

Itô calculus

A summary of some rules

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1. Linearity of the integral:

$$(H(t) + \tilde{H}(t))dX(t) = H(t)dX(t) + \tilde{H}(t)dX(t),$$

$$(cH(t))dX(t) = c(H(t)dX(t)),$$

$$H(t)d(X + \tilde{X})(t) = H(t)dX(t) + \tilde{H}(t)d\tilde{X}(t),$$

$$H(t)d(cX)(t) = c(H(t)dX(t))$$

2. Integral of differentiable processes:

$$dX(t) = \frac{dX(t)}{dt} dt$$

3. Integration of integrals:

$$H(t)(K(t)dX(t)) = (H(t)K(t))dX(t)$$

4. Integration by parts:

$$d(XY)(t) = X(t)dY(t) + Y(t)dX(t) + dX(t)dY(t)$$

5. Linearity of the covariation:

$$d(X + \tilde{X})(t)dY(t) = dX(t)dY(t) + d\tilde{X}(t)dY(t),$$

$$d(cX)(t)dY(t) = cdX(t)dY(t)$$

6. Symmetry of the covariation:

$$dX(t)dY(t) = dY(t)dX(t)$$

7. If X or Y is of finite variation:

$$dX(t)dY(t) = 0$$

8. Itô's formula (uni- and multivariate):

$$df(X(t)) = f'(X(t))dX(t) + \frac{1}{2}f''(X(t))(dX(t))^2$$

$$df(X(t)) = \sum_{i=1}^d \partial_i f(X(t))dX_i(t) + \frac{1}{2} \sum_{i,j=1}^d \partial_{ij} f(X(t))dX_i(t)dX_j(t)$$

9. The Doléans exponential $d\mathcal{E}(X)(t) = \mathcal{E}(X)(t)dX(t)$:

$$\mathcal{E}(X)(t) = \exp \left(X(t) - X(0) - \frac{1}{2}[X, X](t) \right)$$

10. Rules for Wiener processes W :

- Quadratic variation and covariation of independent Wiener processes W, \tilde{W} :

$$(dW(t))^2 = dt,$$

$$dW(t)d\tilde{W}(t) = 0$$

- Measure change: If the density process Z of $Q \sim P$ is of the form $dZ(t) = Z(t)\sigma(t)dW(t)$, then

$$dW(t) = dW^Q(t) + \sigma(t)dt$$

for some Q -Wiener process.

- Moments:

$$E \left(\int_0^t H(s)dW(s) \right) = 0,$$

$$\text{Var}(X(t)) = \int_0^t E(\sigma(s)^2)ds$$

11. Rules for Itô processes $dX(t) = \mu(t)dt + \sigma(t)dW(t)$:

- Quadratic variation:

$$(dX(t))^2 = \sigma(t)^2 dt$$

- Itô's formula:

$$df(t, X(t)) = \left(\dot{f}(t, X(t)) + f'(t, X(t))\mu(t) \right) dt + \frac{1}{2}f''(t, X(t))\sigma(t)^2 dW(t)$$

- Doléans exponential:

$$\mathcal{E}(X)(t) = \exp \left(\int_0^t \left(\mu(s) - \frac{1}{2}\sigma(s)^2 \right) ds + \int_0^t \sigma(s)dW(s) \right)$$

- Stopping at T :

$$dX^T(t) = \mu(t)1_{\{t \leq T\}} dt + \sigma(t)1_{\{t \leq T\}} dW(t)$$