2 Asymptotic Cones and Functions

2.1 Definition of Asymptotic Cones

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We use the book; Asymptotic Cones and Functions in Optimization and Variational Inequalities (author: A.AUSLENDER and M.TEBOULLE), pp.25-31.

The set of natural numbers is denoted by $\mathbb{N}$, so that $k \in \mathbb{N}$ means $k = 1, 2, \ldots$. A sequence $\{x_k\}_{k\in\mathbb{N}}$ or simply $\{x_k\}$ in $\mathbb{R}^n$ is said to converge to x if $\|x_k - x\| \to 0$ as $k \to \infty$, and this will be indicated by the notation $x_k \to x$ or $x = \lim_{k\to\infty} x_k$. We say that x is a cluster point of $\{x_k\}$ if some subsequence converge to x. Recall that every bounded sequence in $\mathbb{R}^n$ converges to x if and only if it is bounded and has x as its unique cluster point.

Let $\{x_k\}$ be a sequence in $\mathbb{R}^n$. We are interested in knowing how to handle convergence properties, we are led to consider direction $d_k \coloneqq x_k \|x_k\|^{-1}$ with $x_k \neq 0$, $k \in \mathbb{N}$. From classical analysis, the Bolzano-Weierstrass theorem implies that we can extract a convergent subsequence $d = \lim_{k\in K} d_k$, $K \subset \mathbb{N}$, with $d \neq 0$. Now suppose that the sequence $\{x_k\} \subset \mathbb{R}^n$ is such that $\|x_k\| \to +\infty$. Then

$$\exists t_k \coloneqq \|x_k\|, k \in K \subset \mathbb{N}, \text{ such that } \lim_{k\in K} t_k = +\infty \text{ and } \lim_{k\in K} \frac{x_k}{t_k} = d.$$

This leads us to introduce the following concepts.

Definition 2.1.1

A sequence $\{x_k\} \subset \mathbb{R}$ is said to converge to a direction $d \in \mathbb{R}^n$ if

$$\exists \{t_k\}, \, with \, t_k \to +\infty \text{ such that } \lim_{k\to\infty} \frac{x_k}{t_k} = d.$$

Definition 2.1.2

Let C be a nonempty set in $\mathbb{R}^n$. Then the asymptotic cone of the set C, denoted by C_∞, is the set of vectors $d \in \mathbb{R}^n$ that are limits in direction of the sequences $\{x_k\} \subset C$, namely

$$C_\infty = \{d \in \mathbb{R}^n \mid \exists t_k \to +\infty, \exists x_k \in C \text{ with } \lim_{k\to\infty} \frac{x_k}{t_k} = d\}.$$

From the definition we immediately deduce the following elementary facts.

Proposition 2.1.1

Let $C \subset \mathbb{R}^n$ be nonempty. Then:

(i) C_∞ is a closed cone.
(ii) $(\operatorname{cl} C)_\infty = C_\infty$.
(iii) If C is a cone, then $C_\infty = \operatorname{cl} C$.

Proof. We will prove each part separately.

(i) C_∞ is a closed cone.

We need to show two propositions: (i-a) C_∞ is a cone and (i-b) C_∞ is a closed set.

(i-a) We show that C_∞ is a cone, that is, $\forall \alpha \geq 0, d \in C_\infty, \alpha d \in C_\infty$.

Since 0 is a element of C_∞, it is clear in the case of $\alpha = 0$.

($\because$ Since C is nonempty, we can take a element x_0 from C. In addition we take a sequence $\{t_k\}_{k=1}^\infty$ with $t_k \to +\infty$ as $k \to \infty$. Of course this sequence exists, for example $t_k := k$. By using $t_k := k$ and $x_k := x_0$, we can obtain 0 as the limit. Hence 0 is a element of C_∞.)

Also we consider the other case $\alpha > 0$. To prove that C_∞ is a cone, we take a any direction d from C_∞. Since d is a element of C_∞,

$$\exists t_k \to +\infty, \exists x_k \in C \text{ with } \lim_{k\to\infty} \frac{x_k}{t_k} = d.$$

Then we define a sequence $\{t'_k\}_{k=1}^\infty := \frac{t_k}{\alpha}$, exactly whose limit becomes $+\infty$ as $k \to \infty$. Accordingly there exist $t'_k \to +\infty$ and $x_k \in C$ with

$$\lim_{k\to\infty} \frac{x_k}{t'_k} = \lim_{k\to\infty} \alpha \cdot \frac{x_k}{t_k} = \alpha d.$$

This means $d \in C_\infty$.

By these results, we can get $\forall \alpha \geq 0, d \in C_\infty, \alpha d \in C_\infty$.

Therefore C_∞ is a cone.

(i-b) We show that C_∞ is a closed set. In order to prove closeness, we consider convergency of a sequence of C_∞. First we take a sequence $\{d_k\}_{k=1}^{\infty} \subset C_\infty$ with $d_k \to d$ as $k \to \infty$ for some d. Then we don't forget that $d \in C_\infty$ is our goal.

For each $k \in \mathbb{N}$,

$$\exists \{x_k^{(n)}\}_{n=1}^{\infty} \subset C \text{ and } \{t_k^{(n)}\}_{n=1}^{\infty} \text{ with } t_k^{(n)} \to \infty \text{ as } n \to \infty.$$

The below figure represents $x_k^{(n)}$ and $t_k^{(n)}$.

Figure:

$k \backslash n$	1	2	$\cdots$	m	$\cdots$	limit
1	$x_1^{(1)},\ t_1^{(1)}$	$x_1^{(2)},\ t_1^{(2)}$	$\cdots$	$x_1^{(m)},\ t_1^{(m)}$	$\cdots$	d_1
2	$x_2^{(1)},\ t_2^{(1)}$	$x_2^{(2)},\ t_2^{(2)}$	$\cdots$	$x_2^{(m)},\ t_2^{(m)}$	$\cdots$	d_2
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$
m	$x_m^{(1)},\ t_m^{(1)}$	$x_m^{(2)},\ t_m^{(2)}$	$\cdots$	$x_m^{(m)},\ t_m^{(m)}$	$\cdots$	d_m
$\vdots$			$\vdots$			

Then we define

$$x_m := x_m^{(m)} \text{ and } t_m := t_m^{(m)}.$$

By the definition of convergence of a sequence,

$$\forall \epsilon > 0, \exists \bar{m} \in \mathbb{N}\ s.t.\ \forall m \geq \bar{m}, ||d_m - d|| < \frac{\epsilon}{2}, \text{and}$$
$$\forall \epsilon > 0, \exists \hat{m} \in \mathbb{N}\ s.t.\ \forall m \geq \hat{m}, ||\frac{x_m}{t_m} - d_m|| < \frac{\epsilon}{2}.$$

Also, we let $\tilde{m} :=\ \max\{\bar{m}, \hat{m}\} \in \mathbb{N}$. By using triangle inequality,

$$\forall \epsilon > 0, \exists \tilde{m} \in \mathbb{N}\ s.t.\ \forall m \geq \bar{m}, ||\frac{x_m}{t_m} - d|| < \epsilon.$$

$(\because ||\frac{x_m}{t_m} - d|| \leq ||\frac{x_m}{t_m} - d_m|| + ||d_m - d|| < \epsilon..)$

Therefore c_∞ is a closed set.

Then (i)'s proof is completed.

(ii) $(\mathrm{cl}\, C)_\infty = C_\infty$.

We need to show two relations: (ii-a) $(\mathrm{cl}\, C)_\infty \supset C_\infty$ (ii-b) $(\mathrm{cl}\, C)_\infty \subset C_\infty$.

(ii-a) We show that C_∞ is included in $(\mathrm{cl}\, C)_\infty$. However it is clear from the definition of asymptotic cone.

(ii-b) We show that $(\mathrm{cl}\, C)_\infty \subset C_\infty$. In order to prove that a element of $(\mathrm{cl}\, C)_\infty$ satisfies the asymptotic cone's relation, we consider convergency of a sequences of $(\mathrm{cl}\, C)_\infty$ and $\mathrm{cl}\, C$. First we take any $d \in (\mathrm{cl}\, C)_\infty$ which satisfies

$$\exists t_k \to +\infty, \exists x_k \in \operatorname{cl} C \;\text{ with }\; \lim_{k\to\infty} \frac{x_k}{t_k} = d.$$

For each $k \in \mathbb{N}$,

$$\exists \{y_k^{(n)}\}_{n=1}^{\infty} \subset C \text{ with } y_k^{(n)} \to x_k \text{ as } n \to \infty.$$

The below figure represents $y_k^{(n)}$.

Figure:

$k \backslash n$	1	2	$\cdots$	m	$\cdots$	limit
1	$y_1^{(1)}$	$y_1^{(2)}$	$\cdots$	$y_1^{(m)}$	$\cdots$	x_1
2	$y_2^{(1)}$	$y_2^{(2)}$	$\cdots$	$y_2^{(m)}$	$\cdots$	x_2
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$
m	$y_m^{(1)}$	$y_m^{(2)}$	$\cdots$	$y_m^{(m)}$	$\cdots$	x_m
$\vdots$				$\vdots$		

Then we define

$$y_m \coloneqq y_m^{(m)}.$$

By the definition of convergence of a sequence,

$$\begin{aligned}
&\forall \epsilon > 0, \exists \bar{m} \in \mathbb{N}\; s.t.\; \forall m \geq \bar{m}, ||d_m - d|| < \frac{\epsilon}{2}, \\
&\forall \epsilon > 0, \exists \hat{m} \in \mathbb{N}\; s.t.\; \forall m \geq \hat{m}, ||y_m^m - x_m|| < \frac{\sqrt{\epsilon}}{2}, \text{ and} \\
&\forall \epsilon > 0, \exists \tilde{m} \in \mathbb{N}\; s.t.\; \forall m \geq \tilde{m}, |\frac{1}{t_m}| < \sqrt{\epsilon}.
\end{aligned}$$

Also, we let $m_0 \coloneqq \max\{\bar{m}, \hat{m}, \tilde{m}\} \in \mathbb{N}$. By using triangle inequality,

$$\forall \epsilon > 0, \exists m_0 \in \mathbb{N}\; s.t.\; \forall m \geq \bar{m}, ||\frac{y_m}{t_m} - d|| < \epsilon.$$

($\because ||\frac{y_m}{t_m} - d|| \leq \frac{1}{|t_m|} \cdot ||y_m - d_m|| + ||\frac{y_m}{t_m} - d|| < \epsilon.$)

Therefore $(\operatorname{cl} C)_\infty \subset C_\infty$.

Then (ii)'s proof is also completed.

(iii) If C is a cone, then $C_\infty = \operatorname{cl} C$.

We need to show two relations: (iii-a) $C_\infty \subset \operatorname{cl} C$ and (iii-b) $C_\infty \supset \operatorname{cl} C$.

(iii-a) We take any direction $d \in C_\infty$ which satisfies

$$\exists t_k \to +\infty, \exists x_k \in C \;\text{ with }\; \lim_{k\to\infty} \frac{x_k}{t_k} = d.$$

Let $d_k := \frac{x_k}{t_k}$ (with $d_k \to d$ as $k \to \infty$). Since C is a cone,

$$d_k = \frac{1}{t_k} \cdot x_k \in C.$$

Due to $d_k \in C$, the limit of d_k is a element of $\operatorname{cl} C$, i.e., $d \in \operatorname{cl} C$.

Therefore $C_\infty \subset \operatorname{cl} C$.

(iii-b) We take any $d \in \operatorname{cl} C$ and show $d \in C_\infty$, that is,

$$\exists t_k \to +\infty, \exists x_k \in C \ \text{ with } \ \lim_{k\to\infty} \frac{x_k}{t_k} = d.$$

By $d \in \operatorname{cl} C$,

$$\exists \{d_k\}_{k=1}^{\infty} \in C \text{ with } d_k \to d \text{ as } \infty,$$

in other words,

$$\lim_{k\to\infty} d_k = d.$$

□

The importance of the asymptotic cone is revealed by the following key property, which is a immediate consequence of its definition.

Proposition 2.1.2

A set $C \subset \mathbb{R}^n$ is bounded if and only if $C_\infty = \{0\}$.

Proof. □

Associated with the asymptotic cone C_∞ is the following related concept, which will help us in simplifying the definition of C_∞ in the particular case where $C \in \mathbb{R}^n$ is assumed convex.

Definition 2.1.3

Let $C \in \mathbb{R}^n$ be nonempty and define

$$C_\infty^1 = \{d \in \mathbb{R}^n \mid \forall t_k \to +\infty, \exists x_k \in C \ \text{ with } \ \lim_{k\to\infty} \frac{x_k}{t_k} = d\}.$$

We say that C is asymptotically regular if $C_\infty = C_\infty^1$

Proposition 2.1.3

Let C be a nonempty convex set in $\mathbb{R}^n$. Then C is asymptotically regular.

Proof. □