Solutions 11

1. (a) Convergence in distribution for X_n: $\lim_{n\to\infty} \mathcal{N}\left(\mu + \frac{1}{n}, \frac{n\sigma^2+2}{n}\right) = \mathcal{N}(\mu,\sigma^2)$

   Convergence in distribution for Y_n: $\lim_{n\to\infty} \mathcal{N}\left(\mu, \frac{1}{n}\right) = \mathcal{N}(\mu, 0)$ (degenerate distribution)

   (b) Convergence in probability for Y_n:
   Using Chebyshevs inequality $P\left(|Y_n - \mathrm{E}(Y_n)| < \epsilon\right) \geq 1 - \frac{\mathrm{Var}(Y_n)}{\epsilon^2}$ yields:

$$\lim_{n\to\infty} P(|Y_n - \mu| < \epsilon) \geq \lim_{n\to\infty} 1 - \frac{\frac{1}{n}}{\epsilon^2} = 1$$

   Thus $Y_n \xrightarrow{p} \mu$

   (c) Using Slutsky's theorems

$$A_n = X_n \cdot Y_n \xrightarrow{d} N(\mu^2, \sigma^2\mu^2)$$
$$B_n = X_n - Y_n \xrightarrow{d} N(0, \sigma^2)$$

2. (a) Since X_i only shows up in the indicator function $\mathcal{I}_{(-\infty,t]}(X_i) \equiv Y_i$, we have that Y_i is Bernoulli distributed with the probability of observing Y_i being the probability that $X_i \in (-\infty,t]$ which is given by the cdf of X evaluated at t. Thus $p = F(t)$. Because $Y_n = \sum_{i=1}^{n} Y_i$ and each Y_i is iid Bernoulli distributed, Y_n is Binomial distributed with size n and probability of "success" $p = F(t)$.

$$E(Z_n) = E\left(\frac{1}{n}Y_n\right) = \frac{1}{n}E(Y_n) = \frac{1}{n}nF(t) = F(t)$$
$$Var(Z_{n)} = Var\left(\frac{1}{n}Y_n\right) = \frac{1}{n^2}Var(Y_n) = \frac{1}{n^2}nF(t)(1-F(t)) = \frac{F(t)(1-F(t))}{n}$$

   (b) Convergence in mean-square:

$$\lim_{n\to\infty} \mathrm{E}(Z_n) = \lim_{n\to\infty} F(t) = F(t)$$
$$\lim_{n\to\infty} \mathrm{Var}(Z_n) = \lim_{n\to\infty} \frac{1}{n^2}nF(t)(1-F(t)) = 0$$

   Thus $Z_n \xrightarrow{m} F(t)$ (Corollary 5.1) which implies convergence in probability.

   (c) Note that $Z_n = \frac{1}{n}\sum_{i=1}^{n} \mathcal{I}_{(-\infty,t]}(X_i) = \frac{1}{n}\sum_{i=1}^{n} Y_i$ is an average, thus by CLT

$$\sqrt{n}(Z_n - \mathrm{E}(Y_i)) \xrightarrow{d} N(0,\mathrm{Var}(Y_i))$$
$$\sqrt{n}(Z_n - F(t)) \xrightarrow{d} N(0,F(t)(1-F(t)))$$

3. For a simple fair die it holds:

$$\mathrm{E}(X_i) = \mu = 3.5;\ \mathrm{Var}(X_i) = \sigma^2 = \frac{35}{12}$$

Examine the average of X_i after 200 tosses:
Lindberg-Levy CLT

$$\frac{(\bar{X} - \mu)}{\sigma/\sqrt{n}} \xrightarrow{d} N(0{,}1);\ \bar{X} \overset{\text{asy.}}{\sim} N(\mu, \sigma^2/n)$$

$$\implies \quad \bar{X} \overset{\text{asy.}}{\sim} N\left(3.5, \frac{35/12}{200}\right)$$

$$P(\bar{X} \leq 3.6) = P\left(\frac{\bar{X} - 3.5}{\sqrt{\frac{1}{200}\frac{35}{12}}} \leq \frac{0.1}{\sqrt{\frac{1}{200}\frac{35}{12}}}\right) = \Phi\left(\frac{0.1}{\sqrt{\frac{1}{200}\frac{35}{12}}}\right) \approx 0.7967$$

4. (a) Due to Lindeberg-Levy CLT:

$$\begin{aligned}
\sqrt{n}(\overline{X}_n - \mathrm{E}(X_i)) &\xrightarrow{d} \mathcal{N}(0,\, \mathrm{Var}(X_i)) \\
\sqrt{n}(\overline{X}_n - \nu) &\xrightarrow{d} \mathcal{N}(0,\, 2\nu) \\
\overline{X}_n &\xrightarrow{a} \mathcal{N}\left(\nu,\, \frac{2\nu}{n}\right)
\end{aligned}$$

(b) With the help of theorems 4.2 and 4.3 (Chi Square is a special case of Gamma), we find

$$\overline{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i \sim \text{Gamma}\left(\frac{n\nu}{2}, \frac{2}{n}\right)$$

(c) Apply the Delta-Method:

$$\begin{aligned}
\sqrt{n}(g(\overline{X}_n) - g(\mu)) &\xrightarrow{d} \mathcal{N}(0,\, G^2\mathrm{Var}(X_i)), \text{ where} \\
G = \frac{\partial g(\overline{X}_n)}{\partial \overline{X}_n}\bigg|_{\overline{X}_n = \mu = \nu} &= 2\overline{X}_n\bigg|_{\overline{X}_n = \nu} = 2\nu \\
\sqrt{n}(\overline{X}_n^2 - \nu^2) &\xrightarrow{d} \mathcal{N}(0,\, 8\nu^3) \\
\overline{X}_n^2 &\xrightarrow{a} \mathcal{N}\left(\nu^2,\, \frac{8\nu^3}{n}\right)
\end{aligned}$$

(d) Again use the Delta-Method with $G = \left.\frac{\partial g(\overline{X}_n)}{\partial \overline{X}_n}\right|_{\overline{X}_n=\nu} = \left.3\overline{X}_n^2 \exp(\overline{X}_n^3)\right|_{\overline{X}_n=\nu} = 3\nu^2 \exp(\nu^3)$:

$$\begin{aligned}
\sqrt{n}(\exp(\overline{X}_n^3) - \exp(\nu^3)) &\xrightarrow{d} \mathcal{N}(0, 18\nu^5 \exp(2\nu^3)) \\
\exp(\overline{X}_n^3) &\xrightarrow{a} \mathcal{N}\left(\exp(\nu^3), \frac{18\nu^5 \exp(2\nu^3)}{n}\right)
\end{aligned}$$

(e) We have

$$\begin{aligned}
\mathrm{E}(X_i^2) &= \mathrm{Var}(X_i) + \mathrm{E}(X_i)^2 = 2\nu + \nu^2 = \nu(\nu+2) \\
\mathrm{E}(X_i^4) &= \nu(\nu+2)(\nu+4)(\nu+6) \\
\mathrm{Var}(X_i^2) &= \mathrm{E}(X_i^4) - E(X_i^2)^2 = 8\nu(\nu+2)(\nu+3)
\end{aligned}$$

Due to Lindeberg-Levy CLT:

$$\begin{aligned}
\sqrt{n}(\overline{X_n^2} - E(X_i^2)) &\xrightarrow{d} \mathcal{N}(0, \mathrm{Var}(X_i^2)) \\
\sqrt{n}(\overline{X_n^2} - \nu(\nu+2)) &\xrightarrow{d} \mathcal{N}(0, 8\nu(\nu+2)(\nu+3)) \\
\overline{X_n^2} &\xrightarrow{a} \mathcal{N}\left(\nu(\nu+2), \frac{8\nu(\nu+2)(\nu+3)}{n}\right)
\end{aligned}$$