Übung 05

Freitag, 12. Mai 2023 11:00

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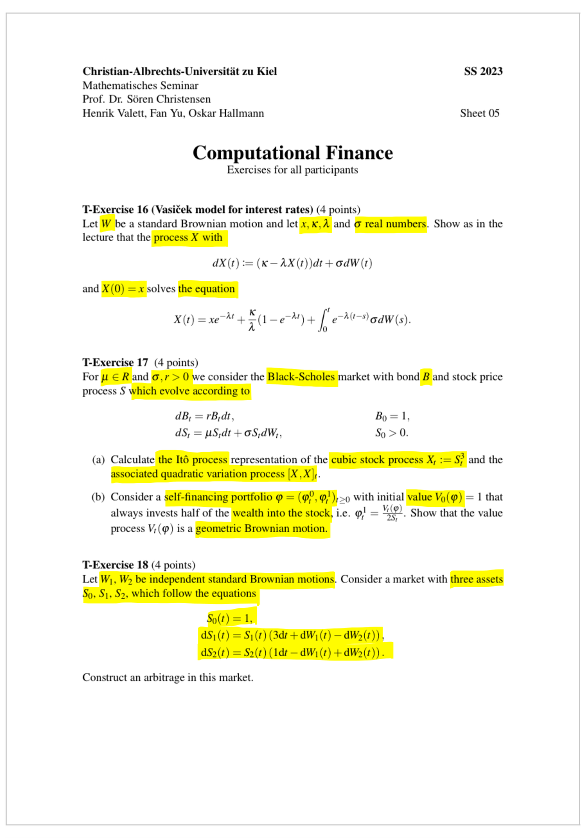


Christian-Albrechts-Universität zu Kiel — SS 2023

Mathematisches Seminar

Prof. Dr. Sören Christensen

Henrik Valett, Fan Yu, Oskar Hallmann — Sheet 05

Computational Finance

Exercises for all participants

T-Exercise 16 (Vasiček model for interest rates) (4 points)
Let W be a standard Brownian motion and let x, κ, λ and σ real numbers. Show as in the lecture that the process X with

$$dX(t) := (\kappa - \lambda X(t))dt + \sigma dW(t)$$

and $X(0) = x$ solves the equation

$$X(t) = xe^{-\lambda t} + \frac{\kappa}{\lambda}(1 - e^{-\lambda t}) + \int_0^t e^{-\lambda(t-s)}\sigma dW(s).$$

T-Exercise 17 (4 points)
For $\mu \in R$ and $\sigma, r > 0$ we consider the Black-Scholes market with bond B and stock price process S which evolve according to

$$dB_t = rB_t dt, \qquad B_0 = 1,$$
$$dS_t = \mu S_t dt + \sigma S_t dW_t, \qquad S_0 > 0.$$

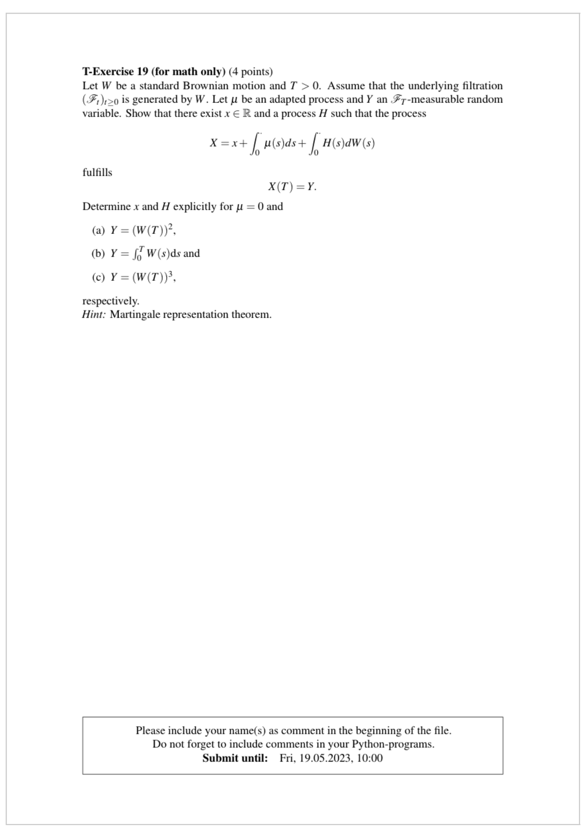
(a) Calculate the Itô process representation of the cubic stock process $X_t := S_t^3$ and the associated quadratic variation process $[X, X]_t$.

(b) Consider a self-financing portfolio $\varphi = (\varphi_t^0, \varphi_t^1)_{t\geq 0}$ with initial value $V_0(\varphi) = 1$ that always invests half of the wealth into the stock, i.e. $\varphi_t^1 = \frac{V_t(\varphi)}{2S_t}$. Show that the value process $V_t(\varphi)$ is a geometric Brownian motion.

T-Exercise 18 (4 points)
Let W_1, W_2 be independent standard Brownian motions. Consider a market with three assets S_0, S_1, S_2, which follow the equations

$$S_0(t) = 1,$$
$$dS_1(t) = S_1(t)(3dt + dW_1(t) - dW_2(t)),$$
$$dS_2(t) = S_2(t)(1dt - dW_1(t) + dW_2(t)).$$

Construct an arbitrage in this market.



T-Exercise 19 (for math only) (4 points)
Let W be a standard Brownian motion and $T > 0$. Assume that the underlying filtration $(\mathscr{F}_t)_{t\geq 0}$ is generated by W. Let μ be an adapted process and Y an $\mathscr{F}_T$-measurable random variable. Show that there exist $x \in \mathbb{R}$ and a process H such that the process

$$X = x + \int_0^T \mu(s)ds + \int_0^T H(s)dW(s)$$

fulfills

$$X(T) = Y.$$

Determine x and H explicitly for $\mu = 0$ and

(a) $Y = (W(T))^2$,

(b) $Y = \int_0^T W(s)ds$ and

(c) $Y = (W(T))^3$,

respectively.
Hint: Martingale representation theorem.

Please include your name(s) as comment in the beginning of the file.
Do not forget to include comments in your Python-programs.
Submit until: Fri, 19.05.2023, 10:00

Sheet: 02

Group: QF27

Group members: Eike Gjeltstoesse, Radislav Fane, Robert Hennings

T-Exercise 16)

T-Exercise 17)

Notes and examples from the lecture:

$$\boxed{d f(X(t)) = \left(f'(X(t)) \cdot \mu(t) + \frac{1}{2} f''(X(t))\, \sigma^2(t)\right) dt + f'(X(t))\, \sigma(t)\, dW(t)}$$

or put:

$$d f(t, X(t)) = \left(\partial_1 f(t, X(t)) + \partial_2 f(t, X(t)) \cdot \mu(t) + \frac{1}{2} \partial_{22} f(t, X(t))\, \sigma^2(t)\right) dt \qquad (3.8)$$
$$+ \partial_2 f(t, X(t))\, \sigma(t)\, dW(t) \qquad (3.9)$$

$f(\overset{1.}{t}, \overset{2.}{X(t)})$

∂_1: first derivative with respect to first parameter in $f(t, X(t))$

∂_2: second derivative —"— —"—

∂_{22}: second derivative with respect to second parameter in $f(t, X(t))$

Consider the example: $\boxed{f(x) = x^2}$ → make it time dependent as X is time dependent!

$f(x) = x^2$
$f'(x) = 2x$
$f''(x) = 2$

time dependent X vanished!

Plug in in the simplified formula:

$$\boxed{d f(X(t)) = \left(f'(X(t)) \cdot \mu(t) + \frac{1}{2} f''(X(t))\, \sigma^2(t)\right) dt + f'(X(t))\, \sigma(t)\, dW(t)}$$

$$\rightarrow d X^2(t) = \left(2X(t) \cdot \mu(t) + \frac{1}{2}\, 2 \cdot \sigma^2(t)\right) dt + 2 \cdot X(t)\, \sigma(t)\, dW(t)$$
$$= \left(2X(t) \cdot \mu(t) + \sigma^2(t)\right) dt + 2 \cdot X(t)\sigma(t)\, dW(t)$$

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a) Calculate Itô process representation, quadratic variation process

Market with 2 assets:

$d B_t = r \cdot B_t\, dt \quad | B_0 = 1$

$d S_t = \mu S_t dt + \sigma S_t dW_t \quad | S_0 > 0$

Cubic stock process: $\underbrace{X_t := S_t^3}$

We want to obtain the Itô process representation of this process

By formulae 3.8 and 3.9 from the lecture notes we have:

$$d f(t, X(t)) = \left(\partial_1 f(t, X(t)) + \partial_2 f(t, X(t)) \cdot \mu(t) + \frac{1}{2} \partial_{22} f(t, X(t))\, \sigma^2(t)\right) dt \qquad (3.8)$$
$$+ \partial_2 f(t, X(t))\, \sigma(t)\, dW(t) \qquad (3.9)$$

$f(\overset{1.}{t}, \overset{2.}{X(t)})$

∂_1: first derivative with respect to first parameter in $f(t, X(t))$

∂_2: first derivative —"— second parameter —"—

∂_{22}: second derivative with respect to second parameter in $f(t, X(t))$

What we will apply here:

$f(t, X(t)) = X(t) = S_t^3$

$\partial_1 f(t, X(t)) = \dfrac{\partial X(t)}{\partial t} = 0 \quad | t$ not directly included!

$\partial_2 f(t, X(t)) = \dfrac{\partial^2 X(t)}{\partial X} = 3 \cdot S_t^2$

$\partial_{22} f(t, X(t)) = \dfrac{\partial^2 X(t)}{\partial X(t)^2} = 6 \cdot S_t$

Plug everything into 3.8 and 3.9:

$$d X(t) = \left(0 + 3 \cdot S_t^2 \cdot \mu(t) + \frac{1}{2} \cdot 6 S_t \cdot \sigma^2(t)\right) dt$$
$$+ 3 \cdot S_t^2\, \sigma(t)\, dW(t)$$

$$\boxed{dX(t) = \left(3 \cdot S_t^2 \cdot \mu(t) + 3 \cdot S_t \cdot \sigma^2(t)\right) dt + 3 \cdot S_t^2\, \sigma(t)\, dW(t)}$$

Quadratic variation

From the lecture notes p.26 we have:

$[X, Y](t)$ denotes the covariation process which is the limit of expressions: $\sum_{k=1}^{n} \left(X(\tfrac{k}{n}t) - X(\tfrac{k-1}{n}t)\right) \cdot \left(Y(\tfrac{k}{n}t) - Y(\tfrac{k-1}{n}t)\right)$ (3.2)

$[X, X]$ sums up products of increments of X over infinitesimal time intervals

For $X = Y$ it is called quadratic variation, in the notation:

$$dX(t)\, dY(t) = d[X, Y](t)$$

In our case we therefore have:

$$dX(t) \cdot dX(t) = d[X, X](t)$$
$$\boxed{(dX(t))^2 = d[X, X](t)}$$

→ Square the obtained Itô process representation:

$$(dX(t))^2 = \left(3 S_t^2 \cdot \mu(t) + 3 \cdot S_t\, \sigma^2(t)) dt + 3 \cdot S_t^2 \sigma(t)\, dW(t)\right)^2 \qquad | \text{ slightly rearranging}$$
$$= \Big(\underbrace{3 \cdot S_t^2 \cdot \mu(t)\, dt}_{I} + \underbrace{3 \cdot S_t \cdot \sigma^2(t)\, dt}_{II} + \underbrace{3 \cdot S_t^2 \sigma(t)\, dW(t)}_{III}\Big)^2 \qquad | \text{ squaring each individual term}$$
$$= 3^2 \cdot S_t^{2\cdot 2} \cdot \mu(t)^2 dt^2 + 3^2 \cdot S_t^2 \cdot \sigma^{2\cdot 2}(t)\, dt^2 + 3^2 \cdot S_t^{2\cdot 2} \sigma(t) (dW(t))^2$$
$$= 9 \cdot S_t^4 \cdot \mu(t)^2 dt^2 + 9 \cdot S_t^2 \cdot \sigma^4(t)\, dt^2 + 9 \cdot S_t^4 \sigma(t)\, (dW(t))^2 \qquad | (dW(t))^2 = dt$$
$$= 9 \cdot S_t^4 \cdot \mu(t)^2 dt^2 + 9 \cdot S_t^2 \cdot \sigma^4(t)\, dt^2 + 9 \cdot S_t^4 \sigma(t)\, dt \qquad | dt^2 = dt \ | \text{ simplify}$$

$$\boxed{(dX(t))^2 = 9 \cdot \left(S_t^4 \cdot \mu(t)^2 + S_t^2 \cdot \sigma^4(t) + S_t^4\, \sigma(t)\right) dt}$$

| from the lecture p. 31 we know:

$dt\, dt = dt\, dW(t) = dW(t)\, dt = 0$

$dW(t)\, dW(t) = dt$

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b) Self-financing portfolio: $\varphi = (\varphi_t^0, \varphi_t^1)_{t \geq 0}$ with initial value $V_0(\varphi) = 1$

- always invest half of the wealth into the stock $\varphi_t^1 = \dfrac{V_t(\varphi)}{2 S_t}$

⇒ Value process $V_t(\varphi)$ is a geometric Brownian motion

Idea: Show that the Value process follows the SDE of a geometric Brownian motion

The value process can be rewritten: $V_t(\varphi) = \varphi_t^0 \cdot B_t + \varphi_t^1 \cdot S_t \quad |$ weight · Bond + weight · stock

The geometric Brownian motion would look like: $dX_t = \mu X_t dt + \sigma X_t dW_t \quad |$ see lecture notes p.

or: $dX_t = X(t) \cdot (\mu dt + \sigma dW(t))$

→ Construct SDE of the value process by using Itô's formula

Again consider Itô's formula:

$$d f(t, X(t)) = \left(\frac{\partial f}{\partial t}(t, X(t)) + \frac{\partial f}{\partial x}(t, X(t))\, \mu(t) + \frac{1}{2} \frac{\partial^2 f}{\partial x^2}(t, X(t))\, \sigma^2(t)\right) dt + \frac{\partial f}{\partial x}(t, X(t))\, \sigma(t)\, dW(t)$$

$V_t(\varphi)$

$$\Rightarrow d V_t(\varphi) = \frac{\partial V}{\partial t}(t, \varphi) + \frac{\partial V}{\partial B}(t, \varphi)\, dB_t + \frac{\partial V}{\partial S}(t, \varphi)\, dS_t + \frac{1}{2} \frac{\partial^2 V}{\partial B^2}(t, \varphi)\, d[B, B]_t + \frac{1}{2} \frac{\partial^2 V}{\partial S^2}(t, \varphi)\, d[S, S]_t$$
$$+ \frac{\partial^2 V}{\partial B \partial S}(t, \varphi)\, dB_t dS_t$$

$d B_t = r \cdot B_t\, dt \quad | B_0 = 1$

$d S_t = \mu S_t dt + \sigma S_t dW_t \quad | S_0 > 0$

Note that for the quadratic variation terms we have:

$[B, B]_t = 0$ since it is just deterministic growth without any randomness

$[S, S]_t = \sigma^2 S_t^2\, dt$ since randomness is involved, see lecture p.30

What simplifies the above equation:

$$d V_t(\varphi) = \frac{\partial V}{\partial t}(t, \varphi) + \frac{\partial V}{\partial B}(t, \varphi)\, dB_t + \frac{\partial V}{\partial S}(t, \varphi)\, dS_t + \frac{1}{2} \frac{\partial^2 V}{\partial S^2}(t, \varphi)\, d[S, S]_t$$

By rearranging the terms:

$$\left(\frac{\partial V}{\partial t}(t, \varphi) + \frac{\partial V}{\partial B}(t, \varphi)\, r \cdot B_t + \frac{\partial V}{\partial S}(t, \varphi)\, \mu S_t + \frac{1}{2} \frac{\partial^2 V}{\partial S^2}(t, \varphi)\, \sigma^2 S_t^2\right) dt + \frac{\partial V}{\partial S}(t, \varphi)\, \sigma S_t\, dW_t$$

⇒ Similar to a BM with drift and diffusion:

$$\mu = \frac{\partial V}{\partial B}(t, \varphi)\, r \cdot B_t + \frac{\partial V}{\partial S}(t, \varphi)\, \mu S_t$$

$$\sigma = \frac{\partial V}{\partial S}(t, \varphi)\, \sigma S_t$$

Considering: $\varphi_t^1 = \frac{1}{2} \cdot V_t(\varphi)$:

$$\mu = \varphi_t^0 \cdot r \cdot B_t + \frac{1}{2} \mu S_t$$

$$\sigma = \frac{1}{2} \sigma S_t$$

Notes and examples from the lecture:

Connection linear SDE and geometric Brownian motion

Linear SDE: $\boxed{dX(t) = \mu X(t)\, dt + \sigma X(t)\, dW(t)}$, $X(0) = x$

with constants μ, σ can be rephrased as:

$\boxed{dX(t) = X(t)\, dY(t), \ X(0) = x}$ $\quad | Y(t) = \mu t + \sigma W(t)$

Using (3.11) $\boxed{\mathcal{E}(X)(t) = \exp\left(\int_0^t \left(\mu(s) - \frac{1}{2}\sigma^2(s)\right) ds + \int_0^t \sigma(s)\, dW(s)\right)}$ for the stochastic exponential we obtain:

$$X(t) = x \cdot \mathcal{E}(Y)(t) = \underbrace{x \cdot \exp\left(\left(\mu - \tfrac{1}{2}\sigma^2\right) t + \sigma W(t)\right)}_{\text{is called geometric BM}}$$