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Advanced Statistics (Winter Term 2023/24)

Problem Set 11

1. Let the sequences of random variables $\{X_n\}$ and $\{Y_n\}$ each follow normal distributions with
$$X_n \sim \mathcal{N}\left(\mu + \frac{1}{n}, \frac{n\sigma^2 + 2}{n}\right) \qquad \text{and} \qquad Y_n \sim \mathcal{N}\left(\mu, \frac{1}{n}\right) .$$
   (a) Derive the limiting distribution of X_n and Y_n.
   (b) Show that $\text{plim}(Y_n) = \mu$.
   (c) Find the asymptotic distributions of
$$A_n = X_n \cdot Y_n \qquad \text{and} \qquad B_n = X_n - Y_n .$$

2. Let $X_1, \ldots, X_n$ be independent and identically distributed random variables with cumulative distribution function (cdf) $F(x)$ and
$$Y_n = \sum_{i=1}^{n} \mathbb{I}_{(-\infty,t]}(X_i), \quad t \in (-\infty, \infty).$$
   (a) Show that Y_n follows a Binomial distribution with parameters n and $p = F(t)$ and find the mean and the variance of $Z_n = \frac{1}{n} Y_n$.
   (b) Check convergence in probability of Z_n and state (if possible) the respective probability limit.
   (c) Derive the asymptotic distribution of Z_n.

3. Consider an ideal die that is rolled n times. The random variable X_i denotes the number of dots facing up on the ith attempt. Find the probability that the average number of dots does not exceed 3.6 after 200 attempts.

4. Let the random variable X_i following a Chi Square distribution
$$f(x) = \frac{1}{2^{\frac{\nu}{2}} \Gamma\left(\frac{\nu}{2}\right)} x^{\frac{\nu}{2}-1} e^{-\frac{x}{2}} \mathcal{I}_{(0,\ \infty)}(x)$$
have stochastically independent realizations.
   (a) Find the asymptotic distribution of $\overline{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$.
   (b) Derive the exact distribution of $\overline{X}_n$.
   (c) Find the asymptotic distribution of $\overline{X}_n^2$.
   (d) Calculate the asymptotic distribution of $\exp\left(\overline{X}_n^3\right)$.
   (e) Give the asymptotic distribution of $\overline{X_n^2} = \frac{1}{n}\sum_{i=1}^{n} X_i^2$.