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Advanced Statistics (Winter Term 2023/24)

Problem Set 7

1. The following moments are given: $E\left(X^2\right) = 2$, $E\left(Y^2\right) = 8$, $E(X) = 1$ and $E(Y) = 2$.
   (a) Use the Cauchy-Schwarz-Inequality to give some bounds for $Cov(X,Y)$.
   (b) What can be said about the correlation coefficient given (a)?
   (c) Use the results of part (b) to bound $Cov(X,Y)$.

2. (a) Assume that the random variables X and Y are independent. What can you say about the joint pdf $f(x,y)$, cdf $F(x,y)$ and the (joint) moments?
   (b) Assume now $Cov(X,Y) = 0$. Are the random variables X and Y correlated? Are they independent?
   (c) Discuss the relationship between independence, $E(X|Y) = E(X)$ and $Cov(X,Y) = 0$.

3. Consider the following probability density functions:
   (a) i. $f(x) = \frac{n!}{(n-x)!x!} p^x (1-p)^{n-x} \mathcal{I}_{\{0,1,\ldots,n\}}(x), \quad p \in [0,1],\ n \in \mathbb{N}$
   ii. $f(x) = \frac{e^{-\lambda}\lambda^x}{x!},\ x \in \mathbb{N}, \quad \lambda > 0$
   iii. $f(x) = \frac{1}{\beta^\alpha \Gamma(\alpha)} x^{\alpha-1} e^{-x/\beta} \mathcal{I}_{(0,\infty)}(x), \quad \alpha,\ \beta > 0$
   iv. $f(x) = \frac{1}{b-a} \mathcal{I}_{[a,b]}(x), \quad a < b$
   v. $f(x) = \frac{1}{B(\alpha,\beta)} x^{\alpha-1} (1-x)^{\beta-1} \mathbb{I}_{(0,1)}(x), \quad \alpha,\ \beta > 0$
   Find the associated moment-generating functions, provided they exist.
   (b) Find the two first non-central moments using the moment-generating functions, provided they exist.
   (c) Find the second central moments, provided they exist.

4. Consider the random variable $X = (X_1,\ldots,X_n)$ with the single elements being each independent random variables with the following moment-generating function:

$$M_{X_i}(t) = (1 - \beta t)^{-\alpha}$$

   Find the moment-generating functions of the following random variables:
   (a) $Z_1 = \frac{1}{n} \sum_{i=1}^{n} X_i$
   (b) $Z_2 = X$