Combinatory Logic
This originates from an article, in my notes, that I wrote in 1984-1985.

Combinatory logic is a mathematical language whose expressions can be built up from

(I) **Combinators** $\mathbf{B}, \mathbf{C}, \mathbf{K}, \mathbf{W}$

(II) **Variable Symbols** $x_0, x_1, x_2, \ldots$

using a binary combination rule called *application*. In practice, terms like
$$(((\alpha\beta)\gamma)\delta)$$
are represented as
$$\alpha\beta\gamma\delta \ .$$
We will follow the convention of removing all the left-most and outer-most brackets.

The basic axioms of the language, through which the combinators are defined, are called *combinatory axioms*. They are
$$\mathbf{C}\alpha\beta\gamma = \alpha\gamma\beta, \quad \mathbf{B}\alpha\beta\gamma = \alpha(\beta\gamma), \quad \mathbf{W}\alpha\beta = \alpha\beta\beta, \quad \mathbf{K}\alpha\beta = \alpha \ .$$
The extensionality axiom

Ext if $\alpha x = \beta x \to \alpha = \beta$, where α, β are expressions not containing x .

will also be adopted.

1. Combinator Types
The constant terms are called combinators. They fall into four basic families: *permuters*, *groupers*, *replicators* and *deleters*. Other, more complex, combinators can be built up from **B**, **C**, **K**, **W**, by decomposition into these basic types.

1.1. Permuters
The basic permuters, $\mathbf{C}, \mathbf{T}$, satisfy the rules
$$\mathbf{T}\beta\gamma = \gamma\beta, \quad \mathbf{C}\alpha\beta\gamma = \alpha\gamma\beta \ .$$
In general, the permuter $\mathbf{C_n}$ is defined by the rule
$$\mathbf{C_n}\alpha_1 \ldots \alpha_n \beta\gamma = \alpha_1 \ldots \alpha_n \gamma\beta \quad (n \geq 0) \ .$$
Thus we find that
$$\mathbf{C_0} = \mathbf{T} = \mathbf{CI}, \quad \mathbf{C_1} = \mathbf{C} \ ,$$
and in general for $n \geq 0$ that
$$\mathbf{C}_{n+1} = \mathbf{BC_n}, \quad \mathbf{C_n} = \mathbf{C}_{n+1}\mathbf{I} \ ,$$
where $\mathbf{I}$ is the *identity combinator* defined by the rule
$$\mathbf{I}\alpha = \alpha \ .$$
It can be expressed in terms of the elementary combinators as
$$\mathbf{I} = \mathbf{WK} \ .$$

1.2. Groupers
The basic groupers $\mathbf{B}, \mathbf{I}'$ satisfy the rules
$$\mathbf{I}'\beta\gamma = \beta\gamma, \quad \mathbf{B}\alpha\beta\gamma = \alpha(\beta\gamma) \ .$$
In general
$$\mathbf{B_n}\alpha_1 \ldots \alpha_n \beta\gamma = \alpha_1 \ldots \alpha_n (\beta\gamma) \quad (n \geq 0) \ ,$$
with $\mathbf{B_n}$ defined analogously to $\mathbf{C_n}$.

1.3. Replicators
The basic replicators are $\mathbf{W}, \mathbf{D}$, given by the rules
$$\mathbf{D}\beta = \beta\beta, \quad \mathbf{W}\alpha\beta = \alpha\beta\beta \ ,$$
with the general replicator given by
$$\mathbf{W_n}\alpha_1 \ldots \alpha_n \beta = \alpha_1 \ldots \alpha_n \beta\beta \quad (n \geq 0) \ .$$
Again, by extensionality, we can write $\mathbf{D} = \mathbf{WI}$.

1.4. Deletors

In a similar way, the basic deletors can be defined by

$$\mathbf{O}\beta = \mathbf{I}, \quad \mathbf{K}\alpha\beta = \alpha\ ,$$

and the general deletors by

$$\mathbf{K_n}\alpha_1 \ldots \alpha_n \beta = \alpha_1 \ldots \alpha_n \quad (n \geq 0)\,.$$

1.5. Summary

Overall, we have basic combinators of each type given by the rules

$$\mathbf{C_n}\boldsymbol{\alpha}\beta\gamma = \boldsymbol{\alpha}\gamma\beta, \quad \mathbf{B_n}\boldsymbol{\alpha}\beta\gamma = \boldsymbol{\alpha}(\beta\gamma), \quad \mathbf{W_n}\boldsymbol{\alpha}\beta = \boldsymbol{\alpha}\beta\beta, \quad \mathbf{K_n}\boldsymbol{\alpha}\beta = \boldsymbol{\alpha} \quad (n \geq 0)$$

where we write $\boldsymbol{\alpha} = \alpha_1 \ldots \alpha_n$. The different orders are related by

$$\alpha_{n+1} = \mathbf{B}\alpha_{\mathbf{n}}, \quad \alpha_{\mathbf{n}} = \alpha_{n+1}\mathbf{I}$$

for $n \geq 0$ and $\alpha = \mathbf{B}, \mathbf{C}, \mathbf{K}, \mathbf{W}$. The combinators $\mathbf{B_1}, \mathbf{C_1}, \mathbf{K_1}, \mathbf{W_1}$ are then defined respectively as $\mathbf{B}, \mathbf{C}, \mathbf{K}, \mathbf{W}$, with the corresponding rules reducing to familiar form; while $\mathbf{C_0}, \mathbf{K_0}, \mathbf{W_0}$ may be defined respectively as $\mathbf{T}, \mathbf{O}, \mathbf{D}$, recapturing the rules of the familiar forms through extensionality with

$$\mathbf{T} = \mathbf{CI}, \quad \mathbf{O} = \mathbf{KI}, \quad \mathbf{D} = \mathbf{WI}\,,$$

and

$$\mathbf{T}\beta\gamma = \gamma\beta, \quad \mathbf{O}\beta = \mathbf{I}, \quad \mathbf{D}\beta = \beta\beta\ .$$

In tabular form, we have, for the lowest orders,

Type	$n = 0$	$n = 1$	$n = 2$	$n = 3$	$\cdots$
Permuter	$\mathbf{T}$	$\mathbf{C}$	$\mathbf{BC}$	$\mathbf{B(BC)}$	$\cdots$
Grouper	$\mathbf{I}'$	$\mathbf{B}$	$\mathbf{BB}$	$\mathbf{B(BB)}$	$\cdots$
Replicator	$\mathbf{D}$	$\mathbf{W}$	$\mathbf{BW}$	$\mathbf{B(BW)}$	$\cdots$
Deleter	$\mathbf{O}$	$\mathbf{K}$	$\mathbf{BK}$	$\mathbf{B(BK)}$	$\cdots$

A combinator, like $\mathbf{S}$, given by the rule

$$\mathbf{S}\alpha\beta\gamma = \alpha\gamma(\beta\gamma)$$

is seen to be a mixture of a replicator, grouper and permuter. It can be constructed successively as follows

$$\alpha\gamma(\beta\gamma) = \mathbf{B_2}\alpha\gamma\beta\gamma = \mathbf{C_2}\mathbf{B_2}\alpha\beta\gamma\gamma = \mathbf{W_3}(\mathbf{C_2}\mathbf{B_2})\alpha\beta\gamma\ .$$

Thus

$$\mathbf{S} = \mathbf{B(BW)(BC(BB))}\,.$$

This result will hold true in general

Combinatory Completeness

Every term $\varphi(\alpha_1, \ldots, \alpha_n)$ can be expressed in the form $\Phi\alpha_1 \ldots \alpha_n$ where

(I) $\Phi = \mathbf{w}(\mathbf{c}(\mathbf{bI}))$ if all terms $\alpha_1 \ldots \alpha_n$ occur in φ

(II) $\Phi = \mathbf{k}(\mathbf{w}(\mathbf{c}(\mathbf{bI})))$ if some $\alpha_1 \ldots \alpha_n$ do not occur in φ

with $\mathbf{b}, \mathbf{c}, \mathbf{k}, \mathbf{w}$ being general grouper, permuter, deleter, replicator strings.

A general α-string (for $\alpha = \mathbf{B}, \mathbf{C}, \mathbf{K}, \mathbf{W}$) has the form

$$\mathbf{\Lambda_n} a_{i_1} a_{i_2} \ldots a_{i_n}$$

where $\mathbf{\Lambda_n}$ is defined by the rule

$$\mathbf{\Lambda_n}\beta_1\beta_2 \ldots \beta_n x = \beta_1(\beta_2(\ldots(\beta_n x)\ldots))\,.$$

Thus

$$\mathbf{\Lambda_0} = \mathbf{B_0}, \quad \mathbf{\Lambda_{n+1}} = \mathbf{B_{n+1}}\mathbf{\Lambda_n} \quad (n \geq 0)\ .$$

Thus, the general alpha-string

$$\mathbf{\Lambda_n} a_{i_1} a_{i_2} \ldots a_{i_n}$$

will reduce to the form

$$\mathbf{B}(\ldots(\mathbf{B}a_{i_1}a_{i_2})\ldots a_{i_{n-1}})a_{i_n}.$$

Example:

The step-wise reduction of the expression $\rho(\alpha x)(\beta x)(\gamma x)$ would proceed as follows:

$$\begin{aligned}
\rho(\alpha x)(\beta x)(\gamma x) &= \mathbf{I}\rho(\alpha x)(\beta x)(\gamma x) \\
&= \mathbf{B_4 I}\rho(\alpha x)(\beta x)\gamma x \\
&= \mathbf{B_3}(\mathbf{B_4 I})\rho(\alpha x)\beta x\gamma x \\
&= \mathbf{B}_2(\mathbf{B_3}(\mathbf{B_4 I}))\rho\alpha x\beta x\gamma x \\
&= \mathbf{bI}\rho\alpha x\beta x\gamma x & (\mathbf{b} = \mathbf{\Lambda_3 B}_2\mathbf{B_3 B_4}) \\
&= \mathbf{C_5}(\mathbf{bI})\rho\alpha x\beta\gamma xx \\
&= \mathbf{C_3}(\mathbf{C_5}(\mathbf{bI}))\rho\alpha\beta x\gamma xx \\
&= \mathbf{C_4}(\mathbf{C_3}(\mathbf{C_5}(\mathbf{bI})))\rho\alpha\beta\gamma xxx \\
&= \mathbf{c}(\mathbf{bI})\rho\alpha\beta\gamma xxx & (\mathbf{c} = \mathbf{\Lambda_3 C}_4\mathbf{C}_3\mathbf{C_5}) \\
&= \mathbf{W_5}(\mathbf{c}(\mathbf{bI}))\rho\alpha\beta\gamma xx \\
&= \mathbf{W_5}(\mathbf{W_5}(\mathbf{c}(\mathbf{bI})))\rho\alpha\beta\gamma x \\
&= \mathbf{w}(\mathbf{c}(\mathbf{bI}))\rho\alpha\beta\gamma x & (\mathbf{w} = \mathbf{\Lambda_2 W_5 W}_5)
\end{aligned}$$

As a function $\varphi(\rho,\alpha,\beta,\gamma,\delta,x) = \rho(\alpha x)(\beta x)(\gamma x)$, where the variable δ is not explicitly involved, this would then be expressed with an additional set of combinators to effect the deletion. The result would be

$$\varphi(\rho,\alpha,\beta,\gamma,\delta,x) = \Phi\rho\alpha\beta\gamma\delta x, \quad \Phi = \mathbf{k}(\mathbf{w}(\mathbf{c}(\mathbf{bI}))) = \mathbf{\Lambda_4 kwcbI},$$

where, in addition, we have

$$\mathbf{k} = \mathbf{\Lambda_2 K_5 K}_5.$$

2. Recursion

Corresponding to each number $n = 0,1,2,\ldots$ is a *numeral* $\mathbf{n}$ formed from the combinators, as follows

$$\mathbf{0} \equiv \mathbf{O}, \quad \mathbf{1} \equiv \mathbf{I}, \quad \mathbf{2} \equiv \mathbf{SBI}, \quad \mathbf{3} \equiv \mathbf{SB}(\mathbf{SBI}), \quad \ldots.$$

Through application of the extensionality axiom, we have

$$\mathbf{SBO}xy = \mathbf{B}x(\mathbf{O}x)y = x(\mathbf{O}xy) = xy = \mathbf{I}xy \rightarrow \mathbf{SBO} = \mathbf{I},$$

therefore, we may write in general

$$\mathbf{n+1} = \mathbf{SBn},$$

therefore $\mathbf{SB}$ represents the successor function. The action of the combinator is given by

$$\mathbf{n}\varphi x = \varphi^n x \quad (n \geq 0),$$

where

$$\varphi^0 x = x, \quad \varphi^1 x = \varphi x, \quad \varphi^2 x = \varphi(\varphi x),$$

and generally

$$\varphi^{n+1}x = \varphi(\varphi^n x), \quad (n \geq 0).$$

These combinators are known as the *Church numerals*. They represent the exponents of repeated function application.

Primitive recursion is a method of producing a new function $p(\mathbf{x}, n)$ from functions

$$f(\mathbf{x}), \quad g(\mathbf{x}, n, y),$$

where $\mathbf{x} = x_1,\ldots,x_k$ with $k \geq 0$. The function is defined in terms of itself at smaller values of $n \geq 0$ by

$$p(\mathbf{x},0) = f(\mathbf{x}), \quad p(\mathbf{x}, n+1) = g(\mathbf{x}, n, p(\mathbf{x}, n)) \quad (n \geq 0).$$

In terms of combinatory logic, primitive recursion can be represented as follows

$$(p\mathbf{x})\mathbf{O} = f\mathbf{x}, \quad (p\mathbf{x})(\mathbf{SBn}) = (g\mathbf{x})\mathbf{n}((p\mathbf{x})\mathbf{n}) \quad (n \geq 0).$$

Writing P, F, G for $p\mathbf{x}, f\mathbf{x}, g\mathbf{x}$ respectively, this becomes

$$P\mathbf{O} = F, \quad P(\mathbf{SBn}) = G\mathbf{n}(P\mathbf{n}) \quad (n \geq 0).$$

2.1. The Recursion Hierarchy

This construction leads to a hierarchy characterized by the function's *order*, based on the number of applications of primitive recursions with respect to n are needed to define the function $p(\mathbf{x},n)$. At the bottom of the hierarchy are the recursive functions of the lowest, or 0^{th} order. These consist of the closure of the following

- The *zero function*: $\mathbf{KO}x = \mathbf{O}$,
- The *successor function*: $\mathbf{SBn} = \mathbf{n}+\mathbf{1}$,
- The *projection functions*: $\mathbf{K}^n\mathbf{K}^m x_0 x_1 \ldots x_n \ldots x_{m+n} = x_n$ for $n > 0$ and $m \geq 0$,

under *composition*

$$(\mathbf{G}_{\mathbf{mn}}\psi\varphi_1 \ldots \varphi_n)x_1 \ldots x_m = \psi(\varphi_1 x_1 \ldots x_m)\ldots(\varphi_n x_1 \ldots x_m) \quad (m,n \geq 0).$$

The recursive functions of order $n+1$ are the closure under composition of these functions and the primitive recursive functions formed with recursive functions $g(\mathbf{x},n)$ of order n or less.

Though not exhaustive of the hierarchy, the following special cases are of separate interest. The ***first-order recursions*** are those formed as solutions to

$$P\mathbf{O} = a_0, \quad P(\mathbf{SBn}) = a_1(P\mathbf{n}) \quad (n \geq 0),$$

where a_1 is independent of $\mathbf{n}$. As previously discussed, the solution involves the Church numerals, themselves

$$P\mathbf{n} = a_1{}^n a_0 = \mathbf{n}a_1a_0 = \mathbf{BCT}a_1a_0\mathbf{n} \rightarrow P = [a_1, a_0] = \langle a_1, a_0 \rangle,$$

wbere

$$\langle x, y \rangle \equiv \mathbf{BCT}xy \,.$$

For ***second-order recursion*** the defining relations are

$$P\mathbf{O} = a_0, \quad P(\mathbf{SBn}) = (a_2{}^n a_1)(P\mathbf{n}) \quad (n \geq 0).$$

From this we get

$$P\mathbf{1} = a_1a_0 = \mathbf{O}a_2a_1a_0, \quad P\mathbf{2} = a_2a_1(a_1a_0) = \mathbf{F}a_2a_1a_0 \;,$$

using the combinator defined by

$$\mathbf{F}xyz = xy(yz)\,.$$

Going further, we have

$$P\mathbf{3} = a_2(a_2a_1)(a_2a_1(a_1a_0)) = \mathbf{F}a_2(a_2a_1)(a_1a_0) = \mathbf{FF}a_2a_1(a_1a_0) = \mathbf{F}(\mathbf{FF}a_2)a_1a_0 = \mathbf{BF}(\mathbf{FF})a_2a_1a_0,$$

and

$$\begin{aligned} P\mathbf{4} &= a_2(a_2(a_2a_1))(a_2(a_2a_1)(a_2a_1(a_1a_0))) \\ &= \mathbf{F}a_2(a_2(a_2a_1))(a_2a_1(a_1a_0)) \\ &= \mathbf{FF}a_2(a_2a_1)(a_2a_1(a_1a_0)) \\ &= \mathbf{F}(\mathbf{FF}a_2)(a_2a_1)(a_1a_0) \\ &= \mathbf{BF}(\mathbf{FF})a_2(a_2a_1)(a_1a_0) \\ &= \mathbf{F}(\mathbf{BF}(\mathbf{FF}))a_2a_1(a_1a_0) \\ &= \mathbf{F}(\mathbf{F}(\mathbf{BF}(\mathbf{FF}))a_2)a_1a_0 \\ &= \mathbf{BF}(\mathbf{F}(\mathbf{BF}(\mathbf{FF})))a_2a_1a_0. \end{aligned}$$

The combinator in the last expression on the right can be written more explicitly as

$$\mathbf{BF}(\mathbf{F}(\mathbf{BF}(\mathbf{FF}))) = \mathbf{BF}(\mathbf{F}(\mathbf{B}(\mathbf{BF})\mathbf{FF})) = \mathbf{B}(\mathbf{BF})\mathbf{F}(\mathbf{B}(\mathbf{BF})\mathbf{FF}) = (\mathbf{B}(\mathbf{BF})\mathbf{F})^2\,\mathbf{F}\,.$$

Noting that

$$(\mathbf{B}(\mathbf{BF})\mathbf{F})^1\,\mathbf{F} = \mathbf{B}(\mathbf{BF})\mathbf{FF} = \mathbf{BF}(\mathbf{FF}),$$

and applying extensionality to

$$\mathbf{B}(\mathbf{BF})\mathbf{FO}xyz = \mathbf{BF}(\mathbf{FO})xyz = \mathbf{F}(\mathbf{FO}x)yz = \mathbf{FO}xy(yz) = \mathbf{O}x(xy)(yz) = xy(yz) = \mathbf{F}xyz,$$

we also have

$$(\mathbf{B}(\mathbf{BF})\mathbf{F})^1\,\mathbf{O} = \mathbf{F}, \quad (\mathbf{B}(\mathbf{BF})\mathbf{F})^{n+1}\,\mathbf{O} = (\mathbf{B}(\mathbf{BF})\mathbf{F})^n\,\mathbf{F} \quad (n \geq 0)\,.$$

Noting that

$$P\mathbf{O} = a_0 = \mathbf{O}a_1 a_o = \mathbf{KO}a_2 a_1 a_o \,,$$

and that, with another application of extensionality to

$$\mathbf{B(BF)F(KO)}xyz = \mathbf{BF(F(KO))}xyz = \mathbf{F(F(KO)}x)\,yz = \mathbf{F(KO)}xy(\,yz) = \mathbf{KO}x(xy)(\,yz) = \,yz = \mathbf{O}yz \,,$$

we have

$$(\mathbf{B(BF)F})^{n+1}(\mathbf{KO}) = (\mathbf{B(BF)F})^{n}\,\mathbf{O}, \quad (\mathbf{B(BF)F})^{n+2}(\mathbf{KO}) = (\mathbf{B(BF)F})^{n}\,\mathbf{F} \quad (n \geq 0)\,,$$

we obtain the apparent result

$$P\mathbf{n} = (\mathbf{B(BF)F})^{n}(\mathbf{KO})a_2 a_1 a_0 \,.$$

Indeed, what we find is that

$$\begin{aligned} \mathbf{B(BF)F}\mu a_2 (a_2{}^n a_1)(P\mathbf{n}) &= \mathbf{BF(F}\mu)a_2 (a_2{}^n a_1)(P\mathbf{n}) \\ &= \mathbf{F(F}\mu a_2)(a_2{}^n a_1)(P\mathbf{n}) \\ &= \mathbf{F}\mu a_2 (a_2{}^n a_1)(a_2{}^n a_1(P\mathbf{n})) \\ &= \mu a_2 (a_2 (a_2{}^n a_1))(a_2{}^n a_1(P\mathbf{n})) \\ &= \mu a_2 (a_2{}^{n+1} a_1)(P(\mathbf{SBn})). \end{aligned}$$

Thus,

$$(\mathbf{B(BF)F})^{n}(\mathbf{KO})a_2 a_1 a_0 = (\mathbf{B(BF)F})^{n}(\mathbf{KO})a_2 a_1 (P\mathbf{O}) = \mathbf{KO}a_2 (a_2{}^{\mathbf{n}} a_1)(P\mathbf{n}) = P\mathbf{n} \,.$$

This leads to the general solution

$$P\mathbf{n} = \mathbf{n(B(BF)F)(KO)}a_2 a_1 a_0 = \mathbf{\Lambda_3 C_3 C_2 C}\langle \mathbf{B(BF)F}, \mathbf{KO}\rangle a_2 a_1 a_0 \mathbf{n} \,,$$

from which we get

$$P = [a_2, a_1, a_0] \equiv \mathbf{\Lambda_3 C_3 C_2 C}\langle \mathbf{B(BF)F}, \mathbf{KO}\rangle a_2 a_1 a_0 \,.$$

Returning to the first-order recursion, and extending this process analogously here, as well, we could write

$$\mathbf{F}\mu a_1 (a_1{}^n a_0) = \mu a_1 (a_1 (a_1{}^n a_0)) = \mu a_1 (a_1{}^{n+1} a_0) \to \mathbf{F}^n \mu a_1 a_0 = \mu a_1 (a_1{}^n a_0)\,.$$

Consequently, the general solution could equivalently be written

$$[a_1, a_0]\mathbf{n} = \mathbf{F}^n \mathbf{O} a_1 a_0 = \mathbf{nFO}a_1 a_0 = \mathbf{\Lambda_2 C_2 C}\langle \mathbf{F}, \mathbf{O}\rangle a_1 a_0 \mathbf{n} \to [a_1, a_0] = \mathbf{\Lambda_2 C_2 C}\langle \mathbf{F}, \mathbf{O}\rangle a_1 a_0 \,.$$

This provides a glimpse into how recursion of order k should be handled. Here, we have

$$P\mathbf{O} = a_0, \quad P(\mathbf{SBn}) = [a_k, \ldots, a_1]\mathbf{n}(P\mathbf{n}) \quad (n \geq 0)\,.$$

The general solution will be

$$P\mathbf{n} = F_k{}^n (\mathbf{K}^k \mathbf{I}) a_k \ldots a_1 a_0 \,,$$

with

$$F_k = (\mathbf{B(CBF)B})^k \,\mathbf{I} \,.$$

This not only includes recursion of order 0, with $F_0 = \mathbf{I}$ and $[a_0](n) = a_0$, but also the case for $k = 1$, through application of extensionality to

$$\mathbf{B(CBF)BI}xy = \mathbf{CBF(BI)}xy = \mathbf{B(BI)F}xy = \mathbf{BI(F}x)\,y = \mathbf{I(F}xy) = \mathbf{F}xy \to F_1 = \mathbf{B(CBF)BI} = \mathbf{F} \,.$$

For $k = 2$ we then have

$$F_2 = \mathbf{B(CBF)BF} = \mathbf{CBF(BF)} = \mathbf{B(BF)F} \,.$$

2.2. Formal Integration and Differentiation

If P and G are functions related by

$$P(n+1) = G(n, P(n)) \quad (n \geq 0)\,,$$

we will write $G = dP$ and call G the ***formal differential*** of P. Applying the converse notion of anti-derivative, we may then write the solution to the equation

$$P(0) = F, \quad P(n+1) = G(n, P(n)) \quad (n \geq 0)$$

formally as an integral

$$P = \int_F G \,,$$

and call P a ***formal integral*** of G. Thus, for primitive recursion of order 1, we have

$$[a_1, a_0] = \int_{a_0} a_1$$

and for order $k+2$,

$$[a_{k+1}, a_k, \ldots, a_1, a_0] = \int_{a_0} \int_{a_1} \cdots \int_{a_k} a_{k+1} \ ,$$

with

$$a_i = d^{k-i}[a_k, \ldots, a_o](0), \quad (0 \le i \le k) \, .$$

Example:
For

$$P = [d, c, b, a] = \int_a \int_b \int_c d$$

we have

$$P(0) = a, \quad dP(0) = b, \quad d^2 P(0) = c, \quad d^3 P(0) = d \, .$$

For the smallest values of n, this leads to the following tabulation

n	$d^3 P(n)$	$d^2 P(n)$	$dP(n)$	$P(n)$
0	d	c	b	a
1	d	dc	cb	ba
2	d	$d(dc)$	$dc(cb)$	$cb(ba)$
3	d	$d(d(dc))$	$d(dc)(dc(cb))$	$dc(cb)(cb(ba))$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

The order-k recursor F_k is cast in such a way as to produce the table row-by-row from the topmost row. This requires

$$F_k \mu a_k \ldots a_1 a_0 = \mu a_k (a_k a_{k-1}) \ldots (a_1 a_0) \ ,$$

and for the topmost row

$$\mu a_k \ldots a_1 a_0 = a_0 = \mathbf{K}^k \mathbf{I} a_k \ldots a_1 a_0 \ ,$$

thus identifying the “seed” function as the projector $\mu = \mathbf{K}^k \mathbf{I}$.

For order 0, the recursor is just $F_0 \mu a_0 = \mu a_0 = \mathbf{I} \mu a_0$. For order $k+1$, the recursor can be resolved inductively as

$$\begin{aligned} F_{k+1} \mu a_{k+1} a_k \ldots a_1 a_0 &= \mu a_{k+1} (a_{k+1} a_k)(a_k a_{k-1}) \ldots (a_1 a_0) \\ &= \mathbf{F} \mu a_{k+1} a_k (a_k a_{k-1}) \ldots (a_1 a_0) \\ &= F_k (\mathbf{F} \mu a_{k+1}) a_k \ldots a_1 a_0 \\ &= \mathbf{B} F_k (\mathbf{F} \mu) a_{k+1} a_k \ldots a_1 a_0 \\ &= \mathbf{B}(\mathbf{B} F_k) \mathbf{F} \mu a_{k+1} a_k \ldots a_1 a_0 \, . \end{aligned}$$

Through application of extensionality, this leads to

$$F_{k+1} = \mathbf{B}(\mathbf{B} F_k) \mathbf{F} = \mathbf{CBF}(\mathbf{B} F_k) = \mathbf{B}(\mathbf{CBF}) \mathbf{B} F_k \, .$$

Thus,

$$F_k = (\mathbf{B}(\mathbf{CBF})\mathbf{B})^k \mathbf{I} \, .$$

The general solution for recursion of order k may thus be written

$$[a_k, \ldots, a_1, a_0] \mathbf{n} = F_k{}^n (\mathbf{K}^k \mathbf{I}) a_k \ldots a_1 a_0 \, ,$$

or in terms of the combinatory numerals

$$F_k{}^n (\mathbf{K}^k \mathbf{I}) a_k \ldots a_1 a_0 = \mathbf{n} F_k (\mathbf{K}^k \mathbf{I}) a_k \ldots a_1 a_0 = \mathbf{\Lambda_{k+1} C_{k+1}} \ldots \mathbf{C_2 C} \langle F_k, \mathbf{K}^k \mathbf{I} \rangle a_k \ldots a_1 a_0 \mathbf{n} \, .$$

Thus,

$$[a_k, \ldots, a_1, a_0] = \mathbf{\Lambda_{k+1} C_{k+1}} \ldots \mathbf{C_2 C} \langle F_k, \mathbf{K}^k \mathbf{I} \rangle a_k \ldots a_1 a_0 \, .$$

2.3. Representation of Primitive Recursion as Integration

The hierarchy is not exhaustive of all the primitive recursive functions, since the integral $P = \int_F G$ may involve a function G formed by compositions, rather than just one of the form $G = [a_k, \ldots, a_1]$. However, by using a similar tabulation method, the more general case can also be resolved by keeping explicit account of $\mathbf{n}$ in the tabulation of $P\mathbf{n}$:

$$\lambda\mu G\mathbf{n}(P\mathbf{n}) = \mu G(\mathbf{SBn})(G\mathbf{n}(P\mathbf{n})) .$$

Assuming a solution of the form $P\mathbf{n} = \lambda^n \mu G\mathbf{O}F$, this leads to the general relation

$$P\mathbf{n} = \lambda^n \mu G\mathbf{O}F = \lambda^n \mu G\mathbf{O}(P\mathbf{O}) = \mu G\mathbf{n}(P\mathbf{n}) ,$$

which is satisfied with the projector $\mu = \mathbf{K}^2\mathbf{I}$. The expression for the update is satisfied if

$$\begin{aligned}
\lambda\mu Gxy &= \mu G(\mathbf{SB}x)(Gxy) \\
&= \mathbf{B}_3 \mu G(\mathbf{SB}x)(Gx)y \\
&= \mathbf{B_3B}_3 \mu G(\mathbf{SB})x(Gx)y \\
&= \mathbf{C}_2(\mathbf{B_3B}_3)\mu(\mathbf{SB})Gx(Gx)y \\
&= \mathbf{C}(\mathbf{C}_2(\mathbf{B_3B}_3))(\mathbf{SB})\mu Gx(Gx)y \\
&= \mathbf{S_3}(\mathbf{C}(\mathbf{C}_2(\mathbf{B_3B}_3))(\mathbf{SB}))\mu GGxy \\
&= \mathbf{W_2}(\mathbf{S_3}(\mathbf{C}(\mathbf{C}_2(\mathbf{B_3B}_3))(\mathbf{SB})))\mu Gxy.
\end{aligned}$$

By extensionality, this leads to the solution

$$\lambda = \mathbf{W_2}(\mathbf{S_3}(\mathbf{C}(\mathbf{C}_2(\mathbf{B_3B}_3))(\mathbf{SB}))) ,$$

to the solution to the recursive system

$$P\mathbf{n} = \left(\int_F G\right)\mathbf{n} = \mathbf{n}\lambda\mu G\mathbf{O}F = \langle\lambda, \mu\rangle\mathbf{n}G\mathbf{O}F = \mathbf{\Lambda_3C_3C_2C}\langle\lambda, \mu\rangle G\mathbf{O}F\mathbf{n} ,$$

and to the formal integral,

$$\int_F G = \mathbf{\Lambda_3C_3C_2C}\langle\lambda, \mathbf{K}^2\mathbf{I}\rangle G\mathbf{O}F = \mathbf{\Lambda}_4\mathbf{CC_3C_2C}\langle\lambda, \mathbf{K}^2\mathbf{I}\rangle\mathbf{O}GF .$$