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Advanced Statistics (Winter Term 2023/24)

Problem Set 6

1. Let the random variable X have the probability density function (Bernoulli distribution)
$$f(x) = \theta^x (1-\theta)^{(1-x)} I_{\{0,1\}}(x) \qquad \text{with } \theta \in (0,1).$$
Find the non-central moments μ_r' and the central moments μ_2 and μ_3 of X.

2. Let X be a continuous random variable with expected value μ, median m and variance σ^2.
   (a) Show that the function $\mathrm{E}[(X-b)^2]$ is minimal for $b = \mu$.
   (b) Show that the median m minimizes the function $\mathrm{E}[|X-b|]$.
   (c) Assume in addition that X is symmetric around $x = b$.
      i. Show that $\mu = b$.
      ii. Prove that $\mu_3 = 0$.
      iii. Verify that the function $\mathrm{E}[(X-b)^4]$ is minimal for $b = \mu$.

3. Let X be a positive random variable. Compare $\mathrm{E}[X^\alpha]$ with $(\mathrm{E}[X])^\alpha$ for all values of $\alpha \in \mathbb{R}$.

4. Let the random variable X be distributed according to the pdf $f(x)$ with $x > 0$.
   (a) Suppose you know that $\mathrm{E}[X] = 8$. Give an estimate for the probability $\mathrm{P}(X < 16)$.
   (b) Suppose you also know that X cannot take negative values and that $\mathrm{Var}(X) = 32$. Include this information and re-estimate the probability $\mathrm{P}(X < 16)$
   (c) The miles per gallon attained by purchases of a line of pickup trucks manufactured in Detroit are outcomes of a random variable with mean of 17 miles per gallon and a standard deviation of 0.25 miles per gallon. How probable is the event that a purchaser attains between 16 and 18 miles per gallon with this line of truck.
   (d) The daily price of a certain penny stock is a random variable with an expected value of 2\$. Is the probability that the stock price will be greater than or equal to 10\$ greater than 20%?

5. Let the random variable $X = (X_1, X_2)$ have the following probability density function:
$$f(x) = \begin{cases} \frac{12}{(1+x_1+x_2)^5}, & \text{if } x_i \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$
   (a) Check if the elements in X are stochastically independent.
   (b) Find the covariance matrix of X.
   (c) Find both regression curves.
   (d) Calculate $\mathrm{Var}(X_1|X_2)$.