Applications of Asymptotic Function to Semidefinite Programming

漸近関数の半正定値計画問題への応用

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Introduction & Motivation

Consider the following general **composite** optimization problem:

$$\begin{aligned} &\inf \phi(x) \\ &\text{s.t. } x \in \mathbb{R}^n \end{aligned} \tag{CM}$$

with

$$\phi(x) = \begin{cases} f_0(x) + H_\infty(f_1(x), \dots, f_m(x)) & \text{if } x \in \bigcap_{i=1}^m \mathrm{dom}\, f_i, \\ +\infty & \text{otherwise,} \end{cases}$$

where $f_0, \dots, f_m$ are real-valued proper l.s.c. functions, H is a proper convex l.s.c. function, and H_∞ is the asymptotic function of H.

Consider the approximate problem for the problem (CM):

$$\begin{aligned}&\inf \phi_r(x)\\&\text{s.t. } x \in \mathbb{R}^n\end{aligned} \tag{CMr}$$

with

$$\phi_r(x) = \begin{cases} f_0(x) + H_r(f_1(x), \ldots, f_m(x)) & \text{if } x \in \bigcap_{i=1}^m \operatorname{dom} f_i, \\ +\infty & \text{otherwise,} \end{cases}$$

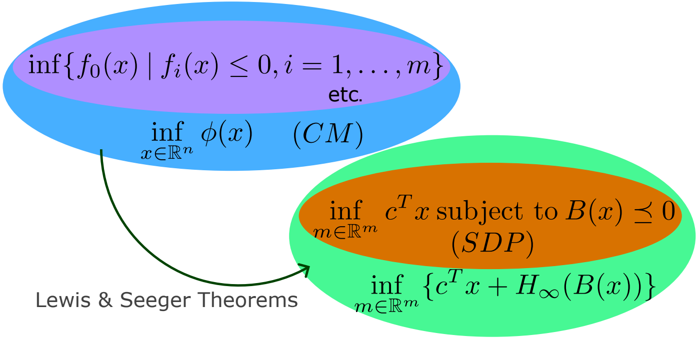
Remark

The asymptotic function H_∞ of a given proper l.s.c. convex function H can be approximated by

$$H_\infty(y) = \lim_{r \to 0^+} H_r(y) := rH(\frac{y}{r}), \forall y \in \operatorname{dom} H.$$

Motivation

In the n-dimensional real Euclidean space $\mathbb{R}^n$, asymptotic cones and functions play a significant role to consider optimization problems. In addition, semidefinite programming (SDP) is one of the most important optimization problems because these problems have appeared in various area of mathematical sciences. The following content explains the relation between asymptotic cones and SDP.



Preliminary

Preliminary

$\mathbb{R}^n$: n-dimensional real Euclidean space.

$\mathbb{S}^n$: n-dimensional real symmetric matrix space.

The inner product of $\mathbb{R}^n$ $\langle \cdot, \cdot \rangle$ is defined by

$$\langle x, y \rangle := \sum_{i=1}^{n} x_i y_i \text{ for } x = (x_1, \ldots, x_n)^T \in \mathbb{R}^n \text{ and } y = (y_1, \ldots, y_n)^T \in \mathbb{R}^n.$$

The norm is defined by $\|x\| := \langle x, x \rangle^{1/2}$. Like that, we can define the inner product and the norm of $\mathbb{S}^n$.

$$\langle X, Y \rangle := \mathrm{tr}(XY) \text{ for } X, Y \in \mathbb{S}^n \quad \text{and} \quad \|X\| := \langle X, X \rangle^{1/2}$$

Definition 1

Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$.

1. The domain of f is defined by

$$\mathrm{dom}\, f := \{x \in \mathbb{R}^n \mid f(x) < +\infty\}.$$

2. The epigraph of f is defined by

$$\mathrm{epi}\, f := \{(x, y) \in \mathbb{R}^n \times \mathbb{R} \mid f(x) \leq y\}.$$

3. f is called proper if $\mathrm{dom}\, f \neq \emptyset$.
4. f is called convex if $\mathrm{epi}\, f$ is convex.
5. f is called lower semicontinuous (l.s.c.) if $\mathrm{epi}\, f$ is closed.

Preliminary

Definition 2
Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$. The conjugate function of f is defined by

$$f^*(y) := \sup_{x \in \mathbb{R}^n} \{\langle x, y \rangle - f(x)\}.$$

Definition 3
Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$. f is said to be symmetric if

$$\forall x \in \mathbb{R}^n \text{ and } P : n \times n \text{ permutation matrix}, f(Px) = f(x).$$

Spectrally defined matrix functions

Spectrally defined matrix functions

Definition 4

The function $\Phi : \mathbb{S}^n \to \mathbb{R} \cup \{+\infty\}$ is said to be spectrally defined if there exists a symmetric function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ such that

$$\Phi(X) = \Phi_f(X) \coloneqq f(\lambda(X)), \forall X \in \mathbb{S}^n$$

where $\lambda(X) \coloneqq (\lambda_1(X), \cdots, \lambda_n(X))^T$ is the vector of eigenvalues of X in nondecreasing order.

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Example

When we define a symmetric function f, a spectrally defined function is deduced:

$$f(\lambda) = \begin{cases} -\Sigma_{i=1}^n \log \lambda_i & \text{if } \lambda > 0; \\ +\infty & \text{otherwise} \end{cases}, \text{and then} \quad \Phi_f(X) = \begin{cases} -\log\det(X) & \text{if } X \succ 0; \\ +\infty & \text{otherwise.} \end{cases}$$

Spectrally defined matrix functions

Theorem 5 (A.S.Lewis (1996))

Suppose that the function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is symmetric, then

$$\Phi_f{}^* = \Phi_{f^*}$$

where $\Phi_f{}^*(Y) := \sup\{\langle X, Y\rangle - \Phi_f(X) \mid X \in \mathbb{S}^n\}, \forall Y \in \mathbb{S}^n$.

Remark

As a result of Theorem 5, the optimization problems $\min\{\Phi_f(X) \mid X \in \mathbb{S}^n\}$ and $\min\{f(x) \mid x \in \mathbb{R}^n\}$ are equivalent. In fact,

$$\begin{aligned}\inf_{X\in\mathbb{S}^n} \Phi_f(X) &= -\sup_{X\in\mathbb{S}^n}\{-\Phi_f(X)\} = -\sup_{X\in\mathbb{S}^n}\{\langle X, 0\rangle - \Phi_f(X)\} \\ &= -\Phi_f^*(0) = -\Phi_{f^*}(0) = -f^*(0) = \inf_{x\in\mathbb{R}^n} f(x).\end{aligned}$$

Asymptotic cones and functions

What is the notion of asymptotic cones?

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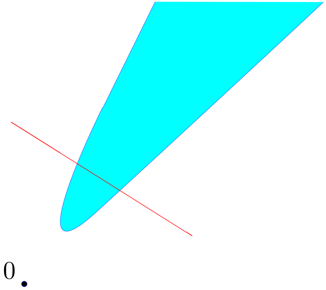

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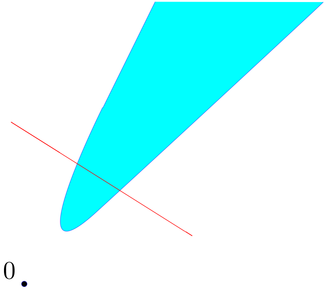

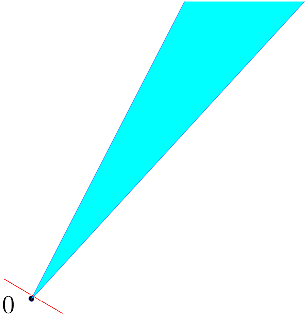

Definition of Asymptotic Cones

Definition 6

$C \subset \mathbb{R}^n$, $C \neq \emptyset$. Then, the asymptotic cone of the set C, denoted by C_∞, is the set below with $\{x_k\} \subset C$;

$$C_\infty = \left\{ d \in \mathbb{R}^n \,\middle|\, \exists t_k \to +\infty, \exists x_k \in C \text{ with } \lim_{k \to \infty} \frac{x_k}{t_k} = d \right\}.$$

Example: $C = \{(x, y) \in \mathbb{R}^2 \mid y = x^2\}$. We let $x_k = (k, k^2)$ and $t_k = \|x_k\|$.

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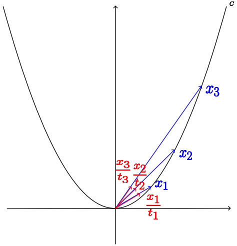

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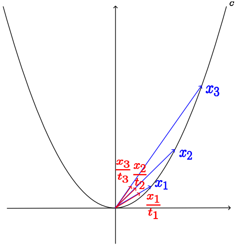

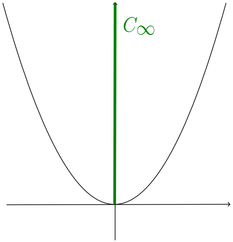

Properties around Asymptotic Cones

Proposition 7

Let C be a nonempty convex set in $\mathbb{R}^n$. Then the asymptotic cone C_∞ is a closed convex cone. Moreover, define the following sets;

$$\begin{aligned} D(x) &:= \{d \in \mathbb{R}^n \mid x + td \in \mathrm{cl}\, C, \forall t > 0\} \forall x \in C, \\ E &:= \{d \in \mathbb{R}^n \mid \exists x \in C \text{ s.t. } x + td \in \mathrm{cl}\, C, \forall t > 0\}, \\ F &:= \{d \in \mathbb{R}^n \mid d + \mathrm{cl}\, C \subset \mathrm{cl}\, C\}. \end{aligned}$$

Then $D(x)$ is in fact independent of x, which is thus now denoted by D, and $C_\infty = D = E = F$.

Remark

When C is a closed convex set, the asymptotic cone is also called the recession cone. In this presentation, we use the term the asymptotic cone.

Definition 8

Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be proper. Then, there exists $f_\infty : \mathbb{R}^n \to \mathbb{R} \cup \{\pm\infty\}$ satisfying $\operatorname{epi} f_\infty = (\operatorname{epi} f)_\infty$, which is called asymptotic function of f.

Definition of Asymptotic Functions

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How to create asymptotic function of f

To gain asymptotic function of f, we have 4 steps including providing the definition of f.

1. provide f
2. consider $\mathrm{epi}\, f$
3. take $(\mathrm{epi}\, f)_\infty$, the asymptotic cone of $(\mathrm{epi}\, f)$
4. define f_∞

How to create Asymptotic Functions

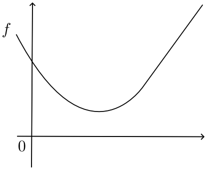

Figure 1: (1) provide f

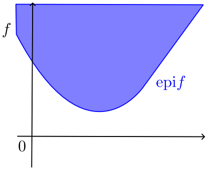

Figure 2: (2) consider $\mathrm{epi}\ f$

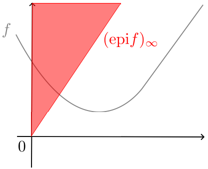

Figure 3: (3) take the asymptotic cone of $(\mathrm{epi}\ f)$

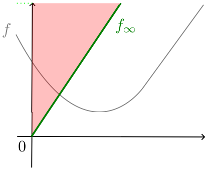

Figure 4: (4) obtain f_∞

Proposition 9
Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a proper, lsc, convex function. Then, the asymptotic function is a lsc, proper, convex function, and one has

$$f_\infty(d) = \lim_{t \to +\infty} \frac{f(x+td) - f(x)}{t} = \sup_{t>0} \frac{f(x+td) - f(x)}{t}, \forall x \in \mathrm{dom}\, f$$

and

$$f_\infty(d) = \sup\{\langle x, d\rangle \mid x \in \mathrm{dom}\, f^*\}.$$

Applications of asymptotic functions to SDP

Application of the notion of asymptotic function

Definition 10

Following Proposition 9, the asymptotic functions of the proper convex lsc function $\Phi : \mathbb{S}^n \to \mathbb{R} \cup \{+\infty\}$ is defined by, for all $D \in \mathbb{S}^n$

$$\Phi_\infty(D) = \sup_{t>0} \frac{\Phi(A + tD) - \Phi(A)}{t}, \forall A \in \mathrm{dom}\, \Phi \quad \text{and}$$

$$\Phi_\infty(D) = \sup\{\langle B, D\rangle \mid B \in \mathrm{dom}\, \Phi^*\}.$$

Theorem 11 (A.Seeger (1997))

Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a symmetric, lsc, proper, convex function with induced spectral function Φ_f. Then

$$(\Phi_f)_\infty = \Phi_{f_\infty}.$$

Conclusion

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Summary:

We figure out why a semidefinite programming with a penalty function like log barrier functions can be solved with using Fenchel duality.

Asymptotic function can be applied to spectrally defined functions.

Issue:

These consequences hardly appears in current studies because the way with using barrier function might not be utilized to obtain an optimal value.

Actually, there are strong relations between spectral functions and composite optimization models.

References

A. Auslender and M. Teboulle, asymptotic cones and functions in optimization and variational inequalities, Springer monographs in Mathematics, Springer-Verlag, New York, 2003.

A.S. Lewis. Convex Analysis on the Hermitian matrices. SIAM J. Optimization,6, 1996, 164-177.

A. Seeger. Convex analysis of spectrally defined matrix functions. SIAM J. Optimization , 7, 1997, 679-696.