Solutions 9

1. (a) $Z_k = \sum_{i=1}^{k} X_i^2$ for $k < n$. Due to theorem 4.7, $Z_k \sim \chi^2(k)$.

   (b) $Y_1 = \delta Z_k$ for $\delta \in (0, \infty)$. Since $Z_k \sim \chi^2(k)$, it holds that $Z_k \sim \text{Gamma}(k/2, 2)$. Then according to theorem 4.3

$$Y_1 \sim \text{Gamma}\left(\frac{k}{2}, 2\delta\right)$$

   (c) $Y_2 = \frac{1}{n}\sum_{i=1}^{n} \frac{X_i + a}{\sqrt{b}}$ for $a, b \in \mathbb{R}_+$.

$$\begin{aligned}
X_i + a &\sim \mathcal{N}(a, 1) \\
\frac{X_i + a}{\sqrt{b}} &\sim \mathcal{N}\left(\frac{a}{\sqrt{b}}, \frac{1}{b}\right) \\
\sum_{i=1}^{n} \frac{X_i + a}{\sqrt{b}} &\sim \mathcal{N}\left(\frac{an}{\sqrt{b}}, \frac{n}{b}\right) \\
Y_2 &\sim \mathcal{N}\left(\frac{a}{\sqrt{b}}, \frac{1}{nb}\right)
\end{aligned}$$

2. Given $X_1 \sim \mathcal{N}(1, 1)$ and $X_2 \sim \mathcal{N}(1, 1)$ with $\rho = 0.6$. Rewrite

$$\mu = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \Sigma = \begin{bmatrix} 1 & \sigma_{12} \\ \sigma_{21} & 1 \end{bmatrix}$$

   Since

$$\rho = \frac{\text{Cov}[X_1 X_2]}{\sqrt{\text{Var}[X_1]\text{Var}[X_2]}} = \frac{\sigma_{21}}{\sigma_1 \sigma_2} = \sigma_{21},$$

   it follows that $\sigma_{21} = \sigma_{12} = 0.6$. Using theorem 4.12 we get

$$X_1 | X_2 = 2 \sim \mathcal{N}(1 + 0.6 \cdot 1(2 - 1), 1 - 0.6^2/1) = \mathcal{N}(1.6, 0.64)$$

3. (a) It belongs to the multivariate (bivariate) normal family.

   (b)

$$\begin{aligned}
\Sigma^{-1} &= \begin{pmatrix} 2 & -3 \\ -3 & 5 \end{pmatrix} \\
\Sigma &= \frac{1}{2 \cdot 5 - (-3)^2} \begin{pmatrix} 5 & 3 \\ 3 & 2 \end{pmatrix} \\
|\Sigma| &= det(\Sigma) = 2 \cdot 5 - (-3)^2 = 1 \\
k &= \frac{1}{\sqrt{(2\pi)^2 |\Sigma|}} = \frac{1}{2\pi}
\end{aligned}$$

   (c) Apply theorem 4.11 for the marginal pdf's to obtain $X_1 \sim \mathcal{N}(1, 5)$ and $X_2 \sim \mathcal{N}(-5, 2)$. For the conditional pdf's use theorem 4.12

$$X_1 | X_2 = x_2 \sim \mathcal{N}(1 + 3 \cdot 2^{-1}(x_2 - (-5)), 5 - 3 \cdot 2^{-1} \cdot 3) = \mathcal{N}\left(\frac{3}{2}x_2 + \frac{17}{2}, \frac{1}{2}\right)$$

$$X_2 | X_1 = x_1 \sim \mathcal{N}(-5 + 3 \cdot 5^{-1}(x_1 - 1), 2 - 3 \cdot 5^{-1} \cdot 3) = \mathcal{N}\left(\frac{3}{5}x_1 - \frac{28}{5}, \frac{1}{5}\right)$$

(d) $E(x_1|x_2) = \frac{3}{2}x_2 + \frac{17}{2}$

4. (a) Theorem 4.10:

$$Y = (Y_1, Y_2, Y_3)' \sim \mathcal{N}(A \cdot 0 + b,\, AIA') = \mathcal{N}\left(\begin{pmatrix}1\\2\\3\end{pmatrix}, \begin{pmatrix}1&0&0\\0&1&1\\0&0&1\end{pmatrix}\begin{pmatrix}1&0&0\\0&1&0\\0&1&1\end{pmatrix}\right)$$

$$= \mathcal{N}\left(\begin{pmatrix}1\\2\\3\end{pmatrix}, \begin{pmatrix}1&0&0\\0&2&1\\0&1&1\end{pmatrix}\right)$$

Special case: $A = (w_1, ..., w_m) \Rightarrow \quad Y = \sum_{i=1}^m w_i x_i + b$
$\rightarrow$ Linear transformations of normal is again normal!

(b) Reorder Y :

$$\widetilde{Y} = \begin{pmatrix}Y_1\\Y_3\\Y_2\end{pmatrix} \sim \mathcal{N}\left(\begin{pmatrix}1\\3\\2\end{pmatrix}, \begin{pmatrix}1&0&0\\0&1&1\\0&1&2\end{pmatrix}\right)$$

$$= \begin{pmatrix}Y_{(1)}\\Y_{(2)}\end{pmatrix} \sim \mathcal{N}\left(\begin{pmatrix}\boldsymbol{\mu_{(1)}}\\\boldsymbol{\mu_{(2)}}\end{pmatrix}, \begin{pmatrix}\boldsymbol{\Sigma_{11}}&\boldsymbol{\Sigma_{12}}\\\boldsymbol{\Sigma_{21}}&\boldsymbol{\Sigma_{22}}\end{pmatrix}\right)$$

$\boldsymbol{\mu_{(1)}} = \begin{pmatrix}1\\3\end{pmatrix}, \quad \boldsymbol{\mu_{(2)}} = 2$

$\boldsymbol{\Sigma_{11}} = \begin{pmatrix}1&0\\0&1\end{pmatrix}, \quad \boldsymbol{\Sigma_{12}} = \begin{pmatrix}0\\1\end{pmatrix}, \quad \boldsymbol{\Sigma_{21}} = \begin{pmatrix}0&1\end{pmatrix}, \quad \boldsymbol{\Sigma_{22}} = 2$

By theorem 4.11:

$Y_{(1)} \sim \mathcal{N}(\boldsymbol{\mu_{(1)}}, \boldsymbol{\Sigma_{11}}) = \mathcal{N}\left(\begin{pmatrix}1\\3\end{pmatrix}, \begin{pmatrix}1&0\\0&1\end{pmatrix}\right)$

$Y_{(2)} \sim \mathcal{N}(\boldsymbol{\mu_{(2)}}, \boldsymbol{\Sigma_{22}}) = \mathcal{N}(2, 2)$

(c) Theorem 4.12

$$Y_{(1)}|(Y_{(2)} = \alpha) \sim \mathcal{N}\left(\boldsymbol{\mu_{(1)}} + \boldsymbol{\Sigma_{12}\Sigma_{22}^{-1}}\left[\alpha - \boldsymbol{\mu_{(2)}}\right], \boldsymbol{\Sigma_{11}} - \boldsymbol{\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}}\right)$$

$$= \mathcal{N}\left(\begin{pmatrix}1\\3\end{pmatrix} + \begin{pmatrix}0\\1\end{pmatrix} \cdot 2^{-1} \cdot [\alpha - 2], \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} - \begin{pmatrix}0\\1\end{pmatrix} \cdot 2^{-1} \cdot \begin{pmatrix}0 & 1\end{pmatrix}\right)$$

$$= \mathcal{N}\left(\begin{pmatrix}1\\2 + 0.5\alpha\end{pmatrix}, \begin{pmatrix}1 & 0\\0 & 0.5\end{pmatrix}\right)$$

$$Y_{(2)}|(Y_{(1)} = \boldsymbol{\beta}) \sim \mathcal{N}\left(\boldsymbol{\mu_{(2)}} + \boldsymbol{\Sigma_{21}\Sigma_{11}^{-1}}\left[\boldsymbol{\beta} - \boldsymbol{\mu_{(1)}}\right], \boldsymbol{\Sigma_{22}} - \boldsymbol{\Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12}}\right)$$

$$= \mathcal{N}\left(2 + \begin{pmatrix}0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}\begin{pmatrix}\beta_1 - 1\\\beta_2 - 3\end{pmatrix}, 2 - \begin{pmatrix}0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}\right)$$

$$= \mathcal{N}\left(\beta_2 - 1, 1\right)$$

5. $X \sim \mathcal{N}(0, 1)$, thus

$$f(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}\mathcal{I}_{(-\infty,\infty)}(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}\mathcal{I}_{(-\infty,0)}(x) + \frac{1}{\sqrt{2\pi}}e^{-x^2/2}\mathcal{I}_{(0,\infty)}(x)$$
$$\equiv f_-(x) + f_+(x)$$

Use the transformation theorem for $f_\pm(x)$ and check the requirements:

i) $Y = X^2 = g(x) \Rightarrow \frac{\partial g(x)}{\partial x} \neq 0 \;\forall\; x$ ✓

ii) $X = \pm\sqrt{Y} = g_\pm^{-1}(Y)$ exists $\forall\; y$ ✓

Note that $Y \in (0, \infty)$ for both parts of the pdf.

$$\frac{\partial g_\pm^{-1}(y)}{\partial y} = \pm\frac{1}{2\sqrt{y}} \Rightarrow \left|\frac{\partial g_\pm^{-1}(y)}{\partial y}\right| = \frac{1}{2\sqrt{y}}$$

$$h(y) = h_-(y) + h_+(y) = f_-(g_-^{-1}(y))\left|\frac{\partial g_-^{-1}(y)}{\partial y}\right| + f_+(g_+^{-1}(y))\left|\frac{\partial g_+^{-1}(y)}{\partial y}\right|$$
$$= \frac{1}{\sqrt{2\pi}}e^{-y/2}\frac{1}{2\sqrt{y}} + \frac{1}{\sqrt{2\pi}}e^{-y/2}\frac{1}{2\sqrt{y}} = \frac{1}{\sqrt{2\pi}}y^{-1/2}e^{-y/2}\mathcal{I}_{(0,\infty)}(y),$$

which is the same pdf as of a chi-squared with one degree since $\Gamma(0.5) = \sqrt{\pi}$.