H.W - 07 / Comp Fin

Group 27

T- Exo 25

Digital call option, $T \geq 0$, $k > 0$

$$V(T) = 1_{\{S(T) \geq K\}}$$

1- Find a formula for the initial price of a digital call opt- in the BS- model

we know that the general formula of the fair price of a call opt- in B-S model is $V(0) = B_0 E_Q(V(T) \cdot B(T)^{-1} | F_0)$

$$= e^{-r \cdot 0} E_Q(1_{\{S(T) \geq K\}} \cdot e^{-rT} | F_0) = e^{-rT} E_Q(1_{\{S(T) - K \geq 0\}} | F_0)$$

$$\Rightarrow V(T) = X_T = (S(T) - K)^+ = 1_{\{S(T) - K \geq 0\}}$$

Using Pricing by integration P-37

$$V(0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (S(T) - K)^+ e^{-rT} e^{-x^2/2} dx.$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [S(0) \exp((r - \sigma^2/2)T + \sigma\sqrt{T} \cdot x)) - K] e^{-rT} e^{-x^2/2} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp[\lg(S(0) \exp((r - \sigma^2/2)T + \sigma\sqrt{T} \cdot x))) - \lg(K)] e^{-rT} e^{-x^2/2} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (\exp[\lg(\frac{S(0)}{K})] + \exp((r - \frac{\sigma^2}{2})T + \sigma\sqrt{T} \cdot x)) e^{-rT} e^{-x^2/2} dx$$

$$= \frac{1}{\sqrt{2\pi}} e^{-rT} [\int_{-\infty}^{\infty} \exp[\log(\frac{S(0)}{K})] e^{-\frac{x^2}{2}} dx + \int_{-\infty}^{\infty} \exp((r - \frac{\sigma^2}{2})T + \sigma\sqrt{T} \cdot x) e^{-\frac{x^2}{2}} dx]$$

$$= e^{-rT} \exp[\log(\frac{S(0)}{K})] + \frac{1}{\sqrt{2\pi}} e^{-rT} \int_{-\infty}^{\infty} \exp((r - \frac{\sigma^2}{2})T - \frac{1}{2}\sigma^2 T) e^{-1/2(x - \sigma\sqrt{T})^2} dx$$

$$= e^{-rT} \exp[\log(\frac{S(0)}{K})] + e^{-rT} \cdot \exp((r - \frac{\sigma^2}{2})T - \frac{1}{2}\sigma^2 T) \underbrace{\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-1/2(x - \sigma\sqrt{T})^2} dx}_{1}$$

$$= e^{-rT} (\exp[\log(\frac{S(0)}{K}) + rT - \sigma^2 T])$$

denoting this value by ?

$e^{-rT}(\exp(d2))$ $\Rightarrow$ $V(0) = e^{-rT}(\exp(d2))$

If one could take the integral of $e(\cdot)$ over some domain,

then $\int_0^{d2} \exp(s)\,ds = \exp(d2) - \exp(0)$!

You can always integrate $f(t) = e^t$ over every compact domain

2) To compute $\varphi = (\varphi_0, \varphi_1)$, we can use two notions from the lecture note

ⓐ Note that φ must be a self financing strategy which discounted value satisfies:

$$d\hat{V}_\varphi(t) = \varphi_1(t)\, d\hat{S}(t) = \varphi_1(t)\, \hat{S}(t)\, \sigma\, dW^Q(t)$$

What about φ_0?

for $t=0$, $d\hat{V}_\varphi(t) = d\hat{V}_\varphi(0) = \varphi_1(0)\,\hat{S}(0)\,\sigma\, dW^Q(0)$

ⓑ we also note that

$$d\hat{V}(t) = d\hat{V}(t, S(t)) = \partial_2 \hat{V}(t, S(t))\,\sigma\, S(t)\, dW^Q(t) \quad ; t \geq 0$$

?

where the drift term equal 0 (if $\hat{V}$ is a Q-martingale, this condition must hold in order to obtain the replicating portfolio φ)

ⓒ Equating ⓐ and ⓑ

$$\varphi_1(0)\hat{S}(0)\,\sigma\, dW^Q(0) = \partial_2\hat{V}(0, S(0))\,\sigma\, S(0)\, dW^Q(0) \rightarrow \varphi_1(0) = \partial_2 \hat{V}(0, S(0))$$

$$\Rightarrow \varphi_1(0) = e^{-rT} \exp(d2) \frac{\partial d2}{\partial S(0)} = \exp(d2 - rT) \cdot \frac{\partial d2}{\partial S(0)}$$

$$\varphi_0(0) = V(0, S(0)) - \partial_2 V(0, S(0))\, S(0)$$

$$= e^{-rT}(\exp(d2)) - \exp(d2 - rT) \frac{\partial d2}{\partial S(0)} = \exp(d2 - rT)\left(1 - \frac{\partial d2}{\partial S(0)}\right)$$

Donc $\varphi_1(0) = \exp(d2 - rT) \cdot \dfrac{\partial d2}{\partial S(0)}$

$\varphi_0(0) = \exp(d2 - rT)\left(1 - \dfrac{\partial d2}{\partial S(0)}\right)$

Calculations for replicating portfolio not helpful or wrong.

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