

Alexander Georges Gretener, M.Sc.  
Mariia Okuneva, M.Sc.  
Anna Titova, Dr.

## Advanced Statistics (Winter Term 2023/24)

### Recap tutorial

1. Your friend arrives at a bus stop at 5 pm. Unfortunately he didn't check the bus schedule, but he knows that the bus will arrive at some time (measured in minutes) uniformly distributed between 5 and 5:30 pm.
  - (a) Find the cumulative distribution function of this uniform distribution.
  - (b) What is the probability that your friend will have to wait longer than ten minutes?
  - (c) If at 5:15 pm the bus has not yet arrived, what is the probability that your friend will have to wait at least an additional 5 minutes?
  - (d) What is the average waiting time until the next bus arrives?
2. A fair die is rolled repeatedly until a 6 shows up. Let  $S_k$  be the event that a 6 will show up for the first time at the k-th throw.

- (a) Derive  $P(S_k)$ .
- (b) Show that the event S "a 6 will show up eventually" will occur with certainty.

3. The random variables X and Y have the following joint probability density function:

$$f(x,y) = \alpha[1 + (2x - 1)(2y - 1)]\mathbb{I}_{(0,1)}(x)\mathbb{I}_{(0,1)}(y).$$

- (a) What can you say about  $\alpha$ ?
  - (b) Is the density  $f(x,y)$  a member of the exponential class?
  - (c) Find the joint cumulative distribution function  $F(x,y)$ .
  - (d) Calculate  $P(X \geq \frac{1}{2}, Y \leq 1)$ .
  - (e) Are X and Y stochastically independent?
  - (f) Calculate  $P(X > Y)$ .
4. Let  $X_1, \dots, X_n$  be independent and identically Beta distributed, i.e. the pdf for each X is given by

$$f(x) = \frac{1}{B(\alpha, \beta)} x^{\alpha-1} (1-x)^{\beta-1} \mathbb{I}_{(0,1)}(x), \quad \alpha > 0, \beta > 0.$$

- (a) Derive the asymptotic distribution of  $\bar{X}_n = \frac{1}{n} \sum_{i=1}^n X_i$ .
- (b) Define  $Z_n = \exp(-\bar{X}_n^2)$ . Find the asymptotic distribution of  $Z_n$ .
- (c) Discuss convergence in probability of  $Z_n$  and derive - if it exists - the probability limit for the special case  $\alpha = \beta$ .
- (d) Find the distribution of  $Y = 1 - X$ .
- (e) Define  $Z = X_1 + X_2 + X_3$ . What is the MGF of Z?