Mathematical Finance Winter term 2023/2024
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Mathematical Finance: MF

Exercises (for discussion on Monday, 05.02.2024)

Exercise 1. Let $(S^0, ..., S^d)$ be a complete market with end time N, $S^0 = 1$ and EMM Q. Let X be an adapted process. We denote the set of stopping times with respect to the filtration generated by S by $\mathcal{T}$. Show that there is a self financing strategy φ such that $V(\varphi)_0 = \sup_{\tau\in\mathcal{T}} E_Q(X_\tau)$ and $V(\varphi) \geq X$.

Exercise 2. Let $W = (W_t)_{t\in[0,\infty)}$ be a standard Brownian motion and $\mu, \sigma \in \mathbb{R}$. Find all μ, σ such that the process $X = (X_t)_{t\in[0,\infty)}$ given by $X_t := \exp(\mu t + \sigma W_t)$, $t \geq 0$ is a martingale.

Exercise 3. Let W be a standard Brownian motion and $\mu, \sigma \geq 0$. Prove that a Brownian motion with drift X given by

$$X_t = \mu t + \sigma W_t, \quad \mu \in \mathbb{R}, \sigma \geq 0$$

for all $t \geq 0$ is a martingale with respect to the natural filtration $\mathcal{F}_t = \sigma(X_s : s \leq t)$ if and only if $\mu = 0$.

Exercise 4. Let W be a standard Brownian motion.

a) Compute the Itō process representation of the following processes, i.e. write them in the form

$$X = X_0 + \int \dots ds + \int \dots dW_s.$$

(i) $X_t = tW_t$, (ii) $X_t = e^{W_t}$, (iii) $X_t = \sin(-t - W_t)e^{-W_t}$.

b) Compute the covariation $[X, Y]$ for

(i) $X_t = Y_t = W_t^2$, (ii) $X_t = W_t^2$ and $Y_t = tW_t$.

General Remark: An adapted, integrable stochastic process X_t, $t \geq 0$, is called a (continuous time) martingale with respect to the filtration $\mathcal{F}_t$, $t \geq 0$, if and only if

$$\mathbb{E}[X_t|\mathcal{F}_s] = X_s$$

for any $s, t \geq 0$ with $t \geq s$.
Submission of the homework until: Thursday, 01.02.2024, 10.00 a.m. via OLAT.