Solutions to Problem Set 2

1. Given: sample space $S = \{(i,j) : i,j = 1,2,...,6\}$
   Events:
   $A_k = \{(i,j) : i+j \leq k\}$; $k = 2,...,12$
   $B_k = \{(i,j) : i+j > k\}$; $k = 2,...,12$

   $\Rightarrow \overline{A}_k = B_k$ and $\overline{B}_k = A_k$

   (Loose) Definition: a field of sets is a set of subsets that contains all of its complements (i.e. closed under complements) and unions (i.e. closed under union).

   (a) A set of subsets $\mathcal{K}_1$ of the power set of S is given. Do they form a field of sets?

   $\mathcal{K}_1 = \{\emptyset, A_2, B_2, S\}$

   $\mathcal{K}_1$ is a field of sets provided that the set of subsets contains all its complements and unions.

   Complements in $\mathcal{K}_1$:
   $\overline{\emptyset} = S \in \mathcal{K}_1$; $\overline{A_2} = B_2 \in \mathcal{K}_1$; $\overline{S} = \emptyset \in \mathcal{K}_1$; $\overline{B_2} = A_2 \in \mathcal{K}_1$

   Unions in $\mathcal{K}_1$:
   $\emptyset \cup A_2 = A_2 \in \mathcal{K}_1$; $\emptyset \cup B_2 = B_2 \in \mathcal{K}_1$; $\emptyset \cup S = S \in \mathcal{K}_1$;
   $B_2 \cup A_2 = S \in \mathcal{K}_1$; $S \cup B_2 = S \in \mathcal{K}_1$; $A_2 \cup S = S \in \mathcal{K}_1$;

   The set $\mathcal{K}_1$ contains all complements and unions. Thus it is a field of sets.

   (b) $\mathcal{K}_2 = \{A_{12}, B_{12}\}$
   Because A_{12} is equal to the sample space und B_{12} is an empty set, and because S and $\emptyset$ form the smallest possible field of sets, $\mathcal{K}_2$ is a field of sets. ($\emptyset \cup S = S \in \mathcal{K}_2$; $\overline{\emptyset} = S \in \mathcal{K}_2$; $\overline{S} = \emptyset \in \mathcal{K}_2$)

   (c) $\mathcal{K}_3 = \{A_{11}, B_{11}\}$
   $\mathcal{K}_3$ is not a field of sets, as for example $A_{11} \cup B_{11} = S$ is not an element of the set $\mathcal{K}_3$.

   (d) $\mathcal{K}_4 = \{A_k, B_l : k,l = 2,...,12\}$
   $\mathcal{K}_4$ is not a field of sets, as for example $A_2 \cup B_{11} = \{(1,1); (6,6)\}$ is not an element of the set $\mathcal{K}_4$.

2. We have for $\overline{A \cup B} = \overline{A} \cap \overline{B}$:

A	B	$\overline{A}$	$\overline{B}$	$A \cup B$	$\overline{A \cup B}$	$\overline{A} \cap \overline{B}$
T	T	F	F	T	F	F
T	F	F	T	T	F	F
F	T	T	F	T	F	F
F	F	T	T	F	T	T

and for $\overline{A \cap B} = \overline{A} \cup \overline{B}$:

A	B	$\overline{A}$	$\overline{B}$	$A \cap B$	$\overline{A \cap B}$	$\overline{A} \cup \overline{B}$
T	T	F	F	T	F	F
T	F	F	T	F	T	T
F	T	T	F	F	T	T
F	F	T	T	F	T	T

3. (a) sample space: $S = \{1, 2, 3, 4, 5, 6, 7, 8\}$
   event space: $\Upsilon = \{A : A \subset S\}$
   set function: $P(A) = \sum_{x \in A} x/36$ for $A \in \Upsilon$.
   - i $P(A) \geq 0$ because every A is a nonempty subset of S containing only $x > 0$ (non-negativity)
   - ii $P(S) = \sum_{x \in S} x/36 = 1$ (standardization)
   - iii Holds by assumption

   (b) sample space: $S = [0, \infty)$
   event space: $\Upsilon = \{A : A$ is an interval subset of $S \vee$ any set formed by unions, intersections, or complements of these interval subsets$\}$
   set function: $P(A) = \int_{x \in A} e^{-x} dx$ for $A \in \Upsilon$.
   - i $P(A) \geq 0$ because A contains at least one interval of length zero, hence the equality. Otherwise, an integral over a positive function is positive.
   - ii $P(S) = \int_0^\infty e^{-x} dx = | - e^{-x}|_0^\infty = 1$
   - iii Holds by assumption

   (c) sample space: $S = \{x : x$ is a positive integer (1,2,3,...)$\}$
   event space: $\Upsilon = \{A : A \subset S\}$
   set function: $P(A) = \sum_{x \in A} x^2/10^5$ for $A \in \Upsilon$.
   Not a probability set function because $10^5 \in S$ and therefore $P(S) \neq 1$

(d) sample space: $S = (2, 5)$
event space: $\Upsilon = \{A : A$ is an interval subset of $S \vee$ any set formed by unions, intersections, or complements of these interval subsets$\}$
set function: $P(A) = \int_{x \in A} \frac{1}{3} dx$ for $A \in \Upsilon$.

i $P(A) \geq 0$ because A contains at least one interval of length zero, hence the equality. Otherwise, an integral over a positive function is positive.

ii $P(S) = \int_2^5 \frac{1}{3} dx = |\frac{1}{3} x|_2^5 = \frac{5}{3} - \frac{2}{3} = 1$

iii Holds by assumption

4. (a) $A_1 - A_2 = A_1 \cap \bar{A}_2$

Given: $(A_1 \cap \bar{A}_2) \,\&\, (A_1 \cap A_2)$ are disjoint. Therefore: $P\Big((A_1 \cap \bar{A}_2) \cup (A_1 \cap A_2)\Big) = P(A_1 \cap \bar{A}_2) + P(A_1 \cap A_2)$

$$\begin{aligned} \implies \quad & P(A_1) = P\Big([A_1 \cap \bar{A}_2] \cup [A_1 \cap A_2]\Big) \\ & P(A_1) = P(A_1 \cap \bar{A}_2) + P(A_1 \cap A_2) \quad \text{(disjoint sets)} \end{aligned}$$

$$\implies \quad P(A_1 \cap \bar{A}_2) = P(A_1 - A_2) = P(A_1) - P(A_1 \cap A_2)$$

(b) Given:

$$\begin{aligned} (1) \quad & [A_1 \cap \bar{A}_2] \cap [\bar{A}_1 \cap A_2] = \emptyset \\ (2) \quad & [A_1 \cap \bar{A}_2] \cup [\bar{A}_1 \cap A_2] = [A_1 \cup A_2] - [A_1 \cap A_2] \end{aligned}$$

$$\begin{aligned} \implies P\Big(\underbrace{[A_1 \cup A_2] - [A_1 \cap A_2]}_{\text{use (2)}}\Big) &= P\Big[[A_1 \cap \bar{A}_2] \cup [\bar{A}_1 \cap A_2]\Big] \\ &= P(A_1 \cap \bar{A}_2) + P(\bar{A}_1 \cap A_2) \quad \text{(because of 1)} \\ &= P(A_1 - A_2) + P(A_2 - A_1) \quad \text{(because of 3a)} \\ &= P(A_1) - P(A_1 \cap A_2) + P(A_2) - P(A_2 \cap A_1) \\ & \quad \text{q.e.d.} \end{aligned}$$

5. (a)

$$\begin{aligned} P(A) &= 1 - P(\bar{A}) \\ P(B) &= 1 - P(\bar{B}) \end{aligned}$$

$$\begin{aligned} \implies P(A) \leq P(B) &\implies 1 - P(\bar{A}) \leq 1 - P(\bar{B}) \\ & \quad\;\; P(\bar{A}) \geq P(\bar{B}) \end{aligned}$$

(b)

$$[A \cap B] \subset C \implies \bar{C} \subset [\overline{A \cap B}] = \bar{A} \cup \bar{B}$$

$$\begin{aligned} P(\bar{C}) \leq \quad & P(\bar{A} \cup \bar{B}) \\ & P(\bar{A} \cup \bar{B}) \leq P(\bar{A}) + P(\bar{B}), \\ & \text{because } P(\bar{A} \cup \bar{B}) = P(\bar{A}) + P(\bar{B}) - P(\bar{A} \cap \bar{B}) \end{aligned}$$