2 Asymptotic Cones and Functions

2.1 Definition of Asymptotic Cones

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We use the book; Asymptotic Cones and Functions in Optimization and Variational Inequalities (author: A.AUSLENDER and M.TEBOULLE), pp.25-31.

The set of natural numbers is denoted by $\mathbb{N}$, so that $k \in \mathbb{N}$ means $k = 1, 2, \ldots$. A sequence $\{x_k\}_{k\in\mathbb{N}}$ or simply $\{x_k\}$ in $\mathbb{R}^n$ is said to converge to x if $\|x_k - x\| \to 0$ as $k \to \infty$, and this will be indicated by the notation $x_k \to x$ or $x = \lim_{k\to\infty} x_k$. We say that x is a cluster point of $\{x_k\}$ if some subsequence converge to x. Recall that every bounded sequence in $\mathbb{R}^n$ converges to x if and only if it is bounded and has x as its unique cluster point.

Let $\{x_k\}$ be a sequence in $\mathbb{R}^n$. We are interested in knowing how to handle convergence properties, we are led to consider direction $d_k := x_k \|x_k\|^{-1}$ with $x_k \neq 0$, $k \in \mathbb{N}$. From classical analysis, the Bolzano-Weierstrass theorem implies that we can extract a convergent subsequence $d = \lim_{k\in K} d_k$, $K \subset \mathbb{N}$, with $d \neq 0$. Now suppose that the sequence $\{x_k\} \subset \mathbb{R}^n$ is such that $\|x_k\| \to +\infty$. Then

$$\exists t_k := \|x_k\|, k \in K \subset \mathbb{N}, \text{ such that } \lim_{k\in K} t_k = +\infty \text{and} \lim_{k\in K} \frac{x_k}{t_k} = d.$$

This leads us to introduce the following concepts.

Definition 2.1.1

A sequence $\{x_k\} \subset \mathbb{R}$ is said to converge to a direction $d \in \mathbb{R}^n$ if

$$\exists \{t_k\},\ with\, t_k \to +\infty \text{ such that } \lim_{k\to\infty} \frac{x_k}{t_k} = d.$$

Definition 2.1.2

Let C be a nonempty set in $\mathbb{R}^n$. Then the asymptotic cone of the set C, denoted by C_∞, is the set of vectors $d \in \mathbb{R}^n$ that are limits in direction of the sequences $\{x_k\} \subset C$, namely

$$C_\infty = \{d \in \mathbb{R}^n \mid \exists t_k \to +\infty, \exists x_k \in C \text{ with } \lim_{k\to\infty} \frac{x_k}{t_k} = d\}.$$

From the definition we immediately deduce the following elementary facts.

Proposition 2.1.1

Let $C \subset \mathbb{R}^n$ be nonempty. Then:

(i) C_∞ is a closed cone.
(ii) $(\operatorname{cl} C)_\infty = C_\infty$.
(iii) If C is a cone, then $C_\infty = \operatorname{cl} C$.

Proof. We will prove each part separately.

(i) C_∞ is a closed cone.

We need to show two propositions: (i-a) C_∞ is a cone and (i-b) C_∞ is a closed set.

(i-a) We show that C_∞ is a cone, that is, $\forall \alpha \geq 0, d \in C_\infty, \alpha d \in C_\infty$.

Since 0 is a element of C_∞, it is clear in the case of $\alpha = 0$.

($\because$ Since C is nonempty, we can take a element x_0 from C. In addition we take a sequence $\{t_k\}_{k=1}^{\infty}$ with $t_k \to +\infty$ as $k \to \infty$. Of course this sequence exists, for example $t_k := k$. By using $t_k := k$ and $x_k := x_0$, we can obtain 0 as the limit. Hence 0 is a element of C_∞.)

Also we consider the other case $\alpha > 0$. To prove that C_∞ is a cone, we take a any direction d from C_∞. Since d is a element of C_∞,

$$\exists t_k \to +\infty, \exists x_k \in C \text{ with } \lim_{k\to\infty} \frac{x_k}{t_k} = d.$$

Then we define a sequence $\{t'_k\}_{k=1}^{\infty} := \frac{t_k}{\alpha}$, exactly whose limit becomes $+\infty$ as $k \to \infty$. Accordingly there exist $t'_k \to +\infty$ and $x_k \in C$ with

$$\lim_{k\to\infty} \frac{x_k}{t'_k} = \lim_{k\to\infty} \alpha \cdot \frac{x_k}{t_k} = \alpha d.$$

This means $d \in C_\infty$.

By these results, we can get

$$\forall \alpha \geq 0, d \in C_\infty, \alpha d \in C_\infty$$

. Therefore C_∞ is a cone.

(i-b)

(ii) $(\mathrm{cl}\, C)_\infty = C_\infty$.

(iii) If C is a cone, then $C_\infty = \mathrm{cl}\, C$.

□

The importance of the asymptotic cone is revealed by the following key property, which is a immediate consequence of its definition.

Proposition 2.1.2

A set $C \subset \mathbb{R}^n$ is bounded if and only if $C_\infty = \{0\}$.