Christian-Albrechts-Universität zu Kiel **SS 2023**
Mathematisches Seminar
Prof. Dr. Sören Christensen
Henrik Valett, Fan Yu, Oskar Hallmann Sheet 05

Computational Finance

Exercises for all participants

T-Exercise 16 (Vasiček model for interest rates) (4 points)
Let W be a standard Brownian motion and let x, κ, λ and σ real numbers. Show as in the lecture that the process X with

$$dX(t) := (\kappa - \lambda X(t))dt + \sigma dW(t)$$

and $X(0) = x$ solves the equation

$$X(t) = xe^{-\lambda t} + \frac{\kappa}{\lambda}(1 - e^{-\lambda t}) + \int_0^t e^{-\lambda(t-s)} \sigma dW(s).$$

T-Exercise 17 (4 points)
For $\mu \in R$ and $\sigma, r > 0$ we consider the Black-Scholes market with bond B and stock price process S which evolve according to

$$\begin{aligned} dB_t &= rB_t dt, & B_0 &= 1, \\ dS_t &= \mu S_t dt + \sigma S_t dW_t, & S_0 &> 0. \end{aligned}$$

(a) Calculate the Itô process representation of the cubic stock process $X_t := S_t^3$ and the associated quadratic variation process $[X, X]_t$.

(b) Consider a self-financing portfolio $\varphi = (\varphi_t^0, \varphi_t^1)_{t \geq 0}$ with initial value $V_0(\varphi) = 1$ that always invests half of the wealth into the stock, i.e. $\varphi_t^1 = \frac{V_t(\varphi)}{2S_t}$. Show that the value process $V_t(\varphi)$ is a geometric Brownian motion.

T-Exercise 18 (4 points)
Let W_1, W_2 be independent standard Brownian motions. Consider a market with three assets S_0, S_1, S_2, which follow the equations

$$\begin{aligned} S_0(t) &= 1, \\ \mathrm{d}S_1(t) &= S_1(t)\left(3\mathrm{d}t + \mathrm{d}W_1(t) - \mathrm{d}W_2(t)\right), \\ \mathrm{d}S_2(t) &= S_2(t)\left(1\mathrm{d}t - \mathrm{d}W_1(t) + \mathrm{d}W_2(t)\right). \end{aligned}$$

Construct an arbitrage in this market.

T-Exercise 19 (for math only) (4 points)
Let W be a standard Brownian motion and $T > 0$. Assume that the underlying filtration $(\mathscr{F}_t)_{t\geq 0}$ is generated by W. Let μ be an adapted process and Y an $\mathscr{F}_T$-measurable random variable. Show that there exist $x \in \mathbb{R}$ and a process H such that the process

$$X = x + \int_0^{\cdot} \mu(s)ds + \int_0^{\cdot} H(s)dW(s)$$

fulfills

$$X(T) = Y.$$

Determine x and H explicitly for $\mu = 0$ and

(a) $Y = (W(T))^2$,

(b) $Y = \int_0^T W(s)\mathrm{d}s$ and

(c) $Y = (W(T))^3$,

respectively.
Hint: Martingale representation theorem.

Please include your name(s) as comment in the beginning of the file.
Do not forget to include comments in your Python-programs.
Submit until: Fri, 19.05.2023, 10:00