Problem Set 2: Conditional expectations and linear projections

Review the Concepts and Proofs

1. Give some economic reasoning why we are interested in conditional expectations instead of marginal expectations.
2. Prove that $E(u|x) = 0$ implies $E(u) = 0$ but not vice versa.
3. Prove the law of iterated expectations for discrete and continuous random variables.
4. Prove that $E(u|x) = 0$ does not necessarily imply independence of u and x.
5. Discuss the different concepts of population model, linear projection and (OLS) estimation.
6. Show that a linear projection has the smallest MSE among all linear relationships between y and a set of explanatory variables $\mathbf{x}$.

Exercises

1. Based on the past premier league season you obtain the following probability mass function for the number of goals scored by home and away team:

P(home,away)		away						
		0	1	2	3	4	5	6
home	0	$\frac{27}{380}$	$\frac{21}{380}$	$\frac{17}{380}$	$\frac{1}{38}$	$\frac{7}{380}$	$\frac{1}{380}$	0
	1	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{6}{95}$	$\frac{3}{95}$	$\frac{7}{380}$	$\frac{1}{380}$	$\frac{1}{380}$
	2	$\frac{33}{380}$	$\frac{32}{380}$	$\frac{17}{380}$	$\frac{3}{380}$	$\frac{1}{190}$	0	0
	3	$\frac{1}{20}$	$\frac{13}{190}$	$\frac{2}{95}$	$\frac{1}{190}$	$\frac{1}{190}$	0	0
	4	$\frac{11}{380}$	$\frac{1}{76}$	$\frac{3}{190}$	$\frac{1}{190}$	0	0	0
	5	$\frac{3}{380}$	$\frac{1}{190}$	0	0	0	0	0
	6	0	$\frac{1}{190}$	0	$\frac{1}{380}$	0	0	0

(a) Calculate the expected number of goals of a home and away team, i.e., $E(home)$ and $E(away)$.

(b) What is $E(home|away = 2)$ and $E(away|home = 6)$?

2. Given random variables y, x_1 and x_2, consider the model

$$E(y|x_1, x_2) = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 {x_2}^2 + \beta_4 x_1 x_2.$$

   (a) Find the partial effects of x_1 and x_2 on $E(y|x_1, x_2)$.

   (b) Writing the equation in error form,

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 {x_2}^2 + \beta_4 x_1 x_2 + u,$$

   what can be said about $E(u|x_1, x_2)$? What about $E(u|x_1, x_2, {x_2}^2, x_1 x_2)$?

   (c) In the equation of part b, what can be said about $Var(u|x_1, x_2)$.

3. Let y and x be scalars such that $E(y|x) = \delta_0 + \delta_1(x-\mu) + \delta_2(x-\mu)^2$, where $\mu = E(x)$.

   (a) Find $\partial E(y|x)/\partial x$, and comment on how it depends on x.

   (b) Show that δ_1 is equal to $\partial E(y|x)/\partial x$ averaged across the distribution of x. Compare the interpretation of $\partial E(y|x)/\partial x$ and $\int \partial E(y|x)/\partial x \; f(x)dx$.

   (c) Suppose that x has a symmetric distribution, so that $E[(x - \mu_x)^3] = 0$. Show that $L(y|1, x) = \alpha_0 + \delta_1 x$ for some α_0.

   (d) Sketch the meaning of the terms "population model", "linear projection" and "estimated model". What does "linear" mean?

4. Suppose that $E(y|x_1, x_2) = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_1 x_2$.

   (a) Write this expectation in error form. Describe the properties of the error u.

   (b) Suppose that x_1 and x_2 have zero means. Show that β_1 is the expected value of $\partial E(y|x_1, x_2)/\partial x_1$, where the expectation is with respect to the population distribution of x_2.

   (c) Now add the assumption that x_1 and x_2 are independent of each other. Show that the linear projection of y on $(1, x_1, x_2)$ is
   $L(y|1, x_1, x_2) = \beta_0 + \beta_1 x_1 + \beta_2 x_2$.

5. Consider the two representations

$$y = \mu_1(\boldsymbol{x}, \boldsymbol{z}) + u_1, \qquad E(u_1|\boldsymbol{x}, \boldsymbol{z}) = 0$$
$$y = \mu_2(\boldsymbol{x}) + u_2, \qquad E(u_1|\boldsymbol{x}) = 0$$

   Assuming that $Var(y|\boldsymbol{x}, \boldsymbol{z})$ and $Var(y|\boldsymbol{x})$ are both constant, what can we say about the relationship between $Var(u_1)$ and $Var(u_2)$? (**Hint**: Use the property $E[Var(y|\boldsymbol{x})] \geq E[Var(y|\boldsymbol{x}, \boldsymbol{z})$.)

6. Consider the conditional expectation $E(y|\boldsymbol{x}, \boldsymbol{z}) = g(\boldsymbol{x}) + \boldsymbol{z}\boldsymbol{\beta}$, where $g(.)$ is a general function of $\boldsymbol{x}$ and $\boldsymbol{\beta}$ is a $1 \times M$ vector. Typically, this is called a partial linear model. Show that $E(\tilde{y}|\tilde{\boldsymbol{z}}) = \tilde{\boldsymbol{z}}\beta$, where $\tilde{y} = y - E(y|\boldsymbol{x})$ and $\tilde{z} = z - E(z|\boldsymbol{x})$.

7. Let $\boldsymbol{x}$ be a $1 \times K$ vector with $x_1 = 1$, and define $\mu(\boldsymbol{x}) = E(y|\boldsymbol{x})$. Let $\boldsymbol{\delta}$ be the $K \times 1$ vector of linear projection coefficients of y on $\boldsymbol{x}$, so that $\delta = [E(\boldsymbol{x}'\boldsymbol{x})]^{-1}E(\boldsymbol{x}'y)$. Show that $\boldsymbol{\delta}$ is also the vector of coefficients in the linear projection of $\mu(\boldsymbol{x})$ on $\boldsymbol{x}$.