Christian-Albrechts-Universität zu Kiel **SS 2023**
Mathematisches Seminar
Prof. Dr. Sören Christensen
Henrik Valett, Fan Yu, Oskar Hallmann, Nele Rothert Sheet 07

Computational Finance

Exercises for all participants

T-Exercise 24 (Mind-map)(4 bonus points)
Create a mind-map with the following list of (unordered) terms that occur throughout the lecture notes. Connect the terms by drawing lines from one term to another and comment each connection/line with some keyword(s) if possible. Try to find additional Keywords and add them to the Mindmap.

- Monte Carlo
- Mean Squared error
- Variance Reduction
- Acceptance/Rejection method
- Partial Differential Equations
- Black-Scholes Model
- American Options
- European Options
- Black-Scholes Formula
- CRR Model
- Control Variates
- Importance Sampling
- Stochastic Mesh
- Stochastic Differential Equations
- Snell Envelope
- Equivalent Martingale Measure
- Arbitrage
- Portfolio
- Martingale
- Discounted Price Process

T-Exercise 25 (Digital option in the Black-Scholes model) (4 points)
A *digital call option* with maturity $T > 0$ and strike $K > 0$ is a European option with payoff

$$V(T) = 1_{\{S(T) \geq K\}}.$$

Find a formula for the initial price of a digital call option in the Black-Scholes model, and compute the perfect hedging strategy.

Hint: To find the formula for the price, it is useful to work with the integral representation and not with the Black-Scholes PDE.

C-Exercise 26 (Explicit pricing formulas in the BS-model) (4 points)
Write a Python function

```
V_0 = CRR_AmOption (S_0, T, M)
```

that computes and returns an approximation to the initial price $V(0)$ of the American option presented in T-Ex.20 (i.e. an American option with payoff defined by (1) on Sheet 06)) with maturity $T > 0$ using the CRR-Model as an approximation to the Black-Scholes model with initial stock price $S(0) > 0$, interest rate $r = 0$ and volatility $\sigma = \sqrt{2}$.
Compare your result to Exercise 20 by implementing the payoff function $V_2(t, S(t))$ in (2) on Sheet 06 for $t = 0$ in a second python function

```
V_0 = CRR_AmOptionDirect (S_0, T).
```

Print the result of both functions for $S(0) = 1, T = 1$ and $M = 500$ for the CRR-Model.

Hint: Modify C-Exercise 06.

C-Exercise 27 (Stochastic calculus) (2+2 points)

(a) Let W be a standard Brownian motion and suppose that X is a stochastic process which evolves according to the stochastic differential equation

$$dX(t) = adt + bdW(t)$$

with $X(0) = 2$. Let further be $f : \mathbb{R} \to \mathbb{R}$ a twice continuously differentiable function.

- State the integral representation of $f(X(t))$.
- Which conditions on a, b and f have to hold in order for $f(X(t))$ to be a martingale?
- Calculate these conditions for $f(x) = exp(x)$.

(b) For a continuous stochastic process Z with $Z > 0$ we define the *stochastic Logarithm* $\mathscr{L}(Z)$ by

$$\mathscr{L}(Z)(t) = \frac{1}{Z} dZ(t) \ .$$

Show that:

(a) $\mathscr{E}(\mathscr{L}(Z)) = \frac{Z}{Z_0}$,
(b) $\mathscr{L}(\mathscr{E}(X)) = X - X_0$,

where X is a continuous stochastic process with $\mathscr{E}(X) > 0$.

Hint: Use Itô's formula on $\ln(Z)(t)$.

Please include your name(s) as comment in the beginning of the file.
Do not forget to include comments in your Python-programs.
Submit until: Fri, 09.06.2023, 10:00