Mathematical Finance Winter term 2023/2024

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Mathematical Finance: MF

Exercises (for discussion on Monday, 22.01.2024)

Exercise 1. In the CRR model let S^2 be the fair price process of an American put option on S^1 with strike price $K = 100$, hence with payoff process $X = (K - S^1)^+$. Let $\tilde{r} = 0.05$, $u = 1.1$, $d = 0.8$, $N = 2$, $S_0^1 = 100$ and $S_0^0 = 1$.

On all knots of the tree determine the value process S^2 of an American put option with strike price K and compare it to the value process of a European put option with strike K. Further, find a self financing strategy φ such that $V(\varphi) \geq X := (K - S^1)^+$ and $V_0(\varphi) = S_0^2$.

Exercise 2. (*12 Points*) We consider the CRR model with infinite time horizon, i.e. $S_n^0 = r^{-n}$ for some $r \in (0,1)$ and $S_n^1 := \prod_{i=1}^n Y_n$ for an i.i.d. sequence $(Y_i)_{i\in\mathbb{N}}$ ($S_0^1 = 1$ w.l.o.g.) such that $Q(Y_1 = u) = 1 - Q(Y_1 = d) = q$ with $q := \frac{r^{-1}-d}{u-d} \in (0,1)$ defines an EMM Q. Further, we assume $0 < u = d^{-1}$. By $\mathcal{T}$ we denote the set of all stopping times with respect to the filtration generated by S. Moreover let $\mathcal{T}_f := \{\tau \in \mathcal{T} : \tau < \infty\ Q\text{-a.s.}\}$ and $\mathcal{T}_b := \{\tau \in \mathcal{T} : \tau \text{ is bounded}\}$. The aim of the exercise is to find the value of and the optimal stopping time for a *perpetual American put option* with strike $K > 0$, i.e. an American put option with infinite time horizon. Mathematically, we are faced with stopping problem

$$\sup_{\tau\in\mathcal{T}_f} E_Q(r^\tau (K - S_\tau^1)^+).$$

To that end let β be the unique positive solution to the equation

$$E_Q(rY_1^{-\beta}) = 1,$$

$g : (0,\infty) \to \mathbb{R};\ x \mapsto (K - x)^+ x^\beta$ and a the maximizer of g on the set of possible values $\{d^m | m \in \mathbb{Z}\}$. Further assume $E_Q(\log(Y_1)) \leq 0$ and $a \leq 1$. Define

$$\tau^* := \inf\{n \in \mathbb{N}_0 | S_n^1 \leq a\}.$$

Show

(i) $\sup_{\tau\in\mathcal{T}_f} E_Q(r^\tau(K - S_\tau^1)^+) = \sup_{\tau\in\mathcal{T}_b} E_Q(r^\tau(K - S_\tau^1)^+)$,

(ii) $M_n := r^n (S_n^1)^{-\beta}$ defines a Q-martingale and $r^n(K - S_n^1)^+ = M_n g(S_n^1)$,

(iii) $\sup_{\tau\in\mathcal{T}_f} E_Q(r^\tau(K - S_\tau^1)^+) \leq g(a)$,

(iv) $\tau^* < \infty$ Q-a.s. and $S_{\tau^*}^1 = a$ Q-a.s.,

(v) $E_Q(M_{\tau^*}) = 1$,

(vi) $E_Q(r^{\tau^*}(K - S_{\tau^*}^1)^+) = g(a)$.

Submission of the homework until: Thursday, 18.01.2024, 10.00 a.m. via OLAT.