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Advanced Statistics I (Winter Term 2023/24)

Problem Set 9

1. Let $X_1, \ldots, X_n$ be independent and standard normally distributed random variables. Determine the distributions of the following variables and indicate the theorems you are using:

   (a) $Z_k = \sum_{i=1}^{k} X_i^2 \qquad \text{for } k < n$

   (b) $Y_1 = \delta Z_k \qquad \text{for } \delta \in (0, \infty)$

   (c) $Y_2 = \frac{1}{n} \sum_{i=1}^{n} \frac{X_i + a}{\sqrt{b}} \qquad \text{for } a, b \in \mathbb{R}_+$

2. Let X_1 and X_2 be identically normally distributed random variables with mean $\mu = 1$ and variance $\sigma^2 = 1$. Assume that the random variables X_1 and X_2 are correlated with $\rho = 0.6$. Find the conditional distribution of X_1 given $x_2 = 2$.

3. The joint density of the random variable $\boldsymbol{X} = (X_1, X_2)$ is

   $$f(x_1, x_2) = k \exp\left(-\frac{1}{2} \begin{pmatrix} x_1 - 1 & x_2 + 5 \end{pmatrix} \begin{pmatrix} 2 & -3 \\ -3 & 5 \end{pmatrix} \begin{pmatrix} x_1 - 1 \\ x_2 + 5 \end{pmatrix} \right) \mathbb{I}_{(-\infty,\infty)}(x_1) \mathbb{I}_{(-\infty,\infty)}(x_2).$$

   (a) To which parametric family does the distribution of $\boldsymbol{X}$ belong to?

   (b) Determine k such that f is a proper pdf.

   (c) Derive the marginal and conditional pdfs of X_1 and X_2.

   (d) Derive the regression curve of X_1 on X_2.

4. Let $X = (X_1, X_2, X_3)' \sim \mathcal{N}(0, I)$, where I is the identity matrix. Suppose $b = (1, 2, 3)'$ and A $= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$. Define $Y = Ax + b$.

   (a) Derive the distribution of Y.

   (b) Derive the marginal distributions of $Y_{(1)} = (Y_1, Y_3)'$ and $Y_{(2)} = Y_2$.

   (c) Give the conditional distribution of $Y_{(1)}$ on $Y_{(2)}$ and vice versa.

5. Prove theorem 4.6 with the change of variables technique.