Mathematical Finance Winter term 2023/2024

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Mathematical Finance: MF

Exercises (for discussion on Monday, 04.12.2023)

Exercise 1. Let (S^0, S^1, S^2, S^3) be an arbitrage-free market with end-time $N \in \mathbb{N}$ and let $K \in [0, \infty)$. Assume that S^0 is deterministic and that S^2 and S^3 are price processes with the final values $S^2_N = (S^1_N - K)^+$ and $S^3_N = (K - S^1_N)^+$, respectively. Show that:

$$S^2 - S^3 = S^1 - K\frac{S^0}{S^0_N}.$$

Exercise 2. Let (S^0, S^1, S^2) be an arbitrage-free market with end-time $N \in \mathbb{N}$, S^0 deterministic and $S^1 \geq 0$. Assume that S^2 is the price process of a European-call option on S_1 with maturity N and strike $K \in [0, \infty)$, i.e. $S^2_N = (S^1_N - K)^+$. Show:

$$\left(S^1 - K\frac{S^0}{S^0_N}\right)^+ \leq S^2 \leq S^1$$

Exercise 3. Let $N \in \mathbb{N}$. Construct a filtered probability space $(\Omega, (\mathcal{F})_{n \leq N}, \mathcal{F}, P)$, two independent simple random walks $(X_n)_{n \leq N}$ and $(Y_n)_{n \leq N}$ (with finite terminal time N), i.e. $X_1, X_2, ..., Y_N$ and $Y_1, Y_2, ...Y_N$ are iid random variables on $(\Omega, (\mathcal{F})_{n \leq N}, \mathcal{F}, P)$ with $P(X_1 = 1) = P(X_1 = -1) = P(Y_1 = 1) = P(Y_1 = -1) = \frac{1}{2}$ and a probability measure $Q \sim P$ on $(\Omega, \mathcal{F})$ such that X is a strict submartingale w.r.t. Q and Y is a strict supermartingale w.r.t. Q. Determine the density process Z of Q w.r.t. P.

Exercise 4. In a market $(S^0, \ldots, S^d)$ with terminal date $N \in \mathbb{N}$, let S^0, S^1 be strictly positive and let P_0, P_1 probability measures with density $\frac{dP_1}{dP_0} := \frac{S^0_0}{S^1_0} \cdot \frac{S^1_N}{S^0_N}$ of P_1 with respect to P_0. Show that P_0 is an equivalent martingale measure w.r.t. to the numeraire S^0 if and only if P_1 is an equivalent martingale measure w.r.t. to the numeraire S^1.

Submission of the homework until: Thursday, 30.11.2023, 10.00 a.m. via OLAT.