Solutions to Problem Set 4

1. According to the exercise: $P(X=i)=c\cdot i^2$; $i=1,...,6$;
   c: constant
   sample space $S=\{1,...,6\}$ with $P(S)=1$

   where $P(S)=\sum_{i=1}^{6}P(X=i)=c\sum_{i=1}^{6}i^2=c\cdot 91\overset{!}{=}1$

   $c=\frac{1}{91}\Longrightarrow$ pdf:
   $$f(x)=\frac{x^2}{91}\mathcal{I}_{\{1,...,6\}}(x)$$

2. Define $\bar{6}=\{1,\cdots,5\}$. Sample space: $S=\{6;(\bar{6},6);(\bar{6},\bar{6},6);(\bar{6},\bar{6},\bar{6})\}$
   random variable $X:f:S\to\mathcal{R}$ represents the number of throws. Hence, $X\in\{1,2,3\}$
   Which elementary event ω_i refers to which value of the random variable X?

ω_i	$X(\omega_i)$
6	1
$(\bar{6},6)$	2
$(\bar{6},\bar{6},6)$	3
$(\bar{6},\bar{6},\bar{6})$	3

$P(X=1)=P(6)=\frac{1}{6}$
$P(X=2)=P(\bar{6},6)=\frac{5}{6}\cdot\frac{1}{6}=\frac{5}{36}$
$P(X=3)=P(\bar{6},\bar{6},6)+P(\bar{6},\bar{6},\bar{6})=\left(\frac{5}{6}\right)^2\frac{1}{6}+\left(\frac{5}{6}\right)^3=\frac{25}{36}$

$$\text{pdf:}\qquad f(x)=\begin{cases}\frac{1}{6} & \text{if } x=1\\ \frac{5}{36} & \text{if } x=2\\ \frac{25}{36} & \text{if } x=3\\ 0 & \text{otherwise}\end{cases}$$

$$\text{cdf:}\qquad F(x)=\begin{cases}0 & \text{if } x<1\\ \frac{1}{6} & \text{if } 1\le x<2\\ \frac{11}{36} & \text{if } 2\le x<3\\ 1 & \text{if } x\ge 3\end{cases}$$

3. pdf: $f(x)=3(2-x)^2\mathcal{I}_{(1,2)}(x)$

   cdf: $F(x)=\int_1^x f(s)ds=-(2-s)^3\Big|_1^x=[1-(2-x)^3]\mathbb{I}_{(1,2)}(x)+\mathbb{I}_{[2,\infty)}(x)$

   (a) $P(X<1.2)=F(1.2)=0.488$
   (b) $P(X>1.6)=1-P(X<1.6)=1-F(1.6)=1-0.936=0.064$
   (c) $P(1.2<X<1.6)=P(X<1.6)-P(X<1.2)=F(1.6)-F(1.2)=0.448$

4. (a) $x \in [0,1) : F(x) = \int_0^x z dz = \frac{1}{2}x^2$

$x \in [1,2) : F(x) = \int_1^x (2-z)dz + F(1) = -\frac{1}{2}x^2 + 2x - 1 = 1 - \frac{1}{2}(2-x)^2$

$$\Rightarrow F(x) = \begin{cases} 0 & x < 0 \\ \frac{1}{2}x^2 & x \in [0,1) \\ 1 - \frac{1}{2}(2-x)^2 & x \in [1,2) \\ 1 & x \geq 2 \end{cases}$$

Alternatively,

$$F(x) = \frac{1}{2}x^2 \cdot I_{[0,1)} + \left(1 - \frac{1}{2}(2-x)^2\right) \cdot I_{[1,2)} + I_{[2,\infty)}$$

(b) $x \in [1,3) : F(x) = \int_1^x \frac{1}{4}(3-z)dz + F(1) = -\frac{1}{8}x^2 + \frac{3}{4}x - \frac{1}{8} = 1 - \frac{1}{8}(3-x)^2$

$$\Rightarrow F(x) = \begin{cases} 0 & x < 0 \\ \frac{1}{2}x^2 & x \in [0,1) \\ 1 - \frac{1}{8}(3-x)^2 & x \in [1,3) \\ 1 & x \geq 3 \end{cases}$$

Alternatively,

$$F(x) = \frac{1}{2}x^2 \cdot I_{[0,1)} + \left(1 - \frac{1}{8}(3-x)^2\right) \cdot I_{[1,3)} + I_{[3,\infty)}$$

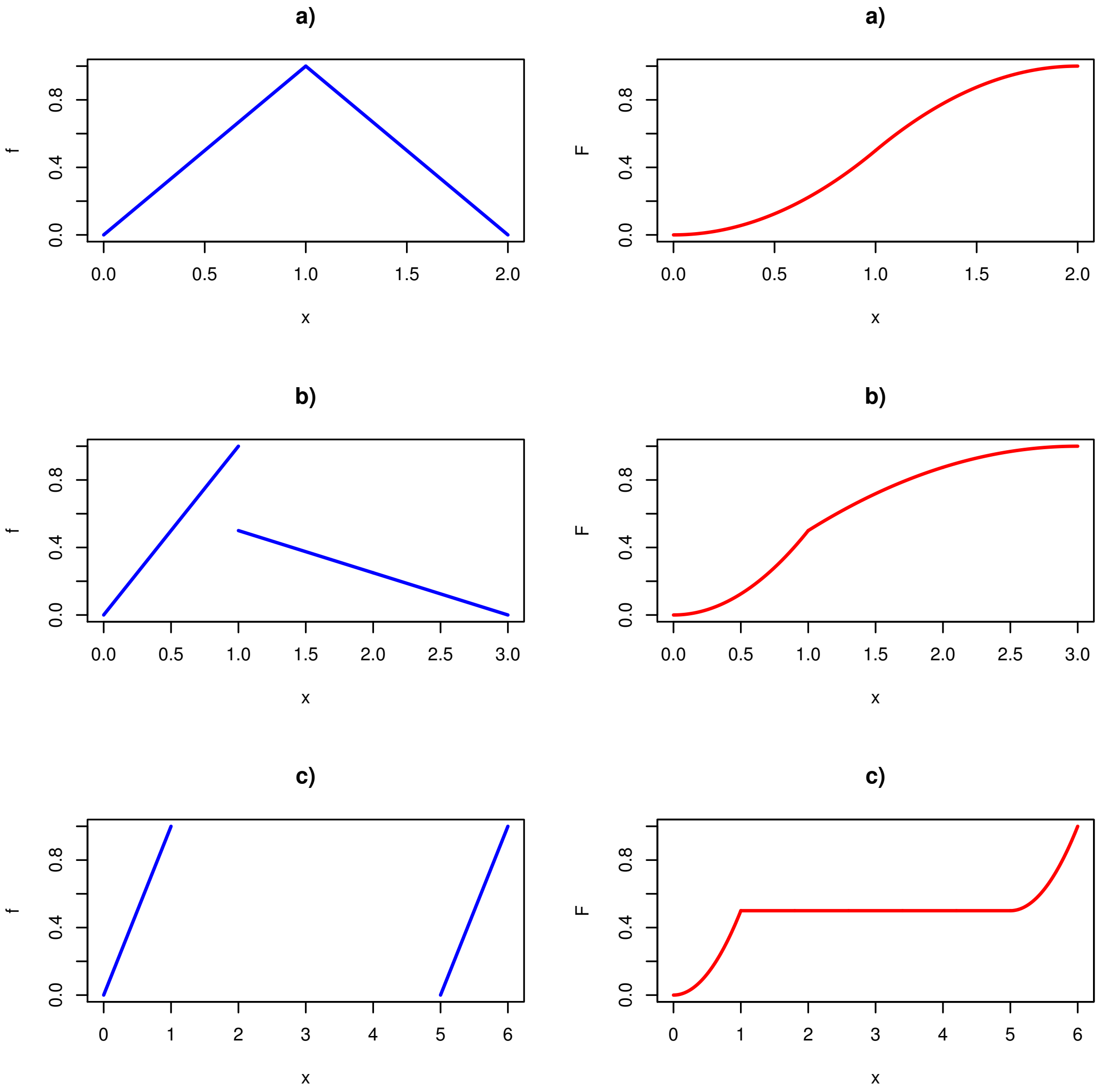
(c) $x \in [5,6) : F(x) = \int_5^x (z-5)dz + F(5) = \frac{1}{2}x^2 - 5x + 13 = \frac{1}{2} + \frac{1}{2}(x-5)^2$

$$\Rightarrow F(x) = \begin{cases} 0 & x < 0 \\ \frac{1}{2}x^2 & x \in [0,1) \\ \frac{1}{2} & x \in [1,5) \\ \frac{1}{2} + \frac{1}{2}(x-5)^2 & x \in [5,6) \\ 1 & x \geq 6 \end{cases}$$

Alternatively,

$$F(x) = \frac{1}{2}x^2 \cdot I_{[0,1)} + \frac{1}{2}I_{[1,5)} + \left(\frac{1}{2} + \frac{1}{2}(x-5)^2\right) \cdot I_{[5,6)} + I_{[6,\infty)}$$

The following graphs show all calculated or given cdfs and pdfs from exercise a) to c).



5. (a) $F' : f(x) = 3(x-2)^2 \mathcal{I}_{[2,3]}(x)$

   (b) $F' : f(x) = \lambda \exp\{-\lambda(x-c)\} \mathcal{I}_{[c,\infty)}(x)$

   (c)

$$F(x) = 1 - \sum_{i=0}^{n-1} \frac{(\lambda x)^i}{i!} e^{-\lambda x} = 1 - e^{-\lambda x} - \sum_{i=1}^{n-1} \frac{(\lambda x)^i}{i!} e^{-\lambda x}$$

$$\begin{aligned} f(x) = F'(x) &= \lambda e^{-\lambda x} - \sum_{i=1}^{n-1} \frac{\lambda^i}{i!} \left(i x^{i-1} e^{-\lambda x} - \lambda e^{-\lambda x} x^i \right) \\ &= e^{-\lambda x} \left[\lambda - \sum_{i=1}^{n-1} \frac{\lambda^i x^{i-1}}{(i-1)!} + \sum_{i=1}^{n-1} \frac{\lambda^{i+1} x^i}{i!} \right] = e^{-\lambda x} \left[\lambda - \sum_{k=0}^{n-2} \frac{\lambda^{k+1} x^k}{k!} + \sum_{i=1}^{n-1} \frac{\lambda^{i+1} x^i}{i!} \right] \\ &= e^{-\lambda x} \left[\lambda - \lambda + \frac{\lambda^n x^{n-1}}{(n-1)!} \right] = \frac{\lambda^n}{(n-1)!} x^{n-1} e^{-\lambda x} \end{aligned}$$

6. (a) $1 \overset{!}{=} k \int_0^1 x_1 dx_1 \int_0^1 x_2 dx_2 = \frac{k}{4} x_1 \Big|_{x_1=0}^{1} x_2 \Big|_{x_2=0}^{1} = \frac{k}{4}$, thus $k = 4$.

   (b) For $X_{1,2} \in (0,1)$ we have
   $F(x_1,x_2) = 4 \int_0^{x_1} s_1 ds_1 \int_0^{x_2} s_2 ds_2 = s_1^2 \Big|_{s_1=0}^{x_1} s_2^2 \Big|_{s_2=0}^{x_2} = x_1^2 x_2^2$ and thus for the complete cdf

$$F(x_1,x_2) = x_1^2 x_2^2 \mathbb{I}_{(0,1)}(x_{1,2}) + x_1^2 \mathbb{I}_{(0,1)}(x_1) \mathbb{I}_{[1,\infty)}(x_2) + x_2^2 \mathbb{I}_{[1,\infty)}(x_1) \mathbb{I}_{(0,1)}(x_2) + \mathbb{I}_{[1,\infty)}(x_{1,2})$$

   (c) We have

$$\begin{aligned} P(0.5 \leq X_1 \leq 1;\ 0.5 \leq X_2 \leq 1) &= F(1,1) - F(1,0.5) - F(0.5,1) + F(0.5,0.5) \\ &= 1 - 0.5^2 - 0.5^2 + 0.5^4 = 0.5625 \end{aligned}$$