Problem Set 5

1. (a) i. Wald test: $H_0 : \boldsymbol{\beta}_0 = 0,\ \boldsymbol{\beta}_1 = 1$ vs. $H_1 : \neg H_0$

ii. Test statistic: $W_N = \left[\mathbf{R}\widehat{\boldsymbol{\beta}} - \mathbf{r}\right]' \left[\mathbf{R} \cdot \widehat{\mathbf{Avar}(\hat{\boldsymbol{\beta}})} \cdot \mathbf{R}'\right]^{-1} \left[\mathbf{R}\widehat{\boldsymbol{\beta}} - \mathbf{r}\right]$

with $\mathbf{R} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \boldsymbol{I}, r = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $\mathbf{Avar}(\hat{\boldsymbol{\beta}}) = N^{-1}\boldsymbol{A}^{-1}\boldsymbol{B}\boldsymbol{A}^{-1} \overset{\text{homosk.}}{=} N^{-1}\sigma^2\boldsymbol{A}^{-1}$, or estimated $\widehat{\mathbf{Avar}}(\hat{\boldsymbol{\beta}}) = N^{-1}\hat{\sigma}^2\boldsymbol{A}^{-1} = \hat{\sigma}^2(\boldsymbol{X}'\boldsymbol{X})^{-1}$.

For the test statistic we need $\hat{\boldsymbol{\beta}}$ and $\hat{\sigma}^2$:

$$\begin{aligned}
\hat{\boldsymbol{\beta}} &= (\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}\boldsymbol{y} = \begin{pmatrix} 100 & -50 \\ -50 & 150 \end{pmatrix}^{-1} \begin{pmatrix} -60 \\ 200 \end{pmatrix} = \begin{pmatrix} 0.08 \\ 1.36 \end{pmatrix} \\
\hat{\sigma}^2 &= \frac{1}{N-k}\left(\boldsymbol{y}'\boldsymbol{y} - \boldsymbol{y}'\boldsymbol{X}\hat{\boldsymbol{\beta}}\right) = \frac{1}{100-2} \cdot \left(300 - \begin{pmatrix} -60 & 200 \end{pmatrix}\begin{pmatrix} 0.08 \\ 1.36 \end{pmatrix}\right) = \\
&= \frac{1}{98}(300 - 267.2) \approx 0.3347 \\
W_N &= \left(\boldsymbol{I}\hat{\boldsymbol{\beta}} - \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)' \left(\boldsymbol{I}\hat{\sigma}^2 (\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{I}\right)^{-1} \left(\boldsymbol{I}\hat{\boldsymbol{\beta}} - \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) = \\
&= \frac{\begin{pmatrix} 0.08 & 0.36 \end{pmatrix}(\boldsymbol{X}'\boldsymbol{X})\begin{pmatrix} 0.08 \\ 0.36 \end{pmatrix}}{\hat{\sigma}^2} \\
&\approx \frac{1}{0.3347}\begin{pmatrix} 0.08 & 0.36 \end{pmatrix}\begin{pmatrix} 100 & -50 \\ -50 & 150 \end{pmatrix}\begin{pmatrix} 0.08 \\ 0.36 \end{pmatrix} \approx 51.3902
\end{aligned}$$

iii. Critical value: $CV = \chi^2_q \overset{\alpha=0.05}{=} \chi^2_{2,0.95} = 5.9915$

iv. Decision: Since $W > CV$, we reject H_0!

(b) $N = 12$, i.e. the sample is "small". For this reason, rather use an F-test instead of a Wald test, since the latter is an asymptotic test and is only suitable for large samples. However, we have to assume normally distributed residuals in order to apply the F-test.

2. (a) We know that

$$\hat{\boldsymbol{\beta}} \overset{a}{\sim} N\left(0, \frac{\boldsymbol{V}}{N}\right) = N\left(0, \frac{\boldsymbol{A}^{-1}\boldsymbol{B}\boldsymbol{A}^{-1}}{N}\right) \left[\overset{\text{homosk.}}{=} N\left(0, \frac{\sigma^2\boldsymbol{A}^{-1}}{N}\right)\right]$$

where $\boldsymbol{A} = \mathrm{E}[\boldsymbol{x}'\boldsymbol{x}], \boldsymbol{B} = E[u^2\boldsymbol{x}'\boldsymbol{x}] \left(\overset{\text{homosk.}}{=} \sigma^2\boldsymbol{A}\right)$.

$$\begin{aligned}
\widehat{\text{AVar}}\left(\hat{\boldsymbol{\beta}}\right)_{\text{homosk.}} &= \frac{\hat{\sigma}^2\hat{\boldsymbol{A}}^{-1}}{N} = \hat{\sigma}^2 \cdot \frac{1}{N} \cdot \left(\frac{1}{N}\sum \boldsymbol{x}_i'\boldsymbol{x}\right)^{-1} \\
&= \hat{\sigma}^2 \cdot \left(\sum \boldsymbol{x}_i'\boldsymbol{x}\right)^{-1} = \begin{pmatrix} 0.0271 & -0.0026 \\ -0.0026 & 0.0003 \end{pmatrix}
\end{aligned}$$

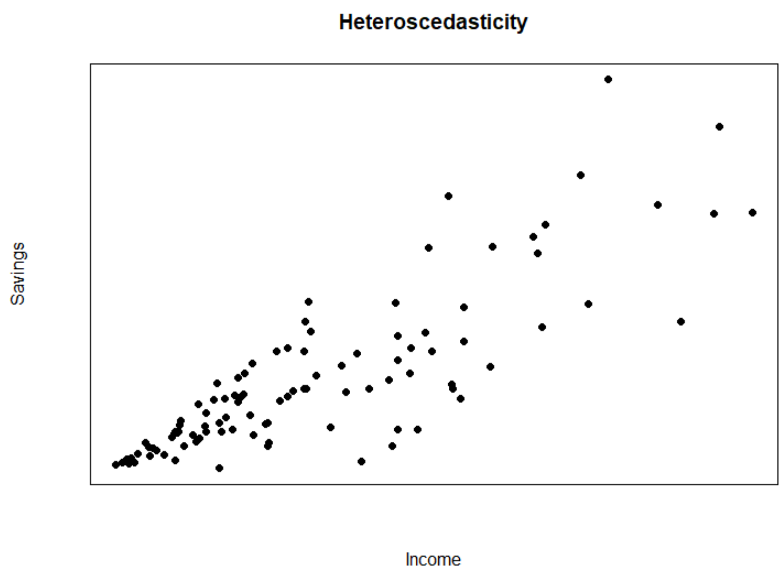
$$
\begin{aligned}
\Rightarrow \widehat{\text{ASE}}\left(\hat{\boldsymbol{\beta}}_1\right)_{\text{homosk.}} &= \sqrt{0.0003} = 0.0165 \\
\widehat{\text{AVar}}\left(\hat{\boldsymbol{\beta}}\right)_{\text{heterosk.}} &= \frac{\left(\frac{1}{N}\sum \boldsymbol{x}_i'\boldsymbol{x}\right)^{-1}\left(\frac{1}{N-K}\sum \hat{u}_i^2\boldsymbol{x}_i'\boldsymbol{x}_i\right)\left(\frac{1}{N}\sum \boldsymbol{x}_i'\boldsymbol{x}_i\right)^{-1}}{N} \\
&= \frac{N}{N-K}\left(\sum \boldsymbol{x}_i'\boldsymbol{x}_i\right)^{-1}\left(\sum \hat{u}_i^2\boldsymbol{x}_i'\boldsymbol{x}_i\right)\left(\sum \boldsymbol{x}_i'\boldsymbol{x}_i\right)^{-1} \\
&= \begin{pmatrix} 0.0691 & -0.0026 \\ -0.0026 & 0.0033 \end{pmatrix} \\
\Rightarrow \widehat{\text{ASE}}\left(\hat{\boldsymbol{\beta}}_1\right)_{\text{heterosk.}} &= \sqrt{0.0033} = 0.0573 \\
\Rightarrow \widehat{\text{ASE}}\left(\hat{\boldsymbol{\beta}}_1\right)_{\text{homosk.}} &< \widehat{\text{ASE}}\left(\hat{\boldsymbol{\beta}}_1\right)_{\text{heterosk.}}
\end{aligned}
$$

(b) $H_0 : \beta_1 = 0.55$ vs. $H_1 : \beta_1 \neq 0.55$

$$
\begin{aligned}
t_{homosk.} &= \frac{\hat{\beta}_1 - 0.55}{\widehat{\text{ASE}}_{homosk.}} = \frac{0.61 - 0.55}{0.0165} = 3.5491 \\
t_{heterosk.} &= \frac{\hat{\beta}_1 - 0.55}{\widehat{\text{ASE}}_{heterosk.}} = \frac{0.61 - 0.55}{0.0573} = 1.0201 \\
CV &= z_{1-\alpha/2} = 1.96
\end{aligned}
$$

Under homoskedasticity we would reject H_0, under heteroskedasticity we would not.

(c) For the given model, we should not assume homoskedasticity. People having higher income have more variation in their savings than people with low income, since people with lower income typically spend much of their income for fixed costs. Hence, variation in savings among people with lower income is smaller than among people with higher income:



Hence, the variation in errors is increasing with income, which corresponds to heteroskedasticity. We should only trust test results obtained under the heteroskedasticity assumption.