Mathematical Finance: MF

Exercises (for discussion on Monday, 06.11.2023)

Exercise 1. Consider a market with two securities, 1 and 2. The prices are $S_0^1 = 6, S_0^2 = 11$. Both securities have a term of one year and make a single payout at the end of the year (at $t = 1$), but the payout is random.

1. In this model, there are two outcomes $\Omega = \{\omega_1, \omega_2\}$. Security 1 pays $S_1^1(\omega_1) = 7$ or $S_1^1(\omega_2) = 5$ and security 2 pays $S_1^2(\omega_1) = 14$ or $S_1^2(\omega_2) = 10$.
2. In another model, $\Omega = \{\omega_3, \omega_4\}$. Security 1 pays $S_1^1(\omega_3) = 7$ or $S_1^1(\omega_4) = 5$ and security 2 pays $S_1^2(\omega_3) = 14$ or $S_1^2(\omega_4) = 8$.

For both models determine if there is an arbitrage strategy. If there is one state it explicitly.

Exercise 2. Many banks offer *reverse convertible bonds*. These are characterized by the maturity T, the underlying asset with prices S_0^1, S_T^1, the nominal amount N, the strike K and the assured interest rate r.
At the beginning the buyer pays the nominal amount. At the end the seller either pays back the nominal amount – in the case $S_T^1 > K$ – or gives $n = \frac{N}{K}$ assets – in the case $S_T^1 \leq K$. In both cases the seller pays the assured interest on the nominal value. This means for the holder the value of S^2 at maturity T is

$$S_T^2 := \begin{cases} Ne^{rT} & \text{if } S_T^1 > K \\ \frac{N}{K} S_T^1 + N(e^{rT} - 1) & \text{if } S_T^1 \leq K. \end{cases}$$

Find a combination of the bond and a call or put option with the same payoff as the reverse convertible bond.

Exercise 3. Let $\lambda, \mu > 0$, let X be a $\exp(\lambda)$ and Y be a $\exp(\mu)$ distributed random variable.

(a) Calculate $E(X + Y)$.

(b) Find the density of $X + Y$ in case, X and Y are independent.

(c) Point out where the assumption of independence comes into play.

Exercise 4. Let $(\Omega, \mathcal{A}, P)$ be a probability space, $X : \Omega \to \mathbb{R}$ a random variable with $E(X^2) < \infty$ and $(B_i)_{i\in\mathbb{N}}$ a partition of Ω. Further let $\mathcal{F} := \sigma(\{B_i : i \in \mathbb{N}\})$ be the sigma algebra generated by $(B_i)_{i\in\mathbb{N}}$. Please show:

$$||X - \sum_{i\in\mathbb{N}, P(B_i)>0} c_i 1_{B_i}||_2$$

is minimal, if for all $i \in \mathbb{N}$ mit $P(B_i) > 0$ the equation $c_i = E(X|B_i) := \frac{E(1_{B_i}X)}{P(B_i)}$ holds.

Submission of the homework until: Thursday, 02.11.2023, 10.00 a.m. via OLAT.