Sheet - 05

Group 27 / Comput Finance

1.

T-Exercise 16 : Show that X

with $dX(t) := (\kappa - \lambda X(t))dt + \sigma dW(t)$ and $X(0) = x$ solves the equation

$$X(t) = xe^{-\lambda t} + \frac{\kappa}{\lambda}(1 - e^{-\lambda t}) + \int_0^t e^{-\lambda(t-s)} \sigma dW(s)$$

(1=) define $u(t) = e^{-\lambda t}$ in $X(t)$ and $v(t) = (x + \frac{\kappa}{\lambda}(\frac{1}{e^{-\lambda t}} - 1) + \int_0^t e^{\lambda s} \sigma dW(s))$

$\frac{1}{e^{-\lambda t}} = e^{\lambda t}$

$\frac{du(t)}{dt} = -\lambda e^{-\lambda t} = -\lambda u(t) \Rightarrow du(t) = -\lambda u(t) dt$

$\frac{dv(t)}{\cancel{dt}} = 0 + \kappa e^{\lambda t} dt + d\left(\int_0^t e^{\lambda s}\sigma dW(s)\right) = \kappa e^{\lambda t} dt + e^{\lambda t} \sigma dW(t)$ notation -0,5

Using Partial differential derivative What is that?

$d(u(t)v(t)) = e^{-\lambda t}(\kappa e^{\lambda t} + e^{\lambda t}\sigma dW(t)) - \lambda e^{-\lambda t}(v(t))dt + d[u,v](t)$, where $d[u,v](t) = 0$

$= e^{-\lambda t}\kappa e^{\lambda t} + e^{-\lambda(t-t)}\sigma dW(t) - \lambda e^{-\lambda t} x d(t) - \lambda e^{-\lambda t} \cdot \kappa \cdot \frac{1}{\lambda e^{-\lambda t}} + \lambda e^{-\lambda t}\frac{\kappa}{\lambda} d(t) - \lambda e^{-\lambda t} d(t) \int_0^t e^{\lambda s} \cdot \sigma dW(s)$

$= e^{-\lambda t}\underline{\kappa} e^{\lambda t} + e^{-\lambda(t-t)}\sigma dW(t) - \lambda e^{-\lambda t} x d(t) - \underline{\kappa} + \kappa e^{-\lambda t} d(t) - \lambda \int_0^t e^{-\lambda(t-s)} \sigma dW(s)$

$= + \sigma dW(t) - \lambda(e^{-\lambda t}x + \frac{1}{\lambda}\kappa e^{-\lambda t} + \int_0^t e^{-\lambda(t-s)}\sigma dW(s))dt$

$= (\kappa - \lambda(e^{-\lambda t}x + \frac{\kappa}{\lambda}(1 - e^{-\lambda t}) + \int_0^t e^{-\lambda(t-s)}\sigma dW(s))dt + \sigma dW(t)$

$= (\kappa - \lambda X(t))dt + \sigma dW(t) = dX(t)$. ✓

You also need to check that $X(0) = x$. -1

2,5/4