スカラー化関数による集合値関数のミニマックス定理の一般化とその応用

Set-Valued Fan-Takahashi Inequalities Via Scalarization

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Contents

Introduction

Background

- Georgiev and Tanaka [2] extended the minimax inequality to the form of set-valued maps.

- Kuwano, Tanaka, and Yamada [5] constructed the result of four types set-valued minimax inequalities with set relations.

- **Our goal is to generalize the result of four types set-valued minimax inequalities which is not related to the specific set-relations and scalarization functions.**

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[2] Pando Gr. Georgiev and Tamaki Tanaka. "Vector-valued set-valued variants of Ky Fan's inequality". In: J. Nonlinear Convex Anal. 1.3 (2000), pp. 245–254.

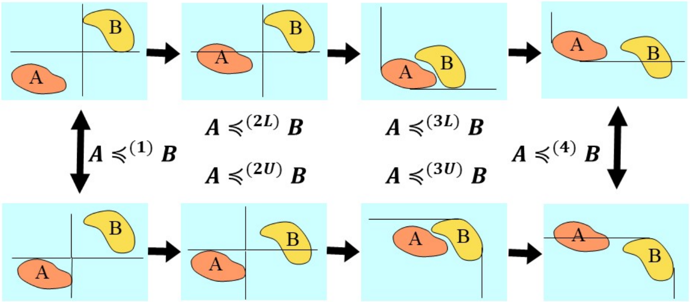
[5] Issei Kuwano, Tamaki Tanaka, and Syuuji Yamada. "Unified scalarization for sets and set-valued Ky Fan minimax inequality". In: J. Nonlinear Convex Anal. 11.3 (2010), pp. 513–525.

Let $(Y, \preccurlyeq)$ be an ordered space, generally.

$A, B \subset Y$: nonempty sets. $A \preccurlyeq^{(j)} B$ $(j = 1, 2L, 2U, 3L, 3U, 4)$ is defined below.

(1) $\forall a \in A, \forall b \in B, a \preccurlyeq b$

(2L) $\exists a \in A$ s.t. $\forall b \in B, a \preccurlyeq b$ **(3L)** $\forall b \in B, \exists a \in A$ s.t. $a \preccurlyeq b$



(2U) $\exists b \in B$ s.t. $\forall a \in A, a \preccurlyeq b$ **(3U)** $\forall a \in A, \exists b \in B$ s.t. $a \preccurlyeq b$

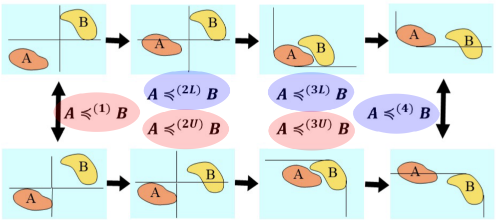
(4) $\exists a \in A, \exists b \in B$ s.t. $a \preccurlyeq b$

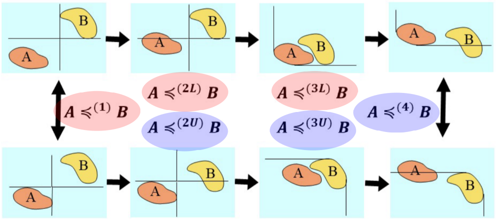
Ordering and Set-relations

Lemma

Let Y be a real topological vector space, C a convex cone with $\operatorname{int} C \neq \emptyset$, and $A, \{b\} \subset Y$. Then

1. $A \preccurlyeq_C^{(1)} \{b\} \Leftrightarrow A \preccurlyeq_C^{(2U)} \{b\} \Leftrightarrow A \preccurlyeq_C^{(3U)} \{b\}$, $A \preccurlyeq_C^{(2L)} \{b\} \Leftrightarrow A \preccurlyeq_C^{(3L)} \{b\} \Leftrightarrow A \preccurlyeq_C^{(4)} \{b\}$
2. $\{b\} \not\preccurlyeq_{\operatorname{int} C}^{(1)} A \Leftrightarrow \{b\} \not\preccurlyeq_{\operatorname{int} C}^{(2L)} A \Leftrightarrow \{b\} \not\preccurlyeq_{\operatorname{int} C}^{(3L)} A$, $\{b\} \not\preccurlyeq_{\operatorname{int} C}^{(2U)} A \Leftrightarrow \{b\} \not\preccurlyeq_{\operatorname{int} C}^{(3U)} A \Leftrightarrow \{b\} \not\preccurlyeq_{\operatorname{int} C}^{(4)} A$





Theorem (Fan-Takahashi [6])
Let X be a nonempty compact convex subset of a Hausdorff topological vector space and $f : X \times X \to \mathbb{R}$. If f satisfies the following conditions:

1. for each fixed $y \in X$, $f(\cdot, y)$ is lower semicontinuous,
2. for each fixed $x \in X$, $f(x, \cdot)$ is quasi concave,
3. $f(x, x) \leq 0$ for all $x \in X$,

then there exists $\bar{x} \in X$ such that $f(\bar{x}, y) \leq 0$ for all $y \in X$.

Ordering and Set-relations

Let $f : X \to \mathbb{R}$ and $F : X \to \mathcal{P}_0(Y)$.

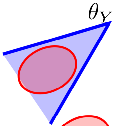
$$f(\bar{x}, y) \leq 0 \iff 0 \not< f(\bar{x}, y)$$

$$\Downarrow \qquad\qquad\qquad\qquad \Downarrow$$

$$\boxed{F(\bar{x}, y) \preccurlyeq_C^{(j)} \{\theta_Y\}} \qquad \boxed{\{\theta_Y\} \not\preccurlyeq_{\operatorname{int} C}^{(j)} F(\bar{x}, y)}$$

Case A1

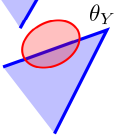
$F(\bar{x}, y) \subset -C$



$\Downarrow$

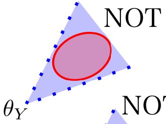
Case A2

$F(\bar{x}, y) \cap (-C) \neq \emptyset$



Case B1

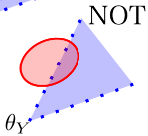
$F(\bar{x}, y) \not\subset \operatorname{int} C$



$\Uparrow$

Case B2

$F(\bar{x}, y) \cap \operatorname{int} C = \emptyset$



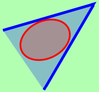
Assumption: F is (j) naturally quasi C-concave
or
(j) properly quasi C-concave

or $I^{(j)}_{k,\{\theta_Y\}}$

$S^{(j)}_{k,\{\theta_Y\}}$

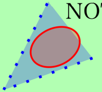
Case A1

$F(\bar{x}, y) \subset -C$



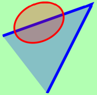
Case B1

$F(\bar{x}, y) \not\subset \mathrm{int} C$



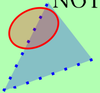
Case A2

$F(\bar{x}, y) \cap (-C) \neq \emptyset$



Case B2

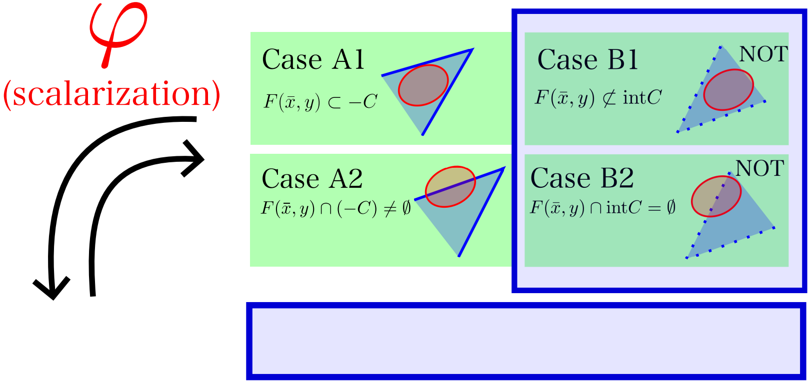
$F(\bar{x}, y) \cap \mathrm{int} C = \emptyset$



proved by Kuwano, Tanaka, Yamada.

$\boxed{\text{Fan-Takahashi minimax inequality}}$

Assumption: F is (j) naturally quasi C-concave



$\boxed{\text{Fan-Takahashi minimax inequality}}$

Theorem [5]
Let X be a nonempty compact convex subset of a Hausdorff topological vector space, Y a real topological vector space, C a proper closed convex cone in Y with $\operatorname{int} C \neq \emptyset$ and $F\colon X \times X \to \mathcal{P}_0(Y)$. If F satisfies the following conditions:

1. F is C-proper and C-closed on $X \times X$,
2. for each fixed $y \in X$, $F(\cdot, y)$ is C-upper continuous,
3. for each fixed $x \in X$, $f(x, \cdot)$ is type (3L) properly C-quasi concave,
4. for all $x \in X$, $F(x, x) \preccurlyeq_C^{(3L)} \{\theta_Y\}$,

then there exists $\bar{x} \in X$ such that $F(\bar{x}, y) \preccurlyeq_C^{(3L)} \{\theta_Y\}$ for all $y \in X$.

Preliminaries

Let X be a topological space, Y a real topological vector space, and θ_Y be a zero vector in Y. Define that $\mathcal{P}_0(Y)$ is the set of all nonempty subsets of Y. The sets of neighborhoods of $x \in X$ and $y \in Y$ is denoted by $\mathcal{N}_X(x)$ and $\mathcal{N}_Y(y)$, respectively.

Definition

For $A, B \in \mathcal{P}_0(Y)$, we define two binary relations on $\mathcal{P}_0(Y)$:

$$A \preccurlyeq_1 B \overset{\text{def}}{\iff} A \cap B \neq \emptyset \quad \text{and} \quad A \preccurlyeq_2 B \overset{\text{def}}{\iff} B \subset A.$$

Definition (set-relations) [4]

For $A, B \in \mathcal{P}_0(Y)$ and a convex cone C, we define;

$$A \preccurlyeq_C^{(3L)} B \overset{\text{def}}{\iff} B \subset A + C \quad \text{and} \quad A \preccurlyeq_C^{(3U)} B \overset{\text{def}}{\iff} A \subset B - C.$$

Definition

Let $f : Y \to \mathbb{R} \cup \{\pm\infty\}$ and $y_0 \in Y$. We say that f is lower semicontinuous (l.s.c. shortly) at y_0 if

$$\forall r < f(y_0), \exists V \in \mathcal{N}_Y(y_0) \text{ s.t. } r < f(y), \forall y \in V;$$

Definition [1]

Let $F\colon X \to \mathcal{P}_0(Y)$, $x_0 \in X$, $\preccurlyeq$ a binary relation on $\mathcal{P}_0(Y)$ and $C \subset Y$ a convex cone. We say that F is $(\preccurlyeq, C)$-continuous at x_0 if

$$\forall W \subset Y, W \text{ open}, W \preccurlyeq F(x_0), \exists V \in \mathcal{N}_X(x_0) \text{ s.t. } W + C \preccurlyeq F(x), \forall x \in V.$$

Remark

As special cases, $(\preccurlyeq_1, C)$-continuity and $(\preccurlyeq_2, C)$-continuity coincide with " C-lower continuity " and " C-upper continuity " for set-valued maps, respectively.

Definition [1]
Let $\varphi\colon \mathcal{P}_0(Y) \to \mathbb{R} \cup \{\pm\infty\}$, $A_0 \in \mathcal{P}_0(Y)$, $\preccurlyeq$ a binary relation on $\mathcal{P}_0(Y)$, and C a convex cone in Y with $C \neq Y$. Then, we say that φ is $(\preccurlyeq, C)$-lower semicontinuous at A_0 if

$$\forall r < \varphi(A_0), \exists W \in \mathcal{P}_0(Y), W \text{ open, s.t. } W \preccurlyeq A_0 \text{ and } r > \varphi(A), \forall A \in U(W + C, \preccurlyeq);$$

where $U(V, \preccurlyeq) := \{A \in \mathcal{P}_0(Y) \mid V \preccurlyeq A\}$.

Theorem [1]
Let $F\colon X \to \mathcal{P}_0(Y)$, $\varphi\colon \mathcal{P}_0(Y) \to \mathbb{R} \cup \{\pm\infty\}$, $x_0 \in X$, $\preccurlyeq$ a binary relation on $\mathcal{P}_0(Y)$, and C a convex cone. If F is $(\preccurlyeq, C)$-continuous at x_0 and φ is $(\preccurlyeq, C)$-lower semicontinuous at $F(x_0)$, then $(\varphi \circ F)$ is lower semicontinuous at x_0.

Definition

Let X be a nonempty set, Y a real topological vector space, C a convex cone in Y, and $F\colon X \to \mathcal{P}_0(Y)$ a set-valued map.

1. F is called type (j) properly quasi C-concave if for each $x, y \in X$ and $\lambda \in (0,1)$,

$$F(x) \preccurlyeq_C^{(j)} F(\lambda x + (1-\lambda)y) \quad \text{or} \quad F(y) \preccurlyeq_C^{(j)} F(\lambda x + (1-\lambda)y)$$

2. F is called type (j) naturally quasi C-concave if for each $x, y \in X$ and $\lambda \in (0,1)$, there exists $\mu \in [0,1]$ such that

$$\mu F(x) + (1-\mu)F(y) \preccurlyeq_C^{(j)} F(\lambda x + (1-\lambda)y).$$

Remark

If F is type (j) properly quasi C-concave, then F is type (j) naturally quasi C-concave.

Preliminaries (Convexity)

Definition [3]
Let $\mathcal{A} \subset \mathcal{P}_0(Y)$. $\mathcal{A}$ is said to be convex if for each $A_1, A_2 \in \mathcal{A}$ and $\lambda \in (0,1)$,

$$\lambda A_1 + (1-\lambda) A_2 \in \mathcal{A}.$$

Definition [3]
Let $\varphi \colon \mathcal{P}_0(Y) \to \mathbb{R} \cup \{\pm\infty\}$. Then,

1. φ is quasi convex if for any $\alpha \in \mathbb{R}$, $\operatorname{lev}(\varphi, \leq, \alpha) \coloneqq \{A \in \mathcal{P}_0(Y) \mid \varphi(A) \leq \alpha\}$ is convex.
2. φ is quasi concave if for any $\alpha \in \mathbb{R}$, $\operatorname{lev}(\varphi, \geq, \alpha) \coloneqq \{A \in \mathcal{P}_0(Y) \mid \varphi(A) \geq \alpha\}$ is convex.

Preliminaries (Convexity)

Definition

Let C be a convex cone in Y. For a given binary relation $\preccurlyeq$, a scalarization function φ is $(\preccurlyeq_C^{(j)})$-monotone if for any $A, B \in \mathcal{P}(Y)\backslash\{\emptyset\}$ with $A \preccurlyeq_C^{(j)} B$, $\varphi(A) \leq \varphi(B)$.

Proposition

Let φ be $(\preccurlyeq_C^{(j)})$-monotone and quasi convex. If F is type (j) naturally quasi C-convex, then $(\varphi \circ F)$ is quasi convex.

Proposition

Let φ be $(\preccurlyeq_C^{(j)})$-monotone and quasi concave. If F is type (j) naturally quasi C-concave, then $(\varphi \circ F)$ is quasi concave.

Main results

Specific scalarization function

To extend Ky Fan inequality for set-valued maps with set relations, consider assumptions of scalarization functions. To begin with, introduce four properties;

1. φ is $(\preccurlyeq, C)$-lower semicontinuous,
2. φ is quasi concave,
3. φ is $(\preccurlyeq_C^{(j)})$-monotone,
4. $\varphi(\{\theta_Y\}) = 0$,

and define the set of functions satisfying these properties as $\varphi \in \Phi(\preccurlyeq_C^{(j)}, \preccurlyeq)$. In addition, establish three vital properties for Ky Fan inequality;

$$\varphi(A) \leq 0 \Rightarrow A \preccurlyeq_C^{(j)} \{\theta_Y\} \quad \text{for any } A \in Y \tag{A}$$

$$\varphi(A) > 0 \Rightarrow \{\theta_Y\} \preccurlyeq_{\mathrm{int}\, C}^{(j)} A \quad \text{for any } A \in Y \tag{B}$$

Main results

Theorem

Let X be a nonempty compact convex subset of a Hausdorff topological vector space, Y a real topological vector space, $\preccurlyeq$ a binary relation on $\mathcal{P}_0(Y)$, C a convex cone in Y, $\varphi\colon \mathcal{P}_0(Y) \to \mathbb{R} \cup \{\pm\infty\}$, and $F\colon X \times X \to \mathcal{P}_0(Y)$ a set-valued map. For the scaralization function $\varphi \in \Phi(\preccurlyeq_C^{(j)}, \preccurlyeq)$ satisfying Assumption (A), if F satisfies the following conditions:

1. $(\varphi \circ F)(x, y) \in \mathbb{R}$ for all $x, y \in X$,
2. for each fixed $y \in X$, $F(\cdot, y)$ is $(\preccurlyeq, C)$-continuous,
3. for each fixed $x \in X$, $F(x, \cdot)$ is j-naturally quasi C-concave,
4. for all $x \in X$, $F(x, x) \preccurlyeq_C^{(j)} \{\theta_Y\}$,

then there exists $\bar{x} \in X$ such that $F(\bar{x}, y) \preccurlyeq_C^{(j)} \{\theta_Y\}$ for all $y \in X$.

Theorem
Let X be a nonempty compact convex subset of a Hausdorff topological vector space, Y a real topological vector space, $\preccurlyeq$ a binary relation on $\mathcal{P}_0(Y)$, C a convex cone in Y, $\varphi\colon \mathcal{P}_0(Y) \to \mathbb{R} \cup \{\pm\infty\}$, and $F\colon X \times X \to \mathcal{P}_0(Y)$ a set-valued map. For the scaralization function $\varphi \in \Phi(\preccurlyeq^{(j)}_{\operatorname{int} C}, \preccurlyeq)$ satisfying Assumption (B), if F satisfies the following conditions:

1. $(\varphi \circ F)(x, y) \in \mathbb{R}$ for all $x, y \in X$,
2. for each fixed $y \in X$, $F(\cdot, y)$ is $(\preccurlyeq, C)$-continuous,
3. for each fixed $x \in X$, $F(x, \cdot)$ is j-naturally quasi C-concave,
4. for all $x \in X$, $\{\theta_Y\} \not\preccurlyeq^{(j)}_{\operatorname{int} C} F(x, x)$,

then there exists $\bar{x} \in X$ such that $\{\theta_Y\} \not\preccurlyeq^{(j)}_{\operatorname{int} C} F(\bar{x}, y)$ for all $y \in X$.

Applications

Tammer's scalarization function

Definition [5]

Let C be a proper closed convex cone in Y with $\mathrm{int}\, C \neq \emptyset$, $V, V' \in \mathcal{P}(Y) \backslash \{\emptyset\}$, and direction $k \in \mathrm{int}\, C$. For each $j = (3U), (3L)$, $I^{(j)}_{k,V'}(V) \colon \mathcal{P}(Y) \to \mathbb{R} \cup \{\pm\infty\}$ are defined by

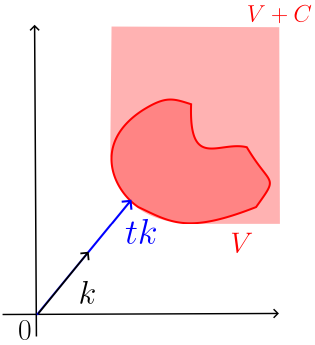
$$I^{(j)}_{k,V'}(V) := \inf\{t \in \mathbb{R} \mid V \preccurlyeq^{(j)}_{C} (tk + V')\}.$$

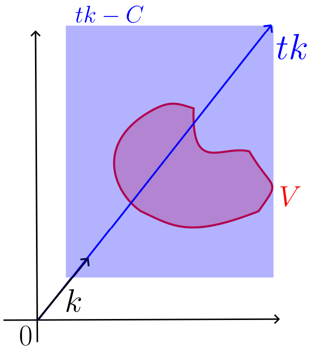
Example

$$I^{(3L)}_{k,\{\theta_Y\}}(V) := \inf\{t \in \mathbb{R} \mid V \preccurlyeq^{(3L)}_{C} (tk + V')\} = \inf\{t \in \mathbb{R} \mid tk \subset V + C\},$$
$$I^{(3U)}_{k,\{\theta_Y\}}(V) := \inf\{t \in \mathbb{R} \mid V \preccurlyeq^{(3U)}_{C} (tk + V')\} = \inf\{t \in \mathbb{R} \mid V \subset tk - C\}.$$

Example

$$I^{(3L)}_{k,\{\theta_Y\}}(V) := \inf\{t \in \mathbb{R} \mid V \preccurlyeq^{(3L)}_{C} \{tk\}\} = \inf\{t \in \mathbb{R} \mid tk \subset V + C\},$$
$$I^{(3U)}_{k,\{\theta_Y\}}(V) := \inf\{t \in \mathbb{R} \mid V \preccurlyeq^{(3U)}_{C} \{tk\}\} = \inf\{t \in \mathbb{R} \mid V \subset tk - C\}.$$





Definition (Hiriart-Urruty Oriented distance) [7]

Let Y be a real normed vector space. For a set $A \subset Y$, let the oriented distance function $\Delta_A \colon Y \to \mathbb{R} \cup \{\pm\infty\}$ be defined by

$$\Delta_A(y) := d_A(y) - d_{Y \setminus A}(y),$$

$d_A(y) = \inf\{\|y - z\| \mid z \in A\}$, $d_\emptyset(y) := +\infty$, and $\|y\|$ denotes the norm of y in Y.

Definition [7]

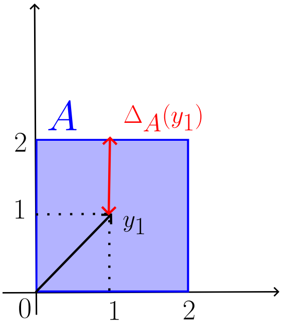
For the set $A \in Y$, let the functions $\mathcal{D}_A^+ \colon \mathcal{P}(Y) \to \mathbb{R} \cup \{\pm\infty\}$ and $\mathcal{D}_A^- \colon \mathcal{P}(Y) \to \mathbb{R} \cup \{\pm\infty\}$ be defined as

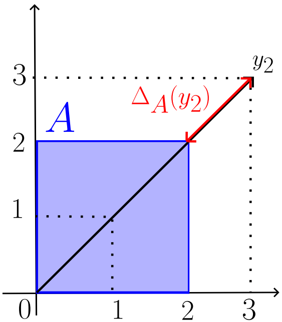
$$\mathcal{D}_A^+(B) := \sup\{\Delta_A(b) \mid b \in B\},\ B \in \mathcal{P}(Y),$$
$$\mathcal{D}_A^-(B) := \inf\{-\Delta_A(b) \mid b \in B\} = -\mathcal{D}_A^+(B),\ B \in \mathcal{P}(Y).$$

Example

Let $Y = \mathbb{R}^2$, $A = [0,2] \times [0.2]$, $y_1 = (1,1)$, and $y_2 = (3,3)$. Then,

$$\Delta_A(y_1) = d_A(y_1) - d_{Y\setminus A}(y_1) = 0 - 1 = -1,$$
$$\Delta_A(y_2) = d_A(y_2) - d_{Y\setminus A}(y_2) = 1 - 0 = \sqrt{2}.$$





Theorem
Let X be a nonempty compact convex subset of a topological vector space, Y a real normed vector space, C a closed convex cone in Y with $\operatorname{int} C \neq \emptyset$, and, $F\colon X \times X \to \mathcal{P}(Y)\backslash\{\emptyset\}$ a set-valued map. If F satisfies the following conditions:

1. there exists $x_0, y_0 \in X$ such that $(\varphi \circ F)(x_0, y_0) \in \mathbb{R}$,
2. for each fixed $y \in X$, $F(\cdot, y)$ is $(\preccurlyeq_2, C)$-continuous (that is, C-upper continuous),
3. for each fixed $x \in X$, $F(x, \cdot)$ is $(3L)$-naturally quasi C-concave,
4. for all $x \in X$, $\{\theta_Y\} \not\preccurlyeq^{(3L)}_{\operatorname{int} C} F(x, x)$,

then there exists $\bar{x} \in X$ such that $\{\theta_Y\} \not\preccurlyeq^{(3L)}_{\operatorname{int} C} F(\bar{x}, y)$ for all $y \in X$.

Conclusion

Conclusion

- We gave a new result of set-valued Fan-Takahashi inequalities via scalarization .
- Kuwano's result which is introduced at first implies the only (3L) type minimax inequality. We can obtain the same type minimax inequality holds while the scalarization function is the oriented distance function.
- We need to check other scalarization functions to satisfy new assumption.

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