Analysis

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Abstract

These are the notes that I took in analysis, my working for class tests and working for tutorial problems, the latter two need to be typed it.

I didn't cover anything up to and beyond Complex Series in Churchill's book, that's a shame and definitely something I want to get around to.

Part I

Lecture Material

Chapter 1

Introduction

1.1 [2.3] Completeness Property of the Reals

1.1.1 Upper and Lower Bounds [2.3.1]

Upper Bound An upper bound is any value greater than or equal to all elements of a set, e.g. u is an upper bound of A if:

$$\forall s \in S, \exists u \in \mathbb{R} : u \geq s \tag{1.1}$$

Lower Bound A lower bound is any value less than or equal to all elements of a set, e.g. w is a lower bound of A if:

$$\forall s \in S, \exists w \in \mathbb{R} : w \leq s \tag{1.2}$$

1.1.2 Supremea and Infima [2.3.2]

Supremum The suprema of a set is the smallest upper bound value of some set.
This value would be the maximum value of the set if the set had a maximum value.
Let V be the set of all upper bound values, u is a suprema iff:

$$u \leq v, \forall v \in V \tag{1.3}$$

Infimum The infimum of a set is the largest lower bound value of some set.
This value would be the maximum value of the set if the set had a maximum value.
Let T be the set of all upper bound values, w is a suprema iff:

$$w \leq t, \forall t \in T \tag{1.4}$$

Chapter 2

Sequences and Functions

2.1 Sequences and Their Limits

A sequence is a type of function that maps from $\mathbb{N} = \{1, 2, 3, \dots\}$ into $\mathbb{R}$

Such that the range is contained in some set S, e.g.

$$S = \left\{1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \dots\right\} \tag{2.1}$$

In thise case (2.1) is a the set of range values of a sequence, this sequence could be described by the function:

$$X : \mathbb{N} \to \mathbb{R} : x \mapsto \frac{1}{2^x} \tag{2.2}$$

Remarks on Sequences Unlike a set, a sequence can have repeated elements and the order of elements does matter.
Sequences are infinite because they are a function from $\mathbb{N}$ to $\mathbb{R}$.
A sequence could be defined to be finite (simply by restricting the domain to an interval or subset of $\mathbb{N}$), however they are defined to be infinite, by nature of their descriptive function, because it is useful to later studies of functions and series (a series is different from a sequence).

2.1.1 Notation

This (2.1) would usually be denoted by the notation:

$$x_n = \frac{1}{2^n} : n \in \mathbb{N} \tag{2.3}$$

However such a sequence can commonly be denoted also:

$$X, \quad \text{or} \quad (x_n), \quad \text{or} \quad (x_n : n \in \mathbb{N}) \tag{2.4}$$

Ordered Sequences Ordered Sequences are denoted with parentheses, e.g.

$$((-1)^n, n \in \mathbb{N}) = (-1, 1, -1, 1, -1, 1, \dots) \tag{2.5}$$

Unordered Sets Unordered Sets are denoted with cages/braces and represent the set of range values of a sequence:

$$\{(-1)^n, n \in \mathbb{N}\} = \{1, -1\} \tag{2.6}$$

Be careful because (2.5) and (2.6) are different things, one is a sequence of ordered values which could be described by a function, the latter is just the range of such a function.

2.1.2 Defining a Sequence

A sequence can be defined by listing the ordered terms until the rule of formation becomes clear or by specifying the formula, both are correct, so for (2.1)

$$X := \left\{1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \dots\right\} = \left\{2^{-n} : n \in \mathbb{N}\right\} \tag{2.7}$$

2.1.3 Limits of a sequence

An informal analogy for the limit of a sequence is:

The limit of $x := (x_n : n \in \mathbb{N})$ is the expected value of x_∞.

The limit value is usualy denoted $\lim(X) = \lim(x_n) = x$

The Precise Definition of a Limit is:

$$\begin{aligned} &\forall \varepsilon > 0, \ \exists K : \\ &\qquad n \geq k \Rightarrow \mid x_n - x \mid < \varepsilon \end{aligned} \tag{2.8}$$

Which says; If for all possible positive values of ε, there is some k value such that if $n \geq K$ then x_n is within the ε-neighborhood of the limit value x.

Limits are Unique a sequence will only approach a single limit value.

Equivalent Limit Definitions

1. X converges to x
2. $\forall \varepsilon > 0, \ \exists K : n \geq K \Rightarrow \mid x_n - x \mid < \varepsilon$
3. $\forall \varepsilon > 0, \ \exists K : n \geq K \Rightarrow (x - \varepsilon) < x_n < (x + \varepsilon)$
4. for every ε-neighborhood $V_\varepsilon(x)$, there exists some $K \in \mathbb{N}$:
   $\forall n \geq K, \ \ x_n \in V_\varepsilon(x)$

The Tail of a Sequence [3.1.9] or the m-tail of a sequence is the sequence starting from the m^{th} term, a tail will aproach the same limit as the original sequence because they will both have the same expected value for x_∞

2.2 [3.2] Limit Theorems

2.2.1 Intervals [2.5]

An **Open Interval** is defined:

$$(a, b) := \{x \in \mathbb{R} : a < x < b\} \tag{2.9}$$

A **Closed Interval** is defined:

$$[a, b] := \{x \in \mathbb{R} : a \leq x \leq b\} \tag{2.10}$$

2.2.2 Bounded Sequences [3.2.1]

If $x_n \in [-M, M]$, for some real positive M, then x_n is said to be bounded.

i.e. if the set of range values of a sequence $\{x_n : n \in \mathbb{N}\}$ is a bounded set then the sequence is said to be a bounded sequence

Bounded Subsets [2.3.1] are subsets of the real numbers that have a maximum and a minimum, Take for example:

$$\{1, 2, 3, 4, \dots 25, 50, 53, 54\} \tag{2.11}$$

$$\{p : p = 2n, n \in \mathbb{Z}^+\} \tag{2.12}$$

In the example above (2.11) is bounded because the minimum is 1 and the maximum value is 54, also notice that (2.11) is not a continuous subset of $\mathbb{R}$, it jumps from 25 to 50 and it doesn't include any quotient values, it is still however a bounded subset of $\mathbb{R}$.

However (2.12) is not a bounded subset because it does not have an upper bound value, it does have a lower bound, anything less than 1, but that means it is only bounded below and it is not a bounded subset of $\mathbb{R}$.

Bounded Subsets and Convergence [3.2.2] If a subset is Convergent it must be bounded because it has a starting value and it approaches another value, e.g.

$$\{1, 1/2, 1/4, 1/8, 1/16, 1/32 \dots\} \tag{2.13}$$

This set (2.13) converges to 0 and starts from 1, so it must have an upper bound of 1 (because all $x_n \leq 1$, $x_n \geq 0$ (it is $\leq$ / $\geq$ not $<$ / $>$)

If a subset is bounded however it doesn't necessarily need to be convergent, e.g.:

$$(1, 2, 1, 2, 1, 2, 1, \dots) \tag{2.14}$$

This set (2.14) does not converge but is clearly bounded by 1 and 2,
(however a monotone series that is bounded will converge but that's in [3.3])

Arithmetic with Sequences In order to manipulate sequences we will define operations that relate to addition and multiplication, this is by definition simply so we can use them.

Let,

$$X = (x_n) \quad Y = (y_n) \quad Z = (z_n) \tag{2.15}$$

We define the following Operations [3.2, p. 63]:

$$X + Y = (x_n + y_n) \tag{2.16}$$
$$X - Y = (x_n - y_n) \tag{2.17}$$
$$c \cdot X = (c \cdot x_n) \tag{2.18}$$
$$X \times Y = (x_n \times y_n) \tag{2.19}$$
$$X/Y = (x_n \div y_n) \tag{2.20}$$

Limits of Sequences for Arithmetic with Sequences [3.2.3] Because the limit of a sequence is essentially the expected value of x_∞ it stands to reason that the limit will distribute over the basic operations:

Let,

$$\lim X = \lim(x_n) = x \quad \lim Y = \lim(y_n) = y \quad \lim Z = \lim(z_n) = z \tag{2.21}$$

Then the limits are:

$$\lim (X + Y) = \lim X + \lim Y = x + y \tag{2.22}$$
$$\lim (X - Y) = \lim X - \lim Y = x - y \tag{2.23}$$
$$\lim (c \cdot X) = c \cdot \lim X = c \cdot x \tag{2.24}$$
$$\lim (X \times Y) = \lim X \times \lim Y = x \times y \tag{2.25}$$
$$\lim (X/Y) = \lim X \div \lim Y = {}^{x}/_{y} \tag{2.26}$$

2.2.3 Limit Theorems

The rest of the chapter provides values of limits, it begins with this simple property of sequence limits:

If X is convergent (i.e. a limit exists) and all $x_n \geq 0$
Then $\lim (x_n) \geq 0$ (3.2.4)

We can build on this theorem by generalising it a little bit:

If X and Y are convergent (i.e. limits exists) and all $x_n \leq y_n$
Then $\lim (x_n) \leq \lim (y_n)$ (3.2.5)

If a sequence has a limit and exists within in an interval, then the limit is also within that interval.

If X is convergent (i.e. a limit exists) and all $x_n \in [a, b]$
Then $\lim (x_n) \in [a, b]$ (3.2.6)

Squeeze Theorem Now if a sequence is always between two other sequences and those sequences have the same limit, then the original sequence must share that limit.

$$\textbf{If } X, Y, Z \text{ are convergent (i.e. limits exist) and all } (x_n) \leq (y_n) \leq (z_n) \text{ and } \lim X = \lim Z$$
$$\textbf{Then } \lim X = \lim Y = \lim Z \tag{3.2.7}$$

Limits Sequence Functions I'd be careful here because the textbook doesn't necessarily imply that all functions will demonstrate this behaviour:

$$|\lim(x_n)| = \lim(|x_n|) \tag{3.2.9}$$
$$\left(\sqrt{\lim(x_n)}\right) = \lim(\sqrt{x_n}) \tag{3.2.10}$$

Ratios The next theorem is useful where a ratio of the next and preceeding term can be reduced into a form that must be less than one $\left(\frac{x_{n+1}}{x_n} < 1\right)$.

$$\textbf{If } \text{all } (x_n) > 0 \text{ and } L := \lim\left(\frac{x_{n+1}}{x_n}\right) \text{ exists}$$
$$\textbf{Then } L < 1 \implies \lim X = 0 \tag{3.2.11}$$

2.3 [3.3] Monotone Sequences

A monotone sequence is a sequence that is either increasing or decreasing, where:

$X = (x_n)$ is said to be **decreasing** if:

$$x_n \geq x_{n+1}, \quad \forall n \in \mathbb{N} \tag{2.27}$$

$X = (x_n)$ is said to be **increasing** if:

$$x_n \leq x_{n+1}, \quad \forall n \in \mathbb{N} \tag{2.28}$$

2.3.1 Monotone Convergence Theorem [3.3.2]

A monotone sequence (x_n) is convergent $\iff$ it is bounded. $(\{x_n\} \in \mathbb{R})$

Whereas an ordinary set is convergent $\implies$ Bounded (at 3.2.2)

Furthermore,

1. If $X = (x_n)$ is *bounded* and *increasing*, then:

$$\lim(x_n) = \sup\{x_n : n \in \mathbb{N}\} \tag{2.29}$$

2. If $Y = (y_n)$ is *bounded* and *decreasing* then:

$$\lim(y_n) = \inf\{y_n : n \in \mathbb{N}\} \tag{2.30}$$

Why this is Important The *Monotone Convergence Theorem* is important because it:

1. Guarantees a Limit exists for a bounded Monotone Sequence
2. Gives us a way to solve that limit if we can evaluate the supremum/infimum

   Supremum and Infimum are defined at [2.3.1]-[2.3.2]

Example Application

Deductive Sequence Evaluate $\lim \left({}^{1}/\sqrt{n}\right)$
Observe that the corresponding sequence is $(x_n) = \left(x : x = \frac{1}{\sqrt{n}}, n \in \mathbb{N}\right)$

The set of range values would be $\{x_n\} = \left\{1, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{3}}, \frac{1}{2}, \dots\right\} \in [1, 0)$

The Infimum (i.e. the largest value smaller than all x_n) of x_n is 0, so the *Monotone Convergence Theorem* provides $\lim(x_n) = 0$

■

Inductive Sequence Evaluate $x_1 = 2, \ \ x_{n+1} = 2 + \frac{1}{x_n}$
Let the limit be:

$$\lim(x_n) = x \tag{2.31}$$

We know that the limit must be equal to the limit of the m-tail of the sequence by [3.1.9], so:

$$\begin{aligned} \lim(x_n) = x &= \lim(x_{n+1}) && (2.32) \\ &= \lim\left(2 + \frac{1}{x_n}\right) \\ &= \lim(2) + \lim\left(\frac{1}{x_n}\right) && \text{Justified by (3.2.3)} \end{aligned}$$

This step is allowable if and only if $x_n \neq 0$, which means $x_n > 0$, hence it is now known that $\lim(x_n) > 0$ by (3.2.4)

$$\begin{aligned} &= 2 + \frac{\lim(1)}{\lim(x_n)} && (2.33) \\ \implies x &= 2 + \frac{1}{x} \\ \implies 0 &= x^2 - 2x - 1 && x > 0 \\ \implies x &= 1 \pm \sqrt{2} \ \ \wedge \ \ x > 0 \\ \implies x &= 1 + \sqrt{2} \end{aligned}$$

Solving Roots with the *Monotone Convergence Theorem*[3.3.5] This can be used to solve square roots, it's laid out in a very convoluted fashion in the text book.

Euler's Number Euler's number is the number $e = \lim(e_n): \ \ e_n = \left(1 + \frac{1}{n}\right)^n$

■

2.4 [3.4] Subsequences

This section [3.4] is all about subsequences, and how they interact with convergence, it also introduces the limit superior/inferior.

2.4.1 Subsequences [3.4.1]

Let $X = (x_n)$ be a sequence of real numbers, from left to right pick values of X (e.g. every third value or perhaps the 2nd, 3rd, 5th, 7th, 11th etc), these values also form a sequence and that sequence is a subsequence of X.

Formally, a subsequence X' of X is a sequence, composed of elements of (x_n), where $n_1 < n_2 < n_3 \cdots \in \mathbb{N}$:

$$(x_{n_1}, x_{n_2}, x_{n_3}, \dots) \tag{3.4.1}$$

Convergence of Subsequences If a sequence converges to some value $x = \lim X$, then the subsequence must also converge to that value (because a subsequence preserves the order of the original sequence).

Non-Converging Subsequences [3.4.4] The following are equivalent statements:

1. The sequence $X = (x_n)$ does not converge to $x \in \mathbb{R}$
2. There exists some value ε_0, such that for any $k \in \mathbb{N}$, there exists $n_k \in \mathbb{N}$ such that $n \geq k$ and $|x_{n_k} - x| \geq \varepsilon_0$
3. There exists an $\varepsilon_0 > 0$ and a subsequence $X' = (x_{n_k})$ of X such that $|x_{n_k} - x| \geq \varepsilon_0$ for all $k \in \mathbb{N}$

Divergence Criteria [3.4.5] If either of the following properties is satisfied, a sequence X can be shown to be divergent.

1. X has two convergent subsequences that converge to different limits

   because there can only be one limit for a subsequence
2. X is unbounded

   If X was convergent it would be necessarily bound by the starting and limit value.

2.4.2 Existence of Monotone Subsequences [3.4.7]

If $X = (x_n)$ is a sequence of real numbers, then there is a subsequence of X that is monotone.
Basically all this says is it is possible to pick values from X in order so that they either increase or decrease (recall that the definition of increasing allows being equal to the previous value by (2.27)).

2.4.3 [3.4.8] The Bolzana-Weierstrass Theorem [3.4.8]

This is Italian-German, so it's pronounced (bolt-tza-no)-(vai-ya-strahzz).

If a sequence is bounded then all subsequences are bounded (this is by definition (3.4.1)),
A monotone subsequence is guaranteed to exist by (3.4.7),
A bounded monotone subsequence must converge by the *MCT* (3.3.2),
Hence a convergent subsequence must exist.

That's the Theorem: *A bounded subsequence must always have a convergent subsequence*

2.4.4 Upper and Lower Bounds [2.3.1]

Upper Bound An upper bound is any value greater than or equal to all elements of a set, e.g. u is an upper bound of A if:

$$\forall s \in S, \exists u \in \mathbb{R} : u \geq s \tag{2.34}$$

Lower Bound A lower bound is any value less than or equal to all elements of a set, e.g. w is a lower bound of A if:

$$\forall s \in S, \exists w \in \mathbb{R} : w \leq s \tag{2.35}$$

2.4.5 Supremum and Infima [2.3.2]

Supremum The suprema of a set is the smallest upper bound value of some set.
This value would be the maximum value of the set if the set had a maximum value.
Let V be the set of all upper bound values, u is a suprema iff:

$$u \leq v, \forall v \in V \tag{2.36}$$

So if a set has a maximum value, the supremum is the maximum value of the set:

$$\exists \max \{(x_n)\} \implies \max(x_n) = \sup \{x_n\} \tag{2.37}$$

If the set doesn't have a maximum, then the supremum is the next largest value, e.g.
$\sup (3, 5) = 5$ and $\sup [3, 5] = 5 = \max ([3, 5])$

Infimum The infimum of a set is the largest lower bound value of some set.
This value would be the maximum value of the set if the set had a maximum value.
Let T be the set of all upper bound values, w is a suprema iff:

$$w \leq t, \forall t \in T \tag{2.38}$$

So if a set has a minimum value, the infimum is the minimum value of the set:

$$\exists \min \{(x_n)\} \implies \min(x_n) = \inf \{x_n\} \tag{2.39}$$

If the set doesn't have a minimum, then the infimum is the next largest value, e.g.
$\inf (3, 5) = 3$ and $\inf [3, 5] = 3 = \min ([3, 5])$

2.4.6 Limit Superior and Limit Inferior [3.4.10]

So the textbook wasn't particularly helpful, instead this video by *SplineGuyMath*[1] was really good

Summary Sometimes it is useful to know the smallest and largest limits that subsequences can have. For this the limit inferior and limit superior are used.

[1] https://www.youtube.com/watch?v=khypO8MQpdc

Limit Inferior The Limit Inferior is the smallest limit that any subsequence of x_n can have; it is denoted:

$$\limsup (X) = \limsup (x_n) = \overline{\lim} (x_n) \tag{2.40}$$

Limit Superior The limit superior is the largest limit that ay subsequence of x_n can have; it is denoted:

$$\liminf (X) = \liminf (x_n) = \underline{\lim} (x_n) \tag{2.41}$$

Definition

Limit Superior
Let $X = x_n$ be bounded above, and
$M_n = \sup \{x_n, x_{n+1}, x_{n+2}, \dots\}$ (This is the $(n-1)$ tail of x_n)
Then the limit superior of x_n is:

$$\limsup \{X\} = \limsup \{x_n\} = \lim \{M_n\} \tag{2.42}$$

This definition of a limit superior works because subsequences preserve order, so if the maximum value of a sequence approaches a limit as we move along that sequence (i.e. take tails), that limit must be the largest of all limits of possible subsequences.

Limit Inferior
Let $X = x_n$ be bounded below, and
$m_n = \inf \{x_n, x_{n+1}, x_{n+2}, \dots\}$ (This is the $(n-1)$ tail of x_n)
Then the limit inferior of x_n is:

$$\liminf \{X\} = \liminf \{x_n\} = \lim \{m_n\} \tag{2.43}$$

This definition of a limit inferior works because subsequences preserve order, so if the minimum value of a sequence approaches a limit as we move along that sequence (i.e. take tails), that limit must be the smallest of all limits of possible subsequences.

Example Find the limit superior and limit inferior of:

$$B = (b_n) = \left(1 + \frac{1}{2^n}\right) = \left(\frac{3}{2}, \frac{5}{4}, \frac{9}{8}, \frac{17}{16}, \dots\right) \tag{2.44}$$

First consider the value of M_n:

$$\begin{aligned} M_n &= \sup \{b_n, b_{n+1}, b_{n+2}, \dots\} && (2.45)\\ &= (1.5, 1.25, 1.0625, 1.03125, 10.015625, \dots) && (2.46)\\ \lim(M_n) &= 1 && (2.47)\\ \limsup(M_n) &= 1 && (2.48)\end{aligned}$$

Now consider the value of m_n:

$$\begin{aligned} m_n &= \inf\{b_n, b_{n+1}, b_{n+2}, \dots\} && (2.49)\\ &= (0,0,0,0,0,\dots) && (2.50)\\ \lim(m_n) &= 0 && (2.51)\\ \liminf(M_n) &= 0 && (2.52) \end{aligned}$$

Hence the largest limit that any subsequence of (b_n) can have (i.e. the **Limit Superior**) is 1 and the smallest limit that any subsequence of (b_n) can have (i.e. the **Limit Inferior**) is 0.

Convergent Sequences and limit Superior/Inferior [3.4.12] If a sequence is bounded it must have a limit superior and a limit inferior, if it is convergent then:

$$\overline{\lim}\,(x_n) = \lim\,(x_n) = \underline{\lim}\,(x_n) \tag{3.4.12}$$

Equivalent Statements of Limit Inferior and Superior [3.4.11] The following are equivalant statements that flow from the definition of the limit superior

1. $x^* = \limsup\,(x_n)$

2. if $\varepsilon > 0$, there are only some values of $n \in \mathbb{N}$ such that $x^* + \varepsilon < x_n$, but there are unlimited numbers of $n \in \mathbb{N}$ such that $x^* - \varepsilon < x_n$.

3. if $u_m = \sup\{x_n : n \geq m\}$, then $x^* = \inf\{u_m : m \in \mathbb{N}\} = \lim\,(u_m)$.

4. if S is the set of subsequential limits of x_n, then $x^* = \sup S$

   This is the definition we used above in (2.42).

Chapter 3

Infinite Series

03 Real Series Autumn 2019 Ryan Greenup

Contents

(03) Series
Wk 4 Material; Topic 3; Due 28 March

3.0.1 The Cauchy Criterion (3.5)

The Cauchy Convergence Criterion

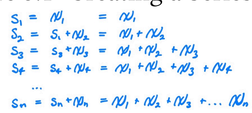
A sequence is convergent if and only if it is a Cauchy sequence

- **Cauchy Sequence** implies **Convergence**
  - Every Cauchy sequence of real numbers is bounded, hence by the Bolzano-Weierstrass theorem the sequence has a convergent subsequence, hence is itself convergent.
- **Convergence** imples **Cauchy Sequence**
  - If two terms can be made arbitrarily close then any term can be made arbitrarily close to another term in the set (which will be the limit point).

3.0.2 Properly Divergent

A series (x_n) is said to be properly divergent if $\lim_{n\to\infty}(x_n) = \pm\infty$

Figure 3.1: Creating a Series from a Sequence



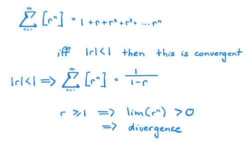


Figure 3.2:

3.0.3 Definition of a Series [3.7.1]

if x_n is a sequence, then the **series** generated by the sequence is $S = (s_k)$:

- The terms of the sequence are $x_n) = (x_1, x_2, x_3, x_4, \ldots s_n)$

  The terms of the series are $(s_n) = (s_1, s_2, s_3, s_4, \ldots s_n)$
  The terms of the series are called the **partial sums** and are defined as such:

3.0.4 Common Series Types

These are series that we are expected to memorise because they so often appear in series problems (and moreover we we will need them for the exam).

Geometric Series (3.7.6 (a))

The Geometric Series is Convergent if and only if $\mid r \mid <$:

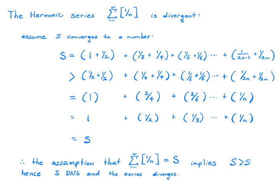


Figure 3.3:

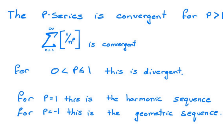


Figure 3.4:

Harmonic Series (3.7.6(b))

P**-Series**

3.0.5 Properties of Series

The n^{th} term test

This is more or less a test for divergence, it is necessary that a sequence (x_n) has a limit of 0 in order for the series to be convergent:

$$\exists L: \quad \sum_{n=1}^{\infty} [x_n] = L \implies \lim(x_n) = 0$$

be careful however because a sequence with a limit of 0 is not sufficient to establish the convergence of a series:

$$\exists L: \quad \sum_{n=1}^{\infty} [x_n] = L \not\Leftarrow \lim(x_n) = 0$$

Cauchy Criterion for series

If a sequence is convergent it must be a Cauchy sequence, hence all convergent series are composed of *Cauchy Sequences* (as a necessary but not sufficient condition).

So to be clear a series converges if and only if it is a *Cauchy Sequence*.

Definitions

- A Cauchy Sequence is:
  - $\forall \varepsilon > 0, \exists M: \quad m, n \geq M \implies |\ s_m - s_n\ | = |x_{n+1} + x_{n+2} + x_{n+3} \dots x_m| < \varepsilon$
- A Series Converges (which is an equivalent statement) if:
  - $\forall \varepsilon > 0, \exists M: \ , n \geq N \implies |\ s_n - s\ | = |x_1 + x_2 + x_3 \dots x_n| < \varepsilon$

3.0.6 Convergence Tests

Types of Convergence

A series $\sum[x_n]$ is ***absolutely convergent*** if and only if $\sum[\ |x_n|\ ]$, otherwise the series is said to be conditionally convergent.

This is important because the convergence of $\sum[\ |x_n|\ ] \implies$ the convergence of $\sum[x_n]$

Below the tests have been split into three categories:

- Comparison Tests
  - These establish non-absolute convergence but are broadly applicable and so are introduced early
- Absolute Convergence Tests
  - These establish absolute convergence.
- Non-Absolute Convergence Tests
  - These are useful for *alternating Series* and series that change sign as they progress (e.g. $\frac{sin(n)}{n}$)

Choosing a Test

Choosing the right test can be difficult, hence I have included an appendix with a flow chart [1] that we should probably memorise for want of the exam

Manipulating Series

Sometimes you'll be given a series in an odd way for example:

$$S_n = \sum_{n=1}^{\infty}\left[\frac{1}{(3n-2)\cdot(3n+1)}\right]$$

Now this could be shown to be convergent using the limit comparison test (which is below) but if you are asked to find the value to which the series converges to there is a bit more work involved.

Generally if you are asked to find what value a series converges to it will be either:

- A Geometric Series (3.7.6(a) of TB), or
- A telescoping Series

Geometric Series have already been shown, but a telescoping series is new and not covered in the textbook, basically, it is a series where most of the terms cancel out by way of rearrangement and grouping to leave only one or two terms left.

[1] Strategy for Series, http://tutorial.math.lamar.edu/Classes/CalcII/SeriesStrategy.aspx

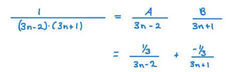


Figure 3.5:

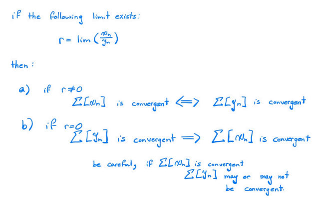


Figure 3.6:

Partial Fractions Often it is necessary to manipulate the terms somewhat in order for them to exhibit the cancelling/telescoping property, often by way of partial fractions (remember from *Mathematics1B*), for an example of this refer to Q3(c) of the corresponding tutorial (tutorial #4 of wk 4 material, due wk. 5, topic 3 from learning guide)

In this case because the provided series is not a geometric series it must be a telescoping series (because otherwise we wouldn't be asked to find the value to which it converges to, we only know how to find the convergence values of those two series, so we know it's telescoping, in order to get it into a form that will work, use partial fractions [2]

From here we would manipulate the series using grouping and rearrangement

Grouping Series Grouping terms in a series does not affect the value to which it converges,

- This flows from the associativity of addition, a property exhibited by the $\mathbb{R}$ which is the codomain of the sequence function

So in the above example the regrouping necessary to demonstrate the telescoping nature:

[2]Partial Fractions, http://tutorial.math.lamar.edu/Classes/CalcII/PartialFractions.aspx

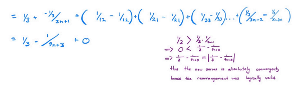


Figure 3.7:

Rearrangements (9.1.5) If a series is absolutely convergent then you can rearrange the terms and the series will converge to the same value (otherwise you can't so be careful)

- So say you have some series and you rearrange it, if this new series is absolutely convergent thet it's fine.
- However, if you rearrange some series and the new series is only conditionally convergent, then the rearrangement wasn't logically valid and this convergence value is erroneous.

So in our example the series is absolutely convergent so we could rearrange it:

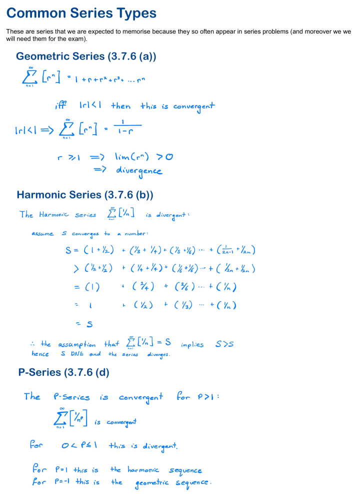


Figure 3.8:

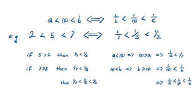


Figure 3.9:

Identities to remember

For the exam We need to remember these identities:

Limit of $e^{\frac{1}{n}}$

Dealing with Inequalities

take positive real sequences and some $k \in \mathbb{N}$:

$$n \geq k \Rightarrow 0 < x_n < y_n$$

so this order only needs to hold for any tail of the sequence.

a) if $\sum[y_n]$ converges, then $\sum[x_n]$ converges

b) if $\sum[x_n]$ diverges, then $\sum[x_n]$ diverges

take positive real sequences and some $k \in \mathbb{N}$:

$$x_n > 0 \text{ and } y_n > 0 \ \forall\, n \in \mathbb{N}$$

Figure 3.10:

take positive real sequences and some $k \in \mathbb{N}$:

$$x_n > 0 \text{ and } y_n > 0 \ \forall\, n \in \mathbb{N}$$

if the following limit exists:

$$r = \lim \left| \frac{x_n}{y_n} \right|$$

then:

a) if $r \neq 0$
$\sum[x_n]$ is absolutely convergent $\Longleftrightarrow$ $\sum[y_n]$ is absolutely convergent

b) if $r = 0$
$\sum[y_n]$ is absolutely convergent $\Longrightarrow$ $\sum[x_n]$ is absolutely convergent

be careful, if $\sum[x_n]$ is convergent $\sum[y_n]$ may or may not be convergent.

Figure 3.11:

Comparison Tests

Comparison Test (3.7.7)

Limit Comparison Test (3.7.8) Sometimes it can be difficult to establish the inequealities of the first test and a ratio would be easier to use, in that case this test can be used:

if the following limit exists:

$$r = \lim \left(\frac{x_n}{y_n} \right)$$

then:

a) if $r \neq 0$
$\sum[x_n]$ is convergent $\Longleftrightarrow$ $\sum[y_n]$ is convergent

b) if $r = 0$
$\sum[y_n]$ is convergent $\Longrightarrow$ $\sum[x_n]$ is convergent

be careful, if $\sum[x_n]$ is convergent $\sum[y_n]$ may or may not be convergent.

Absolute Convergence Tests

If these tests are satisfied they will establish that the series is absolutely convergent.

Limit Comparison Test II (9.2.1) (For Absolute Convergence) This version of the test is useful for establishing absolute convergence, it may be more difficult to establish however.

Ratio Test (9.2.4)

Generalised D'Alambert This can be useful where the ratio test fails for want of $(-1)^{n+1}$ because the $\limsup()$ operator will strip that way for a $(+1)$.

It is worth remembering that a sequence (x_n) is convergent if and only if:

$$\liminf(x_n) = \limsup(x_n) = \lim(x_n)$$

In this test however, we simply need to show that the $\limsup$ exists (which it will if the ratio-sequence has an upper bound), it isn't necessary to show that the ratio-sequence is convergent.

- (However, it is necessary that the sequence which generates the series converges to 0, otherwise the series will be divergent)

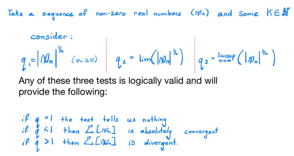


Figure 3.12:

Root Test

Generalised *Cauchy* Test This can be useful where the root test fails for want of $(-1)^n$, the $\limsup()$ operator will strip that away for a $(+1)$.

Integral Test If the series is of a function that is positive and ecreasint, then the series could converge if and only if the integral converges:

Let $f(k)$ be a positive decreasing function and let k be some natural number:

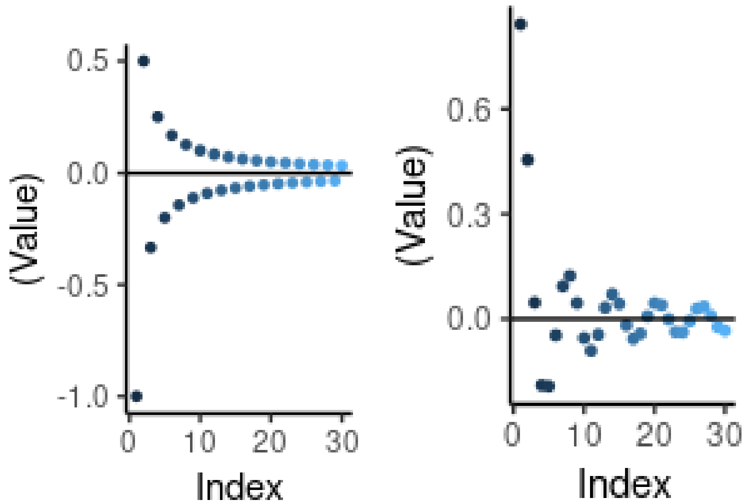
$$\exists L \in \mathbb{R} : \quad L = \sum_{n=k}^{\infty} [f(k)] \iff \int_{\infty}^{k} f(x)\,\mathrm{d}x = \lim_{b\to\infty}\left(\int_{b}^{k} f(x)\,\mathrm{d}x\right)$$

Or basically the series will converge if and only if the corresponding integral converges,

- This flows from the notion that the area under a continuous curve is going to be greater than the various term values, hence by the comparison test it's going to converge.
- this is a test for absolute convergence because the terms of the sequence that generates the series are strictly positive as a prerequisite anyway.

Non-Absolute Convergence Tests

Definition of an Alternating Sequence (9.3.1) An alternating sequence is a sequence that changes sign at each iteration, so for example $(x_n) = \frac{(-1)^{n+1}}{n}$ is an alternating sequence because at each succession the sequence changes sign $(x_n) = \frac{\sin(n}{n}$ is not an alternating sequence because the terms doesn't alternate at each succession:



Alternating Series Test Take a decreasing sequence of positive numbers (Z_n), :

- If the sequence is such that:
  - $Z_{n+1} < Z_n \quad \wedge \quad Z_n > 0 \qquad \forall n \in \ltimes$
- Then the series will be convergent:
  - $\exists L \in (R) : \sum_{n=1}^{\infty} \left[(-1)^{n+1} \cdot Z_n\right]$

So basically if the sequence is decreasing, then the series of the alternating sequence will hence converge.

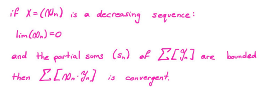


Figure 3.13:

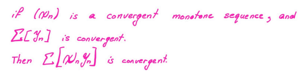


Figure 3.14:

Partial Summation Formula (Abel's Lemma) Let $X := (x_n)$ and $Y := (y_n)$ by sequences in $\mathbb{R}$ and let the partial sums of $\sum(y_n)$ be denoted by $(s_) n)$ with $s_0 := 0$

$$\sum_{k=n+1}^{m}[x_k y_k] = (x_m s_m - x_{n+1} s_n) + \sum k = n + 1^{m-1}(x_k - x_k + 1)\, s_k$$

Dirichlet's Test

3.1 3.5 - Cauchy Criterion Problems (Textbook); Problem 2, (a)

Abel's Test

3.1.1 Problem

Show that the follwoing sequence is a Cauchy Sequence

$$X = \frac{n_+ 1}{n} \tag{3.1}$$

3.2 Solution

Layout the Proof In order to show that the sequence X is a cauchy sequence it must be shown that:

$$\begin{gathered}\forall \varepsilon > 0, \quad \exists H \in \mathbb{N}: \\ m, n > H \implies |\, x_n - x_m \,| < \varepsilon\end{gathered} \tag{3.2}$$

so first we will consider the restriction required by ε and work backwards to find a sufficient value for H.

Consider the ε Restriction

$$\begin{aligned} \mid \frac{n+1}{n} - \frac{m+1}{m} \mid &= \mid \frac{mn+m-mn+n}{mn} \\ &= \mid \frac{m+n}{mn} \mid \\ &= \frac{m+n}{mn} && \text{Because } m,n \in \mathbb{N} \\ &= (m+n) \cdot \frac{1}{mn} && (3.3) \end{aligned}$$

Hence we have:

$$\begin{aligned} &\mid \frac{n+1}{n} - \frac{m+1}{m} \mid < \varepsilon \\ \implies &(m+n) \cdot \frac{1}{mn} < \varepsilon && (3.4) \end{aligned}$$

Assume a Value for H Now assume an arbitrary value for for H, we will use $H \geq 3$, this implies from (3.2):

$$\begin{aligned} m,n &\geq H \\ m,n &\geq 3 && \text{sub } H \geq 3 \\ m \cdot n &\geq 9 \\ \frac{1}{mn} &\leq \frac{1}{9} \\ \frac{1}{mn} &\leq \frac{1}{9} \\ (m+n) \cdot \frac{1}{mn} &\leq \frac{1}{9} \cdot mn \end{aligned}$$

and from (3.4) we have:

$$(m+n) \cdot \frac{1}{mn} \leq \varepsilon$$

Apply the restriction to H So we will choose H:

$$\frac{1}{9}(m+n) > \varepsilon \tag{3.5}$$

So re arranging this to solve some value for m, n, H

$$\begin{aligned} \frac{1}{9}(m+n) &> \varepsilon \\ (m+n) &> 9 \cdot \varepsilon \end{aligned} \tag{3.6}$$

So if we choose a H value such that $H > \frac{9\varepsilon}{2}$ then we will have $m > \frac{9\varepsilon}{2}$ and $n > \frac{9\varepsilon}{2}$ and so $(m+n) > \varepsilon$

Choose the Specific H Value Now there are two values for H, we need a value of $H \geq 3$ and $H > \frac{9\varepsilon}{2}$, this is satisfied by taking $H = \sup\left\{9 \cap \left(\frac{2\varepsilon}{2}, \infty\right)\right\}$

The actual proof

$$\forall \varepsilon, \ \ \exists H = \sup\left\{9 \cap \left(\frac{9\varepsilon}{2}, \infty\right)\right\}$$

Now assume that $m, n > H$, and consider $\mid x_n - x_m \mid$:

$$\begin{aligned} \mid x_n - x_m \mid &= \mid \frac{n+1}{n} - \frac{m+1}{m} \mid && (3.7) \\ &= (m+n) \cdot \frac{1}{mn} && (3.8) \end{aligned}$$

Now because $H \geq 9$ and $m, n \geq H$

$$< \frac{1}{9}(m+n)$$

because $H > \frac{9\varepsilon}{2}$

$$\begin{aligned} &< \frac{1}{9} \cdot \left(\frac{9\varepsilon}{2} + \frac{9\varepsilon}{2}\right) \\ &< \varepsilon && (3.9) \end{aligned}$$

Now because we have shown that $\forall \varepsilon, \ \ \exists H = \sup\left\{9 \cap \left(\frac{9\varepsilon}{2}, \infty\right)\right\}$ such that:

$m, n \geq H \implies \mid x_n - x_m < \varepsilon$
It is established that X must be a Cauchy Sequence.

Chapter 4

Limits

4.1 Proving Limits from First Principles

Prove:

$$\lim_{x \to 1} \left(\frac{1}{x} \right) = 1 \tag{4.1}$$

4.1.1 Precice Definition of a Limit

In order to establish this limit it must be shown that, 1 is a contained in the domain of the function and is a cluster point of the function such that:

$$\begin{aligned} &\forall \epsilon > 0, \ \ \exists \delta > 0 : \\ &\qquad 0 < |x - 1| < \delta \implies |\frac{1}{x} - 1| < \varepsilon \end{aligned} \tag{4.2}$$

Consider the Domain First observe that the domain of the function is $D(f) = \{x \in \mathbb{R} : x \neq 0\}$, and that 1 is contained by that domain and is a cluster point of that set.

4.1.2 How to find a sufficient δ

First consider the restriction on ε and try to deduce the value for δ, in this case the restriction is:

$$|\frac{1}{x} - 1| < \varepsilon \tag{4.3}$$

If we can get the left hand side in the form of $|x - 1| < f(\varepsilon)$ we are done because it would always be possible to find some ε given a δ, so let's try and do that.

4.1.3 Manipulate the ε Restriction

$$
\begin{aligned}
|\frac{1}{x} - 1| &< \varepsilon && (4.3)\\
|\frac{1-x}{x}| &< \varepsilon \\
|-\frac{1-x}{x}| &< \varepsilon \\
|\frac{x-1}{x}| &< \varepsilon \\
\frac{|x-1|}{|x|} &< \varepsilon && (4.4)
\end{aligned}
$$

A Note on getting the factor on the LHS It should always be possible to get a factor for $|x-a|$ on the LHS for typical rational/polynomial functions because it is introduced by subtracting the limit value $|f(x) - L|$; So don't worry about not being able to get the factor to appear by way of algebraic manipulation, worst case scenario you could use polynomial long division to pull the factor out, it will be in there because it is introduced by subtracting L from $f(x)$.

So now we have $|x-1|$ on the LHS but this endeavour has been somewhat upset by the denominator of $|x|$,
now an interval of $|x-1|$ will satisfy the inequality by the nature of absolute values, so we will pick a convenient value for δ and then see how it restricts the values of $|x|$.

Choose $\delta \leq \frac{1}{2}$:

$$
\begin{aligned}
|x-1| &< \frac{1}{2} && (4.2)\\
\implies -\frac{1}{2} < x - 1 &< \frac{1}{2} \\
\implies \frac{1}{2} < x &< \frac{3}{2} \\
\implies \frac{1}{2} < |x| &< \frac{3}{2} && (4.5)
\end{aligned}
$$

So if we choose $\delta = \frac{1}{2}$, then $\frac{1}{2} < |x| < \frac{3}{2}$.

Now let's get this looking like our ε form that we simplified (4.4)

$$\begin{aligned}
\frac{1}{2} < |x| < \frac{3}{2} \qquad (4.5) \\
\frac{3}{2} < \frac{1}{|x|} < 2 \\
\frac{1}{|x|} < 2 \\
|x-1| \cdot \frac{1}{|x|} < 2|x-1| \qquad (4.6)
\end{aligned}$$

So we have what we wanted on the left-side now, but now we also have $|x-1|$ on the right-side.

So again, as we did before we will just choose a convenient value for δ,

So we will choose δ such that:

$$\begin{aligned}
2 \cdot |x-1| < \varepsilon \\
|x-1| < \frac{1}{2} \cdot \varepsilon \\
\implies \delta \leq \frac{\varepsilon}{2} \qquad (4.7)
\end{aligned}$$

Now we have

$$|x-1| \cdot \frac{1}{|x|} < 2|x-1| \qquad (4.6)$$

By the initial assumption/definition at (4.1)

$$|x-1| \cdot \frac{1}{|x|} < 2\delta \qquad (4.8)$$

By using the δ value we chose in (4.7)

$$|x-1| \cdot \frac{1}{|x|} < \varepsilon \qquad (4.9)$$

So this is exactly what we were looking for,

Summarise So to summarise,
If we let δ be some value $\delta \leq \frac{1}{2}$ and also let $\delta \leq \frac{\varepsilon}{2}$ then the restriction is satisfied for all values of ε.

A mild problem here is that we need to chose a single value for ε that hence implies the existence of a ε value, so we will just choose the smallest value:

$$\delta = \min\left\{\frac{1}{2}, \frac{\varepsilon}{2}\right\} = \inf\left\{\frac{1}{2}, \frac{\varepsilon}{2}\right\} \qquad (4.10)$$

4.1.4 The Actual Proof

Let $\varepsilon > 0$, choose $\delta := \inf\left\{\frac{1}{2}, \frac{\varepsilon}{2}\right\}$, then

$$\begin{aligned}
|\frac{1}{x} - 1| &= |\frac{x-1}{x}| \\
&= |\frac{x-1}{x}| \\
&= \frac{|x-1|}{|x|} \\
&< 2 \cdot |x-1| && \text{As implied by } \delta < \frac{1}{2} \text{ at (4.9)} \\
&< 2\delta && \text{As implied by the definition at (4.2)} \\
&< 2\frac{\varepsilon}{2} && \text{As implied by } \delta < \frac{\varepsilon}{2} \text{ at (4.10)} \\
&< \varepsilon && \square \qquad (4.11)
\end{aligned}$$

4.1.5 Conclusion

Hence we have shown that for $\delta = \min\left\{\frac{1}{2}, \frac{\varepsilon}{2}\right\} = \inf\left\{\frac{1}{2}, \frac{\varepsilon}{2}\right\}$:

$$\begin{aligned}
&\forall \epsilon > 0,\ \ \exists \delta > 0 : \\
&\qquad 0 < |x-1| < \delta \implies |\frac{1}{x} - 1| < \varepsilon \qquad (4.2)
\end{aligned}$$

Chapter 5

Continuity

05 Continuity

5.1 (05) Continuity

Wk 4 Material; Topic 3; Due 28 March

f is continuous at c, if,

$\forall \varepsilon > 0, \quad \exists \delta > 0:$

$$|x - c| < \delta \implies |f(x) - f(c)| < \varepsilon$$

Figure 5.1: TODO

f is continuous at c, iff and only if

$\forall \varepsilon > 0, \exists \delta > 0:$

x in δ-neighborhood of c

$\implies$ $f(x)$ is in ε-neighborhood of $f(c)$

which is the same as saying:

$$f\left(A \cap V_\delta(c)\right) \subseteq V_\varepsilon(f(c))$$

Figure 5.2: Notes from the iPad, TODO transcribe

5.1.1 Continuous Functions 5.1

Definition of Continuity

Take some function $f : A \to B$ where $A \subseteq \mathbb{R}$:

- the function f is said to be continuous at some point $c \in A$ if and only if $\lim_{x \to c} = f(c)$

Rigorous Definition So let's phrase that using the ε-δ definition of the limit:

In Terms of Neighborhoods This can be expressed in terms of neighborhoods:

Conditions for Continuity If c is a cluster point of A, then trhee conditions must hold for f to be continuous at c, that is to say that three conditions must hold for $\lim_{x \to c} = f(c)$:

1. f must be defined at c
   - so that $f(c)$ actually has meaning
2. The limit of f at c must exist in $\mathbb{R}$ so that
   - $\lim_{x \to c}$ actually has a meaning
3. These two values are equal
   - $\lim_{x \to c} = f(c)$

   Cluster Points

   A cluster point has infinitely divisible values either side of it, if a value is not a cluster point it's just an isolated point and it is said to be continuous at that point, so generally we just assume points are cluster points because if they're not then they're automatically continuous and so not very interesting.

Sequential Criterion for Continuity [5.1.3]

Just like a limits can be defined in terms of sequences (at (4.1.8) of the TB), continuity can hence be defined in terms of sequences:

A function $f : A \to \mathbb{R}$ is continuous at some point $c \in A$ if and only if:

- for every sequence (x_n) in A that converges to c
  - $f((x_n))$ converges to c

Discontinuity Criterion [5.1.4] Just like limits can have divergence criteria in terms of limits, so can the continuity definition, this is analogous to the *Limit Divergence Criteria* at (4.1.9(a) of the TB).

A function $f : A \to \mathbb{R}$ is **discontinuous** at some point $c \in A$ if and only if:

- there exists some sequence (x_n) in A that converges to c:
  - $f((x_n))$ **does note* converge to c

Example $\lim_{x \to 0} \sin(\frac{1}{x^2})$ is undefined, so a sequence that converges to 0 does is such that $f((x_n))$ **does note* converge to c and so by (5.1.4) we can conclude that the function is discontinuous at 0.

Set Continuity

if B is a subset of A we can say that the function $f : A \to B$ is continuous on B if it is continuous at every point on B.

Defining a function to overcome Discontinuity

So take a function that is discontinuous, e.g. $f(x) = \frac{x^2-1^2}{(x+1)\cdot(x-1)}$ is discontinuous at $x = \pm 1$, to overcome this we can define a anew function $g(x)$:

$$g(x) = \begin{cases} f(x), & x \neq \pm 1 \\ 1 \quad\quad, & x = \pm 1 \end{cases}$$

This function will be continuous because the 'hole' is more or less 'plugged' by a given value.

- if there is no limit value at the discontinuity, then obviously this method won't work because we have no value with which to 'plug' the 'hole'

5.1.2 Combinations of Continuous Functions [5.2]

Absolute Values Preserve Continuity (5.2.4)

Take our function $f : A \to B$ and define the absolute function as :

- abs$(f)(x) = |f|(x) := |f(x)| \qquad \forall x \in A$
  - If f is continuous at c, then $|f(x)|$ is continuous at c
  - If f is continuous on A, then $|f(x)|$ is continuous on A

Square Roots Preserve Continuity

Take our function $f : A \to B$ and define the square root function as :

- sqrt$(f)(x) = \left(\sqrt{(f)}\right)(x) := \sqrt{f(x)} \qquad \forall x \in A$
  - If f is continuous at c, then $\left(\sqrt{(f)}\right)(x)$ is continuous at c
  - If f is continuous on A, then $\left(\sqrt{(f)}\right)(x)$ is continuous on A

Compositions Preserve Continuity

Let:

- $A, B \subseteq R$
  - $f : A \to B$
  - $g : B \to \mathbb{R}$
    - $f(A) \subseteq \mathbb{R}$

If f is continuous at c and g continuous at $f(c)$ then $g(f(x)) = (g \circ f) : A \to \mathbb{R}$ is continuous at c

If f is continuous on A and g continuous on B then $g(f(x)) = (g \circ f) : A \to \mathbb{R}$ is continuous on A

5.1.3 Continuous Functions on Intervals [5.3]

These weren't in the lecture Notes, they're probably not too important.

5.1.4 Uniform continuity [5.4]

These weren't in the lecture Notes, they're probably not too important.

5.1.5 The Mean Value Theorem [6.2]

Maximum / Minimum [6.2.0]

Take some function $f : I \to \mathbb{R}$

- f has a relative **maximum** if there exists some neighbourhood $V := V_\delta(c)$ such that::
  - $f(x) \geq f(c) \quad \forall x \in (V \cap I)$
- f has a relative **minimum** if there exists some neighbourhood $V := V_\delta(c)$ such that:
  - $f(x) \leq f(c) \quad \forall x \in (V \cap I)$

If either of these are satisfied then f is said to have a **relative extrema**

Interior Extrema Theorem [6.2.1]

let c be a point on some interval I at which f has a max/min

- If there is a derivative at c then it must be 0:
  - c is a point of at which f has a *relative extremum* $\implies f'(c) = 0$
- The derivative at c must be 0 or must be undefined.
  - $f'(c) = 0 \lor f'(c) \in \emptyset$
  - $f'(c) = 0 \lor f'(c) \downarrow$

In *Computability: An Introduction to Recursion Theory* 1 the $\downarrow$ symbol is said to mean undefined, it's a useful notation so I've adopted it, $\in \emptyset$ is potentially another technique as well.

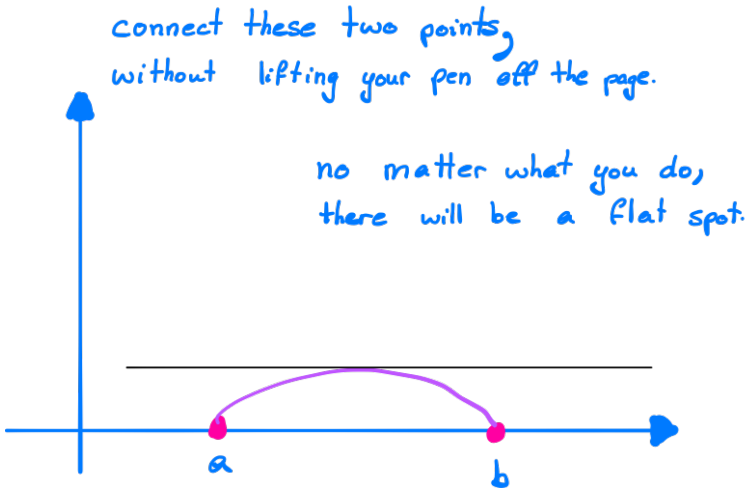


Figure 5.3:

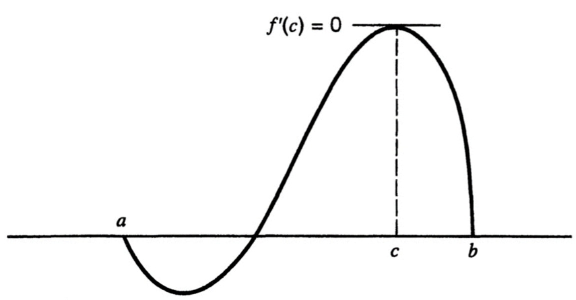


Figure 5.4:

Rolle's Theorem [6.2.3]

Suppose that:

- f is continuous on $I := [a, b]$
- f' exists at every point of the open interval (a, b)
- $f(a) = f(b) = 0$

Then there must exist some value c such that $f'(c) = 0$

- This is the same as saying there must be a point of relative extrema (by the *Interior Extrema Theorem*)

Mean Value Theorem [6.2.4]

This is basically a built up version of Rolle's Theorem,
Suppose that:

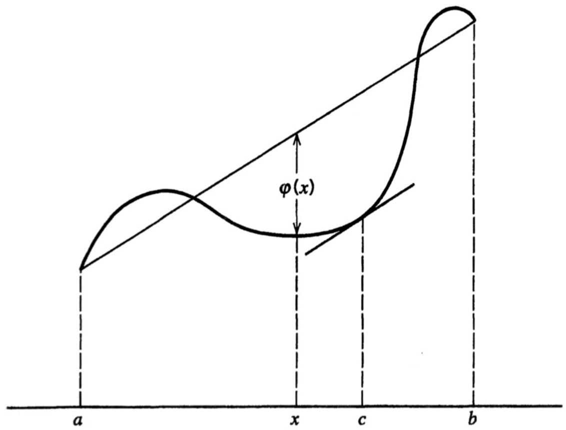


Figure 6.2.2 The Mean Value Theorem

Figure 5.5:

- f is continuous on $I := [a, b]$
- f' exists at every point of the open interval (a, b)

Then, there must exist some c:

$$f'(c) = \frac{f(b) - f(a)}{b - a} \qquad \implies (b - a) \cdot f(c) = f(b) - f(a)$$

5.1.6 L'Hospital's Rules

Cauchy Mean Value Theorem

let f and g be:

- Continuous on [a,b]
- Differentiable on (a,b)

if $g'(x) \neq 0$, then $\exists c \in (a, b)$:

$$\frac{f\left(b\right)-f\left(a\right)}{g\left(b\right)-g\left(a\right)}=\frac{f'\left(c\right)}{g'\left(c\right)}$$

Restrictions By *Rolle's Theorem*, if $g'(x) \neq 0$, then $g(a) \neq g(b)$.

Creating a stricter restriction Let's suppose for some reason you needed to create the restriction:

- $g'(x) \neq 0$, and
- $g(a) \neq g(b)$

An equivalent restriction would be:

$$\left(f'\left(x\right)\right)^2+\left(g'\left(x\right)\right)^2 \neq 0$$

Chapter 6

Complex Values

6.1 Integrals from a Real Domain

To begin this consider the function:

$$w : \mathbb{R} \to \mathbb{C}$$

we can decompose such a function into real and imaginary components:

$$w(t) = u(t) + i \cdot v(t)$$

where u and v are purely real functions.

6.1.1 Differentiation

If $w = f(z)$:

$$f'(z) = \frac{\mathrm{d}\, w}{\mathrm{d}\, z} \quad \text{(As if } z \text{ was a purely real operator)}$$

If $g(x, y) = u(x, y) + i \cdot v(x, y)$:

$$\begin{aligned} f'(z) &= \frac{\partial u}{\partial x} + \frac{\partial v}{\partial x} \\ &= \frac{dw}{dz} \end{aligned}$$

If $w = u(t) + i \cdot v(t)$:

$$w'(t) = \frac{\mathrm{d}\, u}{\mathrm{d}\, t} + i \cdot \frac{\mathrm{d}\, v}{\mathrm{d}\, t}$$

6.1.2 Integration

If $w = u(t) + i \cdot v(t)$:

$$\int_a^b (w(t))\, \mathrm{d}\, t = \int_a^b (u)\, \mathrm{d}\, t + i \cdot \int_a^b (v)\, \mathrm{d}\, t$$

Fundamental Theorem of Calculus
The fundamental theorem of calculus applies here:

$$\int_a^b (w(t))\,\mathrm{d}\,t = [W(t)]_a^b$$

Justification let $W(t) = U(t) + i \cdot V(t)$ be the antiderivative of $w(t)$:

$$\begin{aligned}\int_a^b (w(t))\,\mathrm{d}\,t &= [U(t)]_a^b + i \cdot [v(t)]_a^b \\ &= [U(b) + i \cdot V(b)] - [U(a) + i \cdot V(a)] \\ &= [W(b) - W(a)] \\ &= [W(t)]_a^b\end{aligned}$$

6.2 Contours

Integrals of complex valued functions are of the form:

$$\lim_{n \to \infty} \left(\sum_{i=1}^{n} [f(z_i^*) \cdot \Delta z] \right)$$

Δz could be along any curve in the complex plane rather than just the axis along which we would integrate:

this short curve is what we call a contour, the complex integral will be a line integral along that curve:

Define Contours
A set of points $z = (x, y)$ in the complex plane is said to be an ***arc*** or a curve that we will call C if:

$$x = x(t), \quad y = y(t) \qquad (a \leq t \leq b)$$

where:

- $x(t)$ and $y(t)$ are continuous functions continuous however does not mean smooth/differentiable, sharp points are allowed.
- the output values are ordered corresponding to t (as if t represented time)

It is more convenient to describe the curve at ([arcdef]) as:

$$\begin{aligned}z &= z(t) \\ &= x(t) + i \cdot y(t)\end{aligned}$$

Types of Arcs

- A **Simple Curve** does not cross itself
- a **Closed Curve** meets back up with itself
- a **Simple Closed Curve** is closed and does not cross itself (except where it closes)
- a **Positive Curve** moves counter-clockwise as t increases.

Parametric Representation not unique
Also the parametric representation of such a curve is not unique, the same curve could be represented by lots of different parametric equations, just like a second degree polynomial can also be modelled by a 3rd degree polynomial between intervals.

6.3 Contour Integral

As previously discussed, an integral with respect to z will be a contour integral:

$$\int_C f(z)\,\mathrm{d}\,z$$

If the value of the integral does not depend on the path of the contour and only depends on the endpoints then we would write:

$$\int_C f(z)\,\mathrm{d}\,z = \int_{z_1}^{z_2} f(z)\,\mathrm{d}\,z$$

We can also put the integral in terms of the contour parameters, This is basically *Integration by Substitution* in reverse, there's a more rigorous proof in the textbook by *Osbourne*, just take it as definition.

$$\int_C (f(z))\,\mathrm{d}\,z = \int_a^b (f[z(t)] \cdot z'(t))\,\mathrm{d}\,t$$

Negative Contours
If a contour C is given the contour $-C$ denotes the same contour with the order of the points reversed. Hence we have:

$$\begin{aligned}\int_{-C} (f(z))\,\mathrm{d}\,z &= \int_{-b}^{-a} \left(f[z(-t)] \cdot \frac{\mathrm{d}}{\mathrm{d}\,t}(z(-t))\right)\mathrm{d}\,t \\ &= -\int_a^b (f[z(-t)]\,z'(-t))\,\mathrm{d}\,t \\ &= -\int_C (f(z))\,\mathrm{d}\,z\end{aligned}$$

Refer to p. 128 of *Churchill's 9th* for worked examples.

6.3.1 Notation

If we want to basically i want to know does the symbol:

$$\oint f(z)\,\mathrm{d}\,z$$

- a contour/line integral of any sort
- a contour/line integral around a closed path
- one of the above but specifically in the anticlockwise direction?

6.3.2 Upper Bounds for Moduli

Start with the triangle inequality:

$$|\alpha + \beta| \leq |\alpha| + |\beta|$$

We could prove this, or, by similar reasoning:

Now let:

- C be a contour of length L
- $|f(z)| \leq M$

Using the inequality above it can be rigorously shown:

$$\left| \int (f(z))\,\mathrm{d}\,z \right| \leq M \cdot L$$

This can be somewhat visualised if $f(z)$ is taken to be the height of the function along the contour, and L is taken to be the length of the contour, although the values will be complex and height might become an odd concept, the mathematics should still hold.

Example Evaluate the contour integral:

$$\int_C \left(\frac{1}{z}\right) \mathrm{d}\,z$$

Where:

- $C : |z| = 1$
- For the sake of argument, we say that the circle joins at $z = -1 = \operatorname{cis} \pi = e^{\pi \cdot i}$

Now put the integral in terms of the parametric representation ([tdef]):

$$\begin{aligned}
\int_C \left(\frac{1}{z}\right) \mathrm{d}\, z &= \int_{\pi}^{\pi} \left(\frac{1}{e^{i\theta}} \cdot \frac{\mathrm{d}}{\mathrm{d}\,\theta} \left(e^{i\theta}\right)\right) \mathrm{d}\,\theta \\
&= \int_{\pi}^{\pi} \left(\frac{1}{e^{i\theta}} \cdot i \cdot e^{i\theta}\right) \mathrm{d}\,\theta \\
&= \int_{\pi}^{\pi} (i) \, \mathrm{d}\,\theta \\
&= [i \cdot \theta]_{\pi}^{\pi} \\
&= 0
\end{aligned}$$

6.4 Antiderivatives

Generally the value of a contour integral depends on both the path of the contour and the function. Some contour intergrals however have values independent of the path, for example consider how many integrals over closed paths will have an integral of zero as in the example above.
The antiderivative of a complex function is $F(z)$ such that:

$$F'(z) = f(z)$$

The antiderivative is, of necessity, an analytic function.

The antiderivative is uniqe, there is only one antiderivative for a given funciton (other than the additive constant $(+C)$ component).

6.4.1 Basic Antiderivative Theorem

Suppose $f(z)$ is continuous in a domain, if the function has an antiderivative $F(z)$ through the domain then:

integrals of $f(z)$ along contours lying in the domain depend only on the endpoints of that contour:

$$\int_C f(z) \, \mathrm{d}\, z = \int_{z_1}^{z_2} f(z) \, \mathrm{d}\, z = [F(z)]_{z_1}^{z_2}$$

This also means that integrals around a closed contour will be 0.

Proof if $F'(z) = f(z)$:

$$\begin{aligned}
\int_C f(z) \, \mathrm{d}\, z &= \int_a^b f(z(t)) \, \mathrm{d}\, t \\
&= \int_a^b U(z(t)) \, \mathrm{d}\, t + i \cdot \int_a^b b(z(t)) \, \mathrm{d}\, t
\end{aligned}$$

Which means the Fundamental Theorem of Calculus Applies from (9.1)

$$
\begin{aligned}
&= F\left(z\left(b\right)\right) - F\left(z\left(a\right)\right) \\
&= F\left(z_2\right) - F\left(z_1\right) \\
&= \left[F\left(z\right)\right]_{z_1}^{z_2} \\
&= \left[U\left(t\right)\right]_a^b + i \cdot \left[V\left(t\right)\right]_a^b \\
&= U\left(b\right) - U\left(a\right) + i \cdot \left[V\left(b\right) - V\left(a\right)\right] \\
&= \left[U\left(b\right) + V\left(b\right)\right] - i \cdot \left[U\left(a\right) - V\left(b\right)\right] \\
&= F\left(b\right) - F\left(a\right) \\
&= \left[F\left(z\right)\right]_a^b \\
&= \left[F\left(z\right)\right]_{z_1}^{z_2}
\end{aligned}
$$

6.5 Cauchy Goursat Theorem

This theorem depends on *Green's Theorem* which gives the relationship between a line integral around a simple-closed curve and the double integral over the corresponding bounded region, for purely real functions. Roughly Speaking *Green's Theorem* is a counterpart of the Fundamental Theorem of Calculus for double integrals.
In establishing the *Cauchy-Goursat Theorem*, proving *Green's Theorem* is the tricky part, the *Cauchy-Goursat Theorem* more or less falls out of it.

the *Cauchy-Goursat Theorem* If a function f is analytic at all points interior to, and on, a simple closed contour C, then:

$$\oint_C f\left(z\right) \mathrm{d}\, z = 0$$

6.5.1 Simply-Connected Domains

If $f\left(z\right)$ is analytic through a simply connected domain:

$$\int_C f\left(z\right) \mathrm{d}\, z$$

Where:

- C is any closed contour in that domain (Simple or not) i.e. if the domain is simply connected, the contour can cross itself.

From this it follows:

- If a function is analytic on a simply connected domain, then: there will be an anti-derivative: The contour integral will be independent of path (i.e.$F(b) - F(a)$
- Entire functions always have an antiderivative and will always be independent of path.

6.5.2 Multiply Connected Domains

If C is a closed counter-clockwise contour, inside which are closed Clockwise contours $c_1, c_2, c_3 \dots$ for which the function is analytic interior to C but exterior to $c_1, c_2, c_3 \dots$:

Then we have:

$$\int_C f(z)\,\mathrm{d}\,z + \sum_{n=1}^{k}\left[\int_{c_n} f(z)\,\mathrm{d}\,z\right]$$

This is established by drawing a line through all the interior closed contours and applying the *Cauchy-Goursat* theorem, refer to the *Churchill* textbook.

Principle of Deformation If we had Something like this:

Then it would follow:

$$\begin{aligned}
\int_{C_1} f(z)\,\mathrm{d}\,z + \int_{c_2} f(z)\,\mathrm{d}\,z &= 0 \\
\int_{C_1} f(z)\,\mathrm{d}\,z - \int_{c_2} f(z)\,\mathrm{d}\,z &= 0 \\
\int_{C_1} f(z)\,\mathrm{d}\,z &= \int_{c_2} f(z)\,\mathrm{d}\,z
\end{aligned}$$

So if c_2 can be continuously transformed into C_1 through an analytic region, then the integral is the same.

Spacial Case of Deformation We Should Probably memorise this special case:

$$\oint_{|z-z_0|=r} (z-z_0)^n\,\mathrm{d}\,z = \begin{cases} 0 & ifn \neq -1 \\ 2\pi i & ifn = -12 \end{cases}$$

Example

$$\int_{|z|=1} \frac{1}{z^2+2z+2}\,\mathrm{d}\,z = \int_{|z|=1} \frac{1}{(z+(1+i))\cdot(z+(1-i))}\,\mathrm{d}\,z$$

There is a singularity at $z = -1 \pm i$, which is outside the closed countour, the function is analytic everwhere else inside and on the closed countour hence by the *Cauchy-Goursatt Theorem:*

$$\int_{|z|=1} \frac{1}{z^2+2z+2}\,\mathrm{d}\,z = 0$$

6.6 Cauchy Integral Formula

This has to be memorised.

Cauchy Integral Formula
If f is analytic everwhere inside and on C and z_0 is interior to C:

$$\int_C \frac{f(z)}{z - z_0} \,\mathrm{d}\, z = 2\pi i \cdot f(z_0)$$

Example Solve :

$$\int_{|z|=1} \frac{\cos z}{z^3 + 9z} \,\mathrm{d}\, z$$

First simplify the intergrand somewhat:

$$\int_{|z|=1} \frac{\cos z}{z^3 + 9z} \,\mathrm{d}\, z = \int_{|z|=1} \frac{1}{z} \cdot \frac{\cos z}{z^2 + 9z} \,\mathrm{d}\, z$$

Observe that the intergrand $\frac{\cos z}{z^3+9z}$ is analytic everywhere other than its singularities:

- $z = 0$
- $z = 3i$
- $z = -3i$

because $z = 0$ is inside the contour we cannot use the *Cauchy-Goursatt Theorem*, however, we can use the *Cauchy-Integral Formula* precisely because :

- $z_0 = 0$ is interior to the closed contour
- $f(z) = \frac{\cos z}{z^2+9}$ is analytic everywhere else interior to the closed contour

So by the Cauchy Integral Formula:

$$\begin{aligned}\int_{|z|=1} \frac{\frac{\cos z}{z^2+9}}{(z-0)} \,\mathrm{d}\, z &= 2\pi i \cdot \frac{\cos(0)}{0^2 + 9} \\ &= i \cdot \frac{2\pi}{9}\end{aligned}$$

6.7 Extension of the Cauchy Integral Formula

This has to be memorised:

Extension to Cauchy Integral
if:

- $f(z)$ is analytic on and interior to some closed contour C
- z_0 is interior to C

Then:

$$\int_C \frac{f(z)}{(z - z_0)^{n+1}} \,\mathrm{d}\, z = f^n(z_0) \cdot \frac{2\pi i}{n!}$$

Chapter 7

Complex Variables

(08) Complex Variables Ryan Greenup Autumn 2019

Contents

7.1 Functions of a Complex Variable

A complex function is a function from a complex plane onto another complex + ane:

$$A, B \subseteq \mathbb{C},$$
$$f : A \to B$$

All the usual definitions of functions still apply, e.g.:

- Functions are rigourously defined using sets
- There is a do main, range, codomain, image etc.
- ...

The *Churchill's* Textbook mentions these conventions however

1. Usually the codomain is taken as the set of all complex values.
2. Most of the results concerning real functions are taken as already established without justification
3. x, y, u, v denote real variables where as z and w denote complex variables

   $z = x + iy$

   w = u + i v

   $f(z) = w \implies f(z) = u + iv$
4. Sometimes there won't be a clear distinction between the values of a function and the function itself, e.g.:

   $$g(z) = z^2$$
   $$f(g(z)) = f\left(z^2\right) \qquad \text{Here } z^2 \text{ is shorthand for the function}$$

7.2 Geometric Interpretation

Imagine the real function $y = 3$:

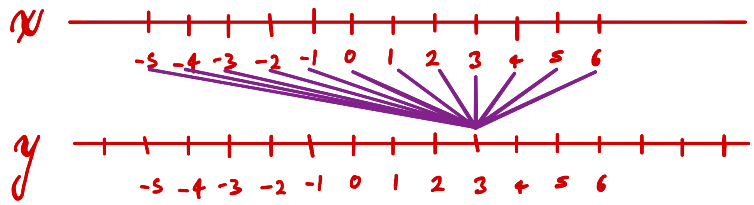


Figure 7.1:

A similar constant complex function would be $f(z) = 3 + 4i$:

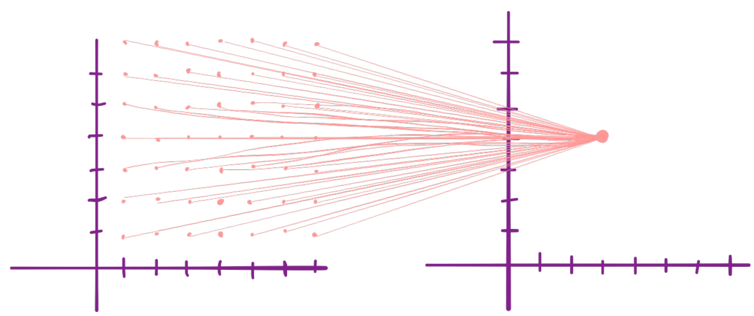


Figure 7.2:

It isn't possible to make a graph like it is with simple single variable real functions because that would require four spacial dimensions in order to plot it, this has the dissapointing consequence that geometric interpretations of derivatives as a slope and integrals as area beneath a curve are no longer helpful.

It isn't uncommon to use a 3D Cartesian plane to illustrate a function from the reals onto the complex, e.g. imagine $y = x^2$, if a complex domain is ullustrated as an x /y plane and a perpendicular z-axis represents the real codomain, the surface representing the values would always have two roots, even if the're not real, the *Welch Labs* video *Imaginary Numbers are real*[1] is really good for getting a visualisation of this but the general visualisation is:

7.3 Complex functions Components

Complex functions can be illustrated as a pair of two-variable real functions. Take a function:

$$f(z) = w$$

The w variable can be expanded:

[1]https://www.youtube.com/watch?v=T647CGsuOVU

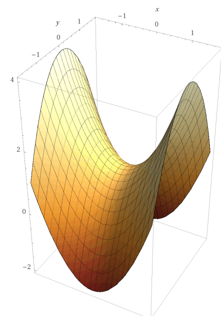


Figure 7.3: This can be generated in *Wolfram* or something else by using $f(x,y) = (x+iy)^2 + 1$

$$f(z) = u + iv$$

the z variable can also be expanded:

$$f(x+iy) = u + iv$$

Observe that the value u is in essence a function of the x and y input variables, the same is true also for v, hence, this can be rewritten:

$$f(x+iy) = u(x,y) + i \times v(x,y)$$

Example

$$\begin{aligned} f(z) &= z^2 \\ &= (x+iy)^2 \\ &= x^2 - y^2 + i \cdot 2xy \\ &= \left(x^2 - y^2\right) + i \cdot (2xy) \end{aligned}$$

So in this case the component functions would be:

$$\begin{aligned} u(x,y) &= \left(x^2 - y^2\right) && \text{and,} \\ v(x,y) &= (2xy) \end{aligned}$$

Essentially a complex-valued function is a pair of two variable real functions.

7.3.1 Limits

if f is defined on all poiints in a *deleted neighbourhood* of α it is written:

$$\lim_{z\to\alpha} f(z) = L, \qquad \text{equivalently,} \qquad f(z) \to w_0 \quad \text{as} \quad z \to z_0.$$

if and only if:

$f(z)$ can be made arbitrarily close to L by making z sufficiently close to α.

In formal notation this is expressed:

Formal Definition of a Complex Limit

If $f : A \to \mathbb{C}$ and $\alpha \in \overline{A}$

$$\forall \varepsilon > 0, \ \ \exists \delta : \\ 0 < |z - \alpha| < \delta \implies |f(z) - L| < \varepsilon$$

Limits in Terms of Sequences A sequence of complex numbers $\{z_n\}_1^\infty$ has a limit z (i.e. it converges to z) if:

Formal Definintion of a Limit to a Complex Sequence

$$\forall \varepsilon > 0, \ \exists N \in \mathbb{Z} : \\ in > N \implies |z_n - z| < \varepsilon$$

So this says, the limit of the sequence is z iif the terms of the sequence can be made arbitrarily close to the limit value by moving sufficiently far along the sequence.

Limit values are unique, a function can only have a single limite value at a point (or no limit value if the limit is undefined at that point).

Limits from multiple directions
A single variable real functions can only approach a variable from the left-hand side or the right-hand side, complex funcitons however can approach a varible along any curve in the complex plane.

So for example consider the limit of a function as $z \to 0$, z could approach zero along:

- the real-axis $(x, 0) : x \to 0 \implies z \to 0$
- the imaginary-axis $(0, y) : y \to 0 \implies z \to 0$
- any straight-line $y = mx : y \to 0 \implies z \to 0$
- along a parabola $y = x^2 : y \to 0 \implies z \to 0$
- any curve whatsoever at all...

What makes this more confusing is that a limit may approach a value along one curve but not another, maybe for example our function approaches $w = f(z) = L$ as the variable approaches 0 on both the x-axis and the y-axis, despite this it's entirely possible that our function approaches the value 33 along a parabola, the value 42 along a straight line and maybe $6\pi + 4i$ along a cubic curve.

So it's really worth noting that as a **necessary but not sufficient condition**, the limit taken along the axis must be equal in order for the limit to exist, if they are equal however, the limit is not guaranteed to exist, it may be another value along a different curve. It's worth reading *Pauls Online Notes* [2]

The reason for often taking limits along the axis (as opposed to some other arbitrary curve), is because the axis zeroes out a term which can be simpler and because the partial derivatives are also taken along the axis, which is used in developing the *Cauchy Riemann* equations later, but, really, there is no difference taking the limit along arbitrary curves or along the axis, the function doesn't necessarily care.

[2] http://tutorial.math.lamar.edu/Classes/CalcIII/Limits.aspx

Theorems on Limits The idea here is to establishh a connection between limits of complex functions and limits of real functions so we can use all the pre-established properties of real limits from calculus.

if:

$$\begin{aligned} z &= x + iy \\ f(z) &= u(x, y) + i \cdot v(x, y) \end{aligned}$$

Then we have:

$$\lim_{z \to \alpha} (f(z)) = L$$

if and only if:

$$\lim_{(x,y) \to (a,b)} [u(x, y)] = \mathrm{Re}(L) \quad \text{and} \quad \lim_{(x,y) \to (a,b)} [v(x, y)] = \mathrm{Im}(L)$$

So now we can break the complex limits up into real components that we already know how to deal with, and all the familiar *Limit Laws* carry over from earlier calculus.

Limit Laws

Distribution over Addition

$$\lim_{z \to z_0} [f(z) g(z)] = \lim_{z \to z_0} [f(z)] + \lim_{z \to z_0} [g(z)]$$

Distribution over Multiplication

$$\lim_{z \to z_0} [f(z) \cdot g(z)] = \lim_{z \to z_0} [f(z)] \cdot \lim_{z \to z_0} [g(z)]$$

Distribution over Division
Assume that $\lim_{z \to z_0} [g(z)] \neq 0$:

$$\lim_{z \to z_0} \left[\frac{f(z)}{g(z)}\right] = \frac{\lim_{z \to z_0} [f(z)]}{\lim_{z \to z_0} [g(z)]}$$

Riemann Sphere Limits at infinity are given a theoretical foundation using an idea called the *Riemann Sphere*, it's interesting but a deep understanding of the theory isn't necessary in order to work with limits at infinity so dont worry about it.

7.4 Continuity

A function f is *continuous* at a point z_0 if for all points $\lim_{z \to z_0} [f(z)] = f(z_0)$.

This is generally broken up into three conditions for want of decomposing problems:

> **Conditions of Continuity**
> A function f is *Continuous* at z_0 if the following three conditions are all satisfied:
>
> 1. $\lim_{z \to z_0} [f(z)]$
> 2. $f(z_0)$ exists
> 3. $\lim_{z \to z_0} [f(z)] = f(z_0)$ (which implies the above 2)

If a function is continuous on some neighbourhood, it's limit value for any point in that neighbourhood is the function value, this means, if we did, for instance, take the limit at a point along both axis (or along any two arbitrary curves), and they were equal, then the limit would be defined at that point, because it would be the function value.

If a function can be differentiated at a point, the function is continuous at that point.

So if we could show that a derivative exists on all points of some neighbourhood, and that the derivative was continuous at some point, then that neighbourhood would be continuous and the limit at that point would certainly exist.

This might seem a little bit contrived, but these are the pieces that are used for the *Cauchy Riemann* equations

Function Composition
A composition of continuous functions is continuous, e.g.

if:

$$f(x) = x^2 \qquad \text{is continuous}$$
$$g(x) = e^x \qquad \text{is contninous}$$

Then:

$$f \circ g = f[g(x)] = e^{x^2} \qquad \text{is continuous}$$

Again, this might seem obvious, but it's useful for complex functions and is necessary in the *Cauchy Riemann* equations.

Continuity of Complex Functions A Complex function is only continuous if the real two-variable components $u(x,y)$ and $v(x,y)$ are continuous:

This is because a composition of continuous functions is continuous

$$f(Z) = u(x,y) + i\,v(x,y)$$

this is continuous if and only if these are continous

Again, remember this for later when we are doing the *Cauchy Riemann* equations.

7.5 Derivatives

Derivatives have the same definition in complex analysis as they do in real calculus, with the difference that the variable is now complex, the derivative of f at a is:

$$\begin{aligned} w &= f(z) \\ \implies \frac{\mathrm{d}\,w}{\mathrm{d}\,z} &= \lim_{\Delta Z \to 0}\left[\frac{\Delta w}{\Delta z}\right] \end{aligned}$$

Or in a more useful fashion:

$$f'(a) = \lim_{\Delta Z \to 0}\left[\frac{f(z) - f(a)}{z-a}\right] = \lim_{\Delta Z \to 0}\left[\frac{f(a+\Delta z) - f(a)}{\Delta z}\right]$$

A function may be differentiable at a point z but not necessarily at any other points in the neighbourhood of z.

The real and imaginary components of a function may have continuous partial derivatives (of all orders) yet this does not imply that the function is differentiable there,

> e.g. $f(z) = |z|$ has continuous partial derivatives away from $z = 0$, but is not differentiable anywhere, because the limits as $\Delta z \to 0$ are different depending on which path is taken.

Conditions of Continuity

- if a function is continuous it may or may not be differentiable

  continuity $\not\Longrightarrow$ differentiability

- if a function is differentiable it must be continuous

  differentiability $\Longrightarrow$ continuity

7.5.1 Derivatives to Memorise

The following complex functions are nowhere differentiable:

- $f(z) = \Re(z)$
- $f(z) = \Im(z)$
- $f(z) = \overline{z}$

The function $f(z) = |z|^2$ is differentiable only at $z = 0$:

$$\frac{\mathrm{d}}{\mathrm{d}\,z}\left(|z|^2\right)\Big|_{z=0} = f'(0) = 0$$

bear in mind however that this function is nowhere analytic, because it is not differentiable on a neighbourhood of 0, this is covered further down.

How to Deal with Derivatives If you are given a function purely of z, then all the familiar differentiation rules carry over, their proofs are different, but the rules come out the same. [3]

If however you are given a function with terms of z, x and y, then you will need to first break the function up into it's components:

$$f(z) = u(x, y) + i \cdot v(x, y)$$

and then you will need to use the *Cauchy Riemann* equations, which we will get to further down.

Example Find the derivative of $w = f(z) = \frac{1}{z}$:
08 Complex Variable Autumn 2019 Ryan Greenup

7.6 Functions of a Complex Variable

A complex functoin is a function from a complex plane onto another complex plane:

$$\begin{aligned} &A, B \subseteq \mathbb{C}, \\ &\qquad f : A \to B \end{aligned}$$

- Functions are rigourously defined using sets
- There is a do main, range, codomain, image etc.
- ...

The text [?]

[3] I mean, be careful with log functions and roots because they have multiple values

Chapter 8

Complex Functions

8.1 Polynomial Functions

an n^{th} degree polynomial is given by:

$$p(z) = w = \alpha_0 + \alpha_1 z_1 + z_2 \ldots \alpha_n z_n$$

where:

- α is a complex constant
- z is a complex variable

8.2 Rational Function

A rational function is composed of two polynomial functions $P(z)$ and $Q(z) \neq 0$:

$$\frac{P(z)}{Q(z)}$$

8.3 Euler's Formula

This is really important, and there are lots of proofs for it, for the most part though, just come to accept it, the best justification for *Euler's Formula* that I've seen is by *3Blue1Brown*, Seriously, watch this, it's really good.

$$\begin{aligned} e^{i\theta} &= \cos\theta + i\cdot\sin\theta \\ \implies r\cdot e^{i\theta} &= r\left(\cos\theta + i\cdot\sin\theta\right) \\ r\cdot e^{i\theta} &= r\cdot\mathrm{cis}(\theta) \\ &= z \end{aligned}$$

It is not uncommon to see Euler's formula used instead of polar notation, particularly in the *Churchill* text, don't stress out, it has the same meaning in effect.

8.4 Exponential Function

The exponential function is defined for all complex values:

$$\begin{aligned} e^z = e^{x+iy} &= e^x \cdot e^i y \\ &= e^x \cdot (\cos y + i\cdot\cos y) \end{aligned}$$

8.4.1 Modulus and Argument

because:

$$e^z = e^x \cdot \mathrm{cis}(y)$$

we have the modulus:

$$|e^z| = e^x$$

and the argument:

$$\arg\left(e^z\right) = y + 2\pi k \qquad (k \in \mathbb{Z}^*)$$

the notation $\mathbb{Z}^*$ means 'non-negative integers' so, $(0, 1, 2, 3 \dots)$

8.5 Properties of the exponential function

Periodicity The complex exponential function is periodic:

$$e^{z+2\pi i} = e^z$$

This corresponds the sliding action of $z + 2\pi i$ to a rotation of $x + 2\pi$ radians of rotation, because the rotation has a period of 2π, the rotation will be x radians.

For this reason we also have:

$$\begin{aligned} e^z &= e^z \\ \left(e^z\right)^{\frac{1}{n}} &= e^{\frac{z+2k\pi}{n}} \end{aligned}$$

Principle Argument We choose the principle argument as $(-\pi, \pi]$, this is useful later on for dealing with logs.

- The Principle Argument of a complex function is denoted by a capital:
  $\mathrm{Arg}\,(r \cdot \mathrm{cis}\theta) = \theta$
- The general Argument of a complex function is denoted by lower case:
  $\arg\,(r \cdot \mathrm{cis}(\theta)) = \theta + 2\pi k \qquad : \quad k \in \mathbb{Z}^+$

Differentiating e^z So if you want to just move on:

$$\frac{\mathrm{d}}{\mathrm{d}\,z}\,[e^z] = e^z$$

If you want to know why:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\,z}\,(e^z) &= \frac{\partial}{\partial x}\left[e^{x+iy}\right] + i \cdot \frac{\partial}{\partial y}\left[e^{x+iy}\right] \\
&= \frac{\partial}{\partial x}\,[e^x \cos y] + \frac{\partial}{\partial x}\,[e^x \sin y] \\
&= \cos y e^x + \sin y e^x \\
&= e^{x+iy} \\
&= e^z
\end{aligned}$$

$\blacksquare$

Polar Form The exponential function can be expressed as:

$$\begin{aligned}
e^z &= e^x \cdot (\cos y + i \sin y) \\
&= e^x \cdot \mathrm{cis} y
\end{aligned}$$

$$\begin{aligned}
\implies \mathrm{Arg}\,(e^z) &= y \\
\implies \arg\,(e^z) &= y + 2\pi \\
\implies |e^z| &= e^x
\end{aligned}$$

Additional Properties Consider that $\forall x, y \in \mathbb{R}\ e^x \neq 0$ and also that $e^{iy} \neq 0$, hence:

$$e^z = e^x \cdot e^{iy} \neq 0$$

Although many things carry over, be careful because:

$$e^x \nleq 0 \qquad \text{however} \qquad e^z < 0 \quad \vee \quad e^z > 0$$

8.6 The Log Function

8.6.1 Deriving the Log Function

With real variables the motivation for the log function is a function for y such that:

$$e^y = x$$

In the complex case the motivation is much the same:

$$e^w = z$$

So in order to solve a value for the complex logarithm express z in polar form and decompose w into a real and complex part:

$$\begin{aligned} e^w = z \implies e^{u+iv} &= re^{i\theta} \\ e^u \cdot e^{iv} &= re^{i\theta} \end{aligned}$$

Equate the real and imaginary parts

$$\begin{aligned} e^u &= r \\ \implies u &= \ln(r) \end{aligned} \qquad \begin{aligned} e^{iv} &= e^{i\theta} \\ \implies v &= \theta + 2\pi k, \qquad k \in \mathbb{Z}^+ \end{aligned}$$

so the equation $e^w = z$ is satisfied if and only if:

$$\begin{aligned} w &= u + iv \\ &= \ln(r) + i \cdot (\theta + 2\pi k), \qquad k \in \mathbb{Z}^+ \\ &= \ln|z| + i \cdot \arg(z) \end{aligned}$$

Complex Logarithm A value in the complex plane $z = r \cdot e^{i \cdot \theta} = r \cdot \mathrm{cis}(\theta) \in \mathbb{C}$ will have a logarithm:

$$\log_e(z) = \ln|z| + i \cdot \arg(z) \tag{8.1}$$

The Complex Logarithm isn't a function, it is what is known in complex analysis as a '*multiple-valued function*' which is merely a *binary relation* [1], this is really ambiguous though because clearly a function can only have one output, the terminology is used to respect the fact that if, for example, $\log_e(z)$, is restricted to a single branch, it is indeed a function.

[1] https://en.wikipedia.org/wiki/Multivalued_function

8.6.2 Notation

Base

Also another really confusing point is that often times in complex analysis $\log(z)$ is used to represent the complex natural logarithm and $\ln(x)$ is used to represent the real natural logarithm, so you might see $\log$ and immediately think base-10, but don't, we deal excusively in e with complex awnalysis.

It's all redundant anyway because the change of base formula applies also to complex logarithm's anyway.

For clarity sake, I'll write $\log_e()$ when dealing with complex natural logarithms and $\ln()$ when dealing with real natural logarithms.

8.6.3 Change of Base

So in this example our desired solution is y :

$$\begin{aligned}
10^y &= e + 4i \\
\text{Log}_e(10^y) &= \text{Log}_e(3+4i) \\
y \cdot \text{Log}_e(10) &= \text{Log}_e(3+4i) \\
y &= \frac{\text{Log}_e(3+4i)}{\text{Log}_e(10)} \\
&\implies \text{Log}_{10}(3+4i) = \frac{\text{Log}_e(3+4i)}{\text{Log}_e(10)}
\end{aligned}$$

■

This works because $\text{Log}_e()$ is a valid function, it should work with other branches but I'm not sure.

Arguments
The principal argument corresponds to $\theta \in (-\pi, \pi]$ and is distinguished by using uppercase:

- Principal Argument:
  $$\mathrm{Arg}\,(z) = \Theta \qquad -\pi < \Theta \leq \pi$$
  So less that or equal to π but $\Theta \neq \pi$
- Argument:
  $$\arg\,(z) = \theta = \Theta + 2\pi k \qquad (k \in \mathbb{Z})$$
  This is periodic.

So for example, we may have $r \cdot \mathrm{cis}(\theta) = r \cdot \mathrm{cis}(\Theta)$ the difference being that

- $\theta \in \mathbb{R}$
- $\Theta \in (-\pi, \pi]$

Logarithms

Notation
Where $z = r \cdot \mathrm{cis}(\theta)$:

The **Principal Value** of the $\log_e()$ function is given by:

$$\mathrm{Log}_e(z) = \ln|z| + i \cdot \mathrm{Arg}\,(z) \tag{8.2}$$
$$= \ln(r) + i \cdot \Theta \tag{8.3}$$

The **multi-valued** log function is given by:

$$\log_e(z) = \ln|z| + i \cdot \arg\,(z) \tag{8.4}$$
$$= \ln(r) + i \cdot (\Theta + 2\pi k) \qquad k \in \mathbb{Z} \tag{8.5}$$

8.6.4 Branches

A branch of a function $f(z)$ that has multiple outputs (e.g. $\log_e(z)$), is a function with a single output that is analytic in the domain.
At every point of the domain, the single-valued function must assume exactly one of the various possible values that the original function might have given as output.

The requirement of analyticity prevents $F(z)$ from taking on a random selection of the values of $f(z)$.
So in the case of logs:

Logarithmic Branches

The **Principal Branch** of the logarithmic function is given by:

$$\text{Log}_e(z) = \ln(r) + i \cdot \Phi \qquad -\pi < \Phi < \pi \tag{8.6}$$

Any **Branch** of the logarithmic function is given by:

$$\log_e(z) = \ln(r) + i \cdot \phi \qquad -\alpha < \phi < \alpha + 2\pi k \tag{8.7}$$

here, the **Principal Value** of the complex log and the **Principal Branch** are denoted ambiguously by $\text{Log}_e()$ but they are both different, for example $\text{Log}_e(-10) = \ln 10 + \pi \cdot i$ as the principal value, however, it is entirely undefined on the principal branch.

So basically, on the principal branch, we delete the entire non-positive x-axis (i.e. including zero), where as the principal value is allowed to take values there using π (but not $-\pi$).

Now one might ask 'why on Earth would we do such a confusing and ambiguous thing?'
The reason is because it is the only way to make the $\log_e()$ function continuous and hence analytic.:

Clearly we couldn't use the *multi-valued* log functions as in (8.5) because that has multiple outputs for a given input and so it is not a function, hence it clearly is not analytic.

Now why can't we restrict the domain to the principal argument of $\theta \in (-\pi, \pi]$?

A function must be continuous in order to be differentiable and hence analytic, the condition for continuity is:

$$\lim_{z \to \alpha} [\mathrm{Log}_e(z)] = \mathrm{Log}_e(\alpha)$$

But if we take a value on the negative x-axis:

$$\lim_{z \to -1} [\mathrm{Log}_e(z)] = +\pi \cdot i \qquad \text{(From Above)}$$
$$\lim_{z \to -1} [\mathrm{Log}_e(z)] = -\pi \cdot i \qquad \text{(From Below)}$$

This implies that the limit does not exist, similarly it can be shown that the limit does not exist as z approaches values on the non-posiitive x-axis, no limit means no continuity, which means no differentiablility, which means not analytic.

The solution here is to remove the negative x-axis and zero from the domain, then the $\mathrm{Log}_e()$ function will be analytic.j

Branch Cuts A branch cut is a curve that is introduced in order to define the multiply defined function $F(z)$, it's basically the line we have to delete.

So imagine $\log_e(z)$:

The principal branch deletes the negative x-axis and 0 as shown in blue below:

So for the Complex Logarithm that we used as a definition at (8.7), the angle α from the origin is used as the branch cut in order to define a branch.

> So for example, as above in Figure 9.1, the line drawn from 0 at $-\pi$ radians corresponds to $\alpha = -\pi$ in the definition; This line is a branch cut of $\log_e(z)$ that we use to define the principal branch of the logarithmic function:

$$\log_e(z) = \ln(r) + i \cdot \Phi \qquad -\pi < \Phi < \pi$$

Basically you just delete that line in the definition of the domain.

Also be aware that the branch cut doesn't have to be a line, it could be a curve, this diagram is an example of a branch cut that would define a valid branch of the $\log_e(z)$ function:

It works because however far the value of z is from the origin, the values of θ will still be restricted to one revolution.

8.6.5 Principal Branch as opposed to Principal Value

Be aware that there is a difference between the *principal value* and the *principal branch* of the complex natural logarithm:

- The **Principal Value** of the complex logarithmic function of $z = r \cdot \text{cis}(\theta)$ is:

  $\log_e(z) = \ln(r) + i \cdot \Theta \qquad \Theta \in (-\pi, \pi]$

- The **Principal Branch** of the complex logarithmic function of $z = r \cdot \text{cis}(\theta)$ is:

  $\log_e(z) = \ln(r) + i \cdot \Phi \qquad -\pi < \Phi < \pi$

So to be clear, the principal value of $\text{Log}_e(-1) = \pi i$, however this is entirely undefined on the principal branch.

8.6.6 Properties of Complex Logs

Familiar properties of logartithms in calculus are sometimes but not aways true in complex analysis, the problem is usually the multiple branches of the log function.

So one of the first things to be careful with is:

$$e^{\log_e(z)} = z \qquad \text{by definition}$$

However:

$$\log_e(e^z) = x + i \cdot (y + 2\pi k) \qquad k \in \mathbb{Z}^+$$

But if we take the principal branch:

$$\begin{aligned}\mathrm{Log}_e(e^z) &= x + i \cdot y \\ &= e^z\end{aligned}$$

Another one to be careful of, for example:

$$\log_e\left(i^2\right) \neq \frac{1}{2} \cdot \log_e(i)$$

because:

$$\begin{aligned}\log_e\left(i^2\right) = \log_e(-1) &= \ln|-1| + i \cdot \arg(-1) \\ &= 0 + \arg(-1) \cdot i \\ &= \arg(-1) \cdot i \\ &= (\pi + 2\pi k) \cdot i \\ &= \pi(1 + 2k) \cdot i\end{aligned}$$

$$\begin{aligned}2 \cdot \log_e(i) &= 2\left[\ln|i| + \arg(i) \cdot i\right] \\ &= 2\left[o + \left(\frac{\pi}{2} + 2\pi k\right) \cdot i\right] \\ &= i \cdot [\pi + 4\pi k] \\ &= \pi \cdot i\,[4k + 1]\end{aligned}$$

If however we choose the principal branch that corresponds to $k = 0$:

$$\log_e\left(i^2\right) = \pi i = 2 \cdot \log_e(i)$$

8.6.7 Log Laws

Addition/Multiplication
because:

$$\begin{aligned}\arg(z_1 \cdot z_2) &= \arg(z_1) + \arg(z_2)\\ \ln(x_1 \cdot x_2) &= \ln(x_1) + \ln(x_2)\end{aligned}$$

We have:

Addition of Logs

$$\log_e(z_1 \cdot z_2) = \log_e(z_1) + \log_e(z_2) \tag{8.8}$$

$$\log_e\left(\frac{z_1}{z_2}\right) = \log_e(z_1) - \log_e(z_2) \qquad (z_2 \neq 0) \tag{8.9}$$

Also because:

$$|z_1 \cdot z_2| = |z_1| \cdot |z_2|$$

we have:

$$\log_e(|z_1 \cdot z_2|) = \log_e(|z_1|) + \log_e(|z_2|)$$

Relationships to Roots

$$\begin{aligned}z^n &= e^{n \cdot \log_e(z)}\\ z^{\frac{1}{n}} &= e^{\frac{1}{n} \cdot \log_e(z)}\\ &= e^{\frac{1}{n}\ln r + \frac{i(\theta 2\pi k)}{n}}\\ &= \sqrt[n]{r} \cdot e^{i\left(\frac{\theta}{n} + \frac{2k\pi}{n}\right)}\\ &= \sqrt[n]{r} \cdot \operatorname{cis}\left(\frac{\theta}{n} + \frac{2\pi k}{n}\right)\end{aligned}$$

Exponentiation

$$\begin{aligned}\log_e(z^n) &= \ln|z^n| + \arg(z^n)\\ &= \ln(|z|^n) + \arg([r \cdot \operatorname{cis}\theta^n])\\ &= n \cdot \ln|z| + \arg(rn \cdot \operatorname{cis}(n\theta))\\ &= n \cdot \ln|z| + \arg(\operatorname{cis}(\theta \cdot n))\\ &= n \cdot \ln|z| + (n\theta + 2\pi k) \qquad \exists k \in \mathbb{Z}\end{aligned}$$

This is only valid for a branch of the logarithmic function, in this case choose the principal branch

$$\begin{aligned}&= n \cdot \ln|z| + n\theta\\ &= n \cdot \operatorname{Log}_e(z^n)\end{aligned}$$

Therefore the exponential log law carries over where we use a single branch of the complex logarithmic function.

8.6.8 Differentiating $\log_e(z)$

Derivative of complex log

$$\frac{\mathrm{d}}{\mathrm{d}\,z}\left(\mathrm{Log}_e(z)\right) = \frac{1}{|z| \cdot \mathrm{cis}(\mathrm{Arg}\,(z))} \tag{8.10}$$

If you're not dealing with the principal branch adjust the argument accordingly, the derivative is only defined for a branch of the logarithmic function.

The working for this is pretty straight-forward.

$$f(z) = \log_e(z) = \ln|z| + i\left(\arg(z) + 2\pi k\right)$$

Let $\theta = \arg(z)$

$$= \ln|z| + (\theta + 2\pi k)$$

Choose a branch of $\log_e(z)$ so that it is a function

$$= \ln|z| + i\Phi \qquad -\alpha < \Phi < \alpha + 2\pi k$$

let $r = |z|$

$$= \ln(r) + i\Phi$$

Let $u = \ln(r)$:

$$u_r = \frac{1}{r}$$
$$u_\Phi = 0$$

Let $v = \Phi$:

$$v_\Phi = 1$$
$$v_r = 0$$

Our domain is $r > 0$, so $\frac{\partial u}{\partial r}$ and $\frac{\partial v}{\partial \Phi}$ are continuous on the entire domain.

because $f(z) = u(r, \Phi) + i \cdot v(r, \Phi)$, use the polar form of the *Cauchy Riemann* equations :

First Condition:

$$\begin{aligned} r \cdot u_r &= v_\Phi \\ r \cdot \frac{\partial}{\partial r} \cdot (\ln(r)) &= \frac{\partial}{\partial \Phi}(\Phi) \\ r \times \frac{1}{r} &= 1 \\ 1 &= 1 \end{aligned}$$

Second Condition:

$$\begin{aligned} u_\Phi &= -r \cdot v_r \\ \frac{\partial}{\partial \Phi}(\ln(r)) &= -r \times \frac{\partial}{\partial r}(\Phi) \\ 0 &= -r \times 0 \\ 0 &= 0 \end{aligned}$$

both the *Cauchy Riemann* equations are satisfied so:

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\,z}\left(\log_e(z)\right) = e^{-i\Phi}\cdot(u_r + iv_r) \\
&= e^{-i\Phi}\cdot\left(\frac{1}{r} + i\times 0\right) \\
&= \frac{1}{re^{i\Phi}} \\
&= \frac{1}{r\cdot\mathrm{cis}(\Phi)} \\
&= \frac{1}{|z|\cdot\mathrm{cis}(Arg\,(z))}
\end{aligned}
$$

Chapter 9

Complex Integrals

Contents

9.1 Integrals from a Real Domain

To begin this consider the function:

$$w : \mathbb{R} \to \mathbb{C}$$

we can decompose such a function into real and imaginary components:

$$w(t) = u(t) + i \cdot v(t)$$

where u and v are purely real functions.

9.1.1 Differentiation

If $w = f(z)$:

$$f'(z) = \frac{\mathrm{d}\, w}{\mathrm{d}\, z} \quad \text{(As if } z \text{ was a purely real operator)}$$

If $g(x, y) = u(x, y) + i \cdot v(x, y)$:

$$\begin{aligned} f'(z) &= \frac{\partial u}{\partial x} + \frac{\partial v}{\partial x} \\ &= \frac{dw}{dz} \end{aligned}$$

If $w = u(t) + i \cdot v(t)$:

$$w'(t) = \frac{\mathrm{d}\, u}{\mathrm{d}\, t} + i \cdot \frac{\mathrm{d}\, v}{\mathrm{d}\, t}$$

9.1.2 Integration

If $w = u(t) + i \cdot v(t)$:

$$\int_a^b (w(t))\, \mathrm{d}\, t = \int_a^b (u)\, \mathrm{d}\, t + i \cdot \int_a^b (v)\, \mathrm{d}\, t$$

Fundamental Theorem of Calculus
The fundamental theorem of calculus applies here:

$$\int_a^b (w(t))\,\mathrm{d}\,t = [W(t)]_a^b \tag{9.1}$$

Justification let $W(t) = U(t) + i \cdot V(t)$ be the antiderivative of $w(t)$:

$$\begin{aligned}\int_a^b (w(t))\,\mathrm{d}\,t &= [U(t)]_a^b + i \cdot [v(t)]_a^b \\ &= [U(b) + i \cdot V(b)] - [U(a) + i \cdot V(a)] \\ &= [W(b) - W(a)] \\ &= [W(t)]_a^b\end{aligned}$$

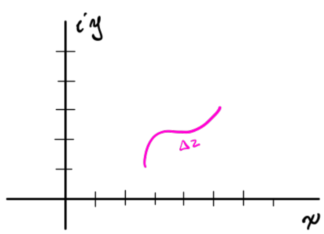
■

9.2 Contours

Integrals of complex valued functions are of the form:

$$\lim_{n\to\infty}\left(\sum_{i=1}^{n}\left[f\left(z_i^*\right)\cdot\Delta z\right]\right)$$

Δz could be along any curve in the complex plane rather than just the axis along which we would integrate:



this short curve is what we call a contour, the complex integral will be a line integral along that curve:

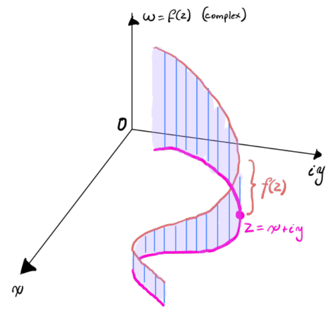


Figure 9.1:

Define Contours
A set of points $z = (x, y)$ in the complex plane is said to be an ***arc*** or a curve that we will call C if:

$$x = x\left(t\right), \quad y = y\left(t\right) \qquad \left(a \leq t \leq b\right) \tag{9.2}$$

where:

- $x\left(t\right)$ and $y\left(t\right)$ are continuous functions

  continuous however does not mean smooth/differentiable, sharp points are allowed.
- the output values are ordered corresponding to t (as if t represented time)

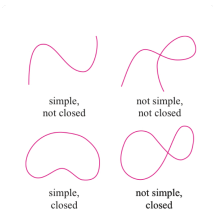
It is more convenient to describe the curve at (9.2) as:

$$\begin{aligned} z &= z\left(t\right) \\ &= x\left(t\right) + i \cdot y\left(t\right) \end{aligned} \tag{9.3}$$

Types of Arcs

- A **Simple Curve** does not cross itself
- a **Closed Curve** meets back up with itself
- a **Simple Closed Curve** is closed and does not cross itself (except where it closes)
- a **Positive Curve** moves counter-clockwise as t increases.

Parametric Representation not unique
Also the parametric representation of such a curve is not unique, the same curve could be represented by lots of different parametric equations, just like a second degree polynomial can also be modelled by a 3rd degree polynomial between intervals.



9.3 Contour Integral

As previously discussed, an integral with respect to z will be a contour integral:

$$\int_C f(z) \, \mathrm{d}\, z$$

If the value of the integral does not depend on the path of the contour and only depends on the endpoints then we would write:

$$\int_C f(z) \, \mathrm{d}\, z = \int_{z_1}^{z_2} f(z) \, \mathrm{d}\, z$$

We can also put the integral in terms of the contour parameters, This is basically *Integration by Substitution* in reverse, there's a more rigorous proof in the textbook by *Osbourne*, just take it as definition.

$$\int_C (f(z)) \, \mathrm{d}\, z = \int_a^b (f[z(t)] \cdot z'(t)) \, \mathrm{d}\, t \tag{9.4}$$

Negative Contours
If a contour C is given the contour $-C$ denotes the same contour with the order of the points reversed. Hence we have:

$$\begin{aligned}
\int_{-C} (f(z)) \, \mathrm{d}\, z &= \int_{-b}^{-a} \left(f[z(-t)] \cdot \frac{\mathrm{d}}{\mathrm{d}\, t} (z(-t)) \right) \mathrm{d}\, t \\
&= -\int_a^b (f[z(-t)] \, z'(-t)) \, \mathrm{d}\, t \\
&= -\int_C (f(z)) \, \mathrm{d}\, z
\end{aligned}$$

Refer to p. 128 of *Churchill's 9th* for worked examples.

9.3.1 Notation

If we want to basically i want to know does the symbol:

$$\oint f(z) \, \mathrm{d}\, z$$

- a contour/line integral of any sort
- a contour/line integral around a closed path
- one of the above but specifically in the anticlockwise direction?

9.3.2 Upper Bounds for Moduli

Start with the triangle inequality:

$$|\alpha + \beta| \leq |\alpha| + |\beta|$$

We could prove this, or, by similar reasoning:

$$\implies \left| \int (w(t)) \, \mathrm{d}\, t \right| \leq \int |w(t|) \, \mathrm{d}\, t$$

Now let:

- C be a contour of length L
- $|f(z)| \leq M$

Using the inequality above it can be rigorously shown:

$$\left| \int (f(z)) \, \mathrm{d}\, z \right| \leq M \cdot L$$

This can be somewhat visualised if $f(z)$ is taken to be the height of the function along the contour, and L is taken to be the length of the contour, although the values will be complex and height might become an odd concept, the mathematics should still hold.

Example Evaluate the contour integral:

$$\int_C \left(\frac{1}{z} \right) \mathrm{d}\, z$$

Where:

- $C : |z| = 1$

Now put the integral in terms of the parametric representation (9.4):

$$\begin{aligned}
\int_C \left(\frac{1}{z} \right) \mathrm{d}\, z &= \int_\pi^\pi \left(\frac{1}{e^{i\theta}} \cdot \frac{\mathrm{d}}{\mathrm{d}\, \theta} \left(e^{i\theta} \right) \right) \mathrm{d}\, \theta \\
&= \int_\pi^\pi \left(\frac{1}{e^{i\theta}} \cdot i \cdot e^{i\theta} \right) \mathrm{d}\, \theta \\
&= \int_\pi^\pi (i) \, \mathrm{d}\, \theta \\
&= [i \cdot \theta]_\pi^\pi \\
&= 0
\end{aligned} \tag{9.5}$$

9.4 Antiderivatives

Generally the value of a contour integral depends on both the path of the contour and the function. Some contour intergrals however have values independent of the path, for example consider how many integrals over closed paths will have an integral of zero as in the example above.

The antiderivative of a complex function is $F(z)$ such that:

$$F'(z) = f(z)$$

The antiderivative is, of necessity, an analytic function.

The antiderivative is uniqe, there is only one antiderivative for a given funciton (other than the additive constant $(+C)$ component).

9.4.1 Basic Antiderivative Theorem

Suppose $f(z)$ is continuous in a domain, if the function has an antiderivative $F(z)$ through the domain then:

> integrals of $f(z)$ along contours lying in the domain depend only on the end-points of that contour:
>
> $$\int_C f(z)\,\mathrm{d}\,z = \int_{z_1}^{z_2} f(z)\,\mathrm{d}\,z = [F(z)]_{z_1}^{z_2}$$
>
> This also means that integrals around a closed contour will be 0.

Proof if $F'(z) = f(z)$:

$$\begin{aligned}\int_C f(z)\,\mathrm{d}\,z &= \int_a^b f(z(t))\,\mathrm{d}\,t \\ &= \int_a^b U(z(t))\,\mathrm{d}\,t + i\cdot\int_a^b b(z(t))\,\mathrm{d}\,t\end{aligned}$$

Which means the Fundamental Theorem of Calculus Applies from (9.1)

$$\begin{aligned}&= F(z(b)) - F(z(a)) \\ &= F(z_2) - F(z_1) \\ &= [F(z)]_{z_1}^{z_2} \\ &= [U(t)]_a^b + i\cdot[V(t)]_a^b \\ &= U(b) - U(a) + i\cdot[V(b) - V(a)] \\ &= [U(b) + V(b)] - i\cdot[U(a) - V(b)] \\ &= F(b) - F(a) \\ &= [F(z)]_a^b \\ &= [F(z)]_{z_1}^{z_2}\end{aligned}$$

9.5 Cauchy Goursat Theorem

This theorem depends on *Green's Theorem* which gives the relationship between a line integral around a simple-closed curve and the double integral over the corresponding bounded region, for purely real functions. Roughly Speaking *Green's Theorem* is a counterpart of the Fundamental Theorem of Calculus for double integrals.

In establishing the *Cauchy-Goursat Theorem*, proving *Green's Theorem* is the tricky part, the *Cauchy-Goursat Theorem* more or less falls out of it.

the *Cauchy-Goursat Theorem* If a function f is analytic at all points interior to, and on, a simple closed contour C, then:

$$\oint_C f(z)\,\mathrm{d}\,z = 0$$

9.5.1 Simply-Connected Domains

If $f(z)$ is analytic through a simply connected domain:

$$\int_C f(z)\,\mathrm{d}\,z$$

Where:

- C is any closed contour in that domain (Simple or not)

  i.e. if the domain is simply connected, the contour can cross itself.

From this it follows:

- If a function is analytic on a simply connected domain, then:

  there will be an anti-derivative:

  The contour integral will be independent of path (i.e.$F(b) - F(a)$

- Entire functions always have an antiderivative

  and will always be independent of path.

9.5.2 Multiply Connected Domains

If C is a closed counter-clockwise contour, inside which are closed Clockwise contours $c_1, c_2, c_3 \ldots$ for which the function is analytic interior to C but exterior to $c_1, c_2, c_3 \ldots$:

Then we have:

$$\int_C f(z)\,\mathrm{d}\,z + \sum_{n=1}^{k}\left[\int_{c_n} f(z)\,\mathrm{d}\,z\right]$$

This is established by drawing a line through all the interior closed contours and applying the *Cauchy-Goursat* theorem, refer to the *Churchill* textbook.

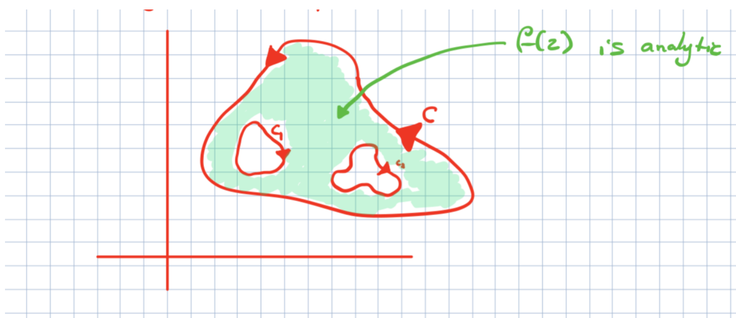


Figure 9.2:

Principle of Deformation If we had Something like this:

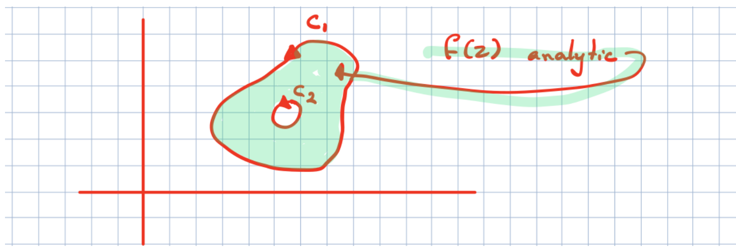


Figure 9.3:

Then it would follow:

$$\int_{C_1} f(z)\,\mathrm{d}\,z + \int_{c_2} f(z)\,\mathrm{d}\,z = 0$$
$$\int_{C_1} f(z)\,\mathrm{d}\,z - \int_{c_2} f(z)\,\mathrm{d}\,z = 0$$
$$\int_{C_1} f(z)\,\mathrm{d}\,z = \int_{c_2} f(z)\,\mathrm{d}\,z$$

So if c_2 can be continuously transformed into C_1 through an analytic region, then the integral is the same.

Spacial Case of Deformation We Should Probably memorise this special case:

$$\oint_{|z-z_0|=r} (z-z_0)^n\,\mathrm{d}\,z = \begin{cases} 0 & if n \neq -1 \\ 2\pi i & if n = -12 \end{cases}$$

Example

$$\int_{|z|=1} \frac{1}{z^2+2z+2}\,\mathrm{d}\,z = \int_{|z|=1} \frac{1}{(z+(1+i))\cdot(z+(1-i))}\,\mathrm{d}\,z$$

There is a singularity at $z = -1 \pm i$, which is outside the closed countour, the function is analytic everwhere else inside and on the closed countour hence by the *Cauchy-Goursatt Theorem:*

$$\int_{|z|=1} \frac{1}{z^2 + 2z + 2} \,\mathrm{d}\, z = 0$$

9.6 Cauchy Integral Formula

This has to be memorised.

Cauchy Integral Formula
If f is analytic everwhere inside and on C and z_0 is interior to C:

$$\int_C \frac{f(z)}{z - z_0} \,\mathrm{d}\, z = 2\pi i \cdot f(z_0)$$

Example Solve :

$$\int_{|z|=1} \frac{\cos z}{z^3 + 9z} \,\mathrm{d}\, z$$

First simplify the intergrand somewhat:

$$\int_{|z|=1} \frac{\cos z}{z^3 + 9z} \,\mathrm{d}\, z = \int_{|z|=1} \frac{1}{z} \cdot \frac{\cos z}{z^2 + 9z} \,\mathrm{d}\, z$$

Observe that the intergrand $\frac{\cos z}{z^3+9z}$ is analytic everywhere other than its singularities:

- $z = 0$
- $z = 3i$
- $z = -3i$

because $z = 0$ is inside the contour we cannot use the *Cauchy-Goursatt Theorem*, however, we can use the *Cauchy-Integral Formula* precisely because :

- $z_0 = 0$ is interior to the closed contour
- $f(z) = \frac{\cos z}{z^2+9}$ is analytic everywhere else interior to the closed contour

So by the Cauchy Integral Formula:

$$\begin{aligned}\int_{|z|=1} \frac{\frac{\cos z}{z^2+9}}{(z-0)} \,\mathrm{d}\, z &= 2\pi i \cdot \frac{\cos(0)}{0^2 + 9} \\ &= i \cdot \frac{2\pi}{9}\end{aligned}$$

9.7 Extension of the Cauchy Integral Formula

This has to be memorised:

Extension to Cauchy Integral

if:

- $f(z)$ is analytic on and interior to some closed contour C
- z_0 is interior to C

Then:

$$\int_C \frac{f(z)}{(z-z_0)^{n+1}} \,\mathrm{d}\, z = f^n(z_0) \cdot \frac{2\pi i}{n!}$$

Chapter 10

Cauchy Integral Formula

10.1 Power Series and Uniform Continuity

10.1.1 Power Series

Convergence

A sequence $x = (x_n)$ converges if:

$$\begin{aligned}&\forall \varepsilon > 0, \quad \exists N : \\ &\qquad n \geq N \implies |x_n - x| < \varepsilon\end{aligned}$$

and it is hence expressed:

$$\lim(x_n) = x$$

A series is generated by a sequence,
If (a_n) is a sequence, the series is (S_n):

$$\begin{aligned}S_1 &= a_1 \\ s_2 &= S_1 + a_2 \\ S_3 &= S_2 + a_3 \\ S_4 &= S_3 + a_4 \\ &\ldots\end{aligned}$$

The series is convergent if:

$$\begin{aligned}&\forall \varepsilon > 0, \exists N : \\ &\qquad n \geq N \implies |S_n - L| < \varepsilon\end{aligned}$$

The series is absolutely convergent if $|S_n|$ is convergent.

The notation for series used is:

$$\sum_{n=1}^{\infty}[(a_n)] = \sum (a_n) = \lim [S_n]$$

Be mindful that this notation is used ambiguously to represent both:

- The infinite Series
- The limit value of the series.

In practice however the ambiguity is a non-issue because context will discern the difference.

Sequences of Functions[1]

We can have sequences of real numbers, and similarly we can have sequences of functions.
Sequences of functions can converge in two ways:

- Pointwise
- uniformly

Uniform convergence is important because it preserves term properties to the limit function which will be seen.

Define a sequence of functions: Let $A \subseteq \mathbb{R}, n \in \mathbb{N}$ and take some function f :

$$f_n : A \to \mathbb{R}$$

It is said that (f_n) is a sequence of functions on A to $\mathbb{R}$.
For every $x \in A$ there will be a sequence of real numbers:

$$f_n(x)$$

For some values of x, the sequence may converge, for others it may not.

- The point of convergence is $\lim \_\{\}[f_n(x_n)]$ which depends on the choice of x.
- Thus we could Create a set of all $x \in A$ forwhich $(f_n(x))$ converges.
  - This set would be a domain for a function $f(x)$ that would act as the limit of the sequence $(f\_n(x))$.

[1]This is in the Bartle and Sherbert Textbook at Chapter [8.1.7] p. 246

Pointwise Convergence Take some function:

$$f : A_0 \to \mathbb{R} \qquad (A_0 \subseteq A \subseteq \mathbb{R})$$

We say that the sequence is pointwise convergent if,

- for every $x \in A_0$
  - The sequence $(f\_n\left(x\right))$ converges to $f\left(x\right)$.

e.g. consider $g_n\left(x\right) := x^n$;

$$\left(g_n\left(x\right)\right) = \left(x, x^2, x^3, x^4 \cdots\right)$$

If $-1 < x < 1$, then x is a fraction or zero so:

$$\left(x, x^2, x^3, x^4 \cdots 0\right) \qquad \text{Converges to } 0$$

if $x = 1$, then:

$$\left(x, x^2, x^3 \cdots\right) = \left(1, 1^2, 1^3, \cdots 1\right) \qquad \text{Converges to } 1$$

if $x = -1$, then:

$$\left(x, x^2, x^3, \cdots\right) = \left(1, -1, 1, -1, \cdots \pm 1\right) \qquad \text{Divergent}$$

if $|x| > 1$, then:

$$\left(x, x^2, x^3, \cdots \infty\right) \qquad \text{Divergent}$$

So $\lim\_\{\}\left[g_n\left(x\right)\right]$ on the set $(-1, 1]$ where:

$$g\left(x\right) = \begin{cases} 0, & \text{for } \ (-1 < x < 1) \\ 1, & \text{for } \ (x = 1) \end{cases}$$

Definition An alternative, but equivalent definition for pointwise convergence is:

$$\forall \varepsilon > 0, \ \ \forall x \in A_0, \ \ \exists N : \tag{10.1}$$

$$\tag{10.2}$$

$$n \geq N \implies \left|f_n\left(x\right) - f\left(x\right)\right| < \varepsilon \tag{10.3}$$

Where N is a function of x and ε, i.e.:

- $N = N(\varepsilon, x)$.

###Definition of Power Series A power series is a series of the form:

$$\sum_{n=0}^{\infty} [c_n x^n] = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots \tag{10.4}$$

More generally a series will be of the form:

$$\sum_{n=0}^{\infty} [c_n (x-a)^n] = c_0 + c_1 (x-a) + c_2 (x-a)^2 + \cdots \tag{10.5}$$

Convergence of Power Series For any given power series of the form $\sum\{\infty\}\_\{n=0\} [c_n (x-a)^n]$ there are only three possibilities:

1. The series converges only when $x = a$
2. The series converges for all x.
3. There is a positive number R such that the series converges if $|x-a| < R$ and diverges if $|x-a| > R$.

Why power Series The whole idea of power series is representing a known function as an infinite series, this is useful for integrating functions that don't have elementary antiderivatives.

Take for example the geometric series:

$$\sum_{n=0}^{\infty} [a x^n] = \frac{a}{1-x} \tag{10.6}$$

$$\implies f(x) = \frac{a}{1-x} = \sum_{n=0}^{\infty} [a x^n] \tag{10.7}$$

Example Find a power series representation for $f(x) = \frac{x^3}{x+2}$

$$\frac{1}{2+x} = \frac{1}{2\left(1+\frac{x}{2}\right)} \tag{10.8}$$

$$= \frac{1}{2\left(1-\left(-\frac{x}{2}\right)\right)} \tag{10.9}$$

$$= \frac{1}{2} \cdot \sum_{n=0}^{\infty} \left[-\frac{x}{2}\right]^n \tag{10.10}$$

$$= \sum_{n=0}^{\infty} \left[\frac{(-1)^n}{2^{n+1}} \cdot x^n\right] \tag{10.11}$$

$$\implies \frac{x^3}{2+x} = \sum_{n=0}^{\infty} \left[\frac{(-1)^n}{2^{n+1}} \cdot x^{n+3}\right] \tag{10.12}$$

because this is a geometric series, it converges when:

$$\left|-\frac{x}{2}\right| < 1 \tag{10.13}$$
$$\implies x \in (-2, 2) \tag{10.14}$$
$$\text{This is known as the radius of convergence } (R = 2) \tag{10.15}$$

So from this we could show something like:

$$f(x) = \frac{x^3}{2+x} = \sum_{n=0}^{\infty}\left[\frac{(-1)^n}{2^{n+1}} \cdot x^{n+3}\right] \tag{10.16}$$
$$\int \frac{x^3}{2+x}\,\mathrm{d}\,x = C + \sum_{n=0}^{\infty}\left[\frac{(-1)^n}{2^{n+1}} \cdot \frac{x^{n+4}}{n+4}\right] \tag{10.17}$$

The important part with all of this is that it works with Taylor Series.[2]

Convergence of Power Series

if ∑{∞}_{n = 0} $[a_n \cdot (z - z_0)^n]$ is convergent for some $z = \alpha$ then, * It will converge absolutely for all values of $|z - z\_0| > |z - \alpha|$

Uniform Convergence of Power Series

if ∑{∞}_{n = 0} $[a_n \cdot (z - z_0)^n]$ converges when $z = \alpha$ but $(\alpha \neq z_0)$: * Then the series converges uniformly in any open-neighbourhood:

$$|z - z_0| \leq r$$
$$\text{Where } r = |z_0 - \alpha|$$

- The sum of the series represents an analytic function, i.e.:

$$f(z) = \sum_{n=0}^{\infty}[a_n \cdot (z - z_0)^n]$$

  Such that $f(z)$ is an analytic function that can also be represented by the power series

Circle of Convergence

∑{∞}_{n = 0} $[a_n \cdot (z - \alpha)^n]$ is convergent only for $|z - \alpha| < R$:

- if $R = 0$, the series is only convergent for $z = \alpha$
- if $r = \infty$, the series is convergent $\forall z \in \mathbb{C}$
- if $R \in \mathbb{R}^+$, the series is convergent on some open disc centred at α of radius R.

[2] Refer to Ch. 58, p. 190 of *Churchill's Complex Variables* 9thEd.

Taylor Series[3]

$$f(z) = \sum_{n=0}^{\infty} \left[a_n \cdot (z-\alpha)^n\right], \qquad |z - z_0| < r$$

$$\text{where: } a_n = \frac{f^{(n)}(z_0)}{n!}$$

This can be shown using the *Cauchy Integral Formula*, however a clearer justification is:

- If a function a power series representation for a function exists:

$$\begin{aligned}
f(z) &= \sum_{n=0}^{\infty} \left[c_n \cdot (z-\alpha)^n\right] && \scriptstyle C_n \text{ and } \alpha \text{ are complex constants} \\
&= c_0 + c_1 (z-\alpha) + c_2 (z-\alpha)^2 \cdots \\
\implies f(\alpha) &= c_0 \times 0^0 + c_1 \times 0 + 0 + \cdots \\
&= c_0
\end{aligned}$$

Consider the first derivative:

$$\begin{aligned}
f'(z) &= c_1 + 2 \cdot c_2 (z-\alpha) + 3 \cdot c_2 \cdot (z\alpha)^2 \cdots \\
\implies f'(\alpha) &= c_1
\end{aligned}$$

Consider the second derivative:

$$\begin{aligned}
f''(z) &= 2 \cdot c_2 + 3 \times 2 \cdot (z-\alpha) + 4 \times 3 \times 2 \cdot (z-\alpha)^2 + \cdots \\
&= 2! c_2 + 3! (z-\alpha) + 4! \cdot (z-\alpha)^2 + \cdots \\
f''(\alpha) &= 2c_2
\end{aligned}$$

By this logic the nth derivative will be:

$$\begin{aligned}
f^{(n)} &= n! \cdot c_n \\
\implies c_n &= \frac{f^{(n)}(\alpha)}{n!}
\end{aligned}$$

We will always be able to find c_n where $f(z)$ is analytic and clearly the power series will be convergent (i.e. to $f(z)$) on a radius of convergence where $f(z)$ is analytic, so:

$$f(z) = \sum_{n=0}^{\infty} \left[\frac{f^{(n)}(\alpha)}{n!} \times (z-\alpha)\right], \qquad \underset{R \text{ is the radius of the open disc of analyticity}}{|z - z_0| < R}$$

[3] *3Blue1Brown* does a nice video on this If a function is analytic at α and in an open disc $|z-\alpha| < R$, there will always be a power series representation of $f(z)$:

Figure 10.1: drawing

Figure 10.2: drawing2

The Cauchy Hadamard Theorem If we have a power series:

$$\sum_{n=0}^{\infty} [a_n \cdot (z - z_0)^n]$$

Then we can find the radius of convergence:

$$l = \lim\sup \left[|a_n|^{\frac{1}{n}}\right]$$

$$\implies R = \frac{1}{l}$$

Further if * |a_n| $\neq$ _0 and $\lim \left[\left|\frac{a_{n+1}}{a_n}\right|\right] = R$ Then: * $\lim \left[|a_n|^{\frac{1}{n}}\right] = \lim \left[\left|\frac{a_{n+1}}{a_n}\right|\right] = R$

Generally the n^{th} root test is more powerful than the ratio test, however the ratio test is the only test that can deal with factorials[4], so it is important to have it in our toolbox.

10.1.2 Uniform Continuity

What is uniform continuity

Imagine the function $y = \frac{1}{x}$ and consider the interval $(0, \infty)$, * Consider the limit value of 2 (e.g. the line $y = 2$)

If I choose some ε value, there is always a corresponding δ-value, this δ-value will depend on the ε-value:

so the function $f(x) = \frac{1}{x}$ is continuous because: * ε can be chosen anywhere * Any ε value will have a corresponding δ value.

The function would be uniformly continuous, if: * ε can be chosen anywhere * The Corresponding δ value will exist AND not change size wherever ε is chosen.

So in this case the function is not uniformly continuous, because if I move ε down, δ would have to get larger, so it IS NOT uniformly continuous:

So basically:

[4] How to choose a test by pauls Online Notes

Figure 10.3: drawing3

- A function is continuous if a δ-value always exists and can be described as a function:
  - $\delta = \delta(\varepsilon, x)$
- A function is uniformly continuous if a δ-value always exists and can be described as a function only of ε :
  - $\delta = \delta(\varepsilon)$

Cantor's Theorem

If a function is continuous on an interval $[a, b]$, it is uniformly continuous on that interval.

- The reasoning being that basically you could chose the smallest δ-value that will work at all points on that interval

Why is uniform continuity important?

Something like,

- if $f(x)$ is uniformly continuous
- Then:

$$\int \lim [f(x)] \,\mathrm{d}\, x \iff \lim \left[\int f(x) \,\mathrm{d}\, x \right]$$

  but I'm not sure about this so don't quote me, we don't do it here so don't worry about it too much, we just need to show that we understand it the idea of uniformly continuous functions.

Problem Example

Prove that the function $f(x) = \frac{1}{1+x^3}$ is uniformly continuous on the interval $[1, \infty)$

So the first thing to notice is that *Cantor's Theorem* cannot be applied because it is an open interval.

State the Definition $f(x)$ is uniformly convergent if:

$$\forall x, y \in [1, \infty), \ \ \forall \varepsilon > 0, \ \exists \delta(\varepsilon) : \\ 0 < |x - y| < \delta \implies |f(x) - f(y)| < \varepsilon$$

Work Backwards from the ε Definition

$$
\begin{aligned}
|f(x) - f(y)| &= \left| \frac{1}{1+x^3} - \frac{1}{1+y^3} \right| \\
&= \frac{\left|y^3 - x^3\right|}{|1+x^3| \times |1+y^3|} \\
&\leq \frac{\left|y^3 - x^3\right|}{|1+x^3|} \\
&= \frac{|y-x| \cdot \left|y^2 + xy + x^2\right|}{|1+x^3|}
\end{aligned}
$$

Without loss of generality assume that $y < x$

$$
\leq |y-x| \times 3 \cdot \frac{\left|x^2\right|}{|1+x^3|}
$$

$$
\text{Recall that } x \geq 1 \implies \frac{1}{|1+x^3|} < \frac{1}{|x|^3}
$$

$$
\begin{aligned}
&\leq |y-x| \cdot \frac{1}{|x|} \cdot 3 \\
&\leq |y-x| \cdot 3 \\
&\leq 3 \cdot \delta
\end{aligned}
$$

So choose δ:

$$
\begin{aligned}
3\delta &\leq \varepsilon \\
\delta &\leq \frac{\varepsilon}{3}
\end{aligned}
$$

$\therefore$ it is sufficient to choose $\delta \leq \frac{\varepsilon}{3}$.

Proof

$$
\forall x, y \in [1, \infty), \ \ \forall \varepsilon > 0, \ \exists \delta \leq \frac{\varepsilon}{3}
$$

$$
\begin{aligned}
|x-y|<\delta \implies |f(x)-f(y)| &\leq \left|\frac{1}{1+x^3}-\frac{1}{1+y^3}\right| \\
&\leq |y-x|\cdot\left(\frac{\left|y^2\right|+|xy|+\left|x^2\right|}{|1+x^3|\cdot|1+y^3|}\right) \\
&\leq |y-x|\cdot\left(\frac{y^2}{|1+y^3|}+\frac{|x|\cdot|y|}{|1+x^3|\cdot 1+y^3}+\frac{\left|x^2\right|}{|1+x^3|}\right) \\
&leq\, |y-x|\cdot\left(\frac{|y|^2}{|y^3|}+\frac{|x|\cdot|y|}{|x|^3\cdot|y|^3}+\frac{|x|^2}{|x|^3}\right) \\
&\leq |y-x|\cdot\left(\frac{1}{|y|}+\frac{1}{|x|\cdot|y|}+\frac{1}{|x|}\right) \\
&\leq |y-x|\cdot\left(\frac{1}{1}+\frac{1}{1\times 1}+\frac{1}{1}\right) \\
&\leq |y-x|\cdot 3 \\
&< 3\cdot\delta \\
&< 3\times\frac{\varepsilon}{3} \\
&< \varepsilon
\end{aligned}
$$

□

Part II

Tutorial Workings

Part III

Assessment Material

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