Question concerning the course *Mathematical Finance*

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Abstract

These are questions that my be helpful for understanding the lecture notes of the course *Mathematical Finance* and in particular for preparing for the oral exam. Some have a clear simple answer, some do not. Not all of them are equally relevant. As a rough guide, red questions are more important than blue ones, which in turn are more relavant than green questions.

1. What is the difference between an outcome and an event?
2. What is the difference between a probability measure and a probability mass function?
3. What is the indicator of an event A?
4. What is the expected value $E(1_A)$ of an indicator of an event A?
5. What is the difference between the conditional probability relative to a set and relative to a σ-field? How are these two notions related?
6. Are conditional expectations additive in the sense that $P(\bigcup_{i=1}^n A_i|B) = \sum_{i=1}^n P(A_i|B)$ for pairwise disjoint $A_1, \dots, A_n$? Do they sum up to 1 if $A_1 \cup \dots \cup A_n = \Omega$, i.e. $\sum P(A|B) = 1$? Why do we care about these properties?
7. What is the conditional expectation of a random variable relative to an event? Does is also make sense to talk about the conditional variance of a random variable relative to an event? How would it be defined?
8. What is a null set?
9. What do absolute continuity and equivalence mean?
10. What is the density of a probability measure relative to another one? Why do we need absolute continuity to define it?
11. Can you tell from the density of an absolute continuous probability measure whether it is actually equivalent to the original one?

12. How does one compute the expectation of a random variable X relative to some probability measure Q whose density $\frac{dQ}{dP}$ is known?

13. If P, Q, R are equivalent probability measures, how can one compute the P-density of R from the P-density of Q and the Q-density of R?

14. What are the relations between the sample space Ω, the power set $\mathscr{P}(\Omega)$, and σ-fields $\mathscr{F}$ on Ω?

15. What do σ-fields on a finite set Ω look like?

16. What is a partition of a set Ω?

17. What does the σ-field generated by a partition look like?

18. Is the power set $\mathscr{P}(\Omega)$ a σ-field? Is it generated by a partition? If yes, why and what is the generating partition?

19. What partition is generating the trivial σ-field?

20. How do σ-fields reflect information?

21. What does $\mathscr{F}$-measurability of a random variable X mean intuitively?

22. What does it mean if X is $\{\emptyset, \Omega\}$-measurable?

23. What does is mean if X is $\mathscr{P}(\Omega)$-measurable?

24. What is $P(A|\{\emptyset, \Omega\})$ for an event A?

25. What is $P(A|\mathscr{P}(\Omega))$ for an event A?

26. How are conditional probabilities and conditional expectations relative to the same σ-field related?

27. What is the tower property for conditional expectations?

28. In which simple limiting cases can we compute the conditional expectation $E(X|\mathscr{F})$ without any calculations?

29. What is a filtration? Why does the requirement $\mathscr{F}_m \subset \mathscr{F}_n$ for $m \leq n$ make sense?

30. What is a stochastic process? What does the index n stand for? Why does the standard assumption of adaptedness make intuitive sense?

31. What does X_- mean?

32. What does predictability of a process X mean intuitively?

33. Give examples (in mathematical finance) of processes that can or cannot typically expected to be predictable.

34. Where do we need predictability?

35. What does is mean if the filtration is generated by a process?

36. What is the intuition behind a stopping time?

37. Do you think that the following times are typically stopping times?

    - the first time when the stock price falls below 10 €
    - the time when the stock price reaches its maximum within a year
    - 12 o'clock on 2 February 2022
    - exactly two days after the stock price reaches the level 10 € for the first time
    - the earlier one of two stopping times
    - the later one of two stopping times
    - exactly one day before the stock price reaches the level 10 € for the first time
    - the average of two stopping times

38. What is the difference between a stopped process X^T and the original process X?

39. What is a martingale and what does it mean intuitively?

40. Are the following deterministic processes martingales/submartingales/supermartingales?

    - $X_n = 7$ for all n
    - $X_n = 3 + n$ for all n
    - $X_n = \sin(n)$ for all n

41. Where do martingales/supermartingales play a role in mathematical finance? Why do they help in these places?

42. Consider a location without seasons, say close to the equator. Is it reasonable to assume that the daily temperature there is a martingale?

43. What are standard ways to construct martingales?

44. Where does the martingale generated by a random variable play a role in mathematical finance?

45. What is the density process of a probability measure?

46. Suppose that Z is the density process of $Q \sim P$. Is it true that $E(\frac{dQ}{dP} 1_A) = Q(A) = E(Z_n 1_A)$ for all events $A \in \mathscr{F}_n$? Does this mean that Z_n is the density of Q relative to P? If yes, why? If no, why not?

47. What is the generalized Bayes formula needed for?

48. What is the idea of the Doob decomposition?

49. What does the compensator tell you about the process?

50. What is the compensator of a martingale?

51. Does the Doob decomposition depend on the probability measure? If no, why not? If yes, how do the martingale part and the compensator change if we replace P by $Q \sim P$?

52. How can the stochastic integral be interpreted intuitively?

53. What is $H \bullet X_n$ if H is a constant?

54. What is $1_{[0,T]} \bullet X_n$ if T is a stopping time?

55. Where does the stochastic integral play a role in mathematical finance?

56. What are the covariation process or the predictable covariation process needed for?

57. What is the difference between the two?

58. Can you think of an illustration of the rule $H \bullet (K \bullet X) = (HK) \bullet X$ in terms of trading strategies?

59. If X is a martingale, so is $H \bullet X$. Do we need predictable H here or is adaptedness enough? Can you illustrate this in terms of trading assets?

60. Suppose that you buy a stock with price process S at time 0 for S_0 € and sell it at a stopping time $T \leq N$ for S_T €. If S is a martingale, is it possible to choose T cleverly such that $E(S_T) > S_0$, i.e. you make a profit on average? If yes, give an example. If no, why not?

61. What is the stochastic exponential?

62. Suppose that $X_0 = 0$ and $Z = \mathscr{E}(X)$. How can we recover X from Z? (X is sometimes called the *stochastic logarithm* of Z.)

63. What is $\mathscr{E}(X)$ for the process defined by $X_n = rn$ for $n = 0, 1, \dots$?

64. What does Yor's formula refer to?

65. Is the martingale property preserved if we change the probability measure form P to Q? If yes, why? If not, how does it change?

66. What is a random walk?

67. Is there a continuous-time counterpart to a random walk?

68. What does the martingale representation theorem refer to?

69. Where can the martingale representation theorem be applied in mathematical finance?

70. How can one represent a filtered probability space in a tree? How is the filtration represented in the tree?

71. How can adapted and predictable processes be represented in this tree?

72. How can stopping times be represented in a tree?

73. In a tree representing a filtered probability space, how can one compute the martingale generated by a random variable?

74. In the tree, how can one compute the density process of $Q \sim P$ from the probability mass function of Q?

75. In the tree, how can one compute the Doob decomposition of a process?

76. In the tree, how does one compute a stochastic integral $H \bullet X$?

77. In the tree, how does one compute the quadratic variation and the predictable quadratic variation of a given process X?

78. In the tree, how does one compute the stochastic exponential $\mathscr{E}(X)$ of a given process X?

79. In a binomial tree with N periods, a martingale S and a random variable X, how can one compute a predictable process φ and a constant v such that $X = v + \varphi \bullet S_N$? Does this also work in trees with more than two branches leaving each mode?

80. What a price processes, trading, strategies, and their value process?

81. What does self-financability mean intuitively?

82. State three equivalent conditions for a strategy to be self-financing?

83. What does *discounting* mean in mathematical finance? How is it related to the ordinary notion of discounting?

84. Why do we consider discounted processes in mathematical finance?

85. What is arbitrage?

86. Where is the notion of arbitrage useful in mathematical finance?

87. What does the law of one price state? Where is it useful in mathematical finance?

88. What is the idea of the proof underlying the law of one price?

89. What is an equivalent martingale measure?

90. Where do equivalent martingale measures play a role in mathematical finance?

91. What does the first FTAP state?

92. Where does the first FTAP prove to be useful in mathematical finance?

93. How can one model dividend payments in mathematical finance?

94. Why is the value of a portfolio defined as $V(\varphi) := \varphi^T(S+\Delta D)$ instead of just $V(\varphi) = \varphi^T S$?

95. What does the self-financing condition look like in the context of dividends? Why does it make sense?

96. What are *discounted* dividends? Why aren't they defined as $\hat{D} = D/S^0$?

97. What does the first FTAP look like in the context of dividends?

98. Where is the first FTAP with dividends applied in mathematical finance?

99. In a tree, how can one compute the self-financing strategy φ corresponding to given $(\varphi^1, \dots, \varphi^d)$ and initial capital V_0?

100. In a tree, how does one compute the discounted wealth process $V(\varphi)$ of the self-financing strategy corresponding to given $(\varphi^1, \dots, \varphi^d)$ and initial capital V_0?

101. In a tree, how can you verify that a given market does not allow for arbitrage?

102. In the previous question, how can you find an arbitrage if showing non-existence actually failed?

103. How can one compute an equivalent martingale measure in a tree?

104. Draw a tree where no equivalent martingale measure exists (without long calculations).

105. Draw a tree where more than one equivalent martingale measure exists (withoud long calculations).

106. In a market tree with dividends, how does one compute the self-financing strategy φ and its value process $V(\varphi)$ if $\varphi^1, \dots, \varphi^d$ and the intial value V_0 are given?

107. Give an example for a simple *realistic* model for two assets. Why is it reasonable? Where are its limits?

108. Why did we study binomial models carefully even though they do not seem very realistic compared to other models with iid logarithmic returns?

109. How could one estimate the parameters of a binomial model if it is meant to provide a reasonable approximation to observed stock price data?

110. How are daily logarithmic returns distributed in the geometric Brownian motion model underlying the Black-Scholes model?

111. How do the liquid derivative and the OTC approach to derivative pricing differ? Where are similarities?

112. What kind of derivative contracts can be distinguished? How does their mathematical treatment differ?

113. What are differences and similarities between forwards and futures?

114. What can be said about price processes of liquid derivatives in markets without arbitrage? In what sense are unique fair prices fair (or the others unfair)?

115. Does the market price of a liquid option depend on supply of and demand for this claim?

116. What can be said about the price process of an attainable claim?

117. Which role do equivalent martingale measures play in derivative pricing?

118. Who chooses the martingale measure in a real market?

119. What does market completeness mean?

120. How can market completeness be shown to hold or not to hold?

121. Are real markets complete?

122. Is market completness rather the rule or the exception in realistic market models?

123. Give examples for complete and incomplete market models.

124. Why do we care about derivative prices and hedges from the individual point of view?

125. What do the lower and the upper price of an option mean?

126. Why do the lower and the upper price of an option constitute reasonable bounds for the OTC price?

127. Are all prices between the lower and the upper price equally reasonable?

128. Who decides which value between the lower and the upper price is actually charged as an OTC price?

129. Is there an arbitrage opportunity for the seller of an option if she receives exactly the upper price?

130. How can the lower and the upper price be determined?

131. What can be said about the risk of the seller if she receives the upper price of the claim?

132. How much money is needed in order to buy the cheapest superhedge?

133. Is it likely that, in practice, the seller can afford the superhedge from the OTC price?

134. What can the seller do if the OTC price does not suffice for buying a superhedge?

135. What can be said about the lower and the upper price of a forward contract?

136. Can a forward contract be hedged perfectly? If yes, how? What do we have to assume about the dynamics of the underlying asset?

137. What can be said about the price process of a futures contract? Does it depend on supply of and demand for the contract? What do we have to assume about the dynamics of the underlying asset?

138. How are the price of a forward and a futures contract on the same asset related? How are the corresponding hedging strategies related?

139. How can the futures price process and the corresponding hedge be derived?
140. Typically, discounted asset prices are Q-martingales for some EMM Q. There is no discounting in the equation for the futures price process. Doesn't this contradict the absence of arbitrage?
141. If the forward and the futures have the same price process, shouldn't they have the same hedge?
142. What is the Cox-Ross-Rubinstein model?
143. It rather looks too simple for a reasonable approximation to real markets. Why is it still so popular in textbooks and even in practice?
144. What can be said about the difference of the lower and the upper price of options in the Cox-Ross-Rubinstein model?
145. What can be said about European call option prices in the Cox-Ross-Rubingstein model?
146. Do they allow for a perfect hedge?
147. A binomial tree with 1000 periods has $2^{1000} \approx 10^{100}$ nodes at the end, which exceeds the number of atoms in the universe and in particular the memory of any conceivable computer. Does this mean that we cannot compute the price of European call/put options in such a large tree?
148. How are the values of European call and put options related if both are liquidly traded?
149. How does the relation between liquid call and put options change if dividends are paid on the underlying asset?
150. What can be said about the range of possible European call option prices regardless of any concrete model for the underlying stock?
151. What can be said about the range of possible European call option prices in the discrete-time model where logarithmic one-period returns are iid Gaussian?
152. What can be said about the range of possible call option prices in the continuous-time extension of the model from question 151?
153. What is an optimal stopping problem?
154. How can one solve optimal stopping problems?
155. What is the Snell envelope and what is it good for?
156. What is an arbitrage in the context of markets with trading constraints?
157. What is an arbitrage in the context of markets involving American options?
158. What does the fundamental theorem of asset pricing look like if assets may not be sold short?

159. How are American option price processes related to optimal stopping problems?

160. How are American and European call prices are related if both are traded liquidly?

161. Does the relation between American and European calls also hold for puts or in the case of divident-paying stocks?

162. It may happen that the discounted American option price process is not a martingale under any equivalent martingale measure. Doesn't this contradict the absence of arbitrage?

163. Is the price process of a liquidly traded American option in a complete market uniquely determined by the absence of arbitrage?

164. In the Cox-Ross-Rubinstein model, what can be said about the American put price if the stock price is very low?

165. As the seller of an American option in a Cox-Ross-Rubinstein market, what is the minimal OTC price allowing to buy a perfect protection against losses?

166. How does one obtain (=compute) such a (super-)hedge?

167. How does one compute the replicating strategy of an arbitrary contingent claim in a binomal tree?

168. How does one compute the unique fair price process of an arbitrary contingent claim in a binomial tree?

169. How does one compute lower and upper prices of a contingent claim in a tree?

170. How does one compute the cheapest superhedge in a tree?

171. How does one compute the Snell envelope of a process X in a tree?

172. How does one compute optimal stopping times for stopping problems in a tree?

173. How does one compute the unique fair American option price process of an arbitrary American option in a binomial tree?

174. What does calibration to observed option prices mean?

175. In the context of optimal investment, why doesn't one typically try to maximize the expected terminal wealth?

176. What does optimality for expected utility of terminal wealth mean?

177. Are optimal strategies for expected utility of terminal wealth unique?

178. Why can it be helpful to consider equivalent martingale measures in the context of expected utility maximization?

179. In a time-homogeneous market and considering power or logarithmic utility, what are qualitative features of the optimal portfolio for terminal wealth?

180. In the context of question 179, how does the optimal portfolio depend on the current wealth?

181. In the context of question 180, how does the optimal portfolio depend on the time horizon N?

182. Suppose that the discounted stock price process is a martingale (under real probabilities). What does the optimal portfolio for expected utility look like in this case?

183. How can one illustrate the optimal portfolio in question 179 in the limiting case of small price changes?

184. What happens in the case of extremely large time horizons?

185. To what extent is expected utility maximization applicable in practice?

186. How can one compute optimal portfolios for arbitrary utility function in a binomal tree?

187. What is the counterpart of a Lévy process in discrete time?

188. Why is Brownian motion the most frequently used process for models in continuous time?

189. What is an Itō process? How can one understand the terms intuitively?

190. What is the covariation of two Itō processes? Where do we need it?

191. If X is an Itō process, is the same true for $Y_t = t^2 X_t^7$? If yes, how can one determine its Itō process representation?

192. Where does the stochastic exponential show up in mathematical finance?

193. What are advantages and limitations of the geometric Brownian motion model underlying the Black-Scholes formula? Is is realistic?

194. What are similarities and differences between the Cox-Ross-Rubinstein and the Black-Scholes model?

195. What does the Black-Scholes formula state?

196. How is the Black-Scholes formula obtained?

197. Why is it surprising that the parameter μ does not show up in the Black-Scholes formula?

198. Intuitively, μ should almost be the most relevant parameter for the option price. Can you solve the paradox?

199. Why is the Black-Scholes model so popular in practice?

200. Do you see problems with applying it in practice?