Solutions to Problem Set 5

1. (a) Check conditions of theorem 2.4:
      i. $\lim\limits_{x\to-\infty} F(x,y) = 0$ and $\lim\limits_{y\to-\infty} F(x,y) = 0$ ✓
      ii. $\lim\limits_{x,y\to\infty} F(x,y) = 1$ ✓
      iii. We have for $x_1 < x_2$ and $y_1 < y_2$ that $x_1 + y_1 < x_2 + y_2$ which implies $F(x_1,y_1) \leq F(x_2,y_2)$ ✓.
      iv. x,y are continuous random variables, but $F(x,y)$ is discontinuous, thus $F(x,y)$ is not a cdf!
      To see why jumps are problematic consider

$$\begin{aligned} P(-1 \leq x \leq 1; 0 \leq y \leq 2) &= F(1,2) - F(-1,2) - F(1,0) + F(-1,0) \\ &= 1 - 1 - 1 + 0 = -1 \end{aligned}$$

   (b) Check conditions of theorem 2.1:
      i. $\lim\limits_{x\to-\infty} F(x) = 0$ and $\lim\limits_{x\to\infty} F(x) = 1$ ✓
      ii. For $x_1 < x_2$ we have $F(x_1) \leq F(x_2)$ ✓
      iii. Every part of the pdf is continuous $\Rightarrow$ $F(x)$ is right continuous ✓

2. (a) Use theorem 2.8 to derive the marginal pdfs:

$$\begin{aligned} f_x(x) &= \int_{y\in R(y)} f(x,y)dy = \int_0^1 3x(1-xy)dy = 3x\left[y - \frac{1}{2}xy^2\right]_0^1 \\ &= 3x\left(1 - \frac{1}{2}x\right)\mathcal{I}_{(0,1)}(x) \\ f_y(y) &= \int_0^1 3x(1-xy)dx = 3\int_0^1 (x - x^2y)dx = 3\left[\frac{1}{2}x^2 - \frac{y}{3}x^3\right]_0^1 \\ &= 3\left(\frac{1}{2} - \frac{y}{3}\right)\mathcal{I}_{(0,1)}(y) \\ F_x(x) &= \int_{s=0}^x f_x(s)ds = \int_0^x 3s\left(1 - \frac{1}{2}s\right)ds = 3\int_0^x \left(s - \frac{1}{2}s^2\right)ds \\ &= 3\left[\frac{1}{2}s^2 - \frac{1}{6}s^3\right]_0^x = 3\left(\frac{1}{2}x^2 - \frac{1}{6}x^3\right)\mathcal{I}_{(0,1)}(x) + \mathcal{I}_{[1,\infty)}(x) \\ F_y(y) &= \int_0^y 3\left(\frac{1}{2} - \frac{s}{3}\right)ds = 3\left[\frac{1}{2}s - \frac{1}{6}s^2\right]_0^y = 3\left(\frac{1}{2}y - \frac{y^2}{6}\right)\mathcal{I}_{(0,1)}(y) + \mathcal{I}_{[1,\infty)}(y) \end{aligned}$$

   (b) Applying theorem 2.10:
   $f_x(x) \cdot f_y(y) = 3x(1-\frac{1}{2}x)3(\frac{1}{2}-\frac{y}{3}) \neq 3x(1-xy) = f(x,y)$, thus x and y are stochastically dependent.

(c)

$$f(y|x) = \frac{f(x,y)}{f(x)} = \frac{3x(1-xy)}{3x(1-\frac{1}{2}x)} = \frac{1-xy}{1-\frac{1}{2}x}\mathcal{I}_{(0,1)}(x)\mathcal{I}_{(0,1)}(y)$$

$$F(y|x) = \int_0^y f(s|x)ds = \int_0^y \frac{1-xs}{1-\frac{1}{2}x}ds = \frac{1}{1-\frac{1}{2}x}\left[s - \frac{x}{2}s^2\right]_0^y$$

$$= \frac{y-\frac{xy^2}{2}}{1-\frac{1}{2}x}\mathcal{I}_{(0,1)}(x)\mathcal{I}_{(0,1)}(y) + \mathcal{I}_{(0,1)}(x)\mathcal{I}_{[1,\infty)}(y)$$

(d) i. $P(X > 0.5) = 1 - P(X \leq 0.5) = 1 - F_x(0.5) = 1 - 3\left(\frac{1}{2}0.5^2 - \frac{1}{6}0.5^3\right) = 0.6875$

ii.

$$P(X > 0.5, Y > 0.5) = \int_{x=0.5}^1 \int_{y=0.5}^1 f(x,y)dx\,dy = \int_{x=0.5}^1 \int_{y=0.5}^1 3x(1-xy)dx\,dy$$

$$= 3\int_{x=0.5}^1 x\left[y - \frac{1}{2}xy^2\right]_{0.5}^1 dx = 3\int_{0.5}^1 \left(0.5 - \frac{3}{8}x^2\right)dx$$

$$= 3\left[\frac{1}{4}x^2 - \frac{1}{8}x^3\right]_{0.5}^1 = \frac{15}{64}$$

iii.

$$P(X > Y) = \int_{x=y}^1 \int_{y=0}^1 f(x,y)dx\,dy = \int_{y=0}^1 3\int_{x=y}^1 (x - x^2y)dx\,dy$$

$$= \int_{y=0}^1 3\left[\frac{1}{2}x^2 - \frac{x^3y}{3}\right]_y^1 dy = 3\int_{y=0}^1 \left(\frac{1}{2} - \frac{y}{3} - \frac{y^2}{2} + \frac{y^4}{3}\right)dy$$

$$= 3\left[\frac{1}{2}y - \frac{1}{6}y^2 - \frac{1}{6}y^3 + \frac{1}{15}y^5\right]_0^1 = 0.7$$

3. We find that $f(x_1,x_2) = k(x_1+1)(x_2+1)\mathcal{I}_{(0,1)}(x_1)\mathcal{I}_{(0,1)}(x_2)$.

(a)

$$1 \stackrel{!}{=} \int_{x_1}\int_{x_2} f(x_1,x_2)dx_1\,dx_2 = k\int_{x_1=0}^1\int_{x_2=0}^1 (x_1+1)(x_2+1)dx_1\,dx_2$$

$$= k\int_{x_1=0}^1 (x_1+1)dx_1 \int_{x_2=0}^1 (x_2+1)dx_2 = k\left[\frac{1}{2}x_1^2 + x_1\right]_0^1\left[\frac{1}{2}x_2^2 + x_2\right]_0^1$$

$$= \frac{9k}{4}$$

$$\Rightarrow k = \frac{4}{9}$$

(b) For $x_1,x_2 \in (0,1)$:

$$F(x_1.x_2) = \int_{s_1=0}^{x_1}\int_{s_2=0}^{x_2} \frac{4}{9}(s_1+1)(s_2+1)ds_1\,ds_2$$

$$= \frac{4}{9}\left[\frac{1}{2}s_1^2 + s_1\right]_0^{x_1}\left[\frac{1}{2}s_2^2 + s_2\right]_0^{x_2} = \frac{4}{9}\left(\frac{1}{2}x_1^2 + x_1\right)\left(\frac{1}{2}x_2^2 + x_2\right)$$

Thus in general we obtain

$$\begin{aligned}
F(x_1,x_2) &= \frac{4}{9}\left(\frac{1}{2}x_1^2+x_1\right)\left(\frac{1}{2}x_2^2+x_2\right)I_{0,1}(x_1)I_{0,1}(x_2) \\
&+\frac{2}{3}\left(\frac{1}{2}x_1^2+x_1\right)I_{(0,1)}(x_1)I_{[1,\infty)}(x_2) \\
&+\frac{2}{3}\left(\frac{1}{2}x_2^2+x_2\right)I_{[1,\infty)}(x_1)I_{(0,1)}(x_2)+I_{(1,\infty)}(x_1)I_{(1,\infty)}(x_2)
\end{aligned}$$

(c) Using again theorem 2.8 we obtain

$$\begin{aligned}
f(x_1) &= \int_{x_2=0}^{1}\frac{4}{9}(x_1+1)(x_2+1)dx_2 = \frac{4}{9}(x_1+1)\left[\frac{1}{2}x_2^2+x_2\right]_0^1 = \frac{2}{3}(x_1+1)\mathcal{I}_{(0,1)}(x_1) \\
F(x_1) &= \int_{s_1=0}^{x_1}\frac{2}{3}(s_1+1)ds_1 = \frac{2}{3}\left[\frac{1}{2}s_1^2+s_1\right]_0^{x_1} = \frac{2}{3}\left(\frac{1}{2}x_1^2+x_1\right)\mathcal{I}_{(0,1)}(x_1)+\mathcal{I}_{[1,\infty)}(x_1)
\end{aligned}$$

(d) Due to symmetry we have $f(x_2)=\frac{2}{3}(x_2+1)\mathcal{I}_{(0,1)}((x_2)$, thus applying theorem 2.9 we obtain $f(x_1)\cdot f(x_2)=\frac{2}{3}(x_1+1)\frac{2}{3}(x_2+1)=\frac{4}{9}(x_1+1)(x_2+1)=f(x_1,x_2)$ thus x_1 and x_2 are stochastically independent.

(e)

$$\begin{aligned}
f(x_2|x_1) &= \frac{f(x_1,x_2)}{f(x_1)} = \frac{\frac{4}{9}(x_1+1)(x_2+1)}{\frac{2}{3}(x_1+1)} = \frac{2}{3}(x_2+1)\mathcal{I}_{(0,1)}(x_1)\mathcal{I}_{(0,1)}(x_2) = f(x_2) \\
F(x_2|x_1) &= \int_0^{x_2} f(s_2|x_1)ds_2 = \int_0^{x_2}\frac{2}{3}(s_2+1)ds_2 = \frac{2}{3}\left[\frac{s_2^2}{2}+s_2\right]_0^{x_2} = \\
&= \frac{2}{3}\left(\frac{x_2^2}{2}+x_2\right)\mathcal{I}_{(0,1)}(x_1)\mathcal{I}_{(0,1)}(x_2)+\mathcal{I}_{(0,1)}(x_1)\mathcal{I}_{[1,\infty)}(x_2)
\end{aligned}$$

4. (a)

$$\begin{aligned}
1 &\overset{!}{=} \int_0^\infty \lambda e^{-\lambda x}dx = -e^{-\lambda x}\Big|_0^\infty = 1-\lim_{x\to\infty}e^{-\lambda x} \\
&\Rightarrow \lim_{x\to\infty}e^{-\lambda x}=0 \Rightarrow \lambda>0
\end{aligned}$$

(b) Use the change of variables technique (theorem 2.12) with $y=ln(x_1)=g(x_1)$. First check the properties:

i. $g'(x_1)=\frac{1}{x_1}\neq 0\ \forall x_1\in(0,\infty)$ ✓

ii. $x_1=e^y=g^{-1}(y)$ exists $\forall y\in(-\infty,\infty)$ ✓

Thus we can apply the theorem

$$h(y)=f(g^{-1}(y))\left|\frac{\partial g^{-1}(y)}{\partial y}\right| = \lambda e^{-\lambda e^y}e^y\mathcal{I}_{(-\infty,\infty)}(y)$$

(c) Due to independence we have $f(x_1,x_2)=\lambda^2e^{-\lambda(x_1+x_2)}\mathcal{I}_{(0,\infty)}(x_1)\mathcal{I}_{(0,\infty)}(x_2)$ with the help of the multivariate version of the change of variables technique (theorem 2.13) we have

i. $Z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ x_1 - x_2 \end{pmatrix} = g(x_1,x_2)$ which is differentiable $\forall x_1,x_2 \in (0,\infty)$✓

ii. $X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \frac{z_1+z_2}{2} \\ \frac{z_1-z_2}{2} \end{pmatrix} = g^{-1}(z_1,z_2) \ \exists$ for $z_1 \in (0,\infty)$ and $z_2 \in (-z_1,z_1)$✓
The ranges of Z_1 and Z_2 are obtained as follows:
Since $X_1,X_2 > 0$ we have $Z_1 + Z_2 > 0$ and $Z_1 - Z_2 > 0$. Adding these Inequalities yields $2Z_1 > 0$ or $Z_1 > 0$. Solving these Inequalities for Z_2 each gives $-Z_1 < Z_2 < Z_1$.

iii. $J = \begin{pmatrix} \frac{\partial g_1^{-1}}{\partial z_1} & \frac{\partial g_1^{-1}}{\partial z_2} \\ \frac{\partial g_2^{-1}}{\partial z_1} & \frac{\partial g_2^{-1}}{\partial z_2} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix} \Rightarrow det(J) = -\frac{1}{4} - \frac{1}{4} = -\frac{1}{2} \neq 0$✓

Thus

$$h(Z) = f(g_1^{-1}(z_1,z_2),g_2^{-1}(z_1,z_2))|det(J)| = \frac{1}{2}\lambda^2 e^{-\lambda z_1} I_{(0,\infty)}(z_1) I_{(-z_1,z_1)}(z_2)$$