Appendix: Equations for Minimum-Time Planar Paths with L_2 Velocity and Acceleration Constraints and a Limited Number of Constant Acceleration Inputs

Victor M. Baez[1], Aaron T. Becker[1]

***Abstract*—Optimal control solutions are often formulated using L_∞ constraints, even for systems better modeled with L_2 constraints. Given starting and ending positions and velocities, L_2 bounds on the acceleration and velocity, and the restriction to no more than two, constant control inputs, this paper provides routines to compute the minimal-time path. Closed form solutions are provided for reaching a position in minimum time with and without a velocity bound. Code is provided on GitHub[1].**

I. Appendix

The equations for the two sextic equations are large, and are placed in this appendix to make the paper easier to read.

If the ending configuration is sufficiently far from the initial configuration, the goal is reachable by a two-phase input which consists of a maximum acceleration input in direction θ_1 for t_1 seconds, followed by a coasting phase for t_2 seconds.

At time t_1 the system reaches velocity v_m under a constant acceleration $a_m[\cos(\theta_1), \sin(\theta_1)]^\top$:

$$\sqrt{(\mathbf{v}_{0x}+a_m\cos(\theta_1)t_1)^2+(\mathbf{v}_{0y}+a_m\sin(\theta_1)t_1)^2} = v_m. \quad (1)$$

This is a quadratic equation with two solutions for t_1, but only the positive value is relevant since we are planning forward in time. We express t_1 as a function of the angle θ_1:

$$t_1(\theta_1) = \frac{\sqrt{v_m^2 - \mathbf{v}_{0x}^2 - \mathbf{v}_{0y}^2 + (\mathbf{v}_{0x}\cos(\theta_1) + \mathbf{v}_{0y}\sin(\theta_1))^2}}{a_m} - \frac{(\mathbf{v}_{0x}\cos(\theta_1) + \mathbf{v}_{0y}\sin(\theta_1))}{a_m}. \quad (2)$$

The position of the particle at time t_1 is

$$\begin{aligned} \mathbf{p}_x(t_1) &= \mathbf{p}_{0x} + \mathbf{v}_{0x}t_1 + \frac{a_m}{2}\cos(\theta_1)t_1^2 \\ \mathbf{p}_y(t_1) &= \mathbf{p}_{0y} + \mathbf{v}_{0y}t_1 + \frac{a_m}{2}\sin(\theta_1)t_1^2, \end{aligned} \quad (3)$$

A. The system reaches terminal velocity

If the ending configuration is sufficiently far from the initial configuration, the goal is reachable by a two-phase input which consists of a maximum acceleration input in direction θ_1 for t_1 seconds, followed by a coasting phase for t_2 seconds.

At time t_1 the system reaches velocity v_m under a constant acceleration $a_m[\cos(\theta_1), \sin(\theta_1)]^\top$:

$$\sqrt{(\mathbf{v}_{0x}+a_m\cos(\theta_1)t_1)^2+(\mathbf{v}_{0y}+a_m\sin(\theta_1)t_1)^2} = v_m. \quad (4)$$

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[1] Department of Electrical Engineering, University of Houston, USA
atbecker@central.uh.edu

[1]https://github.com/RoboticSwarmControl/2023minTimeL2pathsConstraints/

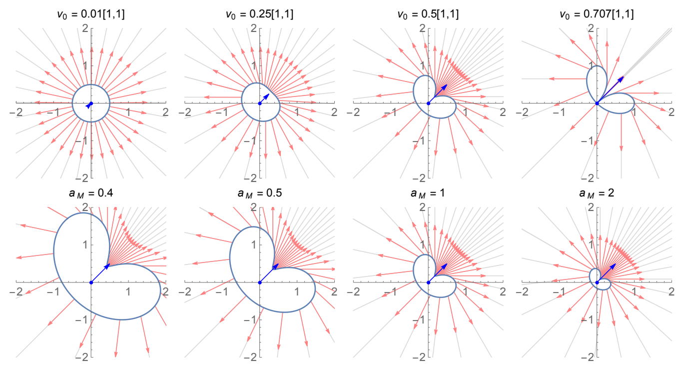

Fig. 1. Locus of positions where the particle reaches $v_m = 1$ (). Top row shows four different starting velocities (blue). The velocity of the particle due to thrust $a_m = 1$ in directions $\theta \in k\pi/16$, $k \in [0, 31]$ is shown with pink arrows, all of length v_m. These arrows point in every direction. The bottom row shows $\mathbf{v}_0 = [1/2, 1/2]$ for four values of a_m. An animation is available at https://youtu.be/2J-p6CDF4FE.

This is a quadratic equation with two solutions for t_1, but only the positive value is relevant since we are planning forward in time. We express t_1 as a function of the angle θ_1:

$$t_1(\theta_1) = \frac{\sqrt{v_m^2 - \mathbf{v}_{0x}^2 - \mathbf{v}_{0y}^2 + (\mathbf{v}_{0x}\cos(\theta_1) + \mathbf{v}_{0y}\sin(\theta_1))^2}}{a_m} - \frac{(\mathbf{v}_{0x}\cos(\theta_1) + \mathbf{v}_{0y}\sin(\theta_1))}{a_m}. \quad (5)$$

The position of the particle at time t_1 is

$$\begin{aligned} \mathbf{p}_x(t_1) &= \mathbf{p}_{0x} + \mathbf{v}_{0x}t_1 + \frac{a_m}{2}\cos(\theta_1)t_1^2 \\ \mathbf{p}_y(t_1) &= \mathbf{p}_{0y} + \mathbf{v}_{0y}t_1 + \frac{a_m}{2}\sin(\theta_1)t_1^2, \end{aligned} \quad (6)$$

and the velocity of the particle at time t_1 is

$$\begin{aligned} \mathbf{v}_x(t_1) &= \mathbf{v}_{0x} + a_m\cos(\theta_1)t_1 \\ \mathbf{v}_y(t_1) &= \mathbf{v}_{0y} + a_m\sin(\theta_1)t_1. \end{aligned} \quad (7)$$

Figure 1 shows eight variations of the locus of positions at the terminal velocity from (6) in light blue, along with arrows showing the velocities along this set from (7) in pink. We want solutions for θ_1 that result in the velocity pointing toward to the goal at time t_1. We could check directly that

$$\arctan(\mathbf{v}_x(t_1), \mathbf{v}_y(t_1)) \equiv \arctan(-\mathbf{p}_x(t_1), -\mathbf{p}_y(t_1)), \quad (8)$$

but this involves solving for inverse trigonometric functions. Instead, we compare the slope of the velocity to the slope of the position error.

$$\frac{\mathbf{v}_y(t_1)}{\mathbf{v}_x(t_1)} \equiv \frac{-\mathbf{p}_y(t_1)}{-\mathbf{p}_x(t_1)} \quad (9)$$

This results in two candidate solutions, but we can check both using (8) and save the correct solution. We will also have to check for zeros of the equation in the same way. The resulting equation is

$$\mathbf{v}_x(t_1)\mathbf{p}_y(t_1) - \mathbf{v}_y(t_1)\mathbf{p}_x(t_1) \equiv 0. \tag{10}$$

Solving for $\sin(\theta_1)$ results in a sextic equation. This equation is long, so it is placed in the Appendix I-B. We solve for the six roots of (13), and discard the complex roots. Finding the root of a sextic equation can be efficiently computed in many programming languages [1], see footnotes for examples[2,3,4].

Each remaining root is a solution for $\sin(\theta_1)$ and provides two possible θ_1 solutions. We substitute each into (5) to get at most 12 candidate t_1 solutions. The smallest, real, non-negative t_1 that satisfies (8) is used.

B. Passing through goal by reaching terminal velocity

We can substitute in the solution to t_1 from (5) into (10), resulting in the inscrutable equality:

$$\begin{aligned}
0 \equiv & \tfrac{1}{2a_m}\big(\cos(\theta_1)\Big(-2\sin(\theta_1)\Big(a_m(-\mathbf{p}_{0x}\mathbf{v}_{0x}+\mathbf{p}_{Gx}\mathbf{v}_{0x}+\mathbf{p}_{0y}\mathbf{v}_{0y}-\mathbf{p}_{Gy}\mathbf{v}_{0y}) \\
& +\mathbf{v}_{0y}^2\sqrt{v_m^2-\mathbf{v}_{0x}^2+(\mathbf{v}_{0x}\cos(\theta_1)+\mathbf{v}_{0y}\sin(\theta_1))^2-\mathbf{v}_{0y}^2}\Big) \\
& +2a_m(\mathbf{p}_{0y}-\mathbf{p}_{Gy})\sqrt{v_m^2-\mathbf{v}_{0x}^2+(\mathbf{v}_{0x}\cos(\theta_1)+\mathbf{v}_{0y}\sin(\theta_1))^2-\mathbf{v}_{0y}^2} \\
& +\mathbf{v}_{0y}\Big(v_m^2-2\mathbf{v}_{0x}^2\Big)-\mathbf{v}_{0y}\cos(2\theta_1)\Big(\mathbf{v}_{0y}^2-3\mathbf{v}_{0x}^2\Big)\Big) \\
& -\sin(\theta_1)\Big(2a_m(\mathbf{p}_{0x}-\mathbf{p}_{Gx})\sqrt{v_m^2-\mathbf{v}_{0x}^2+(\mathbf{v}_{0x}\cos(\theta_1)+\mathbf{v}_{0y}\sin(\theta_1))^2-\mathbf{v}_{0y}^2} \\
& +v_m^2\mathbf{v}_{0x}+\mathbf{v}_{0x}\cos(2\theta_1)(\mathbf{v}_{0x}-\mathbf{v}_{0y})(\mathbf{v}_{0x}+\mathbf{v}_{0y})\Big) \\
& +2\mathbf{v}_{0y}\cos^2(\theta_1)(a_m(\mathbf{p}_{Gx}-\mathbf{p}_{0x})+2\mathbf{v}_{0x}\mathbf{v}_{0y}\sin(\theta_1))+2a_m\mathbf{v}_{0x}\sin^2(\theta_1)(\mathbf{p}_{0y}-\mathbf{p}_{Gy}) \\
& +\mathbf{v}_{0x}(\mathbf{v}_{0x}\sin(2\theta_1)-2\mathbf{v}_{0y}\cos(2\theta_1))\sqrt{v_m^2-\mathbf{v}_{0x}^2+(\mathbf{v}_{0x}\cos(\theta_1)+\mathbf{v}_{0y}\sin(\theta))^2-\mathbf{v}_{0y}^2}\big).
\end{aligned} \tag{11}$$

This function is still to hard to solve, so we need to make some substitutions. First, we rotate our coordinate frame so that $\mathbf{v}_{0y} = 0$. Next, we apply a change of variables to eliminate the two trigonometric functions: $\sin(\theta_1) = s$, and $\cos(\theta_1) = \pm\sqrt{1-s^2}$. This results in the more tractable equation

$$\begin{aligned}
& s\mathbf{v}_{0x}\Big(v_m^2+\mathbf{v}_{0x}\Big(\mathbf{v}_{0x}-2\sqrt{1-s^2}\sqrt{(v_m-s\mathbf{v}_{0x})(v_m+s\mathbf{v}_{0x})}-2s^2\mathbf{v}_{0x}\Big)\Big) \\
\equiv \quad & 2a_m\Big(\mathbf{p}_{0x}\sqrt{1-s^2}s\mathbf{v}_{0x}-\mathbf{p}_{0x}s\sqrt{(v_m-s\mathbf{v}_{0x})(v_m+s\mathbf{v}_{0x})} \\
& +\mathbf{p}_{0y}\sqrt{1-s^2}\sqrt{(v_m-s\mathbf{v}_{0x})(v_m+s\mathbf{v}_{0x})}+\mathbf{p}_{0y}s^2\mathbf{v}_{0x}\Big).
\end{aligned} \tag{12}$$

[2]https://numpy.org/doc/stable/reference/generated/numpy.roots.html
[3]https://www.mathworks.com/help/matlab/ref/roots.html
[4]https://www.alglib.net/equations/polynomial.php

Rewriting (12) gives (13), a sextic equation in s.

$$\begin{aligned}
0 = {} & 16a_m^4v_m^4\mathbf{p}_{0y}^4 \\
& +64a_m^3v_m^4\mathbf{p}_{0y}^3\mathbf{v}_{0x}^2 s \\
& -8a_m^2v_m^2\mathbf{p}_{0y}^2(4a_m^2v_m^2\mathbf{p}_{0x}^2+4a_m^2v_m^2\mathbf{p}_{0y}^2+ \\
& v_m^4\mathbf{v}_{0x}^2+4a_m^2\mathbf{p}_{0x}^2\mathbf{v}_{0x}^2+4a_m^2\mathbf{p}_{0y}^2\mathbf{v}_{0x}^2- \\
& 10v_m^2\mathbf{v}_{0x}^4+\mathbf{v}_{0x}^6)s^2 \\
& -16a_mv_m^2\mathbf{p}_{0y}\mathbf{v}_{0x}^2(6a_m^2v_m^2\mathbf{p}_{0y}^2+v_m^4\mathbf{v}_{0x}^2+ \\
& 6a_m^2\mathbf{p}_{0y}^2\mathbf{v}_{0x}^2-2v_m^2\mathbf{v}_{0x}^4+\mathbf{v}_{0x}^6)s^3 \\
& +(16a_m^4v_m^4\mathbf{p}_{0x}^4+32a_m^4v_m^4\mathbf{p}_{0x}^2\mathbf{p}_{0y}^2+ \\
& 16a_m^4v_m^4\mathbf{p}_{0y}^4-8a_m^2v_m^6\mathbf{p}_{0x}^2\mathbf{v}_{0x}^2- \\
& 32a_m^4v_m^2\mathbf{p}_{0x}^4\mathbf{v}_{0x}^2+8a_m^2v_m^6\mathbf{p}_{0y}^2\mathbf{v}_{0x}^2+ \\
& 32a_m^4v_m^2\mathbf{p}_{0y}^4\mathbf{v}_{0x}^2+v_m^8\mathbf{v}_{0x}^4+8a_m^2v_m^4\mathbf{p}_{0x}^2\mathbf{v}_{0x}^4+ \\
& 16a_m^4\mathbf{p}_{0x}^4\mathbf{v}_{0x}^4-72a_m^2v_m^4\mathbf{p}_{0y}^2\mathbf{v}_{0x}^4+ \\
& 32a_m^4\mathbf{p}_{0x}^2\mathbf{p}_{0y}^2\mathbf{v}_{0x}^4+16a_m^4\mathbf{p}_{0y}^4\mathbf{v}_{0x}^4-4v_m^6\mathbf{v}_{0x}^6+ \\
& 8a_m^2v_m^2\mathbf{p}_{0x}^2\mathbf{v}_{0x}^6-72a_m^2v_m^2\mathbf{p}_{0y}^2\mathbf{v}_{0x}^6+ \\
& 6v_m^4\mathbf{v}_{0x}^8-8a_m^2\mathbf{p}_{0x}^2\mathbf{v}_{0x}^8+8a_m^2\mathbf{p}_{0y}^2\mathbf{v}_{0x}^8- \\
& 4v_m^2\mathbf{v}_{0x}^{10}+\mathbf{v}_{0x}^{12})s^4 \\
& +8a_m\mathbf{p}_{0y}\mathbf{v}_{0x}^2(4a_m^2v_m^4\mathbf{p}_{0x}^2+4a_m^2v_m^4\mathbf{p}_{0y}^2+ \\
& v_m^6\mathbf{v}_{0x}^2-8a_m^2v_m^2\mathbf{p}_{0x}^2\mathbf{v}_{0x}^2+8a_m^2v_m^2\mathbf{p}_{0y}^2\mathbf{v}_{0x}^2- \\
& v_m^4\mathbf{v}_{0x}^4+4a_m^2\mathbf{p}_{0x}^2\mathbf{v}_{0x}^4+4a_m^2\mathbf{p}_{0y}^2\mathbf{v}_{0x}^4- \\
& v_m^2\mathbf{v}_{0x}^6+\mathbf{v}_{0x}^8)s^5 \\
& +16a_m^2\mathbf{v}_{0x}^4(v_m^4\mathbf{p}_{0x}^2+v_m^4\mathbf{p}_{0y}^2-2v_m^2\mathbf{p}_{0x}^2\mathbf{v}_{0x}^2+ \\
& 2v_m^2\mathbf{p}_{0y}^2\mathbf{v}_{0x}^2+\mathbf{p}_{0x}^2\mathbf{v}_{0x}^4+\mathbf{p}_{0y}^2\mathbf{v}_{0x}^4)s^6
\end{aligned} \tag{13}$$

C. Stopping at goal no terminal velocity

If the starting and ending position are sufficiently close such that the velocity never exceeds v_m, then the goal is reachable in minimum time by a two-phase input which consists of a maximum acceleration input in direction θ_1 for t_1 seconds, followed by a maximum acceleration input opposing the current velocity to bring the system to rest in $t_3 = \mathbf{v}(t_1)/a_m$ seconds ($t_2 = 0$).

After applying the constant input $a_m[\cos(\theta_1), \sin(\theta_1)]^\top$ for t_1 seconds, the position and velocity are

$$\begin{aligned}
\mathbf{p}(t_1) &= \begin{bmatrix} \mathbf{p}_{0x}+\mathbf{v}_{0x}t_1+ & \frac{a_m}{2}\cos(\theta_1)t_1^2 \\ \mathbf{p}_{0y} \quad + & \frac{a_m}{2}\sin(\theta_1)t_1^2 \end{bmatrix} \\
\mathbf{v}(t_1) &= \begin{bmatrix} \mathbf{v}_{0x}+ & a_m\cos(\theta_1)t_1 \\ & a_m\sin(\theta_1)t_1 \end{bmatrix}
\end{aligned} \tag{14}$$

The deceleration command is in the opposite direction of $\mathbf{v}(t_1)$ so that $\theta_3 = \arctan(-\mathbf{v}_x(t_1), -\mathbf{v}_y(t_1))$, and lasts for $t_3 = \|\mathbf{v}(t_1)\|/a_m$ seconds. At time t_1+t_3, we want the x and y positions to be zero and the final velocity to be zero. The final position is entirely controlled by the initial conditions and the selected θ_1 and θ_3:

$$
\begin{aligned}
t_3 &= \frac{\|\mathbf{v}(t_1)\|}{a_m} = \sqrt{\left(\frac{\mathbf{v}_{0x}}{a_m} + \cos(\theta_1)t_1\right)^2 + (\sin(\theta_1)t_1)^2} \\
0 &= \mathbf{p}_x(t_1) + \mathbf{v}_x(t_1)\frac{t_3}{2} \\
0 &= \mathbf{p}_y(t_1) + \mathbf{v}_y(t_1)\frac{t_3}{2}.
\end{aligned} \tag{15}
$$

We then scale the starting position and velocity by dividing each by a_m and remove the term a_m from the calculation: $\tilde{\mathbf{p}}_0 = \mathbf{p}_0/a_m$, $\tilde{\mathbf{v}}_0 = \mathbf{v}_0/a_m$. We apply a change of variables to eliminate the two trigonometric functions: $\cos(\theta_1) = c$, and $\sin(\theta_1) = \pm\sqrt{1-c^2}$. The resulting position constraints simplify to:

$$
\begin{aligned}
0 &= 2\tilde{\mathbf{p}}_{0x} + 2\tilde{\mathbf{v}}_{0x}t_1 + ct_1^2 + (\tilde{\mathbf{v}}_{0x} + ct_1)\sqrt{\tilde{\mathbf{v}}_{0x}^2 + 2c\tilde{\mathbf{v}}_{0x}t_1 + t_1^2} \\
0 &= 2\tilde{\mathbf{p}}_{0y} + \sqrt{1-c^2}\,t_1\left(\sqrt{\tilde{\mathbf{v}}_{0x}^2 + 2c\tilde{\mathbf{v}}_{0x}t_1 + t_1^2} + t_1\right).
\end{aligned} \tag{16}
$$

This set of equations can be solved for c as a function of t_1. The calculations are long, but they are included here for completeness.

The set of equations (16) can be solved for c as a function of t_1. Like (12), the resulting equation is sextic, but this time in t_1:

$$
\begin{aligned}
0 = &-4(16\tilde{\mathbf{p}}_{0x}^6 + 8\tilde{\mathbf{p}}_{0x}^4(6\tilde{\mathbf{p}}_{0y}^2 - \tilde{\mathbf{v}}_{0x}^4) + (\tilde{\mathbf{p}}_{0y}\tilde{\mathbf{v}}_{0x}^4 - 4\tilde{\mathbf{p}}_{0y}^3)^2 + \tilde{\mathbf{p}}_{0x}^2(48\tilde{\mathbf{p}}_{0y}^4 + 48\tilde{\mathbf{p}}_{0y}^2\tilde{\mathbf{v}}_{0x}^4 + \tilde{\mathbf{v}}_{0x}^8)) \\
&+ t_1(-4\tilde{\mathbf{p}}_{0x}\tilde{\mathbf{v}}_{0x}(80\tilde{\mathbf{p}}_{0x}^4 + 80\tilde{\mathbf{p}}_{0y}^4 + 72\tilde{\mathbf{p}}_{0y}^2\tilde{\mathbf{v}}_{0x}^4 + \tilde{\mathbf{v}}_{0x}^8 8\tilde{\mathbf{p}}_{0x}^2(20\tilde{\mathbf{p}}_{0y}^2 - 3\tilde{\mathbf{v}}_{0x}^4))) \\
&+ t_1^2(-\tilde{\mathbf{v}}_{0x}^2(464\tilde{\mathbf{p}}_{0x}^4 + 208\tilde{\mathbf{p}}_{0y}^4 + 88\tilde{\mathbf{p}}_{0y}^2\tilde{\mathbf{v}}_{0x}^4 + \tilde{\mathbf{v}}_{0x}^8 + 24\tilde{\mathbf{p}}_{0x}^2(28\tilde{\mathbf{p}}_{0y}^2 - 5\tilde{\mathbf{v}}_{0x}^4))) \\
&+ t_1^3(-256\tilde{\mathbf{p}}_{0x}^3\tilde{\mathbf{v}}_{0x}^3 - 128\tilde{\mathbf{p}}_{0x}\tilde{\mathbf{p}}_{0y}^2\tilde{\mathbf{v}}_{0x}^3 + 64\tilde{\mathbf{p}}_{0x}\tilde{\mathbf{v}}_{0x}^7) \\
&+ t_1^4(64\tilde{\mathbf{p}}_{0x}^4 + 128\tilde{\mathbf{p}}_{0x}^2\tilde{\mathbf{p}}_{0y}^2 + 64\tilde{\mathbf{p}}_{0y}^4 - 64\tilde{\mathbf{p}}_{0x}^2\tilde{\mathbf{v}}_{0x}^4 + 32\tilde{\mathbf{p}}_{0y}^2\tilde{\mathbf{v}}_{0x}^4 + 12\tilde{\mathbf{v}}_{0x}^8) \\
&+ t_1^5(64\tilde{\mathbf{p}}_{0x}^3\tilde{\mathbf{v}}_{0x} + 64\tilde{\mathbf{p}}_{0x}\tilde{\mathbf{p}}_{0y}^2\tilde{\mathbf{v}}_{0x} - 16\tilde{\mathbf{p}}_{0x}\tilde{\mathbf{v}}_{0x}^5) \\
&+ t_1^6(16\tilde{\mathbf{p}}_{0x}^2\tilde{\mathbf{v}}_{0x}^2 - 4\tilde{\mathbf{v}}_{0x}^6).
\end{aligned} \tag{17}
$$

The equation for $c(t_1)$ is inscrutable, but is composed of constants. The variable t_1 appears five times:

$$
\begin{aligned}
c(t_1) = {} \\
&(4\tilde{\mathbf{p}}_{0x}\tilde{\mathbf{v}}_{0x}^2(-4096(\tilde{\mathbf{p}}_{0x}^2 + \tilde{\mathbf{p}}_{0y}^2)^5(15\tilde{\mathbf{p}}_{0x}^4 - 49\tilde{\mathbf{p}}_{0x}^2\tilde{\mathbf{p}}_{0y}^2 + 40\tilde{\mathbf{p}}_{0y}^4) + \\
&2048(\tilde{\mathbf{p}}_{0x}^2 + \tilde{\mathbf{p}}_{0y}^2)^2(49\tilde{\mathbf{p}}_{0x}^8 - 41\tilde{\mathbf{p}}_{0x}^6\tilde{\mathbf{p}}_{0y}^2 - 303\tilde{\mathbf{p}}_{0x}^4\tilde{\mathbf{p}}_{0y}^4 + 529\tilde{\mathbf{p}}_{0x}^2\tilde{\mathbf{p}}_{0y}^6 - 186\tilde{\mathbf{p}}_{0y}^8)\tilde{\mathbf{v}}_{0x}^4 - \\
&256(273\tilde{\mathbf{p}}_{0x}^{10} - 28\tilde{\mathbf{p}}_{0x}^8\tilde{\mathbf{p}}_{0y}^2 - 1114\tilde{\mathbf{p}}_{0x}^6\tilde{\mathbf{p}}_{0y}^4 + 804\tilde{\mathbf{p}}_{0x}^4\tilde{\mathbf{p}}_{0y}^6 \\
&-1263\tilde{\mathbf{p}}_{0x}^2\tilde{\mathbf{p}}_{0y}^8 + 1216\tilde{\mathbf{p}}_{0y}^{10})\tilde{\mathbf{v}}_{0x}^8 + \\
&256(105\tilde{\mathbf{p}}_{0x}^8 - 183\tilde{\mathbf{p}}_{0x}^6\tilde{\mathbf{p}}_{0y}^2 + 163\tilde{\mathbf{p}}_{0x}^4\tilde{\mathbf{p}}_{0y}^4 - 73\tilde{\mathbf{p}}_{0x}^2\tilde{\mathbf{p}}_{0y}^6 - 220\tilde{\mathbf{p}}_{0y}^8)\tilde{\mathbf{v}}_{0x}^{12} + \\
&16(-385\tilde{\mathbf{p}}_{0x}^6 + 1278\tilde{\mathbf{p}}_{0x}^4\tilde{\mathbf{p}}_{0y}^2 - 2233\tilde{\mathbf{p}}_{0x}^2\tilde{\mathbf{p}}_{0y}^4 + 1192\tilde{\mathbf{p}}_{0y}^6)\tilde{\mathbf{v}}_{0x}^{16} + \\
&8(105\tilde{\mathbf{p}}_{0x}^4 - 443\tilde{\mathbf{p}}_{0x}^2\tilde{\mathbf{p}}_{0y}^2 + 542\tilde{\mathbf{p}}_{0y}^4)\tilde{\mathbf{v}}_{0x}^{20} + 7(-9\tilde{\mathbf{p}}_{0x}^2 + 32\tilde{\mathbf{p}}_{0y}^2)\tilde{\mathbf{v}}_{0x}^{24} + 2\tilde{\mathbf{v}}_{0x}^{28}) + \\
&t_1(-4096(\tilde{\mathbf{p}}_{0x}^2 + \tilde{\mathbf{p}}_{0y}^2)^4(159\tilde{\mathbf{p}}_{0x}^6 - 344\tilde{\mathbf{p}}_{0x}^4\tilde{\mathbf{p}}_{0y}^2 + 51\tilde{\mathbf{p}}_{0x}^2\tilde{\mathbf{p}}_{0y}^4 + 74\tilde{\mathbf{p}}_{0y}^6)\tilde{\mathbf{v}}_{0x}^3 + \\
&2048(\tilde{\mathbf{p}}_{0x}^2 + \tilde{\mathbf{p}}_{0y}^2)(497\tilde{\mathbf{p}}_{0x}^{10} - 79\tilde{\mathbf{p}}_{0x}^8\tilde{\mathbf{p}}_{0y}^2 - 1890\tilde{\mathbf{p}}_{0x}^6\tilde{\mathbf{p}}_{0y}^4 + 1510\tilde{\mathbf{p}}_{0x}^4\tilde{\mathbf{p}}_{0y}^6 + 673\tilde{\mathbf{p}}_{0x}^2\tilde{\mathbf{p}}_{0y}^8 - \\
&359\tilde{\mathbf{p}}_{0y}^{10})\tilde{\mathbf{v}}_{0x}^7 - \\
&256(2593\tilde{\mathbf{p}}_{0x}^{10} - 360\tilde{\mathbf{p}}_{0x}^8\tilde{\mathbf{p}}_{0y}^2 - 7914\tilde{\mathbf{p}}_{0x}^6\tilde{\mathbf{p}}_{0y}^4 + 6700\tilde{\mathbf{p}}_{0x}^4\tilde{\mathbf{p}}_{0y}^6 \\
&-2223\tilde{\mathbf{p}}_{0x}^2\tilde{\mathbf{p}}_{0y}^8 + 1988\tilde{\mathbf{p}}_{0y}^{10})\tilde{\mathbf{v}}_{0x}^{11} + \\
&256(905\tilde{\mathbf{p}}_{0x}^8 - 862\tilde{\mathbf{p}}_{0x}^6\tilde{\mathbf{p}}_{0y}^2 - 1020\tilde{\mathbf{p}}_{0x}^4\tilde{\mathbf{p}}_{0y}^4 + 1106\tilde{\mathbf{p}}_{0x}^2\tilde{\mathbf{p}}_{0y}^6 - 201\tilde{\mathbf{p}}_{0y}^8)\tilde{\mathbf{v}}_{0x}^{15} + \\
&16(-2865\tilde{\mathbf{p}}_{0x}^6 + 4204\tilde{\mathbf{p}}_{0x}^4\tilde{\mathbf{p}}_{0y}^2 - 1685\tilde{\mathbf{p}}_{0x}^2\tilde{\mathbf{p}}_{0y}^4 + 1886\tilde{\mathbf{p}}_{0y}^6)\tilde{\mathbf{v}}_{0x}^{19} + \\
&8(617\tilde{\mathbf{p}}_{0x}^4 - 966\tilde{\mathbf{p}}_{0x}^2\tilde{\mathbf{p}}_{0y}^2 + 709\tilde{\mathbf{p}}_{0y}^4)\tilde{\mathbf{v}}_{0x}^{23} + (-239\tilde{\mathbf{p}}_{0x}^2 + 272\tilde{\mathbf{p}}_{0y}^2)\tilde{\mathbf{v}}_{0x}^{27} + 2\tilde{\mathbf{v}}_{0x}^{31} + \\
&t_1(8\tilde{\mathbf{p}}_{0x}(-4096(\tilde{\mathbf{p}}_{0x}^2 - 2\tilde{\mathbf{p}}_{0y}^2)(\tilde{\mathbf{p}}_{0x}^2 + \tilde{\mathbf{p}}_{0y}^2)^6 - 1024(\tilde{\mathbf{p}}_{0x}^2 + \tilde{\mathbf{p}}_{0y}^2)^3(44\tilde{\mathbf{p}}_{0x}^6 - 81\tilde{\mathbf{p}}_{0x}^4\tilde{\mathbf{p}}_{0y}^2 - \\
&10\tilde{\mathbf{p}}_{0x}^2\tilde{\mathbf{p}}_{0y}^4 + 19\tilde{\mathbf{p}}_{0y}^6)\tilde{\mathbf{v}}_{0x}^4 + 256(299\tilde{\mathbf{p}}_{0x}^{10} + 45\tilde{\mathbf{p}}_{0x}^8\tilde{\mathbf{p}}_{0y}^2 - 1018\tilde{\mathbf{p}}_{0x}^6\tilde{\mathbf{p}}_{0y}^4 + 434\tilde{\mathbf{p}}_{0x}^4\tilde{\mathbf{p}}_{0y}^6 \\
&+559\tilde{\mathbf{p}}_{0x}^2\tilde{\mathbf{p}}_{0y}^8 - 127\tilde{\mathbf{p}}_{0y}^{10})\tilde{\mathbf{v}}_{0x}^8 - 128(400\tilde{\mathbf{p}}_{0x}^8 - 317\tilde{\mathbf{p}}_{0x}^6\tilde{\mathbf{p}}_{0y}^2 - 715\tilde{\mathbf{p}}_{0x}^4\tilde{\mathbf{p}}_{0y}^4 + \\
&753\tilde{\mathbf{p}}_{0x}^2\tilde{\mathbf{p}}_{0y}^6 - 17\tilde{\mathbf{p}}_{0y}^8)\tilde{\mathbf{v}}_{0x}^{12} + 16(1125\tilde{\mathbf{p}}_{0x}^6 - 1602\tilde{\mathbf{p}}_{0x}^4\tilde{\mathbf{p}}_{0y}^2 - 275\tilde{\mathbf{p}}_{0x}^2\tilde{\mathbf{p}}_{0y}^4 + 852\tilde{\mathbf{p}}_{0y}^6)\tilde{\mathbf{v}}_{0x}^{16} \\
&-4(884\tilde{\mathbf{p}}_{0x}^4 - 1401\tilde{\mathbf{p}}_{0x}^2\tilde{\mathbf{p}}_{0y}^2 + 255\tilde{\mathbf{p}}_{0y}^4)\tilde{\mathbf{v}}_{0x}^{20} + 9(41\tilde{\mathbf{p}}_{0x}^2 - 47\tilde{\mathbf{p}}_{0y}^2)\tilde{\mathbf{v}}_{0x}^{24} - 16\tilde{\mathbf{v}}_{0x}^{28}) + \\
&4t_1\tilde{\mathbf{v}}_{0x}(4096(\tilde{\mathbf{p}}_{0x}^2 + \tilde{\mathbf{p}}_{0y}^2)^5(4\tilde{\mathbf{p}}_{0x}^4 - 11\tilde{\mathbf{p}}_{0x}^2\tilde{\mathbf{p}}_{0y}^2 + 6\tilde{\mathbf{p}}_{0y}^4) - \\
&1024(\tilde{\mathbf{p}}_{0x}^2 + \tilde{\mathbf{p}}_{0y}^2)^2(45\tilde{\mathbf{p}}_{0x}^8 - 31\tilde{\mathbf{p}}_{0x}^6\tilde{\mathbf{p}}_{0y}^2 - 167\tilde{\mathbf{p}}_{0x}^4\tilde{\mathbf{p}}_{0y}^4 + 219\tilde{\mathbf{p}}_{0x}^2\tilde{\mathbf{p}}_{0y}^6 - 58\tilde{\mathbf{p}}_{0y}^8)\tilde{\mathbf{v}}_{0x}^4 + \\
&256(189\tilde{\mathbf{p}}_{0x}^{10} + 6\tilde{\mathbf{p}}_{0x}^8\tilde{\mathbf{p}}_{0y}^2 - 702\tilde{\mathbf{p}}_{0x}^6\tilde{\mathbf{p}}_{0y}^4 + 732\tilde{\mathbf{p}}_{0x}^4\tilde{\mathbf{p}}_{0y}^6 - 359\tilde{\mathbf{p}}_{0x}^2\tilde{\mathbf{p}}_{0y}^8 + \\
&182\tilde{\mathbf{p}}_{0y}^{10})\tilde{\mathbf{v}}_{0x}^8 - 128(205\tilde{\mathbf{p}}_{0x}^8 - 217\tilde{\mathbf{p}}_{0x}^6\tilde{\mathbf{p}}_{0y}^2 - 290\tilde{\mathbf{p}}_{0x}^4\tilde{\mathbf{p}}_{0y}^4 + 447\tilde{\mathbf{p}}_{0x}^2\tilde{\mathbf{p}}_{0y}^6 \\
&-109\tilde{\mathbf{p}}_{0y}^8)\tilde{\mathbf{v}}_{0x}^{12} + 16(510\tilde{\mathbf{p}}_{0x}^6 - 883\tilde{\mathbf{p}}_{0x}^4\tilde{\mathbf{p}}_{0y}^2 + 255\tilde{\mathbf{p}}_{0x}^2\tilde{\mathbf{p}}_{0y}^4 + 56\tilde{\mathbf{p}}_{0y}^6)\tilde{\mathbf{v}}_{0x}^{16} \\
&-4(369\tilde{\mathbf{p}}_{0x}^4 - 681\tilde{\mathbf{p}}_{0x}^2\tilde{\mathbf{p}}_{0y}^2 + 308\tilde{\mathbf{p}}_{0y}^4)\tilde{\mathbf{v}}_{0x}^{20} + (145\tilde{\mathbf{p}}_{0x}^2 - 188\tilde{\mathbf{p}}_{0y}^2)\tilde{\mathbf{v}}_{0x}^{24} - 6\tilde{\mathbf{v}}_{0x}^{28} + \\
&4t_1\tilde{\mathbf{p}}_{0x}\tilde{\mathbf{v}}_{0x}(4(\tilde{\mathbf{p}}_{0x}^2 + \tilde{\mathbf{p}}_{0y}^2) - \tilde{\mathbf{v}}_{0x}^4)(512(\tilde{\mathbf{p}}_{0x}^2 + \tilde{\mathbf{p}}_{0y}^2)^3(3\tilde{\mathbf{p}}_{0x}^4 - \\
&7\tilde{\mathbf{p}}_{0x}^2\tilde{\mathbf{p}}_{0y}^2 + 3\tilde{\mathbf{p}}_{0y}^4) - 64(32\tilde{\mathbf{p}}_{0x}^8 - 13\tilde{\mathbf{p}}_{0x}^6\tilde{\mathbf{p}}_{0y}^2 - 119\tilde{\mathbf{p}}_{0x}^4\tilde{\mathbf{p}}_{0y}^4 + \\
&169\tilde{\mathbf{p}}_{0x}^2\tilde{\mathbf{p}}_{0y}^6 - 45\tilde{\mathbf{p}}_{0y}^8)\tilde{\mathbf{v}}_{0x}^4 + 16(68\tilde{\mathbf{p}}_{0x}^6 - 95\tilde{\mathbf{p}}_{0x}^4\tilde{\mathbf{p}}_{0y}^2 - 40\tilde{\mathbf{p}}_{0x}^2\tilde{\mathbf{p}}_{0y}^4 + \\
&75\tilde{\mathbf{p}}_{0y}^6)\tilde{\mathbf{v}}_{0x}^8 - 4(72\tilde{\mathbf{p}}_{0x}^4 - 135\tilde{\mathbf{p}}_{0x}^2\tilde{\mathbf{p}}_{0y}^2 + 61\tilde{\mathbf{p}}_{0y}^4)\tilde{\mathbf{v}}_{0x}^{12} + \\
&19(2\tilde{\mathbf{p}}_{0x}^2 - 3\tilde{\mathbf{p}}_{0y}^2)\tilde{\mathbf{v}}_{0x}^{16} - 2\tilde{\mathbf{v}}_{0x}^{20}) + t_1^2\tilde{\mathbf{v}}_{0x}^2(-2\tilde{\mathbf{p}}_{0x} + \tilde{\mathbf{v}}_{0x}^2)(2\tilde{\mathbf{p}}_{0x} + \\
&\tilde{\mathbf{v}}_{0x}^2)(-256(\tilde{\mathbf{p}}_{0x}^2 + \tilde{\mathbf{p}}_{0y}^2)^3(7\tilde{\mathbf{p}}_{0x}^4 - 15\tilde{\mathbf{p}}_{0x}^2\tilde{\mathbf{p}}_{0y}^2 + 6\tilde{\mathbf{p}}_{0y}^4) + \\
&128(18\tilde{\mathbf{p}}_{0x}^8 - 11\tilde{\mathbf{p}}_{0x}^6\tilde{\mathbf{p}}_{0y}^2 - 51\tilde{\mathbf{p}}_{0x}^4\tilde{\mathbf{p}}_{0y}^4 + 83\tilde{\mathbf{p}}_{0x}^2\tilde{\mathbf{p}}_{0y}^6 - 23\tilde{\mathbf{p}}_{0y}^8)\tilde{\mathbf{v}}_{0x}^4 - \\
&32(37\tilde{\mathbf{p}}_{0x}^6 - 59\tilde{\mathbf{p}}_{0x}^4\tilde{\mathbf{p}}_{0y}^2 - 11\tilde{\mathbf{p}}_{0x}^2\tilde{\mathbf{p}}_{0y}^4 + 37\tilde{\mathbf{p}}_{0y}^6)\tilde{\mathbf{v}}_{0x}^8 + \\
&8(38\tilde{\mathbf{p}}_{0x}^4 - 73\tilde{\mathbf{p}}_{0x}^2\tilde{\mathbf{p}}_{0y}^2 + 31\tilde{\mathbf{p}}_{0y}^4)\tilde{\mathbf{v}}_{0x}^{12} + (-39\tilde{\mathbf{p}}_{0x}^2 + 56\tilde{\mathbf{p}}_{0y}^2)\tilde{\mathbf{v}}_{0x}^{16} + 2\tilde{\mathbf{v}}_{0x}^{20} \\
&)))))/(8(4(\tilde{\mathbf{p}}_{0x}^2 + \tilde{\mathbf{p}}_{0y}^2) - \tilde{\mathbf{v}}_{0x}^4)(16(\tilde{\mathbf{p}}_{0x}^2 + \tilde{\mathbf{p}}_{0y}^2)^3 - \\
&8(\tilde{\mathbf{p}}_{0x}^4 - 6\tilde{\mathbf{p}}_{0x}^2\tilde{\mathbf{p}}_{0y}^2 + \tilde{\mathbf{p}}_{0y}^4)\tilde{\mathbf{v}}_{0x}^4 + (\tilde{\mathbf{p}}_{0x}^2 + \tilde{\mathbf{p}}_{0y}^2)\tilde{\mathbf{v}}_{0x}^8)(64\tilde{\mathbf{p}}_{0x}^8 + 16\tilde{\mathbf{p}}_{0x}^6(4\tilde{\mathbf{p}}_{0y}^2 + 5\tilde{\mathbf{v}}_{0x}^4) - \\
&2(8\tilde{\mathbf{p}}_{0y}^4 + 10\tilde{\mathbf{p}}_{0y}^2\tilde{\mathbf{v}}_{0x}^4 + \tilde{\mathbf{v}}_{0x}^8)^2 - 12\tilde{\mathbf{p}}_{0x}^4(16\tilde{\mathbf{p}}_{0y}^4 + 56\tilde{\mathbf{p}}_{0y}^2\tilde{\mathbf{v}}_{0x}^4 + 7\tilde{\mathbf{v}}_{0x}^8) + \\
&\tilde{\mathbf{p}}_{0x}^2(-320\tilde{\mathbf{p}}_{0y}^6 + 976\tilde{\mathbf{p}}_{0y}^4\tilde{\mathbf{v}}_{0x}^4 + 324\tilde{\mathbf{p}}_{0y}^2\tilde{\mathbf{v}}_{0x}^8 + 23\tilde{\mathbf{v}}_{0x}^{12})))
\end{aligned} \tag{18}
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References

[1] W. H. Press, S. A. Teukolsky, W. T. Vetterling, and B. P. Flannery, *Numerical recipes 3rd edition: The art of scientific computing*. Cambridge university press, 2007.