Problem Set 3: Asymptotics

Review the Concepts and Proofs

1. Why do we rely for statistical inference on asymptotic analysis? Discuss possible advantages and disadvantages.

2. Discuss the relationship between CLT and WLLN.

3. Show that the conditions $\lim_{n\to\infty} E(\hat{\boldsymbol{\theta}}) = \boldsymbol{\theta}$ and $\lim_{n\to\infty} Var(\hat{\boldsymbol{\theta}}) = 0$ imply convergence in probability.

4. Explain the difference between asymptotic and approximate distribution.

5. Show that the Wald test for $H_0 : \boldsymbol{R\theta} = \boldsymbol{r}$ is asymptotically χ^2_Q distributed if $\hat{\boldsymbol{\theta}}$ is a consistent estimator for $\boldsymbol{\theta}$. Which additional assumption is needed?

6. Consider the simple t test for a single parameter restriction in small samples. Would you rely on asymptotics there? Why or why not?

Exercises

1. Let $\{y_i : i = 1, 2, ...\}$ be an independent, identically distributed sequence with $E(y_i^2) < \infty$. Let $\mu = E(y_i)$ and $\sigma^2 = Var(y_i)$.

   (a) Let $\overline{y}_N$ denote the sample average based on a sample size of N. Find $\text{Var}[\sqrt{N}(\overline{y}_N - \mu)]$.

   (b) What is the asymptotic variance of $\sqrt{N}(\overline{y}_N - \mu)$?

   (c) What is the asymptotic variance of $\overline{y}_N$? Compare this with $\text{Var}(\overline{y}_N)$!

   (d) What is the asymptotic standard deviation of $\overline{y}_N$? How would you estimate the asymptotic standard deviation?

2. Let $\{z_i : i = 1, 2, ...\}$ be an independent, identically distributed sequence with $E(z_i) = 5$ and $Var(z_i) = 10$.

   (a) Sketch the asymptotic distribution of the sample average $\overline{z}_N$ for
   $N = 10,\ 30,\ 100$ and 1000.

(b) Based on this example discuss the difference between the WLLN and CLT, i.e. discuss the difference between convergence in probability and convergence in distribution.

3. Consider the Cobb-Douglas production function $Y = AK^{\alpha}L^{\beta}$ and its linear relationship in logs $log(Y) = log(A)+\alpha \cdot log(K)+\beta \cdot log(L)$. To estimate this linear relationship a regression $log(Y_i) = \gamma_0+\gamma_1 \cdot log(K_i)+\gamma_2 \cdot log(L_i)+\varepsilon_i$,where $\gamma_0 = log(A)$, $\gamma_1 = \alpha$ and $\gamma_2 = \beta$ for a cross section of firms $i = 1, ..., N$ is conducted. ε_i has mean zero and variance σ^2. Assume all estimates are consistent and asymptotically normal.

   (a) Can we get a consistent estimator for A?

   (b) Find the asymptotic variance of $\sqrt{N}(\hat{A}-A)$ in terms of the asymptotic variance of $\sqrt{N}(\hat{\gamma}_0 - \gamma_0)$.

   (c) Suppose that, for the sample at hand, $\hat{\gamma}_0 = 0.1$ and $se(\hat{\gamma}_0) = 0.075$. What is $\hat{A}$ and its asymptotic standard error?

   (d) Consider the null hypothesis $H_0 : \gamma_0 = 0$. What does it economically mean? Choose a sensible alternative hypothesis. What is the asymptotic t statistic for testing H_0, given the numbers from part (c)? Conduct the test at the 5% level.

   (e) Now state H_0 from part (d) equivalently in terms of A, and use $\hat{A}$ and $se(\hat{A})$ to test H_0. Conduct the test at the 5% level. What do you conclude?

4. Let $\hat{\boldsymbol{\theta}}$ and $\tilde{\boldsymbol{\theta}}$ be two consistent, $\sqrt{N}$-asymptotically normal estimators of the $P \times 1$ parameter vector $\boldsymbol{\theta}$, with Avar$\sqrt{N}(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}) = \boldsymbol{V}_1$ and Avar$\sqrt{N}(\tilde{\boldsymbol{\theta}} - \boldsymbol{\theta}) = \boldsymbol{V}_2$. Define a $Q \times 1$ parameter vector by $\boldsymbol{\gamma} = \boldsymbol{g}(\boldsymbol{\theta})$, where $\boldsymbol{g}(.)$ is a continuously differentiable function. Show that, if $\hat{\boldsymbol{\theta}}$ is asymptotically more efficient than $\tilde{\boldsymbol{\theta}}$, then $\hat{\boldsymbol{\gamma}} = \boldsymbol{g}(\hat{\boldsymbol{\theta}})$ is asymptotically more efficient relative to $\tilde{\boldsymbol{\gamma}} = \boldsymbol{g}(\tilde{\boldsymbol{\theta}})$.

5. Consider the estimation of exercise 3 again. The vector $\boldsymbol{\Delta}$ is defined as $(\alpha, \beta)'$.

   (a) Can we get a consistent estimate for $\delta = \alpha/\beta$?

   (b) Find Avar$(\hat{\delta})$ in terms of $\boldsymbol{\Delta}$ and Avar$(\hat{\boldsymbol{\Delta}})$ using the delta method.

   (c) Assume, for the sample at hand, $\hat{\boldsymbol{\Delta}} = (0.42, 0.63)'$ and Avar$(\hat{\boldsymbol{\Delta}})$ is estimated as $\begin{pmatrix} 0.030 & -0.033 \\ -0.033 & 0.045 \end{pmatrix}$. Find the asymptotic standard error of $\hat{\delta}$.

   (d) How could you test for constant returns to scale? State R, r, X and q for the respective F-Test.