Problems for QR Factorization

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1. *(Easy)*Consider the normal equation
$$X^T X\beta = X^T Y$$
where $X \in \mathbb{R}^{m\times n}, m \geqslant n$ and $Y \in \mathbb{R}^m$. Prove that the solution set of the equation is non-empty. *Hint: you may have to show that the ranks of the coefficient matrix and the augmented matrix are equal.*

2. *(Medium)*Review the Householder QR factorization and answer the following questions.
   (a) Let $X \in \mathbb{R}^{m\times n}, m \geqslant n$, then what's the complexity of Householder QR in terms of arithmetic operations? Only the highest degree terms are needed.
   (b) Let $X \in \mathbb{R}^{2n\times n}$, and be composed of two upper-triangular matrices, i.e.
   $$X = \begin{pmatrix} R_1 \\ R_2 \end{pmatrix}$$
   where R_1 and R_2 are upper-triangular. Write down an efficient algorithm to perform Householder QR factorization on X. What's the complexity of your algorithm? *Hint: you should make use of the special structure of X.*

3. *(Hard, Programming)* Practical Householder QR Factorization. In most numerical algebra softwares such as LAPACK, QR factorization is implemented in a slightly different way. Recall the definition of Householder transformation, $H = I - 2vv^T$, where $v \in \mathbb{R}^m$ is a vector and has unit Euclidean norm. Most softwares use a different definition of H instead of $H = I - 2vv^T$. They prefer $H = I - \tau uu^T$ where u is a vector in $\mathbb{R}^m$ and satisfies $u_1 = 1$, τ is a scalar called Householder multiplier. In practical QR, the Q factor is stored implicitly, by overwriting the strict lower trapezoidal part of X with the scaled Householder vector u_i. And the upper triangular part is overwritten by the R factor exactly. Note that the first element of u_i is 1 so there's no need for storing it. Also, an extra array of τ is needed to store those Householder multipliers. For example, if X has 5 rows and 4 columns, after QR factorization, X should be overwritten by
$$X = \begin{pmatrix} r_{11} & r_{12} & r_{13} & r_{14} \\ u_{21} & r_{22} & r_{23} & r_{24} \\ u_{31} & u_{32} & r_{33} & r_{34} \\ u_{41} & u_{42} & u_{43} & r_{44} \\ u_{51} & u_{52} & u_{53} & u_{54} \end{pmatrix}$$

Also, the τ array should be $(\tau_1, \tau_2, \tau_3, \tau_4)$.

(a) Derive the expression of τ and u such that $Hx = \alpha e_1$, where x is given and $e_1 = (1, 0, \ldots, 0)^T$.

(b) Implement the practical Householder QR.

(c) Suppose we need the Q factor explicitly, show how to obtain the explicit Q factor from the implicitly stored Householder vectors. Implement your algorithm.

4. *(Medium, Programming)*Consider the Hilbert matrix of order 20. Orthogonalize the matrix using Gram-Schmidt process and QR factorization respectively and check your results. Are these computed matrices numerically orthogonal?