Solutions 10

1. (a)

$$f(x;\alpha,\beta) = \frac{1}{B(\alpha,\beta)} x^{\alpha-1}(1-x)^{\beta-1} \qquad \underbrace{\mathbb{I}_{(0,1)}(x)}_{\text{independent of } \alpha \text{ and } \beta!}$$

$$= \exp\left[\underbrace{-\ln(B(\alpha,\beta))}_{d(\Theta)} + \underbrace{(\alpha-1)\ \ln(x)}_{c_1(\Theta)\, g_1(x)} + \underbrace{(\beta-1)\ \ln(1-x)}_{c_2(\Theta)\, g_2(x)}\right]$$

$\Rightarrow$ exponential class

(b)

$$f(x;a,b) = \frac{1}{b-a} \quad \underbrace{\mathbb{I}_{(a,b)}(x)}_{\text{depends on a and b!}}$$

the support must not depend on the parameters to be a member of the exponential class!
$\Rightarrow$ not a member of the exponential class

2. (a) Generalized normal : $f(x) = \dfrac{\beta}{2\alpha\Gamma(\frac{1}{\beta})} \exp\left(-\left|\dfrac{x-\mu}{\alpha}\right|^{\beta}\right), \quad \mu \in \mathbb{R}, \quad \alpha,\beta > 0$

Here: $f(x) = C\ \exp\left(-\dfrac{1}{2}\lambda|x|^{\gamma}\right) = C\ \exp\left(-\left|\dfrac{x}{(\frac{2}{\lambda})^{\frac{1}{\gamma}}}\right|^{\gamma}\right)$

Hence: $\beta = \gamma, \quad \mu = 0,$ and $\alpha = (\frac{2}{\lambda})^{\frac{1}{\gamma}}$

Therefore: $C = \dfrac{\beta}{2\alpha\Gamma(\frac{1}{\beta})} = \dfrac{\gamma}{2(\frac{2}{\lambda})^{\frac{1}{\gamma}}\Gamma(\frac{1}{\gamma})} = \dfrac{\gamma\,\lambda^{\frac{1}{\gamma}}}{2^{1+\frac{1}{\gamma}}\Gamma(\frac{1}{\gamma})}$

(b) $f(x) = C\ \exp\left(-\dfrac{1}{2}\lambda|x|^{\gamma}\right) = \exp\left(\ln(C) - \dfrac{1}{2}\lambda|x|^{\gamma}\right)$

$\Rightarrow$ It is not a member of the exponential class!

(c) Location-scale: $f(x) = \frac{1}{\sigma} g\left(\frac{x-\mu}{\sigma}\right)$, $\quad g$ is a pdf, moments of f are μ, σ^2

Here: $\sigma = \alpha \sqrt{\frac{\Gamma(\frac{3}{\beta})}{\Gamma(\frac{1}{\beta})}}$

$$
\begin{aligned}
f(x) &= \frac{\beta}{2\alpha\Gamma(\frac{1}{\beta})} e^{-\left|\frac{x-\mu}{\alpha}\right|^{\beta}} = \frac{\beta\sqrt{\Gamma(\frac{3}{\beta})}}{2\sigma\Gamma(\frac{1}{\beta})^{\frac{3}{2}}} \exp\left(-\left(\frac{\Gamma(\frac{3}{\beta})}{\Gamma(\frac{1}{\beta})}\right)^{\frac{\beta}{2}} \left|\frac{x-\mu}{\sigma}\right|^{\beta}\right) \\
g(u) &= \frac{\beta\sqrt{\Gamma(\frac{3}{\beta})}}{2\Gamma(\frac{1}{\beta})^{\frac{3}{2}}} \exp\left(-\left(\frac{\Gamma(\frac{3}{\beta})}{\Gamma(\frac{1}{\beta})}\right)^{\frac{\beta}{2}} |u|^{\beta}\right) \\
\int_{-\infty}^{\infty} g(u)du &= \underbrace{\frac{\beta\sqrt{\Gamma(\frac{3}{\beta})}}{2\Gamma(\frac{1}{\beta})^{\frac{3}{2}}}}_{\widetilde{C}} \int_{-\infty}^{\infty} \exp\left(-\left(\frac{\Gamma(\frac{3}{\beta})}{\Gamma(\frac{1}{\beta})}\right)^{\frac{\beta}{2}} |u|^{\beta}\right) du \\
&= 2 \cdot \widetilde{C} \int_{0}^{\infty} \exp\left(-\left(\frac{\Gamma(\frac{3}{\beta})}{\Gamma(\frac{1}{\beta})}\right)^{\frac{\beta}{2}} u^{\beta}\right) du \\
&= \beta \frac{\Gamma(\frac{3}{\beta})^{\frac{1}{2}}}{\Gamma(\frac{1}{\beta})^{\frac{3}{2}}} \int_{0}^{\infty} \frac{1}{\beta} \frac{\Gamma(\frac{1}{\beta})^{\frac{1}{2}}}{\Gamma(\frac{3}{\beta})^{\frac{1}{2}}} x^{\frac{1}{\beta}-1} e^{-x} dx = \frac{1}{\Gamma\left(\frac{1}{\beta}\right)} \Gamma\left(\frac{1}{\beta}\right) = 1
\end{aligned}
$$

Since $g()$ is a proper pdf the claim is verified.

3. (a) $Y_n \sim \mathcal{N}\left(\mu, \frac{\sigma^2+\sqrt{n}}{n+1}\right)$

   i. Convergence in distribution: $\lim_{n\to\infty} Y_n = \lim_{n\to\infty} \mathcal{N}\left(\mu, \frac{\sigma^2+\sqrt{n}}{n+1}\right) = \mathcal{N}(\mu,0) = \mu$. Thus $Y_n \xrightarrow{d} \mu$, which is a degenerate random variable.

   ii. Convergence in probability:

$$
\begin{aligned}
P\left(\left|Y_n - \mu\right| < \epsilon\right) &\geq 1 - \frac{\frac{\sigma^2+\sqrt{n}}{n+1}}{\epsilon^2} \\
\lim_{n\to\infty} P\left(\left|Y_n - \mu\right| < \epsilon\right) &\geq \lim_{n\to\infty} 1 - \frac{\frac{\sigma^2+\sqrt{n}}{n+1}}{\epsilon^2} \\
\lim_{n\to\infty} P\left(|Y_n - \mu| < \epsilon\right) &\geq 1 - 0 = 1
\end{aligned}
$$

   Therefore $Y_n \xrightarrow{p} \mu$. Note that convergence in probability implies convergence in distribution.

iii. Convergence in mean square (Corollary 5.1):

$$\begin{aligned}
\lim_{n\to\infty} \mathrm{E}(Y_n) &= \lim_{n\to\infty} \mu = \mu \checkmark \\
\lim_{n\to\infty} \mathrm{Var}(Y_n) &= \lim_{n\to\infty} \frac{\sigma^2 + \sqrt{n}}{n+1} = 0 = \mathrm{Var}(\mu) \checkmark \\
\lim_{n\to\infty} \mathrm{Cov}(Y_n,Y) &= \lim_{n\to\infty} \mathrm{Cov}(Y_n,\mu) = 0 = Var(\mu) \checkmark
\end{aligned}$$

Thus $Y_n \xrightarrow{m} \mu$. Note that convergence in mean square implies convergence in probability.

(b) $Z_n \sim \Gamma\left(n,\frac{1}{2+n}\right)$

i. Convergence in distribution:
We use the fact that $\Gamma(\alpha,\beta) \xrightarrow{d} \mathcal{N}(\alpha\beta,\alpha\beta^2)$ if $\alpha \to \infty$. Thus $Z_n \sim \mathcal{N}\left(\frac{n}{2+n}, \frac{n}{(2+n)^2}\right)$.
$\lim\limits_{n\to\infty} \mathcal{N}\left(\dfrac{n}{2+n}, \dfrac{n}{(2+n)^2}\right) = \mathcal{N}(1, 0) = 1$. Therefore $Z_n \xrightarrow{d} 1$.

ii. Convergence in probability:

$$\begin{aligned}
P\left(\left|Z_n - \frac{n}{n+2}\right| < \epsilon\right) &\geq 1 - \frac{\frac{n}{(2+n)^2}}{\epsilon^2} \\
\lim_{n\to\infty} P\left(\left|Z_n - \frac{n}{n+2}\right| < \epsilon\right) &\geq \lim_{n\to\infty} 1 - \frac{\frac{n}{(2+n)^2}}{\epsilon^2} \\
\lim_{n\to\infty} P\left(|Z_n - 1| < \epsilon\right) &\geq 1 - 0 = 1 \checkmark
\end{aligned}$$

Thus $Z_n \xrightarrow{p} 1$.

iii. Convergence in mean square:

$$\begin{aligned}
\lim_{n\to\infty} \mathrm{E}(Z_n) &= \lim_{n\to\infty} \frac{n}{2+n} = 1 \checkmark \\
\lim_{n\to\infty} \mathrm{Var}(Z_n) &= \lim_{n\to\infty} \frac{n}{(2+n)^2} = 0 = \mathrm{Var}(1) \checkmark \\
\lim_{n\to\infty} \mathrm{Cov}(Z_n, Z) &= \lim_{n\to\infty} \mathrm{Cov}(Z_n, 1) = 0 = \mathrm{Var}(1) \checkmark
\end{aligned}$$

Thus $Z_n \xrightarrow{m} 1$.

4. (a)

$$
\begin{aligned}
\mathrm{E}\left[\overline{X}_n^2\right] &= \mathrm{E}\left[\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right)^2\right] = \frac{1}{n^2}E\left[\left(\sum_{i=1}^{n} X_i\right)^2\right] \\
&= \frac{1}{n^2}\mathrm{E}\left[\sum_{i=1}^{n} X_i^2 + \sum_{\substack{i=1 \\ i\neq j}}^{n}\sum_{j=1}^{n} X_iX_j\right] = \frac{1}{n^2}\left(\sum_{i=1}^{n}\mathrm{E}[X_i^2] + \sum_{\substack{i=1 \\ i\neq j}}^{n}\sum_{j=1}^{n}\mathrm{E}[X_iX_j]\right) \\
&= \frac{1}{n^2}\left(n\cdot\lambda(\lambda+1) + n(n-1)\lambda^2\right) \\
&= \lambda^2 + \frac{\lambda}{n} \\
&\Rightarrow \mathrm{E}\left[\overline{X}_n^2\right] \neq \lambda^2 \qquad \text{but} \qquad \lim_{n\to\infty}\mathrm{E}\left[\overline{X}_n^2\right] = \lambda^2
\end{aligned}
$$

$$
\begin{aligned}
\text{Note:}\quad \mathrm{Var}[X_i] &= \mathrm{E}[X_i^2] - \mathrm{E}[X_i]^2 \\
\Rightarrow \quad \mathrm{E}[X_i^2] &= \mathrm{Var}[X_i] + \mathrm{E}[X_i]^2 = \lambda^2 + \lambda \\
\mathrm{Cov}[X_iX_j] &= \mathrm{E}[X_iX_j] - \mathrm{E}[X_i]\mathrm{E}[X_j] = 0 \quad \text{(because of independence)} \\
\Rightarrow \quad \mathrm{E}[X_iX_j] &= \mathrm{E}[X_i]\mathrm{E}[X_j] = \lambda^2
\end{aligned}
$$

(b) It holds due to the continuous mapping theorem that

$$\mathrm{plim}(g(x)) = g(\mathrm{plim}(x))$$

Due to Khichin's WLLN

$$\overline{X}_n \xrightarrow{p} \mathrm{E}[X_i] \qquad \text{or} \qquad \mathrm{plim}(\overline{X}_n) = \lambda.$$

Then, $\mathrm{plim}(\overline{X}_n^2) = (\mathrm{plim}(\overline{X}_n))^2 = \lambda^2$.