A relation between Asymptotic Cones and Painlevé-Kuratowski Convergence

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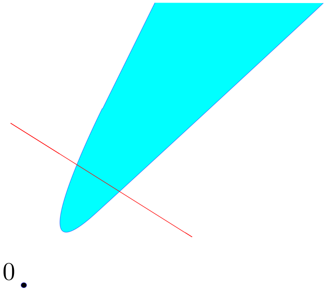
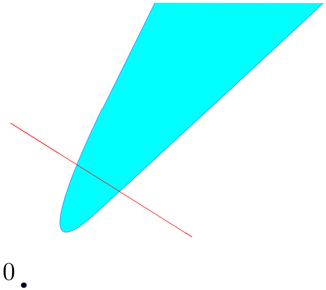
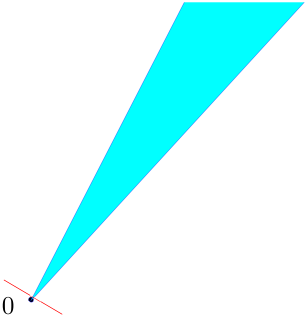


What is asymptotic?

$\rightarrow$ To look at something from a distance, that is, to zoom out.





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Preliminary

$\mathbb{R}^n$: n-dimensional real Euclidean space.
The inner product of $\mathbb{R}^n$ $\langle \cdot, \cdot \rangle$ is defined by

$$\langle x, y \rangle := \sum_{i=1}^{n} x_i y_i, \text{ for } x = (x_1, \ldots, x_n)^T \in \mathbb{R}^n \text{ and } y = (y_1, \ldots, y_n)^T \in \mathbb{R}^n.$$

The norm is defined by $\|x\| := \langle x, x \rangle^{1/2}$.

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Motivation for Asymptotic Cones

Regarding asymptotic cones, we recall the concept of cluster points for bounded sets.

Definition 1 (cluster point)

x is a cluster point of $\{x_k\}$ if some subsequence converges to x.

Proposition 1

The following statements are equivalent:

1. a sequence in $\mathbb{R}^n$ converges to x,
2. a sequence is bounded and has x as its unique cluster point.

Generally, we need to consider the uniqueness of cluster point and the boundedness of the given sequence to determine the convergent point in $\mathbb{R}^n$.

Motivation for Asymptotic Cones

Remark

If the given sequence is bounded, Bolzano-Weierstrass theorem implies that there exists a subsequence which converges to a point.

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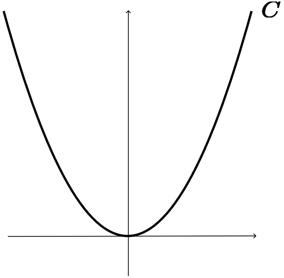
If the given sequence is bounded, Bolzano-Weierstrass theorem implies that there exists a subsequence which converges to a point.

How do we consider the convergence of a sequence, which is not bounded?

Example: $C = \{(x, y) \in \mathbb{R}^2 \mid y = x^2\}$



Definition 2 (Asymptotic Cones)

$C \subset \mathbb{R}^n$, $C \neq \emptyset$. Then, the asymptotic cone of the set C, denoted by C_∞, is the set below with $\{x_k\} \subset C$;

$$C_\infty = \left\{ d \in \mathbb{R}^n \;\middle|\; \exists t_k \to +\infty, \exists x_k \in C \text{ with } \lim_{k \to \infty} \frac{x_k}{t_k} = d \right\}.$$

Example: $C = \{(x, y) \in \mathbb{R}^2 \mid y = x^2\}$. We let $x_k = (k, k^2)$ and $t_k = \|x_k\|$.

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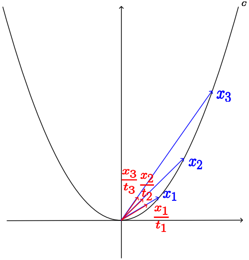

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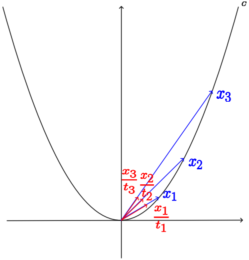

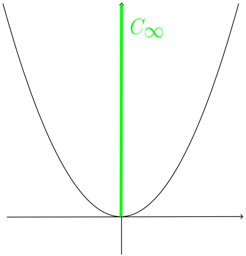

Properties around Asymptotic Cones

Proposition 2

A set $C \subset \mathbb{R}^n$ is bounded if and only if $C_\infty = \{0\}$.

Definition 3

Let $C \subset \mathbb{R}^n$ be nonempty and define

$$C_\infty^1 = \left\{ d \in \mathbb{R}^n \;\middle|\; \forall t_k \to +\infty, \exists x_k \in C \text{ with } \lim_{k \to \infty} \frac{x_k}{t_k} = d \right\}.$$

We say that C is asymptotically regular if $C_\infty = C_\infty^1$.

Proposition 3

Let C be a nonempty convex set in $\mathbb{R}^n$. Then C is asymptotically regular.

Example of Asymptotically Regular

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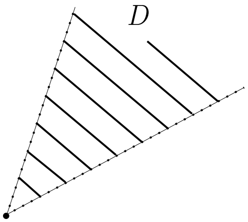
Example: D is NOT asymptotically regular.



Example of Asymptotically Regular

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$$C^1_\infty = \left\{ d \in \mathbb{R}^n \,\middle|\, \forall t_k \to +\infty, \exists x_k \in C \text{ with } \lim_{k\to\infty} \frac{x_k}{t_k} = d \right\}.$$

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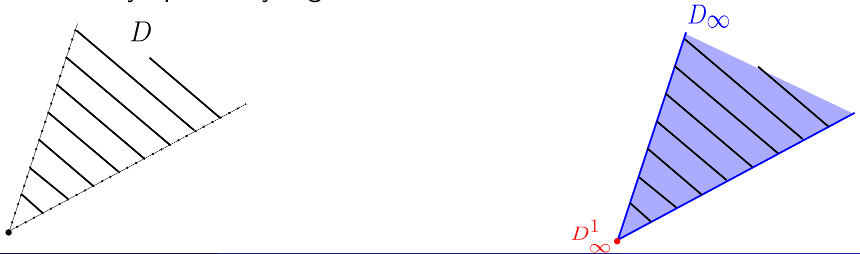

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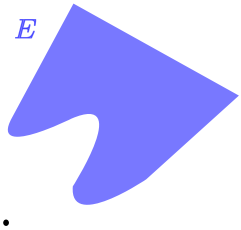

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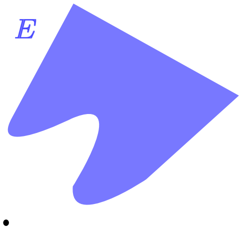

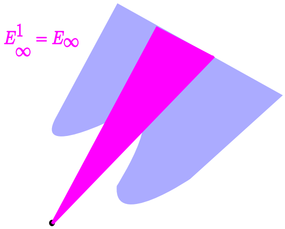

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Definition of the Painlevé-Kuratowski Convergence

Definition 4

Y: a topological vector space.
$\mathcal{P}(Y)$: a family of subset in Y.
Let $(A_n)_{n\in\mathbb{N}} \subset \mathcal{P}(Y)$. We define the inner limit and the outer limit as

$$\liminf_{n\to\infty} A_n \coloneqq \{y \in Y \mid \exists(y_n) \to y \text{ s.t. } y_n \in A_n \text{ for } n \geq n_0\},$$
$$\limsup_{n\to\infty} A_n \coloneqq \{y \in Y \mid \exists(y_{n(k)}) \to y \text{ s.t. } y_{n(k)} \in A_{n(k)} \text{ for } k \in \mathbb{N}\}.$$

If it holds that $\liminf_{n\to\infty} A_n \supset \limsup_{n\to\infty} A_n$, we say that (A_n) converges in the sense of Painlevé-Kuratowski.

Painlevé-Kuratowski Convergence Examples

Example:

$$A_n = \begin{cases} [0, 1], & \text{if } n \text{ is odd} \\ [0, \frac{1}{n}], & \text{if } n \text{ is even} \end{cases}$$

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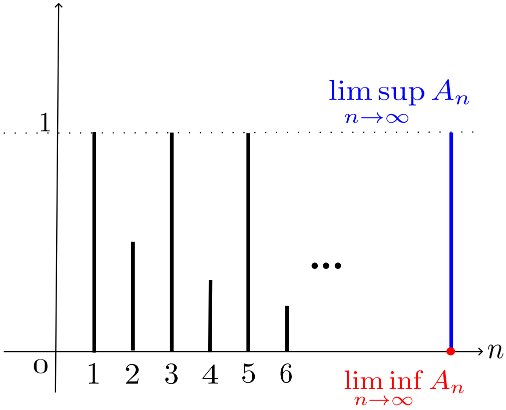

Relations between Asymptotic cones and Painlevé-Kuratowski Convergence

Remark

Providing that the given A_n converges in the sense of Painlevé-Kuratowski, we can let $\Gamma(n) = A_n$ and $\Gamma(\infty) = A$ where $A, A_n \subset Y$. Soon we can find that Γ implies a set-valued mapping.

C: a nonempty set
Γ: $\mathbb{N} \to \mathbb{R}^n$
We let $\Gamma(n) = \frac{C}{n}$.

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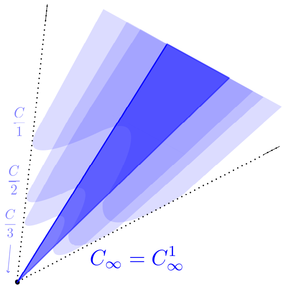

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Conclusions

- Considering a set-valued mappings, we can replace the definition of asymptotic cones as the inner limit.
- The relation allows us to connect the notion of asymptotic cones with some results of continuities in t.v.s.

References

- A. Göpfert, H. Riahi, C. Tammer, and C. Zălinescu, Variational methods in partially ordered spaces, vol. 17 of CMS Books in Mathematics, Springer-Verlag, New York, 2003.
- A. Alfred and M. Teboulle, asymptotic cones and functions in optimization and variational inequalities, Springer monographs in Mathematics, Springer-Verlag, New York, 2003.