Mathematical Finance: MF

Exercises (for discussion on Monday, 13.11.2023)

Exercise 1. Let X and Y be integrable random variables on some probability space $(\Omega, \mathcal{F}, P)$. Let $E[X|Y] = Y$ and $E[Y|X] = X$, show that it holds $X = Y$ $\mathbb{P}$-almost surely.
Hint: Consider $E[(X-Y)(h(X)-h(Y))]$ with a suitably chosen auxiliary function h.

Exercise 2. We consider the probability space $([0,1), \mathcal{B}, \lambda)$, where $\mathcal{B}$ denotes the Borel-σ-field on $[0,1)$ and λ is the Lebesgue measure. We define for each $n \in \mathbb{N}$:

$$\mathcal{F}_n := \sigma\left(\left\{\left[\frac{k}{2^n}, \frac{k+1}{2^n}\right) : 0 \leq k \leq 2^n - 1\right\}\right)$$

Let $X : [0,1) \to \mathbb{R}$ be an integrable random variable.

1. Calculate the random variable $E[X|\mathcal{F}_n]$ for all $n \in \mathbb{N}$.
2. Additionally assume that the mapping X is continuous. To which random variable does the limit $\lim_{n\to\infty} E[X|\mathcal{F}_n]$ converge pointwise?

Exercise 3. Let X and Y be two integrable, independent and identical distributed random variables.

1. Show that $E[X|\sigma(X+Y)] = E[Y|\sigma(X+Y)]$.
2. Calculate $E[X|\sigma(X+Y)]$.

Exercise 4. Let $(\Omega, \mathcal{A}, (\mathcal{F}_n)_{n\in\mathbb{N}}, \mathcal{F}, P)$ be a filtered probability space and let σ and τ be $(\mathcal{F}_n)_{n\in\mathbb{N}}$-stopping times. Show that

1. $\max\{\tau, \sigma\}$, $\min\{\tau, \sigma\}$ and $\tau + \sigma$ are $(\mathcal{F}_n)_{n\in\mathbb{N}}$-stopping times.
2. $\mathcal{F}_\tau := \{A \in \mathcal{F} | A \cap \{\tau \leq n\} \in \mathcal{F}_n \text{ for all } n \in \mathbb{N}\}$ is a σ-algebra.

Submission of the homework until: Thursday, 09.11.2023, 10.00 a.m. via OLAT.