Christian-Albrechts-Universität zu Kiel **SS 2023**
Mathematisches Seminar
Prof. Dr. Sören Christensen
Henrik Valett, Fan Yu, Oskar Hallmann Sheet 04

Computational Finance

Exercises for all participants

C-Exercise 12 (Pricing a deep out-of-the-money European call option by Monte-Carlo with importance sampling) (4 points)
Consider a Black-Scholes model with parameters $S(0)$, r, $\sigma > 0$. The goal is to approximate by the Monte-Carlo method the fair price $V(0)$ of an European call option on the stock with strike $K \gg S(0)$ at maturity T.

Write a Python function

```
[V0, CIl, CIr] = BS_EuCall_MC_IS (S0, r, sigma, K, T, mu, N,
                                  alpha)
```

that approximates the price of the European call option via Monte-Carlo based on $N \in \mathbb{N}$ samples and additionally returns the left and right boundary of an asymptotic α-level confidence interval. Use a new random variable $Y \sim N(\mu, 1)$ for the importance sampling method.

Test your function for $S(0) = 100$, $r = 0.05$, $\sigma = 0.3$, $K = 220$, $T = 1$, $N = 10000$, $\alpha = 0.95$ and plot your estimator for $V(0)$ in dependence on μ against the true value.

Hint: Experiment with the range of μ such that you can see visible changes in the variance of your estimator. For the true value use the Black-Scholes formula provided on sheet 02.

C-Exercise 13 (Using control variables to reduce the variance of MC-estimators) (4 points)
Write a Python function

```
V0 = BS_EuOption_MC_CV (S0, r, sigma, T, K, M)
```

that computes the initial price of a European self-quanto call, i.e. an option with payoff $(S(T)-K)^+S(T)$ for some strike price K at maturity, in the Black-Scholes model via the Monte-Carlo approach with $M \in \mathbb{N}$ samples. Use a European call option with the same strike price K as control variate to reduce the variance of the estimator. To this end, estimate in a first Monte-Carlo simulation with M samples the optimal value

$$\frac{\mathrm{Cov}((S(T)-K)^+S(T), (S(T)-K)^+)}{\mathrm{Var}((S(T)-K)^+)}.$$

Test your function for the parameters

$$S(0) = 100, \quad r = 0.05, \quad \sigma = 0.3, \quad T = 1, \quad K = 110, \quad M = 100000,$$

and compare the result to the plain Monte-Carlo simulation (cf. C-Exercise 10).

Useful Python commands: `numpy.cov`

T-Exercise 14 (Exchange rates) (4 points)
Assume that the exchange rate $D(t)$ of the US-Dollar in Euro at time $t > 0$ follows the equation

$$dD(t) = D(t)\mu \mathrm{d}t + D(t)\sigma \mathrm{d}W(t)$$

with $D(0) > 0$ and $\mu,\ \sigma \in \mathbb{R}$. Hence, the exchange rate of the Euro in US-Dollar at time $t > 0$ is given by $E(t) := \frac{1}{D(t)}$. Represent the process E as Itō process, i.e. in the form

$$dE(t) = \ldots \mathrm{d}t + \ldots \mathrm{d}W(t).$$

Interpret your result economically in the case $\mu = \frac{1}{2}\sigma^2$.

T-Exercise 15 (for math only) (4 points)
Let W be a standard Brownian motion. Show that the process

$$X(t) := \mathscr{E}(W)(t)\left(1 + \int_0^t \frac{1}{\mathscr{E}(W)(s)} \mathrm{d}s\right), \quad t \in \mathbb{R}_+,$$

solves the stochastic differential equation

$$\mathrm{d}X(t) = 1\mathrm{d}t + X(t)\mathrm{d}W(t), \quad X(0) = 1.$$

Please include your name(s) as comment in the beginning of the file.
Do not forget to include comments in your Python-programs.
Submit until: Fri, 12.05.2023, 10:00