Applications of Asymptotic Function to Semidefinite Programming

漸近関数の半正定値計画問題への応用

F22A034C 岩本 峻汰

9th, February, 2024

Niigata Univ

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Introduction & Motivation

Introduction & Motivation

Consider the following general **composite** optimization problem:

$$\begin{aligned} &\inf \phi(x) \\ &\text{s.t. } x \in \mathbb{R}^n \end{aligned} \tag{CM}$$

with

$$\phi(x) = \begin{cases} f_0(x) + H_\infty(f_1(x), \ldots, f_m(x)) & \text{if } x \in \bigcap_{i=1}^m \operatorname{dom} f_i, \\ +\infty & \text{otherwise,} \end{cases}$$

where $f_0, \ldots, f_m$ are real-valued proper l.s.c. functions, H is a proper convex l.s.c. function, and H_∞ is the asymptotic function of H.

Introduction & Motivation

Consider the approximate problem for the problem (CM):

$$\begin{aligned} &\inf \phi_r(x) \\ &\text{s.t. } x \in \mathbb{R}^n \end{aligned} \tag{CMr}$$

with

$$\phi_r(x) = \begin{cases} f_0(x) + H_r(f_1(x), \ldots, f_m(x)) & \text{if } x \in \bigcap_{i=1}^m \operatorname{dom} f_i, \\ +\infty & \text{otherwise,} \end{cases}$$

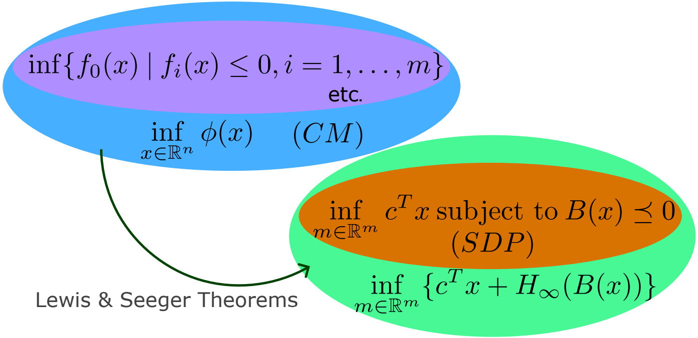
Remark

The asymptotic function H_∞ of a given proper l.s.c. convex function H can be approximated by

$$H_\infty(y) = \lim_{r \to 0^+} H_r(y) := rH(\frac{y}{r}), \forall y \in \operatorname{dom} H.$$

Motivation

In the n-dimensional real Euclidean space $\mathbb{R}^n$, asymptotic cones and functions play a significant role to consider optimization problems. In addition, semidefinite programming (SDP) is one of the most important optimization problems because these problems have appeared in various area of mathematical sciences. The following content explains the relation between asymptotic function and SDP.



Preliminary

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$\mathbb{R}^n$: n-dimensional real Euclidean space.

$\mathbb{S}^n$: n-dimensional real symmetric matrix space.

The inner product of $\mathbb{R}^n$ $\langle \cdot, \cdot \rangle$ is defined by

$$\langle x, y \rangle := \sum_{i=1}^{n} x_i y_i \text{ for } x = (x_1, \ldots, x_n)^T \in \mathbb{R}^n \text{ and } y = (y_1, \ldots, y_n)^T \in \mathbb{R}^n.$$

The norm is defined by $\|x\| := \langle x, x \rangle^{1/2}$. Like that, we can define the inner product and the norm of $\mathbb{S}^n$.

$$\langle X, Y \rangle := \mathrm{tr}(XY) \text{ for } X, Y \in \mathbb{S}^n \quad \text{and} \quad \|X\| := \langle X, X \rangle^{1/2}$$

Definition 1

Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$.

1. The domain of f is defined by

$$\mathrm{dom}\, f := \{x \in \mathbb{R}^n \mid f(x) < +\infty\}.$$

2. The epigraph of f is defined by

$$\mathrm{epi}\, f := \{(x, y) \in \mathbb{R}^n \times \mathbb{R} \mid f(x) \leq y\}.$$

3. f is called proper if $\mathrm{dom}\, f \neq \emptyset$.
4. f is called convex if $\mathrm{epi}\, f$ is convex.
5. f is called lower semicontinuous (l.s.c.) if $\mathrm{epi}\, f$ is closed.

Definition 2
Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$. The conjugate function of f is defined by

$$f^*(y) := \sup_{x \in \mathbb{R}^n} \{\langle x, y \rangle - f(x)\}.$$

Definition 3
Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$. f is said to be symmetric if

$$\forall x \in \mathbb{R}^n \text{ and } P : n \times n \text{ permutation matrix}, f(Px) = f(x).$$

Spectrally defined matrix function

Spectrally defined matrix function

Definition 4

The function $\Phi : \mathbb{S}^n \to \mathbb{R} \cup \{+\infty\}$ is said to be spectrally defined if there exists a symmetric function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ such that

$$\Phi(X) = \Phi_f(X) := f(\lambda(X)), \forall X \in \mathbb{S}^n$$

where $\lambda(X) := (\lambda_1(X), \cdots, \lambda_n(X))^T$ is the vector of eigenvalues of X in nondecreasing order.

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Example

When we define a symmetric function f, a spectrally defined function is deduced:

$$f(\lambda) = \begin{cases} -\Sigma_{i=1}^n \log \lambda_i & \text{if } \lambda > 0; \\ +\infty & \text{otherwise} \end{cases}, \text{and then} \quad \Phi_f(X) = \begin{cases} -\log\det(X) & \text{if } X \succ 0; \\ +\infty & \text{otherwise.} \end{cases}$$

Spectrally defined matrix function

Theorem 5 (A.S.Lewis (1996))
Suppose that the function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is symmetric, then

$$\Phi_f{}^* = \Phi_{f^*}$$

where $\Phi_f{}^*(Y) := \sup\{\langle X, Y\rangle - \Phi_f(X) \mid X \in \mathbb{S}^n\}, \forall Y \in \mathbb{S}^n$.

Remark
As a result of Theorem 5, the optimization problems $\min\{\Phi_f(X) \mid X \in \mathbb{S}^n\}$ and $\min\{f(x) \mid x \in \mathbb{R}^n\}$ are equivalent. In fact,

$$\begin{aligned}\inf_{X\in\mathbb{S}^n} \Phi_f(X) &= -\sup_{X\in\mathbb{S}^n} \{-\Phi_f(X)\} = -\sup_{X\in\mathbb{S}^n} \{\langle X, 0\rangle - \Phi_f(X)\} \\ &= -\Phi_f^*(0) = -\Phi_{f^*}(0) = -f^*(0) = \inf_{x\in\mathbb{R}^n} f(x).\end{aligned}$$

Asymptotic cone and asymptotic function

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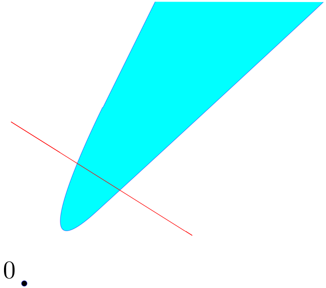

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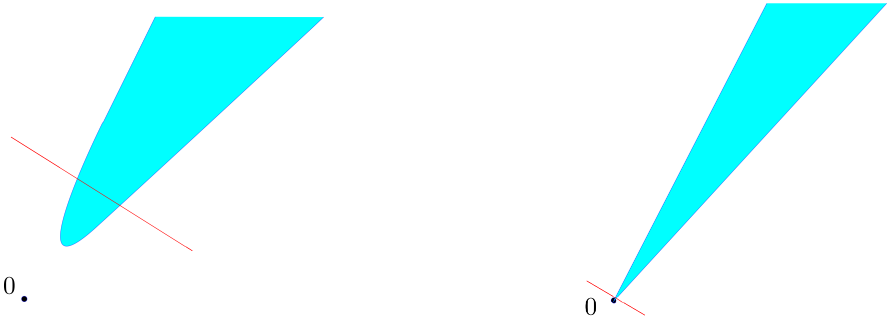

Definition of asymptotic Cone

Definition 6

$C \subset \mathbb{R}^n$, $C \neq \emptyset$. Then, the asymptotic cone of the set C, denoted by C_∞, is the set below with $\{x_k\} \subset C$;

$$C_\infty = \left\{ d \in \mathbb{R}^n \;\middle|\; \exists t_k \to +\infty, \exists x_k \in C \text{ with } \lim_{k \to \infty} \frac{x_k}{t_k} = d \right\}.$$

Example

$C = \{(x, y) \in \mathbb{R}^2 \mid y = x^2\}$. We let $x_k = (k, k^2)$ and $t_k = \|x_k\|$.

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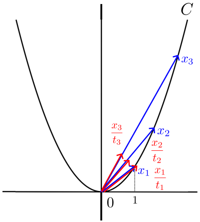

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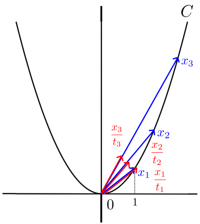

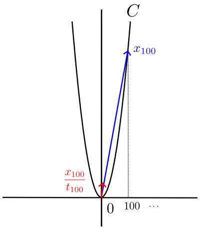

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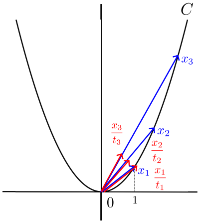

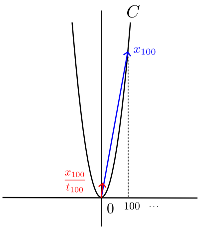

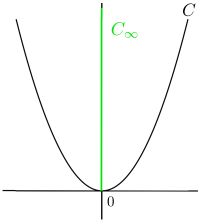

Properties around asymptotic cone

Proposition 7

Let C be a nonempty convex set in $\mathbb{R}^n$. Then the asymptotic cone C_∞ is a closed convex cone. Moreover, define the following sets;

$$\begin{aligned} D(x) &:= \{d \in \mathbb{R}^n \mid x + td \in \mathrm{cl}\, C, \forall t > 0\} \forall x \in C, \\ E &:= \{d \in \mathbb{R}^n \mid \exists x \in C \text{ s.t. } x + td \in \mathrm{cl}\, C, \forall t > 0\}, \\ F &:= \{d \in \mathbb{R}^n \mid d + \mathrm{cl}\, C \subset \mathrm{cl}\, C\}. \end{aligned}$$

Then $D(x)$ is in fact independent of x, which is thus now denoted by D, and $C_\infty = D = E = F$.

Remark

When C is a closed convex set, the asymptotic cone is also called the recession cone. In this presentation, we use the term the asymptotic cone.

Definition of asymptotic function

Definition 8

Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be proper. Then, there exists $f_\infty : \mathbb{R}^n \to \mathbb{R} \cup \{\pm\infty\}$ satisfying $\mathrm{epi}\ f_\infty = (\mathrm{epi}\ f)_\infty$, which is called asymptotic function of f.

Example

$f(x, y) = \sqrt{1 + x^2 + y^2}$, where $x, y \in \mathbb{R}$. Then, the asymptotic function f_∞ is

$$f_\infty(x, y) = \sqrt{x^2 + y^2}$$

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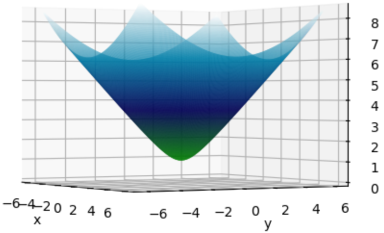

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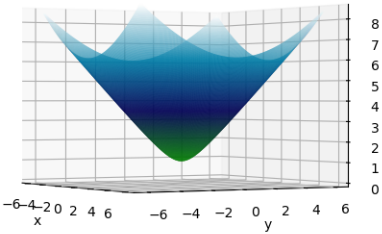

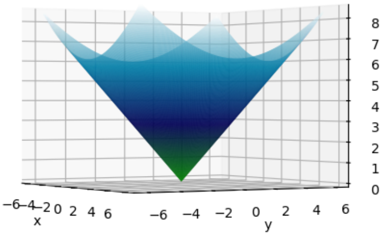

Properties around asymptotic function

Proposition 9

Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a proper, lsc, convex function. Then, the asymptotic function is a lsc, proper, convex function, and one has

$$f_\infty(d) = \lim_{t \to +\infty} \frac{f(x + td) - f(x)}{t} = \sup_{t>0} \frac{f(x + td) - f(x)}{t}, \forall x \in \mathrm{dom}\, f$$

and

$$f_\infty(d) = \sup\{\langle x, d\rangle \mid x \in \mathrm{dom}\, f^*\}.$$

Applications of asymptotic function to SDP

Application of the notion of asymptotic function

Definition 10

Following Proposition 9, the asymptotic functions of the proper convex lsc function $\Phi : \mathbb{S}^n \to \mathbb{R} \cup \{+\infty\}$ is defined by, for all $D \in \mathbb{S}^n$

$$\Phi_\infty(D) = \sup_{t>0} \frac{\Phi(A + tD) - \Phi(A)}{t}, \forall A \in \operatorname{dom} \Phi \quad \text{and}$$

$$\Phi_\infty(D) = \sup\{\langle B, D \rangle \mid B \in \operatorname{dom} \Phi^*\}.$$

Theorem 11 (A.Seeger (1997))

Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a symmetric, lsc, proper, convex function with induced spectral function Φ_f. Then

$$(\Phi_f)_\infty = \Phi_{f_\infty}.$$

Conclusion

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Summary:

We figure out concepts of asymptotic cones and functions. In addition, spectrally defined matrix functions which play a significant role in optimization problems are also introduced.
While the theorem of A.S.Lewis (1996) means we can apply the conjugacy to these functions, the result of A.Seeger (1997) shows the relation between spectral functions and asymptotic functions.

Issue:

We have not yet considered the practical application of asymptotic functions to semidefinite programming in detail.
There exist some other important results in the field of optimization and variational inequalities we do not know.

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