Mathematical Finance Winter term 2023/2024

S. Christensen

P. Le Borne, B. Schroeter, B. Schultz **Sheet MF04**

Mathematical Finance: MF

Exercises (for discussion on Monday, 27.11.2023)

Exercise 1. Let X, Y be adapted processes with $Y_0 = 0$ and $\Delta X \neq -1$. Show that the process

$$Z := \mathcal{E}(X)\left(1 + \frac{1}{\mathcal{E}(X)_-} \bullet Y - \frac{1}{\mathcal{E}(X)} \bullet [X, Y]\right)$$

satisfies:

1. *Regular exercise (2 points):* $Z = \mathcal{E}(X)\left(1 + \frac{1}{\mathcal{E}(X)} \bullet Y\right)$
2. *Bonus exercise (2 points):* $Z = 1 + Z_- \bullet X + Y$

Exercise 2. *(3 points)*

Prove Lemma 3.B.1. from the lecture, that is:

Let $Q \sim P$ denote a probability measure with density process Z. Moreover, let X denote an $\mathcal{F}_n$ -measurable random variable for some n. Then

$$E_Q(X|\mathcal{F}_m) = \frac{E(XZ_n|\mathcal{F}_m)}{Z_m}$$

for any $m \leq n$.

Exercise 3. Let $(\Omega, (\mathcal{F}_n)_{n\in\mathbb{N}}, \mathcal{F}, P)$ be a filtered probability space, $\hat{\tau}$ and τ $(\mathcal{F}_n)_{n\in\mathbb{N}}$-stopping times with $\hat{\tau} \leq \tau$ a.s. and let X be an adapted stochastic process. We denote by $\mathcal{F}_{\hat{\tau}}$ and $\mathcal{F}_\tau$ the σ-algebras of the $\hat{\tau}$- and τ-past, respectively. Show: If X is a martingale, then $E(X_\tau|\mathcal{F}_{\hat{\tau}}) = X_{\hat{\tau}}$ if τ is a.s. bounded.

Exercise 4. Let X be an integrable random variable and $(\mathcal{F}_i)_{i\in I}$ a family of σ-algebras. We set $Z_i := E(X|\mathcal{F}_i)$ for all $i \in I$. Show that the family $(Z_i)_{i\in I}$ is uniformly integrable, i.e.

$$\lim_{R\to\infty} \sup_{i\in I} E(\mathbb{1}_{\{|Z_i|\geq R\}}|Z_i|) = 0.$$

Exercise 5. *(3 points)*

Show that an adapted process X is a martingale if and only if for each bounded stopping time τ we have $E(|X_\tau|) < \infty$ and $E(X_\tau) = E(X_0)$.

Submission of the homework until: Thursday, 23.11.2023, 10.00 a.m. via OLAT.