2 Functional Analysis over Cones

2.5 Continuity Notions of Multifunctions

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We use the book; Variational Methods in Partially Ordered Spaces (author: A.Gopfert, H.Riahi, C.Tammer, and C.Zalinescu).

p.51

> In this section X and Y are separated (in the sense of Hausdorff) topological spaces and $\Gamma : X \rightrightarrows Y$ a multifunction. When mentioned explicitly, Y is a separeated topological vector space (s.t.v.s).

"$\rightrightarrows$" is one of symbols that mean multifunction. Also, "$\twoheadrightarrow$" and "$X \to 2^X$" have the same meaning.

Definition 2.5.1

Let $x_0 \in X$. We say that

(a) Γ is upper continuous (u.c.) at x_0 if

$$\forall D \subset Y, D \text{ open}, \Gamma(x_0) \subset D, \exists U \in \mathcal{V}_X(x_0) \text{ s.t. } \forall x \in U, \Gamma(x) \subset D, \tag{2.33}$$

i.e., $\Gamma^{+1}(D)$ is a neighborhood of x_0 for each open set $D \subset Y$ such that $F(x_0) \subset D$;

(b) Γ is lower continuous (l.c.) at x_0 if

$$\forall D \subset Y, D \text{ open}, \Gamma(x_0) \cap D \neq \emptyset, \exists U \in \mathcal{V}_X(x_0) \text{ s.t. } \forall x \in U, \Gamma(x) \cap D \neq \emptyset, \tag{2.34}$$

i.e., $\Gamma^{-1}(D)$ is a neighborhood of x_0 for each open set $D \subset Y$ such that $\Gamma(x) \cap D \neq \emptyset$.

(c) Γ is continuous at x_0 if Γ is u.c. and l.c. at x_0.

(d) Γ is upper continuous (lower continuous, continuous) at x_0 if Γ is so at every $x \in X$;

(e) Γ is lower continuous at $(x_0, y_0) \in X \times Y$ if

$$\forall V \in \mathcal{V}_Y(y_0), \exists U \in \mathcal{V}_X(x_0) \text{ s.t. } \forall x \in U, \Gamma(x) \cap D \neq \emptyset.$$

It follows from the definition that $x_0 \in \mathrm{int}\,(\mathrm{dom}\,\Gamma)$ and $y_0 \in \mathrm{cl}\,(\Gamma(x_0))$ if Γ is l.c. at (x_0, y_0) and Γ is l.c. at $x_0 \in \mathrm{dom}\,\Gamma$ if and only if Γ is l.c. at every (x_0, y_0) with $y \in \Gamma(x_0)$; moreover, Γ is l.c. at every $x_0 \in X \setminus \mathrm{dom}\,\Gamma$. If $x_0 \in X \setminus \mathrm{dom}\,\Gamma$, then Γ is u.c. at x_0 if and only if $x_0 \in \mathrm{int}\,(X \setminus \mathrm{dom}\,\Gamma)$. So, if Γ is u.c., then $\mathrm{dom}\,\Gamma$ is closed, while if Γ is l.c., then $\mathrm{dom}\,\Gamma$ is open. The next result follows immediately from the definitions.

It means that below.

Note

(i) Γ: l.c. at $(x_0, y_0) \Rightarrow x_0 \in \mathrm{int}\,(\mathrm{dom}\,\Gamma)$, $y_0 \in \mathrm{cl}\,(\Gamma(x_0))$

(ii) Γ: l.c. at $x_0 \in \mathrm{dom}\,\Gamma \Leftrightarrow \Gamma$: l.c. at $\forall (x_0, y)$ with $y \in \Gamma(x_0)$

(iii) Γ: l.c. at $\forall x_0 \in X \setminus \mathrm{dom}\,\Gamma$

(iv) Γ: u.c. at $x_0 \in X \setminus \mathrm{dom}\,\Gamma \Leftrightarrow x_0 \in \mathrm{int}\,(X \setminus \mathrm{dom}\,\Gamma)$

(v) Γ: u.c. $\Rightarrow \mathrm{dom}\,\Gamma$: closed

(vi) Γ: l.c. $\Rightarrow \mathrm{dom}\,\Gamma$: open

Proof. Coming soon... □

Proposition 2.5.2

(i) Γ: u.c. $\Leftrightarrow \forall D \subset Y$: open, $\Gamma^{+1}(D)$: open

(ii) Γ: l.c. $\Leftrightarrow \forall D \subset Y$: open, $\Gamma^{-1}(D)$: open

Proof. Coming soon... □

Definition (limit inferior and limi superior)

The limit inferior of Γ at $x_0 \in X$ is defeined by

$$\liminf_{x \to x_0} \Gamma(x) := \{y \in Y \mid \forall V \in \mathcal{V}_Y(y), \exists U \in \mathcal{V}_X(x_0) \text{ s.t. } \forall x \in U^\bullet, \Gamma(x) \cap D \neq \emptyset\},$$

while the limit superior of Γ at $x_0 \in X$ is defined by

$$\begin{aligned}\limsup_{x \to x_0} \Gamma(x) &:= \{y \in Y \mid \forall V \in \mathcal{V}_Y(y), \forall U \in \mathcal{V}_X(x_0), \exists x \in U^\bullet \text{ s.t. } \Gamma(x) \cap D \neq \emptyset\}, \\ &= \bigcap_{U \in \mathcal{V}_X(x_0)} \mathrm{cl}\,(\Gamma(U^\bullet)),\end{aligned}$$

where for $U \in \mathcal{V}_X(x_0)$, $U^\bullet := U \setminus \{x_0\}$.