Problem Set 3

1. (a)

$$\begin{aligned}\mathrm{Var}(\bar{y}_N) &= \mathrm{Var}\left(\frac{1}{N}\sum_{i=1}^{N} y_i\right) \overset{iid}{=} \frac{1}{N^2}\sum_{i=1}^{N}\mathrm{Var}(y_i) = \frac{1}{N^2}\sum_{i=1}^{N}\sigma^2 \\ &= \frac{1}{N^2}N\sigma^2 = \frac{\sigma^2}{N}\end{aligned}$$

Remember:

$$\mathrm{Var}(y_1+y_2) = \mathrm{Var}(y_1) + \mathrm{Var}(y_2) + 2\underbrace{\mathrm{Cov}(y_1,y_2)}_{=0 \text{ if } y_1,y_2 \text{ iid}} \qquad (*)$$

$$\Rightarrow \mathrm{Var}\left(\sqrt{N}(\bar{y}_N-\mu)\right) = N\,\mathrm{Var}(\bar{y}_N-\mu) \overset{(*)}{=} N\,\mathrm{Var}(\bar{y}_N) = \sigma^2$$

(b) CLT: If w_i is iid, $\mathrm{E}(w_i)=0$ and $\mathrm{E}(w_i^2)<\infty$, then $N^{-1/2}\sum_{i=1}^{N} w_i \overset{a}{\sim} N(0,B)$, where $B=\mathrm{Var}(w_i)=\mathrm{E}(w_i^2)$.
$\Rightarrow$ The $N^{-1/2}$ weighted sum of independent and identically distributed random variables with mean zero and finite variance is asymptotically normally distributed with mean zero and finite variance.

$$\begin{aligned}\sqrt{N}(\bar{y}_N-\mu) &= \sqrt{N}\left(\frac{1}{N}\sum_{i=1}^{N} y_i - \mu\right) = \sqrt{N}\left(\frac{1}{N}\sum_{i=1}^{N}(y_i-\mu)\right) \\ &= N^{-\frac{1}{2}}\underbrace{\sum_{i=1}^{N}(y_i-\mu)}_{\text{iid sum with mean zero}} \overset{a}{\sim} N(0,\mathrm{Var}(y_i-\mu)=\sigma^2)\end{aligned}$$

$$\Rightarrow \mathrm{Avar}(\sqrt{N}(\bar{y}_N-\mu)) = \sigma^2$$

For $N\to\infty$, $\sqrt{N}(\bar{y}_N-\mu)$ is $N(0,\sigma^2)$ distributed. For finite samples, we can approximate using CLT!

(c)

$$\begin{aligned}\mathrm{Avar}\left(\sqrt{N}(\bar{y}_N-\mu)\right) &= N\cdot\mathrm{Avar}(\bar{y}_N-\mu) = N\cdot\mathrm{Avar}(\bar{y}_N) \\ \Rightarrow \mathrm{Avar}(\bar{y}_N) &= \sigma^2/N\end{aligned}$$

$\Rightarrow$ Inaccurate notation since for $N\to\infty$ (asymptotics), $\mathrm{Avar}(\bar{y}_N)=0$. But since we want to approximate finite sample distribution by that we write it like that.

(d) The asymptotic standard deviation of $\bar{y}_N$ is:

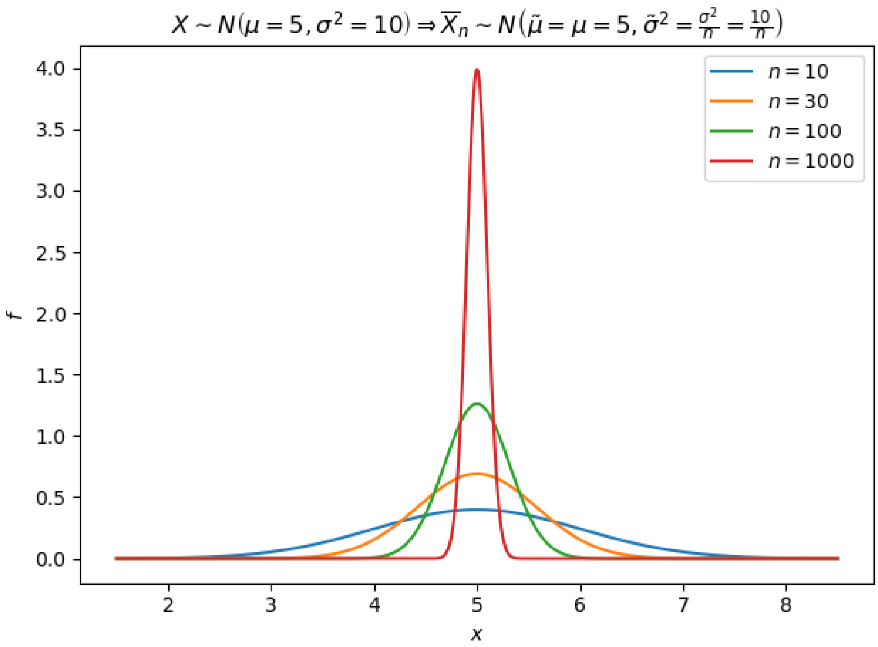
$$\sqrt{\text{Avar}(\bar{y}_N)} = \sigma/\sqrt{N}$$

(e) To estimate the asymptotic standard error of $\bar{y}_N$, we need a consistent estimator of σ. Typically the unbiased estimator of σ^2 is used:

$$\hat{\sigma}^2 = \frac{1}{N-1}\sum_{i=1}^{N}(y_i - \bar{y}_N)^2$$

The estimated asymptotic standard error is simply $\hat{\sigma}/\sqrt{N}$.

2. Consider the following figure:



3. (a)

$$\begin{aligned}
\gamma_0 &= log(A) \Rightarrow A = e^{\gamma_0} \\
\text{plim}(\hat{A}) &= \text{plim}(e^{\hat{\gamma}_0}) = e^{\text{plim}(\hat{\gamma}_0)} = e^{\gamma_0} = A \\
\Rightarrow \hat{A} &= e^{\hat{\gamma}_0} \text{ is a consistent estimator for A!}
\end{aligned}$$

(b) Using the Delta method: If $\sqrt{N}(\hat{\theta} - \theta) \overset{a}{\sim} N(0, V)$ and $c(\theta)$ is continuous and differentiable, then $\sqrt{N}(c(\hat{\theta}) - c(\theta)) \overset{a}{\sim} N(0, C(\theta)VC(\theta)')$ where $C(\theta)$ is the Jacobian of $c(\theta)$ (matrix of first derivatives).
Here:

$$\begin{aligned}
A &= c(\theta) = g(\gamma_0) = e^{\gamma_0} \\
C(\theta) &= g'(\gamma_0) = e^{\gamma_0}. \\
\Rightarrow \text{Avar}(\sqrt{N}(\hat{A} - A)) &= C(\theta)VC(\theta)' = e^{\gamma_0}\,\text{Avar}(\sqrt{N}(\hat{\gamma}_0 - \gamma)) \cdot e^{\gamma_0} \\
&= e^{2\gamma_0}\,\text{Avar}(\sqrt{N}(\hat{\gamma}_0 - \gamma))
\end{aligned}$$

(c)
$$
\begin{aligned}
\hat{A} &= e^{\hat{\gamma}_0} = e^{0.1} \approx 1.1052 \\
\text{Avar}(\sqrt{N}(\hat{A}-A)) &= e^{2\gamma_0}\,\text{Avar}(\sqrt{N}(\hat{\gamma}_0-\gamma_0)) \\
\text{Avar}(\hat{A}-A) &= e^{2\gamma_0}\,\text{Avar}(\hat{\gamma}_0-\gamma_0) \\
\text{Avar}(\hat{A}) &= e^{2\gamma_0}\,\text{Avar}(\hat{\gamma}_0), \quad \text{since } A \text{ and } \gamma_0 \text{ are constant} \\
Ase(\hat{A}) &= e^{\gamma_0}\,Ase(\hat{\gamma}_0) \\
\widehat{Ase}(\hat{A}) &= e^{\hat{\gamma}_0}\,\widehat{Ase}(\hat{\gamma}_0) \\
&= 1.1052 \cdot 0.075 = 0.0829
\end{aligned}
$$

(d) $\gamma_0 = 0 \Rightarrow A = e^0 = 1$. $\gamma_0 = 0$ means that solely capital and labor are responsible for production and technology does not matter! We should choose $H_1 : \gamma_0 > 0$ as alternative since we expect technology to positively affect the production.

$t = \dfrac{\hat{\gamma}_0 - \gamma_0}{\widehat{Ase}(\hat{A})} = \dfrac{0.1 - 0}{0.075} = 1.33$

$CV = 1.28$ (right-sided test, 10% level, standard normal)
$1.33 > 1.28$

$\Rightarrow$ reject H_0
$\Rightarrow$ technology does matter to production!

(e) $H_0 : A = 1$ vs. $H_1 : A > 1$

$t = \dfrac{\hat{A} - A}{\widehat{Ase}(\hat{A})} = \dfrac{1.1052 - 1}{0.0829} = 1.2690 < 1.28$

$\Rightarrow$ do not reject H_0
$\Rightarrow$ technology does <u>not</u> matter for production

$\Rightarrow$ Conclusion: Although both tests describe the same null hypothesis, we once reject and once not. The lesson is that outcomes of tests can be changed by nonlinear transformations (e.g. log(x))

4. $\gamma = g(\theta)$, $\hat{\gamma} = g(\hat{\theta})$, and $\tilde{\gamma} = g(\tilde{\theta})$. Delta method:

$$
\begin{aligned}
Avar\left(\sqrt{N}(\hat{\gamma}-\gamma)\right) &= G(\theta)\underbrace{Avar\left(\sqrt{N}(\hat{\theta}-\theta)\right)}_{V_1}G(\theta)' \\
Avar\left(\sqrt{N}(\tilde{\gamma}-\gamma)\right) &= G(\theta)\underbrace{Avar\left(\sqrt{N}(\tilde{\theta}-\theta)\right)}_{V_2}G(\theta)'
\end{aligned}
$$

Therefore,

$$
Avar\left(\sqrt{N}(\tilde{\gamma}-\gamma)\right) - Avar\left(\sqrt{N}(\hat{\gamma}-\gamma)\right) = G(\theta)(V_2 - V_1)G(\theta)'
$$

By assumption, $V_2 - V_1$ is pos. semidef. ($\hat{\theta}$ is more efficient than $\tilde{\theta}$).
And therefore, $G(\theta)(V_2 - V_1)G(\theta)'$ is pos. semidef. as well (quadratic form).

$$\begin{aligned}&\Rightarrow \quad Avar\left(\sqrt{N}(\tilde{\gamma}-\gamma)\right) \geq Avar\left(\sqrt{N}(\hat{\gamma}-\gamma)\right)\\ &\Rightarrow \quad \hat{\gamma} \text{ is more efficient than } \tilde{\gamma}!\end{aligned}$$

5. (a) Yes!

$$plim(\hat{\delta}) = plim\left(\frac{\hat{\alpha}}{\hat{\beta}}\right) = \frac{plim(\hat{\alpha})}{plim(\hat{\beta})} = \frac{\alpha}{\beta} = \delta$$

$\hat{\delta} = \frac{\hat{\alpha}}{\hat{\beta}}$ is consistent for δ!

(b) $\delta = c(\theta) = \dfrac{\alpha}{\beta} \qquad C(\theta) = \begin{pmatrix}\frac{\delta c}{\delta\alpha} & \frac{\delta c}{\delta\beta}\end{pmatrix} = \begin{pmatrix}\frac{1}{\beta} & -\frac{\alpha}{\beta^2}\end{pmatrix}$

$$\Rightarrow Avar(\hat{\delta}) = \begin{pmatrix}\frac{1}{\beta} & -\frac{\alpha}{\beta^2}\end{pmatrix} Avar(\hat{\Delta}) \begin{pmatrix}\frac{1}{\beta} \\ -\frac{\alpha}{\beta^2}\end{pmatrix}$$

(c)

$$\begin{aligned}Avar(\hat{\delta}) &= \begin{pmatrix}\frac{1}{0.63} & -\frac{0.42}{0.63^2}\end{pmatrix}\begin{pmatrix}0.03 & -0.033\\ -0.033 & 0.045\end{pmatrix}\begin{pmatrix}\frac{1}{0.63}\\ -\frac{0.42}{0.63^2}\end{pmatrix}\\ &\overset{PC}{=} 0.2368\\ \widehat{ASE}\left(\hat{\delta}\right) &= \sqrt{Avar(\hat{\delta})} = \sqrt{0.2368}\end{aligned}$$

(d) $H_0 : R\gamma = r \quad \text{vs.} \quad H_1 : R\gamma \neq r$ constant returns to scale mean

$\alpha + \beta = 1 \Rightarrow \gamma_1 + \gamma_2 = 1$

with $R = \begin{pmatrix}0 & 1 & 1\end{pmatrix} \quad r = 1 \quad q = 1$

and $X = \begin{pmatrix}1 & log(K_1) & log(L_1)\\ \vdots & \vdots & \vdots\\ 1 & log(K_N) & log(L_N)\end{pmatrix}$