Comput finance

sheet 7 | Group 27

Exercice-T 20

we consider a Black-scholes model with $r=0$, $\sigma=\sqrt{2}$, $S(0)=1$, $T>0$

a- show that $V_1(t,S(t))$ of a EU opt. with payoff $f(S(T))=3\sqrt{S(T)}+S(T)^{3/2}$

equals $V_1(t,S(t))=\exp(-\frac{1}{4}(T-t))\,3\sqrt{S(t)}+\exp(\frac{3}{4}(T-t))\,S(t)^{3/2}$

$$f(S(T))=3\cdot\left(S(t)\exp\left(\left(r-\frac{\sigma^2}{2}\right)(T-t)+\sigma\sqrt{T-t}\cdot x\right)\right)^{1/2}+\left(S(t)\exp\left(\left(r-\frac{\sigma^2}{2}\right)(T-t)+\sigma\sqrt{T-t}\cdot x\right)\right)^{3/2}$$

$$=3\cdot(S(t))^{1/2}\left(\exp\left(-(T-t)+\sqrt{2}\sqrt{T-t}\cdot x\right)\right)^{1/2}+(S(t))^{3/2}\left(\exp\left(-(T-t)+\sqrt{2}\sqrt{T-t}\cdot x\right)\right)^{3/2}$$

because $V(t)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f\left(S(t)\exp\left((r-\sigma^2/2)(T-t)+\sigma\sqrt{T-t}\,x\right)\right)e^{-r(T-t)}\,e^{-x^2/2}\,dx$,

$$V_1(t,S(t))=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}3(S(t))^{1/2}\left(\exp(-(T-t)+\sqrt{2}\sqrt{T-t}\cdot x)\right)^{1/2}\cdot e^{-r(T-t)}\,e^{-x^2/2}\,dx \; ① \; +$$

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}(S(t))^{3/2}\left(\exp(-(T-t)+\sqrt{2}\sqrt{T-t}\cdot x)\right)^{3/2}e^{-r(T-t)}\,e^{-x^2/2}\,dx \; ②, \quad e^{-0\cdot(T-t)}=1$$

For ①: $\Rightarrow$

$$\frac{1}{\sqrt{2\pi}}\cdot3\sqrt{S(t)}\int_{-\infty}^{\infty}\exp\left(-\tfrac{1}{2}(T-t)\right)\cdot\exp\left(\tfrac{1}{2}\sqrt{2}\sqrt{T-t}\cdot x-\frac{x^2}{2}\right)dx$$

$$=\frac{1}{\sqrt{2\pi}}\,3\cdot\sqrt{S(t)}\int_{-\infty}^{\infty}\exp\left(-\tfrac{1}{2}(T-t)\right)\cdot\exp\left\{-\tfrac{1}{2}\left(x^2-2\cdot\frac{\sqrt{2}}{2}\sqrt{T-t}\cdot x\pm\left(\frac{\sqrt{2}}{2}\sqrt{T-t}\right)^2\right)\right\}dx$$

$$=3\sqrt{S(t)}\cdot\exp\left(-\tfrac{1}{2}(T-t)\right)\cdot\exp\left(-\tfrac{1}{2}\cdot\left(-\frac{\sqrt{2}}{2}\sqrt{T-t}\right)^2\right)\cdot\underbrace{\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}\exp\left\{-\tfrac{1}{2}\left(x-\frac{\sqrt{2}}{2}\sqrt{T-t}\right)^2\right\}dx}_{=1}$$

$$=3\sqrt{S(t)}\exp\left(-\tfrac{1}{2}(T-t)+\tfrac{2}{8}(T-t)\right)\cdot1=3\sqrt{S(t)}\exp\left(-\tfrac{1}{4}(T-t)\right)$$

For ②: $\frac{1}{\sqrt{2\pi}}\cdot(S(t))^{3/2}\int_{-\infty}^{\infty}\exp\left(-\tfrac{3}{2}(T-t)\right)\exp\left(\tfrac{3}{2}\sqrt{2}\sqrt{T-t}\cdot x-\frac{x^2}{2}\right)dx$

$$=\frac{1}{\sqrt{2\pi}}(S(t))^{3/2}\int_{-\infty}^{\infty}\exp\left(-\tfrac{3}{2}(T-t)\right)\exp\left\{-\tfrac{1}{2}\left(x^2-2\cdot\frac{3\sqrt{2}}{2}\sqrt{T-t}\cdot x\pm\left(\frac{3\sqrt{2}}{2}\sqrt{T-t}\right)^2\right)\right\}dx$$

$$=(S(t))^{3/2}\exp\left(-\tfrac{3}{2}(T-t)+\tfrac{18}{8}(T-t)\right)\underbrace{\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}\exp\left\{-\tfrac{1}{2}\left(x-\frac{3\sqrt{2}}{2}\sqrt{T-t}\right)^2\right\}dx}_{=1}$$

$$\Rightarrow V_1(t,S(t))=\exp\left(-\tfrac{1}{4}(T-t)\right)3\sqrt{S(t)}+\exp\left(\tfrac{3}{4}(T-t)\right)(S(t))^{3/2}$$

✓

Nice, very good solution! 2/2

b) Consider an Am Option with pay off process

$$g(S(t)) := \begin{cases} " & \text{if } S(t) < 1 \\ " & \text{if } S(t) \geq 1 \end{cases} \quad \text{for } t \in T \qquad ①$$

?

show that its fair price $V_2(t)$ equal

$$V_2(t, S(t)) := \begin{cases} g(S(t)) & \text{if } S(t) < e^{-(T-t)} \\ V_1(t, S(t)) & \text{if } S(t) \geq e^{-(T-t)} \end{cases} \quad \text{for } t \in T \qquad ②$$

for an Am option, the value process of the stock is given by its snell envelop.

$$V_2(t, S(t)) : \max \{ g(S(t)) , E_Q(V_2(T) | F_t) \}$$

@ if $S(t) < e^{-r(T-t)}$, the option is deep in the money and therefore,
↳ not correct

$g(S(t)) > V_1(t, S(t))$ and $V_2(t, S(t)) = g(S(t)) = 4S(t)^{3/4}$

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doesn't make any sense !!

② $S(t) \geq e^{-(T-t)}$, the option is out of the money, $\Rightarrow V_1(t, S(t)) > g(S(t))$

$$V_2(t, S(t)) = E_Q(V_2(T) | F_t) = E_Q\left((3\sqrt{S(T)} + S(T)^{3/2}) | F_t \right)$$

$$= \int_{-\infty}^{\infty} (3\sqrt{S(T)} + S(T)^{3/2}) \frac{1}{\sqrt{2\pi}} e^{-r(T-t)} e^{-x^2/2} dx$$

$$= \exp\left(-\tfrac{1}{4}(T-t)\right) 3\sqrt{S(t)} + \exp\left(\tfrac{3}{4}(T-t)\right)(S(t))^{3/2} = V_1(t, S(t))$$

$$\Rightarrow \quad V_2(t, S(t)) = \begin{cases} 4S(t)^{3/4} & \text{if } S(t) < e^{-(T-t)} \\ \exp(-\tfrac{1}{4}(T-t)) 3\sqrt{S(t)} + \exp(\tfrac{3}{4}(T-t))(S(t))^{3/2} & \text{if } S(t) \geq e^{-(T-t)} \end{cases}$$

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Unfortunately, nothing here is helpful for solving the exercise. :(

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