Sheet 05 : Group 27

Comput Finance

I can hardly read this. Please use better scan quality

T- Exercise 18 : W_1, W_2 are standard Brownian motions. S_0, S_1, S_2 are assets "price"

Construct an arbitrage in this market.

We know that there is an arbitrage opportunity in the market if (1) $V(0) = 0$ and $V(t) \geq 0$ and if there exists $T>0$, such that ...

We define 3 strategies φ_0, φ_1 and φ_2 s.t

$$V(t) = \varphi_0 S_0(t) + \varphi_1 S_1(t) + \varphi_2 S_2(t)$$

Using Itô lemma,

(1)
$$dV(t) = S_0(t)\, d(\varphi_0) + d\varphi_1 \cdot S_1(t) + dS_1(t)\cdot \varphi_1 + dS_2(t)\,\varphi_2 + d\varphi_2\, S_2(t) + dS_0(t)\, d(\varphi_0) + d\varphi_1\, dS_1(t) + dS_2(t)\, d\varphi_2$$

$$= dS_1(t)\,\varphi_1 + \varphi_2\, dS_2(t) \,; \quad \left(\frac{d\varphi}{dt} = 0\right)$$

$$= \big(S_1(t)\,(3\,dt + dW_1(t) - dW_2(t))\big)\varphi_1(t) + \big(S_2(t)\,(1\cdot dt - dW_1(t) + dW_2(t))\big)\varphi_2(t)$$

$$= 3\,dt\,\varphi_1(t) S_1(t) + dW_1(t)\,\varphi_1(t) S_1(t) - dW_2(t)\,\varphi_1(t) S_1(t) + S_2(t)\,\varphi_2(t)\,dt - S_2(t)\, dW_1(t)\,\varphi_2(t) + S_2(t)\, dW_2(t)\,\varphi_2(t)$$

(2) Rearranging in terms of W_1 and W_2, we know that if the coefficient of the Wiener processes are zero, $V(t)$ will have no randomness and will be free of risk

$$\Rightarrow dV(t) = 3\,dt\,\varphi_1(t) S_1(t) + (\varphi_1(t) S_1(t) - \varphi_2(t) S_2(t))\, dW_1(t) + S_2(t)\,\varphi_2(t)\,dt + (S_2(t)\,\varphi_2(t) - S_1(t)\,\varphi_1(t))\, dW_2(t)$$

by setting
$$\begin{cases} \varphi_1(t) S_1(t) - \varphi_2(t) S_2(t) = 0 & (a) \\ \varphi_2(t) S_2(t) - \varphi_1(t) S_1(t) = 0 & (b) \end{cases}$$

(3) We can find φ_1, φ_2 that allow riskless gain at

$$\varphi_2(t) S_2(t) = \varphi_1(t) S_1(t) \Rightarrow \varphi_2(t) = \frac{\varphi_1(t) S_1(t)}{S_2(t)} \quad (c)$$

(4) in (a) $\Rightarrow \varphi_1(t) = S_2(t)$ and $\varphi_2(t) = S_1(t)$

Therefore,

$$dV(t) = 3dt + S_2(t)S_1(t) + S_1(t)S_1(t)dt \geq 0.$$

$$= 4dt + S_2(t)S_1(t) \geq 0$$

this is not necessarily true, since S_1 and S_2 can be negative

given that $V(0) = 0$ and $V(t) > 0$, we make riskless gain.

We could short sale $\varphi_0 = S_0(t)$

invest the return into φ_1 and φ_2

At time t we buy φ_0 and sell φ_1 and φ_2 back.

The gain made by keeping φ_1 and φ_2 is our riskless gain

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