Solutions 12

1. (a) Since $X \sim U(0,30)$ we have:

$$F(x) = \int_0^x \frac{1}{30-0} ds = \frac{s}{30}\Big|_0^x = \frac{x}{30}\mathcal{I}_{[0,30]}(x) + \mathcal{I}_{(30,\infty)}(x).$$

(b)

$$\mathrm{P}(X > 10) = 1 - \mathrm{P}(X \le 10) = 1 - F(10) = 1 - \frac{10}{30} = \frac{2}{3}.$$

(c) Use theorem 1.10 to obtain

$$\begin{aligned}\mathrm{P}(X \ge 20 | X \ge 15) &= \frac{\mathrm{P}(X \ge 20 \cap X \ge 15)}{\mathrm{P}(X \ge 15)} = \frac{\mathrm{P}(X \ge 20)}{\mathrm{P}(X \ge 15)} = \frac{1-F(20)}{1-F(15)} \\ &= \frac{1-\frac{20}{30}}{1-\frac{15}{30}} = \frac{\frac{1}{3}}{\frac{1}{2}} = \frac{2}{3}.\end{aligned}$$

(d) $\mathrm{E}(X) = \frac{0+30}{2} = 15$ minutes.

2. (a) We find $\mathrm{P}(S_1) = \frac{1}{6}$, $\mathrm{P}(S_2) = \frac{5}{6}\cdot\frac{1}{6}$, $\mathrm{P}(S_3) = \left(\frac{5}{6}\right)^2\frac{1}{6}, \ldots, \mathrm{P}(S_k) = \left(\frac{5}{6}\right)^{k-1}\frac{1}{6}$.

(b) Note that the events $S_1, \ldots, S_k$ are mutually disjoint, therefore we get

$$\begin{aligned}\mathrm{P}(\text{a 6 will shop up eventually}) &= \lim_{k\to\infty} \mathrm{P}(S_1 \cup \cdots \cup S_k) = \lim_{k\to\infty}\sum_{j=1}^k \mathrm{P}(S_j) \\ &= \frac{1}{6}\sum_{j=1}^\infty \left(\frac{5}{6}\right)^{j-1} = \frac{1}{6}\sum_{m=0}^\infty \left(\frac{5}{6}\right)^m \overset{\text{Hint}}{=} \frac{1}{6}\frac{1}{1-\frac{5}{6}} = 1 \checkmark\end{aligned}$$

3. (a) We have

$$\begin{aligned}\int_0^1\int_0^1 f(x,y)dxdy &= \alpha\int_0^1\int_0^1 (1+(2x-1)(2y-1))dxdy = \alpha\left(1 + (x^2-x)\big|_0^1 (y^2-y)\big|_0^1\right) \\ &= \alpha \overset{!}{=} 1\end{aligned}$$

(b) Yes, since no parameters are involved!

(c) For $x,y \in (0,1)$ we have $F(x,y) = \int_0^x\int_0^y (1+(2x-1)(2y-1))dxdy = xy + xy(x-1)(y-1)$. Therefore,

$$\begin{aligned}F(x,y) = {} & xy(1+(x-1)(y-1))\mathbb{I}_{(0,1)}(x)\mathbb{I}_{(0,1)}(y) + x\mathbb{I}_{(0,1)}(x)\mathbb{I}_{[1,\infty)}(y) \\ & + y\mathbb{I}_{[1,\infty)}(x)\mathbb{I}_{(0,1)}(y) + \mathbb{I}_{[1,\infty)}(x)\mathbb{I}_{[1,\infty)}(y)\end{aligned}$$

(d) $\mathrm{P}\left(X \ge \frac{1}{2}, Y \le 1\right) = \mathrm{P}(X \le 1, Y \le 1) - \mathrm{P}\left(X < \frac{1}{2}, Y \le 1\right) = F(1,1) - F\left(\frac{1}{2}, 1\right) = 1 - \frac{1}{2} = \frac{1}{2}$.

(e) In (c) We found $F(x) = x\mathbb{I}_{(0,1)}(x)$ and $F(y) = y\mathbb{I}_{(0,1)}(y)$ implying the marginal pdf's $f(x) = \mathbb{I}_{(0,1)}(x)$ and $f(y) = \mathbb{I}_{(0,1)}(y)$. Since $f(x,y) \neq f(x) \cdot f(y)$ theorem 2.9 indicates that X and Y are not stochastically independent.

(f)

$$\begin{aligned}
\mathrm{P}(X > Y) &= \int_{x=y}^{1}\int_{y=0}^{1} f(x,y)dxdy = \int_{y=0}^{1}\int_{x=y}^{1}(1 + (2x-1)(2y-1))dxdy \\
&= \int_{y=0}^{1}[(1-y) - (2y-1)(y^2-y))]dy = \int_0^1 (1-y) - (2y^3 - 3y^2 + y)dy \\
&= \int_0^1 1 - 2y^3 + 3y^2 - 2y\; dy = \left[y - \frac{y^4}{2} + y^3 - y^2\right]_0^1 = \frac{1}{2}.
\end{aligned}$$

4. (a) Using the Lindeberg-Levy-CLT (thm. 5.13):

$$\sqrt{n}\frac{\overline{X}_n - \mu}{\sigma} \xrightarrow{d} \mathcal{N}(0,1) \text{ with } \mu = \frac{\alpha}{\alpha+\beta} \text{ and } \sigma^2 = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$$

yields

$$\overline{X}_n \overset{a}{\sim} \mathcal{N}\left(\frac{\alpha}{\alpha+\beta}, \frac{\alpha\beta}{n(\alpha+\beta)^2(\alpha+\beta+1)}\right)$$

(b) Use the Delta-Method (thm. 5.18) by defining $Z_n = g(\overline{X}_n) = \exp(-\overline{X}_n^2)$ with $G = g'(\overline{X}_n)\big|_{\overline{X}_n=\mu} = -2\overline{X}_n\exp(-\overline{X}_n^2)\big|_{\overline{X}_n=\mu} = -2\frac{\alpha}{\alpha+\beta}\exp\left(-\frac{\alpha^2}{(\alpha+\beta)^2}\right)$. Thus

$$Z_n \overset{a}{\sim} \mathcal{N}\left(\exp\left(-\frac{\alpha^2}{(\alpha+\beta)^2}\right), \frac{4\alpha^3\beta}{n(\alpha+\beta)^4(\alpha+\beta+1)}\exp\left(-\frac{2\alpha^2}{(\alpha+\beta)^2}\right)\right)$$

(c) Applying Corollary 5.1 we have

$$\mathrm{E}(Z_n) \overset{n\to\infty}{\to} \exp\left(-\frac{\alpha^2}{(\alpha+\beta)^2}\right) \text{ and } \mathrm{Var}(Z_n) \overset{n\to\infty}{\to} 0, \text{ which implies}$$

$Z_n \xrightarrow{m} \exp\left(-\frac{\alpha^2}{(\alpha+\beta)^2}\right)$ which i turn implies by thm. 5.9 that $Z_n \xrightarrow{p} \exp\left(-\frac{\alpha^2}{(\alpha+\beta)^2}\right) \overset{\alpha=\beta}{=} \exp\left(-\frac{1}{4}\right)$.

(d) Check the requirements of the change of variables technique (thm. 2.12) and apply said theorem for $y = g(x) = 1 - x$:

- $g'(x) = -1 \neq 0\ \forall\, x$ ✓
- $g^{-1}(y) = 1 - y$ exists $\forall\, y \in (0,1)$ ✓, therefore

$$h(y) = f(g^{-1}(y))\left|\frac{dg^{-1}(y)}{dy}\right| = \frac{1}{B(\alpha,\beta)}(1-y)^{\alpha-1}y^{\beta-1}\mathbb{I}_{(0,1)}(y),$$

which is the distribution of a $Beta(\beta,\alpha)$ because $B(\alpha,\beta) = B(\beta,\alpha)$.

(e)

$$\begin{aligned}
M_Z(t) &= \mathrm{E}(e^{tZ}) = \mathrm{E}(e^{t(X_1+X_2+X_3)}) = \mathrm{E}(e^{tX_1}e^{tX_2}e^{tX_3}) \overset{\text{ind.}}{=} \mathrm{E}(e^{tX_1})\mathrm{E}(e^{tX_2})\mathrm{E}(e^{tX_3}) \\
&= M_{X_1}(t)M_{X_2}(t)M_{X_3}(t) = \left(\sum_{r=0}^{\infty}\frac{B(r+\alpha,\beta)}{B(\alpha,\beta)}\frac{t^r}{r!}\right)^3
\end{aligned}$$