Alexander Georges Gretener, M.Sc.
Mariia Okuneva, M.Sc.
Anna Titova, Dr.

Advanced Statistics I (Winter Term 2023/24)

Problem Set 2

1. A field of sets $\mathcal{K}$ is defined by the following properties:

   (i) $A \in \mathcal{K} \Rightarrow \bar{A} \in \mathcal{K}$

   (ii) $A \in \mathcal{K}$ and $B \in \mathcal{K} \Rightarrow A \cup B \in \mathcal{K}$

   The sample space of the experiment of rolling a die twice and summing up the dots facing up, is given by $S = \{(i, j) : i, j = 1, 2, ..., 6\}$. Let A_k $(k = 2, ..., 12)$ be the event "The sum is less than or equal to k" and let B_k $(k = 2, ..., 12)$ be the event "The sum is greater than k". Which of the following subsets of the power set of S form a field of sets?

   (a) $\mathcal{K}_1 = \{\emptyset, A_2, B_2, S\}$;

   (b) $\mathcal{K}_2 = \{A_{12}, B_{12}\}$;

   (c) $\mathcal{K}_3 = \{A_{11}, B_{11}\}$;

   (d) $\mathcal{K}_4 = \{A_k, B_l : k, l = 2, ..., 12\}$.

2. Proof De Morgan's laws, i.e. show that

$$\overline{A \cup B} = \overline{A} \cap \overline{B}$$
$$\overline{A \cap B} = \overline{A} \cup \overline{B}$$

3. For each case below, determine whether or not the real-valued set function $P(A)$ is in fact a probability set function.

   (a) sample space: $S = \{1, 2, 3, 4, 5, 6, 7, 8\}$
   event space: $\Upsilon = \{A : A \subset S\}$
   set function: $P(A) = \sum_{x \in A} x/36$ for $A \in \Upsilon$

   (b) sample space: $S = [0, \infty)$
   event space: $\Upsilon = \{A : A$ is an interval subset of $S \vee$ any set formed by unions, intersections, or complements of these interval subsets $\}$
   set function: $P(A) = \int_{x \in A} e^{-x} dx$ for $A \in \Upsilon$

   (c) sample space: $S = \{x : x$ is a positive integer (1,2,3,...)$\}$
   event space: $\Upsilon = \{A : A \subset S\}$
   set function: $P(A) = \sum_{x \in A} x^2/10^5$ for $A \in \Upsilon$.

   (d) sample space: $S = (2, 5)$
   event space: $\Upsilon = \{A : A$ is an interval subset of $S \vee$ any set formed by unions, intersections, or complements of these interval subsets $\}$
   set function: $P(A) = \int_{x \in A} \frac{1}{3} dx$ for $A \in \Upsilon$.

4. Let $P(\cdot)$ be a probability set function with the event space Υ with $A_i \in \Upsilon$ $(i = 1, ..., r)$. Show that the following relationships hold:

   (a) $$P(A_1 - A_2) = P(A_1) - P(A_1 \cap A_2);$$

   (b) $$P[(A_1 \cup A_2) - (A_1 \cap A_2)] = P(A_1) + P(A_2) - 2P(A_1 \cap A_2)\ .$$

5. Let A, B, C be elements of the event space Υ and let $P(\cdot)$ be an associated probability set function. Show that the following relationships hold:

   (a) $$P(A) \leq P(B) \Rightarrow P(\bar{B}) \leq P(\bar{A});$$

   (b) $$(A \cap B) \subset C \Rightarrow P(\bar{C}) \leq P(\bar{B}) + P(\bar{A})\ .$$