Computational Finance

sheet - 9 , Group 27

Some good approaches, but sadly nothing that works.

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T- Exo 32

Prove that $y(\tilde{t},\tilde{x})$ solves the heat equation

$$\frac{\partial y}{\partial \tilde{t}} = \frac{\partial^2 y}{\partial \tilde{x}^2}$$

if we have $y(t,x)$ instead of $y(\tilde{t},\tilde{x})$ that might make the partial derivative a bit easier to take!!

$$\frac{\partial y}{\partial \tilde{t}} = v(t,x)\,k^{-1}\,\exp\left(\tfrac{1}{2}(q-1)\tilde{x} + \tfrac{1}{4}(q-1)^2+q)\tilde{t}\right)\cdot\left(\tfrac{1}{4}(q-1)^2+q\right)$$

$$\frac{\partial y}{\partial \tilde{x}} = v(t,x)\,k^{-1}\,\exp\left(\tfrac{1}{2}(q-1)\tilde{x} + (\tfrac{1}{4}(q-1)^2+q)\tilde{t}\right)\cdot\tfrac{1}{2}(q-1)$$

$$\frac{\partial^2 y}{\partial \tilde{x}^2} = v(t,x)\,k^{-1}\,\exp\left(\tfrac{1}{2}(q-1)\tilde{x} + (\tfrac{1}{4}(q-1)^2+q)\tilde{t}\right)\cdot\tfrac{1}{4}(q-1)^2$$

Apparently $\frac{\partial y}{\partial \tilde{t}} = \frac{\partial^2 y}{\partial \tilde{x}^2} = \frac{1}{4}(q-1)^2+q = \frac{1}{4}(q-1)^2 \Leftrightarrow$

$\frac{1}{4}(q^2-2q+1)+q = \frac{1}{4}(q-1)^2 \Rightarrow \frac{1}{4}(q+1)^2 = \frac{1}{4}(q-1)^2$ which is not plausible!

Let keep v as it is

What about the derivatives of v?

$$\frac{\partial y}{\partial \tilde{x}} = \frac{\partial v}{\partial \tilde{x}} \cdot k^{-1} \cdot k \exp(\tilde{x}) \exp\left(\tfrac{1}{2}(q-1)\tilde{x} + (\tfrac{1}{4}(q-1)^2 + q)\tilde{t}\right)$$

$$+ \exp\left(\tfrac{1}{2}(q-1)\tilde{x} + (\tfrac{1}{4}(q-1)^2+q)\tilde{t}\right)\left(\tfrac{1}{2}(q-1)\right) v \cdot k^{-1}$$

$$= \frac{\partial v}{\partial \tilde{x}} \exp\left(\tfrac{1}{2}(q+1)\tilde{x} + (\tfrac{1}{4}(q-1)^2+q)\tilde{t}\right) + \exp\left(\tfrac{1}{2}(q-1)\tilde{x} + \right.$$

$$\left.(\tfrac{1}{4}(q-1)^2+q)\tilde{t}\right)\left(\tfrac{1}{2}(q-1)\right) \cdot v \cdot k^{-1}$$

$$\frac{\partial^2 y}{\partial \tilde{x}^2} = k \frac{\partial^2 v}{\partial \tilde{x}^2} \exp\left(\tfrac{1}{2}(q+1)\tilde{x} + (\tfrac{1}{4}(q-1)^2+q)\tilde{t}\right)^{①} + \tfrac{1}{2}(q+1) \cdot$$

$$\exp\left(\tfrac{1}{2}(q+1)\tilde{x}(\tfrac{1}{4}(q-1)\tilde{x}+q)\tilde{t}\right)^{②} \cdot k \cdot \frac{\partial^2 v}{\partial \tilde{x}^2} + \tfrac{1}{4}(q-1)^2 \cdot \exp('')^{③}$$

$$\cdot v \cdot k^{-1} + \underbrace{\frac{\partial v}{\partial x} \cdot k \exp(\tilde{x}) \cdot k^{-1} \exp('')}_{} \left(\tfrac{1}{2}(q-1)\right)^{④}$$

$$= \quad '' \quad + '' + \frac{\partial v}{\partial x} \cdot \exp\left(\tfrac{1}{2} q(+1)\tilde{x}(\tfrac{1}{4}(q-1)x^2 + q)\tilde{t}\right)\left(\tfrac{1}{2}(q-1)\right)$$

$$= \exp\left(\tfrac{1}{2}(q+1)\tilde{x} + (\tfrac{1}{4}(q-1)^2+q)\tilde{t}\right) \cdot \left(k \cdot \frac{\partial^2 v}{\partial x^2} + \tfrac{1}{2}(q+1)k \frac{\partial^2 v}{\partial \tilde{x}^2} + \right.$$

$$\left.\frac{\partial v}{\partial x} \cdot \tfrac{1}{2}(q-1)\right) + \tfrac{1}{4}(q-1)^2 \exp\left(\tfrac{1}{2}(q-1)\tilde{x} + (\tfrac{1}{4}(q-1)^2+q)\tilde{t}\right) \cdot v\, k^{-1}$$

if $\frac{\partial^2 v}{\partial x^2} = \frac{\partial v}{\partial x} = v \Rightarrow$

$$= \exp\left(\tfrac{1}{2}(q+1)\tilde{x} + \dots\right) \cdot \left(kv + \tfrac{1}{2}(q+1)kv + \tfrac{1}{2}(q-1)v\right) + \tfrac{1}{4}(q-1)^2 \exp\left(\right.$$

$$\tfrac{1}{2}(q-1)\tilde{x} + (\tfrac{1}{4}(\dots)\, v\, k^{-1}$$

This is also not equal to $\frac{\partial y}{\partial \tilde{t}} = \frac{\partial v}{\partial \tilde{t}} \cdot k^{-1}\left(-\frac{2}{\sigma^2}\right) \exp\left(\tfrac{1}{2}(q-1)\tilde{x}\right.$

$$+ (\tfrac{1}{4}(q-1)^2+q)\tilde{t}^2) + \exp\left(\tfrac{1}{2}(q-1)\tilde{x} + (\tfrac{1}{4}(q-1)^2+q)\tilde{t}\right) \cdot ($$

$$\tfrac{1}{4}(q-1)^2+q) \cdot v \cdot k^{-1} \;!$$

What if the function of interest was $L = \exp(\frac{1}{2}(q-1)\tilde{x} + (\frac{1}{4}(q+1)^2\tilde{t}$

$$\frac{\partial L}{\partial \tilde{t}} = \left(\frac{1}{4}(q-1)^2+q\right)\exp \frac{1}{2}(q-1)\tilde{x} + \left(\frac{1}{4}(q-1)^2+q)\tilde{t}\right)$$

do not consider special cases

$$\frac{\partial L}{\partial x} = \frac{1}{2}(q-1)\exp(\ ) \Rightarrow \frac{\partial^2 L}{\partial x^2} = \frac{1}{4}(q-1)(q-1)\exp('')$$

$$= \frac{1}{4}(q-1)^2\exp('')$$

Moreover, $\left(\frac{1}{4}(q-1)^2+q\right) = \frac{1}{4}(q^2-2q+1)+q = \frac{1}{4}q^2 - \frac{q}{2} + \frac{1}{4} + q$

$$\frac{1}{4}q^2 + \frac{q}{2} + \frac{1}{4} = \frac{1}{4}(q^2+1)^2$$

$$\frac{1}{4}(q+1)^2\exp\left(\frac{1}{2}(q-1)\tilde{x} + \left(\frac{1}{4}(q-1)^2+q\right)\tilde{t}\right) = \frac{1}{4}(q-1)^2\exp(q-1)\tilde{x}\ldots$$

$(q+1)^2 = (q-1)^2$ Is my final trial...

$$Y(t,x) = v(t,x)K^{-1}\exp\left(\frac{1}{2}(q-1)\log\left(\frac{x}{K}\right) + \left(\frac{1}{4}(q-1)^2+q\right)\frac{\sigma^2(T-t)}{2}\right)$$

$$= v(t,x)K^{-1}\exp\left(\frac{1}{2}q\log\left(\frac{x}{K}\right) - \frac{1}{2}\log\left(\frac{x}{K}\right) + \ldots\right)$$

$$\frac{\partial Y(t,x)}{\partial t} = \frac{\partial v(t,x)}{\partial t}\cdot K^{-1}\exp('') - \frac{\sigma^2}{2}\left(\frac{1}{4}(q-1)^2+q\right)\exp('')\,v(t,x)K^{-1}$$

$$\frac{\partial Y(t,x)}{\partial x} = \frac{\partial v(t,x)}{\partial x}K^{-1}\cdot\exp('') + \frac{1}{x}\frac{1}{2}(q-1)\cdot\exp('')\,v(t,x)K^{-1}$$

$$\frac{\partial^2 Y(t,x)}{\partial x^2} = \frac{\partial^2 v(t,x)}{\partial x^2}K^{-1}\exp('') - \frac{1}{x^2}\frac{1}{2}(q-1)\exp('')\,v(t,x)K^{-1} +$$
$$\frac{1}{x^2}\frac{1}{4}(q-1)^2\exp('')\,v(t,x)K^{-1} + \frac{1}{x}\frac{1}{2}(q-1)\exp('')\,v'(t,x)K^{-1}$$

$$= \frac{\partial^2 v(t,x)}{\partial x^2}K^{-1}\exp'' + \exp('')\,v(t,x)K^{-1}\left(-\frac{1}{x^2}\frac{1}{2}(q-1) + \frac{1}{x^2}\frac{1}{4}(q-1)^2 + \frac{1}{x}\frac{1}{2}(q-1)\right)$$

$$\frac{\partial Y(t,x)}{\partial t} = \frac{\partial^2 v}{\partial x^2} \Rightarrow -\frac{\sigma^2}{2}\left(\frac{1}{4}(q-1)^2+q\right) = +\frac{1}{x}\frac{1}{2}(q-1)\left(-\frac{1}{x} + \frac{1}{x}\frac{1}{2} + 1\right)$$