Solutions 8

1. (a) MGF of binomial: $M_X(t) = (1-p+pe^t)^n$ with $n=1 \Rightarrow M_X(t) = 1-p+pe^t$ which is the MGF of a Bernoulli distribution.

   (b) MGF of a Gamma distribution: $M_X(t) = (1-\beta t)^{-\alpha}$ with $\beta = \theta$ and $\alpha = 1$ one gets $M_X(t) = (1-\theta t)^{-1}$ which is the MGF of an exponential distribution.

   (c) Using the MGF of a Gamma distribution with $\beta = 2$ and $\alpha = \frac{\nu}{2}$ one gets $M_X(t) = (1-2t)^{-\frac{\nu}{2}}$ which is the MGF of a Chi squared distribution.

   (d) The MGF of a Beta distribution is

$$\begin{aligned}
M_X(t) &= 1 + \sum_{r=1}^{\infty} \frac{B(r+\alpha,\beta)}{B(\alpha,\beta)} \frac{t^r}{r!} \quad \text{with } \alpha=\beta=1 \\
&= 1 + \sum_{r=1}^{\infty} \frac{B(r+1,1)}{B(1,1)} \frac{t^r}{r!} \\
&= 1 + \sum_{r=1}^{\infty} \frac{\Gamma(r+1)\Gamma(1)}{\Gamma(r+2)} \frac{\Gamma(2)}{\Gamma(1)\Gamma(1)} \frac{t^r}{r!} \\
&= 1 + \sum_{r=1}^{\infty} \frac{\Gamma(r+1)}{(r+1)\Gamma(r+1)} \frac{t^r}{r!} = 1 + \sum_{r=1}^{\infty} \frac{t^r}{(r+1)!} \quad \text{with } k = r+1 \\
&= 1 + \sum_{k=2}^{\infty} \frac{t^{k-1}}{k!} = 1 + \frac{1}{t}\sum_{k=2}^{\infty} \frac{t^k}{k!} = 1 + \frac{1}{t}\left[\sum_{k=0}^{\infty} \frac{t^k}{k!} - t - 1\right] \\
&= \frac{1}{t}(e^t - 1)
\end{aligned}$$

   which is the MGF of a standard uniform distribution.

   (e) The MGF of a Poisson distributed random variable X is given by $M_X(t) = e^{\lambda(e^t-1)}$. Define the "standardized" random variable $Z = \frac{X-\lambda}{\sqrt{\lambda}}$. The MGF of Z is

$$\begin{aligned}
M_Z(t) &= \mathrm{E}\left[e^{tZ}\right] = \mathrm{E}\left[e^{t\frac{X-\lambda}{\sqrt{\lambda}}}\right] = e^{-t\sqrt{\lambda}}\mathrm{E}\left[e^{\frac{t}{\sqrt{\lambda}}X}\right] = e^{-t\sqrt{\lambda}} M_X\left(\frac{t}{\sqrt{\lambda}}\right) \\
&= e^{-t\sqrt{\lambda}} \exp\left[\lambda\left(\exp\left(\frac{t}{\sqrt{\lambda}}\right) - 1\right)\right] = \exp\left[-t\sqrt{\lambda} + \lambda\sum_{\zeta=0}^{\infty} \frac{t^\zeta}{\zeta!}\lambda^{-\frac{\zeta}{2}} - \lambda\right] \\
&= \exp\left[-t\sqrt{\lambda} + \sum_{\zeta=1}^{\infty} \frac{t^\zeta}{\zeta!}\lambda^{-\frac{\zeta-2}{2}}\right] = \exp\left[\sum_{\zeta=2}^{\infty} \frac{t^\zeta}{\zeta!}\lambda^{-\frac{\zeta-2}{2}}\right] \\
&= \exp\left[\frac{t^2}{2} + \sum_{\zeta=3}^{\infty} \frac{t^\zeta}{\zeta!}\lambda^{-\frac{\zeta-2}{2}}\right]
\end{aligned}$$

$$\lim_{\lambda\to\infty} M_Z(t) = \exp\left(\frac{t^2}{2}\right)$$

   which is the MGF of a standard normal distributed random variable. Thus also X itself is normally distributed.

2.

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\begin{aligned}
M_{\mathbf{Z}}(\mathbf{t}) &= \mathrm{E}\left(e^{t_1 z_1 + t_2 z_2}\right) = \mathrm{E}\left(e^{t_1(x_1+x_2)+t_2(x_1-x_2)}\right) = \\
&= \mathrm{E}\left(e^{x_1(t_1+t_2)+x_2(t_1-t_2)}\right) = \mathrm{E}\left(e^{x_1(t_1+t_2)}e^{x_2(t_1-t_2)}\right) = \\
&= \mathrm{E}\left(e^{x_1(t_1+t_2)}\right)\mathrm{E}\left(e^{x_2(t_1-t_2)}\right) = M_{X_1}(t_1+t_2)M_{X_2}(t_1-t_2) = \\
&= \frac{\lambda}{\lambda - t_1 - t_2}\cdot\frac{\lambda}{\lambda - t_1 + t_2} = \frac{\lambda^2}{(\lambda - t_1 - t_2)(\lambda - t_1 + t_2)}
\end{aligned}
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3. (a)

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\begin{aligned}
M_\omega(t) = \mathrm{E}(e^{t\omega}) &= \int e^{t\omega}\frac{1}{\sqrt{2\pi}\sigma}\exp\left(-\frac{(\omega-\mu)^2}{2\sigma^2}\right)d\omega = \int \frac{1}{\sqrt{2\pi}\sigma}e^{t\omega}\exp\left(-\frac{\omega^2-2\mu\omega+\mu^2}{2\sigma^2}\right)d\omega \\
&= \int \frac{1}{\sqrt{2\pi}\sigma}\exp\left(-\frac{\omega^2-2\omega(\mu+t\sigma^2)+\mu^2}{2\sigma^2}\right)d\omega \\
&= \int \frac{1}{\sqrt{2\pi}\sigma}\exp\left(-\frac{\omega^2-2\omega(\mu+t\sigma^2)+(\mu+t\sigma^2)^2-(\mu+t\sigma^2)^2+\mu^2}{2\sigma^2}\right)d\omega \\
&= \int \exp\left(\frac{(\mu+t\sigma^2)^2-\mu^2}{2\sigma^2}\right)\frac{1}{\sqrt{2\pi}\sigma}\exp\left(-\frac{(\omega-\mu-t\sigma^2)^2}{2\sigma^2}\right)d\omega \\
&= \exp\left(\frac{(\mu+t\sigma^2)^2-\mu^2}{2\sigma^2}\right) = \exp\left(\frac{\mu^2+2\mu t\sigma^2+t^2\sigma^4-\mu^2}{2\sigma^2}\right) \\
&= \exp\left(\mu t + \frac{1}{2}t^2\sigma^2\right)
\end{aligned}
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(b)

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\begin{aligned}
F_X(x) &= P(X \le x) = P(|\omega| < x) = P(-x \le \omega \le x) = P(\omega \le x) - P(\omega \le -x) \\
&= \Phi(x) - \Phi(-x) \\
f(x) &= \frac{\partial F_X(x)}{\partial x} = \phi(x) + \phi(-x) = 2\phi(x) = \frac{2}{\sqrt{2\pi}}e^{-\frac{x^2}{2}} = \sqrt{\frac{2}{\pi}}e^{-\frac{x^2}{2}}\mathcal{I}_{(0,\infty)}(x)
\end{aligned}
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