A relation between Asymptotic Cones and Painlevé-Kuratowski Convergence

漸近錐と集合値写像の半連続性の関係について

Ryota Iwamoto* and Tamaki Tanaka

Niigata Univ

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3 Painlevé-Kuratowski Convergence

4 Continuities of set-valued mappings

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Preliminary

$\mathbb{R}^n$: n-dimensional real Euclidean space.
The inner product of $\mathbb{R}^n$ $\langle \cdot, \cdot \rangle$ is defined by

$$\langle x, y \rangle := \sum_{i=1}^{n} x_i y_i, \text{ for } x = (x_1, \ldots, x_n)^T \in \mathbb{R}^n \text{ and } y = (y_1, \ldots, y_n)^T \in \mathbb{R}^n.$$

The norm is defined by $\|x\| := \langle x, x \rangle^{1/2}$.

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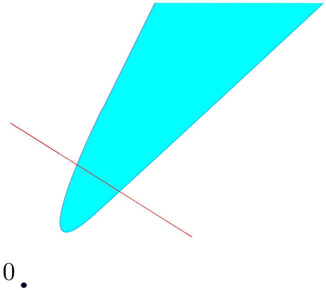

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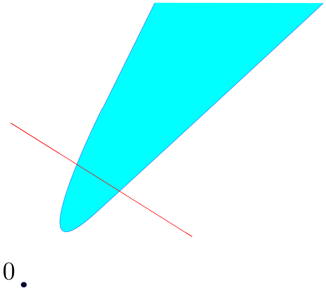

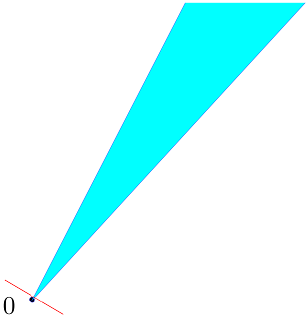

Definition of Asymptotic Cones

Definition 1(Asymptotic Cones)

$C \subset \mathbb{R}^n$, $C \neq \emptyset$. Then, the asymptotic cone of the set C, denoted by C_∞, is the set below with $\{x_k\} \subset C$;

$$C_\infty = \left\{ d \in \mathbb{R}^n \,\middle|\, \exists t_k \to +\infty, \exists x_k \in C \text{ with } \lim_{k \to \infty} \frac{x_k}{t_k} = d \right\}.$$

Example: $C = \{(x, y) \in \mathbb{R}^2 \mid y = x^2\}$. We let $x_k = (k, k^2)$ and $t_k = \|x_k\|$.

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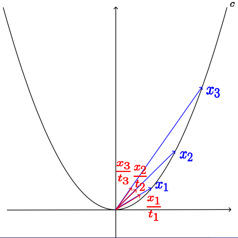

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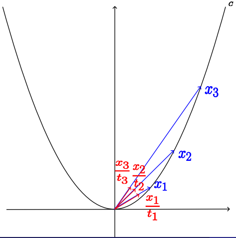

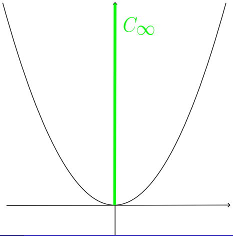

Properties around Asymptotic Cones

Proposition 1

A set $C \subset \mathbb{R}^n$ is bounded if and only if $C_\infty = \{0\}$.

Definition 2

Let $C \subset \mathbb{R}^n$ be nonempty and define

$$C_\infty^1 = \left\{ d \in \mathbb{R}^n \;\middle|\; \forall t_k \to +\infty, \exists x_k \in C \text{ with } \lim_{k\to\infty} \frac{x_k}{t_k} = d \right\}.$$

We say that C is asymptotically regular if $C_\infty = C_\infty^1$.

Proposition 2

Let C be a nonempty convex set in $\mathbb{R}^n$. Then C is asymptotically regular.

Example of Asymptotically Regular

Definition 2

Let $C \subset \mathbb{R}^n$ be nonempty and define

$$C^1_\infty = \left\{ d \in \mathbb{R}^n \;\middle|\; \forall t_k \to +\infty, \exists x_k \in C \text{ with } \lim_{k\to\infty} \frac{x_k}{t_k} = d \right\}.$$

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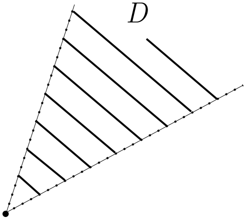

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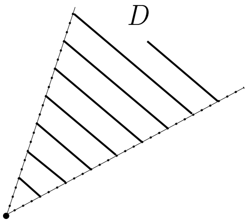

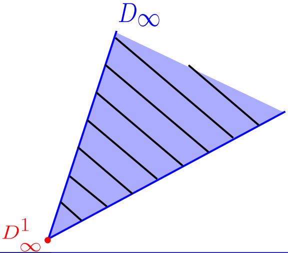

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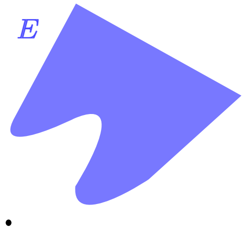

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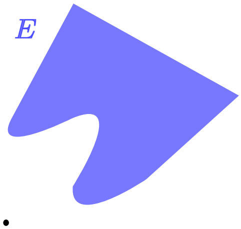

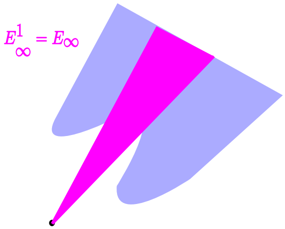

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Definition of the Painlevé-Kuratowski Convergence

Definition 3

Y: a topological vector space.
$\mathcal{P}(Y)$: a family of subset in Y.
Let $(A_n)_{n\in\mathbb{N}} \subset \mathcal{P}(Y)$. We define the inner limit and the outer limit as

$$\liminf_{n\to\infty} A_n := \{y \in Y \mid \exists (y_n) \to y \text{ s.t. } y_n \in A_n \text{ for } n \geq n_0\},$$
$$\limsup_{n\to\infty} A_n := \{y \in Y \mid \exists (y_{n(k)}) \to y \text{ s.t. } y_{n(k)} \in A_{n(k)} \text{ for } k \in \mathbb{N}\}.$$

If it holds that $\liminf_{n\to\infty} A_n \supset \limsup_{n\to\infty} A_n$, we say that (A_n) converges in the sense of Painlevé-Kuratowski.

Painlevé-Kuratowski Convergence Examples

Example:

$$A_n = \begin{cases} [0,1], & \text{if } n \text{ is odd} \\ [0,\frac{1}{n}], & \text{if } n \text{ is even} \end{cases}$$

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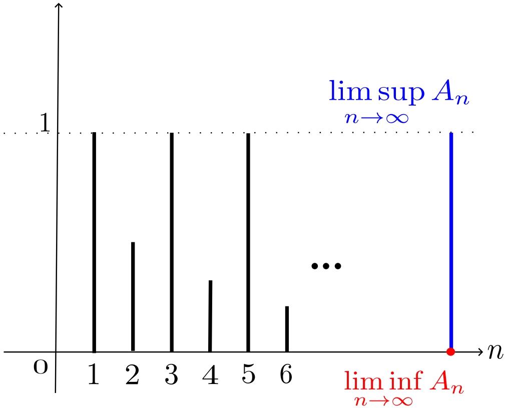

Relations between Asymptotic cones and Painlevé-Kuratowski Convergence

Remark

Providing that the given A_t converges in the sense of Painlevé-Kuratowski, we can let $\Gamma(t) = A_t$ and $\Gamma(\infty) = A$ where $A, A_t \subset Y$. Then, Γ implies a set-valued mapping from $\mathbb{R}_+ \backslash \{0\}$ to $\mathcal{P}(\mathbb{R}^n)$ and it follows that $\Gamma(\infty) = C^1_\infty = C_\infty$

C: a nonempty subset in $\mathbb{R}^n$
Γ: $\mathbb{R}_+ \backslash \{0\} \to \mathcal{P}(\mathbb{R}^n)$
We let $\Gamma(t) = \frac{C}{t}$.

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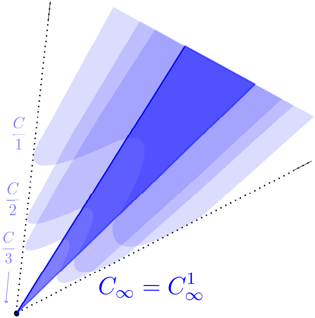

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Definition of upper continuity

Definition 4

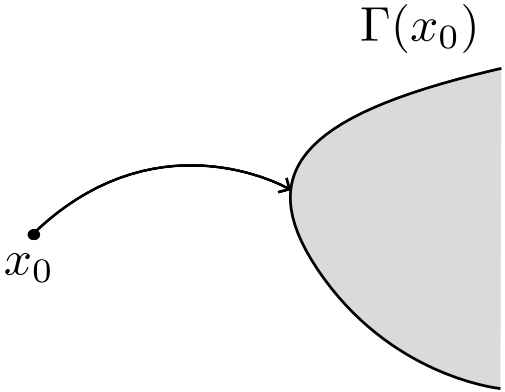
X, Y: topological vector spaces, particularly X is a real t.v.s
Γ: $X \to \mathcal{P}(Y)$
$x_0 \in X$

We say that
(a) Γ is upper continuous (u.c.) at x_0 if

$$\forall D \subset Y, D \text{ open }, \Gamma(x_0) \subset D, \exists U \in \mathcal{V}_X(x_0) \text{ s.t. } \forall x \in U, \Gamma(x) \subset D.$$

(b) Γ is upper continuous (u.c.) if Γ is so at every $x_0 \in X$.

Figure of upper continuity



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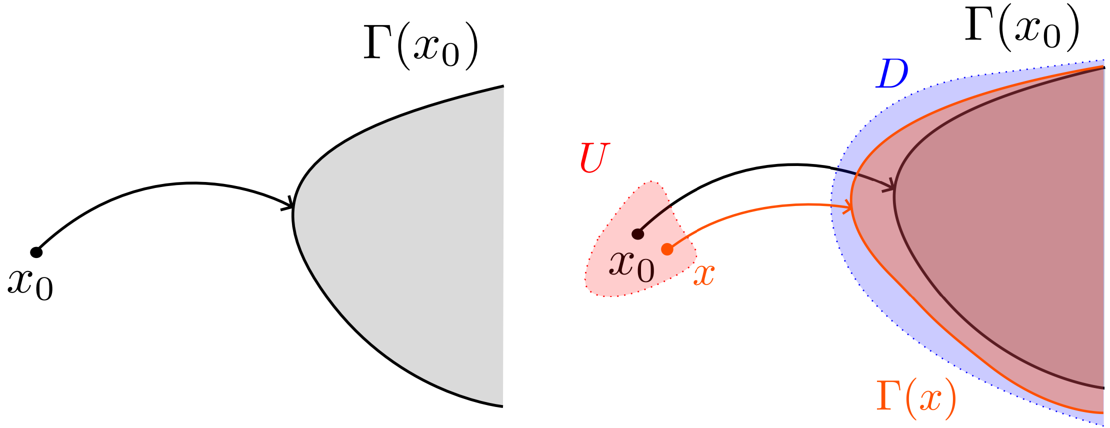

Definition of Hausdorff upper continuity

Definition 5

X, Y: topological vector spaces, particularly X is a real t.v.s
Γ: $X \to \mathcal{P}(Y)$
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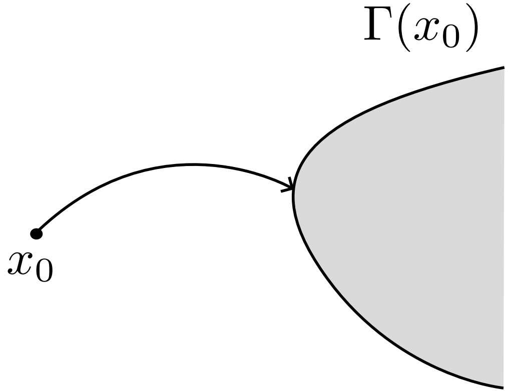
We say that
(c) Γ is Hausdorff upper continuous (H-u.c.) at x_0 if

$$\forall V \subset \mathcal{V}_Y, \exists U \in \mathcal{V}_X(x_0) \text{ s.t. } \forall x \in U, \Gamma(x) \subset \Gamma(x_0) + V.$$

(d) Γ is Hausdorff upper continuous (H-u.c.) if Γ is so at every $x_0 \in X$.

Figure of Hausdorff upper continuity



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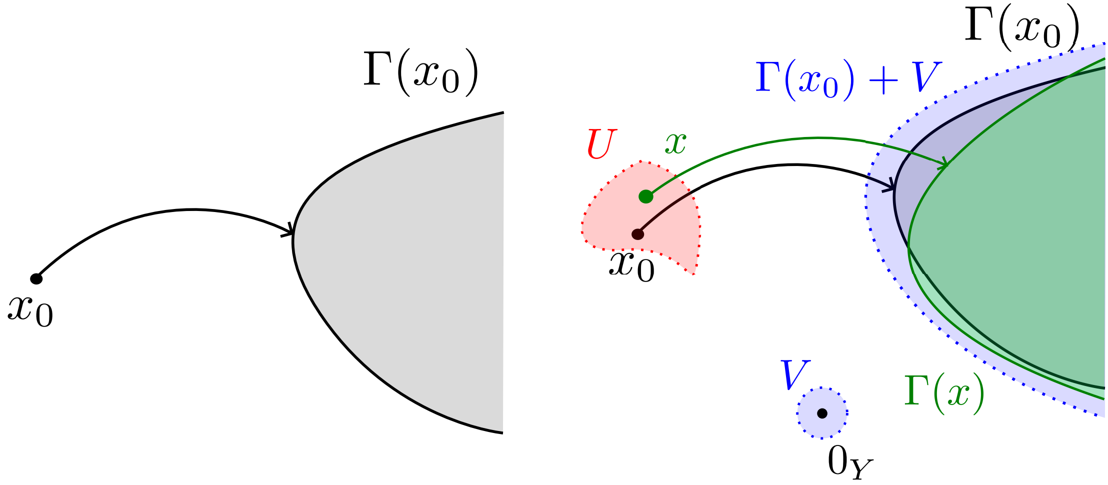

Relations between Asymptotic cones and these continuities

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C: a nonempty convex subset in $\mathbb{R}^n$
$\Gamma(t) = \frac{C}{t}$ where $t > 0$.
Then, we can see that

1. It does not always hold that Γ is upper continuous.
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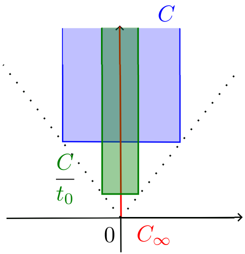

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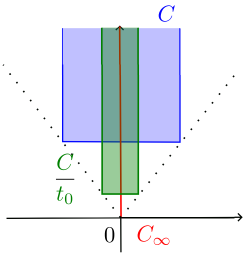

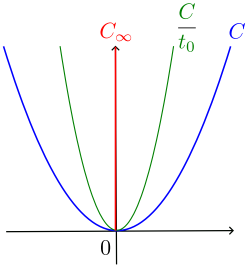

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Conclusion

Recap:

- Considering a set-valued mappings, we can replace the definition of asymptotic cones as the outer limit.
- Using set-valued mappings allows us to connect the notion of asymptotic cones with some results of continuities in t.v.s.

Issues:

- These last results are not enough because we can consider other continuity such as lower continuity.
- Particularly, the relation is a case that a given set is convex.
- I never combine the relation with applications of asymptotic cones.

References

- A. Alfred and M. Teboulle, asymptotic cones and functions in optimization and variational inequalities, Springer monographs in Mathematics, Springer-Verlag, New York, 2003.
- A. Göpfert, H. Riahi, C. Tammer, and C. Zălinescu, Variational methods in partially ordered spaces, vol. 17 of CMS Books in Mathematics, Springer-Verlag, New York, 2003.