Mathematical Finance: MF

Exercises (for discussion on Monday, 08.01.2023)

Exercise 1. (*8 points*)

(a) Let $f : [0,\infty) \to \mathbb{R}$ be a convex (continuous) function that is bounded from below. Show that there is a non-decreasing convex function $f_1 : [0,\infty) \to \mathbb{R}$ with $f_1(0) = 0$ and a non-increasing convex function $f_2 : [0,\infty) \to \mathbb{R}$ with $\lim_{x\to\infty} f_2(x) = 0$ and an $a \in \mathbb{R}$ such that for all $x \geq 0$ we have $f(x) = f_1(x) + f_2(x) + a$.

(b) Let $f : [0,\infty) \to \mathbb{R}$ be a convex (continuous) function with right-hand derivative $f' : [0,\infty) \to \mathbb{R}$, i.e. $f'(x) := \lim_{y\searrow x} \frac{f(x)-f(y)}{x-y}$ for all $x \in [0,\infty)$. Show

(i) If f is non-decreasing with $f(0) = 0$. Then

$$f(x) = \int_{[0,\infty)} (x-z)^+ df'(z) \quad \forall x \geq 0.$$

(ii) If f is non-increasing with $\lim_{z\to\infty} f(z) = 0$. Then

$$f(x) = \int_{[0,\infty)} (z-x)^+ df'(z) \quad \forall x \geq 0.$$

Remark: The measure df' is defined by $df'(\{0\}) = f'(0)$, $df'((a,b]) = f'(b) - f'(a)$ for all $a, b \geq 0$ with $a < b$. If you are not familiar with Stieltjes-integration, you may additionally assume that f is twice continuously differentiable, in that case one can use the density $\frac{df'(z)}{dz} = f''(z)$ in $(0,\infty)$.

(c) Let $S = (S^0, ..., S^d)$ be a market with end time N such that S^0 is deterministic and $S^0, ..., S^d > 0$. We assume that for each $z \in \mathbb{R}$ there are strategies $\varphi_{c,z}$, $\varphi_{p,z}$ for hedging the call option with payoff $(S^1_N - z)^+$ and the put option with payoff $(z - S^1_N)^+$. Let $f : [0,\infty) \to \mathbb{R}$ be convex (continuous) and bounded from below such that $\int_{(0,\infty)} |\varphi_{c,z}(\omega)| + |\varphi_{p,z}(\omega)| df'(z) < \infty$ for P-almost all $\omega \in \Omega$. Show that there is a hedge for $f(S^1_N)$.

Exercise 2. Suppose $|\Omega| < \infty$, consider a one period arbitrage-free market (S^0, S^1) with $S^1 > 0$. Suppose that for all $K \geq 0$ European call options $(S^1_N - K)^+$ are attainable. Show that assets with payoff $f(S^1_N)$ are attainable for measurable functions $f : \mathbb{R} \to (0,\infty)$.

Exercise 3. Let $X = (X_n)_{n\in\{0,\dots,N\}}$ be a stochastic process with $X_0 = 0$. Assume that $X_1, \dots, X_N$ are independent and uniformly distributed on $[0, 1]$. Let $(\mathcal{F}_n)_{n\in\{0,\dots,N\}}$ be the filtration generated by X and let $\mathcal{T}$ denote the set of $\{0, ..., N\}$-valued stopping times associated to the filtration. Find a $\tau \in \mathcal{T}$ such that $E(X_\tau) = \sup_{\tau\in\mathcal{T}} E(X_\tau)$ (in the sense that you give the most explicit characterization of τ can find) and for $n \in \{N-3, N-2, N-1\}$ calculate the threshold values of X_n such that τ calls for stopping.

Submission of the homework until: Thursday, 21.12.2023, 10.00 a.m. via OLAT.