① 池化层反向传播. ⑴已知 pooling 的 δ^l，推导 δ^{l-1}

已知 pooling 的 δ^l，推导 δ^{l-1}

将 δ^l 的所有子矩阵还原回池化前大小 (unsample).

假设池化区 2×2， δ^l 第 k 个子矩阵为:

$$\delta_k^l = \begin{bmatrix} 2 & 8 \\ 4 & 6 \end{bmatrix}.$$

$\downarrow$ 还原大小 (unsample)

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & 8 & 0 \\ 0 & 4 & 6 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(1) 若为 average，则平均. 转换为:

$$\begin{bmatrix} 0.5 & 0.5 & 2 & 2 \\ 0.5 & 0.5 & 2 & 2 \\ 1 & 1 & 1.5 & 1.5 \\ 1 & 1 & 1.5 & 1.5 \end{bmatrix}$$

(2) 若为 max. 则为最大值位置还原(类似 Relu)

若原最大值分别位于 左上，右下，右上，左下

则
$$\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 8 \\ 0 & 4 & 0 & 0 \end{bmatrix}$$

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$$\delta^{l-1} = \text{unsample}(\delta^{l}) \odot \sigma'(z^{l-1})$$

$\sigma(z)$ 为池化层激活函数。

$\sigma(z) = z$

② 卷积层反向传播推导。

假设 $l-1$ 层的输出 a^{l-1} 是 3×3 矩阵，第 l 层卷积核 2×2，采用 1 像素步幅，则输出 z^{l} 为 2×2 矩阵。简化 $b^{l}=0$，则有：

$$a^{l-1} * W^{l} = z^{l}$$

$$\begin{bmatrix} a_{11}^{l-1} & a_{12}^{l-1} & a_{13}^{l-1} \\ a_{21}^{l-1} & a_{22}^{l-1} & a_{23}^{l-1} \\ a_{31}^{l-1} & a_{32}^{l-1} & a_{33}^{l-1} \end{bmatrix} * \begin{bmatrix} W_{11}^{l} & W_{12}^{l} \\ W_{21}^{l} & W_{22}^{l} \end{bmatrix} = \begin{bmatrix} z_{11}^{l} & z_{12}^{l} \\ z_{21}^{l} & z_{22}^{l} \end{bmatrix}$$

$$\Longrightarrow \begin{aligned} z_{11} &= a_{11}W_{11} + a_{12}W_{12} + a_{21}W_{21} + a_{22}W_{22} \\ z_{12} &= a_{12}W_{11} + a_{13}W_{12} + a_{22}W_{21} + a_{23}W_{22} \\ z_{21} &= a_{21}W_{11} + a_{22}W_{12} + a_{31}W_{21} + a_{32}W_{22} \\ z_{22} &= a_{22}W_{11} + a_{23}W_{12} + a_{32}W_{21} + a_{33}W_{22} \end{aligned}$$

令 $\nabla a^{l-1} = \frac{\partial J}{\partial a^{l-1}} = \frac{\partial J}{\partial z^{l}} \cdot \frac{\partial z^{l}}{\partial a^{l-1}} = \delta^{l} \frac{\partial z}{\partial a^{l-1}}$

$$\because \quad \nabla a_{11} = \frac{\partial J}{\partial z_{11}} \cdot \frac{\partial z_{11}}{\partial a_{11}} = \delta_{11} \cdot W_{11}$$

$$\nabla a_{12} = \frac{\partial J}{\partial z_{12}} \cdot \frac{\partial z_{12}}{\partial a_{12}} + \frac{\partial J}{\partial z_{11}} \cdot \frac{\partial z_{11}}{\partial a_{12}} = \delta_{12} \cdot W_{11} + \delta_{11} W_{12}$$

同理 $\nabla a_{13} = \delta_{12} \cdot W_{12}$

$$\nabla a_{21} = \delta_{11} \cdot W_{21} + \delta_{21} \cdot W_{11}$$

$$\nabla a_{22} = \delta_{11} \cdot W_{22} + \delta_{12} \cdot W_{21} + \delta_{21} \cdot W_{12} + \delta_{22} \cdot W_{11}$$

$$\nabla a_{23} = \delta_{12} \cdot W_{22} + \delta_{22} \cdot W_{12}$$

$$\nabla a_{31} = \delta_{21} \cdot W_{21}$$

$$\nabla a_{32} = \delta_{21} \cdot W_{22} + \delta_{22} \cdot W_{21}$$

$$\nabla a_{33} = \delta_{22} \cdot W_{22}$$

$$\Updownarrow$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \delta_{11} & \delta_{12} & 0 \\ 0 & \delta_{21} & \delta_{22} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} * \underbrace{\begin{bmatrix} W_{22} & W_{21} \\ W_{12} & W_{11} \end{bmatrix}}_{rot180(W^l)} = \begin{bmatrix} \nabla a_{11} & \nabla a_{12} & \nabla a_{13} \\ \nabla a_{21} & \nabla a_{22} & \nabla a_{23} \\ \nabla a_{31} & \nabla a_{32} & \nabla a_{33} \end{bmatrix}$$

② 已知conv的δ^l，推导δ^{l-1}

$$a^l = \sigma(z^l) = \sigma(a^{l-1} * W^l + b^l)$$

而 $z^{l+1} = a^l * W^{l+1} + b^{l+1} = \sigma(z^l) * W^{l+1} + b^{l+1}$

计算

$$\delta^{l-1} = \delta^l * rot180(W^l) \odot \sigma'(z^{l-1})$$

③ 已知conv的δ^l，推导$\frac{\partial J}{\partial W^l}$与$\frac{\partial J}{\partial b^l}$

$$z^l = a^{l-1} * w^l + b^l$$

$$\frac{\partial J}{\partial w^l} = \frac{\partial J}{\partial z^l} \cdot \frac{\partial z^l}{\partial w^l} = \delta^l \cdot \frac{\partial z^l}{\partial w^l}$$

$$= \delta^l * \mathrm{rot180}(a^{l-1})$$

(同上).

$$\frac{\partial J}{\partial b^l} = \sum_{u,v} (\delta^l)_{u,v}$$ (b为向量，而 δ^l 为高维张量，需将 δ^l 全部相加)。