Deriving the Normal Distribution

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1 Power Series and Uniform Continuity

1.1 Power Series

1.1.1 Convergence

A sequence $x = (x_n)$ converges if:

$$\begin{aligned}&\forall \varepsilon > 0, \quad \exists N : \\ &\qquad n \geq N \implies |x_n - x| < \varepsilon\end{aligned}$$

and it is hence expressed:

$$\lim(x_n) = x$$

A series is generated by a sequence,
If (a_n) is a sequence, the series is (S_n):

$$\begin{aligned}S_1 &= a_1 \\ s_2 &= S_1 + a_2 \\ S_3 &= S_2 + a_3 \\ S_4 &= S_3 + a_4 \\ &\ldots\end{aligned}$$

The series is convergent if:

$$\forall \varepsilon > 0, \exists N : \\ n \geq N \implies |S_n - L| < \varepsilon$$

The series is absolutely convergent if $|S_n|$ is convergent.
The notation for series used is:

$$\sum_{n=1}^{\infty} [(a_n)] = \sum (a_n) = \lim [S_n]$$

Be mindful that this notation is used ambiguously to represent both:

- The infinite Series
- The limit value of the series.

In practice however the ambiguity is a non-issue because context will discern the difference.

1.1.2 Sequences of Functions[1]

We can have sequences of real numbers, and similarly we can have sequences of functions.
Sequences of functions can converge in two ways:

- Pointwise
- uniformly

Uniform convergence is important because it preserves term properties to the limit function which will be seen.

Define a sequence of functions: Let $A \subseteq \mathbb{R}, n \in \mathbb{N}$ and take some function f :

$$f_n : A \to \mathbb{R}$$

It is said that (f_n) is a sequence of functions on A to $\mathbb{R}$.
For every $x \in A$ there will be a sequence of real numbers:

$$f_n (x)$$

For some values of x, the sequence may converge, for others it may not.

- The point of convergence is $\lim \_\{\} [f_n (x_n)]$ which depends on the choice of x.
- Thus we could Create a set of all $x \in A$ forwhich $(f_n (x))$ converges.
  - This set would be a domain for a function $f(x)$ that would act as the limit of the sequence $(f\_n (x))$.

[1]This is in the Bartle and Sherbert Textbook at Chapter [8.1.7] p. 246

Pointwise Convergence Take some function:

$$f : A_0 \to \mathbb{R} \qquad (A_0 \subseteq A \subseteq \mathbb{R})$$

We say that the sequence is pointwise convergent if,

- for every $x \in A_0$
  - The sequence $(f\_n\,(x))$ converges to $f\,(x)$.

e.g. consider $g_n\,(x) := x^n$;

$$(g_n\,(x)) = \left(x, x^2, x^3, x^4 \cdots\right)$$

If $-1 < x < 1$, then x is a fraction or zero so:

$$\left(x, x^2, x^3, x^4 \cdots 0\right) \qquad \text{Converges to } 0$$

if $x = 1$, then:

$$\left(x, x^2, x^3 \cdots\right) = \left(1, 1^2, 1^3, \cdots 1\right) \qquad \text{Converges to } 1$$

if $x = -1$, then:

$$\left(x, x^2, x^3, \cdots\right) = (1, -1, 1, -1, \cdots \pm 1) \qquad \text{Divergent}$$

if $|x| > 1$, then:

$$\left(x, x^2, x^3, \cdots \infty\right) \qquad \text{Divergent}$$

So lim _{} $[g_n\,(x)]$ on the set $(-1, 1]$ where:

$$g\,(x) = \begin{cases} 0, & \text{for} \quad (-1 < x < 1) \\ 1, & \text{for} \quad (x = 1) \end{cases}$$

Definition An alternative, but equivalent definition for pointwise convergence is:

$$\forall \varepsilon > 0,\ \ \forall x \in A_0,\ \ \exists N : \tag{1}$$
$$\tag{2}$$
$$n \geq N \implies |f_n\,(x) - f\,(x)| < \varepsilon \tag{3}$$

Where N is a function of x and ε, i.e.:

- $N = N\,(\varepsilon, x)$.

###Definition of Power Series A power series is a series of the form:

$$\sum_{n=0}^{\infty}[c_n x^n] = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots \tag{4}$$

More generally a series will be of the form:

$$\sum_{n=0}^{\infty}[c_n (x-a)^n] = c_0 + c_1 (x-a) + c_2 (x-a)^2 + \cdots \tag{5}$$

Convergence of Power Series For any given power series of the form $\sum\{\infty\}\_\{n=0\}\left[c_n (x-a)^n\right]$ there are only three possibilities:

1. The series converges only when $x = a$
2. The series converges for all x.
3. There is a positive number R such that the series converges if $|x-a| < R$ and diverges if $|x-a| > R$.

Why power Series The whole idea of power series is representing a known function as an infinite series, this is useful for integrating functions that don't have elementary antiderivatives.

Take for example the geometric series:

$$\sum_{n=0}^{\infty}[a x^n] = \frac{a}{1-x} \tag{6}$$

$$\implies f(x) = \frac{a}{1-x} = \sum_{n=0}^{\infty}[a x^n] \tag{7}$$

Example Find a power series representation for $f(x) = \frac{x^3}{x+2}$

$$\frac{1}{2+x} = \frac{1}{2\left(1+\frac{x}{2}\right)} \tag{8}$$

$$= \frac{1}{2\left(1-\left(-\frac{x}{2}\right)\right)} \tag{9}$$

$$= \frac{1}{2} \cdot \sum_{n=0}^{\infty}\left[-\frac{x}{2}\right]^n \tag{10}$$

$$= \sum_{n=0}^{\infty}\left[\frac{(-1)^n}{2^{n+1}} \cdot x^n\right] \tag{11}$$

$$\implies \frac{x^3}{2+x} = \sum_{n=0}^{\infty}\left[\frac{(-1)^n}{2^{n+1}} \cdot x^{n+3}\right] \tag{12}$$

because this is a geometric series, it converges when:

$$\left|-\frac{x}{2}\right| < 1 \tag{13}$$

$$\implies x \in (-2, 2) \tag{14}$$

$$\text{This is known as the radius of convergence } (R = 2) \tag{15}$$

So from this we could show something like:

$$f(x) = \frac{x^3}{2+x} = \sum_{n=0}^{\infty}\left[\frac{(-1)^n}{2^{n+1}} \cdot x^{n+3}\right] \tag{16}$$

$$\int \frac{x^3}{2+x}\,\mathrm{d}x = C + \sum_{n=0}^{\infty}\left[\frac{(-1)^n}{2^{n+1}} \cdot \frac{x^{n+4}}{n+4}\right] \tag{17}$$

The important part with all of this is that it works with Taylor Series.[2]

1.1.3 Convergence of Power Series

if $\sum\{\infty\}\text{\_}\{n=0\}\left[a_n \cdot (z-z_0)^n\right]$ is convergent for some $z=\alpha$ then, * It will converge absolutely for all values of $|z - z\text{\_}0| > |z-\alpha|$

1.1.4 Uniform Convergence of Power Series

if $\sum\{\infty\}\text{\_}\{n=0\}\left[a_n \cdot (z-z_0)^n\right]$ converges when $z=\alpha$ but $(\alpha \neq z_0)$: * Then the series converges uniformly in any open-neighbourhood:

$$|z - z_0| \leq r$$

$$\text{Where } \ r = |z_0 - \alpha|$$

- The sum of the series represents an analytic function, i.e.:

$$f(z) = \sum_{n=0}^{\infty}\left[a_n \cdot (z-z_0)^n\right]$$

Such that $f(z)$ is an analytic function that can also be represented by the power series

1.1.5 Circle of Convergence

$\sum\{\infty\}\text{\_}\{n=0\}\left[a_n \cdot (z-\alpha)^n\right]$ is convergent only for $|z-\alpha| < R$:

- if $R=0$, the series is only convergent for $z=\alpha$
- if $r=\infty$, the series is convergent $\forall z \in \mathbb{C}$
- if $R \in \mathbb{R}^+$, the series is convergent on some open disc centred at α of radius R.

1.1.6 Taylor Series[3]

$$f(z) = \sum_{n=0}^{\infty}\left[a_n \cdot (z-\alpha)^n\right], \qquad |z-z_0| < r$$

$$\text{where: } \ a_n = \frac{f^{(n)}(z_0)}{n!}$$

This can be shown using the *Cauchy Integral Formula*, however a clearer justification is:

- If a function a power series representation for a function exists:

[2] Refer to Ch. 58, p. 190 of *Churchill's Complex Variables* 9thEd.
[3] *3Blue1Brown* does a nice video on this If a function is analytic at α and in an open disc $|z-\alpha| < R$, there will always be a power series representation of $f(z)$:

$$\begin{aligned}
f(z) &= \sum_{n=0}^{\infty} \left[c_n \cdot (z-\alpha)^n\right] \qquad {\scriptstyle C_n \text{ and } \alpha \text{ are complex constants}} \\
&= c_0 + c_1 (z-\alpha) + c_2 (z-\alpha)^2 \cdots \\
\implies f(\alpha) &= c_0 \times 0^0 + c_1 \times 0 + 0 + \cdots \\
&= c_0
\end{aligned}$$

Consider the first derivative:

$$\begin{aligned}
f'(z) &= c_1 + 2 \cdot c_2 (z-\alpha) + 3 \cdot c_2 \cdot (z\alpha)^2 \cdots \\
\implies f'(\alpha) &= c_1
\end{aligned}$$

Consider the second derivative:

$$\begin{aligned}
f''(z) &= 2 \cdot c_2 + 3 \times 2 \cdot (z-\alpha) + 4 \times 3 \times 2 \cdot (z-\alpha)^2 + \cdots \\
&= 2! c_2 + 3! (z-\alpha) + 4! \cdot (z-\alpha)^2 + \cdots \\
f''(\alpha) &= 2c_2
\end{aligned}$$

By this logic the nth derivative will be:

$$\begin{aligned}
f^{(n)} &= n! \cdot c_n \\
\implies c_n &= \frac{f^{(n)}(\alpha)}{n!}
\end{aligned}$$

We will always be able to find c_n where $f(z)$ is analytic and clearly the power series will be convergent (i.e. to $f(z)$) on a radius of convergence where $f(z)$ is analytic, so:

$$f(z) = \sum_{n=0}^{\infty} \left[\frac{f^{(n)}(\alpha)}{n!} \times (z-\alpha)\right], \qquad |z - z_0| < R$$

R is the radius of the open disc of analyticity

1.1.7 The Cauchy Hadamard Theorem If we have a power series:

$$\sum_{n=0}^{\infty} \left[a_n \cdot (z-z_0)^n\right]$$

Then we can find the radius of convergence:

$$l = \limsup \left[|a_n|^{\frac{1}{n}}\right]$$

$$\implies R = \frac{1}{l}$$

Further if * $|a\_n| \neq \_0$ and $\lim \left[\left|\frac{a_{n+1}}{a_n}\right|\right] = R$ Then: * $\lim \left[|a_n|^{\frac{1}{n}}\right] = \lim \left[\left|\frac{a_{n+1}}{a_n}\right|\right] = R$

Generally the n^{th} root test is more powerful than the ratio test, however the ratio test is the only test that can deal with factorials[4], so it is important to have it in our toolbox.

[4] How to choose a test by pauls Online Notes

Figure 1: drawing

Figure 2: drawing2

1.2 Uniform Continuity

1.2.1 What is uniform continuity

Imagine the function $y = \frac{1}{x}$ and consider the interval $(0, \infty)$, * Consider the limit value of 2 (e.g. the line $y = 2$)

If I choose some ε value, there is always a corresponding δ-value, this δ-value will depend on the ε-value:

so the function $f(x) = \frac{1}{x}$ is continuous because: * ε can be chosen anywhere * Any ε value will have a corresponding δ value.

The function would be uniformly continuous, if: * ε can be chosen anywhere * The Corresponding δ value will exist AND not change size wherever ε is chosen.

So in this case the function is not uniformly continuous, because if I move ε down, δ would have to get larger, so it IS NOT uniformly continuous:

So basically:

- A function is continuous if a δ-value always exists and can be described as a function:
  - $\delta = \delta(\varepsilon, x)$
- A function is uniformly continuous if a δ-value always exists and can be described as a function only of ε :
  - $\delta = \delta(\varepsilon)$

1.2.2 Cantor's Theorem

If a function is continuous on an interval $[a, b]$, it is uniformly continuous on that interval.

- The reasoning being that basically you could chose the smallest δ-value that will work at all points on that interval

1.2.3 Why is uniform continuity important?

Something like,

- if $f(x)$ is uniformly continuous
- Then:

$$\int \lim [f(x)] \, \mathrm{d}\, x \iff \lim \left[\int f(x) \, \mathrm{d}\, x \right]$$

  but I'm not sure about this so don't quote me, we don't do it here so don't worry about it too much, we just need to show that we understand it the idea of uniformly continuous functions.

Figure 3: drawing3

Problem Example

Prove that the function $f(x) = \frac{1}{1+x^3}$ is uniformly continuous on the interval $[1, \infty)$

So the first thing to notice is that *Cantor's Theorem* cannot be applied because it is an open interval.
State the Definition $f(x)$ is uniformly convergent if:

$$\begin{aligned}&\forall x, y \in [1, \infty),\ \ \forall \varepsilon > 0,\ \exists \delta(\varepsilon) : \\ &\qquad 0 < |x - y| < \delta \implies |f(x) - f(y)| < \varepsilon\end{aligned}$$

Work Backwards from the ε Definition

$$\begin{aligned}|f(x) - f(y)| &= \left| \frac{1}{1+x^3} - \frac{1}{1+y^3} \right| \\ &= \frac{\left|y^3 - x^3\right|}{|1+x^3| \times |1+y^3|} \\ &\le \frac{\left|y^3 - x^3\right|}{|1+x^3|} \\ &= \frac{|y-x| \cdot \left|y^2 + xy + x^2\right|}{|1+x^3|}\end{aligned}$$

Without loss of generality assume that $y < x$

$$\le |y - x| \times 3 \cdot \frac{\left|x^2\right|}{|1+x^3|}$$

Recall that $x \ge 1 \implies \frac{1}{|1+x^3|} < \frac{1}{|x|^3}$

$$\begin{aligned}&\le |y-x| \cdot \frac{1}{|x|} \cdot 3 \\ &\le |y-x| \cdot 3 \\ &\le 3 \cdot \delta\end{aligned}$$

So choose δ:

$$\begin{aligned}3\delta &\le \varepsilon \\ \delta &\le \frac{\varepsilon}{3}\end{aligned}$$

$\therefore$ it is sufficient to choose $\delta \le \frac{\varepsilon}{3}$.
Proof

$$\forall x, y \in [1, \infty),\ \ \forall \varepsilon > 0,\ \exists \delta \le \frac{\varepsilon}{3}$$

$$
\begin{aligned}
|x-y|<\delta \implies |f(x)-f(y)| &\leq \left|\frac{1}{1+x^3}-\frac{1}{1+y^3}\right| \\
&\leq |y-x|\cdot\left(\frac{\left|y^2\right|+|xy|+\left|x^2\right|}{|1+x^3|\cdot|1+y^3|}\right) \\
&\leq |y-x|\cdot\left(\frac{y^2}{|1+y^3|}+\frac{|x|\cdot|y|}{|1+x^3|\cdot 1+y^3}+\frac{\left|x^2\right|}{|1+x^3|}\right) \\
leq\, &|y-x|\cdot\left(\frac{|y|^2}{|y^3|}+\frac{|x|\cdot|y|}{|x|^3\cdot|y|^3}+\frac{|x|^2}{|x|^3}\right) \\
&\leq |y-x|\cdot\left(\frac{1}{|y|}+\frac{1}{|x|\cdot|y|}+\frac{1}{|x|}\right) \\
&\leq |y-x|\cdot\left(\frac{1}{1}+\frac{1}{1\times 1}+\frac{1}{1}\right) \\
&\leq |y-x|\cdot 3 \\
&< 3\cdot\delta \\
&< 3\times\frac{\varepsilon}{3} \\
&< \varepsilon
\end{aligned}
$$

□