

2 Asymptotic Cones and Functions

2.1 Definition of Asymptotic Cones

Ryota Iwamoto

March 27, 2023

We use the book; Asymptotic Cones and Functions in Optimization and Variational Inequalities (author: A.AUSLENDER and M.TEBOULLE), pp.25-31.

The set of natural numbers is denoted by \mathbb{N} , so that $k \in \mathbb{N}$ means $k = 1, 2, \dots$. A sequence $\{x_k\}_{k \in \mathbb{N}}$ or simply $\{x_k\}$ in \mathbb{R}^n is said to converge to x if $\|x_k - x\| \rightarrow 0$ as $k \rightarrow \infty$, and this will be indicated by the notation $x_k \rightarrow x$ or $x = \lim_{k \rightarrow \infty} x_k$. We say that x is a cluster point of $\{x_k\}$ if some subsequence converge to x . Recall that every bounded sequence in \mathbb{R}^n converges to x if and only if it is bounded and has x as its unique cluster point.

Let $\{x_k\}$ be a sequence in \mathbb{R}^n . We are interested in knowing how to handle convergence properties, we are led to consider direction $d_k := x_k \|x_k\|^{-1}$ with $x_k \neq 0, k \in \mathbb{N}$. From classical analysis, the Bolzano-Weierstrass theorem implies that we can extract a convergent subsequence $d = \lim_{k \in K} d_k$, $K \subset \mathbb{N}$, with $d \neq 0$. Now suppose that the sequence $\{x_k\} \subset \mathbb{R}^n$ is such that $\|x_k\| \rightarrow +\infty$. Then

$$\exists t_k := \|x_k\|, k \in K \subset \mathbb{N}, \text{ such that } \lim_{k \in K} t_k = +\infty \text{ and } \lim_{k \in K} \frac{x_k}{t_k} = d.$$

This leads us to introduce the following concepts.

Definition 2.1.1

A sequence $\{x_k\} \subset \mathbb{R}^n$ is said to converge to a direction $d \in \mathbb{R}^n$ if

$$\exists \{t_k\}, \text{ with } t_k \rightarrow +\infty \text{ such that } \lim_{k \rightarrow \infty} \frac{x_k}{t_k} = d.$$

Definition 2.1.2

Let C be a nonempty set in \mathbb{R}^n . Then the asymptotic cone of the set C , denoted by C_∞ , is the set of vectors $d \in \mathbb{R}^n$ that are limits in direction of the sequences $\{x_k\} \subset C$, namely

$$C_\infty = \{d \in \mathbb{R}^n \mid \exists t_k \rightarrow +\infty, \exists x_k \in C \text{ with } \lim_{k \rightarrow \infty} \frac{x_k}{t_k} = d\}.$$

From the definition we immediately deduce the following elementary facts.

Proposition 2.1.1

Let $C \subset \mathbb{R}^n$ be nonempty. Then:

- (i) C_∞ is a closed cone.
- (ii) $(\text{cl } C)_\infty = C_\infty$.
- (iii) If C is a cone, then $C_\infty = \text{cl } C$.

Proof. We will prove each part separately.

- (i) C_∞ is a closed cone.

We need to show two propositions: (i-a) C_∞ is a cone and (i-b) C_∞ is a closed set.

(i-a) We show that C_∞ is a cone, that is, $\forall \alpha \geq 0, d \in C_\infty, \alpha d \in C_\infty$.

Since 0 is an element of C_∞ , it is clear in the case of $\alpha = 0$.

(\because Since C is nonempty, we can take an element x_0 from C . In addition we take a sequence $\{t_k\}_{k=1}^\infty$ with $t_k \rightarrow +\infty$ as $k \rightarrow \infty$. Of course this sequence exists, for example $t_k := k$. By using $t_k := k$ and $x_k := x_0$, we can obtain 0 as the limit. Hence 0 is an element of C_∞ .)

Also we consider the other case $\alpha > 0$. To prove that C_∞ is a cone, we take any direction d from C_∞ . Since d is an element of C_∞ ,

$$\exists t_k \rightarrow +\infty, \exists x_k \in C \text{ with } \lim_{k \rightarrow \infty} \frac{x_k}{t_k} = d.$$

Then we define a sequence $\{t'_k\}_{k=1}^\infty := \frac{t_k}{\alpha}$, exactly whose limit becomes $+\infty$ as $k \rightarrow \infty$. Accordingly there exist $t'_k \rightarrow +\infty$ and $x_k \in C$ with

$$\lim_{k \rightarrow \infty} \frac{x_k}{t'_k} = \lim_{k \rightarrow \infty} \alpha \cdot \frac{x_k}{t_k} = \alpha d.$$

This means $d \in C_\infty$.

By these results, we can get

$$\forall \alpha \geq 0, d \in C_\infty, \alpha d \in C_\infty$$

. Therefore C_∞ is a cone.

(i-b) We show that C_∞ is a closed set. In order to prove closeness, we consider convergence of a sequence of C_∞ . First we take a sequence $\{d_k\}_{k=1}^\infty \subset C_\infty$ with $d_k \rightarrow d$ as $k \rightarrow \infty$ for some d . Then we don't forget that $d \in C_\infty$ is our goal.

For each $k \in \mathbb{N}$,

$$\exists \{x_k^{(n)}\}_{n=1}^\infty \subset C \text{ and } \{t_k^{(n)}\}_{n=1}^\infty \text{ with } t_k^{(n)} \rightarrow \infty \text{ as } n \rightarrow \infty.$$

The below figure represents $x_k^{(n)}$ and $t_k^{(n)}$.

Figure:

$k \setminus n$	1	2	\dots	m	\dots	limit
1	$x_1^{(1)}, t_1^{(1)}$	$x_1^{(2)}, t_1^{(2)}$	\dots	$x_1^{(m)}, t_1^{(m)}$	\dots	d_1
2	$x_2^{(1)}, t_2^{(1)}$	$x_2^{(2)}, t_2^{(2)}$	\dots	$x_2^{(m)}, t_2^{(m)}$	\dots	d_2
\vdots	\vdots	\vdots	\ddots	\vdots	\vdots	\vdots
m	$x_m^{(1)}, t_m^{(1)}$	$x_m^{(2)}, t_m^{(2)}$	\dots	$x_m^{(m)}, t_m^{(m)}$	\dots	d_m
\vdots			\vdots			

Then we define

$$x_m := x_m^{(m)} \text{ and } t_m := t_m^{(m)}.$$

By the definition of convergence of a sequence,

$$\begin{aligned} \forall \epsilon > 0, \exists \bar{m} \in \mathbb{N} \text{ s.t. } \forall m \geq \bar{m}, \|d_m - d\| < \frac{\epsilon}{2}, \text{ and} \\ \forall \epsilon > 0, \exists \hat{m} \in \mathbb{N} \text{ s.t. } \forall m \geq \hat{m}, \left\| \frac{x_m}{t_m} - d_m \right\| < \frac{\epsilon}{2}. \end{aligned}$$

Also, we let $\tilde{m} := \max \{\bar{m}, \hat{m}\} \in \mathbb{N}$. By using triangle inequality,

$$\forall \epsilon > 0, \exists \tilde{m} \in \mathbb{N} \text{ s.t. } \forall m \geq \tilde{m}, \left\| \frac{x_m}{t_m} - d \right\| < \epsilon.$$

$$(\because \left\| \frac{x_m}{t_m} - d \right\| \leq \left\| \frac{x_m}{t_m} - d_m \right\| + \|d_m - d\| < \epsilon..)$$

Therefore c_∞ is a closed set.

Then (i)'s proof is completed.

(ii) $(\text{cl } C)_\infty = C_\infty$.

We need to show two relations: (ii-a) $(\text{cl } C)_\infty \supset C_\infty$ (ii-b) $(\text{cl } C)_\infty \subset C_\infty$.

(ii-a) We show that C_∞ is included in $(\text{cl } C)_\infty$. However it is clear from the definition of asymptotic cone.

(ii-b) We show that $(\text{cl } C)_\infty \subset C_\infty$. In order to prove that a element of $(\text{cl } C)_\infty$ satisfies the asymptotic cone's relation, we consider convergency of a sequences of $(\text{cl } C)_\infty$ and $\text{cl } C$. First we take any $d \in (\text{cl } C)_\infty$ which satisfies

$$\exists t_k \rightarrow +\infty, \exists x_k \in \text{cl } C \text{ with } \lim_{k \rightarrow \infty} \frac{x_k}{t_k} = d.$$

For each $k \in \mathbb{N}$,

$$\exists \{y_k^{(n)}\}_{n=1}^\infty \subset C \text{ with } y_k^{(n)} \rightarrow x_k \text{ as } n \rightarrow \infty.$$

The below figure represents $y_k^{(n)}$.

Figure:

$k \setminus n$	1	2	\dots	m	\dots	limit
1	$y_1^{(1)}$	$y_1^{(2)}$	\dots	$y_1^{(m)}$	\dots	x_1
2	$y_2^{(1)}$	$y_2^{(2)}$	\dots	$y_2^{(m)}$	\dots	x_2
\vdots	\vdots	\vdots	\ddots	\vdots	\vdots	\vdots
m	$y_m^{(1)}$	$y_m^{(2)}$	\dots	$y_m^{(m)}$	\dots	x_m
\vdots				\vdots		

Then we define

$$y_m := y_m^{(m)}.$$

By the definition of convergence of a sequence,

$$\begin{aligned} \forall \epsilon > 0, \exists \bar{m} \in \mathbb{N} \text{ s.t. } \forall m \geq \bar{m}, \|d_m - d\| &< \frac{\epsilon}{2}, \\ \forall \epsilon > 0, \exists \hat{m} \in \mathbb{N} \text{ s.t. } \forall m \geq \hat{m}, \|y_m^m - x_m\| &< \frac{\sqrt{\epsilon}}{2}, \text{ and} \\ \forall \epsilon > 0, \exists \tilde{m} \in \mathbb{N} \text{ s.t. } \forall m \geq \tilde{m}, \left| \frac{1}{t_m} \right| &< \sqrt{\epsilon}. \end{aligned}$$

Also, we let $m_0 := \max \{\bar{m}, \hat{m}, \tilde{m}\} \in \mathbb{N}$. By using triangle inequality,

$$\forall \epsilon > 0, \exists m_0 \in \mathbb{N} \text{ s.t. } \forall m \geq \bar{m}, \left\| \frac{y_m}{t_m} - d \right\| < \epsilon.$$

$$(\because \left\| \frac{y_m}{t_m} - d \right\| \leq \frac{1}{|t_m|} \cdot \|y_m - d_m\| + \left\| \frac{y_m}{t_m} - d \right\| < \epsilon.)$$

Therefore $(\text{cl } C)_\infty \subset C_\infty$.

Then (ii)'s proof is also completed.

(iii) If C is a cone, then $C_\infty = \text{cl } C$.

□

The importance of the asymptotic cone is revealed by the following key property, which is an immediate consequence of its definition.

Proposition 2.1.2

A set $C \subset \mathbb{R}^n$ is bounded if and only if $C_\infty = \{0\}$.

Proof.

□

Associated with the asymptotic cone C_∞ is the following related concept, which will help us in simplifying the definition of C_∞ in the particular case where $C \in \mathbb{R}^n$ is assumed convex.

Definition 2.1.3

Let $C \in \mathbb{R}^n$ be nonempty and define

$$C_\infty^1 = \{d \in \mathbb{R}^n \mid \forall t_k \rightarrow +\infty, \exists x_k \in C \text{ with } \lim_{k \rightarrow \infty} \frac{x_k}{t_k} = d\}.$$

We say that C is asymptotically regular if $C_\infty = C_\infty^1$.

Proposition 2.1.3

Let C be a nonempty convex set in \mathbb{R}^n . Then C is asymptotically regular.

Proof.

□