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Advanced Statistics (Winter Term 2023/24)

Problem Set 5

1. Show that the function
   (a) $F(x,y) = \begin{cases} 0 & x+y \leq 0 \\ 1 & x+y > 0 \end{cases}$ is not a valid cumulative distribution function.
   (b) $F(x) = \begin{cases} 0 & x < 0 \\ \frac{1}{4}x & 0 \leq x < 1 \\ \frac{1}{2} + \frac{1}{4}x & 1 \leq x < 2 \\ 1 & x \geq 2 \end{cases}$ is a valid cumulative distribution function.

2. Let the random variable (X,Y) have a joint probability density function given by

$$f(x,y) = 3x(1-xy)\mathcal{I}_{(0,1)}(x)\mathcal{I}_{(0,1)}(y).$$

   (a) Find the marginal probability density functions $f(x)$ and $f(y)$ and the marginal cumulative distribution functions $F(x)$ and $F(y)$.
   (b) Check if X and Y are stochastically independent.
   (c) Derive the conditional probability density function $f(y|x)$ and the corresponding cumulative distribution function $F(y|x)$.
   (d) Determine the following probabilities
      i. $P(X > 0.5)$
      ii. $P(X > 0.5, Y > 0.5)$
      iii. $P(X > Y)$

3. Let the random variable (X_1,X_2) have the following probability density function

$$f(x_1,x_2) = k(x_1x_2 + x_1 + x_2 + 1)\mathcal{I}_{(0,1)}(x_1)\mathcal{I}_{(0,1)}(x_2).$$

   (a) Determine k.
   (b) Find the joint cumulative distribution function $F(x_1,x_2)$.
   (c) Derive the marginal probability density function $f(x_1)$ and its corresponding cumulative distribution function $F(x_1)$.
   (d) Check if X_1 and X_2 are stochastically independent.
   (e) Find the conditional pdf $f(x_2|x_1)$ and its cdf $F(x_2|x_1)$.

4. Let $X = (X_1,X_2)$, where X_i are independent and identical distributed according to the pdf

$$f(x) = \lambda e^{-\lambda x}\mathcal{I}_{(0,\infty)}(x).$$

(a) Specify the constraints on the choice of the parameter λ for $f(x)$ to be a probability density function.

(b) Suppose $Y = \ln(X_1)$. Derive the pdf of Y.

(c) Derive the pdf of the random vector $Z = (Z_1,Z_2)$, where $Z_1 = X_1 + X_2$ and $Z_2 = X_1 - X_2$.