - \gamma}, \tag{15}$$

where B_i is also a positive constant to be determined.

By the homogeneity of the value functions, there exists a function $\psi_i : (\alpha_i : -1, \infty) \to \mathbb{R}$ in regime i satisfying

$$v_i(x, y) \equiv y^{1-\gamma} \psi_i\left(\frac{x}{y}\right). \tag{16}$$

This implies that

$$\psi_i(z) = \begin{cases} A_i \dfrac{(z + (1 - \alpha_i))^{1-\gamma}}{1 - \gamma} & z < \underline{z}_i \\ B_i \dfrac{(z + (1 + \theta_i))^{1-\gamma}}{1 - \gamma} & z > \bar{z}_i. \end{cases} \tag{17}$$

Using equation (16) and the ratio z, we can simplify the partial differential equation (PDE) to obtain the following ordinary differential equation (ODE) in NT_i:

$$\begin{aligned} &\beta_2^i z^2 \psi_i''(z) + \beta_1^i z \psi_i'(z) + \beta_0^i \psi_i(z) + \frac{\gamma(1 - k)^{1/\gamma}}{1 - \gamma} \psi_i'(z)^{1-1/\gamma} \\ &\quad + k\delta \frac{(z + (1 - \alpha_i))^{1-\gamma}}{1 - \gamma} + \lambda_i \psi_j(z) = 0, \end{aligned} \tag{18}$$

$i \neq j$ for $z \in (\underline{z}_i, \bar{z}_i)$, where $\beta_2^i = \frac{1}{2}\sigma_i^2, \beta_1^i = \gamma\sigma_i^2 - (\mu_i - r_i)$, and $\beta_0^i = \frac{1}{2}\sigma_i^2\gamma(\gamma - 1) + (1 - \gamma)\mu_i - \rho - \delta - \lambda_i$. We then have the following verification theorem:[7]

THEOREM 2: *For $i, j \in \{B, b\}$ and $j \neq i$, suppose we have concave, increasing, and homothetic $C^{2,2}$ solutions to*

[7] The conditions in Theorem 2 assume that the value function has the required smoothness and that the investor always optimally buys some stock. The smoothness issue is examined in Shreve and Soner (1994) for the case $k = 0$ and in Liu and Loewenstein (2002) for $k = 1$ in models with no regime shifting. The main difficulty encountered is when the $x = 0$ axis is contained in the no-transaction region and the value function might not have the required smoothness on the $x = 0$ axis. It is also known that in the case $k = 1$ a short horizon might lead an investor to never buy stock (see Liu and Loewenstein (2002)). Accommodating both of these possibilities does not present much difficulty although it requires some additional technical arguments. For our purposes Theorem 2 is sufficient.

$$\frac{1}{2}\sigma_i^2 y^2 v_{iyy} + r_i x v_{ix} + \mu_i y v_{iy} + \frac{\gamma(1-k)^{1/\gamma}}{1-\gamma} v_{ix}^{1-\frac{1}{\gamma}} - (\rho+\delta)v_i$$

$$+ k\delta \frac{(x+(1-\alpha_i)y)^{1-\gamma}}{1-\gamma} + \lambda_i(v_j - v_i) \le 0, \tag{19}$$

with equality for $\frac{x}{y} \in [\underline{z}_i, \bar{z}_i]$ *and with* $\alpha_i - 1 < \underline{z}_i < \bar{z}_i < \infty$, *which satisfy*

$$(1+\theta_i)v_{ix} \ge v_{iy} \tag{20}$$

with equality for $\frac{x}{y} > \bar{z}_i$ *and*

$$(1-\alpha_i)v_{ix} \le v_{iy} \tag{21}$$

with equality for $\frac{x}{y} < \underline{z}_i$. *Then* v_i *is the value function, the optimal consumption is given by*

$$c^* = \left(\frac{v_{ix}}{1-k}\right)^{-\frac{1}{\gamma}}, \tag{22}$$

and the optimal transaction policy is to transact the minimum amount to keep $\frac{x}{y}$ *between* $\underline{z}_i$ *and* $\bar{z}_i$.

Proof: See the Appendix.

Since functions ψ_B and ψ_b need to be solved simultaneously through the coupled ODEs for the two regimes, the value function and the optimal trading strategy in one regime are affected by the investment opportunity set in the *other* regime. Therefore, in the presence of transaction costs, the optimal trading strategy is no longer myopic, and the investor smooths not only consumption but also trading strategies across regimes.

The following proposition provides some bounds on the value functions and on the optimal no-transaction boundaries for the leading cases we analyze later. Similar bounds can be developed for a more general specification of our model. These bounds facilitate numerical computation by suggesting useful starting values for the boundaries. They also provide useful information on the width of the no-transaction regions. In addition, the proof suggests that similar bounds can be obtained using the same approach when there are more than two regimes.

PROPOSITION 1: *Suppose* $\gamma > 1, k = \delta = 0, r_B = r_b = r, \theta_B = \theta_b = \theta$, *and* $\alpha_B = \alpha_b = \alpha$. *Then assuming we have a solution satisfying the conditions of Theorem 2, we have the following bounds.*

(i) *For any* $1-\alpha \le \zeta \le 1+\theta$ *and for* $i \in \{b, B\}$,

$$\frac{C}{1-\gamma}(x+(1-\alpha)y)^{1-\gamma} \le v_i(x,y) \le \frac{M_i}{1-\gamma}(x+\zeta y)^{1-\gamma}, \tag{23}$$

where $C = (\frac{\rho-(1-\gamma)r}{\gamma})^{-\gamma}$ *and* M_i *is as given in Lemma 1.*

(ii) *If* $\sigma_i^2 \le \sigma_j^2$ *and* $\mu_i \ge \mu_j$, *then* $v_i(x,y) \ge v_j(x,y)$ *and*

$$\underline{z}_i \ge (1-\alpha)\left(\frac{\gamma\sigma_i^2}{2(\mu_i - r)} - 1\right). \tag{24}$$

(iii) *For* $i, j \in \{b, B\}, i \neq j$, *if* $\underline{z}_i \le \underline{z}_j$, *then*

$$\underline{z}_i \le (1-\alpha)\left(\frac{\gamma\sigma_i^2}{\mu_i - r} - 1\right). \tag{25}$$

(iv) *For* $i, j \in \{b, B\}, i \neq j$, *if* $\bar{z}_i \ge \bar{z}_j$, *then*

$$\bar{z}_i \ge (1+\theta)\left(\frac{\gamma\sigma_i^2}{\mu_i - r} - 1\right). \tag{26}$$

Proof: See the Appendix.

II. Liquidity Premium, Trading Strategy, and Transaction Costs

Constantinides (1986) introduces the concept of liquidity premia to measure the effect of transaction costs on expected returns. He defines the liquidity premium to be the maximum expected return an investor is willing to exchange for zero transaction cost. He concludes that transaction costs have only a second-order effect on investors' utility, that is, the LPTC ratio is an order of magnitude smaller than one. However, he obtains this result under the assumption of a constant investment opportunity set, which tends to decrease the effect of transaction costs because the investor trades infrequently. In contrast to that of Constantinides (1986), the investment opportunity set in our model changes stochastically. This stochastic opportunity set may induce an investor to trade more frequently or to alter trading strategies more significantly and thus may produce a first-order effect of the transaction cost.

Following Constantinides (1986), we define the liquidity premium to be the maximum expected return that an investor is willing to give up in both regimes in exchange for zero transaction costs.

DEFINITION 1: *Let* $\Delta_i(x_0, y_0)$ *be the liquidity premium in regime* i *at* (x_0, y_0) *for* $i \in \{B, b\}$. *Then* Δ_i *is such that*

$$v_i(x_0,\, y_0) = M_i\frac{(x_0 + y_0)^{1-\gamma}}{1-\gamma}, \tag{27}$$

where M_i is the unique solution of (10) with

$$\kappa_i = \frac{(\mu_i - \Delta_i - r_i)^+}{\sigma_i}, \quad \kappa_j = \frac{(\mu_j - \Delta_i - r_j)^+}{\sigma_j}, \quad \text{and} \quad j \neq i. \tag{28}$$

In this measure, we take the positive parts because an investor's utility is minimized when the risk premium is zero.

By the homogeneity of the value function, the liquidity premium only depends on the initial fraction $z_0 = \frac{x_0}{y_0}$. We use the steady-state distribution of z_t to compute the average liquidity premium.[8] This can be interpreted as a cross-sectional liquidity premium average for different investors. Let $\Delta_i(x_0, y_0)$ be the liquidity premium in regime i at (x_0, y_0) for $i \in \{B, b\}$, as defined in Definition 1. Then the average liquidity premium $\bar{\Delta}_i$ in regime i for $i \in \{B, b\}$ is equal to

$$\bar{\Delta}_i = \int_{\underline{z}_i}^{\bar{z}_i} \Delta_i(z, 1) \frac{\lambda_B + \lambda_b}{\lambda_j} \phi_i(z)\, dz, \tag{29}$$

where $j \neq i$ and $\phi_i(z)$ is the steady-state density of z in regime i, which is characterized in Proposition 2 in the Appendix. The average liquidity premium across both regimes is equal to

$$\bar{\Delta} = \frac{\lambda_b}{\lambda_B + \lambda_b} \bar{\Delta}_B + \frac{\lambda_B}{\lambda_B + \lambda_b} \bar{\Delta}_b. \tag{30}$$

In the following analysis we use parameter values similar to those estimated for the U.S. equity market by Ang and Bekaert (2002) for the base case: $\mu_B = 0.1394$, $\mu_b = 0.1394$, $\sigma_B = 0.1313$, $\sigma_b = 0.2600$, $\lambda_B = 0.2353$, $\lambda_b = 1.7391$, $\rho = 0.1$, $\gamma = 2.0$, $r_B = r_b = 0.05$, $\theta_B = \alpha_B = 0.01$, $\theta_b = \alpha_b = 0.01$, $k = 0$, and $\delta = 0$. These parameter values reflect the fact that volatility is significantly higher in the bear market regime than in the Bull market regime, and that the null hypothesis that the mean returns are the same across regimes cannot be rejected by the data.[9] As in Constantinides (1986), we set $k = 0$ and $\delta = 0$ so that the utility is derived only from consumption and the investor has an infinite horizon.[10] The Poisson jump intensities are chosen to be consistent with the observation that a Bull regime typically lasts longer (on average 4.25 years) than a bear regime (on average 0.58 years). In our calibrated model, an investor can infer the state of the current regime from observing stock return volatility, which is the only parameter that varies across regimes in our benchmark case.

[8] We have also computed liquidity premia at the Merton line and on $y_0 = 0$ as in Constantinides (1986) and obtained similar results.

[9] To address any concerns that the estimation for the expected return μ_b in the bear regime might be imprecise, we vary μ_b in Figure 9.

[10] It is well known that the investor's horizon can affect liquidity premia (see Liu and Loewenstein (2002), for example). We focus on an infinite horizon case to approximate the case with a long horizon.

A. Optimal Trading Strategies

In the absence of transaction costs, it is straightforward to calculate the optimal trading strategies by solving the system of equations given by (10). As discussed before, the optimal portfolio strategies are myopic: The investor optimally maintains a ratio of stock investment to wealth of 2.5923 in the Bull regime and of 0.6612 in the bear regime. The optimal consumption across regimes displays cross-regime smoothing with a consumption-to-wealth ratio of 12.793% in the Bull market and 12.530% in the bear market.[11]

In the presence of transaction costs, we use an iterative method to solve the coupled free boundary value problem described in equations (17) and (18). Table I depicts the optimal trading strategies for the case with transaction costs, as well as how these strategies change with parameter values. With transaction costs, the optimal consumption to liquidated wealth ratios are around 12.5%. For the base case, we also see cross-regime smoothing in the portfolio holdings. The investor optimally maintains the ratio of the dollar amount invested in the stock to liquidated wealth between 1.9469 and 2.8934 in the Bull regime and between 0.6005 and 0.9183 in the bear regime. Compared with the optimal trading policy in the bear regime with no transaction costs, the investor tends to hold more stock in the bear regime to reduce transaction cost payments upon regime switching because on average the duration of the bear regime is relatively short. This cross-regime smoothing is indicated in Figure 4, which plots the behavior of the bear regime transaction boundaries against changes in transaction costs in the Bull market. The bear market optimal portfolio policy is clearly sensitive to the Bull market transaction costs. In fact, when the Bull market transaction cost is relatively high compared to the bear market transaction cost, the optimal transaction boundary in the bear market is entirely above the myopic policy (the "Merton Line") in the no-transaction-cost case. This possibility arises because it is cheaper to buy the asset in the bear regime, which is expected to last a short time, and enjoy the benefits in the Bull regime.

Table I provides further information on how the optimal policies change with parameter values. While the general comparative statics follow intuitively, changes in the Bull regime parameter values have a relatively greater impact on the optimal policies than those in the bear regime because the Bull regime lasts longer on average.

Table I also displays the liquidity premia averaged across regimes using the steady-state distribution of portfolio holdings. Regime switching increases the liquidity premia by four to five times those reported in Constantinides (1986). We examine this finding in more detail in the subsequent analysis.

[11] In a model with no regime switching, the optimal consumption to wealth ratio for parameter values equal to those in the Bull regime is 13.295%, and the optimal consumption to wealth ratio for parameter values equal to those in the bear regime is 8.978%.

Table I

Comparative Statics

The base case parameter values are: Interest rates $r_B = r_b = 0.05$, expected returns $\mu_B = \mu_b = 0.1394$, volatilities $\sigma_B = 0.1313, \sigma_b = 0.2600$, risk aversion $\gamma = 2.0$, regime-switching intensities $\lambda_B = 0.2353, \lambda_b = 1.7391$, time discount rate $\rho = 0.1$, bequest weight $k = 0$, mortality rate $\delta = 0$, and transaction cost rates $\theta_B = \alpha_B = \theta_b = \alpha_b = 0.01$. $W_i = x + (1 - \alpha_i)y$ is the value of liquidated wealth, c_i^* is the optimal consumption, y_i^* is the optimal holding in the stock account, $\alpha = (\lambda_b\alpha_B + \lambda_B\alpha_b)/(\lambda_B + \lambda_b)$, and $\bar{\Delta}_i$ represents the average liquidity premium for the no-transaction region in regime $i \in \{B, b\}$. $\bar{\Delta}$ is the average liquidity premium across both regimes.

Parameters		c_B^*/W_B at		c_b^*/W_b at		y_B^*/W_B at		y_b^*/W_b at		Average LPTC Ratio		
		$\underline{z}_B$	$\bar{z}_B$	$\underline{z}_b$	$\bar{z}_b$	$\underline{z}_B$	$\bar{z}_B$	$\underline{z}_b$	$\bar{z}_b$	$\bar{\Delta}_B/\alpha_B$	$\bar{\Delta}_b/\alpha_b$	$\bar{\Delta}/\alpha$
Base case		0.1244	0.1264	0.1204	0.1210	2.8934	1.9469	0.9183	0.6005	0.4399	0.5040	0.4475
μ_B	$\times 1.05$	0.1320	0.1343	0.1271	0.1276	3.1584	2.0923	0.9163	0.5981	0.4824	0.5531	0.4909
	$\times 0.95$	0.1174	0.1191	0.1143	0.1148	2.6312	1.7986	0.9202	0.6027	0.3980	0.4555	0.4049
σ_B	$\times 1.05$	0.1203	0.1221	0.1169	0.1175	2.5905	1.7847	0.9194	0.6018	0.4046	0.4634	0.4116
	$\times 0.95$	0.1292	0.1314	0.1245	0.1250	3.2489	2.1321	0.9171	0.5990	0.4771	0.5467	0.4854
σ_b	$\times 1.03$	0.1242	0.1262	0.1202	0.1207	2.8936	1.9470	0.8690	0.5623	0.4491	0.5155	0.4570
	$\times 0.97$	0.1246	0.1266	0.1208	0.1214	2.8932	1.9467	0.9738	0.6423	0.4293	0.4909	0.4366
θ_b	$\times 1.05$	0.1244	0.1264	0.1204	0.1210	2.8934	1.9469	0.9183	0.5942	0.4392	0.5034	0.4469
	$\times 0.95$	0.1244	0.1264	0.1204	0.1210	2.8934	1.9469	0.9183	0.6068	0.4399	0.5040	0.4475
λ_b	$\times 1.05$	0.1246	0.1266	0.1207	0.1213	2.8932	1.9467	0.9304	0.6027	0.4386	0.5029	0.4459
	$\times 0.95$	0.1242	0.1262	0.1201	0.1207	2.8936	1.9471	0.9063	0.5981	0.4405	0.5043	0.4484
γ	$\times 1.05$	0.1233	0.1251	0.1194	0.1198	2.7292	1.8664	0.8768	0.5697	0.4194	0.4836	0.4270
	$\times 0.95$	0.1256	0.1278	0.1216	0.1222	3.0774	2.0342	0.9651	0.6347	0.4429	0.5071	0.4506
ρ	$\times 1.05$	0.1270	0.1290	0.1229	0.1235	2.8905	1.9437	0.9183	0.5998	0.4399	0.5051	0.4477
	$\times 0.95$	0.1219	0.1238	0.1180	0.1185	2.8964	1.9501	0.9183	0.6013	0.4392	0.5022	0.4467

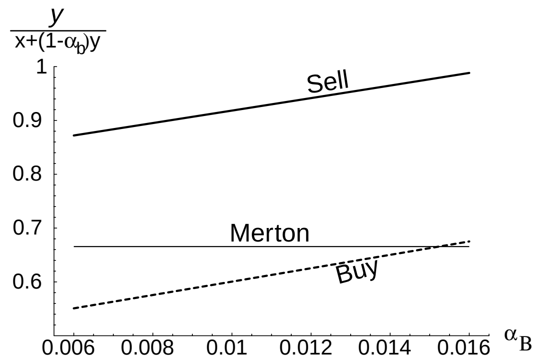

Figure 4. The fraction of wealth invested in the stock in the bear regime as a function of the transaction cost rate in the Bull regime. Parameters: Expected returns $\mu_B = \mu_b = 0.1394$, volatilities $\sigma_B = 0.1313$, $\sigma_b = 0.2600$, regime-switching intensities $\lambda_B = 0.2353$, $\lambda_b = 1.7391$, time-discount rate $\rho = 0.1$, risk aversion $\gamma = 2.0$, interest rates $r_B = r_b = 0.05$, transaction cost rates $\theta_B = \alpha_B$, $\theta_b = \alpha_b = 0.01$, bequest weight $k = 0$, and mortality rate $\delta = 0$.

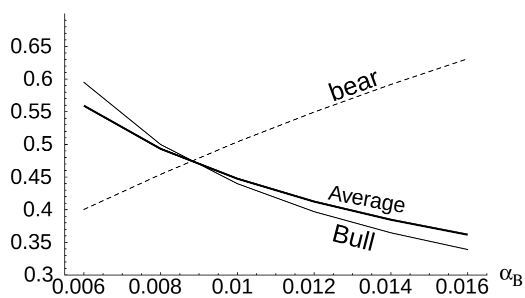

Figure 5. The average liquidity premium to transaction cost ratio as a function of the transaction cost rate in the Bull regime. Parameters: Expected returns $\mu_B = \mu_b = 0.1394$, volatilities $\sigma_B = 0.1313$, $\sigma_b = 0.2600$, regime-switching intensities $\lambda_B = 0.2353$, $\lambda_b = 1.7391$, time-discount rate $\rho = 0.1$, risk aversion $\gamma = 2.0$, interest rates $r_B = r_b = 0.05$, transaction cost rates $\theta_B = \alpha_B$, $\theta_b = \alpha_b = 0.01$, bequest weight $k = 0$, and mortality rate $\delta = 0$.

B. Liquidity Premia in the Regime-switching Model

Figure 5 plots the steady-state average liquidity premia against the transaction cost in the Bull regime. This figure shows that when the transaction cost is small, the liquidity premium to transaction cost ratio is four to six times higher than that found in Constantinides (1986). This suggests that the stochastic nature of the investment opportunity set is a much more important determinant of the liquidity premium than previously thought.

The fundamental intuition that leads to the small liquidity premium in Constantinides (1986) is that an investor optimally reduces trading frequency and in turn reduces transaction cost payments. We examine how regime shifting changes this behavior in Figures 6 and 7. For both of these figures, the Constantinides model corresponds to a model with no regime shifts and with volatility

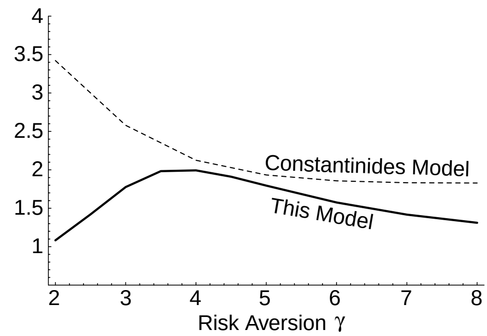

Figure 6. The expected time from purchase to sale as a function of the risk aversion. Parameters: Expected returns $\mu_B = \mu_b = 0.1394$, volatilities $\sigma_B = 0.1313, \sigma_b = 0.2600$, regime-switching intensities $\lambda_B = 0.2353, \lambda_b = 1.7391$, time-discount rate $\rho = 0.1$, interest rates $r_B = r_b = 0.05$, transaction cost rates $\theta_B = \alpha_B = 0.01, \theta_b = \alpha_b = 0.01$, bequest weight $k = 0$, and mortality rate $\delta = 0$. The Constantinides model corresponds to the one-regime case with volatility $\sigma = 0.1466$.

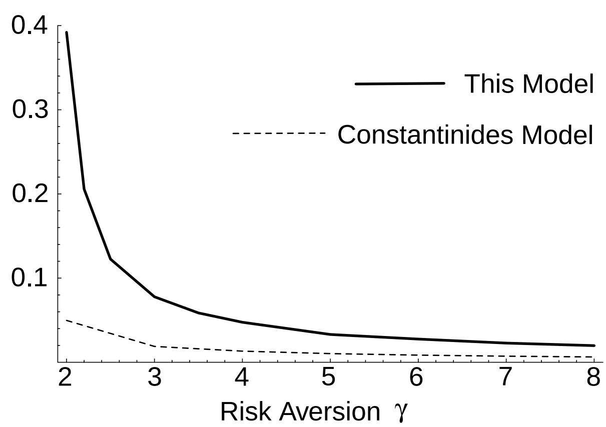

Figure 7. The expected discounted transaction costs as the fraction of the liquidated wealth against the risk aversion. Parameters: Expected returns $\mu_B = \mu_b = 0.1394$, volatilities $\sigma_B = 0.1313, \sigma_b = 0.2600$, regime-switching intensities $\lambda_B = 0.2353, \lambda_b = 1.7391$, time-discount rate $\rho = 0.1$, interest rates $r_B = r_b = 0.05$, transaction cost rates $\theta_B = \alpha_B = 0.01, \theta_b = \alpha_b = 0.01$, bequest weight $k = 0$, and mortality rate $\delta = 0$. The Constantinides model corresponds to the one-regime case with volatility $\sigma = 0.1466$.

set to the unconditional volatility across the regimes in our model. Figure 6 shows the Bull regime's expected time from purchase to sale (see the Appendix for the derivation of this measure) as a function of risk aversion in both the Constantinides model and our model. When the investment opportunity set is not constant, the expected holding period for stock is significantly shorter, consistent with the higher liquidity premium found in Figure 5. For the range of risk aversion displayed, the expected time to sale after a purchase is monotonically decreasing in the risk aversion in the Constantinides model but nonmonotonic in the regime-switching model. Intuitively, by changing the no-transaction region boundaries, an investor can change both the average stock holding and the trading frequency. As the risk aversion decreases, on average the investor keeps more invested in the stock. In the single-regime model, lower risk aversion reduces frequency of trade in part through a higher consumption to wealth ratio and in part through a modification to the relative position of the boundaries to reduce rebalancing costs. However, if there are two regimes, the investor needs to also take into account the transaction costs to be paid at future regime-switching times. Less frequent rebalancing prior to the regime shift can lead to the possibility of larger revisions when the regime changes. Thus, loosely speaking, in the regime-shifting model the investor optimally trades off the large lump-sum transaction costs incurred at future regime-switching times against more frequent but smaller rebalancing costs within a regime. When risk aversion is low, the concern over large lump-sum transaction costs is especially serious because the investor tends to hold more stock and turnover more of the stock when the regime shifts.

To confirm that the changing opportunity set leads to higher transaction costs, Figure 7 displays the Bull regime's expected discounted lifetime transaction costs as a fraction of the liquidated wealth (see the Appendix for the derivation of this measure) against risk aversion in our model and in the Constantinides model. Using a continuously compounded discount rate of 12%, we see that for low levels of risk aversion, the expected discounted lifetime transaction costs can amount to 40% of initial wealth versus only about 5% in the Constantinides model. The expected discounted transaction costs decrease with risk aversion because a more risk-averse investor holds less and trades less stock.

One might wonder how large the liquidity premia can become in our model. To obtain very large liquidity premia, our previous analysis suggests that we need a fairly large difference between the investment opportunity sets, a relatively low transaction cost, and relatively frequent transitions between regimes. Figure 8 shows how the liquidity premium to transaction cost ratio changes as we vary the frequency of switching from the Bull regime to the bear regime. The figure suggests that the changing opportunity set alone cannot generate a liquidity premium much greater than 1.6% with reasonable parameter values. To generate a 1.6% liquidity premium, one would need to assume that the Bull regime lasts only about 0.6 years, which is much shorter than the time frame empirically observed. This finding suggests that transaction costs alone cannot fully explain the equity premium puzzle even in the presence of a stochastic investment opportunity set. Nevertheless, it also shows that transaction costs, along with a stochastic investment opportunity set, can play a more important role than previously thought.

To address any concerns that the estimation for the expected return μ_b in the bear regime might be imprecise, in Figure 9 we plot the steady-state average liquidity premia against μ_b. This figure shows that our main finding that a stochastic investment opportunity set can produce a much higher liquidity premium is robust to changes in the bear regime expected return. In addition,

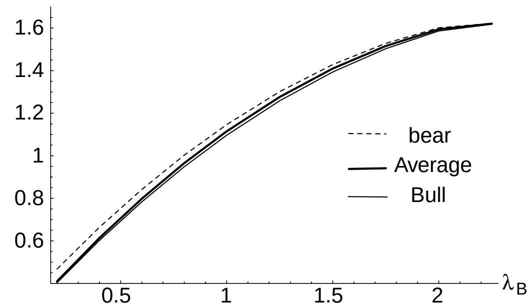

Figure 8. The liquidity premium to transaction cost ratio, as a function of the Bull regime switch intensity. Parameters: Expected returns $\mu_B = \mu_b = 0.1394$, volatilities $\sigma_B = 0.1313, \sigma_b = 0.2600$, bear-regime-switching intensity $\lambda_b = 1.7391$, time-discount rate $\rho = 0.1$, risk aversion $\gamma = 2.0$, interest rates $r_B = r_b = 0.05$, transaction cost rates $\theta_B = \alpha_B = 0.01, \theta_b = \alpha_b = 0.01$, bequest weight $k = 0$, and mortality rate $\delta = 0$.

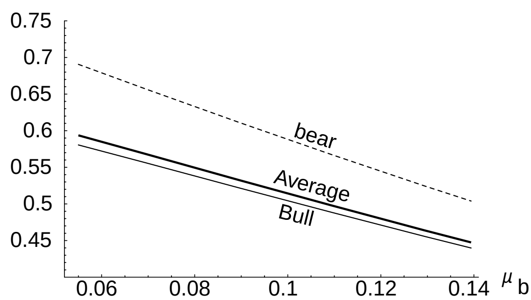

Figure 9. The average liquidity premium to transaction cost ratio as a function of the expected return in the bear regime. Parameters: Bull-regime expected return $\mu_B = 0.1394$, volatilities $\sigma_B = 0.1313, \sigma_b = 0.2600$, bear-regime-switching intensity $\lambda_b = 1.7391$, time-discount rate $\rho = 0.1$, risk aversion $\gamma = 2.0$, interest rates $r_B = r_b = 0.05$, transaction cost rates $\theta_B = \alpha_B = 0.01, \theta_b = \alpha_b = 0.01$, bequest weight $k = 0$, and mortality rate $\delta = 0$.

when the expected return in the bear regime decreases, the liquidity premium becomes even greater. This is because as the investment opportunity sets differ more across regimes, the no-transaction regions become more separated, and an investor incurs higher transaction costs at the regime-switching time.

III. Conclusion

Standard portfolio selection literature (e.g., Constantinides (1986)) finds that although transaction costs alter trading strategies significantly, they only have a second-order effect on liquidity premia. In this paper, we show that this conclusion depends crucially on the assumption of a constant investment opportunity set. In contrast, in a stochastic regime-switching model with transaction costs, we show that transaction costs can have a *first-order* effect on liquidity premia for a wide range of parameter values. This suggests that transaction costs can be very important for asset pricing when an investment opportunity set is stochastic. Our analysis therefore suggests that a stochastic investment opportunity set with small transaction costs may help partially explain the equity premium puzzle and other related anomalies. We believe that further insights on liquidity premia can be generated based on our regime-switching model. For example, one can model liquidity crashes as a regime with extremely high transaction costs. In this setting it is possible to generate significant liquidity premia because the optimal portfolio composition can be dramatically changed by liquidity risk.

Appendix

In this Appendix, we first provide results on the computation of the steady-state distribution, the expected holding period, and the expected lifetime transaction costs. We then present proofs for the propositions and theorems in this paper.

A.1. Steady-State Distribution, Expected Holding Period, and Expected Lifetime Transaction Costs

A.1.1. Steady-State Distribution

It can be verified that

$$dz_t = \mu_{zi}(z_t)\,dt - \sigma_i z_t\,dw_t, \tag{A1}$$

where

$$\mu_{zi}(z) = \left(r_i - \mu_i + \sigma_i^2\right)z - \left(\frac{\psi_i'(z)}{1-k}\right)^{-1/\gamma}. \tag{A2}$$

For simplicity, we focus on the case with separated no-transaction regions. The corresponding results for other cases can be derived using the same method. We have the following result for computing the steady-state distribution of z_t.

PROPOSITION 2: *Suppose $\underline{z}_B < \bar{z}_B < \underline{z}_b < \bar{z}_b$. Let $\phi_i(z)$ be the steady-state density function. Then we have*

$$\phi(z) = \begin{cases} \phi_B(z) & \underline{z}_B < z < \bar{z}_B \\ \phi_b(z) & \underline{z}_b < z < \bar{z}_b \\ 0 & \textit{otherwise}, \end{cases} \tag{A3}$$

where $\phi_B(z)$ and $\phi_b(z)$ solve

$$\frac{1}{2}\sigma_B^2 z^2 \phi_B''(z) - \left(\mu_{zB}(z) - 2\sigma_B^2 z\right)\phi_B'(z) - \left(\lambda_B - \sigma_B^2 + \mu_{zB}'(z)\right)\phi_B(z) = 0$$

subject to

$$\frac{1}{2}\sigma_B^2 \underline{z}_B^2 \phi_B'(\underline{z}_B) - \big(\mu_{zB}(\underline{z}_B) - \sigma_B^2 \underline{z}_B\big)\phi_B(\underline{z}_B) = 0,$$

$$\frac{1}{2}\sigma_B^2 \bar{z}_B^2 \phi_B'(\bar{z}_B) - \big(\mu_{zB}(\bar{z}_B) - \sigma_B^2 \bar{z}_B\big)\phi_B(\bar{z}_B) - \frac{\lambda_b\lambda_B}{\lambda_b + \lambda_B} = 0,$$

and

$$\frac{1}{2}\sigma_b^2 z^2 \phi_b''(z) - \big(\mu_{zb}(z) - 2\sigma_b^2 z\big)\phi_b'(z) - \big(\lambda_b - \sigma_b^2 + \mu_{zb}'(z)\big)\phi_b(z) = 0$$

subject to

$$\frac{1}{2}\sigma_b^2 \underline{z}_b^2 \phi_b'(\underline{z}_b) - \big(\mu_{zb}(\underline{z}_b) - \sigma_b^2 \underline{z}_b\big)\phi_b(\underline{z}_b) + \frac{\lambda_b\lambda_B}{\lambda_b + \lambda_B} = 0,$$

$$\frac{1}{2}\sigma_b^2 \bar{z}_b^2 \phi_b'(\bar{z}_b) - \big(\mu_{zb}(\bar{z}_b) - \sigma_b^2 \bar{z}_b\big)\phi_b(\bar{z}_b) = 0.$$

Proof: See Section A.2 in this Appendix.

Note that Proposition 2 implies that

$$\int_{\underline{z}_B}^{\bar{z}_B} \phi_B(z)\,dz = \frac{\lambda_b}{\lambda_b + \lambda_B} \text{ and } \int_{\underline{z}_b}^{\bar{z}_b} \phi_b(z)\,dz = \frac{\lambda_B}{\lambda_b + \lambda_B}.$$

A.1.2. Expected Holding Period

Next, to investigate the frequency of trade in our model, we show how to compute the expected time to the next sale from a given initial position. Again, we focus on the case in which the no-transactions regions are separated.

In this case, we can define the next sale time to be

$$\tau_s = \inf\{t \geq 0 : z_t = \underline{z}_B \text{ or } z_t = \underline{z}_b\}$$

and the expected time to the next sale starting from z in regime i to be

$$T_i(z) = E[\tau_s \mid z_0 = z].$$

The following proposition provides results on computing the expected time to the next sale.

PROPOSITION 3: *Suppose $\underline{z}_B < \bar{z}_B < \underline{z}_b < \bar{z}_b$. Then*

(i) $T_B(z)$ solves

$$\frac{1}{2}\sigma_B^2 z^2 T_B''(z) + \mu_{zB}(z)T_B'(z) - \lambda_B T_B(z) + 1 = 0,$$

with boundary conditions

$$T_B(\underline{z}_B) = 0, \quad T'_B(\bar{z}_B) = 0;$$

(ii) $T_b(z)$ *solves*

$$\frac{1}{2}\sigma_b^2 z^2 T''_b(z) + \mu_{zb}(z)T'_b(z) + \lambda_b(T_B(\bar{z}_B) - T_b(z)) + 1 = 0,$$

with boundary conditions

$$T_b(\underline{z}_b) = 0, \quad T'_b(\bar{z}_b) = 0,$$

where μ_{zi} *is as defined in (A2).*

Proof: See Section A.2 in this Appendix.

A.1.3. Expected Transaction Costs

Another measure of the effect of transaction costs is the expected discounted transaction costs an investor expects to pay over the entire investment horizon. The following proposition shows one way of computing these costs for the case with separated no-transaction regions.

PROPOSITION 4: *Suppose* $\nu > \max_{i\in\{B,b\}}[-\lambda_i + r_i + (\mu_i - r_i)/(\underline{z}_B + 1)]$ *and* $\underline{z}_B < \bar{z}_B < \underline{z}_b < \bar{z}_b$. *Let* $C_i(x_t, y_t)(i \in \{B, b\})$ *be the expected discounted transaction costs in regime i starting from* (x_t, y_t). *Then, given the optimal policy* (C_t^*, I_t^*, D_t^*),

$$C_i(x, y) \equiv E\left[\int_0^{\tau_i} e^{-\nu t}\left(\theta_i\, dI_t^* + \alpha_i\, dD_t^*\right) + e^{-\nu\tau_i}C_j(x_{\tau_i}, y_{\tau_i})\right] = y g_i(x/y),$$

where for $\underline{z}_i \le z \le \bar{z}_i$, $g_i(\cdot)$ *solves*

$$\frac{1}{2}\sigma_i^2 z^2 g''_i(z) - \left((\mu_i - r_i)z + \left(\frac{\psi'_i(z)}{1-k}\right)^{-1/\gamma}\right) g'_i(z)$$
$$-(\nu + \lambda_i - \mu_i)g_i(z) + \lambda_i g_j(z) = 0, \tag{A4}$$

subject to

$$g_i(\bar{z}_i) - (\bar{z}_i + 1 + \theta_i)g'_i(\bar{z}_i) + \theta_i = 0 \tag{A5}$$

and

$$g_i(\underline{z}_i) - (\underline{z}_i + 1 - \alpha_i)g'_i(\underline{z}_i) - \alpha_i = 0. \tag{A6}$$

In addition,

$$g_B(z) = \frac{1}{\bar{z}_B + 1 + \theta_B}((z + 1 + \theta_B)g_B(\bar{z}_B) + (z - \bar{z}_B)\theta_B) \tag{A7}$$

for $z > \bar{z}_B$, and

$$g_b(z) = \frac{1}{\underline{z}_b + 1 - \alpha_b}((z + 1 - \alpha_b)g_b(\underline{z}_b) + (\underline{z}_b - z)\alpha_b) \tag{A8}$$

for $z < \underline{z}_b$.

Proof: See Section A.2 in this Appendix.

A.2. Proofs

Proof of Lemma 1: We conduct the proof for the case $\eta_i < \eta_j$. The other case can be proven using a similar argument. Equation (10) implies that

$$M_j = f(M_i) = \frac{\eta_i + \lambda_i}{\lambda_i} M_i - \frac{\gamma}{\lambda_i}(1-k)^{1/\gamma} M_i^{1-1/\gamma} - \frac{k\delta}{\lambda_i}.$$

For $i \in \{B, b\}$, let $\bar{M}_i > 0$ be the unique solution of

$$-\eta_i M_i + \gamma(1-k)^{1/\gamma}(M_i)^{1-1/\gamma} + k\delta = 0$$

and let $\underline{M}_i > 0$ be the unique solution of

$$-(\eta_i + \lambda_i)M_i + \gamma(1-k)^{1/\gamma}(M_i)^{1-1/\gamma} + k\delta = 0.$$

Define

$$g(M_i) = -(\eta_j + \lambda_j) f(M_i) + \gamma(1-k)^{1/\gamma} f(M_i)^{1-1/\gamma} + \lambda_j M_i + k\delta.$$

After simplification, we have

$$g(\bar{M}_i) = (\eta_i - \eta_j)\bar{M}_i < 0.$$

In addition, it can be easily verified that $g(\underline{M}_i) = \lambda_j \underline{M}_i + k\delta \geq 0$. By continuity, there exists an M_i such that $g(M_i) = 0$. This implies that $\bar{M}_i \geq M_i \geq \underline{M}_i$. Finally, if $\gamma < 1$, then $f'(x) > 1$ and thus $g'(x) < 0, \forall x > 0$. The solution is therefore unique. If $\gamma > 1$, direct computation reveals that $g''(x) < 0, g'(0) > 0$, and $g(\underline{M}_i) > 0$, which also implies that the solution is unique. Q.E.D.

Proof of Theorem 2: We first state some properties of the candidate value function that satisfies the conditions in Theorem 2.

LEMMA 2: *Suppose $v_i(x, y)$ and $v_j(x, y)$ are as in Theorem 2. Then for any $\epsilon > 0$, we have*

(i) $v_i(x, y) \geq v_i(x + (1 - \alpha_i)y, 0)$.

(ii) For $\frac{x}{y} > \alpha_i - 1 + \epsilon$, there exist finite constants k_i and K_i such that

$$k_i(x + y)^{1-\gamma} \leq v_i(x, y) \leq K_i(x + y)^{1-\gamma}. \tag{A9}$$

(iii) $\frac{y v_{iy}}{v_i}$ *is bounded for* $\frac{x}{y} > \alpha_i - 1 + \epsilon$.

(iv) $\frac{v_{ix}^{\frac{\gamma-1}{\gamma}}}{v_i}$ *is bounded for* $\frac{x}{y} > \alpha_i - 1 + \epsilon$.

Proof of Lemma 2: Part 1 follows from the well-known inequality for any concave function U:

$$\nabla U(z_1) \cdot (z_1 - z_2) \leq U(z_1) - U(z_2). \tag{A10}$$

$$[v_{ix}(x, y) v_{iy}(x, y)][x - (x + (1 - \alpha_i) y) y]^\top \leq v_i(x, y) - v_i(x + (1 - \alpha_i) y, 0). \tag{A11}$$

Using condition (21) and assumptions in Theorem 2, we have

$$0 \leq y(v_{iy}(x, y) - (1 - \alpha_i) v_{ix}(x, y)) \leq v_i(x, y) - v_i(x + (1 - \alpha_i) y, 0), \tag{A12}$$

which proves Part 1.

Part 2 follows from the fact that $v_i(x, y) = (x + y)^{1-\gamma} v_i(\frac{x}{x+y}, \frac{y}{x+y})$ and v_i is continuous in the interior of the solvency region, so the function $v_i(h, 1 - h)$ attains its finite maximum and minimum for $\frac{x}{y} > \alpha_i - 1 + \epsilon$.

Part 3 follows from the fact that $v_i(x, y) = y^{1-\gamma} \psi_i(\frac{x}{y})$, $v_{iy} = (1 - \gamma) y^{-\gamma} \psi_i(\frac{x}{y}) - y^{-\gamma} \frac{x}{y} \psi_i'(\frac{x}{y})$, and thus $\frac{y v_{iy}}{v_i} = (1 - \gamma) - \frac{\frac{x}{y} \psi_i'(\frac{x}{y})}{\psi_i(\frac{x}{y})}$. This quantity is bounded for $\frac{x}{y} > \alpha_i - 1 + \epsilon$, since in NT_i this is a continuous function on a compact set so it attains its maximum and minimum, and outside NT_i the conclusion is implied by equation (17).

Part 4 follows from the fact that $v_{ix} = y^{-\gamma} \psi_i'(\frac{x}{y})$, $v_{ix}^{1-\frac{1}{\gamma}} = y^{1-\gamma} (\psi_i'(\frac{x}{y}))^{1-\frac{1}{\gamma}}$, and thus $\frac{v_{ix}^{\frac{\gamma-1}{\gamma}}}{v_i} = \frac{(\psi_i'(\frac{x}{y}))^{1-\frac{1}{\gamma}}}{\psi_i(\frac{x}{y})}$, which can be shown to be bounded using the same logic as in the proof of Part 3. Q.E.D.

Now we are ready to prove Theorem 2.

The proof relies on results first proved in Davis and Norman (1990). We repeat many of their arguments here, after making adaptations to our particular setting. Applying Itô's lemma to $\log[e^{-(\rho+\delta+\lambda_i)t}(1 - \gamma) v_i(x(t), y(t))]$ leads to

$$\begin{aligned} & e^{-(\rho+\delta+\lambda_i)t} v_i(x_t, y_t) \\ &= v_i(x, y) \exp\Bigg(\int_0^t \frac{1}{v_i} \left(G v - (1 - k) \frac{c_s^{1-\gamma}}{1 - \gamma} - k\delta \frac{(x_s + (1 - \alpha_i) y_s)^{1-\gamma}}{1 - \gamma} - \lambda_i v_j \right) ds \\ &\quad + \int_0^t \frac{1}{v_i} [(v_{iy} - (1 + \theta_i) v_{ix}) \, dI_s + ((1 - \alpha_i) v_{ix} - v_{iy}) \, dD_s] \\ &\quad + \int_0^t \frac{1}{v_i} v_{iy} y_s \sigma \, dw_s - \frac{1}{2} \int_0^t \frac{v_{iy}^2}{v_i^2} \sigma^2 y_s^2 \, ds \Bigg), \end{aligned} \tag{A13}$$

where $Gv \equiv \frac{1}{2}\sigma^2 y^2 v_{iyy} + r_i x v_{ix} - c v_{ix} + \mu_i y v_{iy} - (\rho+\delta)v_i + \lambda_i(v_j - v_i) + (1-k)\frac{c^{1-\gamma}}{1-\gamma} + k\delta\frac{(x+(1-\alpha_i)y)^{1-\gamma}}{1-\gamma}$.

In particular, setting $c = c^* \equiv (\frac{v_{ix}}{1-k})^{-\frac{1}{\gamma}}$ and following the candidate transaction policy, we have $Gv \equiv 0$ in the no-transaction cost region, the terms involving dI and dD are zero, and $\frac{1}{v_i}((1-k)\frac{c^{1-\gamma}}{1-\gamma} + k\delta\frac{(x+(1-\alpha_i)y)^{1-\gamma}}{1-\gamma} + \lambda_i v_j)$ is a positive bounded function bounded away from zero. Moreover, $\frac{v_{iy} y}{v_i}$ is a bounded function for the candidate transaction policy. These properties are proved in Lemma 2. Notice that using Itô's Lemma we also have for a sequence of stopping times $\tau_n \to \infty$

$$
\begin{aligned}
&v_i(x, y)\\
&= E\left[\int_0^{\tau_n\wedge t} e^{-(\rho+\delta+\lambda_i)s}\left((1-k)\frac{c_s^{*1-\gamma}}{1-\gamma} + k\delta\frac{\left(x_s^* + (1-\alpha_i)y_s^*\right)^{1-\gamma}}{1-\gamma} + \lambda_i v_j\left(x_s^*, y_s^*\right)\right) ds\right.\\
&\quad \left. + e^{-(\rho+\delta+\lambda_i)\tau_n\wedge t} v_i\left(x^*_{\tau_n\wedge t}, y^*_{\tau_n\wedge t}\right)\right]. \qquad \text{(A14)}
\end{aligned}
$$

From (A13) we see that if $\gamma > 1$,

$$
\begin{aligned}
0 &\geq v_i\left(x^*_{\tau_n\wedge t}, y^*_{\tau_n\wedge t}\right)\\
&\geq v_i(x, y)\exp\left(\int_0^{\tau_n\wedge t}\frac{1}{v_i}v_{iy}y_s^*\sigma\, dw_s - \frac{1}{2}\int_0^{\tau_n\wedge t}\frac{v_{iy}^2}{v_i^2}\sigma^2 y_s^{*2}\, ds\right) e^{(\rho+\delta+\lambda_i)t}, \qquad \text{(A15)}
\end{aligned}
$$

while if $0 < \gamma < 1$,

$$
\begin{aligned}
0 &\leq v_i\left(x^*_{\tau_n\wedge t}, y^*_{\tau_n\wedge t}\right)\\
&\leq v_i(x, y)\exp\left(\int_0^{\tau_n\wedge t}\frac{1}{v_i}v_{iy}y_s^*\sigma\, dw_s - \frac{1}{2}\int_0^{\tau_n\wedge t}\frac{v_{iy}^2}{v_i^2}\sigma^2 y_s^{*2}\, ds\right) e^{(\rho+\delta+\lambda_i)t}. \qquad \text{(A16)}
\end{aligned}
$$

We remind the reader that the exponential local martingales in equations (A15) and (A16) are in fact Class D martingales since $\frac{1}{v_i}v_{iy}y$ is bounded. Letting $n \to \infty$ in equation (A14), observe that random variables $v_i(x^*_{\tau_n\wedge t}, y^*_{\tau_n\wedge t})$ are bounded by uniformly integrable random variables and using the dominated convergence theorem (Royden (1988), Proposition 18, p. 270),

$$
\begin{aligned}
&v_i(x, y)\\
&= E\left[\int_0^{t} e^{-(\rho+\delta+\lambda_i)s}\left((1+k)\frac{c_s^{*1-\gamma}}{1-\gamma} + k\delta\frac{\left(x_s^* + (1-\alpha_i)y_s^*\right)^{1-\gamma}}{1-\gamma} + \lambda_i v_j\left(x_s^*, y_s^*\right)\right) ds\right.\\
&\quad \left. + e^{-(\rho+\delta+\lambda_i)t} v_i\left(x^*_t, y^*_t\right)\right]. \qquad \text{(A17)}
\end{aligned}
$$

Using (A13), we also have

$$
\begin{aligned}
&e^{-(\rho+\delta+\lambda_i)t} v_i(x_t,\, y_t) \\
&\quad = v_i(x,\, y) \exp\left(\int_0^t \frac{1}{v_i}\left(-(1-k)\frac{c_s^{*1-\gamma}}{1-\gamma} - k\delta \frac{\left(x_s^* + (1-\alpha_i) y_s^*\right)^{1-\gamma}}{1-\gamma} - \lambda_i v_j\right) ds\right) \\
&\qquad \times \exp\left(\int_0^t \frac{1}{v_i} v_{iy} y_s^* \sigma\, dw_s - \frac{1}{2}\int_0^t \frac{v_{iy}^2}{v_i^2}\sigma^2 y_s^{*2} ds\right). \qquad \text{(A18)}
\end{aligned}
$$

Since $\frac{1}{v_i}(-(1-k)\frac{-(1-k)c_s^{*1-\gamma}}{1-\gamma} - k\delta\frac{(x_s^*+(1-\alpha_i)y_s^*)^{1-\gamma}}{1-\gamma} - \lambda_i v_j)$ is a negative function bounded away from zero, and the exponential local martingale is a Class D martingale, we have $\lim_{t\to\infty} e^{-(\rho+\delta+\lambda_i)t} E\left[v_i(x^*(t),\, y^*(t))\right] = 0$. As a result,

$$
\begin{aligned}
&v_i(x,\, y) \\
&= E\left[\int_0^\infty e^{-(\rho+\delta+\lambda_i)s}\left((1-k)\frac{c_s^{*1-\gamma}}{1-\gamma} + k\delta\frac{\left(x_s^* + (1-\alpha_i)y_s^*\right)^{1-\gamma}}{1-\gamma} + \lambda_i v_j\left(x_s^*,\, y_s^*\right)\right) ds\right]. \\
&\qquad\qquad \text{(A19)}
\end{aligned}
$$

Next we show that given v_j, v_i is the value function and vice versa. We start by considering the case $\gamma > 1$. Consider trading strategies that start with $(x+\epsilon, y)$, follow an admissible consumption and trading strategy for initial endowments (x, y), say $(c, \hat{x}, \hat{y}) \in \Theta(x, y)$, and maintain additional ϵe^{rt} in the risk-free account. For these strategies, a simple application of Itô's lemma for a set of stopping times $\tau_n \to \infty$ leads to

$$
\begin{aligned}
&v_i(x+\epsilon,\, y) \\
&\quad \ge E\left[\int_0^{\tau_n\wedge t} e^{-(\rho+\delta+\lambda_i)s}\left((1-k)\frac{c_s^{1-\gamma}}{1-\gamma} + k\delta\frac{\left(\hat{x}_s + \epsilon e^{rs} + (1-\alpha_i)\hat{y}_s\right)^{1-\gamma}}{1-\gamma}\right.\right. \\
&\qquad \left.\left. + \lambda_i v_j\left(\hat{x}_s + \epsilon e^{rs},\, \hat{y}_s\right)\right) ds + e^{-(\rho+\delta+\lambda_i)\tau_n\wedge t} v_i\left(\hat{x}_{\tau_n\wedge t} + \epsilon e^{r\tau_n\wedge t},\, \hat{y}_{\tau_n\wedge t}\right)\right]. \qquad \text{(A20)}
\end{aligned}
$$

From monotonicity and Part 1 of Lemma 2, we have

$$
\begin{aligned}
0 &\ge e^{-(\rho+\delta+\lambda_i)\tau_n\wedge t} v_i\left(\hat{x}_{\tau_n\wedge t} + \epsilon e^{r\tau_n\wedge t},\, \hat{y}_{\tau_n\wedge t}\right) \\
&\ge e^{-(\rho+\delta+\lambda_i)\tau_n\wedge t} v_i\left(\hat{x}_{\tau_n\wedge t} - (1-\alpha_i)\hat{y}_{\tau_n\wedge t} + \epsilon e^{r\tau_n\wedge t}, 0\right) \\
&\ge e^{-(\rho+\delta+\lambda_i)\tau_n\wedge t} v_i(\epsilon, 0)
\end{aligned}
$$

and so by the dominated convergence theorem, we can let $n \to \infty$ to obtain

$$v_i(x+\epsilon,\, y) \ge E\left[\int_0^t e^{-(\rho+\delta+\lambda_i)s}\left((1-k)\frac{c_s^{1-\gamma}}{1-\gamma} + k\delta\frac{\left(\hat{x}_s + \epsilon e^{rs} + (1-\alpha_i)\,\hat{y}_s\right)^{1-\gamma}}{1-\gamma}\right.\right.$$
$$\left.\left. + \lambda_i v_j\left(\hat{x}_s + \epsilon e^{rs},\, \hat{y}_s\right)\right)ds + e^{-(\rho+\delta+\lambda_i)t} v_i\left(\hat{x}_t + \epsilon e^{rt},\, \hat{y}_t\right)\right]. \tag{A21}$$

Letting $t \to \infty$ we have $0 \ge e^{-(\rho+\delta+\lambda_i)t} v_i(\hat{x}_t + \epsilon e^{rt},\, \hat{y}_t) \ge e^{-(\rho+\delta+\lambda_i)t} v_i(\epsilon, 0) \to 0$. Using the monotone convergence theorem, we then have

$$v_i(x+\epsilon,\, y) \ge E\left[\int_0^\infty e^{-(\rho+\delta+\lambda_i)s}\left((1-k)\frac{c_s^{1-\gamma}}{1-\gamma} + k\delta\frac{\left(\hat{x}_s + \epsilon e^{rt} + (1-\alpha_i)\hat{y}_s\right)^{1-\gamma}}{1-\gamma}\right.\right.$$
$$\left.\left. + \lambda_i v_j\left(\hat{x}_s + \epsilon e^{rs},\, \hat{y}_s\right)\right)ds\right]. \tag{A22}$$

Next, letting $\epsilon \downarrow 0$ and using the continuity of v_i and the monotone convergence theorem, we have

$$v_i(x,\, y)$$
$$\ge E\left[\int_0^\infty e^{-(\rho+\delta+\lambda_i)s}\left((1-k)\frac{c_s^{1-\gamma}}{1-\gamma} + k\delta\frac{(x_s + (1-\alpha_i)y_s)^{1-\gamma}}{1-\gamma} + \lambda_i v_j(x_s,\, y_s)\right)ds\right] \tag{A23}$$

for all feasible consumption trading strategies in $\Theta(x, y)$. This implies v_i is the value function given v_j.

In the case $0 < \gamma < 1$, we have

$$v_i(x,\, y) \ge E\left[\int_0^{\tau_n \wedge t} e^{-(\rho+\delta+\lambda_i)s}\left((1-k)\frac{c_s^{1-\gamma}}{1-\gamma} + k\delta\frac{(x_s + (1-\alpha_i)y_s)^{1-\gamma}}{1-\gamma}\right.\right.$$
$$\left.\left. + \lambda_i v_j(x_s,\, y_s)\right)ds + e^{-(\rho+\delta+\lambda_i)\tau_n \wedge t} v_i(x_{\tau_n \wedge t},\, y_{\tau_n \wedge t})\right] \tag{A24}$$

and $v_i(x_t, y_t) \ge 0$. This leads immediately to the conclusion

$$v_i(x,\, y)$$
$$\ge E\left[\int_0^\infty e^{-(\rho+\delta+\lambda_i)s}\left((1-k)\frac{c_s^{1-\gamma}}{1-\gamma} + k\delta\frac{(x_s + (1-\alpha_i)y_s)^{1-\gamma}}{1-\gamma} + \lambda_i v_j(x_s,\, y_s)\right)ds\right] \tag{A25}$$

for all feasible consumption trading strategies from the initial position (x, y). This implies v_i is the value function given v_j. Similar arguments show that v_j is the value function given v_i.

Finally, we show that v_i and v_j are the value functions in regimes i and j, respectively. Let $v_n(x, y) = v_j(x, y)$ if n is odd and $v_n(x, y) = v_i(x, y)$ if n is even. Let τ_n be the time of the nth regime change. The above proof implies

$$v_i(x,\, y) = E\left[\int_0^{\tau_1} e^{-(\rho+\delta)t}\left((1-k)\frac{c_t^{*1-\gamma}}{1-\gamma} + k\delta\frac{\left(x_t^* + (1-\alpha_i)y_t^*\right)^{1-\gamma}}{1-\gamma}\right)dt \right.$$
$$\left. + e^{-(\rho+\delta)\tau_1} v_1\left(x_{\tau_1}^*,\, y_{\tau_1}^*\right)\right] \tag{A26}$$

and for all feasible consumption–investment strategies

$$v_i(x,\, y) \geq E\left[\int_0^{\tau_1} e^{-(\rho+\delta)t}\left((1-k)\frac{c_t^{1-\gamma}}{1-\gamma} + k\delta\frac{(x_t + (1-\alpha_i)y_t)^{1-\gamma}}{1-\gamma}\right)dt \right.$$
$$\left. + e^{-(\rho+\delta)\tau_1} v_1(x_{\tau_1},\, y_{\tau_1})\right]. \tag{A27}$$

Given time τ_1 information, we also know

$$v_1(x_{\tau_1},\, y_{\tau_1})$$
$$= E_{\tau_1}\left[\int_{\tau_1}^{\tau_2} e^{-(\rho+\delta)(t-\tau_1)} e^{-(\rho+\delta)t}\left((1-k)\frac{c_t^{*1-\gamma}}{1-\gamma} + k\delta\frac{\left(x_t^* + (1-\alpha_i)y_t^*\right)^{1-\gamma}}{1-\gamma}\right)dt \right.$$
$$\left. + e^{-(\rho+\delta)(\tau_2-\tau_1)} v_2\left(x_{\tau_2}^*,\, y_{\tau_2}^*\right)\right] \tag{A28}$$

and for all feasible consumption–investment strategies

$$v_1(x_{\tau_1},\, y_{\tau_1}) \geq E\left[\int_{\tau_1}^{\tau_2} e^{-(\rho+\delta)(t-\tau_1)} e^{-(\rho+\delta)t}\left((1-k)\frac{c_t^{1-\gamma}}{1-\gamma} + k\delta\frac{(x_t + (1-\alpha_i)y_t)^{1-\gamma}}{1-\gamma}\right)dt \right.$$
$$\left. + e^{-(\rho+\delta)(\tau_2-\tau_1)} v_2(x_{\tau_2},\, y_{\tau_2})\right]. \tag{A29}$$

Inserting these expressions into equations (A26) and (A27) yields

$$v_i(x,\, y) = E\left[\int_0^{\tau_2} e^{-(\rho+\delta)t}\left((1-k)\frac{c_t^{*1-\gamma}}{1-\gamma} + k\delta\frac{\left(x_t^* + (1-\alpha_i)y_t^*\right)^{1-\gamma}}{1-\gamma}\right)dt \right.$$
$$\left. + e^{-(\rho+\delta)\tau_2} v_2\left(x_{\tau_2}^*,\, y_{\tau_2}^*\right)\right] \tag{A30}$$

and

$$v_i(x,\,y) \geq E\left[\int_0^{\tau_2} e^{-(\rho+\delta)t}\left((1-k)\frac{c_t^{1-\gamma}}{1-\gamma} + k\delta\frac{(x_t+(1-\alpha_i)y_t)^{1-\gamma}}{1-\gamma}\right)dt \right.$$
$$\left. + e^{-(\rho+\delta)\tau_2} v_2(x_{\tau_2},\, y_{\tau_2})\right]. \tag{A31}$$

Continuing in this manner, we have

$$v_i(x,\,y) = E\left[\int_0^{\tau_n} e^{-(\rho+\delta)t}\left((1-k)\frac{c_t^{*1-\gamma}}{1-\gamma} + k\delta\frac{\left(x_t^*+(1-\alpha_i)y_t^*\right)^{1-\gamma}}{1-\gamma}\right)dt \right.$$
$$\left. + e^{-(\rho+\delta)\tau_n} v_n\left(x_{\tau_n}^*,\, y_{\tau_n}^*\right)\right] \tag{A32}$$

and

$$v_i(x,\,y) \geq E\left[\int_0^{\tau_n} e^{-(\rho+\delta)t}\left((1-k)\frac{c_t^{1-\gamma}}{1-\gamma} + k\delta\frac{(x_t+(1-\alpha_i)y_t)^{1-\gamma}}{1-\gamma}\right)dt \right.$$
$$\left. + e^{-(\rho+\delta)\tau_n} v_n(x_{\tau_n},\, y_{\tau_n})\right]. \tag{A33}$$

We now consider the case $\gamma > 1$. As before, consider strategies that start with an initial position $(x+\epsilon, y)$, follow a feasible consumption and trading strategy for an initial position (x, y), and always maintain additional ϵe^{rt} in the riskless account. Similar arguments as those above lead to the conclusion

$$v_i(x+\epsilon,\,y) \geq E\left[\int_0^{\infty} e^{-(\rho+\delta)t}\left((1-k)\frac{c_t^{1-\gamma}}{1-\gamma} + k\delta\frac{\left(x_t+\epsilon e^{rt}+(1-\alpha_i)y_t\right)^{1-\gamma}}{1-\gamma}\right)dt\right], \tag{A34}$$

and letting $\epsilon \to 0$, we have

$$v_i(x,\,y) \geq E\left[\int_0^{\infty} e^{-(\rho+\delta)t}\left((1-k)\frac{c_t^{1-\gamma}}{1-\gamma} + k\delta\frac{(x_t+(1-\alpha_i)y_t)^{1-\gamma}}{1-\gamma}\right)dt\right] \tag{A35}$$

for all feasible consumption and trading strategies.

Since $v_n < 0$ from equation (A32), it follows that

$$v_i(x,\,y) \leq E\left[\int_0^{\tau_n} e^{-(\rho+\delta)t}\left((1-k)\frac{c_t^{*1-\gamma}}{1-\gamma} + k\delta\frac{\left(x_t^*+(1-\alpha_i)y_t^*\right)^{1-\gamma}}{1-\gamma}\right)dt\right] \tag{A36}$$

and since $\tau_n \to \infty$ almost surely as $n \to \infty$, we have from the monotone convergence theorem that

$$v_i(x,\, y) \le E\left[\int_0^\infty e^{-(\rho+\delta)t}\left((1-k)\frac{c_t^{*1-\gamma}}{1-\gamma} + k\delta\frac{\left(x_t^* + (1-\alpha_i)y_t^*\right)^{1-\gamma}}{1-\gamma}\right)dt\right]. \quad \text{(A37)}$$

Thus, from equations (A35) and (A37), we have

$$v_i(x,\, y) = E\left[\int_0^\infty e^{-(\rho+\delta)t}\left((1-k)\frac{c_t^{*1-\gamma}}{1-\gamma} + k\delta\frac{\left(x_t^* + (1-\alpha_i)y_t^*\right)^{1-\gamma}}{1-\gamma}\right)dt\right] \quad \text{(A38)}$$

and

$$v_i(x,\, y) \ge E\left[\int_0^\infty e^{-(\rho+\delta)t}\left((1-k)\frac{c_t^{1-\gamma}}{1-\gamma} + k\delta\frac{(x_t + (1-\alpha_i)y_t)^{1-\gamma}}{1-\gamma}\right)dt\right] \quad \text{(A39)}$$

for all feasible trading and consumption strategies. We have proved the result for $\gamma > 1$.

When $0 < \gamma < 1, v_n > 0$. From equation (A33), we have

$$v_i(x,\, y) \ge E\left[\int_0^\infty e^{-(\rho+\delta)t}\left((1-k)\frac{c_t^{1-\gamma}}{1-\gamma} + k\delta\frac{(x_t + (1-\alpha_i)y_t)^{1-\gamma}}{1-\gamma}\right)dt\right] \quad \text{(A40)}$$

for all feasible consumption and trading strategies. To conclude the proof, we need to show that

$$\lim_{n\to\infty} E\left[e^{-(\rho+\delta)\tau_n} v_n\left(x_{\tau_n}^*,\, y_{\tau_n}^*\right)\right] = 0, \quad \text{(A41)}$$

since this and equation (A32) imply

$$v_i(x,\, y) = E\left[\int_0^\infty e^{-(\rho+\delta)t}\left((1-k)\frac{c_t^{*1-\gamma}}{1-\gamma} + k\delta\frac{\left(x_t^* + (1-\alpha_i)y_t^*\right)^{1-\gamma}}{1-\gamma}\right)dt\right]. \quad \text{(A42)}$$

Equation (A41) follows from the observation that

$$\lim_{\tau_n\to\infty} E\left[\int_{\tau_n}^\infty e^{-(\rho+\delta)t}\left((1-k)\frac{c_t^{*1-\gamma}}{1-\gamma} + k\delta\frac{\left(x_t^* + (1-\alpha_i)y_t^*\right)^{1-\gamma}}{1-\gamma}\right)dt\right] = 0,$$

so

$$\lim_{n\to\infty} E\left[e^{-(\rho+\delta)\tau_n}\left((1-k)\frac{c_{\tau_n}^{*1-\gamma}}{1-\gamma} + k\delta\frac{\left(x_{\tau_n}^* + (1-\alpha_i)y_{\tau_n}^*\right)^{1-\gamma}}{1-\gamma}\right)\right] = 0$$

and

$$0 \le v_n(x,\, y) \le K\left((1-k)\frac{c_{\tau_n}^{*1-\gamma}}{1-\gamma} + k\delta\frac{\left(x_{\tau_n}^* + (1-\alpha_i)y_{\tau_n}^*\right)^{1-\gamma}}{1-\gamma}\right) \quad \text{(A43)}$$

for a suitable constant K, which can be derived from the homotheticity properties of v_i and v_j (implied by Parts 2 and 4 in Lemma 2). Q.E.D.

LEMMA 3: *Suppose the conditions in Proposition 1 hold and $\phi(x, y)$ satisfies*

$$(1-\alpha)\phi_x \leq \phi_y \leq (1+\theta)\phi_x, \tag{A44}$$

(x_t, y_t) correspond to the stock and bond accounts using the optimal c, I, and D in regime i,

$$\lim_{T\to\infty} E\left[e^{-(\rho+\lambda_i)T}\phi(x_T, y_T)\right] = 0, \tag{A45}$$

and $E[\int_0^T \phi_y y_t^2 \sigma_i^2\, dw_t] = 0$. Let $\mathcal{S}_1$ be the set of $(x, y) \in \mathcal{S}$ such that

$$0 > \frac{1}{2}\sigma_i^2 y^2 \phi_{yy} + rx\phi_x + \mu_i y\phi_y - \rho\phi + \frac{\gamma}{1-\gamma}(\phi_x)^{1-\frac{1}{\gamma}} + \lambda_i(v_j - \phi), \tag{A46}$$

and let $\mathcal{S}_2 = \{(x, y) \in \mathcal{S} \mid \phi(x, y) \geq v_i(x, y)\}$. Then if $\mathcal{S}_1 \cup \mathcal{S}_2 = \mathcal{S}$, $\phi(x, y) \geq v_i(x, y)$ for all $(x, y) \in \mathcal{S}$.

Proof: We just need to consider $(x, y) \in \mathcal{S}_1$. Let (x_t, y_t) correspond to the optimal bond and stock accounts using the optimal c, D, and I in regime i, and let $\tau = \inf\{t \mid \frac{1}{2}\sigma_i^2 y^2\phi_{yy} + rx\phi_x + \mu_i y\phi_y - \rho\phi + \frac{\gamma}{1-\gamma}(\phi_x)^{1-\frac{1}{\gamma}} + \lambda_i(v_j - \phi) \geq 0\}$. In other words, $\tau = \inf\{t \mid (x_t, y_t) \in \mathcal{S}_2 \setminus \mathcal{S}_1\}$. By Ito's lemma we have

$$\begin{aligned}
&e^{-(\rho+\lambda_i)T\wedge\tau}\phi(x_{T\wedge\tau}, y_{T\wedge\tau}) + \int_0^{T\wedge\tau} e^{-(\rho+\lambda_i)t}\left(\frac{c_t^{1-\gamma}}{1-\gamma} + \lambda_i v_j(x_t, y_t)\right) dt \\
&\quad = \phi(x_0, y_0) + \int_0^{T\wedge\tau} e^{-(\rho+\lambda_i)t}\Bigg(\frac{1}{2}\sigma_i^2 y_t^2\phi_{yy} + rx_t\phi_x + \mu_i y_t\phi_y - c_t\phi_x + \frac{c_t^{1-\gamma}}{1-\gamma} \\
&\qquad - \rho\phi + \lambda_i(v_j - \phi)\Bigg) dt + \int_0^{T\wedge\tau} e^{-(\rho+\lambda_i)t}(\phi_y - (1+\theta)\phi_x)\, dI_t \\
&\qquad + \int_0^{T\wedge\tau} e^{-(\rho+\lambda_i)t}((1-\alpha)\phi_x - \phi_y)\, dD_t + \int_0^{T\wedge\tau} e^{-(\rho+\lambda_i)t}\sigma_i y\phi_y\, dw_t.
\end{aligned} \tag{A47}$$

We then have $-c_t\phi_x + \frac{c_t^{1-\gamma}}{1-\gamma} \leq \frac{\gamma}{1-\gamma}(\phi_x)^{1-\frac{1}{\gamma}}$. Taking expectations, we obtain

$$\begin{aligned}
\phi(x, y) \geq E\Bigg[&\int_0^{T\wedge\tau} e^{-(\rho+\lambda_i)\rho t}\left(\frac{c_t^{1-\gamma}}{1-\gamma} + \lambda_i v_j(x_t, y_t)\right) dt + e^{-(\rho+\lambda_i)T}\phi(x_T, y_T)\mathbf{1}_{\{T\leq\tau\}} \\
&+ e^{-(\rho+\lambda_i)\tau} v_i(x_\tau, y_\tau)\mathbf{1}_{\{\tau<T\}}\Bigg].
\end{aligned}$$

Letting $T \to \infty$ gives

$$\phi(x, y) \geq E\left[\int_0^{\infty\wedge\tau} e^{-(\rho+\lambda_i)t}\left(\frac{c_t^{1-\gamma}}{1-\gamma} + \lambda_i v_j(x_t, y_t)\right) dt \right.$$

$$\left. + e^{-(\rho+\lambda_i)\tau} v_i(x_\tau, y_\tau)\mathbf{1}_{\{\tau<\infty\}}\right] = v_i(x, y). \tag{A48}$$

Q.E.D.

Proof of Proposition 1: Proof of statement 1. The left-hand inequality simply expresses the fact that the investor cannot be worse off than following the feasible strategy of liquidating the stock, investing all wealth in the money market, and optimally consuming. To establish the right-hand inequality, we follow the strategy of Shreve and Soner (1994), Theorem 9.9. Consider the function $\phi(x, y) = \frac{A}{1-\gamma}(x + \zeta y)^{1-\gamma}$, where $1 - \alpha \leq \zeta \leq 1 + \theta$. We have $(1 - \alpha)\phi_x \leq \phi_y \leq (1 + \theta)\phi_x$. Moreover, evaluating the PDE (13),

$$\frac{1}{2}\sigma_i^2 y^2\phi_{yy} + rx\phi_x + \mu_i y\phi_y - \rho\phi + \frac{\gamma}{1-\gamma}(\phi_x)^{1-\frac{1}{\gamma}} + \lambda_i(v_j - \phi) \tag{A49}$$

is equal to

$$-A(x + \zeta y)^{1-\gamma} \times \left[\frac{\rho + \lambda_i - (1-\gamma)\left(r + \frac{\kappa_i^2}{2\gamma}\right)}{1-\gamma} + \left(\sqrt{\frac{\gamma}{2}}\sigma_i\frac{\zeta y}{x + \zeta y} - \kappa_i\sqrt{\frac{1}{2\gamma}}\right)^2 \right.$$

$$\left. -\frac{\gamma A^{-\frac{1}{\gamma}}}{1-\gamma} - \frac{\lambda_i}{A}\frac{v_j}{(x + \zeta y)^{1-\gamma}}\right], \tag{A50}$$

where $\kappa_i = \frac{\mu_i - r}{\sigma_i}$. To apply the argument of Shreve and Soner (1994), we want to choose A to guarantee the bracketed term in (A50) is nonnegative. A sufficient condition for this is

$$\frac{\rho + \lambda_i - (1-\gamma)\left(r + \frac{\kappa_i^2}{2\gamma}\right)}{1-\gamma} - \frac{\gamma A^{-\frac{1}{\gamma}}}{1-\gamma} - \frac{\lambda_i}{A}\frac{v_j}{(x + \zeta y)^{1-\gamma}} \geq 0, \tag{A51}$$

and since $v_j < 0$, we can choose A so that

$$\frac{\rho + \lambda_i - (1-\gamma)\left(r + \frac{\kappa_i^2}{2\gamma}\right)}{1-\gamma} - \frac{\gamma A^{-\frac{1}{\gamma}}}{1-\gamma} = 0 \tag{A52}$$

or $A = (\frac{\eta_i + \lambda_i}{\gamma})^{-\gamma} = \underline{M}_i$ where η_i is defined in (11) and $\underline{M}_i$ is defined in the proof of Lemma 1. It follows from using Lemma 3 that $v_i(x, y) \leq \frac{\underline{M}_i}{1-\gamma}(x + \zeta y)^{1-\gamma}$. Also note that this implies $v_j(x, y) \leq \frac{\underline{M}_j}{1-\gamma}(x + \zeta y)^{1-\gamma}$. Having obtained this bound, we can improve on it by substituting inequality (A51) for v_j and producing a new bound. Iterating in this manner leads to the result.

Proof of statement 2. We have

$$0 \geq \frac{1}{2}\sigma_i^2 y^2 v_{iyy} + rxv_{ix} + \mu_i y v_{iy} - \rho v_i + \frac{\gamma}{1-\gamma}(v_{ix})^{1-\frac{1}{\gamma}} + \lambda_i(v_j - v_i), \quad \text{(A53)}$$

which can be written as

$$0 \geq \frac{1}{2}\sigma_j^2 y^2 v_{iyy} + rxv_{ix} + \mu_j y v_{iy} - \rho v_i + \frac{\gamma}{1-\gamma}(v_{ix})^{1-\frac{1}{\gamma}}$$
$$+ \frac{1}{2}\left(\sigma_i^2 - \sigma_j^2\right) y^2 v_{iyy} + (\mu_i - \mu_j) y v_{iy} + \lambda_i(v_j - v_i). \quad \text{(A54)}$$

Thus, if the first line is greater than or equal to zero, $v_i(x,y) \geq v_j(x,y)$. If the first line is less than zero, then expanding v_i using the parameters in regime j and applying Lemma 3 also lead to $v_i(x,y) \geq v_j(x,y)$. Using this result and the fact that (A53) holds with equality at $\frac{x}{y} = \underline{z}_i$, we can rewrite the PDE in regime i at $\frac{x}{y} = \underline{z}_i$ as

$$0 \leq -\frac{\gamma}{2}\sigma_i^2(1-\alpha)^2 + (\mu_i - r)(1-\alpha)(\underline{z}_i + 1 - \alpha)$$
$$+ \left\{-(\rho - (1-\gamma)r)\frac{1}{1-\gamma} + \frac{\gamma}{1-\gamma}A_i^{-\frac{1}{\gamma}}\right\}(\underline{z}_i + 1 - \alpha)^2, \quad \text{(A55)}$$

so

$$0 \leq -\frac{\gamma}{2}\sigma_i^2(1-\alpha)^2 + (\mu_i - r)(1-\alpha)(\underline{z}_i + 1 - \alpha), \quad \text{(A56)}$$

from which the bound on $\underline{z}_i$ follows.

Proof of statement 3. Suppose $\underline{z}_i < \underline{z}_j$. Then for $\frac{x}{y} = \underline{z}_i$, $v_i(x, y) = \frac{A_i}{1-\gamma}(x + (1-\alpha)y)^{1-\gamma}$ and $v_j(x, y) = \frac{A_j}{1-\gamma}(x + (1-\alpha)y)^{1-\gamma}$. Using these expressions in the PDE (13) leads to

$$0 = -\frac{\gamma}{2}\sigma_i^2(1-\alpha)^2 A_i + (\mu_i - r)(1-\alpha)(\underline{z}_i + 1 - \alpha)A_i$$
$$+ \left\{-(\rho - (1-\gamma)r)\frac{A_i}{1-\gamma} + \frac{\gamma}{1-\gamma}A_i^{1-\frac{1}{\gamma}} + \lambda_i\frac{A_j - A_i}{1-\gamma}\right\}(\underline{z}_i + 1 - \alpha)^2, \quad \text{(A57)}$$

which can be written as

$$f(\underline{z}_i) + g(\underline{z}_i) = 0, \quad \text{(A58)}$$

where

$$f(z) = -A_i\left(\sqrt{\frac{\kappa_i^2}{2\gamma}}(z+1-\alpha) - \sigma_i(1-\alpha)\sqrt{\frac{\gamma}{2}}\right)^2 \leq 0, \tag{A59}$$

$$g(z) = \left(-\frac{(\rho-(1-\gamma)\left(\frac{\kappa_i^2}{2\gamma}+r\right)A_i}{1-\gamma} + \frac{\gamma}{1-\gamma}A_i^{1-\frac{1}{\gamma}} + \lambda_i\frac{A_j - A_i}{1-\gamma}\right)(z+1-\alpha)^2, \tag{A60}$$

with $\kappa_i = \frac{(\mu_i - r)}{\sigma_i}$. Also from PDE (13), it follows that for $z \leq \underline{z}_i$,

$$f(z) + g(z) \leq 0. \tag{A61}$$

If $f(\underline{z}_i) < 0$, then from (A58) $g(\underline{z}_i) > 0$ which implies $g(z) > 0$ for all $z > \alpha - 1$. Thus, for (A61) to hold for all $\alpha - 1 < z \leq \underline{z}_i$, we must have

$$\underline{z}_i < (1-\alpha)\left(\frac{\gamma\sigma_i^2}{\mu_i - r} - 1\right). \tag{A62}$$

If $f(\underline{z}_i) = 0$, then the above inequality holds with equality, and it follows that $g(z) = 0$ for all z.

Proof of statement 4. This is similar to the proof of statement 3. Q.E.D.

Proof of Proposition 2: Given the optimal transaction policy, any steady-state density function ϕ_i of z_t in regime i must have the form of (A3). In addition, for any C^2 functions $f(z, B)$ and $f(z, b)$ such that $f'(\underline{z}_B, B) = f'(\bar{z}_B, B) = f'(\underline{z}_b, b) = f'(\bar{z}_b, b) = 0$, we must have

$$\int_{\underline{z}_B}^{\bar{z}_B}\left(\frac{1}{2}\sigma_B^2 z^2 f''(z,B) + \mu_{zB}(z)f'(z,B) + \lambda_B(f(\underline{z}_b, b) - f(z,B))\right)\phi_B(z)\,dz$$
$$+ \int_{\underline{z}_b}^{\bar{z}_b}\left(\frac{1}{2}\sigma_b^2 z^2 f''(z,b) + \mu_{zb}(z)f'(z,b) + \lambda_b(f(\bar{z}_B, B) - f(z,b))\right)\phi_b(z)\,dz = 0.$$

By the property of the continuous-time Markov chain, we must have

$$\int_{\underline{z}_B}^{\bar{z}_B}\phi_B(z)\,dz = \frac{\lambda_b}{\lambda_b + \lambda_B} \quad \text{and} \quad \int_{\underline{z}_b}^{\bar{z}_b}\phi_b(z)\,dz = \frac{\lambda_B}{\lambda_b + \lambda_B}.$$

Then, by integration by parts, we have

$$
\begin{aligned}
&\int_{\underline{z}_B}^{\bar{z}_B}\left[\frac{1}{2}\sigma_B^2\big(z^2\phi_B(z)\big)'' - (\mu_{zB}(z)\phi_B(z))' - \lambda_B\phi_B(z)\right] f(z,B)\,dz\\
&+\int_{\underline{z}_b}^{\bar{z}_b}\left[\frac{1}{2}\sigma_b^2\big(z^2\phi_b(z)\big)'' - (\mu_{zb}(z)\phi_b(z))' - \lambda_b\phi_b(z)\right] f(z,b)\,dz\\
&+f(\underline{z}_B,B)\left[\frac{1}{2}\sigma_B^2\big(\underline{z}_B^2\phi_B'(\underline{z}_B)+2\underline{z}_B\phi_B(\underline{z}_B)\big)-\mu_{zB}(\underline{z}_B)\phi_B(\underline{z}_B)\right]\\
&-f(\bar{z}_B,B)\left[\frac{1}{2}\sigma_B^2\big(\bar{z}_B^2\phi_B'(\bar{z}_B)+2\bar{z}_B\phi_B(\bar{z}_B)\big)-\mu_{zB}(\bar{z}_B)\phi_B(\bar{z}_B)-\frac{\lambda_b\lambda_B}{\lambda_b+\lambda_B}\right]\\
&+f(\underline{z}_b,b)\left[\frac{1}{2}\sigma_b^2\big(\underline{z}_b^2\phi_b'(\underline{z}_b)+2\underline{z}_b\phi_b(\underline{z}_b)\big)-\mu_{zb}(\underline{z}_b)\phi_b(\underline{z}_b)+\frac{\lambda_b\lambda_B}{\lambda_b+\lambda_B}\right]\\
&-f(\bar{z}_b,b)\left[\frac{1}{2}\sigma_b^2\big(\bar{z}_b^2\phi_b'(\bar{z}_b)+2\bar{z}_b\phi_b(\bar{z}_b)\big)-\mu_{zb}(\bar{z}_b)\phi_b(\bar{z}_b)\right]\\
&=0.
\end{aligned}
$$

Since $f(z,B)$ and $f(z,b)$ are arbitrary C^2 functions (only need to satisfy $f'(\underline{z}_B,B)=f'(\bar{z}_B,B)=f'(\underline{z}_b,b)=f'(\bar{z}_b,b)=0$), we must have that each bracketed term is equal to zero, which implies that Proposition 2 holds after some simplification. Q.E.D.

Proof of Proposition 3: Let $\tau_i (i\in\{B,b\})$ be the first regime-switching time and τ_{is} be the first time reaching $\underline{z}_i$. Since the no-transaction region is separated, upon regime switching from regime B to regime b the investor immediately sells, and thus in regime B we have

$$
E[\tau_s\,|\,z_0=z]=E[\tau_{Bs}\wedge\tau_B\,|\,z_0=z]=E\left[\int_0^{\tau_{Bs}} e^{-\lambda_B t}\,dt\,|\,z_0=z\right].
$$

Hence, Part (i) follows from Proposition 6 in Liu and Loewenstein (2002). In regime b, upon regime switching from regime b to regime B the investor always buys enough to reach $\bar{z}_B$, and the expected time to the next sale starting from $\bar{z}_B$ is $T_B(\bar{z}_B)$. Thus, we have

$$
\begin{aligned}
E[\tau_s\,|\,z_0=z]&=E[\tau_{bs}\wedge\tau_b\,|\,z_0=z]+T_B(\bar{z}_B)E[1_{\{\tau_b\le\tau_{bs}\}}\,|\,z_0=z]\\
&=E\left[\int_0^{\tau_{bs}}(1+\lambda_b T_B(\bar{z}_B))e^{-\lambda_b t}\,dt\,|\,z_0=z\right],
\end{aligned}\tag{A63}
$$

and in turn Part (ii). Q.E.D.

Proof of Proposition 4: Define $W_t\equiv x_t+y_t$ and $\pi_t\equiv\frac{y_t}{x_t+y_t}$. Direct application of Itô's lemma to $e^{-(\nu+\lambda_i)t}C_i(x_t,y_t)(i\in\{B,b\})$ yields

$$
\begin{aligned}
&e^{-(\nu+\lambda_i)s}C_i(x_s,\, y_s) - C_i(x,\, y) \\
&= \int_0^s e^{-(\nu+\lambda_i)t}\left(\frac{1}{2}\sigma_i^2 z^2 g_i''(z) - \left((\mu_i - r_i)z + \left(\frac{\psi_i'(z)}{1-k}\right)^{-1/\gamma}\right)\right. \\
&\quad \left. \times\, g_i'(z) - (\nu + \lambda_i - \mu_i) g_i(z)\right) y_t\, dt \\
&\quad + \int_0^s e^{-(\nu+\lambda_i)t}(g_i(z_t) - (z_t + 1 + \theta_i) g_i'(z_t))\, dI_t^* \\
&\quad - \int_0^s e^{-(\nu+\lambda_i)t}(g_i(z_t) - (z_t + 1 - \alpha_i) g_i'(z_t))\, dD_t^* \\
&\quad + \int_0^s e^{-(\nu+\lambda_i)t}(g_i(z_t) - z_t g_i'(z_t))\sigma_i y_t\, dw_t \\
&= -\int_0^s e^{-(\nu+\lambda_i)t}\left[\theta_i\, dI_t^* + \alpha_i\, dD_t^* + \lambda_i C_j(x_t,\, y_t)\, dt\right] \\
&\quad + \int_0^s e^{-(\nu+\lambda_i)t}(g_i(z_t) - z_t g_i'(z_t))\sigma_i \pi_t W_t\, dw_t, \qquad \text{(A64)}
\end{aligned}
$$

where the second equality follows from (A4) through (A6). We then have

$$
dW_t = r_i W_t\, dt + (\mu_i - r_i)\pi_t W_t\, dt + \sigma_i \pi_t W_t d w_t - c_t\, dt - \alpha_i dD_t^* - \theta_i dI_t^* \qquad \text{(A65)}
$$

and in turn

$$
W_t \le W_0 e^{\int_0^t (r_i + (\mu_i - r_i)\pi_s) ds} N_t, \qquad \text{(A66)}
$$

where

$$
N_t \equiv e^{-\frac{1}{2}\int_0^t (\pi_s \sigma_i)^2 ds + \int_0^t \pi_s \sigma_i d w_s}
$$

is a martingale. Thus,

$$
\begin{aligned}
0 \le E\left[e^{-(\nu+\lambda_i)t} C_i(x_t,\, y_t)\right] &= E\left[e^{-(\nu+\lambda_i)t}\pi_t W_t g_i(x_t / y_t)\right] \\
&\le E\left[e^{-(\nu+\lambda_i)t} M W_t\right] \\
&\le M E\left[e^{\int_0^t (r_i + (\mu_i - r_i)\pi_s - \nu - \lambda_i)\, ds} N_t\right] \\
&\le M e^{-(\nu + \lambda_i - r_i - (\mu_i - r_i)/(\underline{z}_B + 1))t} \to 0,\ as\ t \to \infty, \qquad \text{(A67)}
\end{aligned}
$$

where M is a constant, the first inequality follows from the boundedness of the optimal π_t and $g(.)$, the second inequality follows from (A66), the third inequality holds because the optimal π_t is bounded below by $1/(\underline{z}_B + 1)$ and $E[N_t] = 1$, and the convergence follows from the first assumption in the proposition.

In addition, (A66) also implies that the last term in (A64) is a martingale. Therefore, taking the expectation and limit as $s \to \infty$, and using (A64) and (A67), we have that

$$yg_i(x/y) = E\left[\int_0^{\tau_i} e^{-\nu t}\left(\theta_i\, dI_t^* + \alpha_i\, dD_t^*\right) + e^{-\nu\tau_i} C_j(x_{\tau_i}, y_{\tau_i})\right].$$

The expressions in (A7) and (A8) follow from the fact that in these transaction regions, the investor immediately transacts to the corresponding boundaries, incurring the costs represented by the first terms. This completes the proof. Q.E.D.

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