Topology

The mathematics of shape

Roderic Guigó Corominas

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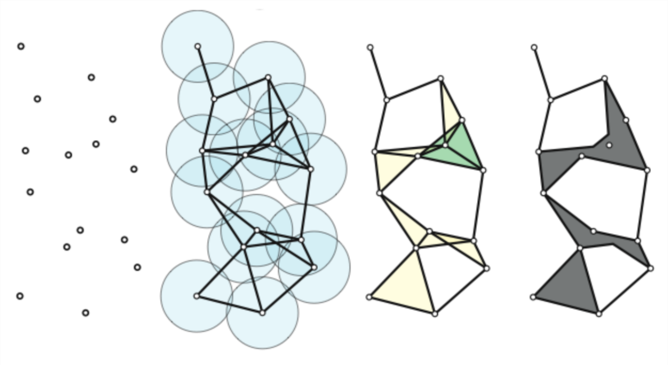
The Shape of Data

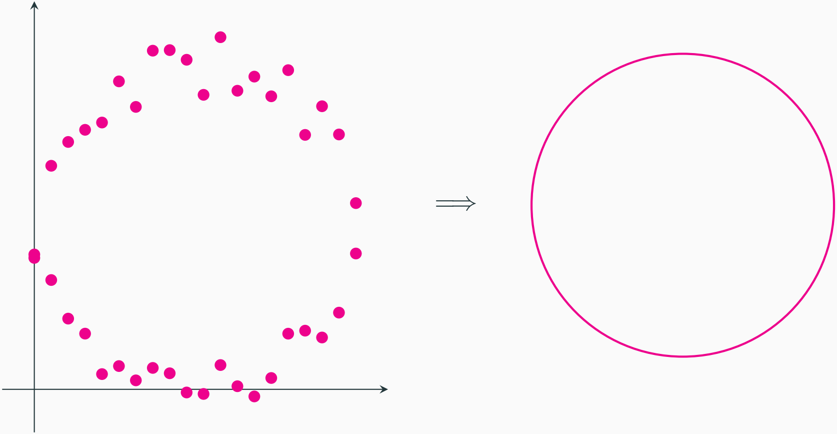
A data point is a vector of numbers $(d_1, \ldots, d_n)$.

- The size of the vector, n, is called the dimension.
- It can be represented as a point in an n-dimensional space.
- Points can be grouped (connected) into spaces.



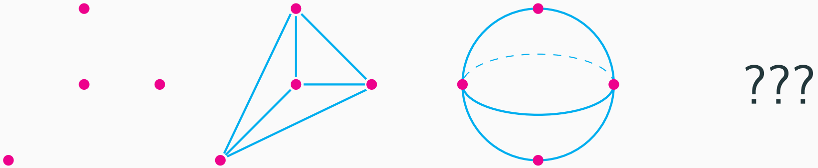
The Shape of Data

Goal: Study a data set from a Topological perspective. Obtain information from its "shape".



Example (Lotka-Volterra): Predator-prey observations suggested that there were periodic solutions.

Flow Chart



Data $\Rightarrow$ $\underbrace{\text{Topology} \Rightarrow \text{Properties}}_{\text{Today}}$ $\Rightarrow$ Meaning

Today: Topology

Can be very abstract, but here we have a provisional definition:

Topology is the area of mathematics that studies shape.

Topological Spaces and Their Properties

Spaces of real numbers $\mathbb{R}^n$

- Line $\mathbb{R}$: 1 dimension.

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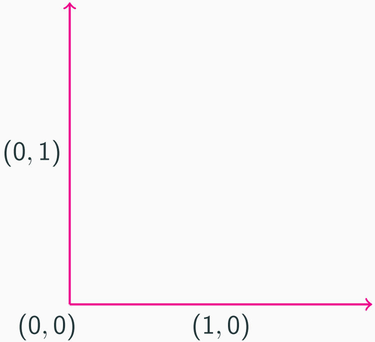
- Plane $\mathbb{R}^2$: 2 dimensions.



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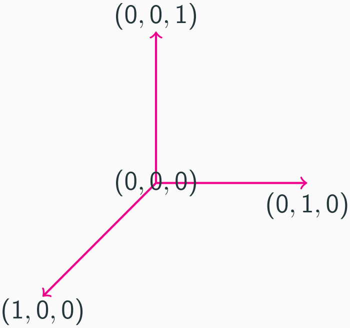

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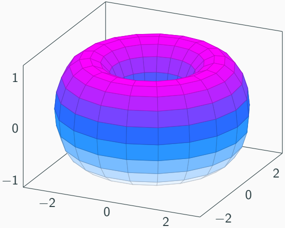
Can only visualize up to 3 dimensions $\Rightarrow$ first issue.

Topological Spaces

A Topological Object is a subset of $\mathbb{R}^n$.

Some examples:

- The whole $\mathbb{R}^n$.
- Interval: $a \leq x \leq b$ in $\mathbb{R}^1$.
- Discrete set of points: $(x_1, y_1, z_1), (x_2, y_2, z_2), \ldots, (x_n, y_n, z_n)$ in $\mathbb{R}^3$.
- Circle or Sphere: $x^2 + y^2 = 1$ in $\mathbb{R}^2$ and $x^2 + y^2 + z^2 = 1$ in $\mathbb{R}^3$.
- A torus (holllow donut) in $\mathbb{R}^3$.



Topological Spaces

How can they be defined?

- Equality/inequality.
- Picture.
- By their properties.
- By some geometric construction.

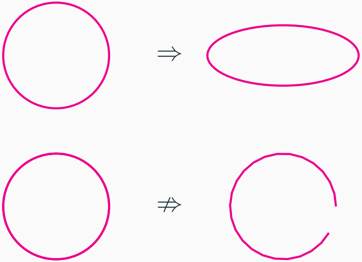
Topological Deformations

We want to introduce the notion of continuous deformation. What is allowed?

- Bend
- Strech
- Twist
- Crumple

What is **not** allowed?

- Tear
- Glue



Topological Deformations

Example: deforming a circle. Consider the equation $x^2 + y^2 = 1$. It defines a circle of radius 1.

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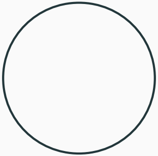
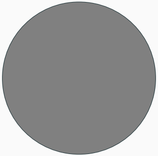
- Translate circle: $(x - a)^2 + (y - b)^2 = 1$.
- Increase the radius: $x^2 + y^2 = r^2$.
- Deform into an ellipse: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Topological Deformations

Similar but different. The circle, on the left is a different object than the disk, on the right. They are both inside $\mathbb{R}^2$.



$x^2 + y^2 = 1$



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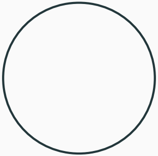

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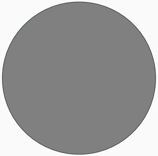

$x^2 + y^2 \leq 1$

One cannot be deformed into the other.

Topology

So we now have an actual definition:

Topology is the study of the properties of a geometric object that are preserved under continuous deformations.

Topological Properties

A topological property or invariant is a property of a topological space that doesn't change when applying a continuous deformations. Which of the following are invariant and which properties vary?

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Topological Properties

Invariant	**Non-invariant**
Components	Length
Dimension	Surface Area
Cardinality	Volume
Holes	Distance between points
Euler Characteristic	Symmetry
Homotopy Type	
Homology Groups	

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Today we focus on holes.

Topological Properties: Components

Figure 1: Mobius Strip

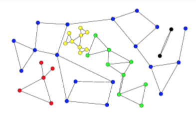


Figure 2: Graph

Figure 3: Network

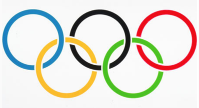


Figure 4: Olympic Rings

Topological Properties: Holes

A loop in a topological space is a function from a circle:

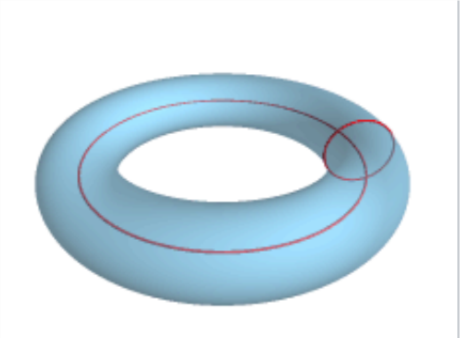
$$\Longleftarrow$$

Think of a rubber band that can stretch and move around the space.

A topological space has a hole if one can place a rubber band that can't be shrinked.

Topological Properties: Holes

Spaces **with** holes: circle, torus, cylinder, punctured disk, etc.



Spaces **without** holes: sphere, disk, etc.

Topological Properties: Voids

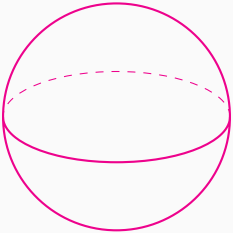
According to our definition, a sphere has no holes, this seems wrong. How do we detect the empty space inside the sphere? Instead of a rubber band, think of a balloon.

Topological Invariants: Higher Dimensions Holes

The notion of hole

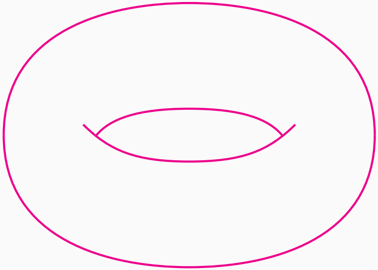
- dimension 1: components
- dimension 2: holes
- dimension 3: voids
- ...

Examples: Sphere



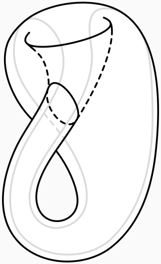
- Dimension: 1
- Components: 1
- Holes: 0
- Voids: 1
- Higher Dimensional Holes: 0

Examples: Torus



- Dimension: 1
- Components: 1
- Holes: 2
- Voids: 1
- Higher Dimensional Holes: 0

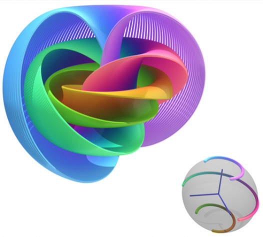
Examples: Klein Bottle



- Dimension: 1
- Components: 1
- Holes: 1
- Voids: 0
- Higher Dimensional Holes: 0

Examples: Sphere of 3 dimensions

The sphere of 3 dimensions of radius 1: $x^2 + y^2 + z^2 + w^2 = 1$.



- Dimension: 3
- Components: 1
- Holes: 0
- Voids: 0
- Holes of dimension 4: 1

Next Time

How can we represent a topological object on a computer?

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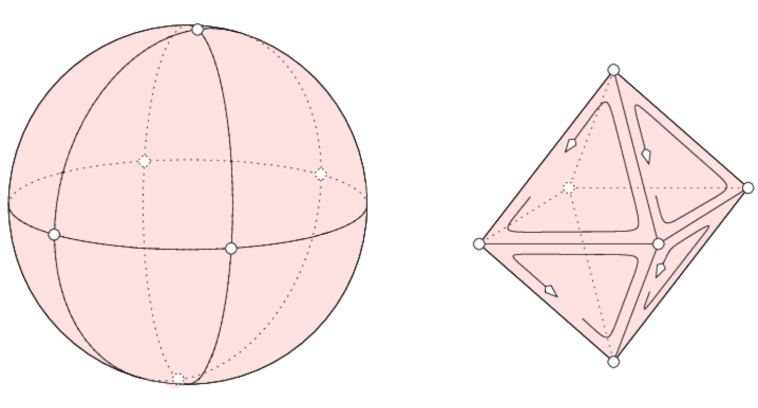
Simplices



Questions?