Solutions to Problem Set 6

1.

$$
\begin{aligned}
\mu'_r &= E[x^r] = 1^r \cdot \theta + 0^r(1-\theta) = \theta \\
\mu_2 &= E[(x-E(x))^2] = E\left(x^2 - 2E(x)x + E(x)^2\right) = \theta - 2\theta^2 + \theta^2 = \theta(1-\theta) \\
\mu_3 &= E[(x-E(x))^3] = E\left(x^3 - 3x^2E(x) + 3xE(x)^2 - E(x)^3\right) \\
&= \theta - 3\theta^2 + 3\theta^3 - \theta^3 = \theta - 3\theta^2 + 2\theta^3 = \theta(1-\theta)(1-2\theta)
\end{aligned}
$$

2. (a)

$$
\begin{aligned}
E(x-b)^2 &= \int_{-\infty}^{\infty} (x-b)^2 f(x)dx = \int x^2 f(x)dx - 2b\int xf(x)dx + b^2\int f(x)dx \\
\frac{\partial E[\cdot]}{\partial b} &= -2\int xf(x)dx + 2b\int f(x)dx = -2E(x) + 2b \overset{!}{=} 0 \\
b &= E(x)
\end{aligned}
$$

(b) We will need the following rule:

Leibniz rule: applicable to limits of integration that depend on a parameter of the function

$$
\begin{aligned}
I &= \int_{l(z)}^{h(z)} \phi(s,z)ds \\
\frac{\partial I}{\partial z} &= \int_{l(z)}^{h(z)} \frac{\partial \phi}{\partial z}ds + \frac{\partial h}{\partial z}\phi(h(z),z) - \frac{\partial l}{\partial z}\phi(l(z),z)
\end{aligned}
$$

$$
\begin{aligned}
E(|x-b|) &= \int_{-\infty}^{\infty} |x-b| f(x)dx = \int_{b}^{\infty}(x-b)f(x)dx - \int_{-\infty}^{b}(x-b)f(x)dx \\
\frac{\partial E(|x-b|)}{\partial b} &= -\int_{b}^{\infty} f(x)dx + 0 - 1\cdot(b-b)f(b) \\
&\quad - \left(-\int_{-\infty}^{b} f(x)dx + 1\cdot(b-b)f(b) - 0\right) \\
&= -\int_{b}^{\infty} f(x)dx + \int_{-\infty}^{b} f(x)dx = -1 + \int_{-\infty}^{b} f(x)dx + \int_{-\infty}^{b} f(x)dx
\end{aligned}
$$

$$= -1 + 2 \int_{-\infty}^{b} f(x)dx = -1 + 2F(b) \stackrel{!}{=} 0$$

$$F(b) = \frac{1}{2} \quad \Rightarrow b = x_{\text{Median}}$$

(c) A symmetric density around $x = b$ implies: $f(b + x_0) = f(b - x_0) \quad \forall x_0 \in R(x)$.

i.

$$\begin{aligned}
E(X) &= \int_{-\infty}^{\infty} xf(x)dx = \int_{-\infty}^{b} xf(x)dx + \int_{b}^{\infty} xf(x)dx \\
&= -\int_{+\infty}^{0} (b - x_0)f(b - x_0)dx_0 + \int_{0}^{\infty} (b + x_0)f(b + x_0)dx_0 \\
&= \int_{0}^{\infty} bf(b - x_0)dx_0 - \int_{0}^{\infty} x_0 f(b - x_0)dx_0 + \int_{0}^{\infty} bf(b + x_0)dx_0 \\
&\quad + \int_{0}^{\infty} x_0 f(b + x_0)dx_0 \\
&= 2b \int_{0}^{\infty} f(b + x_0)dx_0, \qquad \text{because} f(b + x_0) = f(b - x_0) \\
&= 2b \int_{b}^{\infty} f(x)dx = 2b(1 - F(b)) = 2b\left(1 - \frac{1}{2}\right) = b
\end{aligned}$$

ii.

$$\begin{aligned}
\mu_3 &= \int_{-\infty}^{\infty} (x - E(x))^3 f(x)dx = \int_{-\infty}^{\infty} (x - b)^3 f(x)dx \\
&= \int_{-\infty}^{b} (x - b)^3 f(x)dx + \int_{b}^{\infty} (x - b)^3 f(x)dx \\
&= -\int_{\infty}^{0} (-x_0)^3 f(b - x_0)dx_0 + \int_{0}^{\infty} x_0^3 f(b + x_0)dx_0 \\
&= \int_{0}^{\infty} (-x_0)^3 f(b - x_0)dx_0 + \int_{0}^{\infty} x_0^3 f(b - x_0)dx_0 \\
&= \int_{0}^{\infty} ((-x_0)^3 + x_0^3) f(b - x_0)dx_0 = 0
\end{aligned}$$

iii. We have

$$\begin{aligned}
\frac{\partial E[(x - b)^4]}{\partial b} &= -4E[(x - b)^3] \stackrel{!}{=} 0 \\
E[(x - b)^3] &= 0
\end{aligned}$$

Since $E[(x - b)^4]$ is a convex function a unique minimum exists, from part ii we know that $\mu_3 = E[(x - b)^3] = 0$, thus it follows that $b = E(x)$.

3. First note

$$\begin{aligned} E[x^\alpha] &= 1 = (E[x])^\alpha && \text{if } \alpha = 0 \\ E[x^\alpha] &= E[x] = (E[x])^\alpha && \text{if } \alpha = 1. \end{aligned}$$

Assume now that $\alpha \neq 0{,}1$. Let $g(x) = x^\alpha$. Then we have

$$g''(x) = \alpha(\alpha - 1)x^{\alpha-2}.$$

On $(0{,}\infty)$, we can say $g''(x)$ is positive, if $\alpha < 0$ or $\alpha > 1$. It is negative, if $0 < \alpha < 1$. Therefore $g(x)$ is strictly concave for $\alpha \in (0{,}1)$ and strictly convex otherwise. Using Jensen's inequality we have

$$\begin{aligned} E[x^\alpha] &\geq (E[x])^\alpha && \text{if } \alpha < 0 \text{ or } \alpha > 1 \\ E[x^\alpha] &\leq (E[x])^\alpha && \text{if } 0 < \alpha < 1. \end{aligned}$$

4. (a) Given: $E[x] = 8$
   Consider Markov's inequality:

$$P(x \geq a) \leq \frac{E[x]}{a}$$

$$P(x \geq 16) \leq \frac{8}{16} = \frac{1}{2}$$

$$P(x < 16) = 1 - P(x \geq 16) \geq 1 - \frac{1}{2} = \frac{1}{2}$$

   (b) Given: $x > 0$ and $var[x] = 32$
   Consider Chebyshev's inequality:

$$P(|x - E[x]| \geq a) \leq \frac{1}{a^2} var[x]$$

$$P(|x - 8| \geq 8) \leq \frac{1}{8^2} \cdot 32 = \frac{1}{2}$$

$$P(|x - 8| < 8) = P(-8 < x - 8 < 8) = P(0 < x < 16) \geq \frac{1}{2}$$

   Alternatively, use Markov's inequality:

$$E(x^2) = E(x)^2 + var(x) = 96$$

$$P(x \geq 16) = P(x^2 \geq 16^2) \leq \frac{E[x^2]}{16^2} = 0.375$$

   Hence,

$$P(x < 16) = 1 - P(x \geq 16) \geq 0.625$$

   Markov's inequality gives us a tighter lower bound on $P(x < 16)$ and therefore is more informative than Chebyshev's inequality.

(c) $\mu = 17 \qquad \sigma = 0.25$

$P(16 \leq X \leq 18) = P(16 - \mu \leq X - \mu \leq 18 - \mu) = P(-1 \leq X - \mu \leq 1)$

$P(|X - \mu| \leq 1) \overset{\text{CI}}{\geq} 1 - \frac{\sigma^2}{1} = 1 - \frac{1}{16} = \frac{15}{16}$

(d) $\mu = 2$

$P(x \geq 10) \overset{\text{MI}}{\leq} \frac{E[x]}{10} = \frac{2}{10} = 0.2 = 20\% \rightarrow$ No!

5. (a)

$$\begin{aligned} f_1(x_1) &= \int_0^\infty f(x)dx_2 = \int_0^\infty \frac{12}{(1+x_1+x_2)^5} dx_2 \; I_{[0,\infty)}(x_1) \; I_{[0,\infty)}(x_2) \\ &= \frac{-3}{(1+x_1+x_2)^4}\bigg|_0^\infty I_{[0,\infty)}(x_1) = \frac{3}{(1+x_1)^4} \; I_{[0,\infty)}(x_1) \end{aligned}$$

Due to symmetry:

$$f_2(x_2) = \frac{3}{(1+x_2)^4} \; I_{[0,\infty)}(x_2)$$

Obviously:

$$f_1(x_1) \cdot f_2(x_2) \neq f(x_1,x_2) \;\Rightarrow \quad \text{not independent}$$

(b)

$$\begin{aligned} E(x_1) &= \int_0^\infty x_1 f_1(x_1) dx_1 = \int_0^\infty \frac{3x_1}{(1+x_1)^4} dx_1 \\ &\overset{\text{PI}}{=} \underbrace{\frac{-x_1}{(1+x_1)^3}\bigg|_0^\infty}_{=0} - \int_0^\infty \frac{-1}{(1+x_1)^3} dx_1 = -\frac{1}{2(1+x_1)^2}\bigg|_0^\infty = \frac{1}{2} \\ E(x_1^2) &= \int_0^\infty x_1^2 f_1(x_1) dx_1 = \int_0^\infty \frac{3x_1^2}{(1+x_1)^4} dx_1 \\ &\overset{\text{PI}}{=} \underbrace{\frac{-x_1^2}{(1+x_1)^3}\bigg|_0^\infty}_{=0} - \int_0^\infty \frac{-2x_1}{(1+x_1)^3} dx_1 \\ &\overset{\text{PI}}{=} \underbrace{\frac{-x_1}{(1+x_1)^2}\bigg|_0^\infty}_{=0} - \int_0^\infty \frac{-1}{(1+x_1)^2} dx_1 = \frac{-1}{(1+x_1)}\bigg|_0^\infty = 1 \end{aligned}$$

$$Var(x_1) = E(x_1^2) - E(x_1)^2 = 1 - \left(\frac{1}{2}\right)^2 = \frac{3}{4}$$

Due to symmetry:

$$E(x_2) = \frac{1}{2} \; , E(x_2^2) = 1 \; , Var(x_2) = \frac{3}{4}$$

$$
\begin{aligned}
E(x_1 \cdot x_2) &= \int_{x_1=0}^{\infty} \int_{x_2=0}^{\infty} x_1 x_2 f(x_1,x_2) dx_1 dx_2 = \int_0^\infty \int_0^\infty \frac{12 x_1 x_2}{(1+x_1+x_2)^5} dx_1 dx_2 \\
&\overset{\text{PI}}{=} \int_0^\infty \left[\underbrace{\frac{-3x_1x_2}{(1+x_1+x_2)^4}\bigg|_0^\infty}_{=0} - \int_0^\infty \frac{-3x_1}{(1+x_1+x_2)^4} dx_2 \right] dx_1 \\
&= \int_0^\infty \left[\frac{-x_1}{(1+x_1+x_2)^3} \right]_{x_2=0}^\infty dx_1 = \int_0^\infty \frac{x_1}{(1+x_1)^3} dx_1 \\
&\overset{\text{PI}}{=} \underbrace{\frac{-x_1}{2(1+x_1)^2}\bigg|_0^\infty}_{=0} - \int_0^\infty \frac{-1}{2(1+x_1)^2} dx_1 = \frac{-1}{2(1+x_1)}\bigg|_0^\infty = \frac{1}{2}
\end{aligned}
$$

$Cov(x_1,x_2) = E(x_1 \cdot x_2) - E(x_1) \cdot E(x_2) = \frac{1}{2} - \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$. Thus $\Rightarrow Cov(X) = \begin{pmatrix} \frac{3}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{3}{4} \end{pmatrix}$

(c) Regression curve from x_1 on x_2 is $E(x_1|x_2)$.

$$
\begin{aligned}
f(x_1|x_2) &= \frac{f(x_1,x_2)}{f_2(x_2)} = \frac{\frac{12}{(1+x_1+x_2)^5}}{\frac{3}{(1+x_2)^4}} I_{[0,\infty)}(x_1)\, I_{[0,\infty)}(x_2) \\
&= \frac{4(1+x_2)^4}{(1+x_1+x_2)^5} I_{[0,\infty)}(x_1)\, I_{[0,\infty)}(x_2)
\end{aligned}
$$

$$
\begin{aligned}
E(x_1|x_2) &= \int_0^\infty x_1 f(x_1|x_2) dx_1 = 4(1+x_2)^4 \int_{x_1=0}^\infty \frac{x_1}{(1+x_1+x_2)^5} dx_1\, I_{[0,\infty)}(x_2) \\
&\overset{\text{PI}}{=} (1+x_2)^4 \cdot \left[\underbrace{\frac{-x_1}{(1+x_1+x_2)^4}\bigg|_0^\infty}_{=0} - \int_{x_1=0}^\infty \frac{-1}{(1+x_1+x_2)^4} dx_1 \right] I_{[0,\infty)}(x_2) \\
&= (1+x_2)^4 \cdot \left[\frac{-1}{3(1+x_1+x_2)^3} dx_1 \right]_0^\infty I_{[0,\infty)}(x_2) \\
&= \frac{1}{3}(1+x_2)\, I_{[0,\infty)}(x_2)
\end{aligned}
$$

Due to symmetry:

$$
E(x_2|x_1) = \frac{1}{3}(1+x_1)\, I_{[0,\infty)}(x_1)
$$

(d)

$$
\begin{aligned}
E(x_1^2|x_2) &= \int_0^\infty x_1^2 f(x_1|x_2) dx_1 = 4(1+x_2)^4 \int_{x_1=0}^\infty \frac{x_1^2}{(1+x_1+x_2)^5} dx_1\, I_{[0,\infty)}(x_2) \\
&\overset{\text{PI}}{=} (1+x_2)^4 \cdot \left[\underbrace{\frac{-x_1^2}{(1+x_1+x_2)^4}\bigg|_0^\infty}_{=0} - \int_{x_1=0}^\infty \frac{-2x_1}{(1+x_1+x_2)^4} dx_1 \right] I_{[0,\infty)}(x_2)
\end{aligned}
$$

$$\overset{\text{PI}}{=} (1+x_2)^4 \cdot \left[\underbrace{\frac{-2x_1}{3(1+x_1+x_2)^3}\Bigg|_0^\infty}_{=0} - \int_{x_1=0}^{\infty} \frac{-2}{3(1+x_1+x_2)^3} dx_1 \right]_0^\infty I_{[0,\infty)}(x_2)$$

$$= (1+x_2)^4 \cdot \left[\frac{-1}{3(1+x_1+x_2)^2} dx_1 \right]_0^\infty I_{[0,\infty)}(x_2)$$

$$= \frac{1}{3}(1+x_2)^2 \, I_{[0,\infty)}(x_2)$$

$$Var(x_1|x_2) = E(x_1^2|x_2) - E(x_1|x_2)^2 = \frac{1}{3}(1+x_2)^2 - (\frac{1}{3}(1+x_2))^2 = \frac{2}{9}(1+x_2)^2$$