Topology and Combinatorics

Graphs & Simplices

Roderic Guigó Corominas

August 14th 2019

SMTB 2019

Recall

- Topological properties are unchanged under continuous deformations: bending, stretching, twisting and crumbling. No tearing or gluing.
- Components, holes and voids are topological properties.

Recall: Betti Numbers

The numbers of components and holes are called Betti numbers.

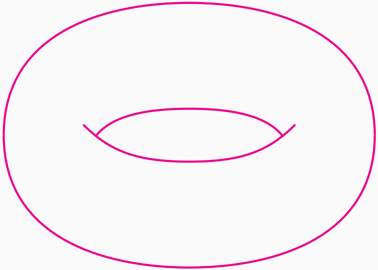
$b_0 = \#$ of components

$b_1 = \#$ of holes

$b_2 = \#$ of voids

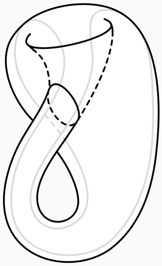
$\vdots$

Correction: Torus



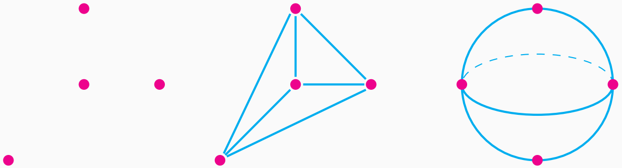
- Components: $b_0 = 1$
- Holes: $b_1 = 2$
- Voids: $b_2 = 1$
- Higher Dimensional Holes: 0

Correction: Klein Bottle



- Components: $b_0 = 1$
- Holes: $b_1 = 1$
- Voids: $b_2 = 0$. There isn't an interior or exterior!
- Higher Dimensional Holes: 0

Flow Chart



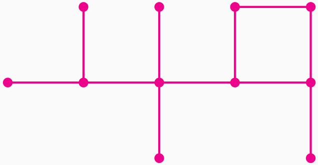
???

Data $\Rightarrow$ $\underbrace{\text{Topology}}_{\text{Today}}$ $\Rightarrow$ Properties $\Rightarrow$ Meaning

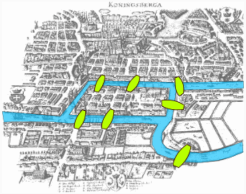
Graphs

Graphs

A graph is a set of vertices and edges between the vertices.

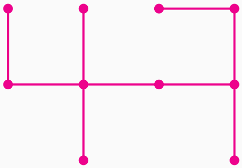


Used to model, one of the most famous problems is **Königsberg bridges**.

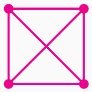


Types of Graphs

A graph with no loops is called tree.



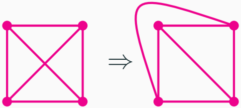
A graph where all vertices are connected to all other vertices is called complete.

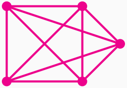


A graph is connected if it has only one component.

Planar Graph

A planar graph is a graph that can be drawn in the plane $\mathbb{R}^2$ without any crossings.

Planar Graphs 

Non Planar Graph 

Euler Characteristic

The Euler characteristic χ of a planar graph G is defined as

$$\chi(G) = \#\text{vertices} - \#\text{edges} + \#\text{faces}.$$

The outer face is also counted.

Euler Characteristic

Theorem: The Euler characteristic of a connected planar graph is always $\chi = 2$.

Proof: The proof is by induction, noticing that the following actions don't change the value of χ.

Euler Characteristic

Theorem: The Euler characteristic of a connected planar graph is always $\chi = 2$.

Proof: The proof is by induction, noticing that the following actions don't change the value of χ.

- Add a vertex and an edge. Adds 1 and subtracts 1 from χ, so the value remains constant.

Euler Characteristic

Theorem: The Euler characteristic of a connected planar graph is always $\chi = 2$.

Proof: The proof is by induction, noticing that the following actions don't change the value of χ.

- Add a vertex and an edge. Adds 1 and subtracts 1 from χ, so the value remains constant.
- Adding an edge between to existing vertices. This adds a face, again adding and subtracting 1 from χ.

Euler Characteristic

Theorem: The Euler characteristic of a connected planar graph is always $\chi = 2$.

Proof: The proof is by induction, noticing that the following actions don't change the value of χ.

- Add a vertex and an edge. Adds 1 and subtracts 1 from χ, so the value remains constant.
- Adding an edge between to existing vertices. This adds a face, again adding and subtracting 1 from χ.

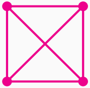
If the graph has n components, then $\chi = 1 + n$

Euler Characteristic for graphs

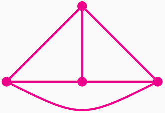
The number of holes b_1 of a planar graph is equal to the number of faces minus 1. We have the following relation

$$b_1 = \#\text{faces} - 1 = \#\text{edges} - \#\text{vertices} + 1.$$

Example:



$b_1 = 6 - 4 + 1 = 3$



This graph has 3 loops.

Graphs as surfaces

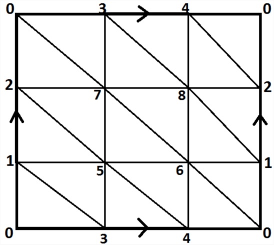
- A planar graph is a graph that can be drawn in a sphere. Has $\chi = 2$.
- What is the Euler characteristic of a graph in a torus?
- What about other surfaces?

A graph on a surface such that all the faces are triangles, then this is called a triangulation.

Graphs as surfaces

- A planar graph is a graph that can be drawn in a sphere. Has $\chi = 2$.
- What is the Euler characteristic of a graph in a torus?
- What about other surfaces?

A graph on a surface such that all the faces are triangles, then this is called a triangulation.

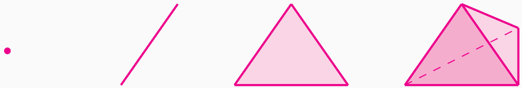


This is a graph on a torus, it has $\chi = 9 - 27 + 18 = 0!!!$

Simplicial Spaces

Standard Simplices

Standard simplices are the building blocks (pieces) of simplicial spaces.



They are defined by the equation

$$\{(x_0, \ldots, x_n) \in \mathbb{R}^{n+1} \mid x_0 + \cdots + x_n = 1, x_i \geq 0\}.$$

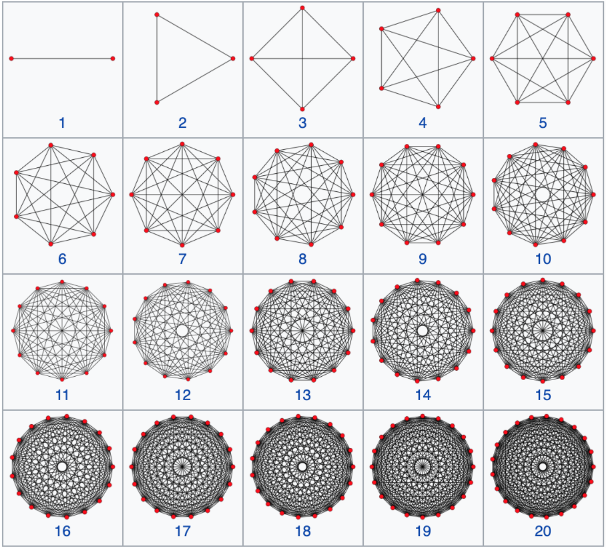
For example:

$$\{x_0 \in \mathbb{R} \mid x_0 = 1, x_0 \geq 0\}$$

$$\{(x_0, x_1) \in \mathbb{R}^2 \mid x_0 + x_1 = 1, x_0 \geq 0, x_1 \geq 0\}$$

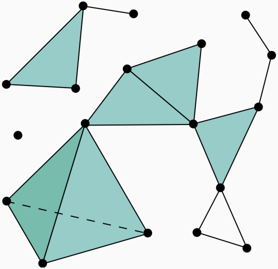
$$\{(x_0, x_1, x_2) \in \mathbb{R}^3 \mid x_0 + x_1 + x_2 = 1, x_0 \geq 0, x_1 \geq 0, x_0 \geq 0\}$$

Standard Simplices



Simplicial Spaces

A simplicial space (or simplicial complex) is a topological object that is made out of pieces that are standard simplices.



Claim: Every (nice) topological space is equivalent to a simplicial complex.

Easy to believe but hard to prove.

Topological Property: Euler Characteristic

Suppose that a simplicial space X is made out of n_i i-simplices. The Euler characteristic is defined as the sum

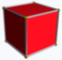
$$\begin{aligned}\chi(X) &= n_0 - n_1 + n_2 - n_3 + n_4 - n_5 + \dots \\ &= \sum_{i=0}^{d} (-1)^i n_i\end{aligned}$$

Euler Characteristic Standard Simplicies

Δ^n	Name	Schläfli Coxeter	0-faces (vertices)	1-faces (edges)	2-faces	3-faces	4-faces	5-faces	6-faces	7-faces	8-faces	9-faces	10-faces	Sum $= 2^{n+1} - 1$
Δ^0	0-*simplex* (point)	( )	1											**1**
Δ^1	1-*simplex* (line segment)	{ } = ( ) ∨ ( ) = 2 · ( )	2	1										**3**
Δ^2	2-*simplex* (triangle)	{3} = 3 · ( )	3	3	1									**7**
Δ^3	3-*simplex* (tetrahedron)	{3,3} = 4 · ( )	4	6	4	1								**15**
Δ^4	4-*simplex* (5-cell)	$\{3^3\}$ = 5 · ( )	5	10	10	5	1							**31**
Δ^5	*5-simplex*	$\{3^4\}$ = 6 · ( )	6	15	20	15	6	1						**63**
Δ^6	*6-simplex*	$\{3^5\}$ = 7 · ( )	7	21	35	35	21	7	1					**127**
Δ^7	*7-simplex*	$\{3^6\}$ = 8 · ( )	8	28	56	70	56	28	8	1				**255**
Δ^8	*8-simplex*	$\{3^7\}$ = 9 · ( )	9	36	84	126	126	84	36	9	1			**511**
Δ^9	9-simplex	$\{3^8\}$ = 10 · ( )	10	45	120	210	252	210	120	45	10	1		**1023**
Δ^{10}	10-*simplex*	$\{3^9\}$ = 11.( )	11	55	165	330	462	462	330	165	55	11	1	**2047**

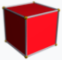
Sphere

In fact, we can relax the condition on simplices. Replace triangles by polygons.

Name	Image	Vertices *V*	Edges *E*	Faces *F*
Tetrahedron		4	6	4
Hexahedron or cube		8	12	6
Octahedron		6	12	8
Dodecahedron		20	30	12
Icosahedron		12	30	20

Sphere

In fact, we can relax the condition on simplices. Replace triangles by polygons.

Name	Image	Vertices V	Edges E	Faces F
Tetrahedron		4	6	4
Hexahedron or cube		8	12	6
Octahedron		6	12	8
Dodecahedron		20	30	12
Icosahedron		12	30	20

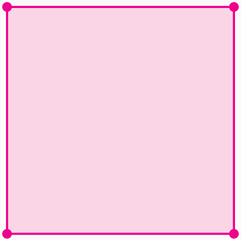
They all have Euler characteristic 2, all these are spheres (topologically).

Euler Characteristic

Claim: the Euler characteristic is a topological invariant. Any two simplices that are topologically equivalent have the same Euler characteristic.

Euler Characteristic and Betti Numbers

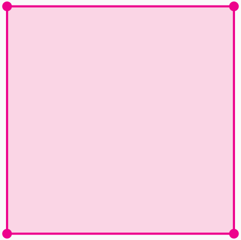
Remember the magic square? Use it to compute the Euler characteristic of some of the objects we constructed.



Compare it to the sum $\chi = b_0 - b_1 + b_2 - b_3 + \ldots$.

Euler Characteristic and Betti Numbers

Remember the magic square? Use it to compute the Euler characteristic of some of the objects we constructed.



Compare it to the sum $\chi = b_0 - b_1 + b_2 - b_3 + \ldots$. We have an alternative definition of the Euler characteristic in terms of Betti numbers!

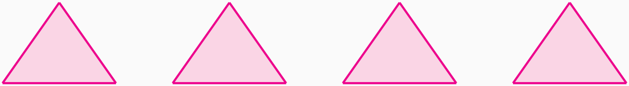
Computer representation

Computers can store the information of a simplicial complex in a very efficient way using lists.

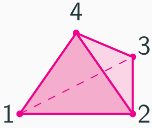
- List of vertices $[v_1, v_2, \dots, v_n]$.
- List of edges $[(v_1, v_2), (v_1, v_4), \dots]$.
- List of faces $[(v_1, v_2, v_3), (v_1, v_4, v_5), \dots]$.
- $\vdots$

Computer representation

A sphere can be contructed using four 2-simplices



- List of vertices $[1, 2, 3, 4]$.
- List of edges $[(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4)]$.
- List of faces $[(1, 2, 3), (1, 2, 4), (1, 3, 4), (2, 3, 4)]$.

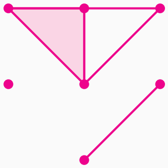


If we also include the 3-face $(1, 2, 3, 4)$, then we have the filled sphere.

Computer representation

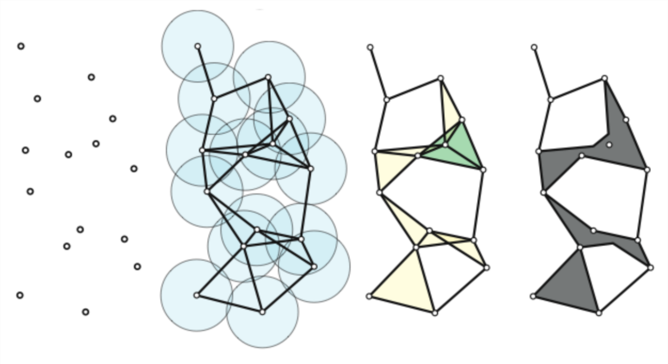
Lists and associated complex.

- $[1, 2, 3, 4, 5, 6, 7]$
- $[(2, 3), (4, 5), (3, 5), (3, 4), (4, 2), (5, 7)]$
- $[(2, 3, 4)]$



Next Time

Real Deal: Homology



Questions?