Set-Valued Fan-Takahashi Inequalities Via Scalarization

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Introduction

Introduction

Let us consider a common scalar optimization problem

$$\min g(x) \quad \text{s.t.} \quad x \in C \tag{1}$$

where C is a given nonempty set in a space X and $g\colon C \to \mathbb{R}$ a given function. Let $x_0 \in C$ be a solution of the problem (1), which implies

$$g(x_0) \leq g(y) \quad \forall y \in C$$

Setting $f(x, y) := g(x) - g(y)$ for $x, y \in C$, x_0 also solves

$$\text{find} \quad x_0 \in C \quad \text{such that} \quad f(x_0, y) \leq 0 \quad \forall y \in C. \tag{2}$$

Theorem (Takahashi [5])
Let X be a nonempty compact convex subset of a Hausdorff topological vector space and $f : X \times X \to \mathbb{R}$. If f satisfies the following conditions:

1. for each fixed $y \in X$, $f(\cdot, y)$ is lower semicontinuous,
2. for each fixed $x \in X$,$f(x, \cdot)$ is quasi concave,
3. $f(x, x) \leq 0$ for all $x \in X$,

then there exists $\bar{x} \in X$ such that $f(\bar{x}, y) \leq 0$ for all $y \in X$.

Motivation

Theorem (Fan [2])
Let X be a nonempty compact convex set in a Hausdorff topological vector space and $f : X \times X \to \mathbb{R}$. If f satisfies the following conditions:

1. for each fixed $y \in X$, $f(\cdot, y)$ is lower semicontinuous,
2. for each fixed $x \in X$,$f(x, \cdot)$ is quasi concave,

then the minimax inequality

$$\min_{x \in X} \sup_{y \in X} f(x, y) \leq \sup_{x \in X} f(x, x)$$

holds.

Background

Background

- Georgiev and Tanaka [3] extended the minimax inequality to the form of set-valued maps.

- Kuwano, Tanaka, and Yamada [4] constructed the result of four types set-valued minimax inequalities with set relations.

- Our goal is to generalize the result of four types set-valued minimax inequalities without set-relations and the scalarization functions.

Preliminaries

Preliminaries

Let X be a topological space, Y a real topological vector space, and θ_Y be a zero vector in Y. Define that $\mathcal{P}(Y)$ is the set of all nonempty subsets of Y. The sets of neighborhoods of $x \in X$ and $y \in Y$ is denoted by $\mathcal{N}_X(x)$ and $\mathcal{N}_Y(y)$, respectively.

Preliminaries

Definition [1]

Let $F\colon X \to \mathcal{P}(Y)$, $x_0 \in X$, $\preccurlyeq$ a binary relation on $\mathcal{P}(Y)$ and $C \subset Y$ a convex cone. We say that F is $(\preccurlyeq, C)$-continuous at x_0 if

$$\forall W \subset Y, W \text{ open}, W \preccurlyeq F(x_0), \exists V \in \mathcal{N}_X(x_0) \text{ s.t. } W + C \preccurlyeq F(x), \forall x \in V.$$

Definition [1]

Let $\varphi\colon \mathcal{P}(Y) \to \mathbb{R} \cup \{\pm\infty\}$, $A_0 \in \mathcal{P}(Y)$, $\preccurlyeq$ a binary relation on $\mathcal{P}(Y)$, and C a convex cone in Y with $C \neq Y$. Then, we say that φ is $(\preccurlyeq, C)$-lower semicontinuous at A_0 if

$$\forall r < \varphi(A_0), \exists W \in \mathcal{P}(Y), W \text{ open, s.t. } W \preccurlyeq A_0 \text{ and } r > \varphi(A), \forall A \in U(W + C, \preccurlyeq);$$

where $U(V, \preccurlyeq) := \{A \in \mathcal{P}(Y) \mid V \preccurlyeq A\}$.

Theorem [1]
Let $F\colon X \to \mathcal{P}(Y)$, $\varphi\colon \mathcal{P}(Y) \to \mathbb{R} \cup \{\pm\infty\}$, $x_0 \in X$, $\preccurlyeq$ a binary relation on $\mathcal{P}(Y)$, and C a convex cone. If F is $(\preccurlyeq, C)$-continuous at x_0 and φ is $(\preccurlyeq, C)$-lower semicontinuous at $F(x_0)$, then $(\varphi \circ F)$ is lower semicontinuous at x_0.

Definition
Let $\mathcal{A} \subset \mathcal{P}(Y) \backslash \{\emptyset\}$. $\mathcal{A}$ is said to be convex if for each $A_1, A_2 \in \mathcal{A}$ and $\lambda \in (0,1)$,

$$\lambda A_1 + (1-\lambda) A_2 \in \mathcal{A}.$$

Definition
Let $\varphi\colon \mathcal{P}(Y) \to \mathbb{R} \cup \{\pm\infty\}$. Then,

1. φ is quasi convex if for any $\alpha \in \mathbb{R}$, $\operatorname{lev}(\varphi, \leq, \alpha) := \{A \in \mathcal{P}(Y) \backslash \{\emptyset\} \mid \varphi(A) \leq \alpha\}$ is convex.
2. φ is quasi concave if for any $\alpha \in \mathbb{R}$, $\operatorname{lev}(\varphi, \geq, \alpha) := \{A \in \mathcal{P}(Y) \backslash \{\emptyset\} \mid \varphi(A) \geq \alpha\}$ is convex.

Preliminaries

Let X be a nonempty set, Y a real topological vector space, C a convex cone in Y, and $F\colon X \to 2^Y \backslash \{\emptyset\}$ a set-valued map.

1. F is called $(\preccurlyeq)$-naturally quasi convex if for each $x, y \in X$ and $\lambda \in (0,1)$, there exists $\mu \in [0,1]$ such that
$$F(\lambda x + (1-\lambda)y) \preccurlyeq \mu F(x) + (1-\mu)F(y).$$
2. F is called $(\preccurlyeq)$-naturally quasi concave if for each $x, y \in X$ and $\lambda \in (0,1)$, there exists $\mu \in [0,1]$ such that
$$\mu F(x) + (1-\mu)F(y) \preccurlyeq F(\lambda x + (1-\lambda)y).$$

Main results

Specific scalarization function

To extend Ky Fan inequality for set-valued maps with a binary relation, consider assumptions of scalarization fucntions. To begin with, introduce four properties;

1. φ is $(\preccurlyeq, C)$-lower semicontinuous,
2. φ is quasi concave,
3. φ is $(\preccurlyeq)$-monotone,
4. $\varphi(\{\theta\}) = 0$,

and define the set of functions satisfying these properties as $\Phi(\preccurlyeq, C)$. In addition, establish three vaital properties for Ky Fan inequality;

$$\varphi(A) \leq 0 \Rightarrow A \preccurlyeq \{\theta\}. \tag{A1}$$

Main results

Theorem

Let X be a nonempty compact convex subset of a topological vector space, Y a real topological vector space, $\preccurlyeq$ a binary relation on $\mathcal{P}(Y)$, C a convex cone in Y, $\varphi\colon \mathcal{P}(Y) \to \mathbb{R} \cup \{\pm\infty\}$, and $F\colon X \times X \to \mathcal{P}(Y)\setminus\{\emptyset\}$ a set-valued map. For the scaralization function $\varphi \in \Phi(\preccurlyeq, C)$ satisfying Assumption (A1), if F satisfies the following conditions:

1. $(\varphi \circ F)(x, y) \in \mathbb{R}$ for all $x, y \in X$,
2. for each fixed $y \in X$, $F(\cdot, y)$ is $(\preccurlyeq, C)$-continuous,
3. for each fixed $x \in X$, $F(x, \cdot)$ is $(\preccurlyeq)$-naturally quasi concave,
4. for all $x \in X$, $F(x, x) \preccurlyeq \{\theta\}$,

then there exists $\bar{x} \in X$ such that $F(\bar{x}, y) \preccurlyeq \{\theta\}$ for all $y \in X$.

Main results

Corollary

- 具体的な二項関係とスカラー化関数を与えた結果を紹介する。

Conclusion

Conclusion

1. Fan-Takahashi minimax inequalitiy and its beckground are introduced.
2. We give a new result of set-valued Fan-Takahashi inequalities via scalarization.
3. One of the next steps is to find a scalarization function satisfying the properties $(A1)$ in $\Phi(\preccurlyeq, C)$.

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