Christian-Albrechts-Universität zu Kiel **SS 2023**
Mathematisches Seminar
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Computational Finance

Exercises for all participants

T-Exercise 20 (Explicit pricing formulas in the BS-model) (2 + 4 points)
We consider a Black-Scholes model with interest rate $r = 0$, volatility $\sigma = \sqrt{2}$, initial stock price $S(0) = 1$, and maturity $T > 0$.

(a) Show that the fair price $V_1(t, S(t))$ of a European option with payoff

$$f(S(T)) := 3\sqrt{S(T)} + S(T)^{3/2}$$

at maturity equals

$$V_1(t, S(t)) = \exp\Big(-\frac{1}{4}(T-t)\Big)3\sqrt{S(t)} + \exp\Big(\frac{3}{4}(T-t)\Big)S(t)^{3/2}.$$

(b) Consider an American option with payoff process

$$g(S(t)) := \begin{cases} 4S(t)^{3/4} & \text{if } S(t) < 1, \\ 3\sqrt{S(t)} + S(t)^{3/2} & \text{if } S(t) \geq 1 \end{cases} \tag{1}$$

for $t \leq T$. Show that its fair price $V_2(t)$ equals

$$V_2(t, S(t)) = \begin{cases} g(S(t)) & \text{if } S(t) < e^{-(T-t)}, \\ V_1(t, S(t)) & \text{if } S(t) \geq e^{-(T-t)} \end{cases} \tag{2}$$

for $t \leq T$.

Hint: This is one of the rare examples where an American option price can be computed explicitly in the Black-Scholes model. For the computation recall that $E(e^X) = \exp(\mu + \sigma^2/2)$ for any Gaussian random variable X with mean μ and variance σ^2.

T-Exercise 21 (Black-Scholes price of a forward start call) (4 points)
A *forward start option* is an option that transforms at time T_0 to a European call option with strike $S(T_0)$, i.e., it pays off at maturity $T > T_0$ the amount

$$V(T) = (S(T) - S(T_0))^+.$$

Determine the fair price process $v(t, S(t))$ and the perfect hedging strategy $\varphi(t) = (\varphi_0(t), \varphi_1(t))$ of the forward start option in the Black-Scholes model for all $t \in [0, T]$.

Hint: Recall the basic properties of conditional expectations.

C-Exercise 22 (Greeks of a European option in the Black-Scholes model) (4 points)
In the 'Material' folder of the OLAT you find a python function

```
V0 = BS_Price_Int (r, sigma, S0, T, g)
```

which computes the price of a European option with payoff $g(S(T))$ at maturity $T > 0$ in a Black-Scholes model with initial stock price $S(0) > 0$, interest rate $r > 0$ and volatility $\sigma > 0$. (This is formula (3.21) from the lecture notes)
The first order greeks for a European option in the Black-Scholes model are given by the first order derivatives

$$\Delta(r,\sigma,S(0),T,g) = \frac{\partial}{\partial S(0)} V_{BS}(r,\sigma,S(0),T,g),$$
$$\nu(r,\sigma,S(0),T,g) = \frac{\partial}{\partial \sigma} V_{BS}(r,\sigma,S(0),T,g),$$
$$\gamma(r,\sigma,S(0),T,g) = \frac{\partial^2}{\partial S(0)\partial S(0)} V_{BS}(r,\sigma,S(0),T,g),$$

where $V_{BS}(r,\sigma,S(0),T,g)$ denotes the Black-Scholes price of the European option.

a) Write a Python function

```
[Delta, vega, gamma]=BS_Greeks_num(r, sigma, S0, T, g ,eps)
```

that computes the greeks described above numerically using the approximations

$$\frac{\partial}{\partial x} f(x,y) \approx \frac{f(x+\varepsilon x,y) - f(x,y)}{\varepsilon x},$$
$$\frac{\partial^2}{\partial x \partial x} f(x,y) \approx \frac{f(x+\varepsilon x,y) - 2f(x,y) + f(x-\varepsilon x,y))}{(\varepsilon x)^2}.$$

For this you can use the function `BS_Price_Int`.

b) Plot $\Delta(r,\sigma,S(0),T,g)$ for the European call with payoff function $g(x) = (x-110)^+$ and parameters $r = 0.05$, $\sigma = 0.3$, $T = 1$ for $S(0) \in [60,140]$. Use $\varepsilon = 0.001$.

T-Exercise 23 (Hedging error in the BS-model) (for math only) (4 points)
Consider a stock with risk-neutral dynamics

$$\begin{aligned} B(t) &= e^{rt}, \\ S(t) &= S(0)\exp\left((r-\tfrac{\sigma^2}{2})t+\sigma W(t)\right). \end{aligned}$$

Denote by $V(t)=v(t,S(t))$ the Black-Scholes price of a European call if the volatility equals $\tilde{\sigma}$ instead of σ, i.e. with

$$\begin{aligned} v(t,x) = x\Phi &\left(\frac{\log\frac{x}{K}+r(T-t)+\frac{\tilde{\sigma}^2}{2}(T-t)}{\tilde{\sigma}\sqrt{T-t}}\right) \\ &-Ke^{-r(T-t)}\Phi\left(\frac{\log\frac{x}{K}+r(T-t)-\frac{\tilde{\sigma}^2}{2}(T-t)}{\tilde{\sigma}\sqrt{T-t}}\right). \end{aligned}$$

Suppose that the bank uses the incorrect volatility estimate $\tilde{\sigma}$. It sells a call option for the wrong price $V(0)$, and tries to hedge it with a self-financing portfolio $\varphi=(\varphi_0,\varphi_1)$ containing

$$\varphi_1(t)=\partial_2 v(t,S(t))$$

shares of stock. Determine the Itô process representation of the *observed hedging error* $\varepsilon(t) := V(t)-(V(0)+\int_0^t \varphi_0(s)dB(s)+\int_0^t \varphi_1(s)dS(s))$. What do you observe?

Hint: The computation of φ_0 can be avoided by working with discounted prices.

Please include your name(s) as comment in the beginning of the file.
Do not forget to include comments in your Python-programs.
Submit until: Fri, 26.05.2023, 10:00