Mathematical Finance Winter term 2023/2024
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Mathematical Finance: MF

Exercises (for discussion on Monday, 11.12.2023)

Exercise 1. Consider an inhomogeneous market with time horizon N. S^0 is the riskless asset given by $S^0_n = 1 + nr$ with a constant $r > 0$. Let the stock S^1 be given by $S^1_n = \prod_{i=1}^n (1 + \Delta\tilde{X}_i)$ with the $\Delta\tilde{X}_i$, $i = 1, 2, ..., N$ being independent random variables with

$$P(\Delta\tilde{X}_i = u_i - 1) = p_i > 0,$$
$$P(\Delta\tilde{X}_i = d_i - 1) = 1 - p_i > 0,$$

for real numbers $u_i > d_i > 0$. We consider the filtration is generated by $\mathcal{F}_n := \sigma(\Delta\tilde{X}_1, \dots, \Delta\tilde{X}_n)$, $\mathcal{F}_0 = \{\Omega, \emptyset\}$ and set $S^0_0 = S^1_0 = 1$. Find a condition on u_i, d_i, $i \in \{1, ..., N\}$ that holds if and only if the market is free of arbitrage.

Exercise 2. Let (S^0, S^1) be the price process in a market with end-time 1. Assume that $S^0_0 = S^1_0 = S^0_1 = 1$ and that

$$S^1_1(\omega) = \begin{cases} x_1, & \text{if } \omega = \omega_1 \\ x_2, & \text{if } \omega = \omega_2 \\ x_3, & \text{if } \omega = \omega_3 \end{cases}$$

with

$$p_1 := P(\{\omega_1\}) > 0,$$
$$p_2 := P(\{\omega_2\}) > 0,$$
$$p_3 := P(\{\omega_3\}) = 1 - p_1 - p_2 \geq 0.$$

For which x_1, x_2, x_3 and p_1, p_2, p_3 is the latter market ...

1. ... arbitrage-free?
2. ... arbitrage-free and complete?

Exercise 3. Let $S = (S^0, \dots, S^d)$ be a market with endpoint $N \in \mathbb{N}$. We additionally assume $S^0 > 0$. Show that the following statements are equivalent:

a) There exists a *one period arbitrage*, meaning an arbitrage strategy φ such that $\widehat{V}_0(\varphi) = \ldots = \widehat{V}_{n-1}(\varphi) = 0$ and $\widehat{V}_n(\varphi) = \ldots = \widehat{V}_N(\varphi)$ hold for some $n \in \{1, \dots, N\}$.

b) There exists an arbitrage φ with non negative value process $V(\varphi)$.

c) There exists an arbitrage.

Exercise 4. Let $(S^0, ..., S^d)$ be a market with end point $N \in \mathbb{N}$ and let $S^0 > 0$. Let Y be an adapted process such that for all $n \in \{1, \dots, N\}$ holds

1. $Y_n > 0$,
2. $E(Y_n | \mathcal{F}_{n-1}) = 1$,
3. $E(Y_n \Delta \hat{S}_n | \mathcal{F}_{n-1}) = 0$.

Show that $\frac{dQ}{dP} := \prod_{n=1}^{N} Y_n$ defines an equivalent martingale measure Q.

Submission of the homework until: Thursday, 07.12.2023, 10.00 a.m. via OLAT.