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Advanced Statistics I (Winter Term 2023/24)

Problem Set 4

1. Consider the experiment of rolling a manipulated die and observing the number of dots facing up. The die is manipulated in the way that the probability of i dots facing up is proportional to i^2, (i = 1,...,6). Let X denote the number of dots facing up. Find the probability density function of X.

2. A fair die is thrown until 6 occurs for the first time, but it is thrown no more than three times. The random variable X counts the number of throws. Find the probability density function and the cumulative distribution function of X.

3. The probability density function of the random variable X is given by

$$f(x) = 3(2-x)^2 \mathbb{I}_{(1,2)}(x).$$

Calculate the following probabilities

(a) $P(X < 1.2)$
(b) $P(X > 1.6)$
(c) $P(1.2 < X < 1.6)$.

4. For each of the probability density functions given below, find the associated cumulative distribution function and sketch both functions for each problem.

(a) $f(x) = \begin{cases} x & \text{if} \quad 0 \le x < 1 \\ 2-x & \text{if} \quad 1 \le x < 2 \\ 0 & \text{otherwise} \end{cases}$

(b) $f(x) = \begin{cases} x & \text{if} \quad 0 \le x < 1 \\ \frac{1}{4}(3-x) & \text{if} \quad 1 \le x < 3 \\ 0 & \text{otherwise} \end{cases}$

(c) $f(x) = \begin{cases} x & \text{if} \quad 0 \le x < 1 \\ (x-5) & \text{if} \quad 5 \le x < 6 \\ 0 & \text{otherwise} \end{cases}$.

5. For each of the cumulative distribution functions given below, find the associated probability density function.

(a) $F(x) = \begin{cases} 0 & \text{if} \quad x < 2 \\ (x-2)^3 & \text{if} \quad 2 \leq x \leq 3 \\ 1 & \text{otherwise} \end{cases}$

(b) $F(x) = \Big[1 - \exp\Big\{ -\lambda(x-c)\Big\}\Big]\ \mathbb{I}_{[c,\infty)}(x)\ , \quad \lambda > 0\ .$

(c) $F(x) = 1 - \sum_{i=0}^{n-1} \frac{(\lambda x)^i}{i!} e^{-\lambda x} \mathbb{I}_{(0,\infty)}(x), \quad n \in \mathbb{N},\ \lambda > 0$

6. The joint probability density function of the random variable (X_1,X_2) is given by

$$f(x_1,x_2) = k x_1 x_2\ \mathbb{I}_{[0,1]}(x_1)\ \mathbb{I}_{[0,1]}(x_2)\ .$$

(a) Determine k.

(b) Find the the joint cumulative distribution function of (X_1,X_2).

(c) Calculate the probability $P(0.5 \leq X_1 \leq 1; 0.5 \leq X_2 \leq 1)$.