

Investment, Uncertainty, and Liquidity

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ABSTRACT

We analyze the dynamic investment decision of a firm subject to an endogenous financing constraint. The threat of future funding shortfalls lowers the value of the firm's timing options and encourages acceleration of investment beyond the first-best optimal level. As well as highlighting another way by which capital market frictions can distort investment behavior, this result implies that (1) the sensitivity of investment to cash flow can be greatest for high-liquidity firms and (2) greater uncertainty has an ambiguous effect on investment.

CORPORATE FINANCE RESEARCH of the last 20 years has highlighted the role of market frictions in firm investment and financing decisions. One important insight of this work is that when there are informational asymmetries (e.g., Greenwald, Stiglitz, and Weiss (1984), Myers and Majluf (1984)) or agency problems (e.g., Stulz (1990), Hart and Moore (1994)), profitable projects may not proceed if the firm has insufficient internal funds to pay for them. As a result, investment is less than its first-best level.

In these models, projects are essentially "now-or-never" investments, so the investment decision is determined solely by the project's current net present value (NPV). In such a setting, the relationship between investment and liquidity is static: A shortage of internal funds reduces the number of states in which the firm can invest and thus has an adverse effect on today's investment opportunities. However, the investment-liquidity relationship can also have dynamic features. For example, if projects can be delayed, then the trade-off between immediate and future investment may also be constrained by financing considerations. In particular, delay of investment exposes the firm to the risk of an adverse cash flow shock that eliminates its ability to finance the project.

To analyze the dynamic relationship between investment and liquidity, we introduce a simple financing constraint into the investment timing model of

* Boyle is at the University of Otago and Guthrie is at Victoria University of Wellington. For helpful comments, we are grateful to Louis Ederington, John Handley, Jayant Kale, Peter MacKay, John Powell, and seminar participants at Otago, Melbourne, and the 2002 New Zealand Finance Colloquium. Valuable suggestions from Rick Green (the editor) and an anonymous referee also significantly improved the paper, while Peter Grundy provided useful research assistance. Any remaining errors or ambiguities are our responsibility.

McDonald and Siegel (1986).¹ In that model, the firm has perpetual rights to a project and seeks to choose the investment date that provides the highest expected payoff. Because the project has an uncertain future value and an irreversible investment cost, the optimal policy is to invest only when the project's NPV exceeds a positive threshold reflecting the value of further delay.

The McDonald and Siegel (1986) model assumes that the investment decision can be made independently of the financing decision.² Mauer and Triantis (1994) extend this setup to allow for the simultaneous determination of investment and the debt/equity mix, but they do not consider the possibility that the firm may face funding restrictions. In contrast, we consider the investment timing decision of a firm facing a capital market friction that constrains its investment choices.

We find that this friction affects both the value of the project rights and the optimal policy for exercising those rights. In particular, potential future financing restrictions encourage acceleration of investment beyond the first-best level. Thus, our model highlights another way by which financing constraints can distort investment behavior: The threat of a future cash shortfall reduces the value of a firm's timing options and leads to suboptimal early exercise of those options. This feature of our model is consistent with the empirical finding of Whited (2002) that small (and presumably more financially constrained) firms invest more often than large firms. It also fits the standard folklore that smaller firms are more aggressive about entering new markets or launching new products than bigger, safer, and less financially constrained firms. Such a phenomenon is typically attributed to differences in risk attitudes (i.e., more caution on the part of large firms) or to differences in management and bureaucracy structure (i.e., slower decision making by large firms), but our model suggests another explanation.

Our analysis identifies an additional determinant of the relationship between firm investment and liquidity. More cash not only loosens the restrictions on current investment, but also makes waiting less risky and thus increases the opportunity cost of current investment. The first effect encourages investment, but the second effect does the opposite. Moreover, because the effect on the risk of waiting is greatest for low-liquidity firms, increases in cash flow can have a greater positive effect on the investment of high-liquidity firms, consistent with the evidence of Kaplan and Zingales (1997) and Cleary (1999).

Our model also has implications for the relationship between investment and uncertainty. Although greater uncertainty about project value increases the value of investment delay and lowers current investment, greater uncertainty about firm liquidity has the opposite effect: More volatility in the firm's future cash flow distribution raises the risk of future funding shortfalls, thereby lowering the value of waiting and increasing current investment. Most feasible measures of

¹ For examples of models similar to McDonald and Siegel (1986), see Bernanke (1983) and Brennan and Schwartz (1985). Majd and Pindyck (1987), Triantis and Hodder (1990), and Ingersoll and Ross (1992), among others, extend the McDonald and Siegel model in various ways. See Dixit and Pindyck (1994) for an excellent overview of this literature.

² The same is true of more complex models that allow for partial reversibility, expandability, and industry structure effects. See, for example, Dixit (1989), Dixit and Pindyck (1994), and Abel et al. (1996).

uncertainty are likely to incorporate both types of volatility, so our model suggests one reason why empirical research finds little or no short-term relationship between investment and uncertainty.³

Our paper builds on a long tradition in finance research. Early work on the effects of financing constraints (by, e.g., Charnes, Cooper, and Miller (1959) and Baumol and Quandt (1965)) uses mathematical programming techniques to analyze the firm's optimal investment policy in the presence of an exogenous restriction on capital market funding, concentrating primarily on the appropriate choice of objective function and discount rate when capital must be rationed. In response, Elton (1970), Myers (1972), and Weingartner (1977) employ simple models of economic equilibrium to show that the firm's investment objective is unaffected by a financing constraint: Any firm that wishes to maximize the economic welfare of its owners should maximize the market value of its assets subject to the limitations imposed by the constraint. Moreover, Weingartner points out that a focus on market-imposed external funding limits is ill conceived in the absence of plausible and explicitly modeled market frictions, so subsequent work by Myers and Majluf (1984) and others endogenizes the constraint as a function of the firm's reaction to specific market frictions.

Continuing this strand of research, but utilizing more recently developed modeling methods, Mello and Parsons (2000) derive the optimal operating and hedging policies of a firm subject to a dynamic financing constraint. Like us, they emphasize the role of the constraint in restricting the firm's options. However, because they focus on the operating policy of a firm that has already invested, they do not consider the effects of the financing constraint on the initial decision to invest. Our focus on the investment timing decision therefore complements their analysis.

In the first section, we set out our model and offer a detailed description of the investment environment. In Section II, we compare the investment problem of the unconstrained firm with that of the financially constrained firm. Section III outlines the optimal investment policy for the constrained firm. In Sections IV and V, we examine the respective effects of liquidity and uncertainty on the constrained firm's investment policy. Section VI contains some concluding remarks.

I. The Investment Environment

A firm owns the rights to an investment project and has the option to invest in this project at any time. If the firm exercises this option at time t , it pays a fixed amount I and receives a project worth V_t . Project value follows the geometric Brownian motion process

$$dV = \mu V dt + \sigma V d\varepsilon, \quad (1)$$

where μ and σ are constant parameters and ε is a Wiener process. The rights are perpetual: At each date, the firm can exercise the rights and invest, or delay investment and retain the rights, or sell the rights to another firm.

³Minton and Schrand (1999) report a negative long-run relationship between cash flow volatility and investment.

In addition to the project rights, the firm has a time t cash stock X_t and some existing assets with perpetual life and market value G . For convenience, we assume that the existing assets do not reinvest any cash flow, so G is a constant. At each date, investors can freely trade existing shares, and the firm can sell new securities in order to fund investment in the project. However, the latter capacity is limited by the willingness of investors to provide additional capital at a reasonable price. For example, the firm may be subject to credit and bank loan constraints, restricted access to bond markets, or prohibitively expensive equity costs. We do not derive from first principles the source of such restrictions (for a recent example of this approach, see Clementi and Hopenhayn (2002)), but instead focus on their effects by assuming that investment is possible at date t if and only if

$$I \leq X_t + G + \alpha V_t, \quad (2)$$

for some constant $\alpha \in [0, 1]$. The right side of equation (2) comprises the potential sources of funds available to the firm at date t : cash plus the realizable value of the firm's assets. The friction embedded in equation (2) captures the idea that uncertainty about the firm's ability or willingness to extract full project value for outside investors limits the amount of funding that these investors are prepared to supply at a nonprohibitive price. For example, the moral hazard and agency problems discussed by Jensen and Meckling (1976) and Stulz (1990) make it difficult for firms to issue claims against the full project value.⁴ Alternatively, the friction might arise because of the contractual problems analyzed by Hart and Moore (1994): The inability to commit vital human capital to the project restricts the supply of new funding. In this case, $(1 - \alpha)V_t$ is the portion of project value attributable to the firm's unique human capital input. Regardless of the source of the friction, the constraint is endogenous: The greater is project value V_t , the greater the firm's funding capacity.

The friction α imposes restrictions on the firm's investment policy. For (X_t, V_t) such that $X_t + G + \alpha V_t < I < V_t$, the investment payoff $V_t - I$ is positive, but the firm's financial resources do not allow it to invest.⁵ The lower the firm's cash balance, the more likely this is to occur, that is, the more severe is the financing constraint. Conversely, as the cash balance becomes very large, the financing constraint becomes immaterial.

Over time, the constraint can be relaxed by augmenting the initial cash balance X . This occurs in two ways. First, if the firm delays investment in the project, X is invested in riskless securities and earns the rate of return r . Second, the

⁴ Issuing debt encourages risk shifting from stockholders to bondholders, so investors are reluctant to lend beyond a certain point. Issuing equity dilutes the ownership stake of current managers and owners, thereby reducing their respective incentives to work and monitor diligently. Investors may also find it difficult to distinguish between good and bad projects when managers pursue their own objectives. These considerations limit the supply of equity funding.

⁵ When $\alpha = 1$, the firm can invest in any project with positive $V_t - I$ so long as doing so would not require it to liquidate, that is, so long as $X_t + G + V_t - I \geq 0$. See the discussion below equation (3).

firm's existing physical assets generate uncertain operational cash flow with dynamics $vdt + \phi d\zeta$, where v and ϕ are constant parameters, and ζ is a Wiener process with $d\epsilon d\zeta = \rho dt$.⁶ Thus, prior to investment in the project, the cash stock follows the process

$$dX = rXdt + vdt + \phi d\zeta. \quad (3)$$

Note that (3) allows the firm to issue protected debt to cover a cash deficit (i.e., $X < 0$). However, this too is limited. If the deficit exceeds the realizable value of the firm's noncash assets, then the firm must liquidate and sell the project rights.⁷

Once investment commences, the value of the project is V . Prior to investment, the value of the project rights is $F^c(X, V)$. That is, $F^c(X, V)$ denotes the expected present value of the payoff from exercising or selling the project rights (at a date determined by the chosen investment policy) when the firm's current cash balance is X and the project value is currently V . For those (X, V) where investment is optimal

$$F^c(X, V) = V - I.$$

That is, if the firm invests, the owners receive the payoff $V - I$. Also

$$\lim_{X \rightarrow \infty} F^c(X, V) = F^u(V), \quad (4)$$

where $F^u(V)$ is the value of the rights if these are held by a firm that faces no financing frictions. Equation (4) states that because the restrictions imposed by the friction disappear as the cash stock becomes large, the value of the project rights to a constrained firm approaches the value available to an unconstrained firm.

As the firm's owners can freely trade their shares, they wish the firm to adopt an investment policy that maximizes its market value. At each date prior to investment, the market value of the firm is equal to the sum of its cash X , the value of its existing assets G , and the value of the project rights $F^c(X, V)$. Thus, the optimal investment policy maximizes $F^c(X, V)$. Our primary interest is in characterizing the effect of financial slack on this policy. To provide a benchmark, we begin by reviewing the case where the investment policy is unconstrained by financial considerations.

⁶ Although it plays no formal part in our analysis, it may be helpful to think of this framework as describing a firm with a "lumpy" investment schedule. Additions or extensions to its existing stock of physical assets can take place only in indivisible units with extensive capital requirements, and while the firm is waiting for sufficient financing capacity to accumulate, the existing cash stock is placed in short-term securities. In the meantime, the firm's existing assets continue to augment (or deplete) the cash stock.

⁷ We assume that the project possesses some feature that is unique to the firm, thereby ensuring that the rights are not fully transferable. Specifically, we assume that the value of the rights to an unconstrained firm is $F^u(\gamma V)$ for some constant $\gamma \in [0, 1)$. This makes selling less attractive and ensures that the firm must confront the investment policy consequences of a lack of financial slack.

II. The Optimal Investment Policy: The Problem

A. The Unconstrained Firm

When the firm has a sufficiently large cash balance, the financing friction is irrelevant and investment expenditure is unconstrained. In this case, the optimal investment policy can be found using the model developed by McDonald and Siegel (1986) and simplified by Dixit and Pindyck (1994). Essentially, this model solves for F^u as a function of the investment policy, then chooses the policy that maximizes the value of this function.

With project value given by equation (1), the firm invests if and only if V exceeds some fixed threshold \hat{V}^u . The optimal investment policy consists of choosing the threshold \hat{V}^u that maximizes F^u . Standard replication arguments imply that, prior to investment, F^u satisfies the differential equation

$$\frac{1}{2}\sigma^2 V^2 F_{VV}^u + (r - \delta) V F_V^u - r F^u = 0, \quad (5)$$

where subscripts denote partial derivatives, r is the riskless interest rate, and δ is the opportunity cost of cash flows forgone due to waiting (henceforth the project's "dividend yield"). Given the boundary conditions

$$F^u(0) = 0, \quad F^u(\hat{V}^u) = \hat{V}^u - I,$$

equation (5) has the unique solution

$$F^u(V) = (\hat{V}^u - I) \left(\frac{V}{\hat{V}^u} \right)^\beta \quad (6)$$

where

$$\beta = \frac{1}{2} - \frac{r - \delta}{\sigma^2} + \sqrt{\frac{2r}{\sigma^2} + \left(\frac{1}{2} - \frac{r - \delta}{\sigma^2} \right)^2}.$$

Maximizing (6) with respect to \hat{V}^u yields the optimal investment threshold

$$\hat{V}^u = \frac{\beta I}{\beta - 1},$$

and investment option value

$$F^u(V) = \left(\frac{I}{\beta - 1} \right)^{1-\beta} \left(\frac{V}{\beta} \right)^\beta.$$

It is straightforward to show that $\beta > 1$, so $\hat{V}^u > I$. That is, there are positive payoff ($V - I > 0$) states in which the firm does not invest, but instead chooses to wait. In doing so, the firm retains the opportunity to receive potentially higher payoffs should project value rise (or avoid losses if project value falls). This additional flexibility also ensures that $F^u(V) \geq \max\{0, V - I\}$. However, these outcomes assume that project financing is guaranteed at all dates and thus ignore the possibility that the firm's timing flexibility may be restricted by its financing capabilities.

B. The Constrained Firm

The problem facing the constrained firm is more difficult. Again, investment is justified if and only if V exceeds some minimum threshold. However, because F^c is a function of X as well as V , the optimal threshold $\hat{V}^c(X)$ must also be a function of X , rather than a constant as in the case of the unconstrained firm.

For those (X, V) where it is optimal to delay investment, F^c satisfies the differential equation (see the Appendix for details)

$$\frac{1}{2}\sigma^2 V^2 F_{VV}^c + \rho\sigma\phi VF_{XV}^c + \frac{1}{2}\phi^2 F_{XX}^c + (r - \delta)VF_V^c + r(X + G)F_X^c - rF^c = 0. \quad (7)$$

The greater complexity of this equation means that an analytical solution for F^c is unknown, as, therefore, is the investment threshold. In the next section, we first offer an intuitive solution to this problem. Subsequently, we generate a numerical solution that both verifies the intuitive version and provides some insight into its economic significance.

III. The Optimal Investment Policy: The Solution

A. An Intuitive Solution

Because the friction α potentially subjects the firm with finite X to a binding financing constraint, it is clear that

$$F^c(V, X) \leq F^u(V).$$

The reason for this difference is that the constrained firm is forced to follow a suboptimal investment policy compared to that of the unconstrained firm. First, there are states in which the unconstrained firm would exercise the investment rights, but the constrained firm has insufficient funding available and so must continue to wait. Second, there are states in which the unconstrained firm would choose to wait, but the constrained firm invests because the benefits of delay are outweighed by the risks of losing the ability to finance the project. Thus, the additional investment restrictions faced by the financially constrained firm are two-fold: In some states it cannot *begin* investment when it wishes to do so; in other states it cannot afford to *delay* investment when it wishes to do so. The potential for these outcomes means that the value of the project rights is lower if held by a constrained firm than if held by an unconstrained firm.

What can we say about the investment policy itself? Note first that equation (2) implies

$$\hat{V}^c(X) \geq \hat{V}^{cm}(X) \equiv (I - G - X)/\alpha.$$

When X is low, $\hat{V}^{cm}(X)$ is high as project value has to be high in order for the firm to have sufficient funds for investment to proceed. But when X is low, the firm has little financial slack and so the risk of losing the ability to finance the project is high. Consequently, there is little to be gained from delaying investment. This

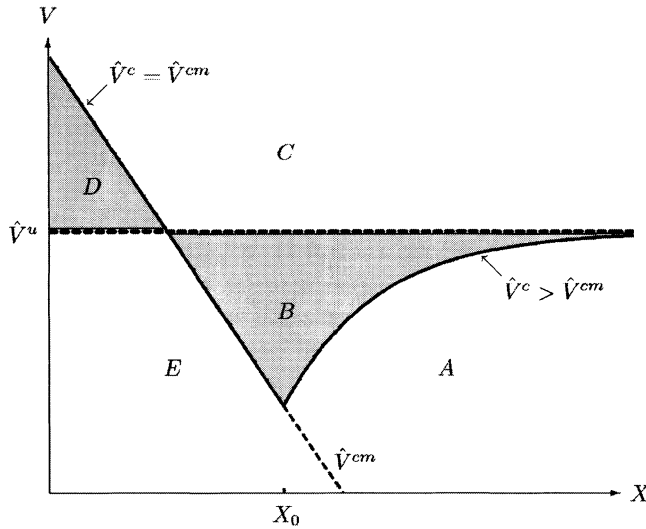


Figure 1. Optimal investment threshold for a constrained firm. The unconstrained firm chooses to invest if (X, V) lies above the line labeled \hat{V}^u . For low X , the constrained firm is able to invest if and only if V is sufficiently high, that is, when (X, V) lies above the line labeled \hat{V}^{cm} . However, once X rises above X_0 , the lower risk of future funding shortfalls increases the value of investment delay, so the constrained firm chooses to invest if (X, V) lies above the curve labeled \hat{V}^c . As a result, the firm does not wish to invest in region A, irrespective of whether it is constrained or unconstrained. In region C, project value V exceeds both \hat{V}^c and \hat{V}^u , so it invests regardless. In region E, the firm does not invest. In region D, the firm invests if unconstrained, but it cannot afford to do so if constrained. In region B, the value of waiting is sufficient for the unconstrained firm to delay investment, but the constrained firm invests immediately because of the risk of future funding shortfalls.

suggests that, for low X

$$\hat{V}^c(X) = \hat{V}^{cm}(X).$$

By contrast, as X becomes higher, the risks of investment delay recede. Simultaneously, $\hat{V}^{cm}(X)$ falls. This suggests that for all X above some critical value X_0

$$\hat{V}^c(X) > \hat{V}^{cm}(X).$$

Finally, we know that as X becomes very large, the firm is effectively unconstrained. Thus

$$\hat{V}^c(X) = \hat{V}^u.$$

These observations imply the approximately “V-shaped” form for $\hat{V}^c(X)$ illustrated in Figure 1. For very low X , the risks of delaying investment are sufficiently high that the optimal policy is to invest as soon as it is possible to do so. In this case, the threshold equals the minimum project value satisfying equation (2) and so is a *decreasing* function of X . As X rises above X_0 , the risk that the firm will have insufficient funds to finance the project in the future falls, thereby

increasing the incentive to wait and raising the investment threshold. Initially, this effect is strong as increases in X from a low level significantly reduce the probability of future funding shortfalls.⁸ Eventually however, the risk of such shortfalls becomes trivial, so further increases in X have little effect and the constrained firm's threshold converges on that of the unconstrained firm.

Figure 1 also illustrates the effect of the financing constraint on the investment decision. In regions A , C , and E , the constraint does not alter the decision. In region A , project value V is below both \hat{V}^c and \hat{V}^u , so the firm does not wish to invest, irrespective of whether it is constrained or unconstrained. In region C , project value V exceeds both \hat{V}^c and \hat{V}^u , so it invests regardless. In region E , the firm does not invest: If unconstrained, it does not wish to do so; if constrained, it cannot afford to do so. By contrast, the constraint leads the firm to make different decisions in regions B and D . In region D , the firm invests if unconstrained, but it cannot afford to do so if constrained. In region B , the value of waiting is sufficient for the unconstrained firm to delay investment, but the constrained firm invests because the risk of subsequently entering region D outweighs the value of waiting.

B. A Numerical Solution

We confirm the above intuitive solution by undertaking a concrete numerical analysis that solves equation (7) using a procedure based on finite difference methods, the details of which are provided in the Appendix. To do so, we use the set of parameter values appearing in Table I. Most of these are similar to those used by other authors, for example, Milne and Whalley (2000) and Mauer and Triantis (1994). The additional parameters are G , ρ , and ϕ . The choice of G is necessarily arbitrary, so we simply set it equal to the investment cost of \$100.⁹ Setting the correlation between X and V to 0.5 is consistent with the investment project having similar, but not identical, characteristics to the firm's existing assets, thereby exposing the firm to the risk of a cash shortfall when it wishes to invest. Finally, given $G=100$ and $r=0.03$, $\phi=60$ is chosen to correspond with actual corporate data.¹⁰

Table II provides an initial indication of the effects of financing constraints on the investment timing decision. For various values of the cash balance X , we

⁸ This occurs because higher X results in greater interest income, but has no effect on future cash flow volatility.

⁹ This means that firms that must currently use external funds to invest ($X < 100$) expect to receive a greater proportion of future increments to their cash stocks from the cash flow generated by their existing physical assets than from the interest return on their existing cash stocks. This seems reasonable insofar as these firms should be motivated to improve the efficiency of their existing assets in order to break free of the financing constraint. For firms with higher liquidity ($X \geq 100$), the primary expected contribution to their cash stocks is from the return on their existing cash. Again, this seems reasonable if, for example, firms that have accumulated high X have done so by skimping on additions to their stock of physical assets.

¹⁰ As G is the market value of the firm's existing assets, rG must equal the certainty-equivalent cash flow generated by these assets. The choice of $r=0.03$ and $G=100$ yields certainty-equivalent cash flow of \$3. Assuming no systematic cash flow risk, the choice of $\phi=60$ implies a ratio of cash flow mean to cash flow standard deviation of 1/20, approximately the value found for U.S. firms listed in the COMPUSTAT database between 1995 and 1999.

Table I
Baseline Parameter Values Used in the Numerical Solution Procedure

This table outlines the parameter values used to numerically solve equation (7) for the optimal investment timing policy in the presence of a financing constraint. Most values are similar to Milne and Whalley (2000) and Mauer and Triantis (1994). The principal additional parameters—the market value of existing assets G and cash flow volatility ϕ —are chosen to correspond with COMPUSTAT data.

Parameter	Value
Project investment cost (\$)	$I = 100$
Project value volatility	$\sigma = 0.2$
Project dividend yield	$\delta = 0.03$
Riskless interest rate	$r = 0.03$
Project value-firm cash flow correlation	$\rho = 0.5$
Cash flow volatility (\$)	$\phi = 60$
Market value of existing assets (\$)	$G = 100$
Market friction	$\alpha = \gamma = 0.8$

Table II
Investment Threshold and Option Values for Unconstrained and Constrained Firms

This table reports the investment thresholds (\hat{V}^u and \hat{V}^c) and project rights values (F^u and F^c) for unconstrained and constrained firms, respectively. X denotes the firm's current cash stock. Parameter values used in generating the threshold and option values are those given in Table I. In addition, for calculating F^u and F^c , we assume initial project value V equals 100.

Cash Stock X	Investment Thresholds		Option Values	
	\hat{V}^u	\hat{V}^c	F^u	F^c
– 200	222	250	28.51	18.98
– 100	222	168	28.51	21.07
– 50	222	188	28.51	24.41
0	222	200	28.51	26.30
50	222	208	28.51	27.35
100	222	212	28.51	27.92
150	222	214	28.51	28.22
200	222	216	28.51	28.37

calculate the investment thresholds and the values of the project rights for both constrained and unconstrained firms.¹¹ For the unconstrained firm with parameters as given in Table I, investment should be delayed until the current project value of \$100 reaches \$222; given this policy, the value of the project rights is

¹¹ For the purposes of calculating the rights values, the initial project value V is set equal to the investment cost $I = 100$, so the project has significant waiting value.

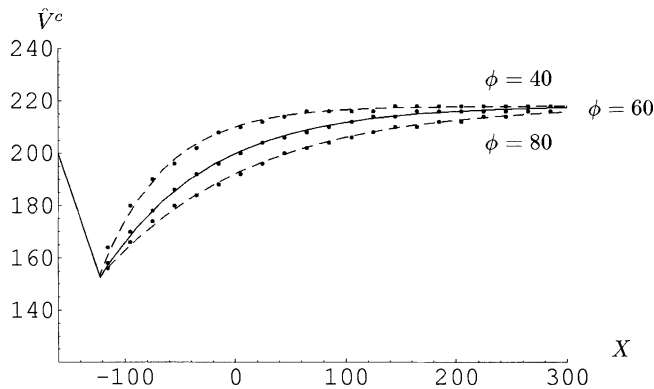


Figure 2. Cash flow volatility and the constrained firm's investment threshold function. The value of the constrained firm's investment threshold is plotted for different values of initial cash stock (X) and cash flow volatility (ϕ). Parameter values are given in Table I. For given ϕ , a rise in the initial cash stock (beyond the minimum necessary to permit investment delay) decreases the risk that the firm will have insufficient funds to finance the project in the future, thereby increasing the value of waiting and raising the investment threshold. For given X , a rise in cash flow volatility increases the risk that the firm will have insufficient funds to finance the project in the future, thereby decreasing the value of waiting and lowering the investment threshold.

\$28.51. However, delay for the constrained firm incurs the risk that the firm's available financing will drop below \$100, thereby making investment (temporarily, at least) impossible. This additional risk makes waiting less valuable, so the optimal investment policy generally requires a lower threshold. For example, when the firm has a moderate cash deficit ($X = -50$), the threshold is \$188, 15 percent below that of an unconstrained firm. Not surprisingly, this lowers the value of the investment rights, in this case to \$24.41, a level some 14 percent below its unconstrained value. For higher values of X , the risks of delay are lower, so the differences are smaller, but the constrained firm nevertheless adopts a lower threshold (and generates a lower rights value) than its unconstrained counterpart. By contrast, when the firm has a significant cash deficit, the risk of losing the ability to fund the project is so great that it cannot afford to delay investment at all. In this case, the project proceeds as soon as the firm has sufficient external funding to make investment possible, so the threshold can be greater than its unconstrained counterpart. For example, when $X = -200$, the threshold is \$250 and the value of the rights is \$18.98. Overall, by inducing a suboptimal investment policy, the existence of a financing constraint can eliminate a significant proportion of a project's potential value.

A more general picture of these effects appears in Figures 2 and 3. Figure 2 confirms the "V-shaped" form for $\hat{V}^c(X)$ illustrated in Figure 1. Overall, the financing constraint encourages the constrained firm to invest earlier than the unconstrained firm. For very low X , the constrained firm invests immediately upon obtaining the necessary funding, even in states where delay is optimal for the unconstrained firm. For intermediate values of X , the constrained firm adopts a

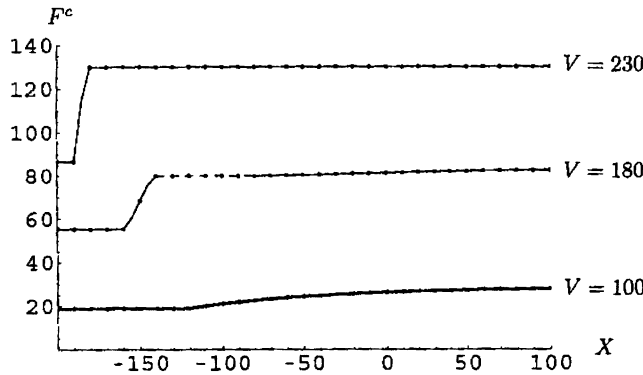


Figure 3. The constrained firm's investment option value. The value of the project rights for the constrained firm (F^c) is plotted for different values of initial cash stock (X) and project value (V). Parameter values are given in Table I. For given V , a rise in X decreases the risk that the firm will have insufficient funds to finance the project in the future, thereby increasing the value of waiting and raising F^c . If V is low, the expected waiting time is high and so F^c increases monotonically with X . For intermediate V , an increase in X raises F^c for values of X below that needed to permit investment. When X only slightly exceeds this minimum level, the risk that the firm will have insufficient funds to finance the project in the future offsets the potential gains from waiting, so the project rights are exercised and additional increments in X have no effect on F^c until X is sufficiently high to significantly reduce the risk of future funding shortfalls. If V is above the unconstrained threshold, then immediate investment is optimal, so F^c increases sharply with X for values of X below the minimum required for investment, but is independent of X thereafter.

lower threshold than the unconstrained firm, so the former continues to invest in states where the latter would choose to wait. Only for high values of X , when the financing constraint is immaterial, does the constrained firm adopt the same investment policy as the unconstrained firm.

Figure 2 also illustrates the influence of cash flow volatility ϕ . For each X , greater cash flow volatility increases the amount by which the constrained firm's threshold deviates from its unconstrained counterpart. The greater is ϕ , the greater the likelihood that adverse cash flow shocks will eliminate the firm's ability to finance the project when it wishes to invest. In response, the firm reduces its exposure to this risk by lowering the investment threshold. We can think of this as a formalized "bird-in-the-hand" strategy; a relatively small payoff received soon and with low risk is preferable to a potentially large payoff received later if there is significant risk of the latter payoff becoming zero due to a funding shortfall.

Figure 3 displays the relationship between the value of the project rights (F^c) and the firm's initial cash stock (X) for different project values (V). The effect of X on F^c is not strictly monotonic, nor is it independent of V . In general, lower X increases the risk that the firm will be unable to finance the project in the future, thereby decreasing the value of waiting and lowering F^c . However, when X becomes very low, the firm must liquidate and sell the project rights, so any further changes in X have no impact on F^c . This explains the initial horizontal

component of each of the curves in Figure 3. Otherwise, for X sufficiently high to preclude the need for liquidation, the nature of the relationship between X and F^c depends on V . If V is low (the bottom curve in Figure 3), the expected waiting time is long and so the funding shortfall risk is significant even if X is currently well above the investment cost I . In this case, F^c increases monotonically with X until it converges on the unconstrained option value. By contrast, if V exceeds the unconstrained investment threshold (the top curve in Figure 3), then immediate investment is optimal, so F^c rises sharply with X for values of X that take the firm out of the liquidation zone (because additional cash reduces the expected suboptimal delay in investment), but is then independent of X once this rises sufficiently to allow investment (because investment occurs and $F^c = V - I$).

If V is greater than I , but less than the unconstrained investment threshold (the middle curve in Figure 3), the relationship between F^c and X becomes more complex. First, in the range of X that is sufficient to allow the firm to remain in business, but insufficient to permit investment, F^c is rapidly increasing in X because each additional dollar reduces the probability that the firm will face a funding shortfall when the optimal investment date arrives. Second, for X equal to or slightly greater than the minimum amount needed for investment, the potential benefits of delaying investment are outweighed by the risk of subsequently losing the ability to finance the project, so the firm invests and additional increments to X have no effect on F^c (this is the dashed line component of the $V = 180$ curve). Third, for X sufficiently above the minimum amount needed for investment, the funding shortfall risk is small enough for investment delay to again become the optimal strategy. Additional increases in X then raise the value of the investment option until it converges on the unconstrained option value.

Finally, the effect of cash on the optimal investment policy can also be depicted in terms of a hurdle rate outcome. For a project with perpetual life, let m denote the hurdle rate margin, measured by the difference between the project's required rate of return and its cost of capital. Thus, m represents the portion of the project hurdle rate attributable to investment timing considerations. In the Appendix, we show that

$$m = \delta \left(\frac{\hat{V}^c(X)}{I} - 1 \right).$$

The relationship between m and X has the same general form as the relationship between \hat{V}^c and X : increasing strongly when X is low, but leveling off at higher X as the risk of future shortfalls becomes insignificant. To give some idea of the quantitative nature of this relationship, we calculate the hurdle rate margin as a function of the firm's ability to finance investment internally and summarize the results in Table III. When the cash position is just sufficient to make waiting viable (which, for a firm with the distributional parameters assumed in Table I, occurs when the cash coverage ratio $X/I = -1.2$), the hurdle rate premium is 1.6 percent. At this point, a positive cash shock equal to the investment cost raises the hurdle rate premium by a further 1.26 percentage points. By contrast, when the cash position is equal to 50 percent of the investment cost, the hurdle rate

Table III
Cash and the Hurdle Rate Margin

This table calculates the effect of cash on the investment timing decision in terms of a hurdle rate margin, measured by the difference between the optimal investment hurdle rate and the unconstrained cost of capital for an infinite-life project owned by a firm with parameters as in Table I. The cash coverage ratio equals the ratio of cash to investment cost. The margin-cash flow sensitivity is the effect (in percentage points) on the hurdle rate margin of a cash flow injection equal to the investment cost. For example, when the firm has a cash overdraft equal to 50 percent of the investment cost, the hurdle rate premium is 2.6 percentage points and a cash shock equal to the investment cost raises this by 0.6 percentage points.

Cash coverage ratio	Hurdle rate margin	Margin-cash flow sensitivity
- 1.2	0.016	1.26
- 1.0	0.020	1.02
- 0.5	0.026	0.60
0.0	0.030	0.36
0.5	0.032	0.21
1.0	0.033	0.13
1.5	0.034	0.07
2.0	0.035	0.04
3.0	0.036	0.02

premium is 3.2 percent, but an additional cash shock raises this premium by only 21 basis points. Overall, the difference between the hurdle rate premium for an unconstrained firm and that for a strongly constrained firm is about two percentage points. Although the level of this difference varies with parameter values and types of project, it is roughly constant in proportionate terms. For example, if the project has a finite life, then the hurdle premia for constrained and unconstrained firms are greater than 1.6 percent and 3.6 percent respectively, but the ratio is still approximately 1:2.

IV. Underinvestment, Accelerated Investment, and Liquidity

In an early analysis of the underinvestment problem, Myers and Majluf (1984) demonstrate that when firm insiders have information that outsiders do not, the latter interpret any attempt by the firm to raise external funds as an indication that the firm is overvalued. Consequently, they lower their estimate of firm value, thereby raising the firm's cost of external financing and lowering the project's NPV. As a result, a fall in internal cash increases the cost of capital and lowers investment. Similarly, Hart and Moore (1994) show that the inability of human capital to credibly commit to a project can cause a firm to forgo positive-NPV projects when internal funds fall below a critical level. Again, a fall in internal cash lowers investment.

In our model, cash has two effects on investment. First, similar to Hart and Moore (1994), a fall in internal cash increases the need for limited external financing and thus reduces the number of states in which investment can occur. That

is, it raises $\hat{V}^{cm}(X)$. Second, because the project does not evaporate if not taken immediately, today's cash position also has implications for the possibility of future investment. In particular, a fall in internal cash today also makes it less likely that sufficient funding will be available in the future, thereby raising the risks of delaying investment. The first effect causes firms to forgo investment in projects they would otherwise have taken; the second effect causes firms to invest in projects earlier than they would otherwise have done. Figure 1 illustrates this point: A fall in internal funds not only makes it more likely that the firm will end up in region *D*, but also in region *B*. In static investment models, where investment opportunities disappear if not taken today, only the first effect operates. In a dynamic framework, by contrast, external financing restrictions can distort investment decisions in two ways. On the one hand, they lead to underinvestment; on the other hand, they encourage accelerated investment.

This observation has interesting implications for the recent debate on whether or not the sensitivity of investment to firm liquidity is a useful measure of financing constraints. Beginning with an influential paper by Fazzari, Hubbard, and Petersen (1988), the standard approach has been to divide a sample of firms into groups reflecting *a priori* rankings of likely financial constraints and then compare the investment-cash flow sensitivities of these different groups. Most studies find that the firms deemed most likely to be financially constrained exhibit greater sensitivity of investment to cash flow, thereby suggesting that this sensitivity is indeed a useful measure of the severity of financing constraints. However, this approach has been questioned by Kaplan and Zingales (1997, 2000) and Cleary (1999). Most tellingly, Kaplan and Zingales find that within the sample of firms Fazzari et al. argue are most likely to be financially constrained, the firms with significant liquidity problems exhibit a *lower* investment-cash flow sensitivity than the firms that appear unlikely to have been financially constrained. Fazzari et al. counter this by pointing out that firms are likely to accumulate high liquidity precisely because they face significant financing constraints, so liquidity may provide no information about the extent to which the firm is constrained. However, as Dasgupta and Sengupta (2002) point out, this does not explain why the sensitivity of investment to cash flow shocks is greater for firms with high liquidity. Kaplan and Zingales stress that it is important to understand the source of this result and speculate that it may be due either to a nonlinearity in the external finance cost function or to currently unknown aversions to raising external finance.

Our model suggests another source. When the firm has limited access to external funding, greater cash flow not only relaxes the constraint on current investment, but also decreases the likelihood that future investment will be constrained, thereby increasing the value of the timing option and thus the opportunity cost of current investment. Consequently, the sensitivity of investment to cash flow is inversely related to the effect of cash flow on the value of the timing option. When the firm is relatively cash rich, the risks of delay are small, so further increases in cash have little effect on the option value. By contrast, when the firm has a smaller amount of cash, the risks of delay are higher, so additional

cash leads to a larger increase in the value of the timing option.¹² As a result, a given cash injection can have a smaller positive impact on the investment of low-liquidity firms than on the investment of high-liquidity firms, consistent with the findings of Kaplan and Zingales (1997) and Cleary (1999).¹³

Of course, this is unlikely to be the full story as current project value, and therefore the attractiveness of immediate investment, may also be affected by firm liquidity. For example, the models of Myers and Majluf (1984) and Greenwald et al. (1984) suggest that project value increases with liquidity because of lower financing costs: More cash reduces the need for costly external financing and thereby lowers the cost of capital. The net effect of a given cash change on investment thus depends on the relative magnitude of the change in financing costs vis-à-vis the change in timing option value; an increase in cash encourages immediate investment via the former, but discourages it via the latter. If the financing cost effect is significantly greater than the timing option effect, then the latter is unlikely to be a major source of the Kaplan and Zingales (1997) and Cleary (1999) results.

However, additional empirical evidence suggests that timing option effects do matter for the relationship between investment and liquidity. To see why, consider a group of high-liquidity firms with similar investment opportunities and constraints. As we have seen, the timing option effect is minor for high-liquidity firms, so the financing cost effect is likely to dominate and the investment-cash flow sensitivity should increase with liquidity in this group of firms. By contrast, for a similar group of low-liquidity firms, the timing option effect is relatively large, so the investment-cash flow sensitivity should increase much more slowly, or even decline, with liquidity. These predictions are consistent with the results obtained by Huang (2002) in a study of 1,235 U.S. firms listed on COMPUSTAT between 1988 and 1998. He calculates the investment-cash flow sensitivity of subgroups of both high and low dividend payout firms. For the high payout firms, the investment-cash flow sensitivity is higher for the more constrained subgroups, but the reverse is true for the low payout firms.

Several other recent studies have also examined the theoretical relationship between investment and liquidity. Almeida and Campello (2001) and Povel and Raith (2001) offer stories similar to ours, in that both identify offsetting effects of liquidity on investment. Almeida and Campello show that more cash not only directly allows more investment, but also allows it indirectly by increasing borrowing capacity and, moreover, that this indirect effect is greater for firms that face smaller frictions (in our model, this corresponds to the case where α is an increasing function of X). Povel and Raith point out that higher cash not only allows the constrained firm to invest more, but also *encourages* it to do so: More investment generates greater revenues that alleviate the constraint. As the firm's liquidity position worsens, the latter effect becomes stronger, so investment has a

¹² This can also be seen in Figure 1. A decrease in X moves more projects into region B when X is low than when X is high.

¹³ For very low X , a rise in X affects investment only if it is large enough to allow the firm to invest, that is, move it to the $\hat{V}^{cm}(X)$ line in Figure 1. This is consistent with the empirical evidence that the investment of firms with very low liquidity is insensitive to cash flow.

relatively weak relationship with cash flow. Moyen (2002) suggests that the investment of high-liquidity firms is more sensitive to cash flow changes for two reasons. First, the value of future investment opportunities increases with current cash flow, but less constrained firms are better placed to take advantage of these opportunities. Second, some of the greater cash flow is absorbed by dividends in constrained firms, but this effect is smaller in firms with higher liquidity. Dasgupta and Sengupta (2002) examine the investment decision of a firm subject to moral hazard problems. More cash diminishes the extent of this problem and thereby permits greater investment. In general, this effect is stronger for firms with low liquidity, but over certain ranges of liquidity, higher-liquidity firms may benefit more. Thus, the relationship between investment and liquidity is not monotonic. Whited (2002) develops a model of dynamic investment decision making in which the risk of future funding shortfalls is negligible (because, in the terminology of our model, the correlation ρ between X and V equals one), but lower liquidity increases the fixed adjustment costs of changing the capital stock. As a result, firms with less slack have, on average, a greater length of time between large, indivisible, investment projects. Finally, Altı (2001) shows that an increase in cash can lead to higher investment even in the absence of capital market frictions. In his model, changes in cash provide additional information about the quality of a firm's investment opportunities. Moreover, this information effect is stronger for low-cash firms because such firms are typically younger and have greater ex ante uncertainty about their future prospects. Consequently, the sensitivity of investment to cash flow is greater for low-cash firms.

These models identify a variety of mechanisms by which cash flow can influence investment. More cash can signal greater project quality or investment opportunities, reduce moral hazard and capital stock adjustment costs, increase borrowing capacity, and decrease investment benefits. However, although all these stories focus on dynamic investment issues, none explicitly considers the effect of cash on the value of the firm's real options. By contrast, our model emphasizes the value of cash for optimal investment timing: More cash lowers the cost of delaying investment. Thus, none of these stories is incompatible with ours, although which, if any, best identifies the primary drivers of the investment-liquidity relationship remains an empirical question.¹⁴

V. Investment and Uncertainty

Our model also has implications for the relationship between investment and uncertainty. When investment is unconstrained, all uncertainty emanates from the stochastic evolution of project value, so greater uncertainty increases the value of waiting and lowers investment. However, the empirical evidence for this

¹⁴ Whited's (2002) finding that higher liquidity increases the investment hazard rate might initially seem inconsistent with our model, but the two are easily reconciled. Consider, for example, two firms with the same V , one in region E of Figure 1, the other in region B . The former waits to invest, but the latter invests immediately. Thus, the less liquid firm has the lower hazard, as Whited's results suggest.

relationship is inconclusive; Ghoshal and Loungani (1996, 2000) find that investment is a decreasing function of price or profit uncertainty in industries with large numbers of small firms, but not for other industry structures; Caballero and Pindyck (1996) report a statistically significant relationship between investment and the variance of the marginal revenue product of capital, but one that is substantially smaller than predicted by the unconstrained firm model. Our model suggests one possible reason for these ambiguous results. In the presence of a financing constraint, the firm faces not only uncertainty about future project value, but also uncertainty about its ability to finance desirable investments. These have opposite effects on the investment threshold; payoff uncertainty raises the threshold, while financing uncertainty lowers it. Thus, any attempt to empirically identify the relationship between uncertainty and investment will pick up offsetting uncertainty effects unless the exact nature of the uncertainty is carefully identified. For example, the uncertainty measures used in the above studies are based on historical estimates of volatility in some aspect of firm performance, and thus seem likely to include aspects of both payoff and financing uncertainty. Consequently, it is unsurprising that the estimated relationship between investment and uncertainty is small or nonexistent.

The conflicting effects of payoff and financing uncertainty can also have implications for the interpretation of other empirical results. Shin and Stulz (2000) find that shareholder wealth is negatively related to equity price volatility (as a proxy for cash flow volatility), a result they attribute to financial distress costs. However, our model makes it clear that such a result is also consistent with dynamic investment decision making; greater cash flow volatility reduces the value of the investment option (and therefore shareholder wealth) because it increases the likelihood of a future funding shortfall and therefore leads to suboptimal investment timing.

VI. Conclusion

When access to external funding is restricted, firms become more reliant on internal funds to finance investment. Although this fundamental point has long been recognized, its implications for optimal investment timing have not previously been analyzed. In this paper, we consider the implications of liquidity for the investment policy of a firm that owns the perpetual rights to a project. Our principal conclusions are as follows:

1. A financing constraint lowers the value of the rights because of the risk that the firm will face a funding shortfall when it wishes to invest.
2. The risk of future funding shortfalls lowers the optimal investment threshold, thereby resulting in suboptimal early investment. Thus, a financing constraint may cause the firm to sacrifice a significant proportion of a project's value not only by forcing it to forgo investment (as would be the case in static models), but also by encouraging it to accelerate investment.
3. Greater cash flow permits more investment, but also raises the threshold required to justify investment. Since the latter effect is greatest for more

constrained firms, such firms can have lower investment-cash flow sensitivities than firms that are less constrained. This contrasts with the conventional view that high sensitivities indicate strong constraints, but is consistent with the evidence of Kaplan and Zingales (1997) and Cleary (1999).

4. Greater payoff uncertainty increases the threshold required to justify investment, but greater financing uncertainty decreases it. Since most empirical measures of volatility are likely to contain elements of both payoff and financing uncertainty, their offsetting effects can help explain why the observed short-term relationship between investment and volatility is weak.

Our analysis has focused on single stand-alone projects that have no effect on the financing constraints faced by other projects. An obvious extension of our work would consider the more general situation where the firm has a number of competing projects, each of which has different implications for the financing constraints faced by all others. For example, suppose a firm has two projects with the same positive NPV and cost I . If the firm has insufficient financial resources to launch both projects, then starting one of them now augments the future cash stock and thereby increases the likelihood of being able to subsequently launch the other. Thus, there is an additional incentive for the constrained firm to invest now, similar to the mechanism identified by Povel and Raith (2001). By contrast, if the payoffs from the two projects are negatively correlated, the unconstrained firm has more incentive to delay investment in order to see which turns out best. In this case, the differences that we have identified between constrained and unconstrained firms seem likely to be accentuated, but other cases may yield different outcomes, so further analysis of this complex problem has the potential to yield additional insights.

APPENDIX

A. Derivation of Equation (7)

We assume that the risks inherent in V and X are spanned by the market of existing securities. Specifically, suppose that there are traded assets or portfolios with prices v and x that evolve according to

$$dv = \mu_v v dt + \sigma_v v d\varepsilon, \quad (\text{A1})$$

$$dx = \mu_x x dt + \sigma_x x d\zeta. \quad (\text{A2})$$

Then a long position in the investment option can be combined with short positions of $\sigma V F_V / (\sigma_v v)$ units of asset v and $\phi F_X / (\sigma_x x)$ units of asset x to produce a total return dR over the time interval dt such that (for shorthand, we drop the “ c ” superscript on F since it is obvious that we are referring only to the

constrained firm)

$$dR = dF - \left(\frac{\sigma VF_V}{\sigma_v v} \right) dv - \left(\frac{\phi F_X}{\sigma_x x} \right) dx.$$

Using Itô's Lemma to obtain an expression for dF , substituting (A1) and (A2) for dv and dx , respectively, and simplifying, we obtain

$$dR = \left(\frac{1}{2} \sigma^2 V^2 F_{VV} + \frac{1}{2} \phi^2 F_{XX} + \rho \sigma \phi VF_{XV} + \left(\mu - \frac{\mu_v \sigma}{\sigma_v} \right) VF_V + \left(rX + v - \frac{\mu_x \phi}{\sigma_x} \right) F_X \right) dt.$$

Since this return is riskless, the portfolio must earn the riskless rate of return. Therefore,

$$dR = r \left(F - \frac{\sigma VF_V}{\sigma_v} - \frac{\phi F_X}{\sigma_x} \right) dt.$$

Equating this to the above expression for dR means that F satisfies the differential equation

$$\begin{aligned} 0 = & \frac{1}{2} \sigma^2 V^2 F_{VV} + \frac{1}{2} \phi^2 F_{XX} + \rho \sigma \phi VF_{XV} \\ & + \left(\mu - \frac{\mu_v \sigma}{\sigma_v} + \frac{r \sigma}{\sigma_v} \right) VF_V + \left(rX + v - \frac{\mu_x \phi}{\sigma_x} + \frac{r \phi}{\sigma_x} \right) F_X - rF. \end{aligned} \quad (\text{A3})$$

Further simplification can most readily be obtained if we assume the expected returns μ_v and μ_x are given by some equilibrium model such as the CAPM. If the latter holds, then

$$\begin{aligned} \mu_x &= r + \rho_{xm} \sigma_x \lambda, \\ \mu_v &= r + \rho_{vm} \sigma_v \lambda, \end{aligned}$$

where $\rho_{xm}(=\rho_{xm})$ and $\rho_{vm}(=\rho_{vm})$ are the correlation coefficients of the market return with dx and dv respectively, and λ is the market price of risk. If $\delta \equiv \mu_v - \mu$ is the project's dividend yield, then

$$\mu + \delta = r + \rho_{vm} \sigma_v \lambda.$$

Hence, the (A3) coefficient on VF_V , $\mu - (\mu_v \sigma / \sigma_v) + (r \sigma / \sigma_v)$, becomes

$$\mu - \rho_{vm} \sigma \lambda = r - \delta.$$

Now let G denote the market value of a claim to the future cash flow generated by the firm's existing physical assets. Clearly G is independent of X and V , so $dG = 0$ over any time interval dt . Thus, the return on a long position in G consists only of the current cash flow ($vdt + \phi d\zeta$). Hence, using (A2), a long position in G combined with a short position in $\phi/(\sigma_x x)$ units of asset x yields a total return of

$$vdt + \phi d\zeta - \left(\frac{\phi}{\sigma_x x} \right) dx = \left(v - \frac{\phi \mu_x}{\sigma_x} \right) dt.$$

Since this return is riskless, we must have

$$v - \frac{\phi \mu_x}{\sigma_x} = r \left(G - \frac{\phi}{\sigma_x} \right),$$

which implies that the (A3) coefficient on F_X , $rX + v - (\mu_x \phi / \sigma_x) + (r \phi / \sigma_x)$, is equal to $r(X + G)$. Making this substitution back into (A3) yields (7).

B. Numerical Solution Procedure for Equation (7)

The partial differential equation is solved on a grid with nodes $\{(X_k, V_j): j=1, \dots, J, k=1, \dots, K\}$, where $X_k - X_{k-1} = dX$ and $V_j = j dV$. At node (X_k, V_j) , the resulting difference equation can be written in the form

$$0 = a_j F_{j-1,k} + b_j F_{j,k} + c_j F_{j+1,k} + d_k F_{j,k-1} + e_k F_{j,k+1} \\ + f_j (F_{j+1,k+1} + F_{j-1,k-1} - F_{j-1,k+1} - F_{j+1,k-1}),$$

where

$$a_j = \frac{\sigma^2 V_j^2}{2dV^2} - \frac{(r - \delta) V_j}{2dV}, \\ b_j = -\frac{\phi^2}{dX^2} - \frac{\sigma^2 V_j^2}{dV^2} - r, \\ c_j = \frac{\sigma^2 V_j^2}{2dV^2} + \frac{(r - \delta) V_j}{2dV}, \\ d_k = \frac{\phi^2}{2dX^2} - \frac{r(X_k + G)}{2dX}, \\ e_k = \frac{\phi^2}{2dX^2} + \frac{r(X_k + G)}{2dX}, \\ f_j = \frac{\rho \phi \sigma V_j}{4dXdV},$$

and $F_{j,k} = F(X_k, V_j)$. This equation is defined whenever $2 \leq j \leq J-1$ and $2 \leq k \leq K-1$. We extend it to the edges of the grid using four boundary conditions. (i) When $j=1$, we use $F_{0,k}=0$, since $F(X, 0)=0$. (ii) When $j=J$, we use $F_{J+1,k} = 2F_{J,k} - F_{J-1,k}$, motivated by the observation that

$$F(X, V + dV) = 2F(X, V) - F(X, V - dV) + O(dV^2).$$

(iii) When $k=1$, we suppose that the liquidation constraint is binding, so that $F_{j,0} = F^u(\gamma V_j)$. (iv) When $k=K$, we use $F_{j,K+1} = F^u(V_j)$, or the value of the project if there is no financing constraint.

We start by setting $F_{j,k} = 0$ if $X_k + G + \alpha V_j < I$ or $V_j < I$, and set $F_{j,k} = V_j - I$ at all other nodes, and then solve the system using the method of Successive Over Relaxation. During each iteration of this method, we solve the difference equation at each node (X_k, V_j) in turn, replacing the calculated value of $F_{j,k}$ with $F^u(\gamma V_j)$ if $X_k + G + F^u(\gamma V_j) < 0$, and with $V_j - I$ at any node for which $X_k + G + \alpha V_j \geq I$ and

$F_{j,k} < V_j - I$. We stop iterating when the largest change in any $F_{j,k}$, measured relative to its value at the end of the preceding iteration, is less than $I/10000$.

C. The Project's Hurdle Rate

Suppose that the project has a perpetual life and generates cash flow of $z_t = \delta V_t$ if it is operating at date t . These assumptions ensure that the value of the completed project equals V_t at date t , and are consistent with our interpretation in Section II of δ as the project's dividend yield. At time t , the project's internal rate of return (IRR) is the number i_t such that

$$0 = \int_0^\infty e^{-i_t s} E_t[z_{t+s}] ds - I = \frac{z_t}{i_t - \mu} - I = \frac{\delta V_t}{i_t - \mu} - I.$$

The project's IRR is therefore

$$i_t = \mu + \frac{\delta V_t}{I}.$$

Investment should occur at date t if and only if $V_t \geq \hat{V}^c(X_t)$, that is, if and only if

$$i_t \geq \mu + \frac{\delta \hat{V}^c(X_t)}{I}.$$

Since the project's cost of capital equals $\mu + \delta$, the margin between the project's hurdle rate and its cost of capital is

$$m_t = \left(\mu + \frac{\delta \hat{V}^c(X_t)}{I} \right) - (\mu + \delta) = \delta \left(\frac{\hat{V}^c(X_t)}{I} - 1 \right).$$

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