Set-valued Ky Fan inequalities via scalarization

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1 Introduction

In convex analysis and optimization theory, Ky Fan minimax inequality plays a key role. A quarter century ago, Georgiev and Tanaka [3, 4] extended Ky Fan minimax inequality for set-valued maps. After that, Kuwano, Tanaka, and Yamada [7] constructed the result of four types set-valued Ky Fan minimax inequality with set relations [6], which are binary relations depending on a given convex cone. However, this result is limited to the case of the specific scalarization functions. To obtain more practical results, we need to generalize the convexity properties for set-valued maps. In addition, Dechboon and Tanaka [1] proposed generalized continuity to inherit properties of cone continuity for set-valued maps. The aim of this paper is to generalize the convexity properties for set-valued maps and to apply them to the set-valued Ky Fan minimax inequality.

2 Mathematical Preliminaries

Basically, let X be a topological space, Y a real topological vector space, and θ_Y be a zero vector in Y. Define that $\mathcal{P}(Y)$ is the set of all nonempty subsets of Y. The sets of neighborhoods of $x \in X$ and $y \in Y$ is denoted by $\mathcal{N}_X(x)$ and $\mathcal{N}_Y(y)$, respectively.

2.1 Set relations and these scalarization functions

Definition 2.1. For $A, B \in \mathcal{P}(Y)$, we define two binary relations on $\mathcal{P}(Y)$:

$$A \preccurlyeq_1 B \overset{\text{def}}{\Longleftrightarrow} A \cap B \neq \emptyset \quad \text{and} \quad A \preccurlyeq_2 B \overset{\text{def}}{\Longleftrightarrow} B \subset A.$$

2.2 Semicontinuity for set-valued maps

Definition 2.2 ([1])**.** Let $F\colon X \to \mathcal{P}(Y)$, $x_0 \in X$, $\preccurlyeq$ a binary relation on $\mathcal{P}(Y)$ and $C \subset Y$ a convex cone. We say that F is $(\preccurlyeq, C)$-continuous at x_0 if

$$\forall W \subset Y, W \text{ open}, W \preccurlyeq F(x_0), \exists V \in \mathcal{N}_X(x_0) \text{ s.t. } W + C \preccurlyeq F(x), \forall x \in V.$$

Definition 2.3 ([1])**.** Let $\varphi\colon \mathcal{P}(Y) \to \mathbb{R} \cup \{\pm\infty\}$, $A_0 \in \mathcal{P}(Y)$, $\preccurlyeq$ a binary relation on $\mathcal{P}(Y)$, and C a convex cone in Y with $C \neq Y$. Then, we say that φ is $(\preccurlyeq, C)$-lower semicontinuous at A_0 if

$$\forall r < \varphi(A_0), \exists W \in \mathcal{P}(Y), W \text{ open, s.t. } W \preccurlyeq A_0 \text{ and } r > \varphi(A), \forall A \in U(W + C, \preccurlyeq);$$

where $U(V, \preccurlyeq) := \{A \in \mathcal{P}(Y) \mid V \preccurlyeq A\}$.

Theorem 2.4 ([1])**.** *Let $F\colon X \to \mathcal{P}(Y)$, $\varphi\colon \mathcal{P}(Y) \to \mathbb{R} \cup \{\pm\infty\}$, $x_0 \in X$, $\preccurlyeq$ a binary relation on $\mathcal{P}(Y)$, and C a convex cone. If F is $(\preccurlyeq, C)$-continuous at x_0 and φ is $(\preccurlyeq, C)$-lower semicontinuous at $F(x_0)$, then $(\varphi \circ F)$ is lower semicontinuous at x_0.*

Definition 2.5. Let $\mathcal{A} \subset \mathcal{P}(Y) \backslash \{\emptyset\}$. $\mathcal{A}$ is said to be convex if for each $A_1, A_2 \in \mathcal{A}$ and $\lambda \in (0,1)$,

$$\lambda A_1 + (1 - \lambda) A_2 \in \mathcal{A}.$$

Definition 2.6. Let $\varphi\colon \mathcal{P}(Y) \to \mathbb{R} \cup \{\pm\infty\}$. Then,

(1) φ is quasi convex if for any $\alpha \in \mathbb{R}$, $\operatorname{lev}(\varphi, \leq, \alpha) := \{A \in \mathcal{P}(Y) \backslash \{\emptyset\} | \varphi(A) \leq \alpha\}$ is convex.

(2) φ is quasi concave if for any $\alpha \in \mathbb{R}$, $\operatorname{lev}(\varphi, \geq, \alpha) := \{A \in \mathcal{P}(Y) \backslash \{\emptyset\} \mid \varphi(A) \geq \alpha\}$ is convex.

2.3 Quasiconvexity properties for composite functions of set-valued map and scalarization function

Definition 2.7. Let X be a nonempty set, Y a real topological vector space, C a convex cone in Y, and $F\colon X \to 2^Y \backslash \{\emptyset\}$ a set-valued map.

(1) F is called $(\preccurlyeq)$-convex if for each $x, y \in X$ and $\lambda \in (0,1)$,

$$F(\lambda x + (1 - \lambda) y) \preccurlyeq \lambda F(x) + (1 - \lambda) F(y).$$

(2) F is called $(\preccurlyeq)$-properly quasi convex if for each $x, y \in X$ and $\lambda \in (0,1)$,

$$F(\lambda x + (1 - \lambda) y) \preccurlyeq F(x) \quad \text{or} \quad F(\lambda x + (1 - \lambda) y) \preccurlyeq F(y)$$

(3) F is called $(\preccurlyeq)$-naturally quasi convex if for each $x, y \in X$ and $\lambda \in (0,1)$, there exists $\mu \in [0,1]$ such that

$$F(\lambda x + (1-\lambda)y) \preccurlyeq \mu F(x) + (1-\mu)F(y).$$

Definition 2.8. Let X be a nonempty set, Y a real topological vector space, C a convex cone in Y, and $F\colon X \to 2^Y \backslash \{\emptyset\}$ a set-valued map.

(1) F is called $(\preccurlyeq)$-concave if for each $x, y \in X$ and $\lambda \in (0,1)$,

$$\lambda F(x) + (1-\lambda)F(y) \preccurlyeq F(\lambda x + (1-\lambda)y).$$

(2) F is called $(\preccurlyeq)$-properly quasi concave if for each $x, y \in X$ and $\lambda \in (0,1)$,

$$F(x) \preccurlyeq F(\lambda x + (1-\lambda)y) \quad \text{or} \quad F(y) \preccurlyeq F(\lambda x + (1-\lambda)y)$$

(3) F is called $(\preccurlyeq)$-naturally quasi concave if for each $x, y \in X$ and $\lambda \in (0,1)$, there exists $\mu \in [0,1]$ such that

$$\mu F(x) + (1-\mu)F(y) \preccurlyeq F(\lambda x + (1-\lambda)y).$$

Remark 2.9. Obvously, if F is $(\preccurlyeq)$-properly quasi convex, then F is $(\preccurlyeq)$-properly quasi convex.

Theorem 2.10. *Let φ be $(\preccurlyeq)$-monotone and $(\preccurlyeq)$-quasi convex. If F is $(\preccurlyeq)$-naturally quasi convex, then $(\varphi \circ F)$ is quasi convex.*

Theorem 2.11. *Let φ be $(\preccurlyeq)$-monotone and $(\preccurlyeq)$-quasi concave. If F is $(\preccurlyeq)$-naturally quasi concave, then $(\varphi \circ F)$ is quasi concave.*

3 Scalarization functions preserving well properties

To extend Ky Fan inequality for set-valued maps with a binary relation, consider assumptions of scalarization fucntions. To begin with, introduce four properties;

(1) φ is $(\preccurlyeq, C)$-lower semicontinuous,
(2) φ is quasi concave,
(3) φ is $(\preccurlyeq)$-monotone,
(4) $\varphi(\{\theta\}) = 0$,

and define the set of functions satisfying these properties as $\Phi(\preccurlyeq, C)$. In addition, establish three vaital properties for Ky Fan inequality;

(A1) $\varphi(A) \leq 0 \Rightarrow A \preccurlyeq \{\theta\}$,
(A2) there is an open neighborhood G of θ such that $\{\theta\} + G \preccurlyeq A$, then $0 < \varphi(A)$,
(A3) there is an open neighborhood G of θ such that $\{\theta\} \preccurlyeq A + G$, then $0 < \varphi(A)$.

4 Applications for Ky-Fan Minimax Inequality

Recall original Ky Fan inequality and provide main results

Theorem 4.1 ([2]). *Let X be a nonempty compact convex subset of a Hausdorff topological vector space and $f\colon X \times X \to \mathbb{R}$. If f satisfies the following conditions:*

(1) for each fixed $y \in X$, $f(\cdot, y)$ is lower semicontinuous,

(2) for each fixed $x \in X$,$f(x, \cdot)$ is quasi concave,

(3) $f(x, x) \leq 0$ for all $x \in X$,

then there exists $\bar{x} \in X$ such that $f(\bar{x}, y) \leq 0$ for all $y \in X$.

Theorem 4.2. *Let X be a nonempty compact convex subset of a Hausdorff topological vector space, Y a real topological vector space, $\preccurlyeq$ a binary relation on $\mathcal{P}(Y)$, C a convex cone in Y, $\varphi\colon \mathcal{P}(Y) \to \mathbb{R} \cup \{\pm\infty\}$, and $F\colon X \times X \to \mathcal{P}(Y) \backslash \{\emptyset\}$ a set-valued map. For the scaralization function $\varphi \in \Phi(\preccurlyeq, C)$ satisfying Assumption (A1), if F satisfies the following conditions:*

(1) $(\varphi \circ F)(x, y) \in \mathbb{R}$ for all $x, y \in X$,

(2) for each fixed $y \in X$, $F(\cdot, y)$ is $(\preccurlyeq, C)$-continuous,

(3) for each fixed $x \in X$, $F(x, \cdot)$ is $(\preccurlyeq)$-naturally quasi concave,

(4) for all $x \in X$, $F(x, x) \preccurlyeq \{\theta\}$,

then there exists $\bar{x} \in X$ such that $F(\bar{x}, y) \preccurlyeq \{\theta\}$ for all $y \in X$.

Theorem 4.3. *Let X be a nonempty compact convex subset of a Hausdorff topological vector space, Y a real topological vector space, $\preccurlyeq$ a binary relation on $\mathcal{P}(Y)$, C a convex cone in Y, $\varphi\colon \mathcal{P}(Y) \to \mathbb{R} \cup \{\pm\infty\}$, and $F\colon X \times X \to \mathcal{P}(Y) \backslash \{\emptyset\}$ a set-valued map. For the scaralization function $\varphi \in \Phi(\preccurlyeq, C)$ satisfying Assumption (A2), if F satisfies the following conditions:*

(1) $(\varphi \circ F)(x, y) \in \mathbb{R}$ for all $x, y \in X$,

(2) for each fixed $y \in X$, $F(\cdot, y)$ is $(\preccurlyeq, C)$-continuous,

(3) for each fixed $x \in X$, $F(x, \cdot)$ is $(\preccurlyeq)$-naturally quasi concave,

(4) for all $x \in X$, $F(x, x) \preccurlyeq \{\theta\}$,

then for any open neighborhood G of θ there exists $\bar{x} \in X$ such that $\{\theta\} + G \not\preccurlyeq F(\bar{x}, y)$ for all $y \in X$.

Theorem 4.4. *Let X be a nonempty compact convex subset of a Hausdorff topological vector space, Y a real topological vector space, $\preccurlyeq$ a binary relation on $\mathcal{P}(Y)$, C a convex cone in Y, $\varphi\colon \mathcal{P}(Y) \to \mathbb{R} \cup \{\pm\infty\}$, and $F\colon X \times X \to \mathcal{P}(Y) \backslash \{\emptyset\}$ a set-valued map. For the scaralization* *function* $\varphi \in \Phi(\preccurlyeq, C)$ *satisfying Assumption (A3), if* F *satisfies the following conditions:*

(1) $(\varphi \circ F)(x, y) \in \mathbb{R}$ *for all* $x, y \in X$*,*

(2) for each fixed $y \in X$*,* $F(\cdot, y)$ *is* $(\preccurlyeq, C)$*-continuous,*

(3) for each fixed $x \in X$*,* $F(x, \cdot)$ *is* $(\preccurlyeq)$*-naturally quasi concave,*

(4) for all $x \in X$*,* $F(x, x) \preccurlyeq \{\theta\}$*,*

then for any open neighborhood G *of* θ *there exists* $\bar{x} \in X$ *such that* $\{\theta\} \not\preccurlyeq F(\bar{x}, y) + G$ *for all* $y \in X$*.*

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