# Solutions 7

1. (a) We have $\text{Cov}(X,Y) = \text{E}(X \cdot Y) - \text{E}(X)\text{E}(Y)$ and by the Cauchy-Schwarz Inequality $[\text{E}(X \cdot Y)]^2 \leq \text{E}\left(X^2\right)\text{E}\left(Y^2\right)$ which is equivalent to $|\text{E}(X \cdot Y)| \leq \sqrt{\text{E}\left(X^2\right)\text{E}\left(Y^2\right)}$, thus

$$-\sqrt{\text{E}\left(X^2\right)\text{E}\left(Y^2\right)} \leq \text{E}(X \cdot Y) \leq \sqrt{\text{E}\left(X^2\right)\text{E}\left(Y^2\right)}$$
$$-\sqrt{\text{E}\left(X^2\right)\text{E}\left(Y^2\right)} - \text{E}(X)\text{E}(Y) \leq \text{Cov}(X,Y) \leq \sqrt{\text{E}\left(X^2\right)\text{E}\left(Y^2\right)} - \text{E}(X)\text{E}(Y)$$
$$-\sqrt{8 \cdot 2} - 1 \cdot 2 \leq \text{Cov}(X,Y) \leq \sqrt{8 \cdot 2} - 1 \cdot 2$$

Therefore
$$\text{Cov}(X,Y) \in [-6; 2]$$

(b)
$$\rho_{X,Y} = \frac{\text{Cov}(X,Y)}{\sqrt{\text{Var}(X)\text{Var}(Y)}} = \frac{\text{Cov}(X,Y)}{\sqrt{\left[\text{E}\left(X^2\right) - \text{E}(X)^2\right] \cdot \left[\text{E}\left(Y^2\right) - \text{E}(Y)^2\right]}}$$

From $-6 \leq \text{Cov}(X,Y) \leq 2$ we obtain

$$\frac{-6}{\sigma_X \sigma_Y} \leq \rho_{X,Y} \leq \frac{2}{\sigma_X \sigma_Y}$$
$$\frac{-6}{1 \cdot 2} \leq \rho_{X,Y} \leq \frac{2}{1 \cdot 2}$$

Thus $\rho_{X,Y} \in [-3; 1]$. Because we know already that $\rho_{X,Y} \in [-1; 1]$ this result gives us no additional information about the correlation coefficient!

(c) Since $\rho_{X,Y} \in [-1; 1]$, we find $\text{Cov}(X,Y) \in [-2; 2]$ reformulating part (b).

2. (a) Iff $X$ and $Y$ are independent the joint pdf and cdf can be rewritten as the product of the marginals, thus $f(x,y) = f(x) \cdot f(y)$ and $F(x,y) = F(x) \cdot F(y)$. For the joint moment $E[g(x) \cdot h(y)]$ where $g$ and $h$ are some functions (e.g. $g(x) = x$, $h(y) = y$) it holds that $E[g(x) \cdot h(y)] = E[g(x)] \cdot E[h(y)]$.

(b) The correlation coefficient is defined as $\rho(x,y) = \frac{Cov(x,y)}{\sqrt{Var(x)Var(y)}} = 0$ since $Cov(x,y) = 0$. Thus the random variables $X$ and $Y$ are not correlated. Independence implies uncorrelatedness but not vice versa in general. Consider the following example: $X$ is a symmetric distribution around 0 (and thus $E(x) = E(x^3) = 0$) and $Y = X^2$. Clearly $X$ and $Y$ are not independent but $Cov(x,y) = E(xy) - E(x)E(y) = E(xy) = E(x^3) = 0$ which implies that $X$ and $Y$ are uncorrelated.

(c) Independence of the random variables $X$ and $Y$ implies that any functions $g(x)$ and $h(y)$ are also independent of each other. It is the strongest concept. $E(X|Y) = E(X)$ is also called conditional mean independence and implies that $X$ is uncorrelated with any function of $Y$ but not vice versa in general. Uncorrelatedness of $X$ and $Y$ is the weakest concept among those three. Thus it holds

$$X, Y \text{ independent} \Rightarrow E(X|Y) = E(X) \Rightarrow Cov(X,Y) = 0$$

Note that the opposite is in general not true (compare e.g. b).

3. (a) i.

$$M_X(t) = \sum_{R(X)} e^{tx} f(x) = \sum_{x=0}^{n} e^{tx} \binom{n}{x} p^x (1-p)^{n-x} = \sum_{x=0}^{n} \binom{n}{x} (e^t p)^x (1-p)^{n-x}$$
$$= (pe^t + 1 - p)^n$$

ii.

$$M_X(t) = \sum_{R(X)} e^{tx} f(x) = \sum_{x=0}^{\infty} e^{tx} \frac{e^{-\lambda}\lambda^x}{x!} = e^{-\lambda} \sum_{x=0}^{\infty} \frac{(e^t\lambda)^x}{x!}$$
$$= e^{-\lambda} e^{\lambda e^t} = e^{\lambda(e^t - 1)}$$

iii.

$$M_X(t) = \int_{R(X)} e^{tx} f(x) dx = \frac{1}{\beta^\alpha \Gamma(\alpha)} \int_0^\infty x^{\alpha-1} e^{-x(1/\beta - t)} dx$$
$$= \frac{1}{\beta^\alpha \Gamma(\alpha)} \int_0^\infty x^{\alpha-1} e^{-x\left(\frac{1-\beta t}{\beta}\right)} dx, \qquad \text{with } \tilde{\beta} \equiv \frac{\beta}{1-\beta t} > 0 \text{ as } t < \beta^{-1} \text{ and } \beta > 0$$
$$= \frac{\tilde{\beta}^\alpha}{\beta^\alpha} \int_0^\infty \frac{1}{\tilde{\beta}^\alpha \Gamma(\alpha)} x^{\alpha-1} e^{-x/\tilde{\beta}} dx = \frac{1}{(1-\beta t)^\alpha} = (1-\beta t)^{-\alpha}$$

iv.

$$M_X(t) = \int_{R(X)} e^{tx} f(x) dx = \int_a^b e^{tx} \frac{1}{b-a} dx = \frac{1}{b-a} \left. \frac{1}{t} e^{tx} \right|_a^b = \begin{cases} \frac{e^{bt} - e^{at}}{t(b-a)} & t \neq 0 \\ 1 & t = 0 \end{cases}$$

v.

$$M_X(t) = \int_{R(X)} e^{tx} f(x) dx = \int_0^1 e^{tx} \frac{1}{B(\alpha,\beta)} x^{\alpha-1} (1-x)^{\beta-1} dx$$
$$= \frac{1}{B(\alpha,\beta)} \int_0^1 \sum_{r=0}^{\infty} \frac{(tx)^r}{r!} x^{\alpha-1} (1-x)^{\beta-1} dx$$
$$= \frac{1}{B(\alpha,\beta)} \sum_{r=0}^{\infty} \frac{t^r}{r!} \int_0^1 x^{r+\alpha-1} (1-x)^{\beta-1} dx$$
$$= \frac{1}{B(\alpha,\beta)} \sum_{r=0}^{\infty} \frac{t^r}{r!} B(r+\alpha,\beta) \int_0^1 \frac{1}{B(r+\alpha,\beta)} x^{r+\alpha-1} (1-x)^{\beta-1} dx$$
$$= \sum_{r=0}^{\infty} \frac{B(r+\alpha,\beta)}{B(\alpha,\beta)} \frac{t^r}{r!}$$

(b) i.

$$M'_X(t) = n(pe^t + 1 - p)^{n-1} pe^t$$
$$M''_X(t) = n(n-1)(pe^t + 1 - p)^{n-2} p^2 e^{2t} + n(pe^t + 1 - p)^{n-1} pe^t$$
$$\mathrm{E}(X) = M'_X(0) = np$$
$$\mathrm{E}(X^2) = M''_X(0) = n(n-1)p^2 + np = np(np - p + 1)$$

ii.

$$\begin{aligned}
M_X'(t) &= e^{\lambda(e^t-1)}\lambda e^t \\
M_X''(t) &= e^{\lambda(e^t-1)}\lambda^2 e^{2t} + e^{\lambda(e^t-1)}\lambda e^t = \lambda e^{\lambda(e^t-1)} e^t[\lambda e^t + 1] \\
\mathrm{E}(X) &= M_X'(0) = \lambda \\
\mathrm{E}(X^2) &= M_X''(0) = \lambda(\lambda+1)
\end{aligned}$$

iii.

$$\begin{aligned}
M_X'(t) &= \alpha\beta(1-\beta t)^{-\alpha-1} \\
M_X''(t) &= \alpha(\alpha+1)\beta^2(1-\beta t)^{-\alpha-2} \\
\mathrm{E}(X) &= M_X'(0) = \alpha\beta \\
\mathrm{E}(X^2) &= M_X''(0) = \alpha(\alpha+1)\beta^2
\end{aligned}$$

iv.

$$\begin{aligned}
M_X'(t) &= \frac{(be^{bt}-ae^{at})t(b-a)-(e^{bt}-a^{at})(b-a)}{t^2(b-a)^2} = \frac{(be^{bt}-ae^{at})t-(e^{bt}-a^{at})}{t^2(b-a)} \\
M_X''(t) &= \frac{(b^2e^{bt}-a^2e^{at})t^3(b-a)-[(be^{bt}-ae^{at})t-(e^{bt}-e^{at})]2t(b-a)}{t^4(b-a)^2} \\
&= \frac{(b^2e^{bt}-a^2e^{at})t^2-2(be^{bt}-ae^{at})t+2(e^{bt}-e^{at})}{t^3(b-a)} \\
\mathrm{E}(X) &= M_X'(0) = \lim_{t\to 0}\frac{(be^{bt}-ae^{at})t-(e^{bt}-a^{at})}{t^2(b-a)} \\
&= \lim_{t\to 0}\frac{(b^2e^{bt}-a^2e^{at})t+(be^{bt}-ae^{at})-(be^{bt}-ae^{at})}{2t(b-a)} = \lim_{t\to 0}\frac{(b^2e^{bt}-a^2e^{at})}{2(b-a)} \\
&= \frac{b+a}{2} \\
\mathrm{E}(X^2) &= M_X''(0) = \lim_{t\to 0} M_X''(t) = \lim_{t\to\infty}\frac{b^3e^{bt}-a^3e^{at}}{3(b-a)} = \frac{b^3-a^3}{3(b-a)}
\end{aligned}$$

v.

$$\begin{aligned}
M_X'(t) &= \sum_{r=1}^{\infty}\frac{B(r+\alpha,\beta)}{B(\alpha,\beta)}\frac{t^{r-1}}{(r-1)!} \\
M_X''(t) &= \sum_{r=2}^{\infty}\frac{B(r+\alpha,\beta)}{B(\alpha,\beta)}\frac{t^{r-2}}{(r-2)!} \\
\mathrm{E}(X) &= M_X'(0) = \frac{B(1+\alpha,\beta)}{B(\alpha,\beta)} = \frac{\Gamma(\alpha+1)\Gamma(\beta)\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(1+\alpha+\beta)} = \frac{\alpha}{\alpha+\beta} \\
\mathrm{E}(X^2) &= M_X''(0) = \frac{B(2+\alpha,\beta)}{B(\alpha,\beta)} = \frac{\Gamma(\alpha+2)\Gamma(\beta)\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(2+\alpha+\beta)} \\
&= \frac{(\alpha+1)\Gamma(\alpha+1)\Gamma(\alpha+\beta)}{(\alpha+\beta+1)\Gamma(\alpha+\beta+1)\Gamma(\alpha)} = \frac{\alpha(\alpha+1)}{(\alpha+\beta)(\alpha+\beta+1)}
\end{aligned}$$

(c) i. $\text{Var}(X) = \text{E}(X^2) - \text{E}(X)^2 = np(np - p + 1) - (np)^2 = np(1 - p)$

ii. $\text{Var}(X) = \text{E}(X^2) - \text{E}(X)^2 = \lambda(\lambda + 1) - \lambda^2 = \lambda$

iii. $\text{Var}(X) = \text{E}(X^2) - \text{E}(X)^2 = \alpha(\alpha + 1)\beta^2 - (\alpha\beta)^2 = \alpha\beta^2$

iv.

$$\begin{aligned}
\text{Var}(X) = \text{E}(X^2) - \text{E}(X)^2 &= \frac{b^3 - a^3}{3(b-a)} - \frac{(a+b)^2}{4} = \frac{4(b^3 - a^3) - 3(b-a)(b+a)^2}{12(b-a)} \\
&= \frac{4b^3 - 4a^3 - 3b^3 - 6ab^2 - 3ba^2 + 3ab^2 + 6a^2b + 3a^3}{12(b-a)} \\
&= \frac{b^3 - 3b^2a + 3a^2b - a^3}{12(b-a)} = \frac{(b-a)^3}{12(b-a)} = \frac{(b-a)^2}{12}
\end{aligned}$$

v.

$$\begin{aligned}
\text{Var}(X) = \text{E}(X^2) - \text{E}(X)^2 &= \frac{\alpha(\alpha+1)}{(\alpha+\beta)(\alpha+\beta+1)} - \left(\frac{\alpha}{\alpha+\beta}\right)^2 \\
&= \frac{\alpha}{\alpha+\beta}\left[\frac{\alpha+1}{\alpha+\beta+1} - \frac{\alpha}{\alpha+\beta}\right] \\
&= \frac{\alpha}{\alpha+\beta}\left[\frac{(\alpha+1)(\alpha+\beta) - \alpha(\alpha+\beta+1)}{(\alpha+\beta)(\alpha+\beta+1)}\right] \\
&= \frac{\alpha}{\alpha+\beta}\left[\frac{\alpha^2 + \alpha\beta + \alpha + \beta - \alpha^2 - \alpha\beta - \alpha}{(\alpha+\beta)(\alpha+\beta+1)}\right] = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}
\end{aligned}$$

4. (a)

$$\begin{aligned}
M_{Z_1}(t) &= \prod_{i=1}^{n} M_{X_i}\left(\frac{t}{n}\right) = \prod_{i=1}^{n}\left(1 - \frac{\beta}{n}t\right)^{-\alpha} = \left(1 - \frac{\beta}{n}t\right)^{-n\alpha} \\
Z_1 &\sim \Gamma\left(\alpha n, \frac{\beta}{n}\right)
\end{aligned}$$

(b)

$$M_{Z_2}(t) = \text{E}\left(e^{\sum t_i X_i}\right) = \prod_{i=1}^{n} \text{E}\left(e^{t_i X_i}\right) = \prod_{i=1}^{n}(1 - \beta t_i)^{-\alpha}$$