(05) Continuity

Wk 4 Material; Topic 3; Due 28 March

Continuous Functions [5.1]

Definition of Continuity

Take some function $f : A \to B$ where $A \subseteq \mathbb{R}$:

- the function f is said to be continuous at some point $c \in A$ if and only if $\lim_{x \to c} = f(c)$

Rigorous Definition So let's phrase that using the ε-δ definition of the limit:

f is continuous at c, iff

$\forall \varepsilon > 0, \quad \exists \delta > 0 :$

$$|x - c| < \delta \implies |f(x) - f(c)| < \varepsilon$$

Figure 1:

In Terms of Neighborhoods This can be expressed in terms of neighborhoods:

f is continuous at c, iff and only if

$\forall \varepsilon > 0, \ \exists \delta > 0 :$

x in δ-neighborhood of c

$\implies$ $f(x)$ is in ε-neigborhood of $f(c)$

which is the same as saying:

$$f\left(A \cap V_\delta(c)\right) \subseteq V_\varepsilon(f(c))$$

Conditions for Continuity If c is a cluster point of A, then trhee conditions must hold for f to be continuous at c, that is to say that three conditions must hold for $\lim_{x \to c} = f(c)$:

1. f must be defined at c
    - so that $f(c)$ actually has meaning
2. The limit of f at c must exist in $\mathbb{R}$ so that
    - $\lim_{x \to c}$ actually has a meaning
3. These two values are equal
    - $\lim_{x \to c} = f(c)$

    Cluster Points

    A cluster point has infinitely divisible values either side of it, if a value is not a cluster point it's just an isolated point and it is said to be continuous at that point, so generally we just assume points are cluster points because if they're not then they're automatically continuous and so not very interesting.

Sequential Criterion for Continuity [5.1.3]

Just like a limits can be defined in terms of sequences (at (4.1.8) of the TB), continuity can hence be defined in terms of sequences:

A function $f : A \to \mathbb{R}$ is continuous at some point $c \in A$ if and only if:

- for every sequence (x_n) in A that converges to c
    - $f((x_n))$ converges to c

Combinations of Continuous Functions [5.2]

Continuous Functions on Intervals [5.3]

Uniform continuity [5.4]