HW - 07 | Comp Fin
Group 27

I- Exo 27

a- Let W be a wiener process and suppose that X is a stochastic process which evolves according to the stochastic diff equat

$$dX(t) = a\,dt + b\,dW(t)$$

1- The integral representation of $f(X(t))$:

since X evolves according to $dX(t) =$ " "

$f(X(t))$ must satisfies the SDE:

$$f(X(t)) = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial X(t)}\, dX(t) + \frac{1}{2}\frac{\partial^2 f}{\partial X(t)^2}\, b^2\, dt. \quad (1)$$

where $\frac{\partial f}{\partial t} = 0$

Given that $dX(t) = X(t) - X(0) = \int_0^t a\,ds + \int_0^t b\,dW(s)$,

$$\Rightarrow X(t) = X(0) - \int_0^t a\,ds + \int_0^t b\,dW(s)$$

Therefore in integral representation,

$$f(X(t)) - f(X(0)) = \int_0^t \frac{\partial f}{\partial X(s)}\, dX(s) + \frac{1}{2}\int_0^t \frac{\partial^2 f}{\partial X(s)^2}\, b^2\, ds$$

$$\Rightarrow f(X(t)) = f(X(0)) + \int_0^t f'(X(s))\,(a\,ds + b\,dW(s)) + \frac{1}{2}\int_0^t f''(X(s))\, b^2\, ds$$

$$f(X(t)) = f(2) + \int_0^t f'(X(s))\,(a\,ds + b\,dW(s)) + \frac{1}{2}\int_0^t f''(X(s))\, b^2\, ds$$

2- For $f(X(t))$ to be a martingale,

a- $E_Q(f(X(t)) | F_0) = f(X(0))$ and f must be a continuous and differentiable function (because we need to take the integral)

b- the drift term a must be equal to zero, So... ? -0,5

c- The volatility term b should be stationary along time. No

3- If $f(x) = \exp(x)$, then $f(X(t))$ must satisfies the SDE:

$$\Rightarrow f(X(t)) = \exp(X(0)) + \int_0^t \exp(X(s))\,(a\,ds + b\,dW(s)) + \frac{1}{2}\int \exp(X(s))\, b^2\, ds$$

for $f(X(t))$ to be a martingale,

$$E_Q(f(X(t)) | F_0) = f(X(0))$$

Please make sure to scan upright and in good quality, this is very hard to read.

$$E[\exp(X(0))] + E[\exp(X(t))(a\,dt + b\,dW(t))] + \frac{1}{2}E[\exp(X(t))\cdot b^2 dt]$$
$$= \exp(X(0))$$

And to calculate these conditions for $f(x) = \exp(x)$, given that $X(t)$ follows the stochastic diff equation $dX(t) = a\,dt + b\,dW(t)$, by aplying the Itô's lemma to $\exp(X(t))$, we obtain its SDE

$$d[\exp(X(t))] = \exp(X(t))(a\,dt + b\,dW(t)) + \frac{1}{2}\exp(X(t))b^2 dt$$
$$\Rightarrow \quad = \exp(X(t))(a + \frac{1}{2}b^2)dt + \exp(X(t))\,b\,dW(t)$$

To be a martingale, the drift term of of the SDE should vanish

$$\Rightarrow \exp(X(t))(a + \frac{1}{2}b^2) = 0$$, so ... ? $-0{,}5$ $1/2$

b/ For a continuous stochastic pros Z with $Z > 0$,

$$\mathcal{L}(Z)(t) = \frac{1}{Z}dZ(t)$$

Show that:

(a) $E(\mathcal{L}(Z)) = \frac{Z}{Z_0}$

Using Itô's formula on $\ln(Z)(t)$:

$$d\ln(Z(t)) = \frac{1}{Z(t)}dZ(t) - \frac{1}{2}\frac{1}{Z^2(t)}dt \quad ①$$

we also know that the stochastic exp is the unique adapted process Z satisfying

$$Z_1 = 1 + Z_- \cdot X \quad , \quad Z_n = Z_{n-1} + Z_{n-1}\Delta X_n$$?

unreadable ?!

$$\Rightarrow \ln(Z_t) = \ln(Z_0) + \ln(Z_0)\,dZ_t \quad ②$$

if one equalizes ① and ② , we obtain $\frac{1}{Z}dZ(t) - \frac{1}{2}\frac{1}{Z^2}dt = \ln(Z_0)dZ_t$

!! rest is missing

b) For a continuous stochastic process Z with $Z>0$, we define the stochastic logarithm by

$\mathcal{L}(Z)$,

s.t., $\mathcal{L}(Z)(t) = \frac{1}{Z} dZ(t)$

Show that:

a- $\mathcal{E}(\mathcal{L}(Z)) = \frac{Z}{Z_0}$

? What's happening here? Doesn't seem to make sense !!

$\mathcal{L}(Z)(t) = \mathcal{L}(Z)(0) + \frac{1}{Z} dZ(t)$, $Z(0) = 1$

$\ln(1)$

$\Rightarrow \mathcal{L}(Z)(t) - \mathcal{L}(Z)(0) = \frac{1}{Z} dZ(t)$

$\mathcal{L}\left(\frac{Z(t)}{Z_0}\right) = \frac{1}{Z} dZ(t) \Rightarrow \frac{Z_t}{Z_0} = \mathcal{E}\left(\frac{1}{Z} dZ(t)\right)$

Therefore, $\frac{Z_t}{Z_0} = \mathcal{E}\left(\frac{1}{Z} dZ(t)\right) = \mathcal{E}(\mathcal{L}(Z))$.

b- $\mathcal{L}(\mathcal{E}(X)) = X - X_0$

$\mathcal{L}(\mathcal{E}(X)) = X - X_0 \Rightarrow \mathcal{E}(X) = \mathcal{E}(X - X_0)$

$\mathcal{E}(X - X_0) = \mathcal{E}(dX(t))$

$\Rightarrow \ln \mathcal{E}(dX(t)) = dX(t) = X(t) - X_0$ That terminates the proo

P-27 $= \cancel{\frac{1}{Z} dZ_t} \to \frac{1}{Z} \frac{1}{Z^2} \cancel{\sigma^2 dt}$

$\mathcal{L}(\mathcal{E}(X)) = \mathcal{L}(\mathcal{E}(X - X_0))$

→ You must not use the claim to show the claim. Only use the assumptions!

0/2

1/4