Advanced Statistics I
- Collection of Exercises -

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Contents

Introduction

This book represents collection of exercises and old exams that have been within the framework of statistics undergraduate courses over the years at the Institute of Statistics and Econometrics. While some of them are still part of the current tutorials, others are no longer presented due to time constraints and changes in the examination regulations. We tried our best to provide detailed solutions and commentaries and included examples from external literature as well[1].

We would like to thank Prof. Roman Liesenfeld, Prof. Uwe Jensen, Dr. Bastian Griebisch, and Dr. Markus Pape, who were kind to provide most of the materials for this book, as well as Sebastian Humberg and Uliana Zaspa for their help.

[1]Pay attention to footnotes. Further on we will refer to *Mittelhammer, R.C. (1996). Mathematical Statistics for Economics and Business. Springer* as Mittelhammer and to *Mood, A. M., F. A. Graybill and D. C. Boes (1974, 3rd ed.). Introduction to the Theory of Statistics. McGraw-Hill* as MGB.

Chapter 1

Elements of Probability Theory

Exercise 1.1

(Sample space and power set) A random experiment consists of drawing $n = 2$ balls from an urn containing $M = 3$ different balls that are numbered from 1 to 3.

a) What is the number of elementary events L of the sample space S while *sampling with replacement* and while *sampling without replacement*?

b) How many subsets are there in the sample space of the experiment while *sampling with replacement*?

Solution to Exercise 1.1

a) Possible elementary events while sampling with replacement:

$$\begin{gathered}\{1,1\},\{1,2\},\{1,3\}\\ \{2,1\},\{2,2\},\{2,3\}\\ \{3,1\},\{3,2\},\{3,3\}\end{gathered}$$

Number of elementary events while sampling with replacement:

$$L = M^n = 3^2 = 9,$$

with L number of elementary events, M number of balls and n number of draws.
Possible elementary events while sampling without replacement:

$$\{1,2\},\{2,1\},\{1,3\},\{3,1\},\{2,3\},\{3,2\}$$

Number of elementary events while sampling without replacement:

$$L = M \cdot (M-1) \cdot (M-2) \cdot ...(M-n+1) = 3 \cdot 2 = 6.$$

b) Number of subsets with k elements:

$$\binom{L}{k} = \frac{L!}{(L-k)!k!}$$

Summing up for every possible k:

$$\sum_{k=0}^{L} \binom{L}{k} = \text{number of subsets.}$$

Consider the binomial theorem:

$$(a+b)^L = \sum_{k=0}^{L} \binom{L}{k} a^{L-k} b^k = \sum_{k=0}^{L} \frac{L!}{(L-k)!k!}\Bigg|_{a=b=1} = 2^L,$$

thus here 2^9.

Exercise 1.2

(Boolean fields of sets as an event space) A field of sets $\mathcal{K}$ is defined by the following properties: (i) $A \in \mathcal{K} \Rightarrow \bar{A} \in \mathcal{K}$; (ii) $A \in \mathcal{K}$ and $B \in \mathcal{K} \Rightarrow A \cup B \in \mathcal{K}$. The sample space of the experiment of rolling a die twice and summing up the dots facing up, is given by $S = \{(i,j) : i,j = 1,2,...,6\}$. Let A_k $(k = 2,...,12)$ be the event "*The sum is less than or equal to k*" and let B_k $(k = 2,...,12)$ be the event "*The sum is greater than k*". Which of the following subsets of the power set of S form a field of sets?

a) $\mathcal{K}_1 = \{\emptyset, A_2, B_2, S\}$;

b) $\mathcal{K}_2 = \{A_{12}, B_{12}\}$;

c) $\mathcal{K}_3 = \{A_{11}, B_{11}\}$;

d) $\mathcal{K}_4 = \{A_k, B_k : k = 2,...,12\}$.

Solution to Exercise 1.2

Sample space: $S = \{(i,j) : i,j = 1,2,...,6\}$
Events:

$A_k = \{(i,j) : i + j \leq k\}$; $k = 2,...,12$
$B_k = \{(i,j) : i + j > k\}$; $k = 2,...,12$

$\Rightarrow \overline{A}_k = B_k$ and $\overline{B}_k = A_k$ (mutually exclusive events)

a) A set of subsets $\mathcal{K}_1$ of the power set of S is given. Do they form a field of sets?

$$\mathcal{K}_1 = \{\emptyset, A_2, B_2, S\}$$

$\mathcal{K}_1$ is a field of sets provided that the set of subsets contains all its complements and unions.

Complements in $\mathcal{K}_1$:
$\overline{\emptyset} = S \in \mathcal{K}_1$; $\overline{A_2} = B_2 \in \mathcal{K}_1$; $\overline{S} = \emptyset \in \mathcal{K}_1$; $\overline{B_2} = A_2 \in \mathcal{K}_1$
Unions in $\mathcal{K}_1$:
$\emptyset \cup A_2 = A_2 \in \mathcal{K}_1$; $\emptyset \cup B_2 = B_2 \in \mathcal{K}_1$; $\emptyset \cup S = S \in \mathcal{K}_1$;
$B_2 \cup A_2 = S \in \mathcal{K}_1$; $S \cup B_2 = S \in \mathcal{K}_1$; $A_2 \cup S = S \in \mathcal{K}_1$;

The set $\mathcal{K}_1$ contains all complements and unions. Thus it is a field of sets.

b) $\mathcal{K}_2 = \{A_{12}, B_{12}\}$
Because A_{12} is equal to the sample space und B_{12} is an empty set, and because S and $\emptyset$ form the smallest possible field of sets, $\mathcal{K}_2$ is a field of sets. ($\emptyset \cup S = S \in \mathcal{K}_2$; $\overline{\emptyset} = S \in \mathcal{K}_2$; $\overline{S} = \emptyset \in \mathcal{K}_2$)

c) $\mathcal{K}_3 = \{A_{11}, B_{11}\}$
$\mathcal{K}_3$ is not a field of sets, as for example $A_{11} \cup B_{11} = S$ is not a subset of the set $\mathcal{K}_3$.

d) $\mathcal{K}_4 = \{A_k, B_k : k = 2,...,12\}$
$\mathcal{K}_4$ is not a field of sets, as for example $A_2 \cup B_{11} = \{(1,1);(6,6)\}$ is not a subset of the set $\mathcal{K}_4$.

Exercise 1.3

(Boolean fields of sets as an event space) Let $S = \{1, 2, 3\}$. Name all nonempty subsets of the power set of S which form a field of sets.

Solution to Exercise 1.3

Power set with $2^3 = 8$ subsets is a field of sets

$$\{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, S\}$$

Field of sets with two elements:

$$\{\emptyset, S\}$$

Field of sets with four elements:

$$\begin{array}{c} \{\emptyset, \{1\}, \{2,3\}, S\} \\ \{\emptyset, \{2\}, \{1,3\}, S\} \\ \{\emptyset, \{3\}, \{1,2\}, S\} \end{array}$$

Besides these 4 fields of sets and the power set there is no other closed field of sets.

Exercise 1.4

(Probability set function) For each case below, determine whether or not the real-valued set function $P(A)$ is in fact a probability set function.

a) sample space: $S = \{1, 2, 3, 4, 5, 6, 7, 8\}$
event space: $\Upsilon = \{A : A \subset S\}$
set function: $P(A) = \sum_{x \in A} x/36$ for $A \in \Upsilon$

b) sample space: $S = [0, \infty)$
event space: $\Upsilon = \{A : A$ is an interval subset of $S \vee$ any set formed by unions, intersections, or complements of these interval subsets $\}$
set function: $P(A) = \int_{x \in A} e^{-x} dx$ for $A \in \Upsilon$

c) sample space: $S = \{x : x$ is a positive integer (1,2,3,...)$\}$
event space: $\Upsilon = \{A : A \subset S\}$
set function: $P(A) = \sum_{x \in A} x^2/10^5$ for $A \in \Upsilon$.

Solution to Exercise 1.4

a)

i $P(A) \geq 0$ because every A is a nonempty subset of S containing only $x > 0$.

ii $P(S) = 1$

iii $P(\cup_{j \in I} A_j) = \sum_{x \in \cup_{j \in I} A_j} \frac{x}{36} = \cup_{j \in I} \underbrace{\sum_{x \in A_j} \frac{x}{36}}_{P(A_j)} = \sum_{j \in I} P(A_j)$

b)

i $P(A) \geq 0$ because A contains at least one interval of length zero.

ii $P(S) = \int_0^\infty e^{-x} dx = | - e^{-x}|_0^\infty = 1$

iii $P(\cup_{j\in I} A_j) = \int_{x\in\cup_{j\in I} A_j} e^{-x} dx = \cup_{j\in I} \underbrace{\int_{x\in A_j} e^{-x} dx}_{P(A_j)} = \sum_{j\in I} P(A_j)$

c) Not a probability set function because $10^5 \in S$ and therefore $P(S) \neq 1$.

Exercise 1.5

(Elementary probability theory) Let $P(\cdot)$ be a probability set function with the event space Υ with $A_i \in \Upsilon$ $(i = 1, ..., r)$. Show that the following relationships hold:

a) $P(A_1 - A_2) = P(A_1) - P(A_1 \cap A_2)$;

b) Given $A_i \cap A_j = \emptyset \ \forall i \neq j$, then $P(\cup_{i=1}^r A_i) = \sum_{i=1}^r P(A_i)$;

c) $P[(A_1 \cup A_2) - (A_1 \cap A_2)] = P(A_1) + P(A_2) - 2P(A_1 \cap A_2)$.

Solution to Exercise 1.5

a) $A_1 - A_2 = A_1 \cap \bar{A}_2$
Given: $(A_1 \cap \bar{A}_2)$ and $(A_1 \cap A_2)$ are disjoint. Therefore:

$$P\Big((A_1 \cap \bar{A}_2) \cup (A_1 \cap A_2)\Big) = P(A_1 \cap \bar{A}_2) + P(A_1 \cap A_2)$$

$$\begin{aligned} \Longrightarrow \quad & P(A_1) = P\Big([A_1 \cap \bar{A}_2] \cup [A_1 \cap A_2]\Big) \\ & P(A_1) = P(A_1 \cap \bar{A}_2) + P(A_1 \cap A_2) \quad \text{(disjoint sets)} \\ \Longrightarrow \quad & P(A_1 \cap \bar{A}_2) = P(A_1 - A_2) = P(A_1) - P(A_1 \cap A_2) \end{aligned}$$

b) By complete induction, consider the following as given:
The induction hypothesis:

$$A_1 \cap A_2 = \emptyset \text{ (disjoint)} \Longrightarrow P(A_1 \cup A_2) = P(A_1) + P(A_2) \text{ (basis/base case)}$$

Inductive step: $(r-1) \longrightarrow r$

$$A_i \cap A_j = \emptyset \quad i, j = 1; \ldots, r-1; \ i \neq j \Longrightarrow P\left(\bigcup_{i=1}^{r-1} A_i\right) = \sum_{i=1}^{r-1} P(A_i)$$

It has to be shown that

$$A_i \cap A_j = \emptyset \quad i, j = 1; \ldots, r; \ i \neq j \Longrightarrow P\left(\bigcup_{i=1}^{r} A_i\right) = \sum_{i=1}^{r} P(A_i)$$

This yields from

$$\begin{aligned} \left[\bigcup_{i=1}^{r-1} A_i\right] \cap A_r &= \bigcup_{i=1}^{r-1} [A_i \cap A_r] = \emptyset \\ \Rightarrow P\left[\bigcup_{i=1}^{r} A_i\right] &= \underbrace{P\left[\left[\bigcup_{i=1}^{r-1} A_i\right] \cup A_r\right] = P\left(\bigcup_{i=1}^{r-1} A_i\right) + P(A_r)}_{\text{application of the induction hypothesis}} \end{aligned}$$

with the induction hypothesis

$$P\left(\bigcup_{i=1}^{r} A_i\right) = \underbrace{\sum_{i=1}^{r-1} P(A_i)}_{\text{induction hypothesis}} \quad +P(A_r) \qquad \text{q.e.d.}$$

c) Given:

$$\begin{array}{ll} (1) & [A_1 \cap \bar{A}_2] \cap [\bar{A}_1 \cap A_2] = \emptyset \\ (2) & [A_1 \cap \bar{A}_2] \cup [\bar{A}_1 \cap A_2] = [A_1 \cup A_2] - [A_1 \cap A_2] \end{array}$$

$$\begin{aligned} \Longrightarrow P\Big(\underbrace{[A_1 \cup A_2] - [A_1 \cap A_2]}_{\text{use (2)}}\Big) &= P\Big[[A_1 \cap \bar{A}_2] \cup [\bar{A}_1 \cap A_2]\Big] \\ &= P(A_1 \cap \bar{A}_2) + P(\bar{A}_1 \cap A_2) \quad \text{(because of (1))} \\ &= P(A_1 - A_2) + P(A_2 - A_1) \qquad \text{(because of 1.5(a))} \\ &= P(A_1) - P(A_1 \cap A_2) + P(A_2) - P(A_2 \cap A_1) \\ &\quad \text{q.e.d.} \end{aligned}$$

Exercise 1.6

(Elementary probability theory) Let A, B, C be elements of the event space Υ and let $P(\cdot)$ be an associated probability set function. Show that the following relationships hold:

a) $P(A) \leq P(B) \Rightarrow P(\bar{B}) \leq P(\bar{A})$

b) $(A \cap B) \subset C \Rightarrow P(\bar{C}) \leq P(\bar{B}) + P(\bar{A})$.

Solution to Exercise 1.6

a)

$$\begin{aligned} P(A) &= 1 - P(\bar{A}) \\ P(B) &= 1 - P(\bar{B}) \\ \Longrightarrow P(A) \leq P(B) &\Longrightarrow 1 - P(\bar{A}) \leq 1 - P(\bar{B}) \\ & \qquad P(\bar{A}) \geq P(\bar{B}) \end{aligned}$$

b)

$$[A \cap B] \subset C \Longrightarrow \bar{C} \subset [\overline{A \cap B}] = \bar{A} \cup \bar{B}$$

$$\begin{aligned} P(\bar{C}) \leq \quad & P(\bar{A} \cup \bar{B}) \\ & P(\bar{A} \cup \bar{B}) \leq P(\bar{A}) + P(\bar{B}) \\ & \text{because } P(\bar{A} \cup \bar{B}) = P(\bar{A}) + P(\bar{B}) - P(A \cap B) \end{aligned}$$

Exercise 1.7

(Elementary probability theory) The events A and B have the probabilities $P(A) = 3/4$ and $P(B) = 3/8$. Show that (a) $P(A \cup B) \geq 3/4$ and (b) $1/8 \leq P(A \cap B) \leq 3/8$.

Solution to Exercise 1.7

a) Given:

$$
\begin{aligned}
P(A \cup B) &\geq P(A) = \frac{3}{4} = \frac{6}{8} \\
P(A \cup B) &\geq P(B) = \frac{3}{8}
\end{aligned}
$$

$$\frac{6}{8} > \frac{3}{8} \Longrightarrow P(A \cup B) \geq \frac{3}{4}$$

b)

$$
\begin{aligned}
P(A \cap B) &\leq P(A) \\
P(A \cap B) &\leq P(B)
\end{aligned}
$$

$$\Longrightarrow P(A \cap B) \leq \frac{3}{8}$$

and

$$P(A \cap B) \geq 1 - P(\bar{A}) - P(\bar{B})$$

(with $P(A \cap B) = 1 - P(\overline{A \cap B}) = 1 - P(\bar{A} \cup \bar{B})$)

$$P(A \cap B) \geq 1 - \frac{2}{8} - \frac{5}{8} = \frac{1}{8}$$

Exercise 1.8

(Elementary probability theory) From an urn containing M balls, n balls are drawn with replacement. What is the probability that at least one ball is drawn more than once?

Solution to Exercise 1.8

Sample space S: set of all n-tuples $(Z_1, \ldots, Z_n)$ containing the numbers 1, 2,..., M. Hence, we have $N(S) = M^n$ number of elementary events. Define

A = at least 1 ball is drawn more than once,
$\bar{A}$ = n different balls are drawn.

$\bar{A}$ consists of all n-tuples without repetition which can be combined from the numbers 1 to M.

If $n = 2$, every number $Z_1 \in \{1, \ldots, M\}$ has $(M-1)$ possible "partners" with $Z_2 \neq Z_1$. Hence, the number of pairs (Z_1, Z_2) with $Z_1 \neq Z_2$ is $(M)(M-1)$.
Number of pairs in general

$$(M)(M-1) \cdot \ldots \cdot M - (n-1)$$

Number of elements in $\bar{A}$

$$N(\bar{A}) = M(M-1) \cdot \ldots \cdot (M - n + 1)$$

It holds that the sample space S while sampling with replacement contains $N(S) = M^n$ equiprobable elementary events. Further, for $\bar{A} \subset S$ according to Laplace that

$$P(\bar{A}) = \frac{N(\bar{A})}{N(S)} = \frac{M(M-1)\cdot \ldots \cdot (M-n+1)}{M^n}$$
$$P(A) = 1 - \frac{N(\bar{A})}{N(S)} = 1 - \frac{M(M-1)\cdot \ldots \cdot (M-n+1)}{M^n}$$

Exercise 1.9

(Conditional probability) Two cards are drawn without replacement from a well-shuffled deck of 52 cards. What is the probability of drawing two aces?

Solution to Exercise 1.9

To determine $P(A \cap B)$, use conditional probabilities. Let B be the event that the first card drawn is an ace (4 aces in 52 cards) with probability $P(B) = \frac{4}{52} = \frac{1}{13}$. Let A be the event that the second card drawn is an ace. Given that the first card drawn is an ace (i.e., given event B), the conditional probability is given by $P(A|B) = \frac{3}{51} = \frac{1}{17}$. Then the probability that both cards drawn are aces is given by

$$P(A \cap B) = P(A|B) \cdot P(B) = \frac{1}{13} \cdot \frac{1}{17} = \frac{1}{221}$$

Exercise 1.10

(Conditional probability) Consider 5 urns which are numbered from 1 to 5. Each of them contains 10 balls. Urn i ($i = 1, ..., 5$) contains i black and $10 - i$ red balls. Consider the following random experiment: In the first step, an urn is chosen and in the second step, a ball is drawn from this urn.

a) What is the probability of drawing a black ball?

b) Assume a black ball has been drawn. What is the probability that the ball comes from urn $i = 5$?

Solution to Exercise 1.10

Given are 5 urns U_i, $i = 1, \ldots, 5$. The probability to pick a particular urn is $P(U_i) = \frac{1}{5}$ and the probabilities to draw a black ball from a given urn are $P(B|U_1) = 0.1$, $P(B|U_2) = 0.2$, etc.

a)

$$P(B) = \sum_{i=1}^{5} P(B|U_i)P(U_i) = \frac{1}{5}\left(\frac{1}{10} + \frac{2}{10} + \frac{3}{10} + \frac{4}{10} + \frac{5}{10}\right) = \frac{3}{10}$$

b)

$$P(U_5|B) = \frac{P(U_5 \cap B)}{P(B)} = \frac{P(B|U_5)P(U_5)}{P(B)} = \frac{\frac{1}{2}\cdot\frac{1}{5}}{\frac{3}{10}} = \frac{1}{3}$$

Exercise 1.11

(Elementary probability theory) Let $A_1, ..., A_n$ be events in the sample space S.

a) If $n = 2$, then $P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 \cap A_2)$. If $n = 3$, show that

$$\begin{aligned} P(A_1 \cup A_2 \cup A_3) &= P(A_1) + P(A_2) + P(A_3) \\ &- P(A_1 \cap A_2) - P(A_1 \cap A_3) - P(A_2 \cap A_3) + P(A_1 \cap A_2 \cap A_3) \end{aligned}$$

holds.

b) Show, by complete induction that the following generalized relationship holds:

$$\begin{aligned} P(\cup_{k=1}^{n} A_k) = \sum_{k=1}^{n} P(A_k) \quad &- \quad \sum_{k_1 < k_2}^{n} P(A_{k_1} \cap A_{k_2}) \\ &+ \quad \sum_{k_1 < k_2 < k_3}^{n} P(A_{k_1} \cap A_{k_2} \cap A_{k_3}) \\ &+ \quad \cdots + (-1)^{n+1} P(\cap_{k=1}^{n} A_k) \ . \end{aligned}$$

Solution to Exercise 1.11

a)

$$\begin{aligned} P(A_1 \cup A_2 \cup A_3) &= P(B_1 \cup A_3) = P(B_1) + P(A_3) - P(B_1 \cap A_3) \\ &= \underbrace{P(A_1) + P(A_2) - P(A_1 \cap A_2)}_{P(B_1)} + P(A_3) - P(B_1 \cap A_3) \end{aligned}$$

$$\begin{aligned} P(B_1 \cap A_3) &= P((A_1 \cup A_2) \cap A_3) = P((A_1 \cap A_3) \cup (A_2 \cap A_3)) \\ &= P(A_1 \cap A_3) + P(A_2 \cap A_3) - \underbrace{P((A_1 \cap A_3) \cap (A_2 \cap A_3))}_{P(A_1 \cap A_2 \cap A_3)} \end{aligned}$$

$\Rrightarrow P(A_1 \cup A_2 \cup A_3) = P(A_1) + P(A_2) + P(A_3) - P(A_1 \cap A_2) - P(A_1 \cap A_3) - P(A_2 \cap A_3) + P(A_1 \cap A_2 \cap A_3)$

b) Basis: See above!
Induction statement:

$$\begin{aligned} P(\cup_{i=1}^{n} A_i) &= \sum_{i=1}^{n} P(A_i) - \sum_{k_1 < k_2}^{n} P(A_{k_1} \cap A_{k_2}) + \sum_{k_1 < k_2 < k_3}^{n} P(A_{k_1} \cap A_{k_2} \cap A_{k_3}) \\ &+ \cdots + (-1)^{n+1} P(\cap_{i=1}^{n} A_i) \end{aligned}$$

We have shown that the formula above holds for $n = 3$! <u>Inductive step</u>: $n \to n + 1$

$$\begin{aligned}
P(\cup_{i=1}^{n+1} A_i) &= P((\cup_{i=1}^{n} A_i) \cup A_{n+1}) \\
&= \underbrace{P(\cup_{i=1}^{n} A_i)}_{\text{use induction hypothesis}} \quad + P(A_{n+1}) - P((\cup_{i=1}^{n} A_i) \cap A_{n+1}) \\
&= \underline{\sum_{i=1}^{n} P(A_i)} - \underbrace{\sum_{k_1<k_2}^{n} P(A_{k_1} \cap A_{k_2})}_{\text{use: *2!}} + \sum_{k_1<k_2<k_3}^{n} P(A_{k_1} \cap A_{k_2} \cap A_{k_3}) + \dots \\
&\quad + (-1)^{n+1} P(\cap_{i=1}^{n} A_i) + \underline{P(A_{n+1})} - \underbrace{P((\cup_{i=1}^{n} A_i) \cap A_{n+1})}_{\text{use: *1!}}
\end{aligned}$$

*1:

$$\begin{aligned}
P((\cup_{i=1}^{n} A_i) \cap A_{n+1}) &= \underbrace{P(\cup_{i=1}^{n} (A_i \cap A_{n+1}))}_{\text{use induction hypothesis!}} \\
&= \underbrace{\sum_{i=1}^{n} P(A_i \cap A_{n+1})}_{\text{use: *2!}} - \sum_{k_1<k_2}^{n} P((A_{k_1} \cap A_{n+1}) \cap (A_{k_2} \cap A_{n+1})) \\
&\quad + \sum_{k_1<k_2<k_3}^{n} P((A_{k_1} \cap A_{n+1}) \cap (A_{k_2} \cap A_{n+1}) \cap (A_{k_3} \cap A_{n+1})) + \dots \\
&\quad + (-1)^{n+1} P(\cap_{i=1}^{n} (A_i \cap A_{n+1}))
\end{aligned}$$

*2

$$-\sum_{k_1<k_2}^{n} P(A_{k_1} \cap A_{k_2}) - \sum_{i=1}^{n} P(A_i \cap A_{n+1}) = -\sum_{k_1<k_2}^{n+1} P(A_{k_1} \cap A_{k_2})$$

$\Rrightarrow$

$$\begin{aligned}
P(\cup_{i=1}^{n+1} A_i) &= \sum_{i=1}^{n+1} P(A_i) - \sum_{k_1<k_2}^{n+1} P(A_{k_1} \cap A_{k_2}) \\
&\quad + \sum_{k_1<k_2<k_3}^{n} P(A_{k_1} \cap A_{k_2} \cap A_{k_3}) + \dots \\
&\quad + \sum_{k_1<k_2}^{n} P(\underbrace{(A_{k_1} \cap A_{n+1}) \cap (A_{k_2} \cap A_{n+1})}_{(A_{k_1} \cap A_{k_2} \cap A_{n+1}) \Rrightarrow k_3 = n+1}) \\
&\quad - \sum_{k_1<k_2<k_3}^{n} P(\underbrace{(A_{k_1} \cap A_{n+1}) \cap (A_{k_2} \cap A_{n+1}) \cap (A_{k_3} \cap A_{n+1})}_{(A_{k_1} \cap A_{k_2} \cap A_{k_3} \cap A_{n+1}) \Rrightarrow k_4 = n+1}) \\
&\quad - \dots \underbrace{-(-1)^{n+1} P(\cap_{i=1}^{n} (A_i \cap A_{n+1}))}_{+(-1)^{n+1+1} P(\cap_{i=1}^{n+1} A_i)} \\
&= \sum_{i=1}^{n+1} P(A_i) - \sum_{k_1<k_2}^{n+1} P(A_{k_1} \cap A_{k_2}) + \sum_{k_1<k_2<k_3}^{n+1} P(A_{k_1} \cap A_{k_2} \cap A_{k_3}) \\
&\quad + \dots + (-1)^{n+2} P(\cap_{i=1}^{n+1} A_i)
\end{aligned}$$

Exercise 1.12

(Elementary Probability Theory) Consider a fair die rolled twice. The elementary events in $S = \{(i,j) : i,j = 1,...,6\}$ have the same probability. Let A be the event "*number of dots facing up in the first cast is less than or equal to 2*" and let B be the event "*number of dots facing up in the second cast is greater than or equal to 5*". Calculate $P(A \cup B)$.

Solution to Exercise 1.12

The sample space S contains 36 elementary events which are all equiprobable (fair die). The events are as follows:

$$A = \{(i,j) : 1 \le i \le 2; j : 1,...,6\}$$
$$B = \{(i,j) : 5 \le j \le 6; i : 1,...,6\}$$

What is the probability $P(A \cup B)$?

$$P(A \cup B) = P(A) + P(B) - P(A \cap B),$$
with $A \cap B = \{(1,5); (1,6); (2,5); (2,6)\}$
$$P(A \cup B) = \tfrac{12}{36} + \tfrac{12}{36} - \tfrac{4}{36} = \tfrac{5}{9}$$

Exercise 1.13

(Elementary Probability Theory) A point in the unit square (side length is equal to one) is chosen at random. The sample space is $S = \{(x,y) : 0 \le x \le 1, 0 \le y \le 1\}$. The event space Υ is given by a Borel field of sets in S. What is the probability of the event

$$A = \{(x,y) : 0 \le x \le 1/2, 1/2 \le y \le 1\}$$

and what is the probability of the event that the point is located within a circle with the centre in $(\frac{1}{2}, \frac{1}{2})$ and the radius $\frac{1}{2}$, that is

$$B = \left\{(x,y) : 0 \le (x-1/2)^2 + (y-1/2)^2 \le \frac{1}{4}\right\} ?$$

Solution to Exercise 1.13

a) $P(A) = \frac{1}{4}$

b) The area of the circle is given by $K = \pi r^2 = \frac{\pi}{4} \Rrightarrow P(B) = \frac{\pi}{4}$

Exercise 1.14

(Conditional probability) A drug is used for the treatment of disease K. The probability that there are no side effects is $2/3$. Given that side effects do not occur, the probability of recovering is $3/4$. Otherwise it is 0. Now 3 test patients take the drug. Assume that the test results are stochastically independent.

a) What is the probability that

i. every patient recovers?

ii. at least one patient recovers?

b) At least how many patients have to take the drug to make sure that with a probability of at least 0.95, it holds that:

i. at least one patient doesn't suffer side effects?

ii. at least one patient recovers?

Solution to Exercise 1.14

Define N_i as the event when patient i suffers side effects and H_i as the event that patient i recovers. Then $P(N_i) = 1 - P(\bar{N}_i) = 1 - \frac{2}{3} = \frac{1}{3}$, and $P(H_i|\bar{N}_i) = \frac{3}{4}$, and $P(H_i|N_i) = 0$. The number of patients is I=3. Treatment and course of disease are independent from each other.

a) i. Consider the law of total probability. N_i and $\bar{N}_i$ form a partition of the sample space. Then

$$P(H_i) = P(H_i|N_i) \cdot P(N_i) + P(H_i|\bar{N}_i) \cdot P(\bar{N}_i) = 0 + \frac{3}{4} \cdot \frac{2}{3} = \frac{1}{2}$$

Probability that all three patients recover

$$P(H_1 \cap H_2 \cap H_3) = \left(\frac{1}{2}\right)^3 = \frac{1}{8}$$

ii. $P(\text{at least one recovers}) = 1 - P(\text{no one recovers})$:

$$1 - P(\bar{H}_1 \cap \bar{H}_2 \cap \bar{H}_3) = 1 - \left(\frac{1}{2}\right)^3 = \frac{7}{8}$$

b) i. Define as B_I the event when at least one patient does not suffer side effects, where I is number of patients in the sample. Then we need to find I such that $P(B_I) \geq 0.95$.

$$P(B_I) = 1 - P(\bar{B}_I) = 1 - P(N_1 \cap N_2 \cap ... \cap N_I) = 1 - \prod_{i=1}^{I} P(N_i) = 1 - (\frac{1}{3})^I$$

$$1 - (\frac{1}{3})^I \geq 0.95 \Rightarrow (\frac{1}{3})^I \leq 0.05$$

$$I \cdot \ln(\frac{1}{3}) \leq \ln(0.05)$$

$$I \geq \frac{\ln(0.05)}{\ln(1/3)} = 2.7268$$

At least three patients have to get a treatment, so that with a probability of at least 95 % at least one patient does not suffer side effects.

ii. Now, H_I is the event when at least one patient recovers with $P(H_I) \geq 0.95$.

$$P(H_I) = 1 - \left(\frac{1}{2}\right)^I$$

$$I \geq \frac{\ln(0.05)}{\ln(1/2)} = 4.322$$

At least five patients have to get a treatment, so that with a probability of 95 % at least one patient recovers.

Exercise 1.15

(Stochastic independence) Let (S, Υ, P) be a probability space with the sample space $S = \{E_1, E_2, E_3, E_4\}$, the event space Υ consisting of the power set of S and a probability set function P with $P(E_i) = 1/4$ $(i = 1, ..., 4)$. Consider the following events:

$$A_1 = \{E_1, E_2\}\ , \quad A_2 = \{E_1, E_3\}\ , \quad A_3 = \{E_1, E_4\}\ ,$$

and show that these events are pairwise stochastically independent but not jointly stochastically independent.

Solution to Exercise 1.15

In case of stochastic independence, it holds that

$$P\left(\bigcap_{i=1}^{n} A_i\right) = \prod_{i=1}^{n} P(A_i).$$

The events A_1, A_2 and A_3 are pairwise stochastically independent, since

$$\begin{aligned} P(A_1 \cap A_2) &= P(E_1) = \frac{1}{4} = P(A_1) \cdot P(A_2) = \frac{2}{4} \cdot \frac{2}{4} \\ P(A_1 \cap A_3) &= P(E_1) = \frac{1}{4} = P(A_1) \cdot P(A_3) \\ P(A_2 \cap A_3) &= P(E_1) = \frac{1}{4} = P(A_2) \cdot P(A_3) \end{aligned}$$

If we take a look at all of the events at once, we can see that they are not jointly stochastically independent

$$P(A_1 \cap A_2 \cap A_3) = P(E_1) = \frac{1}{4} \neq P(A_1) \cdot P(A_2) \cdot P(A_3) = \frac{1}{8}.$$

Exercise 1.16

(Independent vs. disjoint events) Let $P(A_i) = 0.1$ with $i = 1, ..., 10$. What is the probability $P(\cap_{i=1}^{n} A_i)$ if

a) the A_i's are independent;

b) the A_i's are disjoint;

c) $P(A_i | \cap_{j=1}^{i-1} A_j) = 0.2$ with $i = 2, ..., 10$?

Solution to Exercise 1.16

a) $P\left(\bigcap_{i=1}^{10} A_i\right) = \prod_{i=1}^{10} P(A_i) = (0.1)^{10}$

b) $P\left(\bigcap_{i=1}^{10} A_i\right) = P(\emptyset) = 0$

c) $P\left(\bigcap_{i=1}^{10} A_i\right) = P(A_1) \cdot P(A_2|A_1) \cdot P(A_3|A_1 \cap A_2) \cdot ... \cdot P(A_{10}|A_1 \cap A_2 \cap ... \cap A_9) = 0.1 \cdot (0.2)^9$
Explaination: Equation $P(A_3 \cap B) = P(A_3|B) \cdot P(B)$ holds. If $B = A_1 \cap A_2$, it is obvious that $P(A_1 \cap A_2 \cap A_3) = P(A_3|A_1 \cap A_2) \cdot P(A_1 \cap A_2)$. By rearranging we get $P(A_1 \cap A_2 \cap A_3) = P(A_3|A_1 \cap A_2) \cdot P(A_2|A_1) \cdot P(A_1)$. According to the exercise $P(A_1) = 0.1$ and (e.g.) $P(A_5|A_4 \cap A_3 \cap A_2 \cap A_1) = 0.2$, thus the result is $P\left(\bigcap_{i=1}^{10} A_i\right) = 0.1 \cdot (0.2)^9$.

Exercise 1.17[1]

A taxi was involved in a hit and run accident at night. There are two taxi companies in the city, namely Black Taxis and White Taxis. We know that 85% of the taxis in the city are Black and 15% are White. There was a witness to the accident and, according to the witness, the taxi involved in the accident was White. Now further investigation of the reliability of the witness showed that, under similar conditions, the witness was able to identify correctly the color of a taxi 80% of the time.

a) Without any calculation, do you think it is more likely that the taxi involved was Black or White?

b) Calculate the probability that the taxi involved was White.

c) Compare your answers to these two questions.

d) Explore the sensitivity of the answer to Question 18b to the data as follows. Suppose that $0 \leq p \leq 1$ and that $100p\%$ of the taxis are White and $100(1-p)\%$ of the taxis are Black. Leave the reliability of the witness at 80%. Show that the probability that the taxi involved is White, given that the witness claims that the taxi involved was White, exceeds 0.5 if and only if $p > 0.2$.

e) Explore the sensitivity of the answer to Question 18b to the data even further. Suppose that $0 \leq p \leq 1$ and that $100p\%$ of the taxis are White and $100(1-p)\%$ of the taxis are Black. Suppose that $0 \leq q \leq 1$ and that the reliability of the witness is $100q\%$; i.e. the witness is able to identify correctly the color of a taxi $100q\%$ of the time. Determine the region inside the square

$$\{(p,q) : 0 \leq p \leq 1, 0 \leq q \leq 1\}$$

of points for which the probability that the taxi involved is White, given that the witness claims that the taxi involved was White, exceeds 0.5.

[1] The following example is based on material found in the delightful book by Deborah Bennett called *Randomness* [7, pp.2-3] which, in turn, is based on work by D.Kahneman and A.Tversky [69, pp.156-158].

Solution to Exercise 1.17

Considering this sort of question can lead to reading about the connections between probability and psychology; more precisely between the ways in which mathematicians measure risk and the general perceptions of risk.

a) Without any calculation one may be tempted to argue as follows.

In a court of law, the central issue is the reliability of the witness - not the distribution of black and white taxis in the city. As it has been proved that the witness is 80% reliable, we would tend to believe the witness. Hence, it is more likely that the taxi is white.

Below we see the intricate connections between the answer and the parameters of the question.

b) We use Bayes' theorem. Define the following events.
Y_1 = taxi in accident is white.
Y_2 = taxi in accident is black.
X = witness says that taxi is white.
Thus, $P(Y_1) = 0.15$; $P(Y_2) = 0.85$; $P(X|Y_1) = 0.8$; $P(X|Y_2) = 0.2$. By Bayes' theorem

$$\begin{aligned} P(Y_1|X) &= \frac{P(Y_k)P(X|Y_k)}{\sum_{i=1}^{n} P(Y_i)P(X|Y_i)} \\ &= \frac{(0.15)(0.8)}{(0.15)(0.8) + (0.85)(0.2)} \\ &= 0.41; \end{aligned}$$

and similarly

$$P(Y_2|X) = 0.59$$

c) Hence it is more likely that the cab is black. This illustrates the risks associated with "wooly" thinking about probability.

d) Here,

$$P(Y_1) = p; P(Y_2) = 1 - p; P(X|Y_1) = 0.8; P(X|Y_2) = 0.2.$$

By Bayes' theorem

$$\begin{aligned} P(Y_1|X) &= \frac{P(Y_1)P(X|Y_1)}{\sum_{i=1}^{n} P(Y_i)P(X|Y_i)} \\ &= \frac{(p)(0.8)}{(p)(0.8) + (1-p)(0.2)} \\ &= \frac{0.8p}{0.2 + 0.6p}. \end{aligned}$$

So $P(Y_1|X) > 0.5$ if and only if $p > 0.2$.

e) Here,

$$P(Y_1) = p; P(Y_2) = 1 - p; P(X|Y_1) = q; P(X|Y_2) = 1 - q;$$

By Bayes' theorem

$$\begin{aligned}
P(Y_1|X) &= \frac{P(Y_1)P(X|Y_1)}{\sum_{i=1}^{n} P(Y_i)P(X|Y_i)} \\
&= \frac{pq}{pq + (1-p)(1-q)} \\
&= \frac{pq}{1 - p - q + 2pq}
\end{aligned}$$

Thus $P(Y_1|X) > 0.5$ if and only if

$$\begin{aligned}
\frac{pq}{1 - p - q + 2pq} &> \frac{1}{2} \\
2pq &> 1 - p - q + 2pq \\
p + q &> 1.
\end{aligned}$$

Chapter 2

Random Variables and their Probability Distributions

Exercise 2.1

(pdf, univariate) Consider the experiment of rolling a manipulated die and observing the number of dots facing up. The die is manipulated in the way that the probability of i dots facing up is proportional to i, $(i = 1, ..., 6)$. Let X denote the number of dots facing up. Find the probability density function of X.

Solution to Exercise 2.1

According to the exercise, $P(X = i) = c \cdot i$ with $i = 1, ..., 6$ and a constant c. The sample space is given by $S = \{1, ..., 6\}$ with $P(S) = 1$, where $P(S) = \sum_{i=1}^{6} P(X = i) = c \sum_{i=1}^{6} i = c \cdot 21$. Hence, $c = \frac{1}{21}$ and the pdf is

$$f(x) = \frac{x}{21} \mathcal{I}_{\{1,...,6\}}(x)$$

Exercise 2.2

(pdf and cdf, univariate) A fair coin is tossed until heads occurs for the first time, but it is tossed no more than three times. The random variable X counts the number of tosses. Find the probability density function and the cumulative distribution function of X.

Solution to Exercise 2.2

Sample space is given by $S = \{H; (T, H); (T, T, H); (T, T, T)\}$. The random variable X represents a mapping $f : S \to \mathcal{R}$. Which elementary event ω_i refers to which value of the random variable X?

ω_i	$X(\omega_i)$
H	1
(T,H)	2
(T,T,H)	3
(T,T,T)	3

$$P(X = 1) = P(H) = \tfrac{1}{2}$$
$$P(X = 2) = P(T, H) = (\tfrac{1}{2})^2 = \tfrac{1}{4}$$
$$P(X = 3) = P(T, T, H) + P(T, T, T) = \tfrac{1}{8} + \tfrac{1}{8} = \tfrac{1}{4}$$

pdf: $$f(x) = \begin{cases} \frac{1}{2} & \text{if } x = 1 \\ \frac{1}{4} & \text{if } x = 2 \\ \frac{1}{4} & \text{if } x = 3 \\ 0 & \text{otherwise} \end{cases}$$

cdf: $$F(x) = \begin{cases} 0 & \text{if } x < 1 \\ \frac{1}{2} & \text{if } 1 \le x < 2 \\ \frac{3}{4} & \text{if } 2 \le x < 3 \\ 1 & \text{if } x \ge 3 \end{cases}$$

Exercise 2.3

(pdf, univariate) The probability density function of the random variable X is given by

$$f(x) = \frac{1}{2}(2 - x)\mathcal{I}_{(0,2)}.$$

Calculate the following probabilities

a) $P(X < 1.2)$

b) $P(X > 1.6)$

c) $P(1.2 < X < 1.6)$.

Solution to Exercise 2.3

$$\text{pdf:} \qquad f(x) = \frac{1}{2}(2 - x)\mathcal{I}_{(0.2)}(x)$$

$$\text{cdf:} \qquad F(x) = \int_0^x f(X)dX = \Big[X - \frac{X^2}{4}\Big]_0^x = x - \frac{x^2}{4}, \quad 0 \le x \le 2$$

a) $P(X < 1.2) = F(1.2) = 0.84$

b) $P(X > 1.6) = 1 - P(X < 1.6) = 1 - F(1.6) = 1 - 0.96 = 0.04$

c) $P(1.2 < X < 1.6) = P(X < 1.6) - P(X < 1.2) = F(1.6) - F(1.2) = 0.12$

Exercise 2.4

(pdf, univariate) Check whether the following two functions could be probability density functions:

a) $f(x) = \frac{1}{x}\mathcal{I}_{\{2,3,\ldots\}}(x)$

b) $f(x) = \frac{1}{2^x}\mathcal{I}_{\{2,3,\ldots\}}(x)$.

Solution to Exercise 2.4

a) $f(x)$ is a density function if $\sum_2^\infty f(x) = 1$.

$$\sum_2^\infty \frac{1}{x} > \frac{1}{2} + \frac{1}{3} + \frac{1}{4} > 1$$

Hence, it is not a density function.

b) $f(x) = \frac{1}{2^x}\mathcal{I}_{\{2,3,\ldots\}}(x)$.

$$\sum_{x=2}^\infty \frac{1}{2^X} = \sum_{x=0}^\infty \frac{1}{2^X} - (1 + \frac{1}{2}) = \left(\frac{1}{1 - 1/2}\right) - \frac{3}{2} = \frac{1}{2} < 1$$

Hence, it is not a density function.

> **Note:**
> Formula of sums for geometric series with a constant quotient q, $|q| < 1$:
>
> $$\sum_{i=0}^{\infty} q^i = \frac{1}{1-q}$$

Exercise 2.5

(pdf and cdf, univariate) For each of the probability density functions given below find the associated cumulative distribution function and sketch both functions for each problem.

a) $f(x) = \begin{cases} x & \text{if} \quad 0 \le x < 1 \\ 2-x & \text{if} \quad 1 \le x < 2 \\ 0 & \text{otherwise} \end{cases}$

b) $f(x) = \begin{cases} x & \text{if} \quad 0 \le x < 1 \\ \frac{1}{4}(3-x) & \text{if} \quad 1 \le x < 3 \\ 0 & \text{otherwise} \end{cases}$

c) $f(x) = \begin{cases} x & \text{if} \quad 0 \le x < 1 \\ (x-1) & \text{if} \quad 1 \le x < 2 \\ 0 & \text{otherwise} \end{cases}$

d) $f(x) = \begin{cases} x & \text{if} \quad 0 \le x < 1 \\ (x-5) & \text{if} \quad 5 \le x < 6 \\ 0 & \text{otherwise} \end{cases} .$

Solution to Task 2.5

a) $x \in [0, 1)$:

$$F(x) = \int_0^x X dX = \frac{1}{2}x^2$$

$x \in [1, 2)$:

$$F(x) = \int_1^x (2-X)dX + F(1) = -\frac{1}{2}x^2 + 2x - 1 = 1 - \frac{1}{2}(2-x)^2$$

$$\Rightarrow F(x) = \begin{cases} 0 & x < 0 \\ \frac{1}{2}x^2 & x \in [0,1) \\ 1 - \frac{1}{2}(2-x)^2 & x \in [1,2) \\ 1 & x \ge 2 \end{cases}$$

b) $x \in [1, 3)$:

$$F(x) = \int_1^x \frac{1}{4}(3-X)dX + F(1) = -\frac{1}{8}x^2 + \frac{3}{4}x - \frac{1}{8} = 1 - \frac{1}{8}(3-x)^2$$

$$\Rightarrow F(x) = \begin{cases} 0 & x < 0 \\ \frac{1}{2}x^2 & x \in [0,1) \\ 1 - \frac{1}{8}(3-x)^2 & x \in [1,3) \\ 1 & x \geq 3 \end{cases}$$

c) $x \in [1,2)$:

$$F(x) = \int_1^x (X-1)dX + F(1) = \frac{1}{2}x^2 - x + 1 = \frac{1}{2} + \frac{1}{2}(x-1)^2$$

$$\Rightarrow F(x) = \begin{cases} 0 & x < 0 \\ \frac{1}{2}x^2 & x \in [0,1) \\ \frac{1}{2} + \frac{1}{2}(x-1)^2 & x \in [1,2) \\ 1 & x \geq 2 \end{cases}$$

d) $x \in [5,6)$:

$$F(x) = \int_5^x (X-5)dX + F(5) = \frac{1}{2}x^2 - 5x + 13 = \frac{1}{2} + \frac{1}{2}(x-5)^2$$

$$\Rightarrow F(x) = \begin{cases} 0 & x < 0 \\ \frac{1}{2}x^2 & x \in [0,1) \\ \frac{1}{2} & x \in [1,5) \\ \frac{1}{2} + \frac{1}{2}(x-5)^2 & x \in [5,6) \\ 1 & x \geq 6 \end{cases}$$

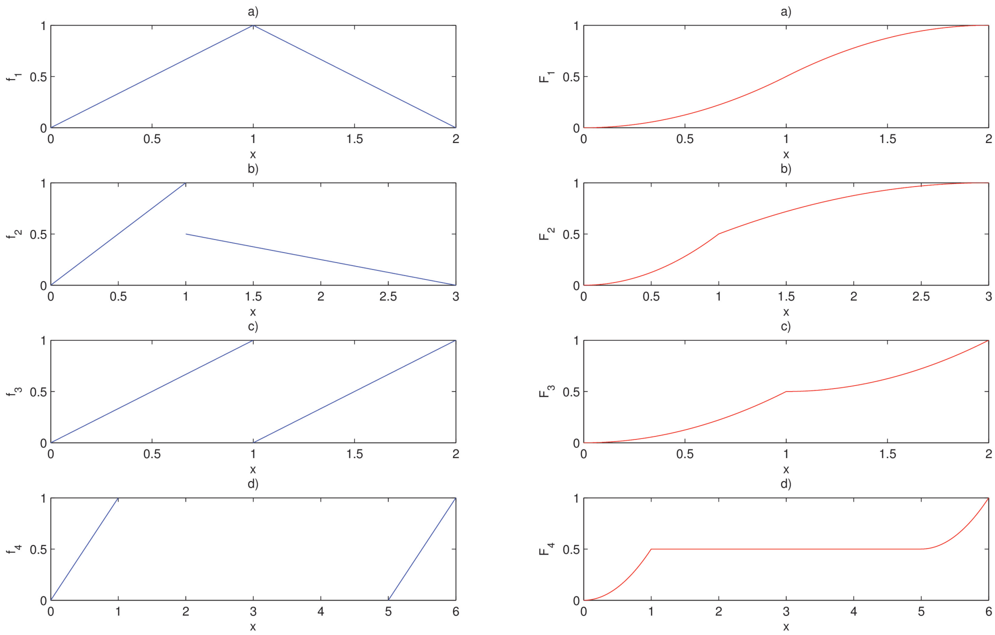


Figure 2.1: Densities and distributions

Exercise 2.6

(pdf and cdf, univariate) For each of the cumulative distribution functions given below find the associated probability density function.

a) $F(x) = \begin{cases} 0 & \text{if} \quad x < 0 \\ x^2 & \text{if} \quad 0 \leq x \leq 1 \\ 1 & \text{otherwise} \end{cases}$

b) $F(x) = \begin{cases} 0 & \text{if} \quad x < 2 \\ (x-2)^3 & \text{if} \quad 2 \leq x \leq 3 \\ 1 & \text{otherwise} \end{cases}$

c) $F(x) = [1 - \exp\{-\lambda(x - c)\}]\mathcal{I}_{[c,\infty)}(x)\ , \quad \lambda > 0\ .$

Solution to Exercise 2.6

a) $F' : f(x) = 2x \cdot \mathcal{I}_{[0,1]}(x)$

b) $F' : f(x) = 3(x-2)^2 \cdot \mathcal{I}_{[2,3]}(x)$

c) $F' : f(x) = \lambda \exp\{-\lambda(x - c)\} \cdot \mathcal{I}_{[c,\infty)}(x)$

Exercise 2.7

(pdf and cdf, univariate) The probability density function of the random variable X is given by:

$$f(x) = a(1-b)^{x-1}\mathcal{I}_{\{1,2,3,\dots\}}(x), \quad \text{where} \quad b \in (0,1)\ .$$

a) Specify the constraints on the choice of the parameter a for $f(x)$ to be a probability density function.

b) Assume that $P(X = 1) = 0.05$. What is the specific functional form of the probability density function $f(x)$? What is the probability $P(X = 10)$?

c) Find the cumulative distribution function of X and determine the probability of the event $X \leq 10$.

Solution to Exercise 2.7

a) For a density function it has to hold that $\sum_{x=1}^{\infty} f(x) = 1$.

$$\begin{aligned}
\sum_{x=1}^{\infty} a(1-b)^{X-1} &= \frac{a}{1-b}\sum_{x=1}^{\infty}(1-b)^x = \\
&= \frac{a}{1-b}\left(\sum_{x=0}^{\infty}(1-b)^x - 1\right) = \\
&= \frac{a}{1-b}\left(\frac{1}{1-(1-b)} - \frac{b}{b}\right) = \\
&= \frac{a}{1-b}\cdot\frac{1-b}{b} = \frac{a}{b} \equiv 1
\end{aligned}$$

After setting the result equal to one we get $a = b$, hence

$$f(x) = b(1-b)^{x-1}\mathcal{I}_{\{1,2,3,\dots\}}(x)$$

b) From $P(X = 1) = f(x = 1) = b(1-b)^0 = 0.05$ it follows that $b = 0.05$. Then

$$f(x) = 0.05 \cdot (0.95)^{x-1}\mathcal{I}_{\{1,2,3,\dots\}}(x).$$

And finally, $f(x = 10) = 0.05 \cdot (0.95)^9$.

c) The cdf can be defined as

$$F(x) = \begin{cases} 0 & x < 1 \\ \sum_{j=1}^{x} b(1-b)^{j-1} & 1 \leq x < \infty \\ 1 & \text{otherwise} \end{cases}$$

Then,

$$F(10) = \sum_{j=1}^{10} b(1-b)^{j-1} = 0.4013$$

Exercise 2.8

(joint pdf and cdf, multivariate) The joint probability density function of the random variable (X_1, X_2) is given by

$$f(x_1, x_2) = \frac{1}{4}\mathcal{I}_{[0,4]}(x_1)\mathcal{I}_{[0,1]}(x_2) \, .$$

a) Find the the joint cumulative distribution function of (X_1, X_2). Calculate the probability $P(2 \leq X_1 \leq 3; 0.5 \leq X_2 \leq 1)$.

b) Find the marginal probability density functions $f(x_1)$ and $f(x_2)$ as well as the conditional density functions $f(x_1|x_2)$ and $f(x_2|x_1)$.

Solution to Exercise 2.8

a) We have to consider the following case distinction:

I. $x_1 \in [0,4],\ x_2 \in [0,1]$

$$\begin{aligned} F(x_1,x_2) &= \int_0^{x_1}\int_0^{x_2}\frac{1}{4}dX_2dX_1 \\ &= \int_0^{x_1}\left[\frac{1}{4}X_2\right]_0^{x_2}dX_1 = \int_0^{x_1}\frac{1}{4}x_2dX_1 = \frac{1}{4}x_1x_2 \end{aligned}$$

II. $x_1 \in [0,4],\ x_2 > 1$

$$\begin{aligned} F(x_1,1) &= \int_0^{x_1}\int_0^{1}\frac{1}{4}dX_2dX_1 = \int_0^{x_1}\left[\frac{1}{4}X_2\right]_0^{1}dX_1 \\ &= \int_0^{x_1}\frac{1}{4}dX_1 = \left[\frac{1}{4}X_1\right]_0^{x_1} = \frac{1}{4}x_1 \end{aligned}$$

III. $x_1 > 4,\ x_2 \in [0,1]$

$$\begin{aligned} F(4,x_2) &= \int_0^{x_2}\int_0^{4}\frac{1}{4}dX_1dX_2 = \int_0^{x_2}\left[\frac{1}{4}X_1\right]_0^{4}dX_2 \\ &= \int_0^{x_2}1\cdot dX_2 = [X_2]_0^{x_2} = x_2 \end{aligned}$$

IV. $x_1 < 0,\ x_2 \in [0,1] \ \rightarrow\ F(x_1,x_2) = 0$

V. $x_1 \in [0,4],\ x_2 < 0 \ \rightarrow\ F(x_1,x_2) = 0$

VI. $x_1 < 0,\ x_2 < 0 \ \rightarrow\ F(x_1,x_2) = 0$

VII. $x_1 < 0,\ x_2 > 1 \ \rightarrow\ F(x_1,x_2) = 0$

VIII. $x_1 > 4,\ x_2 < 0, \rightarrow\ F(x_1,x_2) = 0$

IX. $x_1 > 4,\ x_2 > 1 \ \rightarrow\ F(x_1,x_2) = 1$

Summing up:

$$\begin{cases} 0 & x_1 < 0 \text{ and/or } x_2 < 0 \\ \frac{1}{4}x_1x_2 & x_1 \in [0,4], x_2 \in [0,1] \\ x_2 & x_1 > 4 \text{ and } x_2 \in [0,1] \\ \frac{1}{4}x_1 & x_1 \in [0,4] \text{ and } x_2 > 1 \\ 1 & \text{otherwise} \end{cases}$$

The probability $P(2 \leq X_1 \leq 3; 0.5 \leq X_2 \leq 1)$ can be thought of as the volume above a base that is given by x_1 and x_2. The integral that is equal to $F(3,1)$ is too big, as the corresponding base is given by $x_1 \in [0,3]$ and $x_2 \in [0,1]$. Whereas the relevant base is just given by $x_1 \in [2,3]$ and $x_2 \in [0.5,1]$, so that the following approach applies:

$$\begin{aligned} P(2 \leq X_1 \leq 3; 0,5 \leq X_2 \leq 1) &= F(3,1) - F(2,1) \\ &\quad - F(3;0.5) + F(2;0.5) \\ &= 0.125 \end{aligned}$$

$F(2;0.5)$ is added once because it has previously been subtracted twice ($F(2,1)$ and $F(3;0.5)$ each contain the respective volume).

b) Marginal probability density function:

$f(x_1) = \frac{1}{4}\mathcal{I}_{[0,4]}(x_1)$;
$f(x_2) = \mathcal{I}_{[0,1]}(x_2)$

Conditional probability density function:

$$\begin{aligned} f(x_1|x_2) &= \tfrac{f(x_1,x_2)}{f_2(x_2)} = \tfrac{\frac{1}{4}\mathcal{I}_{[0,4]}(x_1)\mathcal{I}_{[0,1]}(x_2)}{\mathcal{I}_{[0,1]}(x_2)} = \tfrac{1}{4}\mathcal{I}_{[0,4]}(x_1) \\ f(x_2|x_1) &= \tfrac{f(x_1,x_2)}{f_1(x_1)} = \tfrac{\frac{1}{4}\mathcal{I}_{[0,4]}(x_1)\mathcal{I}_{[0,1]}(x_2)}{\frac{1}{4}\mathcal{I}_{[0,4]}(x_1)} = \mathcal{I}_{[0,1]}(x_2) \end{aligned}$$

Exercise 2.9

(joint cdf, multivariate) Examine the function

$$F(x_1, x_2) = \begin{cases} 0 & \text{for} \quad x_1 + x_2 < 0 \\ 1 & \text{for} \quad x_1 + x_2 \geq 0 \end{cases},$$

and show that it cannot represent a joint cumulative distribution function.

Solution to Exercise 2.9

Properties of the cdf:

1. $\lim\limits_{x_i \to -\infty} F(x_1, x_2) = 0, \quad i = 1, 2$
2. $\lim\limits_{x_i \to +\infty} F(x_1, x_2) = 1, \quad i = 1, 2$

Although the function F satisfies the limit properties of a cdf, it does not represent a cdf because of its jump discontinuities. Jump discontinuities lead to conflicts in the calculation of probabilities, e.g.:

$$\begin{aligned} P(-1 \leq X_1 \leq 1; 0 \leq X_2 \leq 2) &= F(1,2) - F(-1,2) \\ &\quad - F(1,0) + F(-1,0) \\ &= -1. \end{aligned}$$

Remember, a cdf has to be continuous!

Exercise 2.10

(joint pdf, multivariate) The joint probability density function of the random variable (X_1, X_2) is given by

$$f(x_1, x_2) = \frac{1}{2}(4x_1x_2 + 1)\mathcal{I}_{[0,1]}(x_1)\mathcal{I}_{[0,1]}(x_2)\ .$$

a) Find the joint cumulative distribution function of (X_1, X_2).

b) Find the marginal probability density functions $f(x_1)$ and $f(x_2)$. Are X_1 and X_2 stochastically independent?

c) Find the conditional density function $f(x_1|x_2)$ as well as the conditional cumulative distribution function $F(x_1|x_2)$.

Solution to Exercise 2.10

a) cdf:

I. $x_1 < 0,\ x_2 < 0\ \rightarrow\ F(x_1, x_2) = 0$

II. $x_1,\ x_2 \in [0,1] \rightarrow\ F(x_1, x_2) = \int_0^{x_1}\int_0^{x_2}\frac{1}{2}(4x_1x_2 + 1)dx_2dx_1 = \frac{1}{2}x_1x_2(x_1x_2 + 1)$

III. $x_1 \in [0,1],\ x_2 > 1 \rightarrow\ F(x_1, x_2) = F(x_1, 1) = \frac{1}{2}(x_1^2 + x_1)$

IV. $x_1 > 1,\ x_2 \in [0,1] \rightarrow\ F(x_1, x_2) = F(1, x_2) = \frac{1}{2}(x_2^2 + x_2)$

V. $x_1 > 1,\ x_2 > 1 \rightarrow\ F(x_1, x_2) = 1$

b) Marginal densities:

$$f(x_1) = \int_0^1 \frac{1}{2}(4X_1X_2 + 1)dX_2 = x_1 + \frac{1}{2}; \quad x_1 \in [0,1]$$

$$f(x_2) = \int_0^1 \frac{1}{2}(4X_1X_2 + 1)dX_1 = x_2 + \frac{1}{2}; \quad x_2 \in [0,1]$$

Check

$$f(x_1)\cdot f(x_2) = (x_1 + \frac{1}{2})(x_2 + \frac{1}{2}) = x_1x_2 + \frac{x_1}{2} + \frac{x_2}{2} + \frac{1}{4} \neq f(x_1, x_2)$$

$\Rightarrow$ stochastically dependent!

c)

$$f(x_1|x_2) = \frac{f(x_1, x_2)}{f_2(x_2)} = \frac{4x_1x_2 + 1}{2x_2 + 1} \qquad \text{for } x_1 \in [0,1], x_2 \in [0,1]$$

$$F(x_1|x_2) = \int_0^{x_1} f(x_1|x_2)dx_1 = \frac{2x_1^2x_2 + x_1}{2x_2 + 1} \qquad \text{for } x_1 \in [0,1], x_2 \in [0,1]$$

Exercise 2.11

(joint pdf, multivariate) The joint probability density function of the random variable (X_1, X_2) is given by

$$f(x_1, x_2) = \frac{1}{2}\mathcal{I}_{[0,4]}(x_1)\mathcal{I}_{[0,\frac{1}{4}x_1]}(x_2) \ .$$

a) Find the cdf of (X_1, X_2). Calculate the probability $P(2 \le X_1 \le 3; 1/2 \le X_2 \le 3/2)$.

b) Find the marginal probability density functions $f_1(x_1)$ and $f_2(x_2)$ as well as the conditional density functions $f(x_1|x_2 = 1/4)$ and $f(x_2|x_1 = 1)$.

Solution to Exercise 2.11

a) In order to find the cdf, consider a case distinction. Note, that the range of x_2 is a function of x_1.

I. For $x_1 \in [0,4]$, $x_2 \in [0, \frac{1}{4}x_1]$: $x_2 \le \frac{1}{4}x_1 \Leftrightarrow x_1 \ge 4x_2$.

$$F(x_1, x_2) = \int_0^{x_2}\int_{4x_2}^{x_1} f(x_1, x_2)dx_1dx_2$$

It follows that:

$$\begin{aligned} F(x_1, x_2) &= \int_0^{x_2}\int_{4x_2}^{x_1} \frac{1}{2}dx_1dx_2 = \int_0^{x_2}\left[\frac{1}{2}x\right]_{4x_2}^{x_1} dx_2 \\ &= \int_0^{x_2}\left(\frac{1}{2}x_1 - 2x_2\right)dx_2 = \frac{1}{2}x_1x_2 - x_2^2 \end{aligned}$$

II. For $x_1 > 4$, $x_2 \in [0,1]$: $F(x_1, x_2) = F(4, x_2) = 2x_2 - x_2^2$

III. If $x_1 \in [0,4]$, $x_2 > \frac{1}{4}x_1$: $F(x_1, x_2) = F(x_1, \frac{1}{4}x_1) = \frac{1}{16}x_1^2$

IV. If $x_1 < 0$, $x_2 < 0$: $F(x_1, x_2) = 0$

V. If $x_1 > 4$, $x_2 > 1$: $F(x_1, x_2) = 1$

$$\begin{aligned} P\left(2 \le X_1 \le 3; \frac{1}{2} \le X_2 \le \frac{3}{2}\right) &= F\left(3, \frac{3}{2}\right) - F\left(3, \frac{1}{2}\right) - F\left(2, \frac{3}{2}\right) + F\left(2, \frac{1}{2}\right) \\ &= \frac{1}{16}\cdot 9 - \left(\frac{1}{2}\cdot 3 \cdot \frac{1}{2} - \frac{1}{2}^2\right) - \frac{1}{16}4 + \frac{1}{16}4 = \frac{1}{16} \end{aligned}$$

b) Marginal density functions:
$x_1 \in [0,4] : f(x_1) = \int_{-\infty}^{\infty} f(x_1, x_2)dx_2 = \int_0^{\frac{1}{4}x_1}\frac{1}{2}dx_2 = \frac{1}{8}x_1$
$x_2 \in [0,1] : f(x_2) = \int_{-\infty}^{\infty} f(x_1, x_2)dx_1 = \int_{4x_2}^{4}\frac{1}{2}dx_1 = 2 - 2x_2$

Conditional density functions:

$$f(x_1|x_2 = \frac{1}{4}) = \frac{f(x_1, x_2 = 1/4)}{f(x_2 = \frac{1}{4})} = \begin{cases} \frac{1}{2}/\frac{3}{2} = \frac{1}{3} & \text{if } x_1 \in [1, 4] \\ 0 & \text{otherwise} \end{cases}$$

$$f(x_2|x_1 = 1) = \frac{f(x_1 = 1, x_2)}{f(x_1 = 1)} = \begin{cases} \frac{1}{2}/\frac{1}{8} = 4 & \text{if } 0 < x_2 \leq \frac{1}{4} \\ 0 & \text{otherwise} \end{cases}$$

Exercise 2.12

(joint pdf, multivariate) The joint probability density function of the random variable (X_1, X_2, X_3) is given by

$$f(x_1, x_2, x_3) = \frac{3}{16}(x_1 x_2^2 e^{-x_3})\mathcal{I}_{[0,2]}(x_1)\mathcal{I}_{[0,2]}(x_2)\mathcal{I}_{[0,\infty)}(x_3) \ .$$

a) Find the marginal probability density functions for X_1, X_2 and X_3.

b) Calculate the probability $P(X_1 \geq 1)$.

c) Are the three random variables stochastically independent?

d) Find the marginal cumulative distribution functions of X_2 and X_3.

e) Calculate the probability $P(x_1 < 1; x_3 > 1)$.

f) Find the joint cumulative distribution function of X_1, X_2 and X_3.

g) Calculate the probability $P(X_1 \leq 1; X_2 \leq 1; X_3 \leq 10)$.

h) Find the conditional density function of X_1, given that $x_2 = 1$ and $x_3 = 0$.

i) Calculate the probability that $X_1 \in [0, 1/2]$, given that $x_2 = 1$ and $x_3 = 0$.

j) Let the two random variables Y_1 and Y_2 be defined by $Y_1 = g_1(x_1, x_2) = x_1^2 x_2$ and $Y_2 = g_2(x_3) = x_3/2$. Are the random variables Y_1 and Y_2 independent?

Solution to Exercise 2.12

a)

$$\begin{aligned} f(x_1) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x_1, x_2, x_3)dx_2dx_3 \\ &= \frac{3}{16}x_1\mathcal{I}_{[0,2]}(x_1)\int_{-\infty}^{\infty} x_2^2\mathcal{I}_{[0,2]}(x_2)dx_2 \int_{-\infty}^{\infty} e^{-x_3}\mathcal{I}_{[0,\infty]}(x_3)dx_3 \\ &= \frac{3}{16}x_1\mathcal{I}_{[0,2]}(x_1)\left(\frac{8}{3}\right)(1) = \frac{1}{2}x_1\mathcal{I}_{[0,2]}(x_1). \end{aligned}$$

Similarly,

$$\begin{aligned} f(x_2) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x_1, x_2, x_3)dx_1dx_3 = \frac{3}{8}x_2^2\mathcal{I}_{[0,2]}(x_2) \\ f(x_3) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x_1, x_2, x_3)dx_1dx_2 = e^{-x_3}\mathcal{I}_{[0,\infty)}(x_3). \end{aligned}$$

b)

$$P(X_1 \geq 1) = \int_1^\infty f(x_1)dx_1 = \int_1^2 \frac{1}{2}x_1 dx_1 = \left.\frac{x_1^2}{4}\right|_1^2 = 0.75.$$

c) Yes, the three random variables are independent. Since we have derived the marginal densities of X_1, X_2, and X_3, it is clear that

$$f(x_1, x_2, x_3) = f(x_1)f(x_2)f(x_3) \qquad \forall\,(x_1, x_2, x_3).$$

d) By definition,

$$\begin{aligned}
F(x_2) &= \int_{-\infty}^{x_2} f(x_2)dx_2 = \int_{-\infty}^{x_2} \frac{3}{8}x_2^2 \mathcal{I}_{[0,2]}(x_2)dx_2 \\
&= \left.\frac{1}{8}x_2^3\right|_0^{x_2} \mathcal{I}_{[0,2]}(x_2) + \mathcal{I}_{[2,\infty)}(x_2) = \frac{1}{8}x_2^3\mathcal{I}_{[0,2]}(x_2) + I_{[2,\infty)}(x_2), \\
F(x_3) &= \int_{-\infty}^{x_3} f(x_3)dx_3 = \int_{-\infty}^{x_3} e^{-x_3}\mathcal{I}_{[0,\infty)}(x_3)dx_3 \\
&= -e^{-x_3}\big|_0^{x_3} I_{[0,\infty)}(x_3) = (1-e^{-x_3})\mathcal{I}_{[0,\infty)}(x_3).
\end{aligned}$$

e)

$$\begin{aligned}
F(x_1) &= \frac{1}{4}x_1^2\mathcal{I}_{[0,2]}(x_1) + \mathcal{I}_{(2,\infty)}(x_1) \\
P(X_1 < 1) &= F(x_1 = 1) = 0.25, \quad P(X_3 > 1) = 1 - F(x_3 = 1) = e^{-1} = 0.3679.
\end{aligned}$$

Now, by independence

$$P(X_1 < 1, X_3 > 1) = P(X_1 < 1)P(X_3 > 1) = F(x_1 = 1)(1 - F(x_3 = 1)) = 0.25 \times 0.37 = 0.0925$$

f) By definition,

$$\begin{aligned}
F(x_1, x_2, x_3) &= \int_{-\infty}^{x_1}\int_{-\infty}^{x_2}\int_{-\infty}^{x_3} f(x_1, x_2, x_3)dx_3\,dx_2\,dx_1 \\
&= \int_{-\infty}^{x_1} \frac{1}{2}x_1\mathcal{I}_{[0,2]}(x_1)dx_1 \int_{-\infty}^{x_2} \frac{3}{8}x_2^2\mathcal{I}_{[0,2]}(x_2)dx_2 \int_{-\infty}^{x_3} e^{-x_3}\mathcal{I}_{[0,\infty)}(x_3)dx_3 \\
&= \left[\frac{x_1^2}{4}\mathcal{I}_{[0,2]}(x_1) + \mathcal{I}_{[2,\infty)}(x_1)\right]\left[\frac{3x_2^3}{24}\mathcal{I}_{[0,2]}(x_2) + \mathcal{I}_{[2,\infty)}(x_2)\right]\left[(1-e^{-x_3})\mathcal{I}_{[0,\infty)}(x_3)\right].
\end{aligned}$$

g) $F(1, 1, 10) = (1/4)(3/24)(1 - e^{-10}) = 0.031.$

h) By definition,

$$f(x_1 \,|x_2 = 1, x_3 = 0) = \frac{f(x_1, 1, 0)}{f(x_2 = 1, x_3 = 0)}.$$

Also,

$$f(x_2, x_3) = \int_{-\infty}^{\infty} f(x_1, x_2, x_3)dx_1 = \frac{3}{8}x_2^2\mathcal{I}_{[0,2]}(x_2)e^{-x_3}\mathcal{I}_{[0,\infty)}(x_3).$$

Thus,

$$f(x_1 \,|x_2 = 1, x_3 = 0) = \frac{\frac{3}{16}x_1\mathcal{I}_{[0,2]}(x_1)}{\frac{3}{8}} = \frac{1}{2}x_1\mathcal{I}_{[0,2]}(x_1)$$

i)

$$
\begin{aligned}
P(x_1 \in [0,1/2] \mid x_2 = 1, x_3 = 0) &= \int_0^{1/2} f(x_1|x_2 = 1, x_3 = 0)dx_1 \\
&= \int_0^{1/2} \frac{1}{2}x_1 I_{[0,2]}(x_1)dx_1 \\
&= \left.\frac{x_1^2}{4}\right|_0^{1/2} = \frac{1}{16}
\end{aligned}
$$

j) Yes, they are independent. The bivariate random variable (X_1, X_2) is independent of the random variable X_3 since $f(x_1, x_2, x_3) = f(x_1, x_2)f(x_3)$, i.e. , the joint density function factors into the product of the marginal density of (X_1, X_2) and the marginal density of X_3. Then since y_1 is a function of only (x_1, x_2) and y_2 is a function of only x_3, y_1 and y_2 are independent random variables.

Exercise 2.13

(joint pdf, multivariate) The joint probability density function of the random variable (X, Y) is given by

$$f(x, y) = k \cdot (x^2 + y^2)I_{[0,1]}(x)I_{[0,1]}(y)$$

a) Calculate k.

b) Find the marginal distribution functions $f(x)$ and $f(y)$.

c) Calculate $P(3x > y)$.

Solution to Exercise 2.13

a)

$$\int_0^1 \int_0^1 k(x^2 + y^2)dxdy \overset{!}{=} 1$$

$$= k \int_0^1 \left[\frac{1}{3}x^3 + xy^2\right]_0^1 dy = k \int_0^1 \frac{1}{3} + y^2 dy = k \left[\frac{1}{3}y + \frac{1}{3}y^3\right]_0^1 = k\left(\frac{1}{3} + \frac{1}{3}\right) \overset{!}{=} 1$$

$$\Rightarrow k = \frac{3}{2}$$

b)

$$f(x) = \int_{-\infty}^{\infty} f(x, y)dy = k \int_0^1 x^2 + y^2 dy = k\left[x^2 y + \frac{1}{3}y^3\right]_0^1 = k\left(x^2 + \frac{1}{3}\right)\mathcal{I}_{(0,1)}(x)$$

analogously:

$$f(y) = k\left(y^2 + \frac{1}{3}\right)\mathcal{I}_{(0,1)}(y)$$

c)

$$\begin{aligned}
P(3x > y) &= P(x > \frac{1}{3}y) = \int_0^1 \int_{\frac{1}{3}y}^1 \frac{3}{2}(x^2+y^2)\ dxdy = \frac{3}{2}\int_0^1 \left[\frac{1}{3}x^3 + xy^2\right]_{\frac{1}{3}y}^1\ dy \\
&= \frac{3}{2}\int_0^1 \frac{1}{3} + y^2 - \frac{1}{3}\frac{1}{27}y^3 - \frac{1}{3}y^3\ dy \\
&= \frac{3}{2}\int_0^1 \frac{1}{3} + y^2 - \frac{28}{81}y^3\ dy \\
&= \frac{3}{2}\left[\frac{1}{3}y + \frac{1}{3}y^3 - \frac{1}{4}\frac{28}{81}y^4\right]_0^1 = \frac{3}{2}\left[\frac{1}{3} + \frac{1}{3} - \frac{7}{81}\right] \\
&= \frac{47}{54}
\end{aligned}$$

Chapter 3

Moments of Random Variables

Exercise 3.1

(Expected value) Let the random variable X have the probability density function (geometric distribution)

$$f(x) = \theta(1-\theta)^{(x-1)}\mathcal{I}_{\{1,2,\dots\}}(x) \qquad \text{with } \theta \in (0,1).$$

Find the expected value $\mathrm{E}(X)$.

Solution to Exercise 3.1

$$\mathrm{E}(X) = \sum_{x=1}^{\infty} x\,\theta(1-\theta)^{x-1}$$

$$\begin{array}{cccccl}
 & x=1 & x=2 & x=3 & \dots & \\
\hline
= & \theta & +\,\theta(1-\theta) & +\,\theta(1-\theta)^2 & +\dots & (= \quad \theta\sum_{x=0}^{\infty}(1-\theta)^x) \\
 & & +\,\theta(1-\theta) & +\,\theta(1-\theta)^2 & +\dots & (= \quad \theta(1-\theta)\sum_{x=0}^{\infty}(1-\theta)^x) \\
 & & & +\,\theta(1-\theta)^2 & +\dots+ & (= \quad \theta(1-\theta)^2\sum_{x=0}^{\infty}(1-\theta)^x)
\end{array}$$

$$\begin{aligned}
\Longrightarrow \quad \mathrm{E}(X) &= \theta\sum_{x=0}^{\infty}(1-\theta)^x + \theta(1-\theta)\sum_{x=0}^{\infty}(1-\theta)^x + \theta(1-\theta)^2\sum_{x=0}^{\infty}(1-\theta)^x + \dots \\
&= (1+(1-\theta)+(1-\theta)^2\dots) \times \theta\sum_{x=0}^{\infty}(1-\theta)^x \\
&= \sum_{x=0}^{\infty}(1-\theta)^x\theta\sum_{x=0}^{\infty}(1-\theta)^x \\
&= \theta\cdot\frac{1}{1-(1-\theta)}\cdot\frac{1}{1-(1-\theta)} = \frac{1}{\theta}
\end{aligned}$$

Exercise 3.2

(Moments) Let the random variable X have the probability density function (Bernoulli distribution)

$$f(x) = \theta^x(1-\theta)^{(1-x)}\mathcal{I}_{\{0,1\}}(x) \qquad \text{with } \theta \in (0,1).$$

Find the non-central moments μ'_r and the central moments μ_2, μ_3 and μ_4 of X.

Solution to Exercise 3.2

Non-central moments

$$\mu_r' = \mathrm{E}[X^r] = 1^r \cdot \theta + 0^r(1-\theta) = \theta$$

Central moments

$$\begin{aligned}
\mu_2 &= \mathrm{E}[(X-\mathrm{E}(X))^2] = \mathrm{E}\left(X^2 - 2\mathrm{E}(X)X + \mathrm{E}(X)^2\right) = \theta - 2\theta^2 + \theta^2 = \theta(1-\theta) \\
\mu_3 &= \mathrm{E}[(X-\mathrm{E}(X))^3] = \mathrm{E}\left(X^3 - 3X^2\mathrm{E}(X) + 3X\mathrm{E}(X)^2 - \mathrm{E}(X)^3\right) \\
&= \theta - 3\theta^2 + 3\theta^3 - \theta^3 = \theta - 3\theta^2 + 2\theta^3 \\
&= \theta(1-\theta)(1-2\theta) \\
\mu_4 &= \mathrm{E}[(X-\mathrm{E}(X))^4] = \mathrm{E}\left(X^4 - 4X^3\mathrm{E}(X) + 6X^2\mathrm{E}(X)^2 - 4X\mathrm{E}(X)^3 + \mathrm{E}(X)^4\right) \\
&= \theta(1-\theta)(1-3\theta+3\theta^2)
\end{aligned}$$

Exercise 3.3

(Moments) Let the random variable X have the probability density function (exponential distribution)

$$f(x) = \lambda e^{-\lambda x}\mathcal{I}_{(0,\infty)}(x) \ .$$

Show via induction that the non-central moments are given by

$$\mu_r' = \frac{r!}{\lambda^r} \ , \ r = 1, 2, ...$$

Solution to Exercise 3.3

a) Base case: Let $r = 1$,

$$\begin{aligned}
\mu_1' &= \mathrm{E}(X) = \underbrace{\int_0^\infty x\lambda e^{-\lambda x} dx}_{\text{integration by parts: } f(x)=x,\ g'(x)=\lambda e^{-\lambda x}} \\
&= [f(x)g(x)]_0^\infty - \int_0^\infty f'(x)g(x)\ dx = \underbrace{[-xe^{-\lambda x}]_0^\infty}_{=0\ \to\ \text{L'Hospital}} - \int_0^\infty -e^{-\lambda x} dx \\
&= \left[-\frac{1}{\lambda}e^{-\lambda x}\right]_0^\infty = \frac{1}{\lambda} = \frac{1!}{\lambda^1}
\end{aligned}$$

Inductive step:

$$\begin{aligned}
\mathrm{E}[X^{r+1}] &= \underbrace{\int_0^\infty x^{r+1}\lambda e^{-\lambda x} dx}_{\text{Integration by parts: } f(x)=x^{r+1},\ g'(x)=\lambda e^{-\lambda x}} \\
&= \underbrace{[x^{r+1}(-e^{-\lambda x})]_0^\infty}_{=0\ \to\ \text{L'Hospital}} - \int_0^\infty (r+1)x^r(-e^{-\lambda x})dx \\
&= (r+1)\int_0^\infty x^r e^{-\lambda x} dx = \frac{r+1}{\lambda}\int_0^\infty x^r \lambda e^{-\lambda x} dx \\
&= \frac{r+1}{\lambda}\mathrm{E}[X^r] = \frac{r+1}{\lambda}\frac{r!}{\lambda^r} = \frac{(r+1)!}{\lambda^{r+1}}
\end{aligned}$$

b) Alternative solution by substitution

$$\mathrm{E}(X^r) = \int_0^\infty x^r \lambda e^{-\lambda x} dx$$

Gamma function:

$$\Gamma(a) = \int_0^\infty x^{a-1} e^{-x} dx \qquad \Gamma(n) = (n-1)! \qquad n \in \mathbb{N}$$

Substitute $\lambda x = y$, then $dx = dy\lambda^{-1}$

$$\begin{aligned} \mathrm{E}(X^r) &= \int_0^\infty \left(\frac{y}{\lambda}\right)^r \lambda e^{-y} \lambda^{-1} dy = \lambda^{-r} \int_0^\infty y^r e^{-y} dy \\ &= \lambda^{-r} \Gamma(r+1) \end{aligned}$$

$$\mathrm{E}(X^r) = \frac{r!}{\lambda^r}$$

Exercise 3.4

(Moments) Let X be a continuous random variable with expected value μ, median m and variance σ^2.

a) Show that the function $\mathrm{E}[(X-b)^2]$ is minimal for $b = \mu$.

b) Show that the median m minimizes the function $\mathrm{E}[|X-b|]$.

Solution to Exercise 3.4

a)

$$\mathrm{E}(X-b)^2 = \int_{-\infty}^{\infty} \underbrace{(x-b)^2}_{} f(x)dx \longrightarrow \text{min!}$$

$$x^2 - 2bx + b^2 \longrightarrow \int x^2 f(x)dx - 2b \int x f(x)dx + b^2 \int f(x)dx$$

$$\frac{\partial \mathrm{E}[\cdot]}{\partial b} = -2 \int x f(x)dx + 2b \int f(x)dx = -2\mathrm{E}(X) + 2b \overset{!}{=} 0$$

Hence, $b = \mathrm{E}(X)$, q.e.d.

b) In order to find a derivative of an integral, which limits depend on a parameter, we need to use Leibnitz rule:

$$\begin{aligned} I &= \int_{l(z)}^{h(z)} \phi(s,z) ds \\ \frac{\partial I}{\partial z} &= \int_{l(z)}^{h(z)} \frac{\partial \phi}{\partial z} ds + \frac{\partial h}{\partial z} \phi(h(z), z) - \frac{\partial l}{\partial z} \phi(l(z), z) \end{aligned}$$

Here, $z = b$ and thus

$$\left.\begin{array}{lcr} h(z) & \Rightarrow & \infty \text{ (1st integral)}, b \text{ (2nd integral)} \\ l(z) & \Rightarrow & b \text{ (1st integral)}, -\infty \text{ (2nd integral)} \end{array}\right| \; (x-b)f(x) = \phi(s,z)$$

$$\begin{aligned}
\mathrm{E}(|X-b|) &= \int\limits_{-\infty}^{\infty} |x-b|f(x)dx = \int\limits_{b}^{\infty} (x-b)f(x)dx - \int\limits_{-\infty}^{b} (x-b)f(x)dx \\
\frac{\partial \mathrm{E}(|X-b|)}{\partial b} &= -\int\limits_{b}^{\infty} f(x)dx + 0 - 1 \cdot (b-b)f(b) - \\
& \quad \left(-\int\limits_{-\infty}^{b} f(x)dx + 1 \cdot (b-b)f(b) - 0\right) \stackrel{!}{=} 0
\end{aligned}$$

Rearranging,

$$\begin{aligned}
-\int\limits_{b}^{\infty} f(x)dx + \underbrace{\int\limits_{-\infty}^{b} f(x)dx}_{=1-\int_b^\infty f(x)dx} &= \frac{\partial \mathrm{E}(|X-b|)}{\partial b} \\
1 - \int\limits_{b}^{\infty} f(x)dx - \int\limits_{b}^{\infty} f(x)dx = 1 - 2\int\limits_{b}^{\infty} f(x)dx \stackrel{!}{=} 0 \\
\frac{1}{2} = \int\limits_{b}^{\infty} f(x)dx = \int\limits_{-\infty}^{b} f(x)dx \Rightarrow b = X_{\text{MEDIAN}}
\end{aligned}$$

Exercise 3.5

(Moments) Consider the density function of a continuous random variable X that is symmetric around the value $x = c$. Show that $\mathrm{E}(X) = c$ and $\mu_3 = 0$.

Solution to Exercise 3.5

A symmetric density around $x = c$ implies:

$$f(c + x_0) = f(c - x_0) \quad \forall x_0 \in D(x)$$

a)

$$\mathrm{E}(X) = \int_{-\infty}^{\infty} xf(x)dx = \int_{-\infty}^{c} xf(x)dx + \int_{c}^{\infty} xf(x)dx$$

$$\begin{array}{llll} x & = c - x_0 & \Leftrightarrow \quad dx = -dx_0 & \text{[1st integral]} \\ x & = c + x_0 & \Leftrightarrow \quad dx = dx_0 & \text{[2nd integral]} \end{array}$$

Substituting,

$$\mathrm{E}(X) = -\int_{+\infty}^{0} (c-x_0)f(c-x_0)dx_0 + \int_{0}^{\infty} (c+x_0)f(c+x_0)dx_0$$

$$= \int_{0}^{\infty} (c-x_0)f(c-x_0)dx_0 + \int_{0}^{\infty} (c+x_0)f(c+x_0)dx_0$$

$$= \overbrace{\int_0^\infty cf(c-x_0)dx_0 - \int_0^\infty x_0 f(c-x_0)dx_0}+\overbrace{\int_0^\infty cf(c+x_0)dx_0 + \int_0^\infty x_0 f(c+x_0)dx_0}$$

$$= \underbrace{\int_0^\infty cf(c-x_0)dx_0 + \int_0^\infty cf(c+x_0)dx_0}\underbrace{-\int_0^\infty x_0 f(c-x_0)dx_0 + \int_0^\infty x_0 f(c+x_0)dx_0}_{=0}$$

$$= 2c\int_0^\infty f(c+x_0)dx_0 \qquad \text{due to symmetry}$$

Substituting back, $x = c + x_0$ yields $dx = dx_0$.

$$\mathrm{E}(X) = 2c\int_c^\infty f(x)dx = 2c(1-F(c)) = 2c\left(1-\frac{1}{2}\right) = c$$

b)

$$\mu_3 = \int_{-\infty}^{\infty} (x - \underbrace{\mathrm{E}(X)}_{=c})^3 f(x)dx = \int_{-\infty}^{c} (x-c)^3 f(x)dx + \int_c^\infty (x-c)^3 f(x)dx$$

Substituting $x = c - x_0$ yields $dx = -dx_0$ for the 1st integral and $x = c + x_0$, $dx = dx_0$ for the 2nd integral.

$$\begin{aligned}
\mu_3 &= -\int_{\infty}^{0} (-x_0)^3 f(c-x_0)dx_0 + \int_0^\infty x_0^3 \underbrace{f(c+x_0)}_{=f(c-x_0)\ \text{(apply symmetry)}} dx_0 \\
&= \int_0^\infty ((-x_0)^3 + x_0^3) f(c-x_0)dx_0 = 0 \quad \text{q.e.d.}
\end{aligned}$$

Exercise 3.6

(Moments) Let the random variable X have the following probability density function (standard normal distribution):

$$f(x) = \frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{x^2}{2}\right\}, \quad -\infty < x < \infty ,$$

with expected value 0 and variance 1. Show that the central moments can be obtained as follows:

a) $\mu_{2k+1} = 0$ for $k = 0, 1, ..$ **b)** $\mu_{2k} = \prod_{i=1}^{k}(2k-2i+1)$ for $k = 1, 2, \ldots$.

Solution to Exercise 3.6

a) Note, that the standard normal distribution is symmetric around the origin: $f(-x_0) = f(x_0), \quad \forall x_0 \in (-\infty, +\infty)$

$$\mu_{2k+1} = \int_{-\infty}^{\infty} x^{2k+1} f(x)dx = \int_{-\infty}^{0} x^{2k+1} f(x)dx + \int_{0}^{\infty} x^{2k+1} f(x)dx$$

In the first integral substitute $x = -x_0$ and $dx = -dx_0$, and in the second integral substitute $x = x_0$ and $dx = dx_0$. It follows that (because of the symmetry of the density function and because $2k+1$ is odd-numbered)

$$\begin{aligned} & -\int_{\infty}^{0} (-x_0)^{2k+1} f(x_0)dx_0 + \int_{0}^{\infty} x_0^{2k+1} f(x_0)dx_0 \\ &= \int_{0}^{\infty} (-x_0^{2k+1} + x_0^{2k+1}) f(x_0)dx_0 = 0. \end{aligned}$$

b) By induction:

Base case: Let $k = 1$, it follows that:

$$\mu_2 = \text{Var}(x) = 1 = \prod_{i=1}^{1} (2 - 2i + 1) = 1.$$

Induction hypothesis: Statement applies to one k.

Inductive step: Concluding from k to $k+1$. Hence,

$$\begin{aligned} \mu_{2k} &= \int_{-\infty}^{\infty} x^{2k} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx = \frac{1}{\sqrt{2\pi}} \underbrace{\int_{-\infty}^{\infty} x^{2k} e^{-\frac{x^2}{2}} dx}_{\text{Integration by parts: } f(x)=e^{-\frac{x^2}{2}},\ g'(x)=x^{2k}} \\ &= \frac{1}{\sqrt{2\pi}} \left([f(x)g(x)]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f'(x)g(x)\ dx \right) \\ &= \frac{1}{\sqrt{2\pi}} \left\{ \left[\frac{x^{2k+1}}{2k+1} e^{-\frac{x^2}{2}} \right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \frac{1}{2k+1} x^{2k+1} (-x) e^{-\frac{x^2}{2}} dx \right\} \\ &= 0 + \frac{1}{2k+1} \int_{-\infty}^{\infty} x^{2k+2} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx = \frac{1}{2k+1} \mu_{2(k+1)}. \end{aligned}$$

Therefore,

$$\begin{aligned} \mu_{2k}(2k+1) &= \mu_{2(k+1)} = \prod_{i=1}^{k} (2k - 2i + 1) \underbrace{(2k+1)}_{2(k+1)-1} \\ &= [(2k-1)(2k-3)\dots 1](2(k+1)-1) \\ &= \prod_{i=1}^{k+1} (2(k+1) - 2i + 1). \end{aligned}$$

Exercise 3.7

(Existence of moments) Let the random variable X have the following probability density function (Cauchy distribution):

$$f(x) = \frac{1}{\pi(1+x^2)} \; , \qquad -\infty < x < \infty \; .$$

Show that the moments of this distribution do not exist.

Solution to Exercise 3.7

$$\begin{aligned}
\int\limits_{-\infty}^{\infty} x f(x) dx &= \int\limits_{-\infty}^{\infty} x \frac{1}{\pi(1+x^2)} dx \\
&= \frac{1}{\pi} \int\limits_{-\infty}^{\infty} \frac{x}{1+x^2} dx = \frac{1}{\pi} \left[\frac{1}{2} \ln(1+x^2) \right]_{-\infty}^{\infty} \\
&= 0
\end{aligned}$$

But, according to the existence condition,

$$\begin{aligned}
\int\limits_{-\infty}^{\infty} |x| f(x) dx &= \int\limits_{-\infty}^{\infty} |x| \frac{1}{\pi(1+x^2)} dx \\
&= 2 \int\limits_{0}^{\infty} \frac{x}{\pi(1+x^2)} dx = \frac{2}{\pi} \int\limits_{0}^{\infty} \frac{x}{1+x^2} dx \\
&= \frac{2}{\pi} \left[\frac{1}{2} \ln(1+x^2) \right]_{0}^{\infty} = \infty
\end{aligned}$$

The existence condition is not fulfilled, meaning, expected value does not exist, therefore higher moments do not exist either.

Exercise 3.8

(Moment-generating function, one-dimensional) Consider the following probability density functions:

i. $f(x) = \mathcal{I}_{[0,1]}(x)$

ii. $f(x) = \lambda e^{-\lambda x} \mathcal{I}_{(0,\infty)}(x) \; , \quad \lambda > 0$

iii. $f(x) = x e^{-x} \mathcal{I}_{(0,\infty)}(x)$

iv. $f(x) = \frac{1}{8} \frac{3!}{(3-x)! x!} \mathcal{I}_{\{0,1,2,3\}}(x)$.

a) Find the associated moment-generating functions, provided they exist.

b) Find the two first non-central moments using the moment-generating functions, provided they exist.

Solution to Exercise 3.8

a)

i.

$$M_X(t) = \int_0^1 e^{xt} dx = \frac{1}{t}(e^t - 1)$$

This is the MGF that is also significant for part b) of the exercise. We have to make sure[1] that the MGF is defined in an interval $(-h, h)$ around the value 0, and that the MGF takes the value 1 at $t = 0$. Consider De L'Hopitals rule,

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)} \text{ if } \lim_{x \to a} \frac{f(x)}{g(x)} = \frac{0}{0} \text{ or } \frac{\infty}{\infty}$$

$$\lim_{t \to 0} \frac{e^t - 1}{t} = \lim_{t \to 0} \frac{e^t}{1} = 1$$

$$M_X(t) = \begin{cases} \frac{1}{t}(e^t - 1) & ; t \neq 0 \\ 1 & ; t = 0 \end{cases}$$

Thus, the necessary conditions are fulfilled so that the function listed above is in fact a MGF.

ii.

$$\begin{aligned} M_X(t) &= \int_0^\infty e^{tx} \lambda e^{-\lambda x} dx = \int_0^\infty \lambda e^{(t-\lambda)x} dx \\ &= \left[\frac{\lambda}{t - \lambda} e^{(t-\lambda)x} \right]_0^\infty \\ &= \begin{cases} \frac{\lambda}{\lambda - t} & \text{if } t < \lambda, \\ \text{undefined} & \text{if } t = \lambda, \\ \infty & \text{otherwise.} \end{cases} \end{aligned}$$

As $\lambda > 0$ is given, it is sufficient to only consider the first case. The MGF is defined for $t = 0$ and takes the value 1 there. It is also defined in an interval around 0.

iii.

$$\begin{aligned} M_X(t) &= \int_0^\infty e^{tx} x e^{-x} dx = \underbrace{\int_0^\infty x e^{(t-1)x} dx}_{\text{Integration by parts: } f(x)=x,\ g'(x)=e^{(t-1)x}} \\ &= \left[x \frac{1}{(t-1)} e^{(t-1)x} \right]_0^\infty - \int_0^\infty \frac{1}{(t-1)} e^{(t-1)x} dx \\ &= \left[\frac{x}{(t-1)} e^{(t-1)x} \right]_0^\infty - \left[\frac{1}{(t-1)^2} e^{(t-1)x} \right]_0^\infty \end{aligned}$$

If $t < 1$, $M_X(t)$ is finite. Thus, the MGF $M_X(t) = \frac{1}{(t-1)^2}$ exists.

[1] See e.g. M96, p. 141

iv. X has a bounded range of values, meaning, all moments exist and therefore so does the MGF.

$$
\begin{aligned}
M_X(t) &= \sum_{x=0}^{3} e^{tx}\frac{1}{8}\binom{3}{x} \\
&= \frac{1}{8}\sum_{x=0}^{3}\binom{3}{x}[e^t]^x \underbrace{\cdot 1^{3-x}}_{\text{binomial theorem}} = \frac{1}{8}(e^t+1)^3
\end{aligned}
$$

b)

i.

$$
M_X(t) = \begin{cases} \frac{1}{t}(e^t-1) & t \neq 0 \\ 1 & t = 0 \end{cases}
$$

$$
M_X'(t) = -\frac{1}{t^2}(e^t-1) + \frac{1}{t}e^t = \frac{(t-1)e^t+1}{t^2}
$$

$$
\mathrm{E}(X) = M_X'(0) = \lim_{t\to 0}\frac{te^t-e^t+1}{t^2} \underset{\substack{\uparrow \\ \text{De L'Hopital}}}{=} \lim_{t\to 0}\frac{te^t}{2t} = \frac{1}{2}
$$

$$
\begin{aligned}
M_X''(t) &= \frac{2}{t^3}(e^t-1) - \frac{e^t}{t^2} - \frac{1}{t^2}e^t + \frac{1}{t}e^t \\
&= \frac{2e^t}{t^3} - \frac{2}{t^3} - \frac{2e^t}{t^2} + \frac{e^t}{t}
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{E}(X^2) &= M_X''(0) = \lim_{t\to 0}\frac{2e^t-2-2te^t+t^2e^t}{t^3} \\
&= \lim_{t\to 0}\frac{2e^t-2e^t-2te^t+2te^t+t^2e^t}{3t^2} \underset{\substack{\uparrow \\ \text{De L'Hopital}}}{=} \frac{1}{3}
\end{aligned}
$$

ii.

$$
\mathrm{E}(X) = M_X'(t)|_{t=0} = \frac{\lambda}{(t-\lambda)^2}|_{t=0} = \frac{1}{\lambda}.
$$

$$
E(X^2) = M_X''(t)|_{t=0} = \frac{-2\lambda}{(t-\lambda)^3}|_{t=0} = \frac{2}{\lambda^2}.
$$

iii.

$$
\begin{aligned}
M_X(t) &= \frac{1}{(t-1)^2} \\
M_X'(t) &= -\frac{2}{(t-1)^3}, \quad \mathrm{E}(X) = 2 \\
M_X''(t) &= \frac{6}{(t-1)^4}, \quad \mathrm{E}(X^2) = 6
\end{aligned}
$$

iv.

$$\begin{aligned}
M_X(t) &= \frac{1}{8}(e^t+1)^3 \\
M_X'(t) &= \frac{3}{8}(e^t+1)^2 e^t, & \mathrm{E}(X) = \frac{3}{2} \\
M_X''(t) &= \frac{3}{8}(e^t(e^t+1)^2 + e^{2t}2(e^t+1)), & \mathrm{E}(X^2) = 3
\end{aligned}$$

Exercise 3.9

(Moment-generating functions, multidimensional) The joint density function of the random variable $\mathbf{X} = (X_1, X_2)$ is given by

$$f(x) = \begin{cases} 2e^{-x_1-x_2} & \text{if} \quad 0 < x_1 < x_2 < \infty \\ 0 & \text{otherwise} \end{cases}$$

a) Find the moment-generating function.

b) Find the two first non-central moments using the moment-generating function.

c) Using the moment-generating function, find the marginal density function of X_1.

Solution to Exercise 3.9

a)

$$\begin{aligned}
M_{\mathbf{X}}(t_1,t_2) &= \int_0^\infty \int_0^{x_2} e^{t_1x_1+t_2x_2} 2e^{-x_1-x_2} dx_1 dx_2 \\
&= 2\int_0^\infty e^{x_2(t_2-1)} \Big[\frac{1}{t_1-1} e^{x_1(t_1-1)}\Big]_0^{x_2} dx_2 \\
&= 2\int_0^\infty e^{x_2(t_2-1)} \Big[\frac{1}{t_1-1} e^{x_2(t_1-1)} - \frac{1}{t_1-1}\Big] dx_2 \\
&= 2\int_0^\infty \Big(e^{x_2(t_1+t_2-2)} \frac{1}{t_1-1} - \frac{1}{t_1-1} e^{x_2(t_2-1)}\Big) dx_2 \\
&= 2\Big[\frac{1}{(t_1-1)(t_2+t_1-2)} e^{x_2(t_2+t_1-2)} - \frac{1}{(t_1-1)(t_2-1)} e^{x_2(t_2-1)}\Big]_0^\infty
\end{aligned}$$

For values $t_1 \neq 1$, $t_2 \neq 1$, $t_2 \neq -t_1+2$, $t_2+t_1-2 < 0$, $t_2 < 1$ we have

$$\begin{aligned}
&= 2\Big[-\frac{1}{(t_1-1)(t_2+t_1-2)} + \frac{1}{(t_1-1)(t_2-1)}\Big] = \dots \\
&= \frac{2}{(t_2-1)} \frac{1}{(t_1+t_2-2)} = 2(t_2^2 + t_1t_2 - 3t_2 - t_1 + 2)^{-1}
\end{aligned}$$

b)

$$\begin{aligned}
\mathrm{E}[X_1X_2] &= M_{\mathbf{X}}'(t_1=0, t_2=0) \\
&= \frac{\partial^2 M_{\mathbf{X}}(t_1,t_2)}{\partial t_1 \partial t_2}
\end{aligned}$$

$$
\begin{aligned}
\frac{\partial^2 M_{\mathbf{X}}(t_1,t_2)}{\partial t_1} &= -2(t_2^2+t_1t_2-3t_2-t_1+2)^{-2}(t_2-1) \\
\frac{\partial^2 M_{\mathbf{X}}(t_1,t_2)}{\partial t_1\partial t_2} &= 4(t_2^2+t_2t_1-3t_2-t_1+2)^{-3}(2t_2+t_1-3)(t_2-1)- \\
&\quad -2(t_2^2+t_2t_1-3t_2-t_1+2)^{-2}\Big|_{t_1=t_2=0}=1
\end{aligned}
$$

c) In general,

$$
\begin{aligned}
M_{\mathbf{X}}(t_1,t_2) &= \frac{2}{(t_2-1)(t_2+t_1-2)} \\
\frac{\partial M_{\mathbf{X}}(\mathbf{t})}{\partial t_1}\Big|_{t_1=t_2=0} &= \mathrm{E}[X_1] \\
\frac{\partial M_{\mathbf{X}}(\mathbf{t})}{\partial t_i^2}\Big|_{t_1=t_2=0} &= \mathrm{E}[X_i^2] \\
\frac{\partial M_{\mathbf{X}}(\mathbf{t})}{\partial t_1\partial t_2}\Big|_{t_1=t_2=0} &= \mathrm{E}[X_1X_2]
\end{aligned}
$$

Now regarding the exercise,

$$
M_{X_1}(x_1)=M_{\mathbf{X}}(t_1,t_2=0)=\frac{2}{2-t_1}
$$

Now, compare the MGF to $X\sim\exp(\lambda)$ with MGF

$$
M_{\mathbf{X}}(t)=\frac{\lambda}{\lambda-t}
$$

Concluding, $X_1\sim\exp(\lambda=2)$ with $f(x_1)=2e^{-2x_1}$.

Exercise 3.10

(Moment-generating function, multidimensional) Find the correlation coefficient for the random variable $\mathbf{X}=(X_1,X_2)$ using the moment-generating function

$$
M_{\mathbf{X}}(\mathbf{t})=e^{t_1-1}+e^{t_2-2}.
$$

Solution to Exercise 3.10

$$
\begin{aligned}
\mathrm{E}[X_1X_2] &= \frac{\partial^2 M_{\mathbf{X}}(\mathbf{t})}{\partial t_1\partial t_2}\Big|_{t_1=t_2=0} \\
\frac{\partial M_{\mathbf{X}}(\mathbf{t})}{\partial t_1} &= e^{t_1-1} \\
\frac{\partial^2 M_{\mathbf{X}}(\mathbf{t})}{\partial t_1\partial t_2} &= 0 \\
\mathrm{E}[X_1X_2] &= 0
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{E}[X_1] &= \frac{\partial M_{\mathbf{X}}(\mathbf{t})}{\partial t_1} = e^{t_1-1}\Big|_{t_1=t_2=0} = \frac{1}{e} \\
\mathrm{E}[X_2] &= \frac{1}{e^2} \\
\mathrm{E}[X_1^2] &= \frac{\partial^2 M_{\mathbf{X}}(\mathbf{t})}{\partial t_1^2} = e^{t_1-1}\Big|_{t_1=0} = \frac{1}{e} \\
\mathrm{E}[X_2^2] &= \frac{\partial^2 M_{\mathbf{X}}(\mathbf{t})}{\partial t_2^2} = \frac{1}{e^2}
\end{aligned}
$$

$$
\mathrm{Corr}(X_1 X_2) = \frac{\mathrm{Cov}(X_1 X_2)}{\sqrt{\mathrm{Var}(X_1)}\sqrt{\mathrm{Var}(X_2)}}
$$

$$
\begin{aligned}
\mathrm{Corr}(X_1 X_2) &= \frac{0 - \frac{1}{e^3}}{\sqrt{\frac{1}{e} - \frac{1}{e^2}}\sqrt{\frac{1}{e^2} - \frac{1}{e^4}}} = \frac{-\sqrt{e^{-6}}}{\sqrt{e^{-1} - e^{-2}}\sqrt{e^{-2} - e^{-4}}} \\
&= \frac{-1}{\sqrt{e^6(e^{-1} - e^{-2})(e^{-2} - e^{-4})}}
\end{aligned}
$$

Exercise 3.11

(Properties of moment-generating functions) Consider the random variable $\mathbf{X} = (X_1, ..., X_n)$ with the single elements being each independent ($\mathcal{N}(\mu, \sigma^2)$-normally distributed) random variables with the following moment-generating function:

$$
M_X(t) = \exp\left\{\mu t + \frac{1}{2}\sigma^2 t^2\right\} \quad , \quad \sigma > 0 \; .
$$

Find the moment-generating functions and distributions of the following random variables:

a) $Z_1 = \frac{1}{n}\sum_{i=1}^n X_i$

b) $Z_2 = \frac{1}{9+n}(10X_1 + \sum_{i=2}^n X_i)$

c) $\mathbf{Y_1} = \mathbf{X}$

d) $\mathbf{Y_2} = (10X_1, X_2, ..., X_n)$

Solution to Exercise 3.11

a)

$$
Z_1 = \frac{1}{n}X_1 + ... + \frac{1}{n}X_n = \frac{1}{n}\sum_{i=1}^n X_i
$$

$$
\begin{aligned}
M_{Z_1}(t) = \mathrm{E}(e^{tz_1}) &= \prod_{i=1}^{n} \mathrm{E}\left(e^{\frac{X_i}{n}t}\right) \\
&= \prod_{i=1}^{n} M_{X_i}\left(\frac{t}{n}\right) \\
&= \prod_{i=1}^{n} \exp\left\{\mu\frac{t}{n} + \frac{1}{2}\sigma^2\left(\frac{t}{n}\right)^2\right\} \\
&= \exp\left\{\mu\frac{t}{n} + \frac{1}{2}\sigma^2\left(\frac{t}{n}\right)^2 + ... + \mu\frac{t}{n} + \frac{1}{2}\sigma^2\left(\frac{t}{n}\right)^2\right\} \\
&= \exp\left\{\mu t + \frac{\sigma^2 t^2}{2n}\right\}
\end{aligned}
$$

$$\Rightarrow Z_1 \sim \mathcal{N}\left(\mu, \frac{\sigma^2}{n}\right)$$

b)

$$
\begin{aligned}
M_{Z_2}(t) &= M_{X_1}\left(\frac{10}{9+n}t\right)\prod_{i=2}^{n} M_{X_i}\left(\frac{t}{9+n}\right) \\
&= e^{\frac{\mu 10}{9+n}t + \frac{1}{2}\sigma^2\frac{100}{(9+n)^2}t^2 + \frac{\mu}{9+n}t + \frac{1}{2}\sigma^2\frac{1}{(9+n)^2}t^2 + ...} \\
&= e^{\mu\left(\frac{10}{9+n} + \frac{n-1}{9+n}\right)t + \frac{t^2\sigma^2}{2}\left(\frac{100}{(9+n)^2} + \frac{n-1}{(9+n)^2}\right)} \\
&= e^{\mu t + \frac{t^2\sigma^2}{2}\left(\frac{n+99}{(9+n)^2}\right)}
\end{aligned}
$$

$$\Rightarrow Z_2 \sim \mathcal{N}\left(\mu, \frac{n+99}{(9+n)^2}\sigma^2\right)$$

c)

$$\mathbf{Y}_1 = \mathbf{X}$$

$$
\begin{aligned}
M_{\mathbf{X}}(\mathbf{t}) &= \int_{-\infty}^{\infty} ... \int_{-\infty}^{\infty} e^{t_1x_1 + ... + t_nx_n} f(x_1, ..., x_n)dx_1...dx_n \\
&= \prod_{i=1}^{n} \int_{-\infty}^{\infty} e^{t_ix_i} f(x_i)dx_i \\
&= \prod_{i=1}^{n} M_{X_i}(t_i) = e^{\mu\sum_{i=1}^{n} t_i + \frac{\sigma^2}{2}\sum_{i=1}^{n} t_i^2}
\end{aligned}
$$

$\Rightarrow$ multivariate random variable $\quad \mathbf{Y_1} \sim \mathcal{N}\left(\underline{\mu}, \Sigma\right)$, with $\sigma_{ij} = 0 \quad \forall i \neq j$.

d)

$$\mathbf{Y_2} = (10X_1, X_2, ..., X_n) \qquad \text{analogous to c)}$$

$$
\begin{aligned}
M_{\mathbf{Y_2}}(\mathbf{t}) &= M_{X_1}(t_1 10)M_{X_2}(t_2)...M_{X_n}(t_n) \\
&= e^{\mu\left(10t_1 + \sum_{i=2}^{n} t_i\right) + \frac{\sigma^2}{2}\left(100t_1^2 + \sum_{j=2}^{n} t_j^2\right)}
\end{aligned}
$$

$\Rightarrow$ multivariate normal random variable.

Exercise 3.12

(Moment-generating functions, multidimensional) Let the (bivariate normally distributed) random variable $\mathbf{X} = (X_1, X_2)$ have the following moment-generating function:

$$M_{\mathbf{X}}(\mathbf{t}) = \exp\left\{\sum_{i=1}^{2} \mu_i t_i + \frac{1}{2}\sum_{i=1}^{2}\sum_{j=1}^{2} \sigma_{ij} t_i t_j\right\}$$

a) Find the joint moment-generating function and the joint density function of $aX_1 + bX_2$ and $cX_1 + dX_2$ with a, b, c and d as constants and $ad - bc \neq 0$.

b) Find the moment-generating function and the density function of $aX_1 + bX_2$ with a and b as constants.

Solution to Exercise 3.12

a)

$$M_X(\mathbf{t}) = \exp\{\sum_{i=1}^{2} \mu_i t_i + \frac{1}{2}\sum_{i=1}^{2}\sum_{i=1}^{2} \sigma_{ij} t_i t_j\}$$

$$\mu_i = \mathrm{E}(X_i), \quad \sigma_{ij} = \mathrm{Cov}(X_i, X_j), \quad \sigma_{ii} = \mathrm{Var}(X_i)$$

X_1, X_2 are jointly normally distributed, marginal densities of X_2, X_2 are also normal.

$$Y_1 = aX_1 + bX_2$$
$$Y_2 = cX_1 + dX_2$$

Linear combinations of normally distributed random variables and therefore also normally distributed.

$$\begin{aligned}
M_{Y_1,Y_2}(t_1,t_2) &= \mathrm{E}[\exp\{(aX_1 + bX_2)t_1 + (cX_1 + dX_2)t_2\}] \\
&= \mathrm{E}[\exp\{(at_1 + ct_2)X_1 + (bt_1 + dt_2)X_2\}] \\
M_{Y_1,Y_2}(t_1,t_2) &= M_{X_1,X_2}(t_1^\star, t_2^\star) \\
&= \exp\{\mu_1 t_1^\star + \mu_2 t_2^\star + \frac{1}{2}[\sigma_{11} t_1^{\star 2} + \sigma_{22} t_2^{\star 2} + 2\sigma_{12} t_1^\star t_2^\star]\} \\
&= \exp\{\mu_1(at_1 + ct_2) + \mu_2(bt_1 + dt_2) + \\
&\quad \frac{1}{2}[\sigma_{11}(at_1 + ct_2)^2 + \sigma_{22}(bt_1 + dt_2)^2 + 2\sigma_{12}(at_1 + ct_2)(bt_1 + dt_2))]\} \\
&= \exp\{\overbrace{(a\mu_1 + b\mu_2)}^{\mu_1^\star} t_1 + \overbrace{(c\mu_1 + d\mu_2)}^{\mu_2^\star} t_2\} \\
&\quad \cdot \exp\{\frac{1}{2}[\overbrace{(a^2\sigma_{11} + b^2\sigma_{22} + 2ab\sigma_{12})}^{\sigma_{11}^\star} t_1^2 + \overbrace{(c^2\sigma_{11} + d^2\sigma_{22} + 2cd\sigma_{12})}^{\sigma_{22}^\star} t_2^2 \\
&\quad +(2\overbrace{(ac\sigma_{11} + bd\sigma_{22} + \sigma_{12}(ad + bc))}^{\sigma_{12}^\star} t_1 t_2]\} \\
M_{Y_1,Y_2}(t_1,t_2) &= \exp\{\sum_{i=1}^{2} \mu_i^\star t_i + \frac{1}{2}\sum_{i=1}^{2}\sum_{j=1}^{2} \sigma_{ij}^\star t_i t_j\}
\end{aligned}$$

This is a MGF of a bivariate normal distribution. The corresponding density function is thus of a bivariate normal random variable.

b)

$$\begin{aligned}
Y_1 &= aX_1 + bX_2 \\
M_{Y_1}(t_1) &= M_{Y_1,Y_2}(t_1, t_2 = 0) = \exp\{\mu_1^\star t_1 + \frac{1}{2}\sigma_{11}^\star t_1^2\}
\end{aligned}$$

This is a MGF of a normal distribution with $\mu_1^\star$ and $\sigma_1^\star$.

$$\begin{aligned}
\Rightarrow Y_1 &\sim \mathcal{N}(\overbrace{a\mu_1 + b\mu_2}^{\mu_1^\star}; \overbrace{a^2\sigma_{11}^2 + b^2\sigma_{22} + 2ab\sigma_{12}}^{\sigma_1^\star}) \\
\mathrm{E}\Big[\sum_{i=1}^n a_i X_i\Big] &= \sum_{i=1}^n a_i \mathrm{E}[X_i] \\
\mathrm{Var}\Big[\sum_{i=1}^n a_i X_i\Big] &= \sum_{i=1}^n a_i^2 \mathrm{Var}[X_i] + \sum_{i=1}^n \sum_{j=1}^n a_i a_j \mathrm{Cov}[X_i, X_j]
\end{aligned}$$

Exercise 3.13

(Properties of moment-generating functions) Consider the series $\{X_i\}_{i=1,2,..}$ of Poisson-distributed random variables with the moment-generating function

$$M_{X_i}(t) = \exp\{i(e^t - 1)\}$$

and $\mathrm{E}(X_i) = \mathrm{Var}(X_i) = i$. For $i \to \infty$, find the limit distribution of the standardized random variable

$$Y_i = \frac{X_i - i}{\sqrt{i}}.$$

Solution to Exercise 3.13

$$\begin{aligned}
M_{X_i}(t) &= \exp\{i(e^t - 1)\} \\
Y_i &= \frac{X_i - i}{\sqrt{i}} = \frac{X_i}{\sqrt{i}} - \sqrt{i} \quad \text{standardized random variable } X_i \\
\Rightarrow M_{Y_i}(t) &= \exp\{-\sqrt{i}t\}\exp\{i(e^{\frac{t}{\sqrt{i}}} - 1)\}
\end{aligned}$$

Taylor expansion around $t = 0$ for $e^{\frac{t}{\sqrt{i}}}$:

$$\begin{aligned}
e^{\frac{t}{\sqrt{i}}} &= \sum_{h=0}^{\infty} \frac{(e^{\frac{t}{\sqrt{i}}})^h(t=0)}{h!}(t-0)^h \\
&= 1 + \frac{t}{\sqrt{i}} + \frac{\frac{t^2}{i}}{2!} + \frac{\frac{t^3}{i^{\frac{3}{2}}}}{3!} + ...
\end{aligned}$$

$$\begin{aligned}
\lim_{i\to\infty} M_{Y_i}(t) &= \lim_{i\to\infty} \exp\{-\sqrt{i}t\}\exp\{i(e^{\frac{t}{\sqrt{i}}}-1)\} \\
&= \lim_{i\to\infty} \exp\Big\{-\sqrt{i}t + i[\underbrace{1+\frac{t}{\sqrt{i}}+\frac{\frac{t^2}{i}}{2!}+\frac{\frac{t^3}{i^{\frac{3}{2}}}}{3!}+\ldots}_{e^{\frac{t}{\sqrt{i}}}}-1]\Big\} \\
&= \lim_{i\to\infty} \exp\left\{\frac{t^2}{2!}+\frac{\frac{t^3}{i^{\frac{1}{2}}}}{3!}+\ldots\right\} \\
&= \exp\left\{\frac{t^2}{2}\right\}
\end{aligned}$$

This is a MGF of a standard normal distribution and therefore

$$\lim_{i\to\infty} Y_i \sim \mathcal{N}(0,1)$$

Exercise 3.14

(Joint moments) The joint probability density function of the random variables $\mathbf{X} = (X_1, X_2)$ is given by the following probability table:

	$x_2 = 1$	$x_2 = 2$	$x_2 = 5$
$x_1 = 2$	0.1	0.1	0
$x_1 = 4$	0.1	0.3	0.1
$x_1 = 8$	0	0.1	0.2

a) Discuss the existence of the moments of $\mathbf{X}$.

b) Find the covariance matrix of $\mathbf{X}$.

c) Find the probability density function of $\mathrm{E}(X_1|X_2)$, $\mathrm{E}(X_2|X_1)$ and $\mathrm{Var}(X_1|X_2)$, and find $\mathrm{E}(X_1) = \mathrm{E}[\mathrm{E}(X_1|X_2)]$ and $\mathrm{Var}(X_1) = \mathrm{Var}[\mathrm{E}(X_1|X_2)] + \mathrm{E}[\mathrm{Var}(X_1|X_2)]$.

d) Find the regression line of a regression of X_1 on x_2.

Solution to Exercise 3.14

a) Since $|X_{ij}| < \infty \ \forall \ (i,j)$, the range of values is bounded. Consequently, moments of any order exist, i.e. defining the corresponding expected values will not result in infinite sums.

b) Covariance matrix:

$$\begin{aligned}
\mathrm{Cov}(\mathbf{X}) &= \mathrm{E}\left[\begin{pmatrix} X_1-\mu_1 \\ X_2-\mu_2 \end{pmatrix}\begin{pmatrix} X_1-\mu_1 & X_2-\mu_2 \end{pmatrix}\right] \\
&= \mathrm{E}\begin{bmatrix} (X_1-\mu_1)^2 & (X_1-\mu_1)(X_2-\mu_2) \\ (X_1-\mu_1)(X_2-\mu_2) & (X_2-\mu_2)^2 \end{bmatrix} \\
&= \Sigma = \begin{pmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{pmatrix}
\end{aligned}$$

The variances can be calculated by using the respective marginal distributions

$$f(x_1) = \begin{pmatrix} 0.2 \\ 0.5 \\ 0.3 \end{pmatrix} \qquad f(x_2) = \begin{pmatrix} 0.2 \\ 0.5 \\ 0.3 \end{pmatrix}$$

$$\begin{aligned}
\mathrm{E}(X_1) &= 2 \cdot 0.2 + 4 \cdot 0.5 + 8 \cdot 0.3 = 4.8 \\
\mathrm{E}(X_1^2) &= 4 \cdot 0.2 + 16 \cdot 0.5 + 64 \cdot 0.3 = 28 \\
\mathrm{E}(X_2) &= 1 \cdot 0.2 + 2 \cdot 0.5 + 5 \cdot 0.3 = 2.7 \\
\mathrm{E}(X_2^2) &= 1 \cdot 0.2 + 4 \cdot 0.5 + 25 \cdot 0.3 = 9.7 \\
\mathrm{E}(X_1 X_2) &= 2 \cdot 0.1 + 4 \cdot 0.1 + 10 \cdot 0 + 4 \cdot 0.1 + 8 \cdot 0.3 \\
&\quad + 20 \cdot 0.1 + 8 \cdot 0 + 16 \cdot 0.1 + 40 \cdot 0.2 = 15
\end{aligned}$$

$$\begin{aligned}
\mathrm{Var}(X_1) &= \mathrm{E}(X_1^2) - \mathrm{E}(X_1)^2 = 28 - 4.8^2 = 4.96 \\
\mathrm{Var}(X_2) &= \mathrm{E}(X_2^2) - \mathrm{E}(X_2)^2 = 9.7 - 2.7^2 = 2.41 \\
\mathrm{Cov}(X_1 X_2) &= \mathrm{E}(X_1 X_2) - \mathrm{E}(X_1)\,\mathrm{E}(X_2) = 15 - 4.8 \cdot 2.7 = 2.04
\end{aligned}$$

$$\mathrm{Cov}(\mathbf{X}) = \begin{pmatrix} 4.96 & 2.04 \\ 2.04 & 2.41 \end{pmatrix}$$

c) Given

$$\mathrm{E}(X_1 \mid x_2) = \begin{cases} 2\frac{0.1}{0.2} + 4\frac{0.1}{0.2} = 3, & x_2 = 1 \\ 2\frac{0.1}{0.5} + 4\frac{0.3}{0.5} + 8\frac{0.1}{0.5} = 4.4, & x_2 = 2 \\ 4\frac{0.1}{0.3} + 8\frac{0.2}{0.3} \approx 6.67, & x_2 = 5 \end{cases}$$

$$f(\mathrm{E}(X_1 \mid x_2)) = \begin{cases} 0.2 & \text{if} \quad E(\cdot) = 3 \\ 0.5 & \text{if} \quad E(\cdot) = 4.4 \\ 0.3 & \text{if} \quad E(\cdot) = 6.67 \end{cases}$$

$$f(E(X_2 \mid x_1)) = \begin{cases} 0.2 & \text{if} \quad E(\cdot) = 1.5 \\ 0.5 & \text{if} \quad E(\cdot) = 2.4 \\ 0.3 & \text{if} \quad E(\cdot) = 4 \end{cases}$$

$$\mathrm{Var}(X_1 \mid x_2) = \begin{cases} 0.5 \cdot (2-3)^2 + 0.5 \cdot (4-3)^2 = 1, & x_2 = 1 \\ 0.2 \cdot (2-4.4)^2 + 0.6 \cdot (4-4.4)^2 + 0.2 \cdot (8-4.4)^2 = 3.84, & x_2 = 2 \\ \frac{1}{3}(4 - \frac{20}{3})^2 + \frac{2}{3}(8 - \frac{20}{3})^2 = 3.56, & x_2 = 5 \end{cases}$$

$$\mathrm{E}(X_1) = \mathrm{E}_{X_2}(\mathrm{E}(X_1 \mid x_2)) = 3 \cdot 0.2 + 4.4 \cdot 0.5 + 6.67 \cdot 0.3 = 4.8$$

$$\mathrm{E}(\mathrm{Var}(X_1 \mid x_2)) = 1 \cdot 0.2 + 3.84 \cdot 0.5 + 3.56 \cdot 0.3 \approx 3.1867$$

$$\begin{aligned}
\mathrm{Var}(\mathrm{E}(X_1 \mid X_2)) &= \mathrm{E}(\mathrm{E}(X_1 \mid X_2)^2) - (\mathrm{E}[\mathrm{E}(X_1 \mid X_2)])^2 \\
&= 3^2 \cdot 0.2 + 4.4^2 \cdot 0.5 + 6.67^2 \cdot 0.3 - 4.8^2 \approx 1.7733
\end{aligned}$$

Law of total variance

$$\mathrm{Var}(X_1) = \mathrm{E}(\mathrm{Var}(X_1 \mid X_2)) + \mathrm{Var}(\mathrm{E}(X_1 \mid X_2)) = 3.1867 + 1.7733 = 4.96$$

Values in calculations and results are rounded.

d) The coordinates of the regression function are given by $[x_2;\ \mathrm{E}(X_1 \mid\ x_2)]$.

Exercise 3.15

(Joint moments) Let the random variable $\mathbf{X} = (X_1, X_2, X_3)$ have the following probability density function:

$$f(\mathbf{X}) = \begin{cases} \frac{36}{(1+x_1+x_2)^5(1+x_3)^4} & \text{if} \qquad x_1, x_2, x_3 > 0 \\ 0 & \text{otherwise} \end{cases}$$

a) Check if the elements in $\mathbf{X}$ are stochastically independent.

b) Find the covariance matrix of $\mathbf{X}$.

c) Find $\mathrm{E}(X_1X_3)$, $\mathrm{E}(X_1|x_3)$ and $\mathrm{Var}(X_1|x_3)$.

d) Find both regression lines for X_1 and X_3.

Solution to Exercise 3.15

$$f(\mathbf{X}) = \begin{cases} \frac{36}{(1+x_1+x_2)^5(1+x_3)^4} & \text{if } x_1, x_2, x_3 > 0 \\ 0 & \text{otherwise.} \end{cases}$$

a) Stochastic independence holds if $f(\mathbf{X}) = f(x_1)f(x_2)f(x_3)$.

$$\begin{aligned}
f(\mathbf{X}) &= \frac{36}{(1+x_1+x_2)^5(1+x_3)^4} \qquad x_1, x_2, x_3 > 0 \\
f(x_1) &= \int_0^\infty \int_0^\infty f(\mathbf{X}) dx_2 dx_3 \\
&= \int_0^\infty \left[\frac{36}{-4}(1+x_1+x_2)^{-4}\right]_0^\infty (1+x_3)^{-4} dx_3 \\
&= \int_0^\infty 9(1+x_1)^{-4}(1+x_3)^{-4} dx_3 \\
&= 9(1+x_1)^{-4}\left[-\frac{1}{3}(1+x_3)^{-3}\right]_0^\infty \\
&= 3(1+x_1)^{-4} \cdot I_{(0,\infty)}(x_1) \\
f(x_2) &= 3(1+x_2)^{-4} \cdot I_{(0,\infty)}(x_2) \\
f(x_3) &= 3(1+x_3)^{-4} \cdot I_{(0,\infty)}(x_3)
\end{aligned}$$

Obviously, $f(\mathbf{X}) \neq f(x_1)f(x_2)f(x_3)$, thus the variables are not stochastically independent. However, since $f(\mathbf{X}) = f(x_1, x_2) \cdot f(x_3)$, it follows that $X_3 \perp (X_1, X_2)$[2].

[2] Meaning also $X_3 \perp X_1$ and $X_3 \perp X_2$

b) Cov[**X**] can be found by calculating the corresponding integrals.

$$\begin{aligned}
\mathrm{E}[X_1] &= \int_0^\infty 3x_1(1+x_1)^{-4}dx_1 \\
\text{Integration by parts:} \quad f(x) &= x_1 \to f'(x) = 1 \\
g'(x) &= (1+x_1)^{-4} \to g(x) = -\frac{1}{3}(1+x_1)^{-3} \\
\mathrm{E}[X_1] &= 3\left\{[f(x)g(x)]_0^\infty - \int_0^\infty f'(x)g(x)dx_1\right\} \\
&= 3\left\{\underbrace{\left[-\frac{1}{3}x_1(1+x_1)^{-3}\right]_0^\infty}_{=0} + \int_0^\infty \frac{1}{3}(1+x_1)^{-3}dx_1\right\} \\
&= 3\left\{0 + \left[-\frac{1}{6}(1+x_1)^{-2}\right]_0^\infty\right\} \\
&= 3\left(\frac{1}{6}\right) = \frac{1}{2} = \mathrm{E}[X_2] = \mathrm{E}[X_3]
\end{aligned}$$

$$\begin{aligned}
\mathrm{E}[X_1^2] &= 3\int_0^\infty x_1^2(1+x_1)^{-4}dx_1 \\
&= 3\left\{\underbrace{\left[-\frac{1}{3}x_1^2(1+x_1)^{-3}\right]_0^\infty}_{=0} + \underbrace{\int_0^\infty \frac{2}{3}x_1(1+x_1)^{-3}dx_1}_{(\star)}\right\} \\
(\star) &= \frac{2}{3}\left\{\underbrace{\left[-\frac{1}{2}x_1(1+x_1)^{-2}\right]_0^\infty}_{=0} + \underbrace{\frac{1}{2}\int_0^\infty (1+x_1)^{-2}dx_1}_{=\frac{1}{2}[-(1+x_1)^{-1}]_0^\infty=\frac{1}{2}}\right\} \\
\mathrm{E}[X_1^2] &= 3\cdot\frac{2}{3}\cdot\frac{1}{2} = 1 \\
\mathrm{Var}[X_1] &= \mathrm{E}[X_1^2] - \mathrm{E}[X_1]^2 = 1 - \left(\frac{1}{2}\right)^2 = \frac{3}{4} = \mathrm{Var}[X_2] = \mathrm{Var}[X_3]
\end{aligned}$$

$$
\begin{aligned}
f(x_1,x_2) &= \frac{36}{(1+x_1+x_2)^5}\cdot \underbrace{\int_0^\infty (1+x_3)^{-4}dx_3}_{\left[-\frac{1}{3}(1+x_3)^{-3}\right]_0^\infty=0+\frac{1}{3}} = \frac{12}{(1+x_1+x_2)^5} \\
\mathrm{E}[X_1,X_2] &= \int_0^\infty\int_0^\infty x_1x_2\frac{12}{(1+x_1+x_2)^5}dx_1dx_2 \\
&= \int_0^\infty 12x_2\int_0^\infty \frac{x_1}{(1+x_1+x_2)^5}dx_1dx_2 \\
&= \int_0^\infty 12x_2\left\{\left[-\frac{1}{4}x_1(1+x_1+x_2)^{-4}\right]_0^\infty + \frac{1}{4}\int_0^\infty (1+x_1+x_2)^{-4}dx_1\right\}dx_2 \\
&= \int_0^\infty 12x_2\left\{0+\frac{1}{4}\left[-\frac{1}{3}(1+x_1+x_2)^{-3}\right]_0^\infty\right\}dx_2 \\
&= \int_0^\infty 12x_2\left\{\frac{1}{4}(0+\frac{1}{3}(1+x_2)^{-3})\right\}dx_2 \\
&= \int_0^\infty x_2(1+x_2)^{-3}dx_2 \\
&= \left[-\frac{1}{2}x_2(1+x_2)^{-2}\right]_0^\infty + \frac{1}{2}\int_0^\infty (1+x_2)^{-2}dx_2 \\
&= 0+\frac{1}{2}\left[-(1+x_2)^{-1}\right]_0^\infty \\
&= \frac{1}{2}(0+1) = \frac{1}{2} \\
\mathrm{Cov}[X_1,X_2] &= \mathrm{E}[X_1,X_2]-\mathrm{E}[X_1]\mathrm{E}[X_2] = \frac{1}{2}-(\frac{1}{2})(\frac{1}{2}) = \frac{1}{4}
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{Cov}[\mathbf{X}] &= \begin{pmatrix} \sigma_1^2 & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_2^2 & \sigma_{23} \\ \sigma_{13} & \sigma_{32} & \sigma_3^2 \end{pmatrix} = \begin{pmatrix} \frac{3}{4} & \frac{1}{4} & 0 \\ \frac{1}{4} & \frac{3}{4} & 0 \\ 0 & 0 & \frac{3}{4} \end{pmatrix} \\
\sigma_{13} &= \sigma_{23} = 0 \qquad \text{due to independence.}
\end{aligned}
$$

c)

$$
\begin{aligned}
\mathrm{E}[X_1X_3] &= \int_0^\infty\int_0^\infty (x_1x_3)f(x_1,x_3)dx_1dx_3 \\
&= \int_0^\infty\int_0^\infty x_1x_3\frac{9}{(1+x_1)^4(1+x_3)^4}dx_1dx_3 \\
&= \frac{1}{4}
\end{aligned}
$$

Due to independence of $X_1 \perp X_3$, it follows,

$$
\begin{aligned}
\mathrm{E}(X_1X_3) &= \mathrm{E}(X_1)\mathrm{E}(X_3) = \frac{1}{4} \\
\mathrm{E}(X_1|X_3) &= \mathrm{E}(X_1) \\
\mathrm{Var}(X_1|X_3) &= \mathrm{Var}(X_1)
\end{aligned}
$$

d) For X_1 the regression line defined as $\mathrm{E}[X_1 \mid X_3] = \frac{1}{2}$ is a constant. In general,

$$
\begin{aligned}
\mathrm{E}[X_1|X_2,X_3] &= \int x_1\frac{f(x_1,x_2,x_3)}{f(x_2,x_3)}dx_1 \\
f(x_2,x_3) &= f(x_2)\,f(x_3), \qquad \text{due to independence.}
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{E}[X_1|X_2,X_3] &= \int_0^\infty x_1 \cdot \frac{36}{(1+x_1+x_2)^5(1+x_3)^4} \cdot \frac{(1+x_2)^4(1+x_3)^4}{9} dx_1 \\
&= \int_0^\infty \frac{4(1+x_2)^4 x_1}{(1+x_1+x_2)^5} dx_1 \\
&= 4(1+x_2)^4 \int_0^\infty \frac{x_1}{(1+x_1+x_2)^5} dx_1 \\
&= 4(1+x_2)^4 \left(\left[-\frac{1}{4} x_1 (1+x_1+x_2)^{-4} \right]_0^\infty + \frac{1}{4} \int_0^\infty (1+x_1+x_2)^{-4} dx_1 \right) \\
&= 4(1+x_2)^4 \left(\frac{1}{4} \left[-\frac{1}{3} (1+x_1+x_2)^{-3} \right]_0^\infty \right) \\
&= 4(1+x_2)^4 \left(\frac{1}{12} (1+x_2)^{-3} \right) \\
&= \frac{1}{3}(1+x_2).
\end{aligned}
$$

For X_3 we have

$$
\begin{aligned}
\mathrm{E}[X_3|X_1,X_2] &= \int_0^\infty x_3 \frac{f(x_1,x_2,x_3)}{f(x_1,x_2)} dx_3 = \mathrm{E}[X_3] = \frac{1}{2} \qquad \text{(due to independence)} \\
\mathrm{E}[X_3|X_1,X_2] &= \int_0^\infty x_3 \frac{f(x_1,x_2,x_3)}{f(x_1,x_2)} dx_3 \\
&= \int_0^\infty x_3 \frac{\frac{36}{(1+x_1+x_2)^5(1+x_3)^4}}{\frac{12}{(1+x_1+x_2)^5}} dx_3 \\
&= \int_0^\infty x_3 \; 3(1+x_3)^{-4} dx_3 \\
&= \left[x_3 \left(-\frac{1}{(1+x_3)^3} \right) \right]_0^\infty + \int_0^\infty \frac{1}{(1+x_3)^3} dx_3 \\
&= \left[-\frac{1}{2} \frac{1}{(1+x_3)^2} \right]_0^\infty \\
&= \frac{1}{2}
\end{aligned}
$$

Exercise 3.16

(Moments) Let X be a random variable that cannot take negative values and has the distribution function $F(x)$. Show that:

$$\mathrm{E}(x) = \int_0^\infty (1 - F(x))dx$$

Illustrate the integral with a graph and use the substitution rule of the integral calculus.

Solution to Exercise 3.16

$$\mathrm{E}(x) \quad = \quad \int\limits_0^\infty \big(1 - F(x)\big)dx$$

As shown in the graphics above this is equivalent to

$$\int\limits_0^1 F^{-1}(z)dz$$

After substituting $z = F(x)$ we get

$$\int\limits_{F(0)=0}^{F(\infty)=1} F^{-1}(z)dz = \int_0^\infty F^{-1}(F(x))f(x)dx = \int_0^\infty xf(x)dx = \mathrm{E}(x)$$

Chapter 4

Parametric Families of Density Functions

Exercise 4.1

(Moments of the hypergeometric distribution) The density of a hypergeometric distribution is defined by

$$f(x; M, K, n) = \frac{\begin{pmatrix} K \\ x \end{pmatrix}\begin{pmatrix} M-K \\ n-x \end{pmatrix}}{\begin{pmatrix} M \\ n \end{pmatrix}} \mathcal{I}_{\{0,1,\ldots,n\}}(x) \ ,$$

with $M \in \mathbb{N}$, $K = 0, 1, ..., M$ and $n = 1, 2, ..., M$. Find the expected value $\mathrm{E}(X)$ and the variance $\mathrm{Var}(X)$.

Solution to Exercise 4.1

$$f(x; M, K, n) = \frac{\binom{K}{x}\binom{M-K}{n-x}}{\binom{M}{n}} \quad \mathcal{I}_{\{0,1,2,\ldots,n\}}(x)$$

$$\text{with} \quad \sum_{x=0}^{n} \frac{\binom{K}{x}\binom{M-K}{n-x}}{\binom{M}{n}} = 1$$

$$\begin{aligned}
\mathrm{E}[X] &= \sum_{x=0}^{n} x \frac{\binom{K}{x}\binom{M-K}{n-x}}{\binom{M}{n}} = \sum_{x=1}^{n} x \frac{\binom{K}{x}\binom{M-K}{n-x}}{\binom{M}{n}} \\
&= \sum_{x=1}^{n} x \frac{\frac{K}{x}\binom{K-1}{x-1}\binom{M-K}{n-x}}{\frac{M}{n}\binom{M-1}{n-1}} = n\frac{K}{M} \sum_{x=1}^{n} \frac{\binom{K-1}{x-1}\binom{M-K}{n-x}}{\binom{M-1}{n-1}}
\end{aligned}$$

$$\text{Note:} \quad \frac{K}{x}\begin{pmatrix} K-1 \\ x-1 \end{pmatrix} = \frac{K(K-1)!}{x(K-1-(x-1))!(x-1)!} = \frac{K!}{(K-x)!x!} = \begin{pmatrix} K \\ x \end{pmatrix}$$

Set $(x-1) = y$,

$$\mathrm{E}[X] = n\frac{K}{M} \underbrace{\sum_{y=0}^{n-1} \frac{\binom{K-1}{y}\binom{M-1-K+1}{n-1-y}}{\binom{M-1}{n-1}}}_{=1} = \frac{nK}{M}$$

Note, the sum is taken over a hypergeometric distribution $f(y, M-1, K-1, n-1)$, hence it equals to one.

Now, to the variance,

$$\begin{aligned}
\mathrm{Var}(X) &= \mathrm{E}(X^2) - \mathrm{E}(X)^2 = \mathrm{E}[X(X-1)] + \mathrm{E}(X) - \mathrm{E}(X)^2 \\
\mathrm{E}(X(X-1)) &= \sum_{x=0}^{n} x(x-1)\frac{\binom{K}{x}\binom{M-K}{n-x}}{\binom{M}{n}} \\
&\quad \text{(if } x=0 \text{ and } x=1 \text{, the sum is equal to 0)} \\
&= \sum_{x=2}^{n} x(x-1)\frac{\binom{K}{x}\binom{M-K}{n-x}}{\binom{M}{n}} \\
&= \sum_{x=2}^{n} x(x-1)\frac{\frac{K(K-1)}{x(x-1)}\binom{K-2}{x-2}\binom{M-K}{n-x}}{\frac{M(M-1)}{n(n-1)}\binom{M-2}{n-2}} \\
&= n(n-1)\frac{K(K-1)}{M(M-1)}\underbrace{\sum_{x=2}^{n}\frac{\binom{K-2}{x-2}\binom{M-K}{n-x}}{\binom{M-2}{n-2}}}_{=1}
\end{aligned}$$

Set $x = y + 2$,

$$\frac{\binom{K-2}{y}\binom{M-2-K+2}{n-2-y}}{\binom{M-2}{n-2}} \longrightarrow f(y; M-2, K-2, n-2)$$

$$\begin{aligned}
\mathrm{Var}(X) &= n(n-1)\frac{K(K-1)}{M(M-1)} + n\frac{K}{M} - n^2\frac{K^2}{M^2} \\
&= n\frac{K}{M}\left((n-1)\frac{K-1}{M-1} + 1 - n\frac{K}{M}\right) \\
&= \frac{nK}{M}\left[\frac{(M-K)(M-n)}{M(M-1)}\right]
\end{aligned}$$

Exercise 4.2

(Moments of the beta distribution) The density of a beta distribution is defined by

$$f(x;\alpha,\beta) = \frac{1}{B(\alpha,\beta)}x^{\alpha-1}(1-x)^{\beta-1}\mathcal{I}_{(0,1)}(x)\ , \qquad \alpha,\beta > 0\ ,$$

where $B(\cdot,\cdot)$ is called the beta function with

$$B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}\ , \qquad \Gamma(z) = \int_0^\infty t^{z-1}e^{-t}dt \quad \text{(gamma function)}\ .$$

Find the expected value $\mathrm{E}(X)$ and the variance $\mathrm{Var}(X)$.

Solution to Exercise 4.2

We will need the following property of the gamma function: $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$.

$$\begin{aligned}
\mathrm{E}[X] &= \int_0^1 \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \underbrace{x^{\alpha}(1-x)^{\beta-1}}_{\text{kernel of Beta}(\alpha+1,\beta)} dx \\
&= \frac{\Gamma(\alpha+\beta)\Gamma(\alpha+1)}{\Gamma(\alpha+\beta+1)\Gamma(\alpha)} \int_0^1 \frac{\Gamma(\alpha+\beta+1)}{\Gamma(\alpha+1)\Gamma(\beta)} x^{(\alpha+1)-1}(1-x)^{\beta-1}dx \\
&= \frac{\alpha}{\alpha+\beta}.
\end{aligned}$$

The procedure to find the variance is similar. First, we calculate $\mathrm{E}[X^2]$.

$$\begin{aligned}
\mathrm{Var}[X] &= \mathrm{E}[X^2] - \mathrm{E}[X]^2 \\
\mathrm{E}[X^2] &= \int_0^1 \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \underbrace{x^{\alpha+1}(1-x)^{\beta-1}}_{\text{kernel of Beta}(\alpha+2,\beta)} dx \\
&= \frac{\Gamma(\alpha+\beta)\Gamma(\alpha+2)}{\Gamma(\alpha)\Gamma(\alpha+\beta+2)} \underbrace{\int_0^1 \frac{\Gamma(\alpha+\beta+2)}{\Gamma(\alpha+2)\Gamma(\beta)} x^{\alpha+1}(1-x)^{\beta-1}dx}_{=1} \\
&= \frac{\Gamma(\alpha+\beta)(\alpha+1)(\alpha)\Gamma(\alpha)}{\Gamma(\alpha)(\alpha+\beta+1)(\alpha+\beta)\Gamma(\alpha+\beta)} \\
&= \frac{(\alpha+1)\alpha}{(\alpha+\beta+1)(\alpha+\beta)} \\
\rightarrow \mathrm{Var}[X] &= \frac{(\alpha+1)\alpha}{(\alpha+\beta+1)(\alpha+\beta)} - \frac{\alpha^2}{(\alpha+\beta)^2} \\
&= \cdots = \frac{\alpha\beta}{(\alpha+\beta+1)(\alpha+\beta)^2}
\end{aligned}$$

Exercise 4.3

(Mixture of distributions) Consider the density of a Poisson distribution

$$f(x;\theta) = \frac{e^{-\theta}\theta^x}{x!}\mathcal{I}_{\{0,1,\ldots,n\}}(x)\,, \qquad \theta > 0\,,$$

and assume that θ follows a gamma distribution with the density

$$g\left(\theta; r, \frac{1}{\lambda}\right) = \frac{\lambda^r}{\Gamma(r)}\theta^{r-1}e^{-\lambda\theta}\mathcal{I}_{(0,\infty)}(\theta)\,, \qquad r, \lambda > 0.$$

Show that the *mixture of distributions* (also: *contagious distribution*) of Poisson distributions, i.e.

$$\int_0^{\infty} f(x;\theta)\cdot g(\theta;\lambda,r)d\theta,$$

is a negative binomial distribution with the parameters r and $p = \lambda/(\lambda+1)$.

Solution to Exercise 4.3[1]

Poisson distribution

$$f(x;\theta) = \frac{e^{-\theta}\theta^x}{x!}\mathcal{I}_{\{0,1,\dots,\}}(x);\ \theta > 0$$

Parameter θ follows a gamma distribution

$$g\left(\theta; r, \frac{1}{\lambda}\right) = \frac{\lambda^r}{\Gamma(r)}\theta^{r-1}e^{-\lambda\theta}\,\mathcal{I}_{(0,\infty)}(\theta);\ r,\lambda > 0$$

We are looking for a mixture of Poisson distributions where the variable of mixture θ follows a gamma distribution

$$f\left(x; r, \frac{1}{\lambda}\right) = \int\limits_0^\infty f(x;\theta)g\left(\theta; r, \frac{1}{\lambda}\right)d\theta$$

Density of the negative binomial distribution

$$f(x;r,p) = \binom{x-1}{r-1}p^r(1-p)^{x-r}\mathcal{I}_{r,r+1,r+2,\dots}(x)$$

This represents the number of repetitions of a Bernoulli experiment until the rth success.

$$\begin{aligned}
\int_0^\infty f(x;\theta)g\left(\theta; r, \frac{1}{\lambda}\right)d\theta &= \int\limits_0^\infty \frac{e^{-\theta}\theta^x}{x!}\frac{\lambda^r}{\Gamma(r)}\theta^{r-1}e^{-\lambda\theta}d\theta \\
&= \frac{\lambda^r}{\Gamma(r)x!}\int\limits_0^\infty \theta^{x+r-1}e^{-\lambda\theta-\theta}d\theta \\
&= \frac{\lambda^r}{\Gamma(r)x!}\underbrace{\int\limits_0^\infty \theta^{x+r-1}e^{-\theta(\lambda+1)}d\theta}_{\text{attempt to extract a gamma function}}
\end{aligned}$$

Consider the gamma function $\Gamma(z) = \int\limits_0^\infty t^{z-1}e^{-t}dt$

Substitute $t = (\lambda+1)\theta$, then $\theta = \frac{t}{\lambda+1}$ and $d\theta = dt\frac{1}{\lambda+1}$.

$$\begin{aligned}
f\left(x; r, \frac{1}{\lambda}\right) &= \frac{\lambda^r}{\Gamma(r)x!}\int\limits_0^\infty \left(\frac{1}{\lambda+1}\right)^{x+r-1} t^{x+r-1}e^{-t}\frac{1}{\lambda+1}dt \\
&= \frac{\lambda^r}{\Gamma(r)x!}\left(\frac{1}{\lambda+1}\right)^{x+r}\underbrace{\int\limits_0^\infty t^{x+r-1}e^{-t}dt}_{=\Gamma(x+r)} \\
&= \left(\frac{\lambda}{\lambda+1}\right)^r\left(\frac{1}{\lambda+1}\right)^x\frac{\Gamma(x+r)}{\Gamma(r)x!}
\end{aligned}$$

[1] Compare Mood, Graybill and Boes (1974), pp. 122

Note,

$$
\begin{aligned}
\Gamma(x+r) &= \overbrace{(x+r-1)\Gamma(x+r-1)}^{\text{repeat } x\text{-times}} \\
&= (x+r-1)(x+r-2)\Gamma(x+r-2) \\
&\vdots \\
&= (x+r-1)\dots(x+r-x)\Gamma(x-x+r) \\
&= (x+r-1)\dots(r)\Gamma(r)
\end{aligned}
$$

$$
\begin{aligned}
f\left(x;r,\frac{1}{\lambda}\right) &= \left(\frac{\lambda}{\lambda+1}\right)^r\left(1-\frac{\lambda}{\lambda+1}\right)^x \frac{(x+r-1)(x+r-2)\cdot\ldots\cdot(r)}{x!} \\
&= \left(\frac{\lambda}{\lambda+1}\right)^r\left(1-\frac{\lambda}{\lambda+1}\right)^x \frac{(x+r-1)!}{x!(r-1)!} \\
&= p^r(1-p)^x\binom{x+r-1}{r-1}, \quad \text{where } p=\frac{\lambda}{\lambda+1}
\end{aligned}
$$

Hence, X follows a negative binomial distribution with p and r. X counts the number of repetitions of the Bernoulli experiment that is greater than r until the rth success.

Exercise 4.4

(Univariate and multivariate normal distributions)

a) The density of a univariate normal distribution is defined by

$$f(x;\mu,\sigma)=\frac{1}{\sqrt{2\pi}\sigma}e^{-(x-\mu)^2/(2\sigma^2)}\,.$$

Show that the area under the density $A=\int_{-\infty}^{\infty}f(x;\mu,\sigma)dx$ is equal to 1. (Note: Make use of $A^2=1\Rightarrow A=1$.)

b) The general definition of the density of a multivariate normal distribution for the n-dimensional random variable $X=(X_1,...,X_n)'$ is

$$f(x;\mu,\Sigma)=\frac{1}{(2\pi)^{n/2}|\Sigma|^{1/2}}e^{-(x-\mu)'\Sigma^{-1}(x-\mu)/2}\,,$$

where $\mathrm{E}(X)=\mu$ and $\mathrm{Var}(X)=\Sigma$. Show that if $n=2$ the density can be defined as follows

$$f(x;\mu_1,\mu_2,\sigma_1,\sigma_2,\rho)=\frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}e^{\left\{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x_1-\mu_1}{\sigma_1}\right)^2-2\rho\frac{(x_1-\mu_1)}{\sigma_1}\frac{(x_2-\mu_2)}{\sigma_2}+\left(\frac{x_2-\mu_2}{\sigma_2}\right)^2\right]\right\}}\,,$$

where ρ is correlation coefficient of X_1 and X_2.

c) Consider the density of the bivariate normal distribution. Show that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(x;\mu_1,\mu_2,\sigma_1,\sigma_2,\rho)dx_2dx_1=1\,.$$

d) Show that the moment generating function of a standard normal distribution is defined by

$$M(t) = \exp\left(\frac{t^2}{2}\right)$$

e) Let $x_1, x_2, ..., x_n$ be independent random variables that follow a normal distribution with $\mu = 1$ and $\sigma = 2$. Find the moment generating function for:

i. X_1

ii. $S_2 = X_1 + X_2$

iii. $S_n = X_1 + X_2 + ... + X_n$

iv. $A_n = S_n/n$

f) Show that the moment generating function of a bivariate normal distribution is defined by

$$M(t_1, t_2) = e^{t_1\mu_1 + t_2\mu_2 + \frac{1}{2}(t_1^2\sigma_1^2 + 2\rho t_1 t_2 \sigma_1\sigma_2 + t_2^2\sigma_2^2)}.$$

Use the moment generating functions to find $\mathrm{E}(X_1)$, $\mathrm{E}(X_2)$, $\mathrm{Var}(X_1)$, $\mathrm{Var}(X_2)$ and $\mathrm{Cov}(X_1, X_2)$.

Solution to Exercise 4.4

a)

$$f(x; \mu, \sigma) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{1}{2\sigma^2}(x-\mu)^2\right\} \mathcal{I}_{(-\infty,\infty)}(x)$$

$$\int_{-\infty}^{\infty} f(x; \mu, \sigma) dx = 1$$

Thus, it holds as well

$$\int_{-\infty}^{\infty} f(x_1; \mu, \sigma) dx_1 \cdot \int_{-\infty}^{\infty} f(x_2; \mu, \sigma) dx_2 \quad =$$

$$\begin{aligned}
&= \frac{1}{2\pi\sigma^2}\left[\int_{-\infty}^{\infty} \exp\left\{-\frac{1}{2\sigma^2}(x_1-\mu)^2\right\} dx_1 \cdot \int_{-\infty}^{\infty} \exp\left\{-\frac{1}{2\sigma^2}(x_2-\mu)^2\right\} dx_2\right] \\
&= \frac{1}{2\pi\sigma^2}\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\left\{-\frac{1}{2\sigma^2}\left\{(x_1-\mu)^2 + (x_2-\mu)^2\right\}\right\} dx_1 dx_2\right]
\end{aligned}$$

$$\begin{array}{ccccccc}
x_1 & = & z_1\sigma + \mu & \text{and} & x_2 & = & z_2\sigma + \mu \\
\frac{dx_1}{dz_1} & = & \sigma & & \frac{dx_2}{dz_2} & = & \sigma \\
\infty & \to & \infty & & \infty & \to & \infty \\
-\infty & \to & -\infty & & -\infty & \to & -\infty
\end{array}$$

Substitution yields

$$\frac{1}{2\pi\sigma^2}\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\left\{-\frac{1}{2}(z_1^2 + z_2^2)\right\} \sigma^2 dz_1 dz_2\right]$$

In order to deal with this improper integral, we will reserve to transformation into polar coordinates. Define further

$$I(z_1, z_2) = \frac{1}{2\pi}\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\left\{-\frac{1}{2}(z_1^2 + z_2^2)\right\} dz_1 dz_2\right],$$

and $z_1 = r\ \cos\theta$, $z_2 = r\ \sin\theta$. Now

$$\frac{1}{2\pi}\left[\int_{-\pi}^{\pi}\int_{0}^{\infty} \exp\left\{-\frac{1}{2}(r\ \cos\theta)^2 + (r\ \sin\theta)^2\right\} |J|\ dr d\theta\right]$$

where $|J|$ is the determinant of the Jacobian matrix (matrix of the first derivatives).

$$|J| = \left|\begin{pmatrix} \frac{\partial z_1}{\partial r} & \frac{\partial z_1}{\partial \theta} \\ \\ \frac{\partial z_2}{\partial r} & \frac{\partial z_2}{\partial \theta} \end{pmatrix}\right|$$

Since $\begin{bmatrix} \frac{d\cos(\theta)}{d\theta} \\ \\ \frac{d\sin(\theta)}{d\theta} \end{bmatrix} = \begin{bmatrix} -\sin(\theta) \\ \\ \cos(\theta) \end{bmatrix}$, it follows that

$$\begin{aligned} |J| &= \left|\begin{pmatrix} \cos(\theta) & -r\ \sin(\theta) \\ \\ \sin(\theta) & r\ \cos(\theta) \end{pmatrix}\right| \\ &= r\ \cos^2(\theta) + r\ \sin^2(\theta) = r\underbrace{(\cos^2(\theta) + \sin^2(\theta))}_{=1} = r \end{aligned}$$

Plug in I,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\int_{0}^{\infty} \exp\{-\frac{1}{2}r^2\underbrace{(\cos^2(\theta) + \sin^2(\theta))}_{=1}\}\ r\ dr\ d\theta$$

Regarding the limits of integration,

$$\begin{array}{ll} z_1 \ \epsilon\ (-\infty, \infty) & z_2 \ \epsilon\ (-\infty, \infty) \\ \\ r \ \epsilon\ (0, \infty) & \underbrace{\theta \ \epsilon\ (-\pi, \pi)}_{\text{runs through the entire spectrum}} \end{array}$$

Finally,

$$\begin{aligned} &\frac{1}{2\pi}\int_{0}^{\infty}\int_{-\pi}^{\pi} \exp\left\{-\frac{1}{2}r^2\right\} r\ d\theta dr = \frac{1}{2\pi}\int_{0}^{\infty}\left[\theta \exp\left\{-\frac{1}{2}r^2\right\} r\right]_{-\pi}^{\pi} dr \\ = &\frac{1}{2\pi}\int_{0}^{\infty} 2\pi \exp\left\{-\frac{1}{2}r^2\right\} r\ dr = \int_{0}^{\infty} \exp\left\{-\frac{1}{2}r^2\right\} r\ dr \\ = &\left[-\exp\left\{-\frac{1}{2}r^2\right\}\right]_{0}^{\infty} = 1 \end{aligned}$$

b) Forming and multiplying out the density function in matrix notation.

$$f\left(\mathbf{X};\mu,\Sigma\right)=(2\pi)^{-\frac{n}{2}}\;|\Sigma|^{-\frac{1}{2}}\exp\left\{-\frac{1}{2}(x-\mu)'\Sigma^{-1}(x-\mu)\right\}.$$

Let $n=2$ and let ρ be the correlation coefficient,

$$\underbrace{\frac{1}{2\pi}\left|\begin{pmatrix}\sigma_1^2 & \rho\sigma_1\sigma_2\\ \rho\sigma_1\sigma_2 & \sigma_2^2\end{pmatrix}\right|^{-\frac{1}{2}}}_{I}\underbrace{\exp\left\{-\frac{1}{2}\begin{pmatrix}x_1-\mu_1\\ x_2-\mu_2\end{pmatrix}'\underbrace{\begin{pmatrix}\sigma_1^2 & \rho\sigma_1\sigma_2\\ \rho\sigma_1\sigma_2 & \sigma_2^2\end{pmatrix}^{-1}}_{=\frac{\text{conjugate transpose}}{|\Sigma|}}\begin{pmatrix}x_1-\mu_1\\ x_2-\mu_2\end{pmatrix}\right\}}_{II}$$

I.

$$\frac{1}{2\pi}\cdot\left|\begin{pmatrix}\sigma_1^2 & \rho\sigma_1\sigma_2\\ \rho\sigma_1\sigma_2 & \sigma_2^2\end{pmatrix}\right|^{-\frac{1}{2}}=\frac{1}{2\pi}\left(\sigma_1^2\sigma_2^2-\rho^2\sigma_1^2\sigma_2^2\right)^{-\frac{1}{2}}=\frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}$$

II.

$$\begin{aligned}
&\exp\left\{-\frac{1}{2}\begin{pmatrix}x_1-\mu_1\\ x_2-\mu_2\end{pmatrix}'\begin{pmatrix}\sigma_2^2 & -\rho\sigma_1\sigma_2\\ -\rho\sigma_1\sigma_2 & \sigma_1^2\end{pmatrix}\cdot\frac{1}{|\Sigma|}\begin{pmatrix}x_1-\mu_1\\ x_2-\mu_2\end{pmatrix}\right\}\\
=\;&\exp\left\{-\frac{1}{2}\Big[\left(\sigma_2^2(x_1-\mu_1)-\rho\sigma_1\sigma_2(x_2-\mu_2)\right)\;;\;\left(-\rho\sigma_1\sigma_2(x_1-\mu_1)+\sigma_1^2(x_2-\mu_2)\right)\Big]\cdot\begin{bmatrix}(x_1-\mu_1)\\ (x_2-\mu_2)\end{bmatrix}\cdot\frac{1}{|\Sigma|}\right\}\\
=\;&\exp\left\{-\frac{1}{2\sigma_1^2\sigma_2^2(1-\rho^2)}[\sigma_2^2(x_1-\mu_1)^2-\rho\sigma_1\sigma_2(x_1-\mu_1)(x_2-\mu_2)-\rho\sigma_1\sigma_2(x_1-\mu_1)(x_2-\mu_2)+\sigma_1^2(x_2-\mu_2)^2]\right\}\\
=\;&\exp\left\{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x_1-\mu_1}{\sigma_1}\right)^2-2\rho\left(\frac{x_1-\mu_1}{\sigma_1}\right)\left(\frac{x_2-\mu_2}{\sigma_2}\right)+\left(\frac{x_2-\mu_2}{\sigma_2}\right)^2\right]\right\}
\end{aligned}$$

Finally,

$$\frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}\exp\left\{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x_1-\mu_1}{\sigma_1}\right)^2-2\rho\left(\frac{x_1-\mu_1}{\sigma_1}\right)\left(\frac{x_2-\mu_2}{\sigma_2}\right)+\left(\frac{x_2-\mu_2}{\sigma_2}\right)^2\right]\right\}$$

c) Making transformations, using the solutions from a).

$$\begin{aligned}
z_1 &= \left(\frac{x_1-\mu_1}{\sigma_1}\right) &\rightarrow\quad \frac{dz_1}{dx_1}&=\frac{1}{\sigma_1}\\
z_2 &= \left(\frac{x_2-\mu_2}{\sigma_2}\right) &\rightarrow\quad \frac{dz_2}{dx_2}&=\frac{1}{\sigma_2}
\end{aligned}$$

$$\begin{aligned}
&\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{1}{2\pi\sigma_1\sigma_2\sqrt{(1-\rho^2)}}\exp\left\{-\frac{1}{2\left(1-\rho^2\right)}\left[z_1^2-2\rho z_1z_2+z_2^2\right]\right\}\sigma_1\sigma_2dz_1dz_2\\
=\;&\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{1}{2\pi\sqrt{(1-\rho^2)}}\exp\left\{-\frac{1}{2\left(1-\rho^2\right)}\left[z_1^2-2\rho z_1z_2+z_2^2\right]\right\}dz_1dz_2\\
=\;&\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{1}{2\pi\sqrt{(1-\rho^2)}}\exp\left\{-\frac{1}{2\left(1-\rho^2\right)}\left[(z_1-\rho z_2)^2+(1-\rho^2)z_2^2\right]\right\}dz_1dz_2\\
=\;&\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{1}{2\pi\sqrt{(1-\rho^2)}}\exp\left\{-\frac{1}{2}\left[\left(\frac{z_1-\rho z_2}{\sqrt{1-\rho^2}}\right)^2+z_2^2\right]\right\}dz_1dz_2
\end{aligned}$$

Substitute $\quad w = \frac{z_1-\rho z_2}{\sqrt{1-\rho^2}}$, $dw = \frac{1}{\sqrt{1-\rho^2}} dz_1$. Then,

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{2\pi} \exp\left\{-\frac{1}{2}\left[w^2+z_2^2\right]\right\} dw dz_2 = 1 \qquad \text{see a)}$$

d) Moment generating function of the standard normal distribution

$$\begin{aligned}
M_X(t) = E[e^{tx}] &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left\{tx - \frac{x^2}{2}\right\} dx \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2}\left(x^2-2tx+t^2\right)+\frac{t^2}{2}\right\} dx \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2}\left(x-t\right)^2\right\} e^{\frac{t^2}{2}} dx \\
&= e^{\frac{t^2}{2}} \underbrace{\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2}(x-t)^2\right\} dx}_{=1 \text{ integral over pdf of a } \mathcal{N}(t,1)} \\
&= e^{\frac{t^2}{2}}
\end{aligned}$$

e) Let $X_1, \ldots, X_n$ be independent random variables with $\mu = 1$ and $\sigma = 2$.

i.

$$M_{X_1}(t) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} \underbrace{\exp\left\{-\frac{1}{2\sigma^2}(x-\mu)^2+tx\right\}}\, dx$$

$$\begin{aligned}
&= -\frac{1}{2\sigma^2}(x^2-2x\mu+\mu^2)+tx \\
&= -\frac{1}{2\sigma^2}(x^2-2x\mu-2\sigma^2tx+\mu^2) \\
&= -\frac{1}{2\sigma^2}(x^2-2x(\mu+\sigma^2t)+\mu^2) \\
&= -\frac{1}{2\sigma^2}(x-(\mu+\sigma^2t))^2+\frac{2\mu\sigma^2t}{2\sigma^2}+\frac{\left(\sigma^2t\right)^2}{2\sigma^2}
\end{aligned}$$

$$\begin{aligned}
M_{X_1}(t) &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{1}{2\sigma^2}(x-(\mu+\sigma^2t))^2+\frac{2\mu\sigma^2t}{2\sigma^2}+\frac{\left(\sigma^2t\right)^2}{2\sigma^2}\right\} dx \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{1}{2\sigma^2}(x-(\mu+\sigma^2t))^2\right\} \cdot \exp\left\{\mu t+\frac{1}{2}\sigma^2t^2\right\} dx \\
&= \underbrace{\int_{-\infty}^{\infty} N(\mu+\sigma^2t;\sigma^2)dx}_{=1} \cdot \exp\left\{\mu t+\frac{1}{2}\sigma^2t^2\right\} \\
&= M_{X_1}(t) = \exp\left\{\mu t+\frac{1}{2}t^2\sigma^2\right\} = \exp\left\{t+2t^2\right\}
\end{aligned}$$

ii. Under independence,

$$
\begin{aligned}
M_{X_1+X_2}(t) &= \prod_{i=1}^{n} M_{X_i}(t) \\
&= \exp\left\{\sum_{i=1}^{2} \mu t + \frac{1}{2} t^2 \sigma^2\right\} = \exp\left\{2\mu t + t^2\sigma^2\right\} \\
&= \exp\left\{2t + 4t^2\right\}
\end{aligned}
$$

iii. Analogous,

$$
\begin{aligned}
M_{\sum_{i=1}^{n} X_i}(t) &= \prod_{i=1}^{n} M_{X_i}(t) \\
&= \exp\left\{\sum_{i=1}^{n} \mu t + \sum_{i=1}^{n} \frac{1}{2} t^2 \sigma^2\right\} \\
&= \exp\left\{n\mu t + \frac{n}{2} t^2 \sigma^2\right\} \\
&= \exp\left\{nt + 2nt^2\right\}
\end{aligned}
$$

iv.

$$
\begin{aligned}
M_{S_n}(t) &= \prod_{i=1}^{n} M_{X_i}\Big(\frac{t}{n}\Big) \\
&= \exp\left\{n\mu \frac{t}{n} + \frac{n}{2}\left(\frac{t}{n}\right)^2 \sigma^2\right\} \\
&= \exp\{t + 2t^2/n\}
\end{aligned}
$$

f)

$$
M_{X_1,X_2}(\mathbf{t}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{t_1x_1+t_2x_2} f(\cdot) dx_1\, dx_2
$$

Substitute $\frac{x_1-\mu_1}{\sigma_1} = u$, $\frac{x_2-\mu_2}{\sigma_2} = \upsilon$,

$$
\begin{aligned}
M_{U\Upsilon}(\mathbf{t}) &= e^{t_1\mu_1+t_2\mu_2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{t_1\sigma_1 u + t_2\sigma_2 \upsilon} \frac{1}{2\pi\sqrt{1-p^2}} e^{-\frac{1}{2(1-p^2)}\{u^2 - 2p\upsilon u + \upsilon^2\}} du\, d\upsilon \\
&= e^{t_1\mu_1+t_2\mu_2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{2\pi\sqrt{1-p^2}} e^{Q} du\, d\upsilon,
\end{aligned}
$$

where

$$
Q = -\frac{1}{2(1-p^2)}\{\underbrace{u^2 - 2pu\upsilon}_{(1)} + \upsilon^2 \underbrace{-2(1-p^2)t_1\sigma_1 u}_{(1)} - 2(1-p^2)t_2\sigma_2\upsilon\}
$$

To (1):

$$\begin{aligned}
& \underbrace{u^2 - 2puv}_{=u^2-2puv+p^2v^2-p^2v^2=(u-pv)^2-p^2v^2} -2(1-p^2)t_1\sigma_1 u = (u-pv)^2 - p^2v^2 - 2(1-p^2)t_1\sigma_1 u \\
= \; & (u-pv)^2 - p^2v^2 - 2(1-p^2)t_1\sigma_1(u-pv) - 2(1-p^2)t_1\sigma_1 pv \\
& \underbrace{[+(1-p^2)^2 t_1^2\sigma_1^2 - (1-p^2)^2 t_1^2\sigma_1^2]}_{\text{expanded}} \\
= \; & (u - pv - (1-p^2)t_1\sigma_1)^2 - p^2v^2 - 2(1-p^2)t_1\sigma_1 pv - (1-p^2)^2 t_1^2\sigma_1^2
\end{aligned}$$

Substituting into Q,

$$\begin{aligned}
Q \; = \; & -\frac{1}{2(1-p^2)}\Big\{(u - pv - (1-p^2)t_1\sigma_1)^2 + (1-p^2) \\
& (\quad \underbrace{v^2 - 2t_1\sigma_1 pv - 2t_2\sigma_2 v}_{=v^2-2(t_1\sigma_1 p+t_2\sigma_2)v+(t_1\sigma_1 p+t_2\sigma_2)^2-(t_1\sigma_1 p+t_2\sigma_2)^2} \\
& -(1-p^2)t_1^2\sigma_1^2)\Big\} \\
= \; & -\frac{1}{2(1-p^2)}\Big\{(u - pv - (1-p^2)t_1\sigma_1)^2 + (1-p^2)(v - t_1\sigma_1 p - t_2\sigma_2)^2 \\
& -(1-p^2)\Big((t_1\sigma_1 p + t_2\sigma_2)^2 + t_1^2\sigma_1^2(1-p^2)\Big)\Big\}
\end{aligned}$$

Back to the problem of integration, we substitute

$$\begin{aligned}
w &= \frac{u - pv - (1-p^2)t_1\sigma_1}{\sqrt{1-p^2}} \\
z &= (v - t_1\sigma_1 p - t_2\sigma_2) \\
\frac{dw}{du} &= \frac{1}{\sqrt{1-p^2}} \\
\frac{dz}{dv} &= 1
\end{aligned}$$

$$\begin{aligned}
Q &= -\frac{1}{2}w^2 - \frac{1}{2}z^2 + \frac{1}{2}\{t_1^2\sigma_1^2 + t_2^2\sigma_2^2 + 2pt_1t_2\sigma_1\sigma_2\} \\
M(t) &= e^{t_1\mu_1 + t_2\mu_2 + \frac{1}{2}\{t_1^2\sigma_1^2 + t_2^2\sigma_2^2 + 2pt_1t_2\sigma_1\sigma_2\}} \underbrace{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{2\pi} e^{-\left(\frac{w^2}{2}+\frac{z^2}{2}\right)} dw\, dz}_{=1,\text{ see a)}}
\end{aligned}$$

Exercise 4.5

(Conditional distribution from a multivariate normal distribution) Let X be a random variable following a bivariate normal distribution $\mathcal{N}(\mu, \Sigma)$, with

$$\mu = \begin{bmatrix} 5 \\ 8 \end{bmatrix} \qquad \text{and} \qquad \Sigma = \begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix}.$$

a) Define the regression curve of X_1 on X_2 and find $\mathrm{E}(X_1|x_2 = 9)$.

b) What is the conditional variance of X_1 given that $x_2 = 9$?

c) Find the probability that $x_1 > 5$ and the conditional probability that $x_1 > 5$ given that $x_2 = 9$.

Solution to Exercise 4.5

The regression function is the indefinite expected value $[X_1|x_2]$. It can be found with the help of the theorem of conditional densities described in the formulary. The result is

$$X_1|x_2 \sim \mathcal{N}(\underbrace{\mu_1 + \Sigma_{12}\Sigma_{22}^{-1}(x_2 - \mu_2)}_{=7\frac{2}{3}-\frac{1}{3}x_2}, \underbrace{\Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}}_{1\frac{2}{3}}).$$

a) $\mathrm{E}[X_1|x_2] = 7\frac{2}{3} - \frac{1}{3}x_2$ and $\mathrm{E}[X_1|x_2 = 9] = 4\frac{2}{3}$.

b) $\mathrm{Var}[X_1|x_2] = 1\frac{2}{3}$. Note that the variance is independent from x_2!

c) $\Pr(X_1 > 5) = 0.5$ and $\Pr(X_1 > 5|x_2 = 9) = 1 - \Phi\left(\frac{5-4\frac{2}{3}}{\sqrt{1\frac{2}{3}}}\right) = 1 - \Phi(0.258) \approx 0.4$

Exercise 4.6

(Exponential family) Consider the probability density functions listed below, and determine for each whether it belongs to the exponential family.

a) (Bernoulli distribution)

$$f(x;p) = p^x(1-p)^{1-x}\mathcal{I}_{\{0,1\}}(x)\,, \qquad p \in [0,1]$$

b) (gamma distribution)

$$f(x;\alpha,\beta) = \frac{1}{\beta^\alpha\Gamma(\alpha)}x^{\alpha-1}e^{-x/\beta}\mathcal{I}_{(0,\infty)}(x)\,, \qquad \alpha,\beta > 0$$

c) (Pareto distribution)

$$f(x;\beta) = \beta x^{-(\beta+1)}\mathcal{I}_{(1,\infty)}(x)\,, \qquad \beta > 0$$

d) (log-normal distribution)

$$f(x;\mu,\sigma) = \frac{1}{\sqrt{2\pi}\sigma x}e^{-[\ln(x)-\mu]^2/(2\sigma^2)}\mathcal{I}_{(0,\infty)}(x)\,, \qquad \mu \in \mathbb{R}, \sigma > 0\,.$$

Solution to Exercise 4.6

A density function belongs to the exponential family if its range is independent from the parameter θ of the density function and if the density function can be defined as follows:

$$f(x;\theta) = \begin{cases} \exp\left(\sum\limits_{i=1}^{k} c_i(\theta)g_i(x) + d(\theta) + z(x)\right) & \text{if } x \in A,\ A \perp \theta \\ 0 & \end{cases}$$

a)

$$\begin{aligned} z(x) &= 0 \\ d(\theta) &= \ln(1-p) \\ c_1(\theta) &= \ln p - \ln(1-p) \\ g_1(x) &= x \\ f(x;p) &= \exp\{(\ln p - \ln(1-p))x + \ln(1-p)\} \end{aligned}$$

b)

$$\begin{aligned} c_1(\theta) &= \alpha - 1 \\ c_2(\theta) &= -\frac{1}{\beta} \\ g_1(x) &= \ln x \\ g_2(x) &= x \\ d(\theta) &= \ln \frac{1}{\beta^\alpha \Gamma(\alpha)} \\ z(x) &= 0 \\ f(x;\theta) &= \exp\left\{ \ln x(\alpha-1) + \left(-\frac{1}{\beta}\right) x + \ln \frac{1}{\beta^\alpha \Gamma(\alpha)} + 0\right\} \\ &= \frac{1}{\beta^\alpha \Gamma(\alpha)} x^{\alpha-1} e^{-\frac{x}{\beta}} \end{aligned}$$

c)

$$\begin{aligned} f(x;\beta) &= \exp(\ln\beta - (\beta+1)\ln x) \\ c(\theta) &= -(\beta+1) \\ g(x) &= \ln x \\ d(\theta) &= \ln \beta \end{aligned}$$

d)

$$\begin{aligned} f(x;\mu,\sigma) &= \exp\left\{ \ln \frac{1}{\sqrt{2\pi}\sigma} - \ln x - \frac{[\ln x - \mu]^2}{2\sigma^2} \right\} = \\ &= \exp\left\{ -\ln\sqrt{2\pi}\sigma - \ln x - \frac{(\ln x)^2 - 2\mu \ln x + \mu^2}{2\sigma^2} \right\} = \\ &= \exp\left\{ \underbrace{-\ln\sqrt{2\pi}\sigma - \frac{\mu^2}{2\sigma^2}}_{d(\theta)} - \underbrace{\ln x}_{z(x)} - \underbrace{\frac{(\ln x)^2}{2\sigma^2}}_{g_1(x)\cdot c_1(\theta)} + \underbrace{\frac{\mu \ln x}{\sigma^2}}_{g_2(x)\cdot c_2(\theta)} \right\} \end{aligned}$$

Exercise 4.7

A random variable X is said to belong to the location-scale family when its cumulative distribution (cdf) is a function only of $(x-\mu)/\sigma$ with (μ, σ) being a location-scale parameter vector,

$$F_X(x|\mu,\sigma) = F\Big(\frac{x-\mu}{\sigma}\Big), \qquad \mu \in \mathbb{R}, \quad \sigma > 0,$$

where $F(\cdot)$ has no other dependencies on (μ, σ) rather than in $\frac{x-\mu}{\sigma}$. Check, if the following densities are members of the location-scale family:

a) The arc-sine distribution with pdf

$$f(x;\mu,\sigma) = \frac{1}{\sigma\pi\sqrt{1-\big(\frac{x-\mu}{\sigma}\big)^2}}, \qquad \mu - \sigma < x < \mu + \sigma$$

b) The exponential distribution with pdf

$$f(x;\mu,\sigma) = \frac{1}{\sigma}\exp\Big(-\frac{x-\mu}{\sigma}\Big), \qquad x \geq \mu,\ \mu \in \mathbb{R},\ \sigma > 0$$

c) The U-shaped parabolic distribution with pdf

$$f(x;\mu,\sigma) = \frac{3}{2\sigma}\Big(\frac{x-\mu}{\sigma}\Big)^2, \qquad \mu - \sigma < x < \mu + \sigma,\ \mu \in \mathbb{R},\ \sigma > 0$$

d) The Weibull distribution with pdf

$$f(x;\mu,\sigma,c) = \frac{c}{\sigma}\Big(\frac{x-\mu}{\sigma}\Big)^{c-1}\exp\Big[-\Big(\frac{x-\mu}{\sigma}\Big)^c\Big], \qquad x \geq \mu,\ \mu \in \mathbb{R},\ \sigma > 0,\ c > 0$$

Solution to Exercise 4.7

a) Given

$$f(x;\mu,\sigma) = \frac{1}{\sigma\pi\sqrt{1-\big(\frac{x-\mu}{\sigma}\big)^2}}, \qquad \mu - \sigma < x < \mu + \sigma$$

Then

$$F(x) = \int_{\mu-\sigma}^{x} \frac{1}{\sigma\pi\sqrt{1-\big(\frac{x-\mu}{\sigma}\big)^2}}$$

Note that $\dfrac{d}{dx}\arcsin x = \dfrac{1}{\sqrt{1-x^2}}$ and further substitute $z = \dfrac{x-\mu}{\sigma}$, such that $dz = \dfrac{1}{\sigma}dx$. Then

$$\begin{aligned} F(x) &= \ldots = \int_{-1}^{z} \frac{1}{\sigma\pi\sqrt{1-z^2}}\sigma \cdot dz = \\ &= \frac{1}{\pi}\Big[\arcsin z\Big]_{-1}^{z} = \frac{1}{\pi}(\arcsin z + \pi/2) \end{aligned}$$

Substituting back we get

$$F(x) = \frac{\pi + 2\arcsin\big(\frac{x-\mu}{\sigma}\big)}{2\pi},$$

which is a function of $\frac{x-\mu}{\sigma}$ and has no further dependencies on (μ, σ). Hence, it belongs to the location-scale family.

b) Given

$$f(x;\mu,\sigma) = \frac{1}{\sigma}\exp\Big(-\frac{x-\mu}{\sigma}\Big), \quad x \geq \mu,\ \mu \in \mathbb{R},\ \sigma > 0$$

Then

$$\begin{aligned} F(x) &= \int_{\mu}^{x} \frac{1}{\sigma}\exp\Big(-\frac{x-\mu}{\sigma}\Big)dx = \frac{1}{\sigma}\int_{\mu}^{x}\exp\Big(-\frac{x-\mu}{\sigma}\Big)dx = \\ &= \frac{1}{\sigma}\Big[-\sigma\exp\Big(-\frac{x-\mu}{\sigma}\Big)\Big]_{\mu}^{x} = 1 - \exp\Big(-\frac{x-\mu}{\sigma}\Big) \end{aligned}$$

which is a function of $\frac{x-\mu}{\sigma}$ and has no further dependencies on (μ,σ). Hence, it belongs to the location-scale family.

c) Given

$$f(x;\mu,\sigma) = \frac{3}{2\sigma}\Big(\frac{x-\mu}{\sigma}\Big)^2, \quad \mu-\sigma < x < \mu+\sigma,\ \mu \in \mathbb{R},\ \sigma > 0$$

Then

$$\begin{aligned} F(x) &= \int_{\mu-\sigma}^{x} \frac{3}{2\sigma}\Big(\frac{x-\mu}{\sigma}\Big)^2 dx = \frac{3}{2\sigma}\Big[\frac{\sigma}{3}\Big(\frac{x-\mu}{\sigma}\Big)^2\Big]_{\mu-\sigma}^{x} = \\ &= \frac{1}{2}\Big(1+\Big(\frac{x-\mu}{\sigma}\Big)^3\Big) \end{aligned}$$

which is a function of $\frac{x-\mu}{\sigma}$ and has no further dependencies on (μ,σ). Hence, it belongs to the location-scale family.

d) Given

$$f(x;\mu,\sigma,c) = \frac{c}{\sigma}\Big(x;\mu,\sigma\Big)^{c-1}\exp\Big[-\Big(x;\mu,\sigma\Big)^c\Big], \quad x \geq \mu,\ \mu \in \mathbb{R},\ \sigma > 0,\ c > 0$$

Then

$$F(x) = \int_{\mu}^{x} \frac{c}{\sigma}\Big(\frac{x-\mu}{\sigma}\Big)^{c-1}\exp\Big[-\Big(\frac{x-\mu}{\sigma}\Big)^c\Big]dx$$

Substitute $z = \dfrac{x-\mu}{\sigma}$, such that $dz = \dfrac{1}{\sigma}dx$. Then

$$F(x) = \ldots = \int_0^z \frac{c}{\sigma} z^{c-1}e^{-z^c}\sigma\, dz = \int_0^z c z^{c-1}e^{-z^c}\, dz$$

Note that $\dfrac{d}{dz}e^{-z^c} = e^{-z^c}(-cz^{c-1})$, then

$$F(x) = \ldots = \Big[-e^{-z^c}\Big]_0^z = 1 - e^{-z^c}$$

Substituting back

$$F(x) = 1 - \exp\Big[-\Big(\frac{x-\mu}{\sigma}\Big)^c\Big]$$

which is a function of $\frac{x-\mu}{\sigma}$ for a given c and has no further dependencies on (μ,σ). Hence, it belongs to the location-scale family.

Exercise 4.8

Consider the cdf of a Gumbel-distributed random variable X

$$F(x;\alpha,\beta) = \exp(-e^{-(x-\alpha)/\beta})$$

with $\alpha \in \mathbb{R}$ and $\beta > 0$. Find the pdf of X and show that its expected value is $\mathrm{E}[X] = \alpha + \beta\gamma$ with $\gamma \approx 0.5772$ (Euler-Mascheroni constant[2]).

Solution to Exercise 4.8

$$\begin{aligned} f(x) &= F'(x;\alpha,\beta) = e^{-e^{-\frac{x-\alpha}{\beta}}} \cdot \Big(- e^{-\frac{x-\alpha}{\beta}} \Big) \cdot \Big(-\frac{1}{\beta} \Big) \\ &= \frac{1}{\beta} e^{-e^{-\frac{x-\alpha}{\beta}}} e^{-\frac{x-\alpha}{\beta}} \end{aligned}$$

Further, note that if $Y \sim \mathrm{Gumbel}(\alpha = 0, \beta = 1)$, then $X = \beta Y + \alpha$. Then

$$\mathrm{E}[Y] = \int_{-\infty}^{\infty} y\ f(y)\ dy = \int_{-\infty}^{\infty} y\ e^{-e^{-y}} dy$$

We substitute $z = e^{-y}$, then $y = -\ln z$ and $dy = -\frac{1}{z}$,

$$\begin{aligned} - \int_0^{\infty} e^{-z} \ln z dz &= -\frac{\partial}{\partial a} \int_0^{\infty} z^a e^{-z} dz \Big|_{a=0} = \\ &= -\frac{\partial}{\partial a} \Gamma(a+1) \Big|_{a=0} = \Gamma'(1) = \gamma \approx 0.5772. \end{aligned}$$

Finally,

$$\mathrm{E}[X] = \mathrm{E}[\beta Y + \alpha] = \beta \mathrm{E}[Y] + \alpha = \beta\gamma + \alpha$$

Exercise 4.9

Consider a Pareto-distributed random variable X with pdf

$$f(x;a,b,c) = \frac{c}{b}\Big(\frac{x-a}{b}\Big)^{-c-1}, x \geq a+b,\ a \in \mathbb{R},\ b > 0,\ c > 0.$$

Show that $\mathrm{E}(X) = a + b\dfrac{c}{c-1}$ for $c > 1$ and $\mathrm{Var}(X) = b^2 \dfrac{c}{(c-1)^2(c-2)}$ for $c > 2$.

Solution to Exercise 4.9

$$\mathrm{E}(X) = \int_{a+b}^{\infty} x \frac{c}{b}\Big(\frac{x-a}{b}\Big)^{-c-1} dx$$

[2]– Hint: $\Gamma'(1) = \gamma$

Substitute $z = (x-a)/b$, such that $x = bz + a$ and $dx = b\,dz$. Then

$$\begin{aligned}
\mathrm{E}(X) &= \frac{c}{b}\int_1^{\infty}(bz+a)z^{-c-1}b\,dz = \\
&= c\int_1^{\infty} bz^{-c} + az^{-c-1}dz = \\
&= c\Big[\frac{b}{1-c}z^{-c+1} - \frac{a}{c}z^{-c}\Big]_1^{\infty} = \\
&= 0 - \frac{cb}{1-c} + a = a + b\frac{c}{c-1} \\
\mathrm{E}(X^2) &= \frac{c}{b}\int_1^{\infty}(bz+a)^2 z^{-c-1}b\,dz = \\
&= c\int_1^{\infty} b^2 z^{-c+1} + 2abz^{-c} + a^2 z^{-c-1}dz = \\
&= c\Big[\frac{b^2}{-c+2}z^{-c+2} + \frac{2ab}{-c+1}z^{-c+1} + \frac{a^2}{-c}z^{-c}\Big]_1^{\infty} = \\
&= \frac{cb^2}{c-2} + \frac{2ab}{c-1} + a^2 \\
\mathrm{Var}(X) &= \mathrm{E}(X^2) - \mathrm{E}(X)^2 = \frac{cb^2}{c-2} + \frac{2ab}{c-1} + a^2 - \Big(a + b\frac{c}{c-1}\Big)^2 = \\
&= \frac{cb^2}{c-2} + \frac{2ab}{c-1} + a^2 - a^2 - 2ab\frac{c}{c-1} - b^2\frac{c^2}{(c-1)^2} = \\
&= \frac{cb^2}{c-2} - b^2\frac{c^2}{(c-1)^2} = b^2\frac{c}{(c-1)^2(c-2)}
\end{aligned}$$

Exercise 4.10

A random variable X has the Laplace distribution with probability density function proportional to $\exp^{-\alpha|x-m|}$ for $(-\infty < X < \infty)$, m being a positive parameter. If c is any given positive number $\leq m$, prove that

$$\mathrm{E}\{|X-c|\} = \frac{1}{\alpha}[\exp^{-\alpha(m-c)} + \alpha(m-c)].$$

Hence deduce that the minimum value of the mean deviation of X is $1/\alpha$, which is attained for $c = m$.

Solution to Exercise 4.10

The proportionality factor is $\alpha/2$ obtained from

$$k\left[\int_{-\infty}^{m}\exp^{-\alpha(m-x)}\,dx + \int_{m}^{\infty}\exp^{-\alpha(x-m)}\,dx\right] = 1$$

$$\mathrm{E}[|X-c|] = \frac{\alpha}{2}\left[\int_{-\infty}^{c}(c-x)\exp^{-\alpha(m-x)}\,dx + \int_{c}^{m}(x-c)\exp^{-\alpha(m-x)}\,dx + \int_{m}^{\infty}(x-c)\exp^{-\alpha(x-m)}\,dx\right],$$

whence the result.

Exercise 4.11

If x_1 and x_2 are independent observations from a rectangular distribution in the $(0,1)$ interval, find the joint distribution of the statistics

$$u = x_1 x_2 \quad \text{and} \quad v = (1 - x_1)(1 - x_2).$$

Solution to Exercise 4.11[3]

The Jacobian of the transformation $u = x_1 x_2$, $v = (1-x_1)(1-x_2)$ is $|x_1 - x_2|$, where $(x_1 - x_2)^2 = (1 - u + v)^2 - 4v \geq 0$. Combining the contributions for $x_1 - x_2 \geq 0$ and $x_2 - x_1 \geq 0$, the joint distribution of u and v is

$$2[(1 - u + v)^2 - 4v]^{-\frac{1}{2}} \mathrm{d}u \mathrm{d}v$$

For fixed v in $(0 \leq v \leq 1)$, $0 \leq u \leq (1 - \sqrt{v})^2$, and so the marginal distribution of v is $-\log v \mathrm{d}v$, $(0 \leq v \leq 1)$.
The conditional distribution of u given v is

$$2[(1 - u + v)^2 - 4v]^{-\frac{1}{2}}(-\log v)^{-1} \mathrm{d}u, \qquad 0 \leq u \leq (1 - \sqrt{v})^2.$$

Directly, $u = \exp^{-\chi^2/2}$, where χ^2 has 4 d.f., and so the marginal distribution of u is $-\log u \mathrm{d}u$, $(0 \leq u \leq 1)$.
The distribution of v is the same, since if x is uniform in $(0,1)$ so is $(1 - x)$ in $(0,1)$.

Exercise 4.12

Given a random variable

$$X : \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ \frac{1}{10} & \frac{3}{10} & \frac{3}{10} & \frac{2}{10} & \frac{1}{10} \end{pmatrix}$$

Characterize a truncated random variable that only has realizations 1, 2 and 3.

Solution to Exercise 4.12

$$\begin{aligned} W(\{1 \leq X \leq 3\}) &= W(\{X = 1\}) + W(\{X = 2\}) + W(\{X = 3\}) \\ &= \frac{3}{10} + \frac{3}{10} + \frac{2}{10} = \frac{8}{10}, \end{aligned}$$

yields

$$\begin{aligned} W(X^* = 1) &= \frac{3}{10} \div \frac{8}{10} = \frac{3}{8}, \\ W(X^* = 2) &= \frac{3}{10} \div \frac{8}{10} = \frac{3}{8}, \\ W(X^* = 3) &= \frac{2}{10} \div \frac{8}{10} = \frac{2}{8}, \end{aligned}$$

[3]– Barlett, M.S. (1936), Proceedings of the Royal Society, London, Series A, 154, 124

The truncated distribution is thus

$$X^* : \begin{pmatrix} 1 & 2 & 3 \\ \frac{3}{8} & \frac{3}{8} & \frac{2}{8} \end{pmatrix}.$$

If X is a continuous random variable and $F^*(x)$ is differentiable, we get a pdf $f^*(x)$ of X^* as a derivative of the cdf:

$$\begin{aligned} f^*(x) &= \frac{dF^*(x)}{dx} = \frac{f(x)}{F(\beta) - F(\alpha)} \\ &= \frac{f(x)}{\int_\alpha^\beta f(x)\,dx}, \quad \alpha \le x \le \beta. \end{aligned}$$

Further we get

$$\begin{aligned} \mathrm{E}(X^*) &= \frac{1}{F(\beta) - F(\alpha)} \int_\alpha^\beta x \cdot f(x)\,dx \\ &= \frac{\int_\alpha^\beta x f(x)\,dx}{\int_\alpha^\beta f(x)\,dx} \\ \mathrm{Var}(X^*) &= \frac{1}{F(\beta) - F(\alpha)} \int_\alpha^\beta [x - E(X^*)]^2 \cdot f(x)\,dx \\ &= \frac{\int_\alpha^\beta [x - E(X^*)]^2 \cdot f(x)\,dx}{\int_\alpha^\beta f(x)\,dx} \end{aligned}$$

Analogously, one could find $F^*(x)$, $f^*(x)$, $\mathrm{E}(X^*)$ as well as $\mathrm{Var}(X^*)$ for a one-side truncated distribution.

Exercise 4.13

Given a random variable $X \sim \mathcal{N}(0,1)$ and an interval $[\alpha, \beta]$, as well as

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{v^2}{2}}\,dv$$

and

$$\varphi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

Find $F^*(x)$, $f^*(x)$, $\mathrm{E}(X^*)$ and $\mathrm{Var}(X^*)$ of a distribution truncated on the interval $[\alpha, \beta]$.

Solution to Exercise 4.13

$$
\begin{aligned}
F^*(x) &= \frac{\Phi(x)-\Phi(\alpha)}{\Phi(\beta)-\Phi(\alpha)}, \\
f^*(x) &= \frac{\varphi(x)}{\Phi(\beta)-\Phi(\alpha)}, \\
\mathrm{E}(X^*) &= \frac{1}{\Phi(\beta)-\Phi(\alpha)}\int_\alpha^\beta x\cdot\varphi(x)\,dx \\
&= \frac{1}{\Phi(\beta)-\Phi(\alpha)}\cdot\frac{1}{\sqrt{2\pi}}\cdot\int_\alpha^\beta x\cdot e^{-\frac{x^2}{2}}\,dx \\
&= \frac{1}{\Phi(\beta)-\Phi(\alpha)}\cdot\frac{1}{\sqrt{2\pi}}\cdot\left[-e^{-\frac{x^2}{2}}\right]\Big|_\alpha^\beta \\
&= \frac{e^{-\frac{\alpha^2}{2}}-e^{-\frac{\beta^2}{2}}}{\sqrt{2\pi}\left[\Phi(\beta)-\Phi(\alpha)\right]} \\
&= \frac{\varphi(\alpha)-\varphi(\beta)}{\Phi(\beta)-\Phi(\alpha)}, \\
\mathrm{Var}(X^*) &= E\left((X^*)^2\right)-\left[E(X^*)\right]^2, \\
\mathrm{E}\left((X^*)^2\right) &= \frac{1}{\Phi(\beta)-\Phi(\alpha)}\cdot\left(\int_\alpha^\beta x^2\cdot\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}\,dx\right) \\
&= \frac{1}{\Phi(\beta)-\Phi(\alpha)}\cdot\left(\int_\alpha^\beta x\cdot\left(\frac{1}{\sqrt{2\pi}}x\,e^{-\frac{x^2}{2}}\right)dx\right) \\
&= \frac{1}{\Phi(\beta)-\Phi(\alpha)}\cdot\left\{\left[-x\cdot\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}\right]_\alpha^\beta+\int_\alpha^\beta\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}\,dx\right\} \\
&= \frac{\alpha\cdot\varphi(\alpha)-\beta\cdot\varphi(\beta)}{\Phi(\beta)-\Phi(\alpha)}+\underbrace{\frac{\Phi(\beta)-\Phi(\alpha)}{\Phi(\beta)-\Phi(\alpha)}}_{=1},
\end{aligned}
$$

It follows

$$
\mathrm{Var}(X^*) = 1+\frac{\alpha\cdot\varphi(\alpha)-\beta\cdot\varphi(\beta)}{\Phi(\beta)-\Phi(\alpha)}-\left[\frac{\varphi(\alpha)-\varphi(\beta)}{\Phi(\beta)-\Phi(\alpha)}\right]^2
$$

Exercise 4.14

(Sum of Poisson random variables) Let $\{X_i, i: 1\to n\}$ be independently Poisson distributed random variables with parameter λ_i. Find the distribution of $\sum_{i=1}^n X_i$.

Solution to Exercise 4.14

$$
\begin{aligned}
M_{X_i}(t) &= \exp\{\lambda_i(e^t-1)\} \\
M_{\sum X_i}(t) &\overset{iid}{=} \prod_{i=1}^n \exp\left\{\lambda_i(e^t-1)\right\} = \exp\left\{\sum_{i=1}^n \lambda_i(e^t-1)\right\}
\end{aligned}
$$

Thus,

$$\sum_{i=1}^{n} \lambda_i = \lambda' \Rightarrow \sum_{i=1}^{n} X_i \sim \text{Poisson}(\lambda')$$

because of the bijective relationship between MGF and pdf[4].

Exercise 4.15

(Square of normally distributed random variables) Let X_1 and X_2 be two independent standard normal random variables. Find the distributions of $Y = (X_1 + X_2)^2/2$.

Solution to Exercise 4.15

$$Y = X_1 + X_2, \quad Y \sim \mathcal{N}(0, 2)$$

What is the distribution of $\frac{Y^2}{2}$?

$$\frac{Y}{\sqrt{2}} \sim \mathcal{N}(0, 1)$$

The square of a standard normal random variable is $\chi^2_{(1)}$-distributed.

$$\frac{Y^2}{2} \sim \chi^2_{(1)}$$

Note, however, it holds that $\sum\limits_{i=1}^{n} X_i^2 \sim \chi^2_{(n)}$, but $\left(\sum X_i\right)^2 \neq \sum X_i^2$.

Exercise 4.16

(Sum and Difference of random variables) Let X_1 and X_2 be two independent random variables and $Y_1 = g_1(X_1, X_2) = X_1 + X_2$ and $Y_2 = g_2(X_1, X_2) = X_2 - X_1$.

a) Assume that X_1 and X_2 follow the standard normal distribution, and find the joint distribution of Y_1 and Y_2.

b) Assume now that X_1 and X_2 each follow the uniform distribution on the interval $(0, 1)$, and find the joint distribution of Y_1 and Y_2.

Solution to Exercise 4.16

a) Solution using the MGF

$$\begin{aligned}
M_{Y_1,Y_2}(t_1, t_2) &= \mathrm{E}\left(\exp\{t_1 \overbrace{(x_1 + x_2)}^{y_1} + t_2 \overbrace{(x_2 - x_1)}^{y_2}\}\right) \\
&= \mathrm{E}\Big(\exp\{x_1(t_1 - t_2) + x_2(t_1 + t_2)\}\Big)\Big|_{X_1 \perp X_2} \\
&= \mathrm{E}\left(e^{x_1(t_1 - t_2)}\right) \mathrm{E}\left(e^{x_2(t_1 + t_2)}\right) \\
&= M_{X_1}(t_1 - t_2) M_{X_2}(t_1 + t_2).
\end{aligned}$$

[4] – every MFG has one distinct pdf and vice versa.

We know that X_1 and X_2 are standard normal random variables, thus

$$\begin{aligned} M_{X_1}(t_1 - t_2) &= \exp\frac{(t_1-t_2)^2}{2} \\ M_{X_2}(t_1 + t_2) &= \exp\frac{(t_1+t_2)^2}{2} \end{aligned}$$

$$\begin{aligned} M_{Y_1,Y_2}(t_1,t_2) &= \exp\frac{(t_1-t_2)^2}{2}\exp\frac{(t_1+t_2)^2}{2} \\ &= \exp\left\{\frac{t_1^2 - 2t_1t_2 + t_2^2 + t_1^2 + 2t_1t_2 + t_2^2}{2}\right\} \\ &= \exp\{t_1^2 + t_2^2\} \\ &= \underbrace{\exp\left\{\frac{2t_1^2}{2}\right\}}_{y_1 \sim \mathcal{N}(0,2)} \underbrace{\exp\left\{\frac{2t_2^2}{2}\right\}}_{y_2 \sim \mathcal{N}(0,2)} \end{aligned}$$

Obviously, $Y_1 \perp Y_2$.

b) Solution using the transformation method

$$f(\underline{x}) = \mathcal{I}_{(0,1)}(x_1)\,\mathcal{I}_{(0,1)}(x_2)$$

$$h(z,s) = f(g^{-1}(y))\,|\det(J)|$$

$$\begin{aligned} g(x_1,x_2) &= \begin{pmatrix} x_1 + x_2 \\ x_2 - x_1 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \\ g^{-1}(y_1,y_2) &= \begin{pmatrix} \frac{y_1-y_2}{2} \\ \frac{y_1+y_2}{2} \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} J &= \begin{bmatrix} \frac{\partial g_1^{-1}(y)}{\partial y_1} & \frac{\partial g_1^{-1}(y)}{\partial y_2} \\ \frac{\partial g_2^{-1}(y)}{\partial y_1} & \frac{\partial g_2^{-1}(y)}{\partial y_2} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \\ \det J &= \left(\frac{1}{2}\right)^2 - \frac{1}{2}\left(-\frac{1}{2}\right) = \frac{1}{2} \\ h(z,s) &= \frac{1}{2}\mathcal{I}_{(0,1)}\left(\frac{y_1-y_2}{2}\right)\mathcal{I}_{(0,1)}\left(\frac{y_1+y_2}{2}\right) \end{aligned}$$

Exercise 4.17

(Cauchy distribution) Let X_1 and X_2 be two independent standard normal random variables. Consider the functions $Y_1 = X_1 + X_2$ and $Y_2 = X_1/X_2$.

a) Find the joint distribution of Y_1 and Y_2.

b) Define the marginal distribution of Y_2.

Solution to Exercise 4.17

a)[5] Given

$$x_1 = g_1^{-1}(y_1, y_2) = \frac{y_1 y_2}{1 + y_2} \quad \text{and} \quad x_2 = g_2^{-1}(y_1, y_2) = \frac{y_1}{1 + y_2}$$

$$J = \begin{vmatrix} \frac{y_2}{1+y_2} & \frac{y_1}{(1+y_2)^2} \\ \frac{1}{1+y_2} & -\frac{y_1}{(1+y_2)^2} \end{vmatrix} = -\frac{y_1(y_2+1)}{(1+y_2)^3} = -\frac{y_1}{(1+y_2)^2}$$

$$\begin{aligned} f(y_1, y_2) &= \frac{|y_1|}{(1+y_2)^2} \frac{1}{2\pi} \exp\left\{-\frac{1}{2}\left[\frac{(y_1 y_2)^2}{(1+y_2)^2} + \frac{y_1^2}{(1+y_2)^2}\right]\right\} \\ &= \frac{1}{2\pi} \frac{|y_1|}{(1+y_2)^2} \exp\left\{-\frac{1}{2} \frac{(1+y_2^2) y_1^2}{(1+y_2)^2}\right\} \end{aligned}$$

b) In order to find the marginal distribution of Y_2, we have to integrate with respect to Y_1:

$$\begin{aligned} f(y_2) &= \int_{-\infty}^{\infty} f_{Y_1,Y_2}(y_1, y_2) dy_1 \\ &= \frac{1}{2\pi} \frac{1}{(1+y_2)^2} \int_{-\infty}^{\infty} |y_1| \exp\left\{-\frac{1}{2} \frac{(1+y_2^2) y_1^2}{(1+y_2)^2}\right\} dy_1 \end{aligned}$$

Let

$$u = \frac{1}{2} \frac{(1+y_2^2)}{(1+y_2)^2} y_1^2,$$

then

$$du = \frac{(1+y_2^2)}{(1+y_2)^2} y_1 dy_1$$

and finally

$$f(y_2) = \frac{1}{2\pi} \frac{1}{(1+y_2)^2} \frac{(1+y_2)^2}{(1+y_2^2)} 2 \int_0^{\infty} e^{-u} du = \frac{1}{\pi} \frac{1}{(1+y_2^2)},$$

a Cauchy density. The ratio of two independent standard normal random variables is thus Cauchy distributed.

Exercise 4.18

(Linear combinations of normally distributed random variables) Let $\{X_i, i : 1 \to 3\}$ be independent random variables following the standard normal distribution. Consider the following linear combinations: (i) $Y_1 = X_1$, (ii) $Y_2 = (X_1 + X_2)/2$ and (iii) $Y_3 = (X_1 + X_2 + X_3)/3$, and find their joint distribution.

Solution to Exercise 4.18

The following functional relationships are given

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = g(x_1, x_2, x_3) = \begin{pmatrix} x_1 \\ (x_1 + x_2)/2 \\ (x_1 + x_2 + x_3)/3 \end{pmatrix}.$$

with the associated inverse functions

$$g^{-1} = \begin{pmatrix} y_1 \\ 2y_2 - y_1 \\ 3y_3 - 2y_2 \end{pmatrix}.$$

Then:

$$|\det(J)| = \begin{vmatrix} 1 & 0 & 0 \\ -1 & 2 & 0 \\ 0 & -2 & 3 \end{vmatrix} = 6.$$

Thus, the joint distribution of the Y's is given by

$$\begin{aligned} f(y_1, y_2, y_3) &= |\det(J)| f_{\mathbf{X}}(g_1^{-1}, g_2^{-1}, g_3^{-1}) \\ &= 6\left(\frac{1}{\sqrt{2\pi}}\right)^3 \exp\left\{-\frac{1}{2}[y_1^2 + (2y_2 - y_1)^2 + (3y_3 - 2y_2)^2]\right\} \\ &= 6\left(\frac{1}{\sqrt{2\pi}}\right)^3 \exp\left\{-\frac{1}{2}[2y_1^2 - 4y_1y_2 + 8y_2^2 - 12y_2y_3 + 9y_3^2]\right\}. \end{aligned}$$

We find the marginal densities by integration. Doing so we form the integral in a way that we get conditional normal distributions which can be integrated to 1, e.g.

$$\begin{aligned} f(y_3) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(y_1, y_2, y_3)dy_1 dy_2 \\ &= 6\left(\frac{1}{\sqrt{2\pi}}\right)^3 \int_{-\infty}^{\infty} \exp\left\{-\frac{1}{2}[6y_2^2 - 12y_2y_3 + 9y_3^2]\right\} \\ &\quad \times \left(\int_{-\infty}^{\infty} \underbrace{\exp\left\{-\frac{1}{2}(2y_1^2 - 4y_1y_2 + 2y_2^2)\right\} dy_1}_{\text{Gaussian kernel in } y_1 \text{ given } y_2}\right) dy_2 \\ &= \frac{6}{\sqrt{2}}\left(\frac{1}{\sqrt{2\pi}}\right)^2 \int_{-\infty}^{\infty} \underbrace{\exp\left\{-\frac{1}{2}[6y_2^2 - 12y_2y_3 + 6y_3^2]\right\}}_{\text{Gaussian kernel in } y_2 \text{ given } y_3} \exp\left\{-\frac{1}{2}[3y_3^2]\right\} dy_2 \\ &= \frac{\sqrt{3}}{\sqrt{2\pi}} \exp\left\{-\frac{3}{2}y_3^2\right\}. \end{aligned}$$

Now, Y_3 is normally distributed with expected value 0 and variance $\frac{1}{3}$.

Exercise 4.19

(Inverted gamma distribution) Let X be gamma distributed with a probability density function

$$f(x) = \frac{1}{(n-1)!\beta^n} x^{n-1} e^{-x/\beta} \mathcal{I}_{(0,\infty)}\ , n \in \mathbb{N}, \quad \beta > 0.$$

Find the probability density function of $Y = 1/X$.

Solution to Exercise 4.19

$$X \sim \text{Gamma}(n, \beta), \;\; 0 < x < \infty$$

$$\begin{aligned} Y &= \frac{1}{X} = g(X) \\ g^{-1}(Y) &= \frac{1}{Y} = X \\ G &= \frac{\partial g^{-1}(Y)}{\partial Y} = -\frac{1}{Y^2} \end{aligned}$$

Using the transformation theorem

$$h(y) = f(g^{-1}(y))|G|$$

$$\begin{aligned} h(y) &= \frac{1}{\Gamma(n)\beta^n}\left(\frac{1}{y}\right)^{n-1} e^{-\frac{1}{y\beta}}\left|\frac{1}{y^2}\right| \\ &= \frac{1}{\Gamma(n)\beta^n}\left(\frac{1}{y}\right)^{n+1} e^{-\frac{1}{y\beta}} \end{aligned}$$

This resembles a pdf of an inverse Gamma distribution with $0 < y < \infty$, since $0 < \frac{1}{x}$.

$$h(y) = \frac{1}{\Gamma(n)\beta^n}\left(\frac{1}{y}\right)^{n+1} e^{-\frac{1}{y\beta}}\mathcal{I}_{(0,\infty)}(y)$$

Chapter 5

Basic Asymptotics

Exercise 5.1

(Convergence of a sequence of functions) Let the function series be defined by

$$f_n(x) = \begin{cases} 4n^2x & \text{if } 0 \le x < 1/(2n) \\ -4n^2x + 4n & \text{if } 1/(2n) \le x \le 1/n \\ 0 & \text{otherwise} \end{cases}$$

for $x \in \mathbb{R}$. Examine whether this function converges pointwise to the function $f(x) = 0$. Does it also converge uniformly?

Solution to Exercise 5.1

The sequence converges pointwise to zero, as, on the one hand, the range in which the sequence takes positive values converges to zero, and as, on the other hand, the sequence is always zero at point zero.

Exercise 5.2

(Convergence in probability) Suppose that the random variables Y_n for $n \in \mathbb{N}$ follow a $\mathcal{N}(\mu, [\sigma^2 + \sqrt{n}]/[n+1])$ distribution. Show that the sequence $\{Y_n\}$ converges in probability to μ.

Solution to Exercise 5.2

Convergence in probability to a random variable or to a deterministic value ξ holds if

$$\lim_{n\to\infty} P(|Y_n - \xi| < \epsilon) = 1.$$

According to Chebyshev,

$$\begin{aligned}
P(|Y_n - \mathrm{E}[Y_n]| < \epsilon) &\ge 1 - \frac{\mathrm{Var}(Y_n)}{\epsilon^2} \\
P(|Y_n - \mu| < \epsilon) &\ge 1 - \frac{\frac{\sigma^2+\sqrt{n}}{n+1}}{\epsilon^2} \\
\lim_{n\to\infty} P(|Y_n - \mu| < \epsilon) &\ge \lim_{n\to\infty} 1 - \frac{\frac{\sigma^2+\sqrt{n}}{n+1}}{\epsilon^2} = 1.
\end{aligned}$$

Thus, it follows that $Y_n \xrightarrow{p} \mu$.

Exercise 5.3

(Convergence in probability) Let the sequence of random variables $\{X_n\}$ have the following probability density function:

$$f_n(x) = \begin{cases} (2n-1)/(3n) & \text{if } x = 1 \\ 1/3 & \text{if } x = 1 + 1/(n+1) \\ 1/(3n) & \text{if } x = 2 \\ 0 & \text{otherwise} \end{cases} .$$

Show that the sequence $\{X_n\}$ converges in probability to 1.

Solution to Exercise 5.3

Here, we have to look at two movements: On the one hand the sequence converges to the points $\{1, 2\}$. On the other hand $P(x = 2)$ converges to zero. In the end, the entire probability mass can be found in point 1.

Exercise 5.4

(Convergence in distribution) Let the random variables X_n have the following distribution functions:

$$F_n(x) = \begin{cases} 0 & \text{if } x < -n \\ 1/2 & \text{if } -n \leq x < n \\ 1 & \text{if } x \geq n \end{cases} .$$

Examine whether the sequence $\{X_n\}$ converges in distribution to a random variable X.

Solution to Exercise 5.4

The sequence of distribution functions indeed converges to the function $F(x) = \frac{1}{2}\mathcal{I}_{(-\infty,\infty)}(x)$ but this function does not have the properties of a distribution function (e.g. $\lim_{x\to\infty} F = 1$). Thus, we have no convergence in distribution.

Exercise 5.5

(WLLN) Let $\{X_n\}$ be a sequence of independent random variables, and suppose $\mathrm{E}(X_n) = \mu_n$. Moreover, let

$$Y_n = \frac{1}{n}\sum_{i=1}^{n}(X_i - \mu_i) .$$

a) Assume that X_n is a Bernoulli random variable with parameter p_n, and show that $Y_n \xrightarrow{p} 0$.

b) Assume now that X_n follows an unknown distribution and that a constant $c \in (0, \infty)$ exists such that

$$\mathrm{Var}\Big(\sum_{i=1}^{n} X_i\Big) \leq c \cdot n , \quad \forall i \in \mathbb{N} .$$

Check whether $Y_n \xrightarrow{p} 0$ applies here as well.

Solution to Exercise 5.5

a) The convergence in probability to zero can be shown with the help of Chebyshev's inequality. Moreover, the necessary and sufficient conditions for WLLN (non-iid case) are fulfilled and can be used in this case as well. Note, that $X_n \sim$Bernoulli(p_n) are independent, but not identical, since every X_n has a corresponding parameter p_n. Also, $\mathrm{E}[X_n] = p_n$ and $\mathrm{Var}[X_n] = p_n(1-p_n)$.

$$P(|Y_n - \mathrm{E}[Y_n]| < \epsilon) \geq 1 - \frac{\mathrm{Var}[Y_n]}{\epsilon^2}$$

$$\begin{aligned}
\mathrm{E}[Y_n] &= \mathrm{E}\left[\frac{1}{n}\sum_{i=1}^{n}(X_i - \mu_i)\right] = \frac{1}{n}\sum_{i=1}^{n}(\mathrm{E}[X_i] - \mu_i) = 0 \\
\mathrm{Var}[Y_n] &= \mathrm{Var}\left[\frac{1}{n}\sum_{i=1}^{n}(x_i - \mu_i)\right] = \frac{1}{n^2}\mathrm{Var}\left[\sum_{i=1}^{n} X_i - \underbrace{\sum_{i=1}^{n}\mu_i}_{\text{a constant}}\right] \\
&= \frac{1}{n^2}\mathrm{Var}\left[\sum_{i=1}^{n} X_i\right] \overset{\text{independence}}{=} \frac{1}{n^2}\sum_{i=1}^{n}\mathrm{Var}[x_i] \\
&= \frac{1}{n^2}\sum_{i=1}^{n} p_i(1-p_i)
\end{aligned}$$

$$\begin{aligned}
\text{Since} \quad p_i(1-p_i) &< 1 \quad \forall \quad p_i \quad \text{it follows:} \\
\frac{1}{n^2}\sum_{i=1}^{n} p_i(1-p_i) &< \frac{1}{n^2}\sum_{i=1}^{n} 1 = \frac{n}{n^2} \underset{n\to\infty}{\to} 0 \\
\text{That is,} \quad \mathrm{Var}[Y_n] &\to 0
\end{aligned}$$

$$\text{Hence,} \quad \lim_{n\to\infty} P(|Y_n - 0| < \epsilon) \geq 1 - \frac{0}{\epsilon^2} = 1 \quad \text{and} \quad Y_n \overset{p}{\to} 0$$

b) Similar to a), but variance only known w.r.t. an upper limit which converges to zero for $n \to \infty$.

$$\mathrm{Var}\left(\sum_{i=1}^{n} X_i\right) \leq c \cdot n \qquad \Leftrightarrow \qquad \underbrace{\frac{1}{n^2}\mathrm{Var}\left(\sum_{i=1}^{n} X_i\right)}_{\mathrm{Var}(Y_n)} \leq \frac{1}{n^2} \cdot c \cdot n$$

$$\text{Since} \quad \frac{1}{n^2} \cdot c \cdot n \underset{n\to\infty}{\to} 0 \qquad \text{it follows} \quad \mathrm{Var}(Y_n) \to 0$$

Hence,

$$\lim_{n\to\infty} P(|Y_n - 0| < \epsilon) \geq 1 - \frac{\mathrm{Var}(Y_n)}{\epsilon^2} = 1 \quad \Rightarrow \quad Y_n \overset{p}{\to} 0$$

Exercise 5.6

(WLLN) Let $\{S, \Upsilon, P\}$ be the probability space of a random variable and let A be any event in S. Also, let N_A be the frequency with which the event A occurs during n independent repetitions of the experiment.
Show that the relative frequency N_A/n converges in probability to the probability $P(A)$.

Solution to Exercise 5.6

The random experiment is repeated n times (identical and independent). The variables $Z_1, \ldots, Z_n$ indicate the result of each round. Then we can define

$$X_i = \mathcal{I}_A(Z_i)$$

i.e. the variable X_i takes the value 1 if event A occurs in the ith round of the random experiment. This happens with the probability of event A, $P(A)$. As the random experiment is repeated identically and independently, the X_i's are thus,

$$X_i \overset{iid}{\sim} \text{Bernoulli}(P(A)).$$

For this reason the X_i's fulfill the requirement of the WLLN according to Khinchin (iid case). Thus,

$$\overline{X}_n \xrightarrow{p} P(A)$$

i.e. the arithmetic mean of the X_i's converges in probability to its expected value. But at the same time the arithmetic mean of the X_i's gives the relative frequency of the event A, N_A/n; therefore,

$$N_A/n \xrightarrow{p} P(A).$$

Exercise 5.7

(WLLN) Let $\{X_n\}$ be a sequence of random variables with

$$X_i \ \sim \mathcal{N}(1, 1 + 1/i) \quad \text{and} \quad \sigma_{ij} = \rho^{|i-j|}\ , \quad \rho \in (0,1)\ , \quad i \neq j\ .$$

Show that $(\sum_{i=1}^{n} X_i)/n \xrightarrow{p} 1$.

Solution to Exercise 5.7

The weak law of large numbers states that the sample mean converges in probability to the expected value as $n \longrightarrow \infty$. Meaning,

$$\bar{X}_n \xrightarrow{p} \mu$$

According to the theorem of Chebyshev,

$$P(|X - \mu| > k\sigma) \leq \frac{1}{k^2} \quad \Longleftrightarrow \quad P(|X - \mu| < \varepsilon) \geq 1 - \frac{\text{Var}(X)}{\varepsilon^2}$$

Since $\text{E}[\bar{X}_n] = 1$ it has to be shown that the variance of $\frac{\sum X_i}{n}$ converges to zero.

$$\text{Var}\left(\frac{\sum X_i}{n}\right) = \frac{1}{n^2}\text{Var}\left(\sum X_i\right) = \frac{1}{n^2}\left(\sum \text{Var}(X_i) + 2\sum_i \sum_j \sigma_{ij}\right) \text{ where } j > i$$

As a result, we get a sum of sums

$$\begin{aligned}
\mathrm{Var}\left(\frac{\sum X_i}{n}\right) &= \frac{1}{n^2}\mathrm{Var}\left(\sum X_i\right) = \frac{1}{n^2}\left(\sum \mathrm{Var}(X_i) + 2\sum_i\sum_j \sigma_{ij}\right) \quad \text{where } j > i \\
&= \frac{1}{n^2}\Big(\sum \mathrm{Var}(X_i) + 2\Big(\underbrace{\sum_{j=2}^{n}\sigma_{1j}}_{*} + \sum_{j=3}^{n}\sigma_{2j} + \sum_{j=4}^{n}\sigma_{3j} + \ldots + \sum_{j=n}^{n}\sigma_{(n-1)j}\Big)\Big)
\end{aligned}$$

Consider now the covariance terms. Assume $i = 1$, representing the greatest partial sum(*) possible.

$$\sum_{j>i}^{n}\rho^{|j-i|} = \sum_{k=1}^{n-1}\rho^k \overset{n\to\infty}{\longrightarrow} \sum_{k=1}^{\infty}\rho^k = \frac{\rho}{1-\rho}$$

Considering the sums of the variances,

$$\sum_{i=1}^{n} 1 + i^{-1} < \sum_{i=1}^{n} 2 = 2n$$

Hence,

$$\mathrm{Var}\left(\frac{\sum X_i}{n}\right) \leq \frac{1}{n^2}\left(2n + 2(n-1)\frac{\rho}{1-\rho}\right)$$

$$\lim_{n\to\infty}\frac{1}{n^2}\left(2n + 2(n-1)\frac{\rho}{1-\rho}\right) = 0$$

$$\lim_{n\to\infty}\mathrm{Var}\left(\frac{\sum X_i}{n}\right) = 0$$

Remark regarding variance-covariance matrix

$$\mathrm{Cov}(\mathbf{X}) = \begin{bmatrix} \sigma_1^2 & \sigma_{12} & \sigma_{13} & \cdots \\ \sigma_{21} & \sigma_2^2 & \sigma_{23} & \vdots \\ \sigma_{31} & \sigma_{32} & \sigma_3^2 & \vdots \\ \vdots & \cdots & \cdots & \ddots \end{bmatrix} \qquad \mathbf{X} = \{X_1, X_2, X_3, \ldots, X_n\}$$

Obviously, $\sigma_{12} = \sigma_{21}$, $\sigma_{31} = \sigma_{13}$, ... Every element above the diagonal has a correspondent below the diagonal; the variance-covariance matrix is always symmetric. Then the variance of $\sum_{i=1}^{n} X_i$ is given by

$$\mathrm{Var}\left(\sum X_i\right) = \sum \mathrm{Var}\,(X_i) + 2\sum_j\sum_i \sigma_{ij} \quad \text{where } j > i$$

Because of the symmetry, we can multiply the double sum by 2. Therefore, we need to add up only the elements above the main diagonal.

$$A = \begin{bmatrix} \sigma_1^2 & \sigma_{12} & \sigma_{13} & \cdots \\ & \ddots & \sigma_{23} & \vdots \\ & & \ddots & \vdots \\ & & & \sigma_n^2 \end{bmatrix} \implies \sum_j \sum_i \sigma_{ij} = \sum_{i=1}^{n-1} \sum_{j>i}^{n} \sigma_{ij}$$

The sum of the covariances above the diagonal is a sum of sums, in which we add up all elements with $j > i$ (in matrix A, the j of the elements above the diagonal are greater than the i). This sum of sums can be written as $\sum_{i=1}^{n-1} \sum_{j>i}^{n} \sigma_{ij}$. If we look at $\sum_{j>i}^{n} \sigma_{ij}$ and keep i constant, e.g. $i = 1$, we get $\sum_{j>i}^{n} \sigma_{1j} = \sum_{j=2}^{n} \sigma_{1j}$.

Exercise 5.8

(CLT) Consider an ideal die that is rolled n times. The random variable X_i denotes the number of dots facing up on the ith attempt. Find the probability that the average number of dots does not exceed 3.6 after 200 attempts.

Solution to Exercise 5.8

According to Lindberg-Levy's CLT,

$$Y_n = \frac{(\bar{X}_n - \mu)}{\sigma_X/\sqrt{n}} \xrightarrow{d} Y \sim \mathcal{N}(0,1)$$

if $\bar{X}_n = \sum X_i/n$, $\mathrm{E}(X_i) = \mu$ and $\mathrm{Var}(X_i) = \sigma_X^2$, then

$$\bar{X}_n \overset{\text{asy}}{\sim} \mathcal{N}(\mu, \sigma^2/n)$$

Here,

$$\mathrm{E}(X_i) = \mu = 3.5; \quad \mathrm{Var}(X_i) = \sigma^2 = \frac{35}{12} \quad \text{for a simple, fair die.}$$

Applying the CLT,

$$\frac{(\bar{X} - \mu)}{\sigma/\sqrt{n}} \xrightarrow{d} \mathcal{N}(0,1)$$
$$\bar{X} \overset{\text{asy}}{\sim} \mathcal{N}(\mu, \sigma^2/n)$$

Plugging in,

$$\bar{X} \overset{\text{asy.}}{\sim} \mathcal{N}\left(3.5, \frac{35/12}{200}\right)$$

Now, to approximate $P(\bar{X} \leq 3.6)$ we will make use of the standard normal cdf Φ and its quantiles. However, we would need to transform our $\bar{X}$ into a standard normal random variable.

$$P(\bar{X} \leq 3.6) = P\left(\frac{\bar{X} - 3.5}{\sqrt{\frac{1}{200}\frac{35}{12}}} \leq \frac{0.1}{\sqrt{\frac{1}{200}\frac{35}{12}}}\right) \approx \Phi\left(\frac{0.1}{\sqrt{\frac{1}{200}\frac{35}{12}}}\right) \approx 0.7967$$

Exercise 5.9

(CLT) With the help of the central limit theorem (Lindberg-Levy), motivate the possibility of approximating a binomial distribution by an associated normal distribution. What means correction for continuity in this context?

Solution to Exercise 5.9[1]

Let Y be a Bernoulli-type (n, p) random variable with

$$Y = \sum_{i=1}^{n} X_i, \quad f(x_i; p) = p^{x_i}(1-p)^{1-x_i}\mathcal{I}_{\{0,1\}}(x_i)$$

Then, by the Lindberg-Levy CLT,

$$\begin{aligned} \frac{\sum_{i=1}^{n} X_i - np}{n^{1/2}(p(1-p))^{1/2}} &\xrightarrow{d} \mathcal{N}(0,1) \quad \text{and} \\ Y = \sum_{i=1}^{n} X_i &\overset{a}{\sim} \mathcal{N}(np, np(1-p)) \end{aligned}$$

Continuity correction: It has been found in practice that such approximations are improved by making a continuity correction, where each outcome x in the range of discrete random variable is associated with the interval event $[x - 0.5, x + 0.5]$. Then:

$$P(x = 20) \approx \int_{20-0.5}^{20+0.5} \mathcal{N}(20, 10)dz$$

if $n = 40$ and $p = 0.5$.

Exercise 5.10

(Convergence in distribution, in quadratic mean and in probability) Consider stochastically independent realizations of the random variable X_i from an exponential distribution with density

$$f(x) = \frac{1}{\theta}e^{-x/\theta}\mathcal{I}_{(0,\infty)}(x).$$

You can use that and $\bar{X}_n^2 = (\sum_{i=1}^{n} X_i/n)^2$ to find an estimate for θ^2.

a) Is it true that $\mathrm{E}(\bar{X}_n^2) = \theta^2$ and $\lim_{n\to\infty} \mathrm{E}(\bar{X}_n^2) = \theta^2$?

b) Does $\mathrm{plim}(\bar{X}_n^2) = \theta^2$?

c) Define the asymptotic distribution for $\bar{X}_n^2$.

[1]– see Mittelhammer p. 272.

Solution to Exercise 5.10

a)

$$\begin{aligned}
\mathrm{E}\left[\overline{X}_n^2\right] &= \mathrm{E}\left[\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right)^2\right] = \frac{1}{n^2}\mathrm{E}\left[\left(\sum_{i=1}^{n} X_i\right)^2\right] \\
&= \frac{1}{n^2}\mathrm{E}\left[\sum_{i=1}^{n} X_i^2 + \sum_{i=1}^{n}\sum_{\substack{j=1\\ i\neq j}}^{n} X_i X_j\right] = \frac{1}{n^2}\left(\sum_{i=1}^{n}\mathrm{E}[X_i^2] + \sum_{i=1}^{n}\sum_{\substack{j=1\\ i\neq j}}^{n}\mathrm{E}[X_i X_j]\right) \\
&= \frac{1}{n^2}\left(n\cdot 2\theta^2 + n(n-1)\theta^2\right) \\
&= \frac{n+1}{n}\theta^2
\end{aligned}$$

Therefore, $\mathrm{E}\left[\overline{X}_n^2\right] \neq \theta^2$ but $\lim_{n\to\infty} E\left[\overline{X}_n^2\right] = \theta^2$

$$\begin{aligned}
\text{Note,}\quad \mathrm{Var}[X_i] &= \mathrm{E}[X_i^2] - \mathrm{E}[X_i]^2 \\
\mathrm{E}[X_i^2] &= \mathrm{Var}[X_i] + \mathrm{E}[X_i]^2 = \theta^2 + \theta^2 = 2\theta^2 \\
\mathrm{Cov}[X_i X_j] &= \mathrm{E}[X_i X_j] - \mathrm{E}[X_i]\mathrm{E}[X_j] = 0 \quad \text{(due to independence)} \\
\mathrm{E}[X_i X_j] &= \mathrm{E}[X_i]\mathrm{E}[X_j] = \theta^2
\end{aligned}$$

$$\begin{aligned}
\mathrm{E}\left[\overline{X}_n^2\right] &\neq \theta^2 \quad \text{can also be shown with the help of Jensen's inequality:} \\
\mathrm{E}[g(x)] &> g(\mathrm{E}[X]) \qquad \text{if } g(x) \text{ is a strictly convex function.} \\
\mathrm{E}[g(x)] &< g(\mathrm{E}[X]) \qquad \text{if } g(x) \text{ is a strictly concave function.} \\
\text{Here,}\quad g(x) &= \overline{X}_n^2 \qquad \text{is a strictly convex function with} \qquad X = \overline{X}_n \\
\mathrm{E}[\overline{X}_n] &= \theta \\
\Rightarrow \mathrm{E}[\overline{X}_n^2] &> (\mathrm{E}[\overline{X}_n])^2 = \theta^2
\end{aligned}$$

b) It holds that $\mathrm{plim}(g(x)) = g(\mathrm{plim}(x))$. Here we have $g(x) = \overline{X}_n^2$ with $x = \overline{X}_n$. Hence,

$$\mathrm{plim}\left(\overline{X}_n^2\right) = \left(\mathrm{plim}(\overline{X}_n)\right)^2$$

Because $X_i \overset{iid}{\sim} \mathrm{Exp}(\theta)$, it follows according to WLLN that

$$\mathrm{plim}(\overline{X}_n) = \mu = \theta \quad \text{and} \quad \mathrm{plim}\left(\overline{X}_n^2\right) = \left(\mathrm{plim}(\overline{X}_n)\right)^2 = \theta^2$$

Alternatively, convergence in probability can also be shown through convergence in mean square, as the latter implies the former. Since X_i is iid,

$$\begin{aligned}
\mathrm{Var}[\overline{X}_n] &= \frac{\mathrm{Var}(X_i)}{n} = \frac{\theta^2}{n} \underset{n\to\infty}{\to} 0 \\
&\Rightarrow \overline{X}_n \overset{m}{\to} \mathrm{E}[\overline{X}_n] = \theta \\
&\Rightarrow \overline{X}_n \overset{p}{\to} \theta
\end{aligned}$$

Finally, we could have used Chebyshev's inequality in order to show $\overline{X}_n \overset{p}{\to} \theta$.

c) According to Lindberg-Levy's CLT,

$$\sqrt{n}(\bar{X} - \mu) \xrightarrow{d} \mathcal{N}(0, \sigma^2)$$

Asymptotic distribution of functions of asymptotically normal random variables (aka the Delta Method) ensures that

$$n^{1/2}\Big(g(X_n) - g(\mu)\Big) \xrightarrow{d} \mathcal{N}(0, G^2\sigma^2)$$
$$\hookrightarrow\; g(X_n) \overset{asy.}{\sim} N\Big(g(\mu), n^{-1}G^2\sigma^2\Big)$$

with $G = \frac{\partial g(\mu)}{\partial x}$ is the derivative of g with respect to x evaluated at $x = \mu$.
Applied to the exercise,

$$g(\bar{X}_n) = \bar{X}_n^2; \qquad \frac{\partial g(\bar{X}_n)}{\partial \bar{X}_n} = 2\bar{X}_n \overset{\bar{X}_n = \mu}{\longmapsto} G = 2\theta$$

$$\text{yields} \qquad n^{1/2}(\bar{X}_n^2 - \theta^2) \xrightarrow{d} \mathcal{N}\Big(0, (2\theta)^2\theta^2\Big)$$
$$\text{and thus} \qquad \bar{X}_n^2 \overset{asy}{\sim} \mathcal{N}\left(\theta^2, \frac{1}{n}4\theta^4\right)$$

Exercise 5.11

(Convergence in distribution, in quadratic mean and in probability) Consider the following random variable

$$Y_i = \beta \cdot X_i + V_i\ ,$$

where

$$|X_i| \in [a, b],\ \forall i \quad \text{is a fixed deterministic factor}$$
$$\beta \quad \text{is an unknown coefficient}$$
$$V_i \sim iid \text{ with } \mathrm{E}(V_i) = 0,\ \mathrm{E}(V_i^2) = \sigma^2 \text{ and } P(|V_i| \leq m) = 1,\ \forall i\ .$$

(a, b, m and σ are finite positive constants.) Both functions

$$B = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{Y} \qquad \text{and} \qquad B_r = (\mathbf{X}'\mathbf{X} + k)^{-1}\mathbf{X}'\mathbf{Y}\ ,$$

with $\mathbf{Y} = (Y_1, ..., Y_n)'$, $\mathbf{X} = (x_1, ..., x_n)'$ and $k > 0$ can be used to calculate estimates for β.

a) Determine the mean and the variance of the two estimators.

b) Examine whether $\lim_{n\to\infty} \mathrm{E}(B) = \beta$ and $\lim_{n\to\infty} \mathrm{E}(B_r) = \beta$

c) Examine whether the two estimators converge in mean square to β.

d) Examine whether the two estimators converge in probability to β.

e) Determine asymptotic distributions for each of the estimators.

Solution to Exercise 5.11

$$\begin{aligned}
B &= \frac{\sum x_i y_i}{\sum x_i^2} = \frac{\sum x_i(x_i\beta + \upsilon_i)}{\sum x_i^2} = \beta + \frac{\sum x_i \upsilon_i}{\sum x_i^2} \\
B_r &= \frac{\sum x_i Y_i}{\sum x_i^2 + k} = \frac{\beta \sum x_i^2}{\sum x_i^2 + k} + \frac{\sum x_i \upsilon_i}{\sum x_i^2 + k}
\end{aligned}$$

a)

$$\begin{aligned}
\mathrm{E}(B) &= \beta + \frac{\sum x_i E(\upsilon_i)(=0)}{\sum x_i} = \beta \quad \text{unbiased} \\
\mathrm{E}(B_r) &= \beta \frac{\sum x_i^2}{\sum x_i^2 + k} \neq \beta \qquad \text{biased}
\end{aligned}$$

$$\begin{aligned}
\mathrm{Var}(B) &= \mathrm{E}\Big((B-\beta)^2\Big) = \mathrm{E}\left[\frac{(\sum x_i \upsilon_i)^2}{(\sum x_i^2)^2}\right] = \frac{E(\upsilon_i)^2 \sum x_i^2}{(\sum x_i^2)^2} = \frac{\sigma^2 \sum x_i^2}{(\sum x_i^2)^2} = \frac{\sigma^2}{\sum x_i^2} \\
& \Big(\text{since } \mathrm{E}\Big[\sum_{i=1}^{n} x_i \upsilon_i\Big]^2 = \mathrm{E}\Big[\sum_{i=1}^{n}\sum_{j=1}^{n} x_i x_j \upsilon_i \upsilon_j\Big] = \sigma^2 \sum_{i=1}^{n} x_i^2\Big) \\
\mathrm{Var}(B_r) &= \mathrm{E}\Big[(B_r - \mathrm{E}(b_r))^2\Big] = \mathrm{E}\frac{(\sum x_i \upsilon_i)^2}{(\sum x_i^2 + k)^2} = \frac{\sigma^2 \sum x_i^2}{(\sum x_i^2 + k)^2}
\end{aligned}$$

b) $\mathrm{E}(B) = \beta \quad \forall n$, thus, $\lim_{n\longrightarrow\infty} \mathrm{E}(B) = \beta$

$$\begin{aligned}
\lim_{n\longrightarrow\infty} \mathrm{E}(B_r) = \lim_{n\longrightarrow\infty} \beta \left[\frac{1}{1 + \frac{k}{\sum x_i^2}}\right] = \beta, \\
\text{because} \qquad \lim_{n\longrightarrow\infty} \frac{k}{\sum x_i^2} \longrightarrow 0, \quad \lim_{n\longrightarrow\infty} \sum_{i=1}^{n} x_i^2 \longrightarrow \infty
\end{aligned}$$

Hence, B_r is asymptotically unbiased.

c) In general, Y_n converges in mean square to Y if $\lim_{n\longrightarrow\infty} \mathrm{E}(Y_n - Y)^2 = 0$.

$$\begin{aligned}
\lim_{n\longrightarrow\infty} \mathrm{E}\Big[(B-\beta)^2\Big] &= \lim_{n\longrightarrow\infty} \Big(\mathrm{E}[B^2 - 2B\beta + \beta^2]\Big) \\
&= \lim_{n\longrightarrow\infty} \mathrm{E}(B^2) - 2\beta E(B) + \beta^2 - \mathrm{E}(B)^2 + \mathrm{E}(B)^2 \\
&= \lim_{n\longrightarrow\infty} \mathrm{Var}(B) + (\mathrm{E}[B] - \beta)^2 = 0 \\
\text{because} \qquad & \lim_{n\longrightarrow\infty} Var(B) = \lim_{n\longrightarrow\infty} \frac{\sigma^2}{\sum x_i^2} = 0 \\
\text{and} \qquad & \lim_{n\longrightarrow\infty} \mathrm{E}(B) = \beta
\end{aligned}$$

For B_r, alternatively with a "helping clause":

$$\underbrace{\lim_{n\longrightarrow\infty} \mathrm{E}(B_r) = \beta; \ \lim_{n\longrightarrow\infty} \mathrm{Var}(B_r) = 0; \ \left(\frac{\sigma^2 \sum x_i^2}{(\sum x_i^2 + k)^2} \overset{n\to\infty}{\longrightarrow} 0\right)}$$

$$B_r \xrightarrow{m} \beta$$

d)

$$
\begin{aligned}
B \xrightarrow{m} \beta \quad &\implies \quad B \xrightarrow{p} \beta \\
B_r \xrightarrow{m} \beta \quad &\implies \quad B_r \xrightarrow{p} \beta
\end{aligned}
$$

e)[2] We need to find the distribution of B, where $B = \beta + \frac{\sum x_i v_i}{\sum x_i^2}$. Define $w_i = x_i v_i$, so that

$$
\begin{aligned}
\mathrm{E}[w_i] &= x_i \cdot \mathrm{E}[v_i] = 0 \\
\mathrm{Var}[w_i] &= \mathrm{E}\left[w_i^2\right] = x_i^2 \mathrm{E}\left[v_i^2\right] = x_i^2 \sigma^2 \\
\sum \mathrm{Var}\left[w_i\right] &= \sigma^2 \sum x_i^2 \overset{n \to \infty}{\to} \infty
\end{aligned}
$$

Here we will use the CLT for bounded random variables. To do so the following has to hold:

1. $\mathrm{E}\left[w_i\right] = \mu_i < \infty \quad \checkmark$
2. $\mathrm{Var}\left[w_i\right] = \sigma_i^2 < \infty \quad \checkmark$
3. $P(|w_i| < m) = 1 \quad \checkmark$
4. $\sum \mathrm{Var}(w_i) = \sigma^2 \sum x_i^2 \to \infty \quad \checkmark$

Therefore,

$$
\sqrt{n} \frac{\frac{1}{n} \sum w_i - \frac{1}{n} \sum \overbrace{\mathrm{E}[w_i]}^{=0}}{\bar{\sigma}_n} \xrightarrow{d} \mathcal{N}(0,1)
$$

The asymptotic distribution of B is thus,

$$
\begin{aligned}
\sqrt{n} \frac{\overline{w}_n}{\bar{\sigma}_n} &\overset{a}{\sim} \mathcal{N}(0,1) \\
\sqrt{n} \overline{w}_n &\overset{a}{\sim} \mathcal{N}\left(0, \bar{\sigma}_n^2\right) \\
\overline{w}_n &\overset{a}{\sim} \mathcal{N}\left(0, \frac{\bar{\sigma}_n^2}{n}\right) \\
\frac{1}{n} \sum x_i v_i &\overset{a}{\sim} \mathcal{N}\left(0, \frac{1}{n^2} \sigma^2 \sum x_i^2\right) \\
\frac{\frac{1}{n} \sum x_i v_i}{\frac{1}{n} \sum x_i^2} &\overset{a}{\sim} \mathcal{N}\left(0, \frac{1}{n^2} \sigma^2 \sum x_i^2 \frac{n^2}{1} \left(\sum x_i^2\right)^{-2}\right) \\
B = \frac{\sum x_i v_i}{\sum x_i^2} + \beta &\overset{a}{\sim} \mathcal{N}\left(\beta, \sigma^2 \left(\sum x_i^2\right)^{-1}\right)
\end{aligned}
$$

Similarly, for B_r, we need to find the distribution, given $B_r = \frac{\beta \sum x_i^2}{\sum x_i^2 + k} + \frac{\sum x_i v_i}{\sum x_i^2 + k}$. Define further $w_i = x_i v_i$. Then, analogously to B

$$
\sqrt{n} \frac{\frac{1}{n} \sum w_i - \frac{1}{n} \sum \overbrace{E[w_i]}^{=0}}{\bar{\sigma}_n} \xrightarrow{d} \mathcal{N}(0,1)
$$

The asymptotic distribution of B_r is thus,

$$\begin{aligned}
\sqrt{n}\frac{\overline{w}_n}{\bar{\sigma}_n} &\overset{a}{\sim} \mathcal{N}(0,1) \\
\sqrt{n}\overline{w}_n &\overset{a}{\sim} \mathcal{N}\left(0,\bar{\sigma}_n^2\right) \\
\overline{w}_n &\overset{a}{\sim} \mathcal{N}\left(0,\frac{\bar{\sigma}_n^2}{n}\right) \\
\frac{1}{n}\sum x_i v_i &\overset{a}{\sim} \mathcal{N}\left(0,\frac{1}{n^2}\sigma^2\sum x_i^2\right) \\
\frac{\frac{1}{n}\sum x_i v_i}{\frac{1}{n}\left(\sum x_i^2+k\right)} &\overset{a}{\sim} \mathcal{N}\left(0,\frac{1}{n^2}\sigma^2\sum x_i^2\frac{n^2}{1}\left(\sum x_i^2+k\right)^{-2}\right) \\
\frac{\sum x_i v_i}{\sum x_i^2+k} &\overset{a}{\sim} \mathcal{N}\left(0,\sigma^2\left(\sum x_i^2+2k+\frac{k^2}{\sum x_i^2}\right)^{-1}\right) \\
B_r = \frac{\sum x_i v_i}{\sum x_i^2+k}+\frac{\beta\sum x_i^2}{\sum x_i^2+k} &\overset{a}{\sim} \mathcal{N}\left(\frac{\beta\sum x_i^2}{\sum x_i^2+k},\sigma^2\left(\sum x_i^2+2k+\frac{k^2}{\sum x_i^2}\right)^{-1}\right)
\end{aligned}$$

Exercise 5.12

Let X_n be a random sequence and let a_n and b_n be nonrandom, nonnegative sequences. The orders of probability $O_p(a_n)$ and $o_p(b_n)$ are defined analogously to the orders of magnitude of a sequence,

$$X_n = O_p(a_n) \quad \text{iff} \quad \frac{X_n}{a_n} \text{ is uniformly bounded in probability,}$$

i.e. $\forall \varepsilon > 0$, $\exists C$ such that $P\left(\left|\frac{X_n}{a_n}\right| > C\right) < \varepsilon$ for all $n \in \mathbb{N}$, and

$$X_n = o_p(b_n) \quad \text{iff} \quad \frac{X_n}{b_n} \xrightarrow{p} 0.$$

1. Prove that if $\mathrm{E}\left(|X_n|\right) = O(a_n)$, then $X_n = O_p(a_n)$.
2. Prove that if $\mathrm{E}\left(X_n^2\right) = O(a_n)$, then $X_n = O_p\left(\sqrt{a_n}\right)$.
3. Prove that if $\mathrm{E}\left(X_n^2\right) = o(b_n)$, then $X_n = o_p\left(\sqrt{b_n}\right)$.
4. Argue that $f\left(X_n + O_p\left(n^{-\alpha}\right)\right) = f\left(X_n\right) + O_p\left(n^{-\alpha}\right)$ for some $\alpha > 0$ and any Lipschitz-continuous function f. Hint: A Lipschitz function f satisfies $|f(x_1) - f(x_2)| \le C\,|x_1 - x_2|$ for some $C > 0$ and all x_1, x_2.

Solution to Exercise 5.12

1. We need to show that $\frac{X_n}{a_n}$ is uniformly bounded in probability, so we must get an upper bound for the probability that $\left|\frac{X_n}{a_n}\right|$ exceeds some C. We know something about $\mathrm{E}\left(|X_n|\right)$, and we resort to Markov's inequality to transform the given upper bound for $\mathrm{E}\left(|X_n|\right)$ in suitable bounds for the desired probabilities. Concretely,

$$P\left(\left|\frac{X_n}{a_n}\right| > C\right) \le \frac{\mathrm{E}\left(\left|\frac{X_n}{a_n}\right|\right)}{C} = \frac{1}{Ca_n}\mathrm{E}\left(|X_n|\right),$$

where the condition $\mathrm{E}(|X_n|) = O(a_n)$ implies that there exists $C^* > 0$ such that

$$0 \leq \mathrm{E}(|X_n|) \leq C^* a_n.$$

Summing up,

$$P\left(\left|\frac{X_n}{a_n}\right| > C\right) \leq \frac{C^*}{C}.$$

Since we may find C such that, for any $\varepsilon > 0$ and $C^* > 0$, $\frac{C^*}{C} < \varepsilon$, uniform boundedness in probability is checked and $X_n = O_p(a_n)$ indeed.

2. The reasoning is the same, but we resort to the (generalized) Markov's inequality implying that

$$P\left(\left|\frac{X_n}{\sqrt{a_n}}\right| > C\right) \leq \frac{\mathrm{E}\left(\left|\frac{X_n}{\sqrt{a_n}}\right|^2\right)}{C^2} = \frac{1}{C^2 a_n}\mathrm{E}\left(X_n^2\right)$$

with $\mathrm{E}\left(X_n^2\right) \leq C^* a_n$ for some $C^* > 0$ thanks to the assumption $\mathrm{E}\left(X_n^2\right) = O(a_n)$. We may then find C such that, for any $\varepsilon > 0$ and $C^* > 0$, $\frac{C^*}{C^2} < \varepsilon$. Therefore, uniform boundedness in probability holds and $X_n = O_p(\sqrt{a_n})$ as required.

3. We resort (again) to the (generalized) Markov's inequality, leading like before for any $C > 0$ to

$$P\left(\left|\frac{X_n}{\sqrt{b_n}}\right| > C\right) \leq \frac{\mathrm{E}\left(\left|\frac{X_n}{\sqrt{b_n}}\right|^2\right)}{C^2} = \frac{1}{C^2 b_n}\mathrm{E}\left(X_n^2\right).$$

But $\mathrm{E}\left(X_n^2\right) = o(b_n)$ implies that $\mathrm{E}\left(X_n^2\right)/b_n \to 0$ as $n \to \infty$, and therefore

$$P\left(\left|\frac{X_n}{\sqrt{b_n}}\right| > C\right) \to 0 \ \forall C > 0, \qquad \text{i.e. } \frac{X_n}{\sqrt{b_n}} \xrightarrow{p} 0,$$

as required.

4. First, the expression $f\left(X_n + O_p\left(n^{-\alpha}\right)\right)$ is to be understood as $f\left(X_n + Y_n\right)$ where $Y_n = O_p\left(n^{-\alpha}\right)$.[3] Now, Lipschitz functions satisfy, for some positive constant C and any x_1, x_2 in the domain of f,

$$|f(x_2) - f(x_1)| \leq C\,|x_2 - x_1|.$$

Therefore,

$$|f(X_n + Y_n) - f(X_n)| \leq C\,|(X_n + Y_n) - X_n| = C\,|Y_n|.$$

Since $Y_n = O_p\left(n^{-\alpha}\right)$, i.e. $\left|\frac{Y_n}{n^{-\alpha}}\right|$ is uniformly bounded in probability, it follows immediately that $C\,|Y_n| = O_p\left(n^{-\alpha}\right)$ as well, and thus

$$|f(X_n + Y_n) - f(X_n)| = O_p\left(n^{-\alpha}\right)$$

or $f(X_n + Y_n) = f(X_n) + O_p\left(n^{-\alpha}\right)$ as required.

[3] Recall, the very reason to introduce o, O notation is to allow for such shorthand notations.

Chapter 6

Old Exams

In this chapter you will find old exams starting from winter term 2010. Please note, that the earlier exams were designed for a 2 hour period, while the latest for a 1 hour period. Some tasks are recycled in several exams, while some might be outside of the present material's scope. You are encouraged to solve them on your own.

Winter Term 2010/11

Exam WS10/11 1

Topic 1

a) Let the random variables (X, Y) have following joint probability density function (pdf):

$$f(x, y) = 3x(1 - xy)\mathcal{I}_{(0,1)}\mathcal{I}(x)_{(0,1)}(y)$$

a1) Find the marginal pdfs $f(x)$ and $f(y)$ and the cumulative density function (cdf) $F(x)$.

a2) Check if X and Y are stochastically independent.

a3) Find the conditional pdf $f(y|x)$ and its cdf $F(y|x)$.

a4) Determine the probabilities $P(X > 0.5)$, $P(X > 0.5, Y > 0.5)$ and $P(X > Y)$.

a5) Give the regression function of a regression of y on x and sketch the function for admissible values of x.

a6) Use the law of iterated expectations to obtain from $\mathrm{E}[Y|X = x]$ the unconditional expectation $\mathrm{E}[Y]$.

b) Let $X_1, \ldots X_n$ be independent and standard normally distributed random variables. Determine the distributions of the following variables and indicate the theorems you are using:

b1) $\quad Z_k = \sum_{i=1}^{k} X_i^2 \qquad \text{for } k < n$

b2) $\quad Y_1 = \delta Z_k \qquad \text{for } \delta \in (0, \infty)$

b3) $\quad Y_2 = \frac{1}{n} \sum_{i=1}^{n} \frac{X_i + a}{\sqrt{b}} \qquad \text{for } a, b \in \mathbb{R}_+$

b4) $\quad Y_3 = \frac{X_1 + \cdots + X_{n-1}}{\sqrt{X_n^2 \cdot (n-1)}}$

b5) $\quad Y_4 = \frac{(n-k) \cdot (X_1^2 + \cdots + X_k^2)}{k \cdot (X_{k+1}^2 + \cdots + X_n^2)}$

c) Let $X_1, \ldots, X_n$ denote a random sample from a population distribution with cdf $F(x)$. Consider the empirical distribution function (edf) at a certain point $t \in (-\infty, \infty)$

$$F_n(t) = \frac{1}{n} \sum_{i=1}^{n} \mathcal{I}_{(-\infty,t]}(X_i)$$

and derive its discrete probability density function.

Topic 2

a) Consider the experiment of tossing a coin and define the random variable X to be 1 if you observe a tail and 0 if you observe a head. Let p denote the probability of observing a tail.
Let

$$Y_n = \sum_{i=1}^{n} X_i$$

for n tosses of the coin.

a1) Let the coin be a fair one, i.e. $p = 0.5$. Name the distribution of Y_n and calculate its mean and variance.

a2) Give the moment generating function $M_Y(t)$ for Y_n and use it to evaluate the mean and variance to verify your results from a1).

a3) Find the probability limit of Y_n for $n \to \infty$ or explain why Y_n does not converge in probability.

a4) Let $Z_n = Y_n/n$. Determine whether Z_n converges in probability. If it does, give the probability limit.

b) Now assume you toss $k = 3$ coins, numbered 1, 2 and 3. Consider the three random variables X_j with

$$X_j = \begin{cases} 1 & \text{if coin } j \text{ lands tails} \\ 0 & \text{if coin } j \text{ lands heads} \end{cases} \qquad \text{for} \quad j = 1, 2, 3 \ ,$$

and let the probabilities of observing a tail for coin j be $p_1 = 0.6$, $p_2 = 0.5$ and $p_3 = 0.4$.

b1) Assume that you toss the three coins exactly once. Give the sample space S of the experiment and evaluate the following probabilities:

i. $P(X_3 \geq X_1 + X_2)$
ii. $P(X_1 + X_2 = 2 | x_3 = 0)$
iii. $P(X_1 \leq X_2 \leq X_3)$

b2) Now assume that you toss the three coins n times. Let

$$Y_{n,j} = \sum_{i=1}^{n} X_{i,j}$$

denote the sum of tails observed for coin j. Find the probability limits of the following random variables, or explain why they do not exist:

i. $S_n = \left(\frac{1}{n} \sum_{j=1}^{3} Y_{n,j} \right)^2$ ii. $A_n = \frac{Y_{n,1}}{Y_{n,2}}$

c) Let (X_1, X_2) be a bivariate random variable having a probability density $\mathcal{N}(\mu, \Sigma)$ with

$$\mu = \begin{pmatrix} 5 \\ 8 \end{pmatrix} \quad \text{and} \quad \Sigma = \begin{pmatrix} 2 & -1 \\ -1 & 3 \end{pmatrix} .$$

c1) Give the marginal densities for X_1 and X_2.

c2) Give the conditional density of X_1 given $X_2 = x_2$.

Topic 3

a) Let X_1 and X_2 be identically independently exponentially distributed with probability density function

$$f(x; \lambda) = \lambda e^{-\lambda x} \mathcal{I}_{(0,\infty)}(x) .$$

a1) Give the density of

$$Y = 1 - e^{-\lambda X_1} .$$

a2) Give the joint density of

$$Z_1 = X_1 + X_2 \quad \text{and} \quad Z_2 = \frac{X_1}{X_1 + X_2} .$$

b) Let $X_1, \ldots X_n$ be independent and identically distributed with probability density function

$$f(x; \lambda) = \lambda e^{-\lambda x} \mathcal{I}_{(0,\infty)}(x) .$$

Find the asymptotic distribution of

$$S = \left(\frac{1}{n} \sum_{i=1}^{n} X_i \right)^2 .$$

c) Let $X_1, \ldots X_n$ be independent and identically distributed with probability density function

$$f(x; \lambda) = \frac{e^{-\lambda} \lambda^x}{x!} \mathcal{I}_{\{0,1,2,3,\ldots\}}(x) .$$

c1) Determine whether the joint pdf $f(x_1, \ldots, x_n)$ is a member of the exponential family of distributions.

c2) Derive the moment generating function $M_X(t)$ of X.

c3) Let

$$W_n = \frac{1}{n} \sum_{i=1}^{n} X_i .$$

Find the asymptotic distribution of W_n.

c4) Let

$$V_n = \frac{\sum_{i=1}^{n} X_i - n\lambda}{\sqrt{n\lambda}} .$$

Find the moment generating function $M_V(t)$ of V and examine its behavior as $n \to \infty$. Determine the asymptotic distribution of V_n from your result.

Hint: Use the Taylor series expansion

$$e^x = \sum_{i=0}^{\infty} \frac{x^i}{i!} .$$

Exam WS10/11 2

Topic 1

a) Let the random variables (X, Y) have following joint probability density function (pdf):

$$f(x, y) = \frac{1}{12}(x + xy + y)\mathcal{I}_{(0,2)}(x)\mathcal{I}_{(0,2)}(y) \ .$$

a1) Find the cumulative density function (cdf) $F(x, y)$.

a2) Find the marginal pdf $f(x)$ and its cdf $F(x)$.

a3) Check if X and Y are stochastically independent.

a4) Find the conditional pdf $f(y|x)$ and its cdf $F(y|x)$.

a5) Give the regression function of a regression of y on x and sketch the function for admissible values of x.

a6) Find two expressions for the probability $P(X > Y)$ and evaluate one of them.

b) Let $X_1, \ldots, X_n$ be independent and identically distributed with probability density function

$$f(x; \lambda) = \lambda e^{-\lambda x}\mathcal{I}_{(0,\infty)}(x), \qquad \lambda > 0 \ .$$

Define

$$Z_1 = X_1 + X_2 \qquad \text{and} \qquad Z_2 = X_1 - X_2 \ .$$

b1) Derive the moment generating function (MGF) $M_X(t)$ of X. Verify that $M_X(0)$ exists and takes the value 1. Use the MGF to find the expected value $\mathrm{E}[X]$ and the variance $\mathrm{Var}(X)$.

b2) Derive the moment generating function $M_{\mathbf{Z}}(t)$ for $\mathbf{Z} = (Z_1, Z_2)$.

b3) Give the conditional pdf $f(z_1|x_2)$ and its expectation $\mathrm{E}[Z_1|X_2 = x_2]$.

Topic 2

a) Let the random variable X be distributed according to the pdf $f(x)$ with $x > 0$.

a1) Suppose you know that $\mathrm{E}[X] = 8$. Give an estimate for the probability $P(X < 16)$.

a2) Suppose you also know that X cannot take negative values nd that $\mathrm{Var}(X) = 32$. Include this information and re-estimate the probability $P(X < 16)$.

a3) Suppose you know that X follows a Gamma distribution with pdf

$$\frac{1}{\beta^\alpha \Gamma(\alpha)} x^{\alpha-1} e^{-x/\beta}\mathcal{I}_{(0,\infty)}(x) \ ,$$

with $\alpha = 2$ and $\beta = 4$ and include the information from before.
Give the exact probability $P(X < 16)$.

b) Let the sequences of random variables $\{X_n\}$ and $\{Y_n\}$ each follow Normal distributions with

$$X_n \sim \mathcal{N}\left(\mu + \frac{1}{n}, \frac{n\sigma^2 + 2}{n}\right) \qquad \text{and} \qquad Y_n \sim \mathcal{N}\left(\mu, \frac{1}{n}\right) \ .$$

b1) Examine which type of convergence applies for X_n and Y_n and give their asymptotic distribution or probability limit.

b2) Find the asymptotic distributions of

$$A_n = X_n \cdot Y_n \qquad \text{and} \qquad B_n = X_n - Y_n \ .$$

c) Let the random variables $X_1, \ldots, X_n$ each be exponentially distributed with

$$f(x_i; \lambda_i) = \lambda_i e^{-\lambda_i x_i} \mathcal{I}_{(0,\infty)}(x_i), \qquad \lambda_i > 0 \ .$$

Assume that

$$\sum_{i=1}^{n} \lambda_i^{-2} \to \infty \quad \text{as} \quad n \to \infty \ .$$

and let

$$Y_n = \frac{1}{n} \sum_{i=1}^{n} X_i \ .$$

c1) Examine the convergence properties of Y_n and find its asymptotic distribution, if it exists.

c2) Now assume that $\lambda_i = \lambda \ \forall \ i$. Find the asymptotic distributions of

$$Z_n = n \cdot Y_n \qquad \text{and} \qquad W_n = \sqrt{Y_n} \ ,$$

if they exist.

d) Let the random variables $X_1, \ldots, X_n$ be independent and identically distributed with $\mathrm{E}[X_i] = \mu$ and $\mathrm{Var}(X) = \sigma^2 \ll \infty \quad \forall \ i$.
Let

$$Y = \frac{1}{n(n+1)} \sum_{i=1}^{n} i X_i \ .$$

Examine Y with regard to convergence in probability.

Hint: Consider that $\displaystyle\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$.

Topic 3

a) Let the random variables $X_1, \ldots, X_n$ be independent and identically distributed with pdf

$$f(x; \mu, \sigma) = \frac{1}{x\sigma\sqrt{2\pi}} \exp\left(-\frac{(\ln x - \mu)^2}{2\sigma^2}\right) \qquad x \ \in \ (-\infty, \infty) \ .$$

a1) Determine whether the pdf is a member of the exponential family of distributions.

a2) Use the change of variables technique to determine the pdf of $Y = \ln X$.

b) Let the random variables $X_1, \ldots, X_n$ be independent and identically Gamma distributed with pdf

$$\frac{1}{\beta^\alpha \Gamma(\alpha)} x^{\alpha-1} e^{-x/\beta} \mathcal{I}_{(0,\infty)}(x) \ , \qquad \alpha \in \mathbb{N}, \beta > 0 \ .$$

b1) Use the moment generating function $M_X(t)$ to determine the distributions of

$$Y_1 = \sum_{i=1}^{n} X_i \qquad \text{and} \qquad Y_2 = \frac{1}{2n} \sum_{i=1}^{n} X_i \ .$$

Name the two theorems that you have proven by deriving the distributions of Y_1 and Y_2.

b2) Derive the distribution of

$$Z_1 = \frac{X_1}{X_2}\ .$$

Hint: Use the auxiliary variable $Z_2 = X_2$ and start with the joint distribution of Z_1 and Z_2.

c) Let (X_1, X_2) be a bivariate random variable having a probability density $\mathcal{N}(\mu, \Sigma)$ with

$$\mu = \begin{pmatrix} -1 \\ 5 \end{pmatrix} \quad \text{and} \quad \Sigma = \begin{pmatrix} 3 & -2 \\ -2 & 8 \end{pmatrix}\ .$$

c1) Give the marginal densities for X_1 and X_2.

c2) Give the conditional density of X_1 given $X_2 = x_2$.

d) Let the random variables $X_1, \ldots, X_n$ be independent and identically distributed with probability density $\mathcal{N}(1, \sigma^2)$. Determine the distribution of the following variables and indicate the theorems you are using:

d1) $Y_1 = \sum_{j=1}^{k} X_j \qquad \text{with} \quad k < n-1$

d2) $Y_2 = \dfrac{Y_1 - k}{\sqrt{k}}$

d3) $Y_3 = \left(\dfrac{Y_2}{\sigma}\right)^2 + \left(\dfrac{X_{k+1} - 1}{\sigma}\right)^2$

d4) $Y_4 = \dfrac{\sqrt{2}X_{k+2} - \sqrt{2}}{\sigma} / \sqrt{Y_3}$

Winter Term 2011/12

Exam WS11/12 1

Topic 1

a) Let the random variables (X, Y) have the following joint probability density function (pdf) with parameter α:

$$f(x, y; \alpha) = [1 + \alpha(2x - 1)(2y - 1)]\ \mathcal{I}_{(0,1)}(x)\mathcal{I}_{(0,1)}(y).$$

a1) Find the range for the parameter α, which ensures that $f(x, y)$ is indeed a probability density function.

a2) Find the marginal pdfs $f(x)$ and $f(y)$ and the joint cumulative density function (cdf) $F(x, y)$.

a3) Check if X and Y are stochastically independent.

a4) Check whether the probability density function $f(x, y)$ is a member of the exponential class of densities.

a5) Find the conditional pdf $f(y|x)$ and the according cdf $F(y|x)$.

a6) Give the regression function of a regression of x on y and use it to derive $\mathrm{E}[X]$.

a7) Write the probability $P(X > Y)$ as an integral over the joint density function $f(x, y; \alpha)$. **Note**: you do not need to evaluate the integral!

b) b1) A fair die is rolled repeatedly until a 6 shows up. Let S_k be the event that a 6 will show up for the first time at the kth throw. Give $P(S_k)$ and show that the event S: "a 6 will eventually show up" is certain to occur.

b2) Let C and D be two events such that $P(C) = p_1 > 0$, $P(D) = p_2 > 0$ and $p_1 + p_2 > 1$. Show that $P(D|C) \geq 1 - [(1 - p_2)/p_1]$.

b3) Prove that $P(A|B) = P(B|A)P(A)/P(B)$ whenever $P(A)P(B) \neq 0$.

b4) Show that if $P(A|B) > P(A)$, then $P(B|A) > P(B)$.

Topic 2

a) Consider the experiment of independently tossing two fair coins and rolling two fair dice. Let the random variables X_1 and X_2 represent whether heads (x_i=1) or tails (x_i=0) appear on the first and second coins, respectively. Let the random variables Y_1 and Y_2 represent the number of dots facing up on each of the two dice, respectively.

a1) Derive the joint pdf of X_1, X_2, Y_1 and Y_2.

a2) Consider the random vectors

$$\mathbf{U} = \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} = \begin{bmatrix} X_1 + X_2 \\ X_1 X_2 \end{bmatrix}, \quad \mathbf{V} = \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} Y_1/Y_2 \\ Y_1 - Y_2 \end{bmatrix}.$$

Check whether the random vectors $\mathbf{U}$ and $\mathbf{V}$ are stochastically independent. Justify your answer and name the according theorem!

a3) Are the elements of the random vector $\mathbf{V}$ stochastically independent? Justify your answer!

a4) Derive the joint pdf of U_1 and U_2.

a5) Derive the covariance matrix of $\mathbf{U}$.

b) The moment generating function (MGF) of a random variable X is given by

$$M_X(t) = \left(1 - \frac{t}{4}\right)^{-3} \quad \text{for } t < 4.$$

b1) Use the MGF $M_X(t)$ to derive the mean and the variance of the random variable X.

b2) Give an upper bound for the probability $P(X^2 \geq 3)$.

b3) Derive the skewness of the random variable X.

b4) Give the exact form of the probability density function for X.

b5) Consider a random variable Y, which is stochastically independent of X and shares the same distribution. Use the according MGF to derive the distribution of the random variable $Z = \delta(X + Y)$, where δ is a positive constant.

c) Let $\mathbf{X}$ be a 2×1 discrete random vector and denote its components by X_1 and X_2. Let the support of $\mathbf{X} = [X_1\ X_2]'$ be $R(\mathbf{X}) = \{[1\ 1]', [2\ 0]', [0\ 0]'\}$. The according probability density function (pdf) is given by

$$f_{\mathbf{X}}(\mathbf{x}) = \begin{cases} 1/3, & \text{if } \mathbf{x} = [1\ 1]' \\ 1/3, & \text{if } \mathbf{x} = [2\ 0]' \\ 1/3, & \text{if } \mathbf{x} = [0\ 0]' \\ 0 & \text{otherwise.} \end{cases}$$

c1) Discuss the existence of moments for the random variable $\mathbf{X}$.

c2) Derive the joint moment generating function of $\mathbf{X}$.

Topic 3

a) Let $X_1, \ldots X_n$ be independent and identically Gamma distributed with probability density function

$$f(x; \alpha, \beta) = \frac{1}{\beta^\alpha \Gamma(\alpha)} x^{\alpha-1} e^{-x/\beta}\, \mathcal{I}_{(0,\infty)}(x).$$

Furthermore let $Z_n = \exp\{\overline{X}_n^2\}$, where $\overline{X}_n = \frac{1}{n}\sum_{i=1}^n X_i$.

a1) Derive the asymptotic distribution of $\overline{X}_n$.

a2) Derive the exact distribution of $\overline{X}_n$ and compare your result with the asymptotic distribution derived in a1).

a3) Derive the asymptotic distribution of Z_n and discuss convergence in probability of Z_n.

a4) Derive the joint pdf of the random variables $Y_1 = 1/X_1$ and $Y_2 = X_1 + X_2$.

b) Let $\{X_n\}$ be a sequence of independently normal distributed random variables with $\mathrm{E}[X_n] = 1/n$ and $\mathrm{Var}[X_n] = 1/(1 + 2/n)$. Furthermore let $\{Y_n\}$ be a sequence of independently normal distributed random variables with $\mathrm{E}[Y_n] = 2 + 1/n$ and $\mathrm{Var}[Y_n] = 1/n$.

b1) Show that X_n^2 converges in distribution to a χ^2 distribution with one degree of freedom.

b2) Discuss convergence in distribution for the random variables

i) $P_n = Y_n X_n$,

ii) $Q_n = X_n^2 / Y_n$.

Name the according limiting distributions!

c) Let $X_1, \dots, X_n$ be a random sample from a normal distribution with mean μ and variance σ^2. Then the t-statistic is given by

$$T = \frac{\bar{X}_n - \mu}{\sqrt{\hat{\sigma}_n^2 / n}},$$

where $\bar{X}_n = \frac{1}{n} \sum_{i=1}^n X_i$ and $\hat{\sigma}_n^2 = \frac{1}{n-1} \sum_{i=1}^n (X_i - \bar{X}_n)^2$.

c1) Show that the random variable T follows a t-distribution with $\nu = n - 1$ degrees of freedom.

c2) Show that the t-distribution converges in distribution to a standard normal distribution for $\nu \to \infty$.

c3) Let $1 < m < n$. Consider the random variable

$$U = \frac{\frac{1}{m-1} \sum_{i=1}^m (X_i - \frac{1}{m} \sum_{i=1}^m X_i)^2}{\frac{1}{n-m-1} \sum_{i=m+1}^n (X_i - \frac{1}{n-m} \sum_{i=m+1}^n X_i)^2}.$$

Derive the distribution of U.

Exam WS11/12 2

Topic 1

a) Let the random variable X have the following probability density function (pdf) with parameter θ:

$$f(x; \theta) = \theta \cdot 2^\theta x^{-(\theta+1)} \, \mathcal{I}_{(2,\infty)}(x),$$

where $\theta > 0$.

a1) Show that $f(x; \theta)$ is indeed a probability density function.

a2) Find the cumulative distribution function (cdf) $F(x)$ and derive the median of X.

a3) Check whether the probability density function $f(x)$ is a member of the exponential class of densities.

a4) Derive the mean and the variance of X and discuss conditions for their existence.

b) The joint probability density function of the random variable (X_1, X_2) is given by

$$f(x_1, x_2) = \begin{cases} \frac{1}{10}(x_1 + x_2), & \text{if } 0 < x_1 < 1, \ 0 < x_2 < 4 \\ 0, & \text{otherwise.} \end{cases}$$

b1) Derive the joint cdf $F(x_1, x_2)$.

b2) Calculate the probability $P(X_1 > X_2)$.

b3) Derive $\mathrm{E}(e^{X_2})$.

b4) Give the regression function of a regression of X_1 on X_2 and use it to derive $\mathrm{E}[X_1]$.

Topic 2

a) Let $P(A)$ be a probability set function with the event space Υ with $A_i \in \Upsilon$ $(i = 1, \ldots, r)$. Show via complete induction that the following relationship holds:

$$\text{Given } A_i \cap A_j = \emptyset \; \forall i \neq j \quad \text{then} \quad P(\cup_{i=1}^{r} A_i) = \sum_{i=1}^{r} P(A_i).$$

b) Let $X_1, \ldots, X_n$ be independent and identically distributed discrete random variables with probability density function (pdf)

$$f(x; p) = \frac{(1-p)^x}{-x \ln(p)} \, \mathcal{I}_{\{1,2,3,\ldots\}}(x),$$

where $p \in (0, 1)$.

b1) Show that the moment generating function (MGF) $M_X(t)$ is given by

$$M_X(t) = \frac{\ln(1 - (1-p)e^t)}{\ln p}$$

for $t < -\ln(1-p)$.

b2) Use the MGF $M_X(t)$ to derive the mean and the variance of X.

b3) Find the MGF of

i) $P = \frac{1}{n} \sum_{i=1}^{n} X_i$

ii) $Q = 2 + \sum_{i=1}^{3} \frac{X_i}{i}$.

c) Let X_1 and X_2 be independent and identically normally distributed random variables with mean $\mu = 1$ and variance $\sigma^2 = 1$.

c1) Consider the random variables $Y_1 = X_2 + X_1$ and $Y_2 = X_2 - X_1$. Derive the joint moment generating function (MGF) of Y_1 and Y_2 and show that Y_1 and Y_2 are independent random variables.

c2) Find the distribution of

i) $P = \frac{(X_2+X_1-2)^2}{(X_2-X_1)^2}$

ii) $Q = \frac{(X_2-X_1)}{(X_2+X_1-2)}$.

c3) Now assume that the random variables X_1 and X_2 are correlated with $\rho = 0.6$. Find the conditional distribution of X_1 given $x_2 = 2$.

Topic 3

a) Consider the random variables X_1 and X_2 with joint density function

$$f(x_1, x_2) = 4x_1 x_2 e^{-(x_1^2 + x_2^2)} \, \mathcal{I}_{(0,\infty)}(x_1) \mathcal{I}_{(0,\infty)}(x_2).$$

a1) Consider the random variable $Z = X_1^2$. Find the probability density function (pdf) $f(z)$ and state the mean and the variance of Z.

Hint: You do not need to derive $\mathrm{E}(Z)$ and $\mathrm{Var}(Z)$.

a2) Derive the probability density function (pdf) of the random variable $Y = \sqrt{X_1^2 + X_2^2}$.

Hint: Introduce the auxiliary variable $S = X_2$.

b) Let the random variable X_n follow an exponential distribution with probability density function (pdf)

$$f(x_n; \theta_n) = \frac{1}{\theta_n} e^{-\frac{x_n}{\theta_n}} \, \mathcal{I}_{(0,\infty)}(x_n),$$

where $\theta_n = 1 + n^{-1}$.

b1) Discuss convergence in distribution for X_n.

b2) Show that $X_n - X = (\theta_n - 1)X$, where X follows an exponential distribution with parameter $\theta = 1$.

b3) Derive the cumulative distribution function (cdf) of $U = X_n - X$ and use it to show that X_n converges in probability to X.

c) Let $X_1, \ldots, X_n$ be independent and identically distributed random variables with cumulative distribution function (cdf) $F(x)$ and

$$Y_n = \sum_{i=1}^{n} \mathcal{I}_{(-\infty,t]}(X_i), \quad t \in (-\infty, \infty).$$

c1) Show that Y_n follows a Binomial distribution with parameters n and $p = F(t)$ and find the mean and the variance of

$$Z_n = \frac{1}{n} Y_n.$$

c2) Check convergence in probability of Z_n and state (if possible) the respective probability limit.

c3) Derive the asymptotic distribution of Z_n and $\exp\{-Z_n\}$.

Winter Term 2012/13

Exam WS12/13 1

Topic 1

a) Let the random variables (X_1, X_2) have the following joint probability density function (pdf):

$$f(x_1, x_2) = \frac{1}{2}(4x_1x_2 + 1)\ \mathcal{I}_{(0,1)}(x_1)\mathcal{I}_{(0,1)}(x_2).$$

- a1) Find the marginal pdfs $f(x_1)$ and $f(x_2)$. Check if X_1 and X_2 are stochastically independent.
- a2) Find the covariance matrix of the random vector $\mathbf{X} = (X_1, X_2)'$.
- a3) Derive the joint cumulative density function (cdf) $F(x_1, x_2)$ and the marginal cdfs $F(x_1)$ and $F(x_2)$.
- a4) Give the regression function of a regression of x_1 on x_2. Sketch the regression function.
- a5) Derive the unconditional mean $\mathrm{E}[X_1]$ from the regression function obtained in a4).

b) Consider the discrete random variable X with pdf

$$f(x) = \frac{1}{6}\mathcal{I}_{\{-3,3\}}(x) + \frac{2}{3}\mathcal{I}_{\{0\}}(x).$$

- b1) Derive the moment generating function (MGF) of X.
- b2) Use the MGF in order to derive $\mathrm{E}[X^r]$ for arbitrary $r = 1, 2, \ldots$. Distinguish the cases where r is even or odd.
- b3) Find the standard deviation of X.

Topic 2

a) Consider a probability space $\{S, Y, P\}$ and let A and B be elements of the event space Y.

- a1) Show that if the events A and B are independent, then events A and $\overline{B}$ are also independent.
- a2) Show that $P(A \cap B) \geq 1 - P(\overline{A}) - P(\overline{B})$ (Bonferroni's Inequality).

  Hint: Apply DeMorgan's Law: $P(\overline{A \cap B}) = P(\overline{A} \cup \overline{B})$.
- a3) Assume $P(B) \neq 0$ and show that the conditional probability

  $$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

  defines a probability set function with domain Y.

  Hint: Show that $P(A|B)$ adheres to the Kolmogorov Axioms 1-3.
- a4) Assume that A and B are disjoint. Does this imply that A and B are independent?

b) Consider an asset which is traded on the time interval $[0, t]$ at a fictitious stock exchange. Assume that the "occurrence" of buy orders follows a Poisson process with intensity γ.

b1) Give the probability that no buy order occurs in $[0, t]$.

b2) Show that the distribution of the random variable X, which is the time elapsing until the first buy order occurs, is an exponential distribution with parameter $\theta = \frac{1}{\gamma}$.

b3) Show that the distribution derived in b2) obtains as a special case of the gamma distribution.

b4) Prove the "memoryless property" of the exponential distribution.

c) Let X_1, X_2 be a random sample from a normal distribution with mean $\mu = 1$ and variance $\sigma^2 = 1$. Derive the distribution of

c1)
$$Y = \frac{(X_1 + X_2 - 2)^2}{(X_2 - X_1)^2}$$

c2)
$$Z = \frac{X_2 - X_1}{X_1 + X_2 - 2}.$$

Topic 3

a) Let $\mathbf{X} = (X_1, X_2)'$ be a bivariate random variable which follows a $\mathcal{N}(\mu, \Sigma)$ distribution with

$$\mu = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} \quad \text{and} \quad \Sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{bmatrix}.$$

Furthermore consider the random vector $\mathbf{Z} = (Z_1, Z_2)'$, where

$$Z_1 = \frac{X_1 - \mu_1}{\sqrt{\sigma_{11}}} \quad \text{and} \quad Z_2 = \frac{X_2 - \mu_2}{\sqrt{\sigma_{22}}}.$$

a1) Give the marginal density for X_1 and derive the moment generating function (MGF) of X_1.

a2) Use the joint MGF of Z_1 and Z_2 in order to derive the joint distribution of Z_1 and Z_2.

a3) Now employ the change of variables technique" in order to derive the joint distribution of Z_1 and Z_2.

b) Let $\{X_n\}$ be a sequence of independent random variables with pdf

$$f_n(x) = \begin{cases} n^2 x, & \text{for } 0 \leq x \leq \frac{1}{n} \\ 2n - n^2 x, & \text{for } \frac{1}{n} < x \leq \frac{2}{n} \\ 0 & \text{else.} \end{cases}$$

b1) Derive the mean $\mathrm{E}[X_n]$ and the variance $\mathrm{Var}[X_n]$.

b2) Consider the random variable $Y_n = \frac{1}{n} \sum_{i=1}^{n} X_i$ and analyze convergence in mean square, convergence in probability and convergence in distribution.

Hint: $\sum_{i=1}^{n} \frac{1}{i} < 1 + \ln n$.

c) Let $X_1, \ldots, X_n$ be a random sample from a chi-square distribution with pdf

$$f(x;\nu) = \frac{1}{2^{\nu/2}\Gamma(\nu/2)} x^{(\nu/2)-1} e^{-x/2} \mathcal{I}_{(0,\infty)}(x), \quad \nu > 0.$$

Consider the arithmetic mean $\bar{X}_n = \frac{1}{n}\sum_{i=1}^n X_i$ which is used in order to obtain with $\bar{X}_n^2$ an estimator for ν^2.

c1) Find the distribution of the random variable $\bar{X}_n = \frac{1}{n}\sum_{i=1}^n X_i$. Show that for large n this distribution can be approximated by a normal distribution.

c2) Check if $\mathrm{E}[\bar{X}_n^2] = \nu^2$ and $\lim_{n\to\infty} \mathrm{E}[\bar{X}_n^2] = \nu^2$.

c3) Derive the asymptotic distribution of $\bar{X}_n^2$.

Exam WS12/13 2

Topic 1

a) Let the joint probability density function (pdf) of the bivariate random variable (X, Y) be given by

$$f(x,y) = \frac{1}{y} e^{-x/y} e^{-y}\ \mathcal{I}_{(0,\infty)}(x)\ \mathcal{I}_{(0,\infty)}(y).$$

a1) Find the marginal density function $f(y)$ and the according cumulative distribution function (cdf).

a2) Derive the conditional density function $f(x|y)$ and the according cdf.

a3) Derive the moment generating function (MGF) for $f(x|y)$.

a4) Give the regression function of a regression of x on y.

a5) Check whether the joint density function $f(x, y)$ belongs to the exponential class of densities.

a6) Give two integrals for the probability $P(X > Y)$ and solve one of the integrals.

a7) Are the random variables X and Y independent? Justify your answer!

b) Let the random variables X_1, X_2 and X_3 have the following moment generating functions (mgfs)

i) $M_{X_1}(t) = (1 - \frac{t}{2})^{-1}, \quad t < 2$

ii) $M_{X_2}(t) = \frac{e^{2t}-1}{2t}$

iii) $M_{X_3}(t) = e^{2(e^t-1)}$.

Calculate the probability $P(X_i \leq 1)$ for $i = 1, 2, 3$.

Topic 2

a) An urn contains three red and two green balls. Two balls are drawn successively

(a) with replacement (b) without replacement.

Define the the following random variables for $i = 1, 2$:

$$X_i = \begin{cases} 0, & \text{if ball no. } i \text{ is green} \\ 1, & \text{if ball no. } i \text{ is red.} \end{cases}$$

Derive the joint density function for X_1 and X_2 as well as the covariance for cases (a) and (b).

b) A fair die is tossed n times. The binary random variable X_i takes the value one if the i'th toss shows heads, otherwise X_i takes the value zero.

   b1) Use an appropriate approximation in order to obtain an upper bound for the probability that the arithmetic mean of the X_i's exceeds 0.6.

   b2) Use an appropriate approximate distribution in order to calculate the probability considered in b1). Assume a sample size of $n = 100$ tosses.

   b3) Give an expression for the exact value of the probability considered in b2).

   b4) Use an appropriate approximation in order to derive the number of tosses n which are necessary in order to obtain an arithmetic mean of the X_i's which lies between 0.49 and 0.51 with a probability of at least 0.95.

c) Let $\{X_i,\ i = 1, \dots, n\}$ be independent $\mathcal{N}(0,1)$ distributed random variables. Consider the following functions:

$$\bar{X}_1 = \frac{1}{k}\sum_{i=1}^{k} X_i \quad \text{and} \quad \bar{X}_2 = \frac{1}{n-k}\sum_{i=k+1}^{n} X_i, \quad 1 \le k < n.$$

Derive the distributions of

   c1) $S_1 = k\bar{X}_1^2$

   c2) $S_2 = X_1/\sqrt{(n-k)\bar{X}_2^2}$

   c3) $S_3 = k\bar{X}_1^2/(n-k)\bar{X}_2^2$.

Topic 3

a) Let the random variables X_1 and X_2 be iid exponentially distributed with pdf

$$f(x;\lambda) = \lambda e^{-\lambda x}\mathcal{I}_{(0,\infty)}(x), \quad \lambda > 0.$$

   a1) Give the joint density function of X_1 and X_2.

   a2) Set $\lambda = 1$. Derive the density function of the random variable $U = 1 - e^{-X_1}$.

   a3) Set $\lambda = 1$. Derive the joint density function of

$$Y_1 = X_1 + X_2 \quad \text{and} \quad Y_2 = \frac{X_1}{X_1 + X_2}.$$

   a4) Check whether Y_1 and Y_2 are stochastically independent. Give the marginal distributions of Y_1 and Y_2.

b) Assume that the series of random variables $\{Y_n\}$ converges for $n \to \infty$ in mean square to the random variable Y. Use Markov's inequality in order to show that $\{Y_n\}$ also converges in probability to Y.

c) The series of random variables $\{Y_n\}$ has the density function

$$f_n(y) = \frac{n}{\theta}\left(\frac{y}{\theta}\right)^{n-1}\mathcal{I}_{[0,\theta]}(y), \quad \theta > 0.$$

   c1) Give the cdf $F_n(y)$ of Y_n.

   c2) Derive $\mathrm{E}[Y_n]$ and $\mathrm{Var}[Y_n]$.

   c3) Check if $\{Y_n\}$ converges in probability for $n \to \infty$.

   c4) Check for $\{Y_n\}$ convergence in distribution. Sketch the behavior of $F_n(y)$ for $n \to \infty$.

Winter Term 2013/14

Exam WS13/14 2

Topic 1

a) Let the random variables (X, Y) have following joint probability density function (pdf):

$$f(x, y) = \frac{4}{3}\left(xy + \frac{x^2}{2}\right)\mathcal{I}_{(0,1)}(x)\mathcal{I}_{(0,2)}(y) \ .$$

- a1) Find the cumulative distribution function (cdf) $F(x, y)$.
- a2) Find the marginal pdf $f(x)$ and its cdf $F(x)$.
- a3) Check if X and Y are stochastically independent.
- a4) Find the conditional pdf $f(x|y)$ and its cdf $F(x|y)$.
- a5) Give the regression function of a regression of x on y.

b) Let X_1 and X_2 be random variables with a joint moment generating function (MGF):

$$M_{X_1,X_2}(t_1, t_2) = \frac{\lambda^5}{(\lambda - t_1)(\lambda^2 - t_2)^2}e^{at_1}$$

with

$$\lambda > 0, a \geq 0, t_1 < \lambda, t_2 < \lambda^2 \ .$$

- b1) Derive the marginal moment generating functions $M_{X_1}(t_1)$ and $M_{X_2}(t_2)$ of X_1 and X_2, respectively.
- b2) Which well-known distribution family does X_2 belong to?
- b3) Check if X_1 and X_2 are stochastically independent and give their correlation coefficient $\rho_{1,2}$ and the conditional expectation $\mathrm{E}[X_2|X_1]$.
  Hint: You don't need to derive the respective pdf's!

c) Let the random variable X be uniformly distributed on the interval $(0, 1)$ and $Y = \lambda \ln(X)$, $\lambda < 0$. Give the probability density function (pdf) of Y using the transformation technique.

Topic 2

a) You are in a court of law. Let G be an event that the defense attorney holds a good final speech. It is known that $P(G) = 0.6$. Let N be an event that the jury finds the defendant A not guilty.

- a1) It holds that $P(N|\overline{G}) = 0.2$ and $P(N|G) = 0.4$. Interpret these probabilities.
- a2) Find the probability that A is found not guilty.
- a3) Are the events G and N stochastically independent?
- a4) Given A is found not guilty, what is the probability that the defense attorney has held a good final speech?
- a5) Find and interpret $P(G \cap N)$.
- a6) Simplify, calculate and interpret $1 - P[(G \cap N) \cup (G \cap \overline{N}) \cup N]$.

a7) There are 5 people selected for the jury, 3 of whom are women. How many possible seating arrangements are there for the 2 men, given there is a total of 5 free seats for the jury?

b) Let A and B be two independent events with $P(A) > 0$ and $P(B) > 0$. Can A and B be disjoint? Justify your answer formally!
Hint: Give the definitions of independence and disjointness!

Topic 3

a) Let $\{U_1, U_2, ..., U_n\}$ be a sequence of stochastically independent random variables from an exponential distribution with

$$f(U;\theta) = \theta \cdot e^{-\theta U}\mathcal{I}_{(0,\infty)}(U), \quad \theta > 0 \ .$$

Define

$$Z_n = \frac{1}{n}\sum_{i=1}^{n} U_i \ .$$

a1) Show that $Z_n \xrightarrow{p} c$, where c is a constant, and give c.

a2) Give the asymptotic distribution of Z_n.

a3) Give the asymptotic distribution of $Y_n = \exp\{-Z_n\}$.

b) Let the random variables X_1 and X_2 be independently and identically distributed with a probability density function (pdf):

$$f(X_i) = \frac{1}{{X_i}^2}\mathcal{I}_{(1,\infty)}(X_i)$$

with $i = 1, 2$.
Find the probability density function of $Y_1 = \sqrt{X_1 X_2}$. Use $Y_2 = X_2$ as an auxiliary variable. Make sure to adjust the indicator functions for the new random variables.
Hint: Recall $\int \frac{1}{x}\, dx = \ln(x)$.

c) Let the random variable X_i following a Poisson distribution

$$f(x) = \frac{e^{-\lambda}\lambda^x}{x!}\mathcal{I}_{\{0,1,2,\ldots\}}(x)$$

have stochastically independent realizations. The latter are used to estimate λ^2 by means of $\overline{X}_n^2 = (\sum_{i=1}^{n} X_i/n)^2$.

c1) Check if $\mathrm{E}(\bar{X}_n^2) = \lambda^2$ and $\lim_{n\to\infty} \mathrm{E}(\overline{X}_n^2) = \lambda^2$ are fulfilled.
Hint: Use $\left(\sum_{i=1}^{n} x_i\right)^2 = \sum_{i=1}^{n} {x_i}^2 + \sum\sum_{i\neq j} x_i x_j$ to simplify.

c2) Does $\mathrm{plim}(\overline{X}_n^2) = \lambda^2$ hold?

c3) Give the asymptotic distribution of $\overline{X}_n^2$.

Winter Term 2014/15

Exam WS14/15 1

Topic 1
Let the random variables (X, Y) have following joint probability density function (pdf):

$$f(x, y) = \frac{1}{y} e^{-x/y} e^{-y} \mathcal{I}_{(0,\infty)}(x) \mathcal{I}_{(0,\infty)}(y).$$

a) Find the marginal pdf $f(y)$ and the cumulative density function (cdf) $F(y)$.

b) Find the conditional pdf $f(x|y)$ and its cdf $F(x|y)$.

c) Give the moment generating function for $f(x|y)$.

Topic 2
Let $X_1, \ldots, X_n$ be independently and identically distributed random variables with the probability density function

$$f(x; \lambda) = \lambda e^{-\lambda x} \mathcal{I}_{(0,\infty)}(x), \qquad \lambda > 0.$$

a) Give the moment generating function $M_{X_i}(t)$ for X_i.

b) Let $S = \sum_{i=1}^{n} X_i$. Find its MGF $M_S(t)$.

c) Find the expected value and the variance of S with the help of the MGF.

Topic 3

a) Let C and D be two events such that $P(C) = p_1 > 0$, $P(D) = p_2 > 0$ and $p_1 + p_2 > 1$. Show that $P(D|C) \geq 1 - [(1 - p_2)/p_1]$ using Bonferroni's inequality.

b) Prove that $P(A|B) = P(B|A)P(A)/P(B)$ whenever $P(A)P(B) \neq 0$.

c) Show that if $P(A|B) > P(A)$, then $P(B|A) > P(B)$.

Topic 4
Let $\{X_i,\ i = 1, \ldots, n\}$ be independent $\mathcal{N}(0, 1)$ distributed random variables. Consider the following functions:

$$\bar{X}_1 = \frac{1}{k} \sum_{i=1}^{k} X_i \quad \text{and} \quad \bar{X}_2 = \frac{1}{n-k} \sum_{i=k+1}^{n} X_i, \quad 1 \leq k < n.$$

Derive the distributions of

a) $S_1 = (\bar{X}_1 + \bar{X}_2)/2$

b) $S_2 = k\bar{X}_1^2 + (n - k)\bar{X}_2^2$

Topic 5
Let X_1 and X_2 be independent and identically distributed random variables following exponential density with a pdf:

$$f(x; \lambda) = \lambda e^{-\lambda x} \mathcal{I}_{(0,\infty)}(x), \quad \lambda > 0.$$

a) Give the joint pdf of X_1 and X_2.

b) Check if the joint pdf $f(x_1, x_2)$ is a member of the exponential class of densities.

c) Let $\lambda = 1$. Derive the pdf of $U = 1 - e^{-X_1}$ using the transformation theorem. Remember to adjust the support of the indicator function!

Topic 6
Let $X_1, \ldots X_n$ be independent and identically distributed with probability density function

$$f(x;\lambda) = \frac{e^{-\lambda}\lambda^x}{x!}\mathcal{I}_{\{0,1,2,3,\ldots\}}(x) \ .$$

Further define

$$W_n = \frac{1}{n}\sum_{i=1}^{n} X_i \ .$$

a) Find the asymptotic distribution of W_n.

b) Find the asymptotic distribution of ${W_n}^2$.

c) Discuss convergence in probability of W_n.

d) Find the probability limit of $\sqrt{W_n}$.

Exam WS14/15 2

Topic 1
Let the random variables (X, Y) have following joint probability density function (pdf):

$$f(x, y) = \frac{1}{y}e^{-x/y}e^{-y}\mathcal{I}_{(0,\infty)}(x)\mathcal{I}_{(0,\infty)}(y).$$

a) Find the marginal pdf $f(y)$ and the cumulative density function (cdf) $F(y)$.

b) Find the conditional pdf $f(x|y)$ and its cdf $F(x|y)$.

c) Give the moment generating function for $f(x|y)$.

Topic 2
Let $X_1, \ldots, X_n$ be independently and identically distributed random variables with the probability density function

$$f(x;\lambda) = \lambda e^{-\lambda x}\mathcal{I}_{(0,\infty)}(x), \qquad \lambda > 0.$$

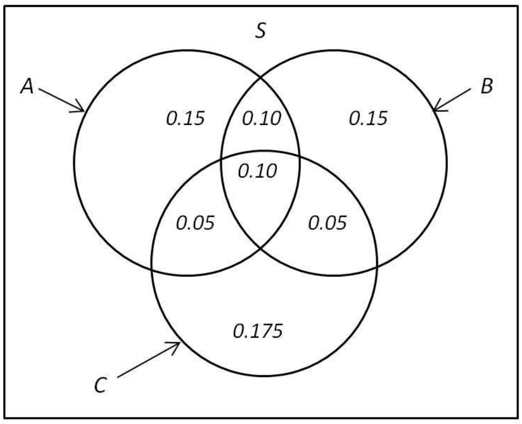
a) Give the moment generating function $M_{X_i}(t)$ for X_i.

b) Let $S = \sum_{i=1}^{n} X_i$. Find its MGF $M_S(t)$.

c) Find the expected value and the variance of S with the help of the MGF.

Topic 3
The following diagram indicates how probabilities have been assigned to various subsets of the sample space S:

a) Are the three events A, B and C pairwise independent?



b) Are the three events A, B and C jointly independent?

c) What is the value of $P((A \cap B)|C)$?

Now, disregard the diagram.

d) Let A and B be events, so that following holds:

$$P(A) = 0.3 \ , \ P(A \cup B) = 0.8 \text{ and } P(B) = p.$$

For which value of p are A and B stochastically independent?

Topic 4

Let $\{X_i, \ i = 1, \ldots, n\}$ be independent $\mathcal{N}(3, 9)$ distributed random variables. Derive the distributions of

a) $Y = \sum_{i=1}^{n} X_i$

b) $S_1 = \frac{1}{9} \sum_{i=1}^{n} (X_i - 3)^2$

c) $S_2 = kS_1$ for $k > 0$.

Topic 5

Let the sequences of random variables $\{X_n\}$ and $\{Y_n\}$ each follow normal distributions with

$$X_n \ \sim \ \mathcal{N}\left(\mu + \frac{1}{n}, \frac{n\sigma^2 + 2}{n}\right) \qquad \text{and} \qquad Y_n \ \sim \ \mathcal{N}\left(\mu, \frac{1}{n}\right) \ .$$

a) Examine which type of convergence applies for X_n and Y_n and give their asymptotic distribution or probability limit.

b) Find the asymptotic distributions of

$$A_n = X_n \cdot Y_n \qquad \text{and} \qquad B_n = X_n - Y_n \ .$$

c) Let $\{X_i,\ i = 1,\ldots,n\}$ be now iid $\mathcal{N}(\mu,\sigma^2)$. Further define

$$W = \frac{1}{n}\sum_{i=1}^{n} X_i \qquad \text{and} \qquad T = W^3.$$

Find the asymptotic distributions of W and T.

Topic 6

Let the random variables $X_1,\ldots,X_n$ be independent and identically distributed with pdf

$$f(x;\mu,\sigma) = \frac{1}{x\sigma\sqrt{2\pi}} \exp\left(-\frac{(\ln x-\mu)^2}{2\sigma^2}\right) \qquad x \in (0,\infty)\ .$$

a) Determine whether the pdf is a member of the exponential family of distributions.

b) Use the change of variables technique to determine the pdf of $Y = \ln X$.

Exam WS14/15 1, 8 credits

Topic 1

Let the random variables (X,Y) have following joint probability density function (pdf):

$$f(x,y) = \frac{1}{y}e^{-x/y}e^{-y}\mathcal{I}_{(0,\infty)}(x)\mathcal{I}_{(0,\infty)}(y)$$

a) Find the marginal pdf $f(y)$ and the cumulative density function (cdf) $F(y)$.

b) Find the conditional pdf $f(x|y)$ and its cdf $F(x|y)$.

c) Give the regression function of a regression of x on y. **Hint**: Substitute $t = \frac{x}{y}$.

Topic 2

Let $X_1,\ldots,X_n$ be independently and identically distributed random variables with the probability density function

$$f(x;\lambda) = \lambda e^{-\lambda x}\mathcal{I}_{(0,\infty)}(x), \qquad \lambda > 0.$$

a) Give the moment generating function $M_{X_i}(t)$ for X_i.

b) Let $S = \sum_{i=1}^{n} X_i$. Find its MGF $M_S(t)$.

c) Find the expected value and the variance of S with the help of the MGF.

Topic 3

a) Let C and D be two events such that $P(C) = p_1 > 0$, $P(D) = p_2 > 0$ and $p_1 + p_2 > 1$. Show that $P(D|C) \geq 1 - [(1-p_2)/p_1]$.

b) Prove that $P(A|B) = P(B|A)P(A)/P(B)$ whenever $P(A)P(B) \neq 0$.

c) A fair die is rolled repeatedly until a 6 shows up. Let S_k be the event that a 6 will show up for the first time at the kth throw. Give $P(S_k)$ and show that the event S: "a 6 will eventually show up" is certain to occur.

Topic 4

Let $\{X_i,\ i = 1,\dots,n\}$ be independent $\mathcal{N}(0,1)$ distributed random variables. Consider the following functions:

$$\bar{X}_1 = \frac{1}{k}\sum_{i=1}^{k} X_i \quad \text{and} \quad \bar{X}_2 = \frac{1}{n-k}\sum_{i=k+1}^{n} X_i, \quad 1 \le k < n.$$

Derive the distributions of

a) $S_1 = k\bar{X}_1^2$

b) $S_2 = X_1/\sqrt{(n-k)\bar{X}_2^2}$

c) $S_3 = k\bar{X}_1^2/(n-k)\bar{X}_2^2$.

Topic 5

Let X_1 and X_2 be independent and identically distributed random variables following exponential density with a pdf:

$$f(x;\lambda) = \lambda e^{-\lambda x}\mathcal{I}_{(0,\infty)}(x), \quad \lambda > 0.$$

a) Give the joint pdf of X_1 and X_2.

b) Check if the joint pdf $f(x_1, x_2)$ is a member of the exponential class of densities.

c) Let $\lambda = 1$. Derive the joint pdf of

$$Y_1 = X_1 + X_2 \quad \text{and} \quad Y_2 = X_1 - X_2$$

using the change of variables technique.

Topic 6

Let $X_1, \dots X_n$ be independent and identically distributed with probability density function

$$f(x;\lambda) = \frac{e^{-\lambda}\lambda^x}{x!}\mathcal{I}_{\{0,1,2,3,\dots\}}(x)\ .$$

Further define

$$W_n = \frac{1}{n}\sum_{i=1}^{n} X_i\ .$$

a) Find the asymptotic distribution of W_n.

b) Find the asymptotic distribution of ${W_n}^2$.

c) Discuss convergence in probability of W_n.

d) Find the probability limit of $\sqrt{W_n}$.

Exam WS14/15 2, 8 credits

Topic 1

Let the random variables (X, Y) have following joint probability density function (pdf):

$$f(x, y) = \frac{1}{y} e^{-x/y} e^{-y} \mathcal{I}_{(0,\infty)}(x) \mathcal{I}_{(0,\infty)}(y)$$

a) Find the marginal pdf $f(y)$ and the cumulative density function (cdf) $F(y)$.

b) Find the conditional pdf $f(x|y)$ and its cdf $F(x|y)$.

c) Give the regression function of a regression of x on y. **Hint**: Substitute $t = \frac{x}{y}$.

Topic 2

Let $X_1, \ldots, X_n$ be independently and identically distributed random variables with the probability density function

$$f(x; \lambda) = \lambda e^{-\lambda x} \mathcal{I}_{(0,\infty)}(x), \qquad \lambda > 0.$$

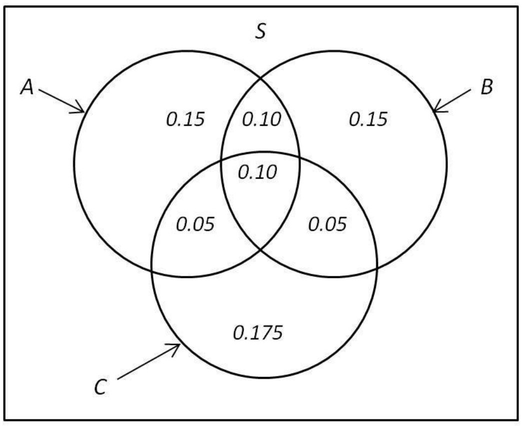
a) Give the moment generating function $M_{X_i}(t)$ for X_i.

b) Let $S = \sum_{i=1}^{n} X_i$. Find its MGF $M_S(t)$.

c) Find the expected value and the variance of S with the help of the MGF.

Topic 3

The following diagram indicates how probabilities have been assigned to various subsets of the sample space S:



a) Are the three events A, B and C pairwise independent?

b) Are the three events A, B and C jointly independent?

c) What is the value of $P((A \cap B)|C)$?

d) Now, disregard the diagram. An urn contains 1 blue, 2 white and 3 red balls. Suppose, you draw three balls without replacement. What is the probability of all three balls being different colors?

Topic 4

Let the random variables $X_1, \ldots, X_n$ be independent and identically distributed with probability density $\mathcal{N}(1, \sigma^2)$. Determine the distribution of the following variables and indicate the theorems you are using:

a) $Y_1 = \sum_{j=1}^{k} X_j \quad \text{with} \quad 0 < k < n-1$

b) $Y_2 = \frac{Y_1 - k}{\sqrt{k}}$

c) $Y_3 = \left(\frac{Y_2}{\sigma}\right)^2 + \left(\frac{X_{k+1} - 1}{\sigma}\right)^2$

d) $Y_4 = \frac{\sqrt{2}X_{k+2} - \sqrt{2}}{\sigma} / \sqrt{X_3}$

Topic 5

Let X_1 and X_2 be identically independently exponentially distributed with probability density function

$$f(x; \lambda) = \lambda e^{-\lambda x} \mathcal{I}_{(0,\infty)}(x) \ .$$

a) Give the joint pdf of X_1 and X_2.

b) Give the joint density of

$$Z_1 = X_1 + X_2 \quad \text{and} \quad Z_2 = \frac{X_1}{X_1 + X_2} \ .$$

Topic 6

Let $X_1, \ldots X_n$ following a Poisson distribution with a pdf

$$f(x) = \frac{e^{-\lambda}\lambda^x}{x!} \mathcal{I}_{\{0,1,2,\ldots\}}(x)$$

have stochastically independent realizations. Further define

$$U_n = \frac{1}{n} \sum_{i=1}^{n} X_i \ .$$

a) Check if the pdf X_i is a member of the exponential class of densities.

b) Find the asymptotic distribution of U_n.

c) Find the asymptotic distribution of $\sqrt{U_n}$.

Winter Term 2015/16

Exam WS15/16 1

Topic 1

Consider a Poisson-distributed population with distribution $P(X_i = k) = e^{-\lambda}\frac{\lambda^k}{k!}$.

1. Show that the Poisson distribution is a member of the exponential class of distributions.
2. Derive its moment generating function. (Hint: $\sum_{j\geq 0}\frac{x^n}{n!} = e^x$ for all $x \in \mathbb{R}$.)
3. Let $X_i \sim \text{Poisson}(\lambda_i)$ independent, and $Y_n = \sum_{i=1}^n X_i$. Show that Y_n is itself Poisson distributed; derive the mean and the variance of the sum.
4. Derive the asymptotic behavior $\bar{X}_n = \frac{1}{n}Y_n$ with Y_n from 3 when $\underline{\lambda} \leq \lambda_i \leq \overline{\lambda}$ for all i. (Hint: you might need to reformulate the problem in terms of $\bar{X}_n - \mathrm{E}(\bar{X}_n)$, and you can't apply Khinchin's LLN.)

Topic 2

Consider an exponentially-distributed population, $P(X \leq x) = 1 - e^{-\lambda x}$ and an iid sample X_i, $i = 1, \ldots, n$.

1. Show that $\bar{X}_n \xrightarrow{p} \frac{1}{\lambda}$.
2. Derive the limiting distribution of $\sqrt{n}(\bar{X}_n - \lambda^{-1})$.
3. Show that $\frac{1}{\bar{X}_n} \xrightarrow{p} \lambda$
4. Derive the limiting distribution of the suitably standardized $\frac{1}{\bar{X}_n}$.

Topic 3

Consider a Gaussian random variable, $X \sim \mathcal{N}(\mu, \sigma^2)$.

1. Express the cdf of X in terms of Φ, the cdf of the standard normal distribution.
2. Based on this, derive the density function of X and double-check the result using Theorem 6.8. (Hint: $\Phi' = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}$.)

Topic 4

Consider the bivariate density function $f(x, y) = \frac{3}{4}x^2\mathcal{I}_{[-1,1]^2}(x, y)$.

1. Derive the regression curve of Y given X and X given Y.
2. Derive the cdf associated to the given density.

Exam WS15/16 2

Topic 1

Consider a Bernoulli-distributed population with success probability $p \in (0, 1)$.

1. Show that the Bernoulli distribution is a member of the exponential class of distributions.

2. Derive its moment generating function.

3. Let $X_i \sim \text{Bernoulli}(p_i)$ be independent (but not necessarily identically) distributed, and $Y_n = \sum_{i=1}^n X_i$. Derive the mean and the variance of Y_n as a function of p_i, $i = 1, \ldots, n$.

4. Derive the asymptotic distribution of the suitably standardized $\bar{X}_n = \frac{1}{n} Y_n$ with Y_n from 3.

Topic 2
Consider the regression curve $\mathrm{E}(Y|X) = aX + b$ and assume that the conditional distribution of Y given X is normal with unity variance. Assume that X is standard normally distributed as well.

1. Derive the unconditional distribution of Y.

2. Derive the regression curve $\mathrm{E}(X|Y)$.

Topic 3
Consider a Gaussian random variable, $X \sim \mathcal{N}(\mu, \sigma^2)$.

1. Express the cumulative distribution function F_X of X as a function of μ and σ.

2. Given $F_X(x)$, you construct the new random variable $Y = F_X(X)$. Using Theorem 6.7, show that Y is uniformly distributed on $[0, 1]$. (**Hint**: for the derivative of an inverse function, we have the very useful identity $\left(g^{-1}\right)'(y) = \frac{1}{g'(g^{-1}(y))}$ when g is monotonic.)

Topic 4
Consider a random variable X taking the values -1, 0 and 1 with probabilities p_1, p_2 and p_3. Assume that you know the values of the first three noncentral moments of X, μ_1, μ_2 and μ_3. Compute p_1, p_2 and p_3 as functions of μ_1, μ_2 and μ_3.

Winter Term 2016/17

Exam WS16/17 1

Topic 1

The probability that a student passes the Statistics I exam is 0.75, the probability of passing the Econometrics I exam is 0.6 and the probability of passing both exams is 0.5.

1. What is the probability of passing any of the two exams?
2. What is the probability of passing Econometrics I given the student has passed Statistics I?
3. What is the probability of passing Statistics I given the student has passed Econometrics I?
4. What is the probability of passing neither of the two exams?

Topic 2

Let $S = [0, 100]$ be the sample space consisting of all possible values of points that can be obtained for this exam. Assume for simplicity that the number of points is continuously distributed. Given the events $A = [0, 80]$, $B = [60, 100]$, $C = [0, 60]$, $D = [0, 50]$ and $E = [40, 60]$ as well as $P(A) = 0.512$ and $P(B) = 0.634$.

1. What is the probability of achieving between 60 and 80 points?
2. What is the probability of the event C?
3. Can $P(D) = 0.6$?
4. Given that $P([50, 100]) = 0.75$, what can you say about $P(E)$?

Topic 3

1. (a) Show that the moment generating function (MGF) of the Gamma distribution
$$f(x, \alpha, \beta) = \frac{1}{\beta^\alpha \Gamma(\alpha)} x^{\alpha-1} e^{-\frac{x}{\beta}}$$
is given by
$$M_X(t) = (1 - \beta t)^{-\alpha}.$$
   (b) Derive expectation as well as variance using the MGF.
   (c) Assume $\{X_n\}$ being randomly sampled from Gamma(α, β). Derive the exact as well as the limiting distribution of $\overline{X}_n = \frac{1}{n} \sum_{i=1}^{n} X_i$. Compare your results.
2. Consider a joint pdf of X and Y
$$f(x, y) = \frac{1}{8}(x + y)\mathcal{I}_{[0,2]}(x)\mathcal{I}_{[0,2]}(y).$$
   (a) Find the joint cdf of X and Y.
   (b) Find the marginal pdf's of X and Y. Are X and Y stochastically independent?

(c) Find the conditional pdf $f(y|x)$ and the regression curve of Y on X.

Topic 4

Consider the case of the sample average $\overline{X}_n$ for a random (iid) sample from population with pdf f such that $\mathrm{E}(X_i) = \mu$ and $\mathrm{Var}(X_i) = \sigma^2$ are finite.

1. Derive $\mathrm{E}(\overline{X}_n)$ and $\mathrm{Var}(\overline{X}_n)$ and show that $\overline{X}_n \xrightarrow{m} \mu$ as $n \to \infty$.
2. Prove that convergence in mean square implies convergence in probability.
3. Assume that $X_i \sim \mathcal{N}(\mu, \sigma^2)$. Using the delta method, derive an approximate distribution for the standardized sample skewness,
$$\hat{\gamma} = \frac{M_3}{\sqrt{M_2^3}} \qquad \text{where} \quad M_r = \frac{1}{n}\sum_{i=1}^{n}\left(X_i - \overline{X}_n\right)^r .$$
Hint: Note that $\mathrm{E}\left((X_i - \mu)^4\right) = 3\sigma^4$ and $\mathrm{E}\left((X_i - \mu)^6\right) = 15\sigma^6$.

Exam WS16/17 2

Topic 1

1. Consider an ideal **20 sided** die called “icosahedron” which is often used for roleplaying games. The random variables X_i denotes the number of dots facing up on the ith attempt. Find the probability that the average number of dots does not exceed 8 after 200 attempts. [Hint: $E(X^2) = 143.5$]
2. A statistician claims that the time T (in minutes) the HSV (a football club) gets a goal in a match can be well approximated by an exponential distribution $f(t) = \lambda e^{-\lambda t}$ with $\lambda = 0.05$. Calculate the probabilities that the HSV gets a goal
   (a) in the complete game (90 mins)
   (b) in the first 10 minutes after the kicking off
   (c) in 10 mins, given that ten minutes have already passed
   (d) Show that the probability of getting a goal after t minutes does not depend on how much time (s) has already passed [i.e. $P(T > s + t|T > s) = P(T > t)$].

Topic 2

Let $X \sim \mathcal{N}(0, 1)$ and consider $Y = |X|$.

1. Find the cdf of Y and show that the density of Y is given by $f(y) = \sqrt{\frac{2}{\pi}} \exp\left(-\frac{y^2}{2}\right)$.
2. Compute the expectation of $|X|$.
3. Express the pth quantile of Y with the help of the quantiles of the standard normal distribution.

Topic 3

1. Find the cdf for the following pdf and sketch the solution:

$$f(x) = \begin{cases} x & \text{if} & 0 \leq x < 1 \\ (x-3) & \text{if} & 3 \leq x < 4 \\ 0 & \text{otherwise} & \end{cases}$$

2. Consider the following pdf

$$f(x_1, x_2) = k \cdot x_1 x_2 \mathcal{I}_{[0,4]}(x_1) \mathcal{I}_{[0,1]}(x_2).$$

   (a) Determine the value of k.

   (b) Find the marginal probability density functions $f(x_1)$ and $f(x_2)$ as well as the conditional density functions $f(x_1|x_2)$ and $f(x_2|x_1)$.

   (c) Find the regression functions of X_1 on X_2 and vice versa.

3. Consider the random variable $\underline{X} = (X_1, ..., X_n)$ with the single elements being each independent ($\mathcal{N}(\mu, \sigma^2)$-normally distributed) random variables with the following moment-generating function:

$$M_X(t) = \exp\left\{\mu t + \frac{1}{2}\sigma^2 t^2\right\} \quad , \quad \sigma > 0.$$

   (a) Find the moment-generating function and distribution of $Z = \frac{1}{n}\sum_{i=1}^{n} 3 \cdot X_i$

   (b) Find the pdf of $Y = \exp(X_1)$.

   (c) Find the asymptotic distribution of Z^2.

Winter Term 2017/18

Exam WS17/18 1

Topic 1

1. A random variable X has a MGF given by
$$M_X(t) = e^{4(e^t-1)}.$$
   (a) Derive the mean and the variance of X.
   (b) Compare the upper bounds for the probability $\mathrm{P}(X < 20)$ obtained using the Markov and the Chebyshev inequalities.
   (c) Which parametric family does the distribution of X belong to?

2. Consider 3 boxes which are numbered from 1 to 3. Each box contains 7 items of fruit (apples and peaches). Box i ($i = 1, 2, 3$) contains i apples and $7 - i$ peaches. First, you choose a box and, second, you take a piece of fruit from that box.
   (a) What is the probability to draw an apple?
   (b) What is the probability to draw a peach?
   (c) Assume you have drawn an apple. What is the probability that this apple comes from the second box?

Topic 2

The joint density of the random variable $\boldsymbol{X} = (X_1, X_2)$ is
$$f(x_1, x_2) = k\frac{x_1}{x_2}\mathcal{I}_{[-5,0]}(x_1)\mathcal{I}_{[1,e)}(x_2)$$

1. Determine k such that f is a proper pdf.
2. Derive the joint cdf $F(x_1, x_2)$.
3. Calculate $P(-2 < X_1 \leq -1, X_2 = 2)$.
4. Derive the marginal pdfs and cdfs of X_1 and X_2.
5. Derive the regression curve of X_1 on X_2.

Topic 3

1. Prove theorem 4.6 with the change of variables technique. [**Hint:** $\Gamma(0.5) = \sqrt{\pi}$]
2. Let $X_i \sim \chi^2(\nu)$ i.i.d.
   (a) Derive the exact distribution of $Y = \dfrac{1}{n}\sum_{i=1}^{n} X_i$.
   (b) Derive an asymptotic distribution of $Y = \dfrac{1}{n}\sum_{i=1}^{n} X_i$ using a suitable central limit theorem.
   (c) Prove that$Y = \dfrac{1}{n}\sum_{i=1}^{n} X_i \xrightarrow{p} \nu$ as $n \to \infty$ using the MGF approach.
   Hint: $\lim_{n\to\infty}(1 + a_n)^{b_n} = \exp(\lim_{n\to\infty} a_n b_n)$ if $a_n \to 0$, $b_n \to \infty$.

Exam WS17/18 2

Topic 1

Let the random variables (X, Y) have the following joint probability density function (pdf):

$$f(x,y) = \frac{2}{11}\left(xy^2 + x^2\right)\mathbb{I}_{(0,1)}(x)\mathbb{I}_{(0,3)}(y) \ .$$

1. Find the marginal pdf's $f(x)$ and $f(y)$ and their corresponding marginal cdf's.
2. Check whether X and Y are stochastically independent.
3. Find the conditional pdf $f(x|y)$ and the cdf $F(x|y)$.
4. Find the joint cumulative distribution function (cdf) $F(x, y)$.
5. Find $P(X \leq 0.5, Y \leq 1)$.

Topic 2

The joint density of the random variable $\boldsymbol{X} = (X_1, X_2)$ is

$$f(x_1,x_2) = k\exp\left(-\frac{1}{2}\begin{pmatrix}x_1-1 & x_2+5\end{pmatrix}\begin{pmatrix}2 & -3\\ -3 & 5\end{pmatrix}\begin{pmatrix}x_1-1\\ x_2+5\end{pmatrix}\right)\mathbb{I}_{(-\infty,\infty)}(x_1)\mathbb{I}_{(-\infty,\infty)}(x_2).$$

1. To which parametric family does the distribution of $\boldsymbol{X}$ belong to?
2. Determine k such that f is a proper pdf.
3. Derive the marginal and conditional pdfs of X_1 and X_2.
4. Derive the regression curve of X_1 on X_2.
5. Determine the pdf of the random variable Y^2, $Y = \boldsymbol{a}\boldsymbol{x}' - 6$ with $\boldsymbol{a} = \begin{pmatrix}1 & -1\end{pmatrix}$.

Topic 3

Suppose the random variable X follows a Beta distribution, i.e.

$$f(x) = \frac{1}{B(\alpha,\beta)}x^{\alpha-1}(1-x)^{\beta-1}\mathbb{I}_{(0,1)}(x).$$

1. Show that $Y = \dfrac{X}{1-X}$ has a pdf $h(y) = \frac{1}{B(\alpha,\beta)}y^{\alpha-1}(1+y)^{-\alpha-\beta}$. What is the range of Y?
2. Show that the MGF of Y is given by $M_Y(t) = \sum_{k=0}^{\infty}\frac{t^k}{k!}\frac{B(\alpha+k,\ \beta-k)}{B(\alpha,\beta)}$.
   Hints: You can use that $e^{tx} = \sum_{k=0}^{\infty}\frac{t^k x^k}{k!}$. Don't forget about the properties of any pdf!
3. Calculate $\mathrm{E}[Y]$ and $\mathrm{Var}[Y]$.
4. Derive the limiting distribution of $Z_n = nX_n$, where X_n is Beta$(1, n)$ distributed.
   Hint: $\lim_{n\to\infty}\left(1+\frac{u}{n}\right)^n = e^u$.

Winter Term 2018/19

Exam WS18/19 1

Topic 1

The following data were given in a study of a group of 1000 subscribers to a certain magazine: in reference to job, marital status, and education there were 312 professionals (P), 470 married persons (M), 525 college graduates (C), 42 professional college graduates, 147 married college graduates, 86 married professionals, and 25 married professional college graduates.

1. Find the probabilities for the events P, M and C.
2. Find the probabilities for the events $P \cap M$, $P \cap C$ and $M \cap C$.
3. Show that the numbers reported in the study must be incorrect.

Topic 2

Consider a poisson-distributed population with distribution $P(X_i = k) = e^{-\lambda} \frac{\lambda^k}{k!}$.

1. Let $X_i \sim \text{Poisson}(\lambda_i)$ be independent. Define $Y_n = \sum_{i=1}^{n} X_i$. Show that Y_n is itself Poisson distributed via the MGF-approach.
2. Derive the mean and the variance of Y_n.
3. Show that $\overline{X}_n = \frac{1}{n} Y_n$ is converging in probability to $E(\overline{X}_n)$ if $\lambda_i \leq \bar{\lambda}$ for all i, where $\bar{\lambda}$ is a finite constant.

Topic 3

Assume X_1 and X_2 are independent standard normal random variables.

1. Derive the joint density $f(x_1, x_2)$.
2. Derive the joint density of the random variables $Y_1 = \frac{X_1}{X_2}$ and $Y_2 = X_2$.
3. Prove that Y_1 is t-distributed with 1 degree of freedom.
   Hints: Y_2 is symmetric around 0 and $\Gamma(0.5) = \sqrt{\pi}$.
4. Show that $\int_{R(y_1)} y_1 f(y_1) dy_1 = 0$, where $R(y_1)$ is the range of Y_1.
5. Verify that $E(Y_1)$ does not exist. What can you conclude for higher order moments, i.e. $E(Y_1^{\zeta})$ for $\zeta > 1$?

Topic 4

Consider an exponentially-distributed population, $P(X \leq x) = 1 - e^{-\varphi x}$ and an iid sample X_i, $i = 1, \ldots, n$. Let $\overline{X}_n = \frac{1}{n} \sum_{i=1}^{n} X_i$.

1. Show that $\overline{X}_n \xrightarrow{p} \frac{1}{\varphi}$. Do **not** use Khinchin's WLLN.
2. Derive the limiting distribution of $\sqrt{n}(\overline{X}_n - \varphi^{-1})$.
3. Show that $\dfrac{1}{\overline{X}_n} \xrightarrow{p} \varphi$
4. Derive the limiting distribution of the suitably standardized $\dfrac{1}{\overline{X}_n}$.

Exam WS18/19 2

Topic 1

Two balls are drawn at random without replacement from a box that contains five balls numbered 1 through 5. If the sum of the numbers on the balls is even, the random variable X equals 7, if the sum is odd, $X = -5$.

1. Find the MGF of X.
2. Find the first and second non-central moments with the help of the MGF of a). Hint: In case you have no result for a) use $M_X(t) = 5e^{20t}$.
3. Find the variance of X.

Topic 2

Suppose X, Y are two jointly distributed random variables with joint probability density function

$$f_{X,Y}(x,y) = \begin{cases} \alpha xy(1-x) & 0 < x, y < 1 \\ 0 & \text{otherwise} \end{cases}$$

1. Verify that $f_{X,Y}(x,y)$ is indeed a pdf and give α.
2. What is the pdf of X? What is the pdf of Y?
3. Derive the conditional pdf $f_{x|y}$.
4. Find the probability $P(Y < 0.5|X > 0.5)$.

Topic 3

A certain component is necessary for the operation of a computer server system and must be replaced immediately upon failure for the server to continue operating. The lifetime of such a component is a random variable X with mean lifetime 100 hours and a standard deviation of 30 hours. Find the number of such components which must be in stock in order to make sure that the probability that the system can run continuously for at least the next 20,000 hours is not less than 95%. Use **asymptotic** arguments and assume that no other possible failures can occur.

Hints: $\Phi(1.645) = 0.95$, $\Phi(x)$ is monotonically increasing and $1 - \Phi(x) = \Phi(-x)$, where $\Phi(x)$ denotes the cdf of a standard normal.

Topic 4

Consider a random variable ω with the following pdf

$$f(\omega) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(\omega-\mu)^2}{2\sigma^2}\right) \mathcal{I}_{(-\infty,\infty)}(\omega).$$

1. Derive the MGF of ω.
2. Assume now $\mu = 0$ and $\sigma = 1$. Find the pdf of the random variable $X = |\omega|$.

Winter Term 2019/20

Exam WS19/20 1

Topic 1

Your friend arrives at a bus stop at 5 pm. Unfortunately he didn't check the bus schedule, but he knows that the bus will arrive at some time (measured in minutes) uniformly distributed between 5 and 5:30 pm.

1. Find the cumulative distribution function of this uniform distribution.
2. What is the probability that your friend will have to wait longer than ten minutes?
3. If at 5:15 pm the bus has not yet arrived, what is the probability that your friend will have to wait at least an additional 5 minutes?
4. What is the average waiting time until the next bus arrives?

Topic 2

Consider a geometric-distributed population with probability mass function $f(x) = (1-p)^x p$, where $x \in \{0, 1, 2, 3, \dots\}$ and $p \in (0, 1)$.

1. Show that the moment generating function is given by $M_X(t) = \dfrac{p}{1-(1-p)e^t}$.

   Hint: $\displaystyle\sum_{k=0}^{a-1} x^k = \frac{1-x^a}{1-x}$ if $x \neq 1$.
2. Derive expectation and variance of X using the MGF.
3. Give the cumulative distribution function of X and calculate $\mathrm{P}(X \leq 10)$ if $p = 0.5$.
4. Derive expectation and variance without using the MGF.

   Hints: $\mathrm{Var}(X) = \mathrm{E}(X(X-1)) + \mathrm{E}(X) - \mathrm{E}(X)^2$ and $\displaystyle\sum_{k=0}^{b} kx^{k-1} = \frac{\partial}{\partial x}\sum_{k=0}^{b} x^k$ for $x > 0$.

Topic 3

Let X and Y have a joint density given by

$$f(x,y) = \frac{\sqrt{\lambda}}{\Gamma(\alpha)\beta^\alpha\sqrt{2\pi}} y^{\alpha-\frac{1}{2}} e^{-\frac{y}{\beta}} \exp\left(-\frac{\lambda y(x-\mu)^2}{2}\right) \mathcal{I}_{(-\infty,\infty)}(x)\mathcal{I}_{(0,\infty)}(y),$$

where $\mu \in \mathbb{R}$ and $\alpha, \beta, \lambda > 0$.

1. Show that $Y \sim \mathit{Gamma}\,(\alpha, \beta)$.
2. Derive the conditional density $f(x|y)$.
3. Is the joint density $f(x,y)$ a member of the exponential class?
4. Are X and Y independent? Argue briefly.

Topic 4

Let $X_i \overset{iid}{\sim} N(0,1)$ for $i = 1, \dots, n$. Define $\overline{X_n^2} = n^{-1}\displaystyle\sum_{i=1}^{n} X_i^2$.

1. Show that the exact distribution of $\overline{X_n^2}$ is given by $Gamma\left(\frac{n}{2}, \frac{2}{n}\right)$.

2. Derive the limiting distribution of $\overline{X_n^2}$.

3. Prove that $\overline{X_n^2} \xrightarrow{p} 1$. Do **not** use Khinchin's WLLN.

4. Find $\operatorname{plim}\left(\frac{1}{\overline{X_n^2}}\right)$.

5. Give the density of $Y = \frac{1}{\overline{X_n^2}}$.

Exam WS19/20 2

Topic 1

Consider the random variable $\underline{X} = (X_1, ..., X_n)$ with the single elements being each independent $Gamma(\alpha, \beta)$ distributed random variables with the following probability density function:

$$f(x) = \frac{1}{\beta^\alpha \Gamma(\alpha)} x^{\alpha-1} e^{-\frac{x}{\beta}} \mathbb{I}_{(0,\infty)}(x), \quad \alpha > 0, \beta > 0.$$

1. Is the density $f(x_1)$ a member of the exponential class?

2. Find the pdf of $Y = 3 \cdot \exp(X_1)$.

3. Find the moment-generating function and distribution of $S = \frac{1}{n} \sum_{i=1}^n 10 \cdot X_i$

4. Find the asymptotic distribution of S^3.

5. Suppose you know that $\alpha = 2, \beta = 4$. Give the exact probability $P(X_1 < 12)$.

Topic 2

Let the sequences of random variables $\{X_n\}$ and $\{Y_n\}$ each follow normal distributions with

$$X_n \sim \mathcal{N}\left(\mu + \frac{1}{\ln(n)}, \frac{n^3\sigma^2 + 10}{n^3}\right) \qquad \text{and} \qquad Y_n \sim \mathcal{N}\left(\mu, \frac{\sigma^2 + \sqrt{n}}{n+1}\right) .$$

1. Examine which type of convergence applies for X_n and Y_n and give their asymptotic distribution and probability limit if it exists.

2. Find the asymptotic distributions of

$$A_n = X_n \cdot Y_n \qquad \text{and} \qquad B_n = X_n + Y_n \ .$$

Topic 3

Assume $X = \begin{pmatrix} X_1 & X_2 & X_3 \end{pmatrix}'$ is normally distributed with parameters $\mu = \begin{pmatrix} 3 & 2 & 1 \end{pmatrix}'$ and $\Sigma = \begin{pmatrix} 2 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{pmatrix}$. Define $Z_1 = X_2$ and $Z_2 = \begin{pmatrix} X_1 & X_3 \end{pmatrix}'$.

1. Find the distribution of Z_1.

2. Derive the distribution of $Z_1 | (Z_2 = z_2)$.

3. Calculate the regression curve of Z_1 on Z_2
4. Are Z_1 and Z_2 independent? Argue shortly.

Topic 4

The probability p_n that a family has at least n children is given by $p_n = \dfrac{\alpha p^n}{1-p}$ for $n \in \mathbb{N}\backslash\{0\}$, where $p \in (0,1)$.

1. Calculate the probability that a family has no child.
2. Calculate the probability that a family has exactly n children.
3. What can you say about α?
4. Assume that the children can be distinguished by sex (i.e. boys and girls) and the probability to have a boy or girl is the same. Show that the probability to have **exactly one** girl is given by $\dfrac{2\alpha p}{(2-p)^2}$.

   Hints: $\displaystyle\sum_{k=0}^{\infty} x^n = \frac{1}{1-x}$ for $|x| < 1$ and $\displaystyle\sum_{k=1}^{\infty} kx^k = x\frac{\partial}{\partial x}\sum_{k=1}^{\infty} x^k$.

Winter Term 2020/21

Exam WS20/21 1

Topic 1

The teaching assistant has known the working group of students A, B and C for a long time and knows that student A handles 80%, student B 15% and student C only 5% of the tasks and they organize it so that no task is done twice. Due to their different experience, they can solve a problem correctly with a probability of 90%, 50% or only 10% respectively. The assistant received a faulty solution from the working group. What is the probability that this comes from student A, B or C? Which of these probabilities is the greatest?

Topic 2

Consider the random variable $\underline{X} = (X_1, \ldots, X_n)$ with the single elements being each independent and identical distributed random variables with the following probability density function:

$$f(x) = \frac{12\left(x - \frac{b+a}{2}\right)^2}{(b-a)^3} \mathcal{I}_{(a,b)}(x), \quad a \in \mathbb{R}, b > a$$

with moments $\mathrm{E}(X) = \frac{a+b}{2}$, $\mathrm{Var}(X) = \frac{3(b-a)^2}{20}$ and MGF

$$M_X(t) = \frac{-3(e^{at}(4 + (a^2 + 2a(-2+b) + b^2)t) - e^{bt}(4 + (-4b + (a+b)^2)t))}{(a-b)^3 t^2}.$$

1. Is the density $f(x)$ a member of the exponential class?
2. Find the probability density function of the random variable $Y = 1 - e^{-5X}$.

Let $N \sim \text{Poisson}(\lambda)$, i.e. $f(n) = \frac{e^{-\lambda}\lambda^n}{n!}$ for $n \in \mathbb{N} \cup \{0\}, \lambda > 0$ and define

$$Z = \sum_{i=1}^{N} X_i$$

3. Given that $\underline{X}$ and N are independent, show that the moment generating function of Z is given by $M_Z(t) = \exp(\lambda(M_X(t) - 1))$. Hint: Use the law of iterated expectations.
4. Find the expectation and variance of Z using the MGF approach.
5. What can you say about $\mathrm{P}(S_1 \cup S_2)$, where the events S_1 and S_2 are given by $S_1 = \{Z \geq \lambda\frac{a+b}{2} + \sqrt{\lambda}\}$ and $S_2 = \{Z \leq \lambda\frac{a+b}{2} - \sqrt{\lambda}\}$?

Topic 3

The joint density of the random variable $\boldsymbol{X} = (X_1, X_2)$ is

$$f(x_1, x_2) = k\,(10x_1x_2 + 6)\,\mathbb{I}_{(0,1)}(x_1)\mathbb{I}_{(0,2)}(x_2)\ .$$

1. Determine k such that f is a proper pdf.
2. Derive the joint cdf $F(x_1, x_2)$.
3. Calculate $P(X_1 \leq 0.1, X_2 \leq 1.5)$.
4. Derive the marginal pdfs of X_1 and X_2 and the conditional pdf $f(x_1|x_2)$.

5. Check whether X_1 and X_2 are stochastically independent.

Topic 4

LThe MGF of a random variable $X \sim Geometric(p)$ is given by

$$M_X(t) = \frac{pe^t}{1-(1-p)e^t} \quad \text{for } t < -\ln(1-p).$$

1. Show that geometric distribution is a special case of the negative binomial distribution. Based on your findings, give the expectation and the variance of X.
2. Let $X_1, \dots X_r$ be independent and identically geometrically distributed random variables. Derive the moment generating function $M_Y(t)$ of the random variable $Y = \sum_{i=1}^r X_i$ and determine its distribution.
3. With the help of the central limit theorem, motivate the possibility of approximating a negative binomial distribution by an associated normal distribution for $r \to \infty$.
4. Give the asymptotic distribution of Y^3 for $r \to \infty$.

Exam WS20/21 2

Topic 1

Let X_1, X_2 be independent and identically distributed with pdf

$$f(x_i) = -x_i \mathbb{I}_{[-1,0)}(x_i) + (2 - x_i)\mathbb{I}_{[1,2)}(x_i).$$

1. Find the cdf of X_1. Sketch your solution.
2. Calculate $\mathrm{P}(X_1 > X_2)$.

Topic 2

Let X, Y be jointly normally distributed and X standard normal. Assume $\mathrm{E}(Y|X) = 5 + X$,

1. Give a condition for the variance of Y.
2. Choose a value for the variance of Y, which is in line with 1. and derive $\mathrm{E}(X|Y)$.
3. Find t in terms of the cdf of a standard normal, $\Phi(\dots)$, such that $\mathrm{P}(Y < t) = 0.6$.
   Hint: If you were not able to solve parts 1.+2. choose $\mu_Y = 10$ and $\sigma_Y^2 = 4$.

Topic 3

An insurance company has 30% women and 70% men as insured persons for motor vehicle liability insurance. In every year the probability that a woman has an insurance claim is 0.3 and for a man 0.4. Calculate the probability that

1. a randomly chosen insured person has an insured event this year.
2. a randomly chosen insured person has no insured event for two consecutive years.
   Hint: For two sets A and B, it holds that $\overline{A \cap B} = \overline{A} \cup \overline{B}$.
3. a person with an insured event is a woman.
4. Let A_i be the event that a randomly chosen person has an insurance event in year i. Show that

$$\mathrm{P}(A_2|A_1) \geq \mathrm{P}(A_1).$$

Topic 4

Let X be a finite random variable with probability mass function $f(x)$ and cardinality N_X.

1. Show that $\mathrm{E}\left[\log\left(\frac{1}{f(X)}\right)\right] \leq \log N_X$.
   Hint: The logarithm is a concave function.

2. Verify that if $X \sim U(a,b)$, i.e. X is uniformly distributed, the bound in 1. is attained.

3. Explain briefly why $\mathrm{E}\left[\log\left(\frac{1}{f(X)}\right)\right]$ is not defined for continuous random variables in general.

Topic 5

Let $X_1, \ldots X_n$ be independent and identically distributed random variables with pdf

$$f(x_i) = \frac{1}{4}\mathbb{I}_{[0,4]}(x_i)$$

and $\mathrm{Var}[(X_i - \mathrm{E}(X_i))^2] = t^2 < \infty$.

1. Show that $\sum_{i=1}^n \left(X_i - \overline{X}_n\right)^2 = \sum_{i=1}^n \left(X_i - \mathrm{E}(X_i)\right)^2 - n(\overline{X}_n - \mathrm{E}(X_i))^2$, where $\overline{X}_n = \frac{1}{n}\sum_{i=1}^n X_i$.

2. Find the asymptotic distribution of $W_n = \frac{1}{n}\sum_{i=1}^n \left(X_i - \mathrm{E}(X_i)\right)^2$.

3. Find the probability limit of $P_n = (\overline{X}_n - \mathrm{E}(X_i))^2$.

4. Give the asymptotic distribution of $S_n^2 = \frac{1}{n}\sum_{i=1}^n \left(X_i - \overline{X}_n\right)^2$.

5. Derive the asymptotic distribution of $Y = \overline{X}_n^2$.

Winter Term 2021/22

Exam WS21/22 1

Topic 1

Let $\mathbf{X} = (X_1, X_2)$ have a joint discrete pdf given by

$\mathbf{x}$	(0,0)	(0,1)	(1,1)	(1,0)
$f_x(\mathbf{x})$	1/4	1/4	1/4	1/4

What is the joint discrete pdf of $\mathbf{Y} = (Y_1, Y_2)$ with $Y_1 = |X_1 - X_2|$ and $Y_2 = X_1 \cdot X_2$?

Topic 2

Let $X_1, \ldots, X_n$ be independent and identically uniformly $U(0, 1)$ distributed random variables.

1. Derive the asymptotic distribution of $\overline{X}_n$.
2. Show that $\frac{1}{B(1,n)} = n$, where B is the beta function, and that a cdf of the uniformly distributed random variable on the interval $[a, b]$ is $F(x) = \frac{x-a}{b-a}$.
3. Find a cumulative distribution function of $X_{(1)} = min(X_1, X_2, \ldots, X_n)$.
   Hint: $P(min(X_1, X_2, \ldots, X_n) > x) = P(X_1 > x, X_2 > x, \ldots, X_n > x)$.
4. Find a pdf of $X_{(1)} = min(X_1, X_2, \ldots, X_n)$ and identify its distribution.

Topic 3

The pdfs of random variables $Z_1 \sim \Gamma(\alpha, \beta)$ and $Z_2 \sim exp(\lambda)$ are given by

$$f_{Z_1}(z_1) = \frac{1}{\Gamma(\alpha)}\beta^{\alpha} z_1^{\alpha-1} e^{-\beta z_1} \mathcal{I}_{(0,\infty)}(z_1), \quad \alpha > 0,\ \beta > 0,$$

$$f_{Z_2}(z_2) = \lambda e^{-\lambda z_2} \mathcal{I}_{(0,\infty)}(z_2).$$

Assume $X_1 \sim \Gamma(3, 5)$ and $X_2 \sim exp(5)$ are independent random variables.

1. Derive the joint density $f(x_1, x_2)$.
2. Derive the joint density of the random variables $Y_1 = \frac{X_2}{X_1+X_2}$ and $Y_2 = X_1 + X_2$.
3. Prove that $Y_1 \sim Beta(1, 3)$.
4. Derive $E(Y_1)$ and $Var(Y_1)$. You are not allowed to use an MGF approach.
5. Obtain a lower bound on $P(\frac{1}{8} \leq Y_1 \leq \frac{3}{8})$ using the results from 4. Is it useful?
6. Compare this lower bound to the exact probability $P(\frac{1}{8} \leq Y_1 \leq \frac{3}{8})$.

Topic 4

Consider a home antigen test for detecting Covid-19 infection. The probability of a positive outcome of a test given a person is sick is 78.9% and the probability of a negative outcome of a test given a person is healthy is 97.1%. Consider that about 9% of the population has the disease.

1. Find the probability that a test on a randomly chosen member of the population gives a false negative result.

2. Find the probability that a randomly chosen person is sick if they tested positive.
3. Further assume a randomly chosen person performs 3 home tests, two of which turn out to be positive and one turns out to be negative. What is the probability that a person is sick?

Topic 5

Let the joint discrete pdf of x and y be

$$f(x,y) = \begin{cases} \frac{x+3y-2xy}{k} & \text{for } x = 0,1,2;\ y = 0,1,2;\ y \leq x \\ 0 & \text{otherwise} \end{cases}$$

1. Determine the value for k such that $f(x,y)$ is a proper pdf.
2. Does the pdf $f(x,y)$ belong to an exponential class?
3. Find the marginal pdf of x.
4. Find the marginal cdf of x and compute the probability that $x \leq 1$.
5. Derive the MGF of x. Using the MGF compute $E(x)$ and $Var(x)$.
6. Find the conditional pdf of y given $x = 1$.

Hint: $\sum_{k=1}^{n} k(k+1)(k+2)\ldots(k+p) = \frac{n(n+1)(n+2)\ldots(n+p+1)}{p+2}$ for $p \in \{0,1,2,\ldots\}$

Exam WS21/22 2

Topic 1

Let $X_1,\ldots,X_n$ be independent and identically distributed random variables with the following probability density function:

$$f(x) = \frac{\gamma}{(1+x)^{\gamma+1}}\mathcal{I}_{(0,\infty)}(x).$$

1. Is the density $f(x_1)$ a member of the exponential class?

From now on, assume that $\gamma = 3$ and the corresponding pdf is

$$f(x) = \frac{3}{(1+x)^4}\mathcal{I}_{(0,\infty)}(x).$$

2. Derive $E(X_1)$ and $Var(X_1)$. You are not allowed to use an MGF approach.
3. Find the asymptotic distribution of $\overline{X}_n$.
4. Find the asymptotic distribution of $\frac{1}{\overline{X}_n}$.
5. Discuss convergence in probability of $\overline{X}_n$.
6. Find the probability limit of $\sqrt{\overline{X}_n}$.
7. Find the pdf of $Y_1 = \ln(X_1 + 1)$ and identify its distribution.

8. Let $Y_1 = \ln(X_1 + 1)$, $Y_2 = \ln(X_2 + 1)$, $W_1 = Y_1 + Y_2$, and $W_2 = Y_1 - Y_2$. Derive the moment-generating function $M_W(t)$ for $W = (W_1, W_2)$.

Topic 2

Let $X = \begin{pmatrix} X_1 & X_2 & X_3 \end{pmatrix}' \sim N(\mu, \Sigma)$, where $\mu = \begin{pmatrix} 1 & -2 & 3 \end{pmatrix}'$ and $\Sigma = \begin{pmatrix} 1 & -1 & 1 \\ -1 & 2 & -2 \\ 1 & -2 & 3 \end{pmatrix}$.

1. Find $P(X_1 \geq X_2)$ in terms of the cdf of a standard normal, $\Phi(\dots)$.
2. Define $Y_2 = X_3 - aX_1 - bX_2$. Derive the distribution of $Y = \begin{pmatrix} X_1 & X_2 & Y_2 \end{pmatrix}'$.
3. Find such constants a and b that $Y_1 = \begin{pmatrix} X_1 & X_2 \end{pmatrix}'$ and $Y_2 = X_3 - aX_1 - bX_2$ are independent.

Topic 3

Consider an urn that contains 5 red balls and 5 green balls. Assume you are conducting an experiment in which a participant draws one ball from an urn 3 times (**with replacement**) and records the sequence of red and green balls she drew.

Let a random variable X denote the number of green balls a participant draws in a single play. Let a random variable Y denote the money a participant wins in a single play following the rules:

- 1.5€ if the first green ball occurs on the first draw.
- 2€ if the first green ball occurs on the second draw.
- 2.5€ if the first green ball occurs on the third draw.
- -0.5€ if no green ball occurs.

1. Define all possible subsets of the sample space $\mathcal{S}$?
2. What is the probability that a participant draws at least two green balls?
3. What is the probability that a participant doesn't draw any green balls?
4. What is the probability that a participant earns 2€?
5. Calculate $Var(X)$ and $Var(Y)$.

Topic 4

Let the continuous joint density of random variables X and Y be

$$f(x, y) = \frac{1}{16}(k - 2x - 2y)\mathbb{I}_{(0,2)}(x)\mathbb{I}_{(2,4)}(y)$$

1. Determine the value for k such that $f(x, y)$ is a proper pdf.
2. Derive the joint cdf $F(x, y)$.
3. Using result obtained in 2, calculate the $P(X \geq 1, Y \leq 3)$.
4. Calculate the $Cov(X, Y)$.
5. Are X and Y stochastically independent? Explain your answer.

Winter Term 2022/23

Exam WS22/23 1

Topic 1

Let the random variable X have the general arcsine density $X \sim arcsine(a, b)$ with pdf

$$f(x) = \frac{1}{\pi\sqrt{(x-a)(b-x)}}\mathbb{I}_{(a,b)}(x).$$

1. Is the general arcsine distribution a member of the exponential class?
2. The standard arcsine distribution is given by $a = 0$ and $b = 1$. Show that the standard arcsine distribution is a special case of the Beta distribution. **Hint:** $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$
3. Set $a = 0$ and $b = 1$. Show that $\int_{R(X)} f(x)dx = 1$ by using the substitution $x = \sin(u)^2$.
   Hint: $\sin(x)^2 + \cos(x)^2 = 1$

Topic 2

Let the random variable X have the (shifted) exponential density

$$f(x) = \lambda e^{-\lambda(x-a)}\mathbb{I}_{(a,\infty)}(x) \text{ with } \lambda > 0.$$

1. Derive the cdf $F(x)$.
2. Find $E(X)$ via integration by parts.
   Hint: $\displaystyle\int_a^b h(x)g'(x)dx = [h(x)g(x)]_a^b - \int_a^b h'(x)g(x)dx$
3. Find $M_X(t)$.
4. Find $E(X)$ via $M_X(t)$.
5. Find the pdf of $Y = -2X$.

Topic 3

Consider the random variable $\underline{X} = (X_1, ..., X_r)$ with the single elements being each independent and identically hypergeometrically distributed. The density of a hypergeometric distribution is defined by

$$f(x; M, K, n) = \frac{\binom{K}{x}\binom{M-K}{n-x}}{\binom{M}{n}}\mathcal{I}_{\{0,1,\dots,n\}}(x) ,$$

with $M \in \mathbb{N}$, $K = 0, 1, ..., M$, and $n = 1, 2, ..., M$.

1. Show that $\frac{K}{x}\binom{K-1}{x-1} = \binom{K}{x}$.
2. Derive the expectation of X_i, a single element of the vector $\underline{X}$, without using the MGF approach.
   Hint: You might want to use the result from 1.
3. Find the asymptotic distribution of $\overline{X}_r$ for $r \to \infty$.
   Hint: $Var(X) = \frac{nK}{M}\frac{(M-K)(M-n)}{M(M-1)}$.
4. Give the asymptotic distribution of $\frac{1}{\overline{X}_r}$ for $r \to \infty$.

5. Discuss convergence in probability of $\overline{X}_r$.

Topic 4

Let $\boldsymbol{X} = (X_1, X_2, X_3)' \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ and define $\boldsymbol{Y} = \boldsymbol{A}\boldsymbol{x}$.

Suppose $\boldsymbol{\mu} = (0, 1, 3)'$, $\boldsymbol{\Sigma} = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & -1 \\ 0 & -1 & 3 \end{pmatrix}$ and $\boldsymbol{A} = \begin{pmatrix} 0 & 2 & 1 \\ 1 & 1 & 0 \end{pmatrix}$.

1. Find the correlation matrix $\boldsymbol{P}$.
2. Derive the distribution of $\boldsymbol{Y}$.
3. Derive the conditional distribution of Y_1 on Y_2.
4. Derive the regression curve of Y_1 on Y_2.
5. Derive the marginal distributions of $X_{(1)} = (X_1, X_3)$ and $X_{(2)} = X_2$.
6. What can you conclude about the relationship between X_1 and X_3?
7. Derive the conditional distribution of $X_{(1)}$ on $X_{(2)}$.

Exam WS22/23 2

Topic 1

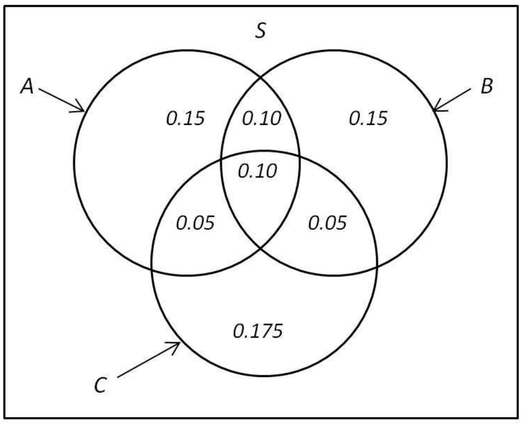
Consider the experiment of independently tossing two fair coins and rolling two fair dice. Let the random variables X_1 and X_2 represent whether heads ($x_i = 0, i = 1, 2$) or tails ($x_i = 1, i = 1, 2$) appear on the first and second coin, respectively. Let the random variables Y_1 and Y_2 represent the number of dots facing up on each of the two dice, respectively.

1. Find the kurtosis of X_1 defined by $\kappa = E\left(\left(\frac{X_1 - \mu}{\sigma}\right)^4\right)$, where μ is the expectation and σ the standard deviation of X_1.
2. State the joint pdf of X_1, X_2, Y_1 and Y_2.
3. Check whether the random vectors $\mathbf{A}$ and $\mathbf{B}$ given below are stochastically independent. Justify your answer and name the according theorem.
$$\mathbf{A} = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = \begin{pmatrix} X_1^2 \\ X_2 - X_1 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} = \begin{pmatrix} Y_1/Y_2 \\ Y_2 + Y_1 \end{pmatrix}$$
4. Derive the joint distribution $f_{\mathbf{A}}(a_1, a_2)$ of A_1 and A_2 by filling in the 12 missing values for the corresponding probabilities in the table.

	$a_1 = 0$	$a_1 = 1$	$\sum_{a_1 \in R(A_1)} f_{\mathbf{A}} = f_{A_2}(a_2)$
$a_2 = -1$			
$a_2 = 0$			
$a_2 = 1$			
$\sum_{a_2 \in R(A_2)} f_{\mathbf{A}} = f_{A_1}(a_1)$			

Topic 2

The diagram shows how probabilities have been assigned to subsets of the sample space S:



1. State the Kolmogorov probability axioms, i.e. for a given sample space S and an associated event space Υ (a sigma algebra on S) state the properties which a probability (set) function P with domain Υ has to fulfill.

2. Find $P(A), P(B)$, $P(C)$, and $P(S)$.

3. Are the events A and C independent? Are they disjoint?

4. Find $P(A \cap B \cap C)$ and $P(A \cup B \cup C)$.

5. Find $P(B \cup C|A)$

6. Find $P(A \cap \overline{B})$.

7. Find $P(\overline{A} \cup \overline{B} \cup \overline{C})$.

Topic 3

Consider the random variable $\underline{X} = (X_1, ..., X_n)$ with the single elements being each independent (but not necessarily identically) $\chi^2(\nu_i)$ distributed random variables, $i = 1, 2, ..., n$.

1. Is the density $f(x_1)$ a member of the exponential class? Explain!

2. Derive the moment-generating function of X_1.

3. Derive the exact distribution of $Y = \frac{1}{n} \sum_{i=1}^{n} X_i$.

4. Derive the mean and variance of Y. Discuss convergence of Y in mean square and probability given that $\underline{\nu} \leq \nu_i \leq \overline{\nu}$ for all i.

5. Derive an asymptotic distribution of Y using a suitable central limit theorem.

Topic 4

Let the random variable X have the Weibull density

$$f(x; k, \lambda) = \frac{k}{\lambda} \left(\frac{x}{\lambda} \right)^{k-1} \exp \left(- \left(\frac{x}{\lambda} \right)^k \right) \mathcal{I}_{[0,\infty)}(x)$$

with $k > 0$ and $\lambda > 0$.

1. Derive the cdf $F(x)$ using integration by substitution.
   Hint: use the substitution $z := \left(\frac{t}{\lambda} \right)^k$.

2. Show that the exponential distribution is a special case of the Weibull distribution.

3. Let $X_1, ..., X_n$ be independent random variables from a Weibull distribution with $k = 1$. Find the moment-generating function of $R = \frac{1}{n}\sum_{i=1}^{n} 5X_i$. Using the MGF identify the distribution of R.

4. Using the MGF from 3. compute $E(R)$ and $Var(R)$.

5. Let $X \sim Exp(\lambda)$. Find the pdf of $Y = X^2$