Problem Set 1

1. (a) 10 observations (upper left corner of $X'X$)

$$X = \underbrace{\left.\begin{pmatrix} 1 & x_{11} & x_{12} \\ 1 & x_{21} & x_{22} \\ \vdots & \vdots & \vdots \\ 1 & x_{N1} & x_{N2} \end{pmatrix}\right\} N}_{K}$$

Building $X'X$, in the upper left corner we have $\sum_{i=1}^{N} 1^2 = 10 \Rightarrow N = 10$

(b)

$$\hat{\beta} = (X'X)^{-1}X'Y = \begin{pmatrix} 0.38 & -0.12 & -0.08 \\ -0.12 & 0.08 & 0.02 \\ -0.08 & 0.02 & 0.03 \end{pmatrix} \begin{pmatrix} 20 \\ 36 \\ 16 \end{pmatrix} = \begin{pmatrix} 2 \\ 0.8 \\ -0.4 \end{pmatrix}$$

Since y_i and x_{i2} are given in logs:
if x_{i2} increases by 1%, y_i on average decreases by 0.4%. This is the elasticity of car sales with respect to gas prices.

(c) The predicted change in car sales is

$$\hat{\Delta} y = \hat{\beta}_1 \Delta x_1 + \hat{\beta}_2 \Delta x_2 = 0.8 \cdot 0 - 0.4 \cdot 7.5 = -3$$

Thus a gas price increase by 7.5% will lead to a decrease in car sales by 3% on average according to our model.

(d)

$$\begin{aligned} \hat{\sigma}^2 &= \frac{1}{N-K} \sum_{i=1}^{N} \hat{u}_i^2 = \frac{1}{N-K} \hat{U}'\hat{U} \\ \hat{U} &= Y - X\hat{\beta} \end{aligned}$$

$$
\begin{aligned}
\Rightarrow \hat{\sigma}^2 &= \frac{1}{N-K}(Y - X\hat{\beta})'(Y - X\hat{\beta}) \\
&= \frac{1}{N-K}(Y'Y - \underbrace{Y'X\hat{\beta}}_{\text{scalar}} - \underbrace{\hat{\beta}'X'Y}_{\text{scalar}} + \hat{\beta}'X'X\hat{\beta}) \\
&= \frac{1}{N-K}(Y'Y - 2y'X\hat{\beta} + \hat{\beta}'X'X\hat{\beta}) \\
&= \frac{1}{N-K}(Y'Y - 2Y'X\hat{\beta} + \underbrace{Y'X(X'X)^{-1}}_{\hat{\beta}'}X'X\hat{\beta}) \\
&= \frac{1}{N-K}(Y'Y - 2Y'X\hat{\beta} + Y'X\hat{\beta}) \\
&= \frac{1}{N-K}(Y'Y - Y'X\hat{\beta}) \\
&= \frac{1}{10-3}\left(63.52 - \begin{pmatrix} 20 & 36 & 16 \end{pmatrix}\begin{pmatrix} 2 \\ 0.8 \\ -0.4 \end{pmatrix}\right) \\
&= 0.16
\end{aligned}
$$

(e)

$$
\begin{aligned}
Var(\hat{\beta}) &= \sigma^2(X'X)^{-1} \\
\widehat{Var}(\hat{\beta}) &= \hat{\sigma}^2(X'X)^{-1} = 0.16\begin{pmatrix} 0.38 & -0.12 & -0.08 \\ -0.12 & 0.08 & 0.02 \\ -0.08 & 0.02 & 0.03 \end{pmatrix} \\
&= \begin{pmatrix} 0.0608 & -0.0192 & -0.0128 \\ -0.0192 & 0.0128 & 0.0032 \\ -0.0128 & 0.0032 & 0.0048 \end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow \hat{\sigma}_{\beta_0} &= \sqrt{0.0608} \approx 0.2466 \\
\hat{\sigma}_{\beta_1} &= \sqrt{0.0128} \approx 0.1131 \\
\hat{\sigma}_{\beta_2} &= \sqrt{0.0048} \approx 0.0693
\end{aligned}
$$

(f)

$$
R^2 = 1 - \frac{\hat{U}'\hat{U}}{Y'Y - N\bar{y}^2} \qquad \text{(see lecture 1, slide 30)}
$$

From Exercise 3:

$$
\hat{\sigma}^2 = \frac{1}{N-K}\hat{U}'\hat{U}
$$

$$\begin{aligned}
\Rightarrow \hat{U}'\hat{U} &= (N-K)\cdot\hat{\sigma}^2 = 7\cdot 0.16 = 1.12\\
\bar{y}^2 &= \left(\frac{1}{N}\sum_{i=1}^{N} y_i\right)^2 = \frac{1}{N^2}\left(\sum_{i=1}^{N} y_i\right)^2
\end{aligned}$$

From $X'y$:

$$\begin{aligned}
X'Y &= \begin{pmatrix} 1 & 1 & \dots & 1\\ x_{11} & x_{21} & \dots & x_{N1}\\ x_{12} & x_{22} & \dots & x_{N2}\end{pmatrix}\cdot Y = \begin{pmatrix}\sum y_i\\ \sum x_{i1}y_i\\ \sum x_{i2}y_i\end{pmatrix}\\
&\Rightarrow \sum_{i=1}^{N} y_i = 20\\
&\Rightarrow \bar{y}^2 = \frac{1}{10^2}\cdot 20^2 = 4\\
&\Rightarrow R^2 = 1 - \frac{1.12}{63.52 - 10\cdot 4} \approx 0.9524
\end{aligned}$$

(g) Hypothesis:

$$H_0 : \beta_1 = 0.6 \text{ vs. } H_1 : \beta_1 \neq 0.6 \qquad \text{with } \alpha = 0.1$$

Test statistic:

$$t = \frac{\hat{\beta}_1 - \beta_1}{\hat{\sigma}_{\hat{\beta}_1}} = \frac{0.8 - 0.6}{0.1131} \approx 1.77$$

Critical value:

$$\begin{aligned}
t &\sim t(N-K)\\
t^* &= t_{(1-\frac{\alpha}{2})}(N-K) = t_{0.95}(7) = 1.895
\end{aligned}$$

Test decision:

$$\text{Since } |t| < t^* \text{ do not reject } H_0! \tag{1}$$

(h) The first test examines whether gas prices have any impact on car sales.
The test $H_0 : \beta_2 = 0$ vs. $H_1 : \beta_2 < 0$ (one-sided) would be economically more meaningful since it answers the question if the gas prices have a negative effect or not.

(i) Notation for joint hypothesis:

Hypothesis:

$$H_0 : R\beta = r \text{ vs. } H_1 : R\beta \neq r \text{ with } \alpha = 0.05, \beta_1 = 0.6, \beta_2 = -0.1$$

and R as a selection matrix.
Here:

$$R = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } r = \begin{pmatrix} 0.6 \\ -0.1 \end{pmatrix}$$

Test statistic:

$$\lambda = \frac{(R\hat{\beta} - r)'(R(X'X)^{-1}R')^{-1}(R\hat{\beta} - r)/q}{\hat{\sigma}^2} = 16.875$$

Critical value:

$$\begin{aligned} \lambda &\sim F_{1-\alpha}(q, N-K) \text{ with } q = \text{number of restrictions} = 2 \\ \Rightarrow \lambda^* &= F_{0.95}(2,7) = 4.74 \end{aligned}$$

Test decision:

$$\lambda > \lambda^* \text{ reject } H_0!$$

2. (a) 50 observations (since we have an intercept in the model, the number of observations is equal to the first entry of the X'X matrix).

$$\hat{\beta} = (X'X)^{-1}X'y = \begin{pmatrix} 50 & 1960 \\ 1960 & 83800 \end{pmatrix}^{-1} \begin{pmatrix} 1680 \\ 73250 \end{pmatrix} = \begin{pmatrix} -7.50 \\ 1.05 \end{pmatrix}$$

Interpretation of $\hat{\beta}_0$: Do not interpret intercepts!

(b)

$$\begin{aligned} \hat{\sigma}^2 &= \frac{1}{N-K}(y'y - \hat{\beta}'X'y) \\ &= \frac{1}{50-2}\left(67000 - \begin{pmatrix} -7.50 & 1.05 \end{pmatrix} \begin{pmatrix} 1680 \\ 73250 \end{pmatrix}\right) \\ &= \frac{1}{48}(67000 - 64312.5) = 55.99 \end{aligned}$$

$$Var(\hat{\beta}) = \hat{\sigma}^2(X'X)^{-1} = 55.99 \begin{pmatrix} 50 & 1960 \\ 1960 & 83800 \end{pmatrix}^{-1} = \begin{pmatrix} 13.56 & -0.32 \\ -0.32 & 0.0081 \end{pmatrix}$$

(c) Hypothesis:

$$H_0: \ \beta_1 \leq 1 \text{ vs. } H_1: \ \beta_1 > 1.$$

Test statistic:

$$t = \frac{\hat{\beta}_1 - 1}{SE(\hat{\beta}_1)} = \frac{1.05 - 1}{\sqrt{0.0081}} = 0.56$$

Critical value:

$$CV = z_{1-\alpha} = 1.28$$

Test decision:

$$\text{Since } t < CV \ \Rightarrow \text{do not reject } H_0.$$

(d) Ability is an OV here since it influences both having an acacemic degree or not and tax revenue through income. Measurement problems occur since an academic degree may be defined differently across countries (more on those topics follows during this course)