2 Asymptotic Cones and Functions
2.1 Definition of Asymptotic Cones

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We use the book; Asymptotic Cones and Functions in Optimization and Variational Inequalities (author: A.AUSLENDER and M.TEBOULLE), pp.25-31.

The set of natural numbers is denoted by $\mathbb{N}$, so that $k \in \mathbb{N}$ means $k = 1, 2, \ldots$. A sequence $\{x_k\}_{k\in\mathbb{N}}$ or simply $\{x_k\}$ in $\mathbb{R}^n$ is said to converge to x if $\|x_k - x\| \to 0$ as $k \to \infty$, and this will be indicated by the notation $x_k \to x$ or $x = \lim_{k\to\infty} x_k$. We say that x is a cluster point of $\{x_k\}$ if some subsequence converge to x. Recall that every bounded sequence in $\mathbb{R}^n$ converges to x if and only if it is bounded and has x as its unique cluster point.

Let $\{x_k\}$ be a sequence in $\mathbb{R}^n$. We are interested in knowing how to handle convergence properties, we are led to consider direction $d_k := x_k \|x_k\|^{-1}$ with $x_k \neq 0$, $k \in \mathbb{N}$. From classical analysis, the Bolzano-Weierstrass theorem implies that we can extract a convergent subsequence $d = \lim_{k\in K} d_k$, $K \subset \mathbb{N}$, with $d \neq 0$. Now suppose that the sequence $\{x_k\} \subset \mathbb{R}^n$ is such that $\|x_k\| \to +\infty$. Then

$$\exists t_k := \|x_k\|, k \in K \subset \mathbb{N}, \text{ such that } \lim_{k\in K} t_k = +\infty \text{ and } \lim_{k\in K} \frac{x_k}{t_k} = d.$$

This leads us to introduce the following concepts.

Definition 2.1.1

A sequence $\{x_k\} \subset \mathbb{R}$ is said to converge to a direction $d \in \mathbb{R}^n$ if

$$\exists \{t_k\},\ with\, t_k \to +\infty \text{ such that } \lim_{k\to\infty} \frac{x_k}{t_k} = d.$$

Definition 2.1.2

Let C be a nonempty set in $\mathbb{R}^n$. Then the asymptotic cone of the set C, denoted by C_∞, is the set of vectors $d \in \mathbb{R}^n$ that are limits in direction of the sequences $\{x_k\} \subset C$, namely

$$C_\infty = \{d \in \mathbb{R}^n \mid \exists t_k \to +\infty, \exists x_k \in C \text{ with } \lim_{k\to\infty} \frac{x_k}{t_k} = d\}.$$

From the definition we immediately deduce the following elementary facts.

Proposition 2.1.1

Let $C \subset \mathbb{R}^n$ be nonempty. Then:

(i) C_∞ is a closed cone.
(ii) $(\operatorname{cl} C)_\infty = C_\infty$.
(iii) If C is a cone, then $C_\infty = \operatorname{cl} C$.

Proof. We will prove each part separately.

(i) C_∞ is a closed cone.

We need to show two propositions: (i-a) C_∞ is a cone and (i-b) C_∞ is a closed set.

(i-a) We show that C_∞ is a cone, that is, $\forall \alpha \geq 0, d \in C_\infty, \alpha d \in C_\infty$.

Since 0 is a element of C_∞, it is clear in the case of $\alpha = 0$.

($\because$ Since C is nonempty, we can take a element x_0 from C. In addition we take a sequence $\{t_k\}_{k=1}^\infty$ with $t_k \to +\infty$ as $k \to \infty$. Of course this sequence exists, for example $t_k := k$. By using $t_k := k$ and $x_k := x_0$, we can obtain 0 as the limit. Hence 0 is a element of C_∞.)

Also we consider the other case $\alpha > 0$. To prove that C_∞ is a cone, we take a any direction d from C_∞. Since d is a element of C_∞,

$$\exists t_k \to +\infty, \exists x_k \in C \text{ with } \lim_{k\to\infty} \frac{x_k}{t_k} = d.$$

Then we define a sequence $\{t'_k\}_{k=1}^\infty := \frac{t_k}{\alpha}$, exactly whose limit becomes $+\infty$ as $k \to \infty$. Accordingly there exist $t'_k \to +\infty$ and $x_k \in C$ with

$$\lim_{k\to\infty} \frac{x_k}{t'_k} = \lim_{k\to\infty} \alpha \cdot \frac{x_k}{t_k} = \alpha d.$$

This means $d \in C_\infty$.

By these results, we can get $\forall \alpha \geq 0, d \in C_\infty, \alpha d \in C_\infty$.

Therefore C_∞ is a cone.

(i-b) We show that C_∞ is a closed set. In order to prove closeness, we consider convergency of a sequence of C_∞. First we take a sequence $\{d_k\}_{k=1}^\infty \subset C_\infty$ with $d_k \to d$ as $k \to \infty$ for some d. Then we don't forget that $d \in C_\infty$ is our goal.

For each $k \in \mathbb{N}$,

$$\exists \{x_k^{(n)}\}_{n=1}^\infty \subset C \text{ and } \{t_k^{(n)}\}_{n=1}^\infty \text{ with } t_k^{(n)} \to \infty \text{ as } n \to \infty.$$

The below figure represents $x_k^{(n)}$ and $t_k^{(n)}$.

Figure:

$k \setminus n$	1	2	$\cdots$	m	$\cdots$	limit
1	$x_1^{(1)},\, t_1^{(1)}$	$x_1^{(2)},\, t_1^{(2)}$	$\cdots$	$x_1^{(m)},\, t_1^{(m)}$	$\cdots$	d_1
2	$x_2^{(1)},\, t_2^{(1)}$	$x_2^{(2)},\, t_2^{(2)}$	$\cdots$	$x_2^{(m)},\, t_2^{(m)}$	$\cdots$	d_2
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$
m	$x_m^{(1)},\, t_m^{(1)}$	$x_m^{(2)},\, t_m^{(2)}$	$\cdots$	$x_m^{(m)},\, t_m^{(m)}$	$\cdots$	d_m
$\vdots$				$\vdots$		

Then we define

$$x_m := x_m^{(m)} \text{ and } t_m := t_m^{(m)}.$$

By the definition of convergence of a sequence,

$$\forall \epsilon > 0, \exists \bar{m} \in \mathbb{N} \; s.t. \; \forall m \geq \bar{m}, ||d_m - d|| < \frac{\epsilon}{2}, \text{and}$$
$$\forall \epsilon > 0, \exists \hat{m} \in \mathbb{N} \; s.t. \; \forall m \geq \hat{m}, ||\frac{x_m}{t_m} - d_m|| < \frac{\epsilon}{2}.$$

Also, we let $\tilde{m} :=$ max $\{\bar{m}, \hat{m}\} \in \mathbb{N}$. By using triangle inequality,

$$\forall \epsilon > 0, \exists \tilde{m} \in \mathbb{N} \; s.t. \; \forall m \geq \bar{m}, ||\frac{x_m}{t_m} - d|| < \epsilon.$$

$(\because ||\frac{x_m}{t_m} - d|| \leq ||\frac{x_m}{t_m} - d_m|| + ||d_m - d|| < \epsilon..)$

Therefore c_∞ is a closed set.

Then (i)'s proof is completed.

(ii) $(\operatorname{cl} C)_\infty = C_\infty$.

We need to show two relations: (ii-a) $(\operatorname{cl} C)_\infty \supset C_\infty$ (ii-b) $(\operatorname{cl} C)_\infty \subset C_\infty$.

(ii-a) We show that C_∞ is included in $(\operatorname{cl} C)_\infty$. However it is clear from the definition of asymptotic cone.

(ii-b) We show that $(\operatorname{cl} C)_\infty \subset C_\infty$. In order to prove that a element of $(\operatorname{cl} C)_\infty$ satisfies the asymptotic cone's relation, we consider convergency of a sequences of $(\operatorname{cl} C)_\infty$ and $\operatorname{cl} C$. First we take any $d \in (\operatorname{cl} C)_\infty$ which satisfies

$$\exists t_k \to +\infty, \exists x_k \in \operatorname{cl} C \ \text{ with } \ \lim_{k\to\infty} \frac{x_k}{t_k} = d.$$

For each $k \in \mathbb{N}$,

$$\exists \{y_k^{(n)}\}_{n=1}^{\infty} \subset C \text{ with } y_k^{(n)} \to x_k \text{ as } n \to \infty.$$

The below figure represents $y_k^{(n)}$.

Figure:

$k \setminus n$	1	2	$\cdots$	m	$\cdots$	limit
1	$y_1^{(1)}$	$y_1^{(2)}$	$\cdots$	$y_1^{(m)}$	$\cdots$	x_1
2	$y_2^{(1)}$	$y_2^{(2)}$	$\cdots$	$y_2^{(m)}$	$\cdots$	x_2
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$
m	$y_m^{(1)}$	$y_m^{(2)}$	$\cdots$	$y_m^{(m)}$	$\cdots$	x_m
$\vdots$				$\vdots$		

Then we define

$$y_m \coloneqq y_m^{(m)}.$$

By the definition of convergence of a sequence,

$$\begin{aligned}
&\forall \epsilon > 0, \exists \bar{m} \in \mathbb{N}\ s.t.\ \forall m \geq \bar{m}, ||d_m - d|| < \frac{\epsilon}{2},\\
&\forall \epsilon > 0, \exists \hat{m} \in \mathbb{N}\ s.t.\ \forall m \geq \hat{m}, ||y_m^m - x_m|| < \frac{\sqrt{\epsilon}}{2}, \text{and}\\
&\forall \epsilon > 0, \exists \tilde{m} \in \mathbb{N}\ s.t.\ \forall m \geq \tilde{m}, |\frac{1}{t_m}| < \sqrt{\epsilon}.
\end{aligned}$$

Also, we let $m_0 \coloneqq \max\{\bar{m}, \hat{m}, \tilde{m}\} \in \mathbb{N}$. By using triangle inequality,

$$\forall \epsilon > 0, \exists m_0 \in \mathbb{N}\ s.t.\ \forall m \geq \bar{m}, ||\frac{y_m}{t_m} - d|| < \epsilon.$$

($\because ||\frac{y_m}{t_m} - d|| \leq \frac{1}{|t_m|} \cdot ||y_m - d_m|| + ||\frac{y_m}{t_m} - d|| < \epsilon.$)

Therefore $(\operatorname{cl} C)_\infty \subset C_\infty$.

Then (ii)'s proof is also completed.

(iii) If C is a cone, then $C_\infty = \operatorname{cl} C$.

We need to show two relations: (iii-a) $C_\infty \subset \operatorname{cl} C$ and (iii-b) $C_\infty \supset \operatorname{cl} C$.

(iii-a) We take any direction $d \in C_\infty$ which satisfies

$$\exists t_k \to +\infty, \exists x_k \in C \ \text{ with } \ \lim_{k\to\infty} \frac{x_k}{t_k} = d.$$

Let $d_k := \frac{x_k}{t_k}$ (with $d_k \to d$ as $k \to \infty$). Since C is a cone,

$$d_k = \frac{1}{t_k} \cdot x_k \in C.$$

Due to $d_k \in C$, the limit of d_k is a element of $\operatorname{cl} C$, i.e., $d \in \operatorname{cl} C$.

Therefore $C_\infty \subset \operatorname{cl} C$.

(iii-b) We take any $d \in \operatorname{cl} C$ and show $d \in C_\infty$, that is,

$$\exists t_k \to +\infty, \exists x_k \in C \ \text{ with } \ \lim_{k\to\infty} \frac{x_k}{t_k} = d.$$

By $d \in \operatorname{cl} C$,

$$\exists \{d_k\}_{k=1}^{\infty} \in C \text{ with } d_k \to d \text{ as } k \to \infty,$$

in other words,

$$\lim_{k\to\infty} d_k = d.$$

We define $y_k = k \cdot d_k$ and $s_k = k$ for each k. Since $d_k \in C$ and C is a cone, y_k is also a element of C.

There exist $\{s_k\}_{k=1}^{\infty}$ with $s_k \to \infty$ as $k \to \infty$ and $\{y_k\}_{k=1}^{\infty} \subset C$ such that

$$\lim_{k\to\infty} \frac{y_k}{s_k} = \lim_{k\to\infty} d_k = d.$$

As d is a element of C_∞, therefore $C_\infty \supset \operatorname{cl} C$.

□

The importance of the asymptotic cone is revealed by the following key property, which is a immediate consequence of its definition.

Proposition 2.1.2

A set $C \subset \mathbb{R}^n$ is bounded if and only if $C_\infty = \{0\}$.

Proof. We show that:

(i) a set $C \subset \mathbb{R}^n$ is bounded $\Rightarrow C_\infty = \{0\}$, and
(ii) a set $C \subset \mathbb{R}^n$ is unbounded $\Rightarrow C_\infty \neq \{0\}$.

(i) By Proposition 2.1.1 (i), $0 \in C_\infty$. Also, by the assumption C is bounded,

$$\exists r > 0, \forall x_k \in C \text{ where } k \in \mathbb{N}, \|x_k\| \leq r.$$

For any sequence $\{t_k\}_{k=1}^{\infty}$ such that $t_k \to \infty$ as $k \to \infty$,

$$\lim_{k\to\infty} \frac{x_k}{t_k} = 0.$$

Thus the limit becomes only 0 for any $\{x_k\}_{k=1}^{\infty} \subset C$ and $\{t_k\}_{k=1}^{\infty}$ with $t_k \to \infty$ as $k \to \infty$.

Therefore if C is bounded then $C_\infty = \{0\}$.

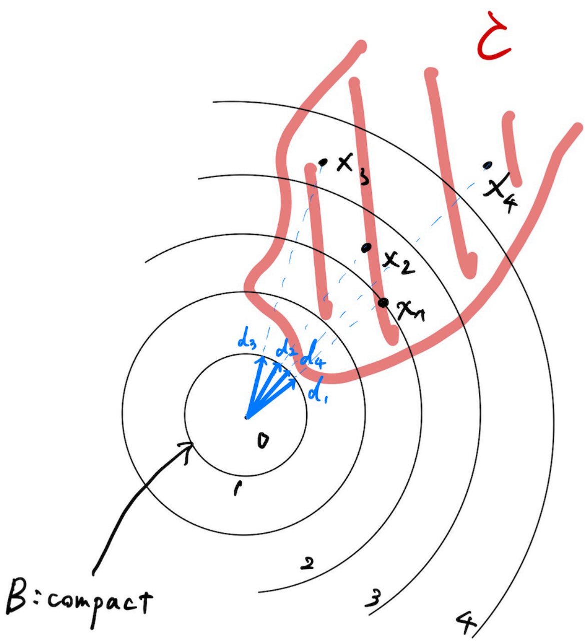
(ii) If C is unbounded, then there exists a sequence $\{x_k\} \subset C$ with $x_k \neq 0$, $\forall k \in \mathbb{N}$, such that $t_k := \|t_k\| \to \infty$ and thus the vectors $d_k = t_k^{-1} x_k \in \{d : \ \|d\| = 1\}$.

By the Bolzano-Weierstrass, we can extract a subsequence of $\{d_k\}$ such that $\lim_{k\in K} d_k = d$, $K \subset \mathbb{N}$, and with $\|d\| = 1$. This nonzero vector d is an element of C_∞ by Definition 2.1.2, a contradiction.

$\square$

Figure:

Why do we take a subsequence of $\{d_k\}$?



Associated with the asymptotic cone C_∞ is the following related concept, which will help us in simplifying the definition of C_∞ in the particular case where $C \in \mathbb{R}^n$ is assumed convex.

Definition 2.1.3

Let $C \in \mathbb{R}^n$ be nonempty and define

$$C_\infty^1 = \{d \in \mathbb{R}^n \mid \forall t_k \to +\infty, \exists x_k \in C \text{ with } \lim_{k\to\infty} \frac{x_k}{t_k} = d\}.$$

We say that C is asymptotically regular if $C_\infty = C_\infty^1$

Remark: A set $D = \{(x, y) \in \mathbb{R}_+ \times \mathbb{R}_+ \mid y = e^x - 1\}$ is not asymptotically regular.

Proposition 2.1.3

Let C be a nonempty convex set in $\mathbb{R}^n$. Then C is asymptotically regular.

Proof. The inclusion $C_\infty^1 \subset C_\infty$ clearly holds from the definition of C_∞^1 and C_∞, respectively. Let $d \in C_\infty$. Then $\exists\{x_k\} \in C$, $\exists s_k \to \infty$ such that $d = \lim_{k\to\infty} s^{-1} x_k$. Let $x \in C$ and define $d_k = s^{-1}(x_k - x)$. Then we have

$$d = \lim_{k\to\infty} d_k, x + s_k d_k \in C.$$

$\because x_k = x + s_k d_k \in C.$

Now note that an arbitrary sequence such that $\lim_{k\to\infty} t_k = +\infty$. For any fixed $m \in \mathbb{N}$, there exists $k(m)$ with $\lim_{m\to\infty} k(m) = +\infty$ such that $t_m \leq s_{k(m)}$, and since C is convex, we have $x'_m = x + t_m + t_m d_{m(k)} \in C$. Hence, $d = \lim_{m\to\infty} t_m^{-1} x'_m$, showing that $d \in C_\infty^1$.

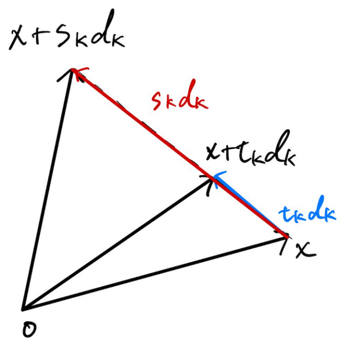
□

Figure:

Why should we consider $k(m)$ with $\lim_{m\to\infty} k(m) = +\infty$ such that $t_m \leq s_{k(m)}$?

If $\{s_k\} = 1, 2, 3, 7, 8, 9, 13, \cdots$ and $\{t_k\} = 1, 3, 4, 6, 8, 10, 11, \cdots$, then we can get

$$\{k(m)\} = 1, 3, 4, 5, 6, 6, \cdots.$$



We note that a set can be nonconvex, yet asymptotically regular Indeed, consider, for example, sets definition by $C := S + K$, with S compact and K a closed convex cone. Then clearly C is not necessarily convex, but it can be easily seen that $C_\infty = C^1_\infty$.

Remark 2.1.1

Note that the definitions of C_∞ and C^1_∞ are related to the theory of set convergence of Painleve-Kuratowski. Indeed, for a family $\{C_t\}_{t>0}$ of subsets of $\mathbb{R}^n$, the outer limit as $t \to +\infty$ is the set.

$$\limsup_{t\to+\infty} C_t = \{x \mid \liminf_{t\to+\infty} d(x, C_t) = 0\},$$

while the inner limit as $t \to +\infty$ is the set

$$\liminf_{t\to+\infty} C_t = \{x \mid \limsup_{t\to+\infty} d(x, C_t) = 0\},$$

It can then be verified that the corresponding asymptotic cones can be written as

$$C_\infty = \limsup_{t\to+\infty} t^{-1}C, \;\; C^1_\infty = \liminf_{t\to+\infty} t^{-1}C.$$