2 Asymptotic Cones and Functions

2.1 Definition of Asymptotic Cones

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We use the book; Asymptotic Cones and Functions in Optimization and Variational Inequalities (author: A.AUSLENDER and M.TEBOULLE), pp.25-31.

The set of natural numbers is denoted by $\mathbb{N}$, so that $k \in \mathbb{N}$ means $k = 1, 2, \ldots$. A sequence $\{x_k\}_{k\in\mathbb{N}}$ or simply $\{x_k\}$ in $\mathbb{R}^n$ is said to converge to x if $\|x_k - x\| \to 0$ as $k \to \infty$, and this will be indicated by the notation $x_k \to x$ or $x = \lim_{k\to\infty} x_k$. We say that x is a cluster point of $\{x_k\}$ if some subsequence converge to x. Recall that every bounded sequence in $\mathbb{R}^n$ converges to x if and only if it is bounded and has x as its unique cluster point.

Let $\{x_k\}$ be a sequence in $\mathbb{R}^n$. We are interested in knowing how to handle convergence properties, we are led to consider direction $d_k := x_k \|x_k\|^{-1}$ with $x_k \neq 0$, $k \in \mathbb{N}$. From classical analysis, the Bolzano-Weierstrass theorem implies that we can extract a convergent subsequence $d = \lim_{k\in K} d_k$, $K \subset \mathbb{N}$, with $d \neq 0$. Now suppose that the sequence $\{x_k\} \subset \mathbb{R}^n$ is such that $\|x_k\| \to +\infty$. Then

$$\exists t_k := \|x_k\|, k \in K \subset \mathbb{N}, \text{ such that } \lim_{k\in K} t_k = +\infty \text{and} \lim_{k\in K} \frac{x_k}{t_k} = d.$$

This leads us to introduce the following concepts.

Definition 2.1.1

A sequence $\{x_k\} \subset \mathbb{R}$ is said to converge to a direction $d \in \mathbb{R}^n$ if

$$\exists\{t_k\}, with t_k \to +\infty \text{ such that } \lim_{x\to\infty} \frac{x_k}{t_k} = d.$$