Mathematical Finance Winter term 2023/2024
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Mathematical Finance: MF

Exercises (for discussion on Monday, 20.11.2023)

Exercise 1.

1. Let $X_1, X_2, \dots$ be i.i.d. square-integrable random variables with $E(X_1) = 0$. Find a $c \in \mathbb{R}$, such that

$$(M_n) := \left(\left(\sum_{i=1}^{n} X_i \right)^2 - nc \right)_{n \in \mathbb{N}}$$

is a martingale with respect to the filtration $\mathcal{F}_n = \sigma(X_1, \dots, X_n)$.

2. Now, additionally assume $P(X_1 = 1) = P(X_1 = -1) = \frac{1}{2}$. Let $k \in \mathbb{N}$ and set $\tau_k := \inf\{n \geq 0 : |\sum_{i=1}^{n} X_i| \geq k\}$. Find $E(\tau_k)$.

Exercise 2. (2 points)
Prove Theorem 3.13 (Doob decomposition) from the lecture notes, i.e. show that for every adapted process X there are a martingale M with $M_0 = 0$ and a predictable process A with $A_0 = 0$ such that

$$X = X_0 + M + A.$$

Exercise 3. (2 points)
For each adapted process $X = (X_n)_{n \in \mathbb{N}}$ define $\mathcal{E}(X) := \prod_{i=1}^{n}(1 + \Delta X_i)$. Let X be an adapted process with $\Delta X \neq -1$. Show that

$$\frac{1}{\mathcal{E}(X)} = \mathcal{E}\left(-\frac{1}{1+\Delta X} \bullet X\right) = \mathcal{E}\left(-X + \frac{1}{1+\Delta X} \bullet [X,X]\right).$$

You may use (without proof) *Yor's formula*

$$\mathcal{E}(X)\mathcal{E}(Y) = \mathcal{E}(X + Y + [X,Y]).$$

Exercise 4. Let X be the binomial random walk, i.e. it holds $X_0 = 0$ and

$$X_n = \sum_{i=1}^{n} R_i \text{ for all } n \in \mathbb{N},$$

where $R_1, R_2, \ldots$ are i.i.d. random variables with $P(R_i = 1) = P(R_i = -1) = \frac{1}{2}$ for all $i \in \mathbb{N}$, and let $f : \mathbb{N} \to \mathbb{R}$ be a function. We define *the discrete first and second derivative of* f as

$$\begin{aligned} f'(x) &:= \frac{f(x+1) - f(x-1)}{2}, \\ f''(x) &:= f(x-1) + f(x+1) - 2f(x) \end{aligned}$$

and in addition

$$F_n' := f'(X_{n-1}) \quad \text{and} \quad F_n'' := f''(X_{n-1}).$$

1. Show that $f(X_n) = f(X_0) + (F' \bullet X)_n + A_n$, where $A := (\frac{1}{2} \sum_i^n F_i'')_{n \in \mathbb{N}}$.
2. Find the Doob decomposition of $f(X) = (f(X_n))_{n \in \mathbb{N}}$.

Exercise 5. Two players agree on a fair coin tossing game where you gain/lose 1 from/to the other player if your side shows up/doesn't show up. The game ends when one of the players is bankrupt. Player A is endowed with a capital of $k_A \in \mathbb{N}$ Euro and player B with $k_B \in \mathbb{N}$ Euro. What is the chance of player A ending up bankrupt? State the mathematical model you use for your answer and give a proof of your answer within your model.

Submission of the homework until: Thursday, 16.11.2023, 10.00 a.m. via OLAT.