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Advanced Statistics I (Winter Term 2023/24)

Solutions to Problemset 1

1. Review the *binomial theorem* as well as the *Taylor series* calculus.

   (i) **Binomial theorem**:
   $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$
   Note that: $\binom{n}{k} = \frac{n!}{(n-k)!k!}$ and $n! = \prod_{i=1}^{n} i$
   A (well known) example: $(a-b)^2 = (a+(-b))^2 = \sum_{k=0}^{2} \binom{2}{k} a^{2-k}(-b)^k = a^2 - 2ab + b^2$

   (ii) **Taylor series**:
   $T_f^\infty(x) = \sum_{j=0}^{\infty} \frac{f^{(j)}(x_0)}{j!}(x-x_0)^j$
   Example: Taylor series of e^x at $x_0 = 0$:
   $e^x = T_f^\infty(x) = \sum_{j=0}^{\infty} \frac{f^{(j)}(0)}{j!} x^j$
   We need $f^{(j)}(x_0)$:

$$\left.\begin{aligned} f^{(0)}(x) &= e^x \\ f^{(1)}(x) &= e^x \\ \vdots &= \vdots \\ f^{(j)}(x) &= e^x \end{aligned}\right\} f^{(j)}(0) = 1$$

   $\Rightarrow e^x = T_f^\infty(x) = \sum_{j=0}^{\infty} \frac{f^{(j)}(0)}{j!} x^j = \sum_{j=0}^{\infty} \frac{x^j}{j!}$

2. Illustrate DeMorgan's laws with the help of Venn's diagrams.
   see here

3. Find the limits of following expressions using l'Hôspital's rule, if those exist:

   l'Hôspital's rule: $\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$ if $f(a) = g(a)$

   (a) $\lim_{x \to \infty} \frac{e^x}{x^2} = \lim_{x \to \infty} \frac{e^x}{2x} = \lim_{x \to \infty} \frac{e^x}{2} \to \infty$

   (b) $\lim_{x \to -\infty} xe^x = \lim_{x \to -\infty} \frac{x}{e^{-x}} = \lim_{x \to -\infty} \frac{1}{-e^{-x}} = 0$

   (c) $\lim_{x \to \infty} \frac{3^x}{x^2 + x - 1} = \lim_{x \to \infty} \frac{\ln(3)3^x}{2x + 1} = \lim_{x \to \infty} \frac{\ln^2(3)3^x}{2} \to \infty$

   (d) $\lim_{x \to \infty} \frac{5x^2 + 4}{8x^2 - 3} = \lim_{x \to \infty} \frac{10x}{16x} = \frac{5}{8}$

4. Calculate the values of the following integrals:

   (a) $\int_0^\infty \lambda e^{-\lambda x} dx = -\frac{\lambda}{\lambda} e^{-\lambda x} \Big|_0^\infty = 1$

   (b) $\int_{\frac{1}{2}}^2 3x^2 + 5e^x - \frac{1}{x} dx = \left[x^3 + 5e^x - \ln(x)\right]_{\frac{1}{2}}^2 = 35.19$

   (c)

$$\int_0^x \sum_{i=2}^n (i-1)s^{i-2} ds = \sum_{i=2}^n (i-1) \int_0^x s^{i-2} ds = \sum_{i=2}^n s^{i-1} \Big|_0^x = \sum_{i=2}^n x^{i-1} = \sum_{k=1}^{n-1} x^k$$

$$= \sum_{k=0}^{n-1} x^k - 1 = \frac{1 - x^n}{1 - x} - \frac{1 - x}{1 - x} = x \frac{1 - x^{n-1}}{1 - x}$$

5. Review the *integration by parts* theorem and use it to find the following integral:

   Formula: $\int_a^b f(x)g'(x)dx = [f(x)g(x)]_a^b - \int_a^b f'(x)g(x)dx$

$$\int e^{x-1}x^3 dx = e^{x-1}x^3 - \int e^{x-1}3x^2 dx = e^{x-1}x^3 - e^{x-1}3x^2 + \int e^{x-1}6x dx$$

$$= e^{x-1}x^3 - e^{x-1}3x^2 + e^{x-1}6x - 6\int e^{x-1} dx$$

$$= x^3 e^{x-1} - 3x^2 e^{x-1} + 6xe^{x-1} - 6e^{x-1} + \mathcal{C}$$

6. If pokémons wanted to form a string quartet, how many combinations would be possible to do this? There are currently 808 listed pokémons. How many possible quartets can there be if the 17 armless pokémons and 34 pokémons that only consist of a head were not allowed to audition?

   i) In case that all n=808 pokémons are allowed to audition, k=4 pokémons are choosen to form the string quartet. The number of combinations depends on whether you allow drawing with replacement (the string quartet might consist of pokémons of the same type, e.g. three pidgeys, or without replacement (each type of pokémon can show up not more than once) and you take permutations into account (the first violin might have a different standing than the second...) or not:

	without replacement	with replacement
with permutation	$\frac{n!}{(n-k)!} = 423.1 \cdot 10^9$	$n^k = 426.2 \cdot 10^9$
without permutation	$\binom{n}{k} = 17.6 \cdot 10^9$	$\binom{n+k-1}{k} = 17.9 \cdot 10^9$

ii) If only n=757 pokémons are allowed to audition the number of combinations is

	without replacement	with replacement
with permutation	$\frac{n!}{(n-k)!} = 325.8 \cdot 10^9$	$n^k = 328.4 \cdot 10^9$
without permutation	$\binom{n}{k} = 13.6 \cdot 10^9$	$\binom{n+k-1}{k} = 13.8 \cdot 10^9$

7. Find all first partial derivatives of following functions:

(a)

$$
\begin{aligned}
f(x,y) &= \sqrt{2x^3} + 3\sqrt{xy^2} \\
f'_x(x,y) &= \frac{3x^2}{\sqrt{2x^3}} + \frac{3y^2}{2\sqrt{xy^2}} \\
f'_y(x,y) &= \frac{3xy}{\sqrt{xy^2}}
\end{aligned}
$$

(b)

$$
\begin{aligned}
f(x,y) &= \ln\left(\frac{y+1}{\sqrt{x}}\right) \\
f'_x(x,y) &= -\frac{1}{2x} \\
f'_y(x,y) &= \frac{1}{y+1}
\end{aligned}
$$

(c)

$$
\begin{aligned}
f(x,y) &= a^x(xy - y^2) \\
f'_x(x,y) &= a^x \ln(a)(xy - y^2) + a^x y \\
f'_y(x,y) &= a^x(x - 2y)
\end{aligned}
$$

(d)

$$
\begin{aligned}
f(x,y) &= \frac{2}{(1+x+y)^3} \\
f'_x(x,y) &= -\frac{6}{(1+x+y)^4} \\
f'_y(x,y) &= -\frac{6}{(1+x+y)^4}
\end{aligned}
$$