(4) Limits

Limits of Functions [4.1]

Intuitively limits of functions are the expected value of a function at points that can't be solved because they are undefined, e.g.

$\frac{(x-2)\left(x+2 \right)}{\left(x-2 \right)}$ would be undefined at x=2, however as x is made sufficiently close to 2, that value will become arbitrarily close to 4.

The Limit Generally

From early calculus the limit of $f(x)$, as x approaches a was said to be some value L, denoted $\lim_{x\to a}(f(x))=L$

$$\begin{aligned} \forall \varepsilon > 0, \enspace, \exists \delta: & \notag \\ & \qquad 0 < \lvert x-a \rvert < \delta \implies \lvert f\left(x \right) - L \rvert < \varepsilon \label{stewartlimdef}\end{aligned}$$

Remarks on this Definition

Observe that the following statements are equivalent:

1. $x\neq c \enspace \wedge \enspace \lvert x-a \rvert < \delta$
2. $0<|xa|<$
3. $|xa|(0,)$

Notation

If L is a limit of f at c, then it is said that:

1. f converges to L at c
2. $f(x)$ approaches L as x approaches c This is sometimes expressed with the symbolism $f(x)\to L$ as $x\to c$

And the following notation is used

1. $\lim_{x\to c}(f(x))=L$

2. $\lim_{x \to c} f$

The Limit Using Cluster Points

In analysis we more or less use the same definition but we introduce the concept of cluster points to make it more rigorous.

Neighborhoods [2.2.7]

A neighborhood is an interval about a value, e.g. the -neighborhood of a is some set $V(a)$:
$$\begin{aligned} V_{\varepsilon}(a) = \left(\varepsilon-a, \varepsilon+a \right) &= \left\{ x : \varepsilon - a < x < \varepsilon + a \right\} \\ &= \left\{ x : -\varepsilon < x - a < \varepsilon \right\} \\ &= \left\{ x: \lvert x-a \rvert < \varepsilon \right\} \label{nebhdef}\end{aligned}$$

Cluster Points

Let c be a real number and let A be a subset of the real numbers, c may or may not be contained by A it doesn't matter.
Take some interval around c, or rather consider the -neighborhood of c,
if, some value (other than c) can be found inside that interval/neighborhood that is also inside A, regardless of how small that interval is made, Then c is said to be a cluster point of A.

i.e., if the following is true
$\forall \varepsilon > 0, \enspace \exists x\neq c \in A\cap V_{\varepsilon}(c)$
then c is said to be a cluster point of A.

It basically means that there are infinitely infinitesimal points between any point in A and the value c.

Example

- The point 4 of the set {3,4,5} is not a cluster point of that set because a 0.1-neighbourhood of 4 would be the set $V_{0.1}(4)$={4}, this set does not contain a value x4 that is also inside the original set.

- The point 6 of (1,6)={x:1<x<6} is a cluster point of (1,6) because no matter how small a neighborhood is made around 6, there will always be values x6 inside that interval that are also inside (1,6) also observe that in this case 6(1,6)

Definition of the Limit [4.1.4]

So this is the definition that we moreso use in this unit and the one to memorise (or the Neighborhoods one seems simpler to memorise).

Let A and let c be a cluster point of A.

Now take some function:
$$\begin{aligned} f: A \rightarrow \mathbb{R} \label{clustlimdeffunc} \end{aligned}$$
It is said that L is a limit of f at c if:
$$\begin{aligned} \forall \varepsilon > 0, &\enspace \exists \delta > 0: \notag \\ \qquad & \left(x \in A \enspace \wedge \enspace 0<\lvert x-c \rvert < \delta \right) \implies \lvert f\left(x \right) - L \rvert < \varepsilon \tag{4.1.4} \label{414} \end{aligned}$$

What's the Distinction

This is more or less the same as the typical definition given in early calculus ([stewartlimdef]), the distinction here is that we have specified that c must be a cluster point of A, this is more rigorous because c is always such that there are infinitely many values in any infinitesimal distance between intself and any xA, So the limit will always mean a continuous approach as we expect, this is just a more thorough definition.

Definition using Neigborhoods [4.1.6]

A value L is said to be the limit of f as $x \to c$, denoted $\lim_{x\to c}(f(x))$ if and only if:

2em0pt *For any* given -neighbourhood of L, $\enspace V_{\varepsilon}(L)$
There exists a -neighbourhood of c, $\enspace V_{\delta}\left(L \right)$

such that:

4em0pt =0.7cm *If* xc is in both A and $V(c)$
=0.5cm *Then* $f(x)$ must be within the neighbourhood $V(L)$

Formally

$$\begin{aligned} \forall \varepsilon > 0, & \enspace \exists \delta > 0: \notag \\ & \qquad x \neq c, \enspace x \in A\cap V_{\varepsilon}\left(L \right) \implies f\left(x \right) \in V_{\delta}\left(c \right) \tag{4.1.6} \label{416}\end{aligned}$$

Defintions ([416]) and ([414]) are equivalent, and are both consistent with the initial less rigorous definition ([stewartlimdef]).

Only one Limit Value [4.1.5]

If f:$A \rightarrow$ and c is a cluster point of A, then there is only one value L: $\lim_{x\rightarrow c}(f(x))=L$

Using Sequences to Define Limits [4.1.8]

Now that limits are defined we can use sequences to define them as well, this will give us more tools to use later and allows a connection to be made between material of Chapter 3 and 4.

Definition

A value L is said to be the limit of f as $x\rightarrow c$, denoted $\lim_{x\rightarrow c}(f(x))$ if and only if:

2em0pt *For every* sequence (x_n) in A,

4em0pt *if* (x_n) converges to c such that $x_n c$,

4em0pt *Then* $(f(x_n))$ converges to L

So basically, again, if x gets close to c, $f(x)$ gets close to L, but we took x from a sequence.

Divergence Criteria [4.1.9]

Now we can use the *Divergence Criteria* from [3.4.5] to determine whether or not a limit exists generally or at a point.

(a) Limit is not a Specific Value

If L, then f does not have a limit at c, if and only if:

There is a sequence (x_n) in A with $x_n c$, such that:
(x_n) converges to c but the sequence $f(x_n)$ does not converge to L

(b) No Limit whatsover

If L, then f does not have a limit at c, if and only if:

There is a sequence (x_n) in A with $x_n c$, such that:
(x_n) converges to c but the sequence $f(x_n)$ does not converge in

The Signum Function

The Signum function returns the sign of the input value:

$$\begin{aligned} sgn\left(x \right) &:= \begin{cases} +&1 \qquad \text{for} \enspace $x \textgreater \enspace 0$ \\ &0 \qquad \text{for} \enspace $x= 0$ \\ -&1 \qquad \text{for} \enspace $x \textless \enspace 0$ \end{cases} \tag{4.1.10} \label{4110} \\ &= \frac{x}{\lvert x \rvert} \notag\end{aligned}$$

Limit Theorems [4.2]

These are useful for calculating limits of functions, they are mostly extensions of [3.2].

Bounded Functions

Definition

Let A, $f{:}A \to$ and let c be a cluster point of A.
It is said that f *is bounded on a neighbourhood of* c if:

2em0pt there exists a -neighborhood $V(c)$ and some constant value $M{>}0$ such that:

4em0pt $|f(x)| M$ for every $x A V(c)$

So basically a function is said to be *bounded on a neighbourhood of* c if:

4em0pt for some interval (It doesn't matter how small) around c,

6em0pt $f(x)$ can be contained in some interval

4em0pt $\exists \delta>0, \enspace \exists M>0:$

6em0pt $x V(c) |f(x)|<M$

So for example:

- $f(x)=x^3$ is *bounded on every neighborhood of every* x whereas,
- $g\left(x \right) = \sfrac{1}{x}$ is ***not*** *bounded on a neighborhood of 0* because $g(x)$ tends to infinity as $x \to 0$,
  - furthermore $g(x)$ is *bounded on **some but notall** neighborhoods of 1*, because an interval around 1 must not be drawn large enough to encapsulate 0.

Limits imply Bounded Neighbourhoods [4.2.2]

A function is bounded on a neighborhood of a point that is a limit of that function.
If a function has a limit at c, then f must be *bounded on some neighborhood of* c,
this flows from the initial definitions because we know that c is a cluster point and that $(f(x))$ moves closer to L,
hence it must be possible to draw a small enough interval (e.g. horizontal lines on the y-axis) to contain all $f(x)$ defined by

Functions and Arithmetic [4.2.3]

Just like with sequences we can define arithmetic operations that relate to addition and multiplication with functions in order to manipulate them:

Let A ,
$$\begin{aligned} f: A \rightarrow \mathbb{R} \qquad g: A \rightarrow \mathbb{R} \qquad h: A \rightarrow \mathbb{R}, \enspace h(x) \neq 0, \enspace \forall x \in A \label{seqdefgen}\end{aligned}$$

We define the following Operations [4.2, p. 111]:

$$\begin{aligned} {1} \left(f+g \right) \left(x \right) &:= f\left(x \right) + g\left(x \right) \label{addfundef} \\ \left(f-g \right) \left(x \right) &:= f\left(x \right) + g\left(x \right) \label{subfundef} \\ \left(fg \right) \left(x \right) &:= f\left(x \right) \times g\left(x \right) \label{multfundef} \\ \left(bf \right) \left(x \right) &:= b \times f\left(x \right) \label{confundef} \\ \left(\frac{f}{h} \right)\left(x \right) &:= \frac{f\left(x \right)}{h\left(x \right)} \end{aligned}$$

Limits of Function Operations [4.2.4]

Because the limit of a function is essentially the expected value of the function around that value, it stands to reason that the limit will distribute over the basic operations:
Let the functions be defined as they were in ([seqdefgen]) and let c be a custer point of A.

$$\begin{aligned} \lim_{x \rightarrow c}\left(f \right) = L \qquad \lim_{x \rightarrow c}\left(g \right) = M \quad \lim_{x\rightarrow c}\left(h \right) = H \neq 0 \label{fundeflim}\end{aligned}$$

Then the limits are:

$$\begin{aligned} {2} \lim_{x\rightarrow c} \left(f+g \right) &= \lim_{x\rightarrow c}\left(f \right) + \lim_{x\rightarrow c}\left(g \right) &= L + M \label{addlimdef} \\[1em] \lim_{x\rightarrow c} \left(f-g \right) &= \lim_{x\rightarrow c} \left(f \right) - \lim_{x\rightarrow c} \left(g \right) &= x - y \label{minlimdef} \\[1em] \lim_{x\rightarrow c} \left(c \cdot f \right) &= c \cdot \lim_{x\rightarrow c} \left(f \right) &= c \cdot x \label{conmultlimdef} \\[1em] \lim_{x\rightarrow c} \left(f\times g \right) &= \lim_{x\rightarrow c} \left(f \right) \times \lim_{x\rightarrow c} \left(g \right) &= x \times y \label{multlimdef} \\[1em] \lim_{x\rightarrow c} \left(f/h \right) &= \lim_{x\rightarrow c} \left(f \right) \div \lim_{x\rightarrow c} \left(h \right) &= \sfrac{x}{y} \label{divlimdef}\end{aligned}$$

Limit Theorems

The rest of the chapter just provides values of varios limits.
Let the functions be defined as they were in ([seqdefgen]) and let c be a custer point of A.

Limits Captured in Intervals [4.2.6]

if $f(x)[a,b]$ for all $x \in A, \enspace x \neq c$, and $\lim_{x\rightarrow c}(f)$ exists,

2em0pt *then* $f(x)[a,b]$

Squeeze Theorem [4.2.7]

if [4.2.6] is extended to functions, then we have the squeeze theorem:
if g is within an interval defined by the functions f and h:

$$\begin{aligned} f\left(x \right) \leq g\left(x \right) \leq h\left(x \right),
\quad \forall x\in A, \enspace x\neq c \label{squeezelimdist}\end{aligned}$$

2em0pt *then* the limit of g must also be 0

$$\begin{aligned} \lim_{x\rightarrow c}\left(g \right)=L \label{limisL}\end{aligned}$$

A Positive Limit implies a neighbourhood with Positive Values

Let A and let c be a cluster point of A as in (3.4.6) above.
If:
$$\begin{aligned} \lim_{x\rightarrow c}\left(f \right) > 0 \label{limneighbourpor}\end{aligned}$$

Then:

4em0pt there is a neighborhood $V(c)$ such that $f\left(x \right) >0, \enspace
\forall x \in A \cap V_{\delta}\left(c \right)$

This also holds for negative values and basically all it says, in more rigorous language, is that if the limit point is above the x-axis then there's gotta be points to the left and right that are above the x-axis as well (because the whole cluster point thing means everything can be arbitrarily small).

Although this may start to seem a little pointless, the idea of making the definitions this rigorous is like writing code in a scripting language, by using this very precise language, the logical consequences give us exactly the concept that we want, even though we need to take a longer or alternate path to get to that concept than we would otherwise would generally take in order to describe the concept.

Extensions of the Limit Concept [4.3]

These are written in a particularly convoluted fashion, however if the preceeding material is understood the textbook can be used more or less as a reference, hence these notes will be brief.

One-Sided Limits [4.3.1]

Definition [4.3.1]

Let c be a cluster point of $A\cap \left(c, \infty \right) = \left\{ x \in A
\enspace : \enspace x > c \right\}$
It is said that L is a *Right-hand limit of* f *at* c and it is written:

$$\begin{aligned} \lim_{x \rightarrow c^+} \left(f \right) = L \tag{4.3.1} \label{431}\end{aligned}$$

This can be extended to left-hand limits as well.

Definition in Term of Sequences [4.3.2]

As above it is said that L is a *Right-hand limit of* f *at* c if:

2em0pt Every sequence (x_n) in A that converges to c is such that $f(x_n)$ converges to L, given that $x_n>c, \enspace \forall n \in \mathbb{N}$

Limit must be equal on both sides

A limit is defined only if the limit is equal from both directions $$\begin{aligned} \lim_{x\rightarrow c}\left(f \right) = L \iff \lim_{x\rightarrow c^+}\left(f \right) = L = \lim_{x\rightarrow c }\left(f \right) \tag{3.4.3} \label{343}\end{aligned}$$

Infinite Limits [4.3.5]

Let c be a cluster point of A,
It is aid that f tends to as $x \rightarrow c$, and it is written:

$$\begin{aligned} \lim_{x\rightarrow c}\left(f \right) = \infty \tag{4.3.5} \label{435}\end{aligned}$$

If , $\enspace \exists \delta > 0$:

6em0pt $0<|xc|<f(x)>,xA$

One-Sided Limits to Infinity [4.3.8]

Let c be a cluster point of $A\cap \left(c, \infty \right) = \left\{ x \in A \enspace : \enspace x > c \right\}$,
It is aid that f tends to as $x \rightarrow c^+$, and it is written:

$$\begin{aligned} \lim_{x\rightarrow c}\left(f \right) = \infty \tag{4.3.8} \label{438}\end{aligned}$$

If , $\enspace \exists \delta > 0$:

6em0pt $0<xc<f(x)>,xA$

Ordered Functions

If $f(x)<g(x)$, then:
$$\begin{aligned} \lim_{x\rightarrow c}\left(f \right) = \infty &\implies \lim_{x \rightarrow c}\left(g \right) = \infty \tag{4.3.7 (a)} \label{437a}\\ \lim_{x\rightarrow c}\left(g \right) = -\infty &\implies \lim_{x \rightarrow c}\left(f \right) = -\infty \tag{4.3.7 (b)} \label{437b}\end{aligned}$$

Limits at Infinity [4.3.10]

It is also useful to talk about limits as x tends to

Let $(a,)A$ for some ain
It is aid that the limit of f as $x\rightarrow$ is L, and it is written:
$$\begin{aligned} \lim_{x\rightarrow \infty}\left(f \right) = L \tag{4.3.10} \label{4310}\end{aligned}$$

If $ \forall \varepsilon >0, \enspace \exists K > 0$:

6em0pt $x>K|f(x)L|<$

Limits at Infinity in Terms of Sequences [4.3.11]

equivalently to ([4310]), the definition can be expressed in terms of sequences:

2em0pt Every sequence (x_n) in $A(a,)$ that has $\lim(x_n)=$ is such that the sequence $(f(x_n))$ converges to L

Infinite Limits at Infinity

So this basically combines [4.3.10] with [4.3.5]

Let $(a,)A$ for some a

It is aid that f tends to as $x\rightarrow$, and it is written:
$$\begin{aligned} \lim_{x\rightarrow \infty}\left(f \right) = \infty \tag{4.3.13} \label{4313}\end{aligned}$$

If $ \forall \varepsilon >0, \enspace \exists K > \alpha$:

6em0pt $x>Kf(x)>$

Infinite Limits at Infinity in Terms of Sequences [4.3.14]

equivalently to ([4313]), the definition can be expressed in terms of sequences:

2em0pt Every sequence (x_n) in $A(a,)$ that has $\lim(x_n)$= is such that the limit of the sequence of function values $\lim(f(x_n))$=

Ratios of Functions

This result uses (4.3.14) to restate (3.6.5) in terms of functions:
If $g\left(x \right) > 0 \enspace \forall x > a$ and $L0$ is defined:

$$\begin{aligned} lim_{x\rightarrow \infty}\left(\frac{f\left(x \right)}{g\left(x \right)} \right) \tag{4.3.15} \label{4315}\end{aligned}$$

then,

$$\begin{aligned} L > 0 \implies \lim_{x\rightarrow \infty}\left(f \right) = \infty \iff \lim_{x \rightarrow \infty \left(g \right) = \infty} \tag{4.3.15 (i)} \label{4315i} \\ L < 0 \implies \lim_{x\rightarrow \infty}\left(f \right) = -\infty \iff \lim_{x \rightarrow \infty \left(g \right) = \infty} \tag{4.3.15 (ii)} \label{4315ii}\end{aligned}$$