Mathematical Finance: MF

Exercises (for discussion on Monday, 29.01.2024)

Exercise 1. Let $N \in \mathbb{N}$, $(X)_{n \le N}$ be an adapted process with $E(|X_n|) < \infty$ for all $n \le N$, $U = (U_n)_{n \le N}$ the Snell-envelope of X, $M = (M_n)_{n \le N}$ a martingale with $M_0 = 0$ and $A = (A_n)_{n \le N}$ a previsible process with $A_0 = 0$ such that

$$U = U_0 + M + A$$

(Doob decomposition). Let $\mathcal{T}_0$ be the set of $\{0, ...N\}$-valued stopping times and $\sigma \in \mathcal{T}_0$. Show that $E(X_\sigma) = \sup_{\tau \in \mathcal{T}_0} E(X_\tau)$ if and only if both $A_\sigma = 0$ and $X_\sigma = U_\sigma$ hold a.s.

Exercise 2. Let $S = (S^0, S^1, S^2, S^3, S^4, S^5)$ denote an arbitrage-free market with time horizon $N \in \mathbb{N}$. We have $S_n^0 = \exp(rn)$ for some $r \ge 0$. Further let S^2 and S^3 denote the value processes for an American call option respectively an American put option with identical strike K and maturity N on the stock S^1 which does not pay dividends. S^4 and S^5 are price processes for the corresponding European call and put options. Prove the following inequalities

$$S_0^1 - K \le S_0^2 - S_0^3 \le S_0^1 - K \exp(-rN)$$

which are an analogue to the Put-Call-parity for European options.

Exercise 3. Suppose a customer wants to buy a European call option on the stock S^1 with strike 101.75€ and maturity $T = 31.03.2024$. On the market European call options with strikes 100€, 101€, 102€, 103€, 104€, 105€ and maturity $T = 31.03.2024$ are liquidly traded. Their prices are given in the following table:

Strike K_i	100 €	101 €	102 €	103 €	104 €	105 €
Price P_i	7.453 €	6.970 €	6.448 €	5.958 €	5.467 €	5.070 €

Suppose that in a very simple model the price of a European call option on S^1 with strike K and maturity $T = 31.03.2024$ is given by

$$P(K, \vartheta_0, \vartheta_1) = \vartheta_0 + \vartheta_1 K$$

for parameters $\vartheta_0, \vartheta_1 \in \mathbb{R}$.

(a) Find ϑ_0, ϑ_1 such that

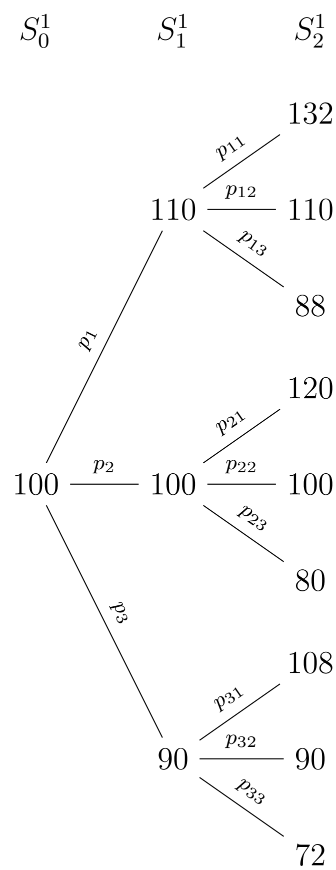
$$\sum_{i=1}^{6} (P_i - P(K_i, \vartheta_0, \vartheta_1))^2$$

is minimized.

(b) Use the result from part (a) to determine the approximated fair price of the European call option with strike 103.5€.

(c) Discuss whether for arbitrary strike K the pricing approach used in this exercise may lead to an arbitrage-free market.

Exercise 4. (*bonus exercise*)
We consider a market $S = (S^0, S^1)$ with three time points with $S^0 \equiv 1$, and the possible developments of S^1 are given by the tree below. As filtration choose $\sigma(S^1)$ and P is given by the transition probabilities $p_1, p_2, \ldots, p_{32}, p_{33}$ which are considered to be > 0.



a) An EMM Q is determined by the transition probabilities $q_1, q_2, \ldots, q_{32}, q_{33}$ in the tree. Determine conditions for Q. Is Q unique?

b) We observe, at time $n = 0$ a liquid European Call option on S^1 with strike 100 and maturity $N = 2$ and price $S^2_0 = 7.5$. Furthermore, we choose the *model assumption* $q_2 = q_{12} = q_{22} = q_{32}$. Show that there exists only one EMM Q of such form fitting to S^2_0. Determine S^2.

c) Show that the market (S^0, S^1, S^2) with S^2 is complete.

Submission of the homework until: Thursday, 25.01.2024, 10.00 a.m. via OLAT.