

## 2 Asymptotic Cones and Functions

### 2.1 Definition of Asymptotic Cones

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We use the book; Asymptotic Cones and Functions in Optimization and Variational Inequalities (author: A.AUSLENDER and M.TEBOULLE), pp.25-31.

The set of natural numbers is denoted by  $\mathbb{N}$ , so that  $k \in \mathbb{N}$  means  $k = 1, 2, \dots$ . A sequence  $\{x_k\}_{k \in \mathbb{N}}$  or simply  $\{x_k\}$  in  $\mathbb{R}^n$  is said to converge to  $x$  if  $\|x_k - x\| \rightarrow 0$  as  $k \rightarrow \infty$ , and this will be indicated by the notation  $x_k \rightarrow x$  or  $x = \lim_{k \rightarrow \infty} x_k$ . We say that  $x$  is a cluster point of  $\{x_k\}$  if some subsequence converge to  $x$ . Recall that every bounded sequence in  $\mathbb{R}^n$  converges to  $x$  if and only if it is bounded and has  $x$  as its unique cluster point.

Let  $\{x_k\}$  be a sequence in  $\mathbb{R}^n$ . We are interested in knowing how to handle convergence properties, we are led to consider direction  $d_k := x_k \|x_k\|^{-1}$  with  $x_k \neq 0, k \in \mathbb{N}$ . From classical analysis, the Bolzano-Weierstrass theorem implies that we can extract a convergent subsequence  $d = \lim_{k \in K} d_k$ ,  $K \subset \mathbb{N}$ , with  $d \neq 0$ . Now suppose that the sequence  $\{x_k\} \subset \mathbb{R}^n$  is such that  $\|x_k\| \rightarrow +\infty$ . Then

$$\exists t_k := \|x_k\|, k \in K \subset \mathbb{N}, \text{ such that } \lim_{k \in K} t_k = +\infty \text{ and } \lim_{k \in K} \frac{x_k}{t_k} = d.$$

This leads us to introduce the following concepts.

#### Definition 2.1.1

A sequence  $\{x_k\} \subset \mathbb{R}^n$  is said to converge to a direction  $d \in \mathbb{R}^n$  if

$$\exists \{t_k\}, \text{ with } t_k \rightarrow +\infty \text{ such that } \lim_{k \rightarrow \infty} \frac{x_k}{t_k} = d.$$

Definition 2.1.2

Let  $C$  be a nonempty set in  $\mathbb{R}^n$ . Then the asymptotic cone of the set  $C$ , denoted by  $C_\infty$ , is the set of vectors  $d \in \mathbb{R}^n$  that are limits in direction of the sequences  $\{x_k\} \subset C$ , namely

$$C_\infty = \{d \in \mathbb{R}^n \mid \exists t_k \rightarrow +\infty, \exists x_k \in C \text{ with } \lim_{k \rightarrow \infty} \frac{x_k}{t_k} = d\}.$$

From the definition we immediately deduce the following elementary facts.

Proposition 2.1.1

Let  $C \subset \mathbb{R}^n$  be nonempty. Then:

- ( i )  $C_\infty$  is a closed cone.
- ( ii )  $(\text{cl } C)_\infty = C_\infty$ .
- ( iii ) If  $C$  is a cone, then  $C_\infty = \text{cl } C$ .

*Proof.* We will prove each part separately.

- ( i )  $C_\infty$  is a closed cone.

We need to show two propositions: (i-a)  $C_\infty$  is a cone and (i-b)  $C_\infty$  is a closed set.

(i-a) We show that  $C_\infty$  is a cone, that is,  $\forall \alpha \geq 0, d \in C_\infty, \alpha d \in C_\infty$ .

Since 0 is a element of  $C_\infty$ , it is clear in the case of  $\alpha = 0$ .

( $\because$  Since  $C$  is nonempty, we can take a element  $x_0$  from  $C$ . In addition we take a sequence  $\{t_k\}_{k=1}^\infty$  with  $t_k \rightarrow +\infty$  as  $k \rightarrow \infty$ . Of course this sequence exists, for example  $t_k := k$ . By using  $t_k := k$  and  $x_k := x_0$ , we can obtain 0 as the limit. Hence 0 is a element of  $C_\infty$ .)

Also we consider the other case  $\alpha > 0$ . To prove that  $C_\infty$  is a cone, we take a any direction  $d$  from  $C_\infty$ . Since  $d$  is a element of  $C_\infty$ ,

$$\exists t_k \rightarrow +\infty, \exists x_k \in C \text{ with } \lim_{k \rightarrow \infty} \frac{x_k}{t_k} = d.$$

Then we define a sequence  $\{t'_k\}_{k=1}^\infty := \frac{t_k}{\alpha}$ , exactly whose limit becomes  $+\infty$  as  $k \rightarrow \infty$ . Accordingly there exist  $t'_k \rightarrow +\infty$  and  $x_k \in C$  with

$$\lim_{k \rightarrow \infty} \frac{x_k}{t'_k} = \lim_{k \rightarrow \infty} \alpha \cdot \frac{x_k}{t_k} = \alpha d.$$

This means  $d \in C_\infty$ .

By these results, we can get  $\forall \alpha \geq 0, d \in C_\infty, \alpha d \in C_\infty$ .

Therefore  $C_\infty$  is a cone.

(i-b) We show that  $C_\infty$  is a closed set. In order to prove closeness, we consider convergence of a sequence in  $C_\infty$ . First we take a sequence  $\{d_k\}_{k=1}^\infty \subset C_\infty$  with  $d_k \rightarrow d$  as  $k \rightarrow \infty$  for some  $d$ . To obtain  $d \in C_\infty$ , we need two sequences like  $\{t_n\}_{n=1}^\infty$  with  $t_n \rightarrow \infty$  as  $n \rightarrow \infty$  and  $\{x_n\}_{n=1}^\infty$  where  $\frac{x_n}{t_n} \rightarrow d$  as  $n \rightarrow \infty$ . Since  $d_k \rightarrow d$  and  $t_k^{m-1} \cdot x_k^m \rightarrow d_k$  as  $m \rightarrow \infty$  for each  $k \in \mathbb{N}$ ,

$$\forall n \in \mathbb{N}, \exists k(n) \in \mathbb{N} \text{ s.t. } \forall j \geq k(n), \|d_j - d\| < \frac{1}{n}, \text{ and}$$

$$\forall k \in \mathbb{N} (1 \leq k \leq k(n)), \exists m(n, k) \in \mathbb{N} \text{ s.t. } \forall m \geq m(n, k), \left\| \frac{x_k^m}{t_k^m} - d_k \right\| < \frac{1}{n}.$$

Now we can rearrange

$$k(n) := \max\{k(n-1), k(n)\} + n \text{ and}$$

$$m(n) := \max_{1 \leq k \leq k(n)} \{m(n, k)\} + n$$

as sequences of  $n \in \mathbb{N}$ . Then it holds that  $k(1) \leq k(2) \leq \dots$  and  $m(1) \leq m(2) \leq \dots$ . Let's define

$$t_n := t_{k(n)}^{m(n)}, \text{ and}$$

$$x_n := x_{k(n)}^{m(n)}.$$

Also we can find that

$$t_n \rightarrow \infty \text{ as } n \rightarrow \infty,$$

$$x_n \in C, \text{ and}$$

$$\frac{x_n}{t_n} = \frac{x_{k(n)}^{m(n)}}{t_{k(n)}^{m(n)}}.$$

Hence we get for each  $n \in \mathbb{N}$

$$0 \leq \left\| \frac{x_n}{t_n} - d \right\| \leq \left\| \frac{x_{k(n)}^{m(n)}}{t_{k(n)}^{m(n)}} - d_{k(n)} \right\| + \|d_{k(n)} - d\| < \frac{1}{2n} \rightarrow 0$$

as  $n \rightarrow \infty$ .

Thus  $d \in C_\infty$ , that is,  $C_\infty$  is a closed set.

Then (i)'s proof is completed.

(ii)  $(\text{cl } C)_\infty = C_\infty$ .

We need to show two relations: (ii-a)  $(\text{cl } C)_\infty \supset C_\infty$  (ii-b)  $(\text{cl } C)_\infty \subset C_\infty$ .

(ii-a) We show that  $C_\infty$  is included in  $(\text{cl } C)_\infty$ . However it is clear from the definition of asymptotic cone.

(ii-b) We show that  $(\text{cl } C)_\infty \subset C_\infty$ . Like (i-b), we'll show that.

Then (ii)'s proof is also completed.

(iii) If  $C$  is a cone, then  $C_\infty = \text{cl } C$ .

We need to show two relations: (iii-a)  $C_\infty \subset \text{cl } C$  and (iii-b)  $C_\infty \supset \text{cl } C$ .

(iii-a) We take any direction  $d \in C_\infty$  which satisfies

$$\exists t_k \rightarrow +\infty, \exists x_k \in C \text{ with } \lim_{k \rightarrow \infty} \frac{x_k}{t_k} = d.$$

Let  $d_k := \frac{x_k}{t_k}$  (with  $d_k \rightarrow d$  as  $k \rightarrow \infty$ ). Since  $C$  is a cone,

$$d_k = \frac{1}{t_k} \cdot x_k \in C.$$

Due to  $d_k \in C$ , the limit of  $d_k$  is a element of  $\text{cl } C$ , i.e.,  $d \in \text{cl } C$ .

Therefore  $C_\infty \subset \text{cl } C$ .

(iii-b) We take any  $d \in \text{cl } C$  and show  $d \in C_\infty$ , that is,

$$\exists t_k \rightarrow +\infty, \exists x_k \in C \text{ with } \lim_{k \rightarrow \infty} \frac{x_k}{t_k} = d.$$

By  $d \in \text{cl } C$ ,

$$\exists \{d_k\}_{k=1}^\infty \in C \text{ with } d_k \rightarrow d \text{ as } k \rightarrow \infty,$$

in other words,

$$\lim_{k \rightarrow \infty} d_k = d.$$

We define  $y_k = k \cdot d_k$  and  $s_k = k$  for each  $k$ . Since  $d_k \in C$  and  $C$  is a cone,  $y_k$  is also a element of  $C$ .

There exist  $\{s_k\}_{k=1}^\infty$  with  $s_k \rightarrow \infty$  as  $k \rightarrow \infty$  and  $\{y_k\}_{k=1}^\infty \subset C$  such that

$$\lim_{k \rightarrow \infty} \frac{y_k}{s_k} = \lim_{k \rightarrow \infty} d_k = d.$$

As  $d$  is a element of  $C_\infty$ , therefore  $C_\infty \supset \text{cl } C$ .

□

The importance of the asymptotic cone is revealed by the following key property, which is a immediate consequence of its definition.

Proposition 2.1.2

A set  $C \subset \mathbb{R}^n$  is bounded if and only if  $C_\infty = \{0\}$ .

*Proof.* We show that:

( i ) a set  $C \subset \mathbb{R}^n$  is bounded  $\Rightarrow C_\infty = \{0\}$ , and

( ii ) a set  $C \subset \mathbb{R}^n$  is unbounded  $\Rightarrow C_\infty \neq \{0\}$ .

( i ) By Proposition 2.1.1 (i),  $0 \in C_\infty$ . Also, by the assumption  $C$  is bounded,

$$\exists r > 0, \forall x_k \in C \text{ where } k \in \mathbb{N}, \|x_k\| \leq r.$$

For any sequence  $\{t_k\}_{k=1}^\infty$  such that  $t_k \rightarrow \infty$  as  $k \rightarrow \infty$ ,

$$\lim_{k \rightarrow \infty} \frac{x_k}{t_k} = 0.$$

Thus the limit becomes only 0 for any  $\{x_k\}_{k=1}^\infty \subset C$  and  $\{t_k\}_{k=1}^\infty$  with  $t_k \rightarrow \infty$  as  $k \rightarrow \infty$ .

Therefore if  $C$  is bounded then  $C_\infty = \{0\}$ .

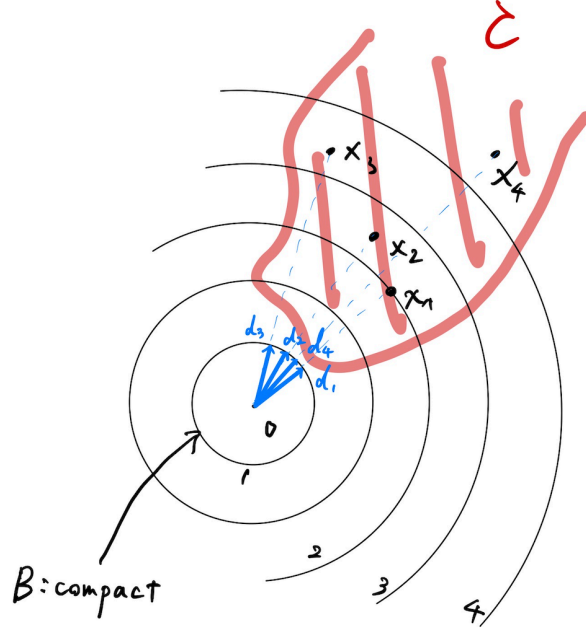
( ii ) If  $C$  is unbounded, then there exists a sequence  $\{x_k\} \subset C$  with  $x_k \neq 0, \forall k \in \mathbb{N}$ , such that  $t_k := \|x_k\| \rightarrow \infty$  and thus the vectors  $d_k = t_k^{-1}x_k \in \{d : \|d\| = 1\}$ .

By the Bolzano-Weierstrass, we can extract a subsequence of  $\{d_k\}$  such that  $\lim_{k \in K} d_k = d, K \subset \mathbb{N}$ , and with  $\|d\| = 1$ . This nonzero vector  $d$  is an element of  $C_\infty$  by Definition 2.1.2, a contradiction.

□

Figure:

Why do we take a subsequence of  $\{d_k\}$ ?



Associated with the asymptotic cone  $C_\infty$  is the following related concept, which will help us in simplifying the definition of  $C_\infty$  in the particular case where  $C \subset \mathbb{R}^n$  is assumed convex.

#### Definition 2.1.3

Let  $C \subset \mathbb{R}^n$  be nonempty and define

$$C_\infty^1 = \{d \in \mathbb{R}^n \mid \forall t_k \rightarrow +\infty, \exists x_k \in C \text{ with } \lim_{k \rightarrow \infty} \frac{x_k}{t_k} = d\}.$$

We say that  $C$  is asymptotically regular if  $C_\infty = C_\infty^1$ .

#### Proposition 2.1.3

Let  $C$  be a nonempty convex set in  $\mathbb{R}^n$ . Then  $C$  is asymptotically regular.

*Proof.* The inclusion  $C_\infty^1 \subset C_\infty$  clearly holds from the definition of  $C_\infty^1$  and  $C_\infty$ , respectively. Let  $d \in C_\infty$ . Then  $\exists \{x_k\} \in C$ ,  $\exists s_k \rightarrow \infty$  such that  $d = \lim_{k \rightarrow \infty} s_k^{-1} x_k$ . Let  $x \in C$  and define  $d_k = s_k^{-1}(x_k - x)$ . Then we have

$$d = \lim_{k \rightarrow \infty} d_k, x + s_k d_k \in C.$$

$$\therefore x_k = x + s_k d_k \in C.$$

Now note that an arbitrary sequence such that  $\lim_{k \rightarrow \infty} t_k = +\infty$ . For any fixed  $m \in \mathbb{N}$ , there exists  $k(m)$  with  $\lim_{m \rightarrow \infty} k(m) = +\infty$  such that  $t_m \leq s_{k(m)}$ , and since  $C$  is convex, we have  $x'_m := x + t_m d_{k(m)} \in C$ . Hence,  $d = \lim_{m \rightarrow \infty} t_m^{-1} x'_m$ , showing that  $d \in C_\infty^1$ .

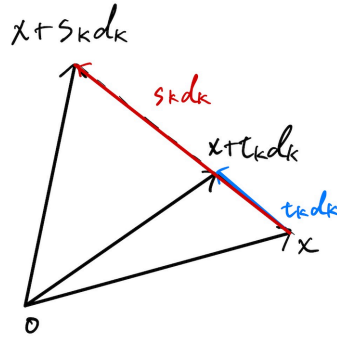
□

Figure:

Why should we consider  $k(m)$  with  $\lim_{m \rightarrow \infty} k(m) = +\infty$  such that  $t_m \leq s_{k(m)}$ ?

If  $\{s_k\} = 1, 2, 3, 7, 8, 9, 13, \dots$  and  $\{t_k\} = 1, 3, 4, 6, 8, 10, 11, \dots$ , then we can get

$$\{k(m)\} = 1, 3, 4, 5, 6, 6, \dots$$



We note that a set can be nonconvex, yet asymptotically regular. Indeed, consider, for example, sets definition by  $C := S + K$ , with  $S$  compact and  $K$  a closed convex cone. Then clearly  $C$  is not necessarily convex, but it can be easily seen that  $C_\infty = C_\infty^1$ .

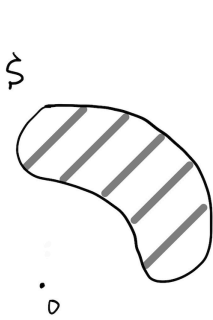


Figure1  $S$ :compact

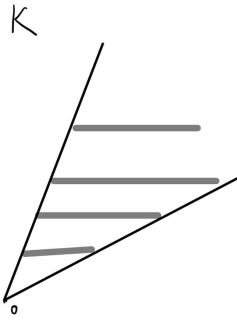


Figure2  $K$ :closed convex cone

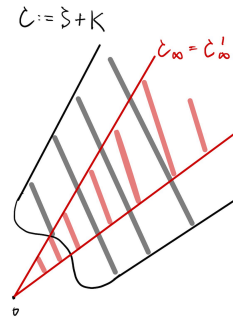


Figure3  $C := S + K$

*Proof.* We show that for any  $d \in C_\infty$ ,  $d \in C_\infty^1$ .

By the definition of the asymptotic cone,

$$\exists t_k \rightarrow \infty, \exists x_k \in C \text{ with } \lim_{k \rightarrow \infty} \frac{x_k}{t_k} = d.$$

As  $S$  is compact,  $S$  is asymptotically regular, that is,

$$\forall t'_l \rightarrow \infty, \exists s_l \in C \text{ with } \lim_{l \rightarrow \infty} \frac{x_l}{t'_l} = 0.$$

For each  $k$ ,

$$\exists s_k \in S, b_k \in K \text{ s.t. } x_k = s_k + b_k.$$

Then we get  $d \in k_\infty$  because it holds that

$$\exists t_k \rightarrow \infty, \exists b_k \in K \text{ with } \frac{b_k}{t_k} \rightarrow d.$$

The convexity of  $K$  and Proposition 2.1.3 lead to  $d \in K_\infty^1$ .

Thus we obtain  $d \in C_\infty^1$  because it holds that

$$\forall t'_l \rightarrow \infty, \exists x_l \in C \text{ where } x_l := b_l + s_l \text{ with } \lim_{l \rightarrow \infty} \frac{x_l}{t'_l} = d.$$

Therefore  $C_\infty = C_\infty^1$ .

□

#### Remark 2.1.1

Note that the definitions of  $C_\infty$  and  $C_\infty^1$  are related to the theory of set convergence of Painleve-Kuratowski. Indeed, for a family  $\{C_t\}_{t>0}$  of subsets of  $\mathbb{R}^n$ , the outer limit as  $t \rightarrow +\infty$  is the set.

$$\limsup_{t \rightarrow +\infty} C_t = \{x \mid \liminf_{t \rightarrow +\infty} d(x, C_t) = 0\},$$

while the inner limit as  $t \rightarrow +\infty$  is the set

$$\liminf_{t \rightarrow +\infty} C_t = \{x \mid \limsup_{t \rightarrow +\infty} d(x, C_t) = 0\},$$

It can then be verified that the corresponding asymptotic cones can be written as

$$C_\infty = \limsup_{t \rightarrow +\infty} t^{-1}C, \quad C_\infty^1 = \liminf_{t \rightarrow +\infty} t^{-1}C.$$

#### Proposition 2.1.4

Let  $C \subset \mathbb{R}^n$  be nonempty and define the normalized sets

$$C_N := \{d \in \mathbb{R}^n \mid \exists \{x_k\} \in C, \|x_k\| \rightarrow +\infty \text{ with } d = \lim_{k \rightarrow \infty} \frac{x_k}{\|x_k\|}\}.$$

Then  $C_\infty = \text{pos } C_N$ , where for any set  $C$ ,  $\text{pos } C = \{\lambda x \mid x \in C, \lambda \geq 0\}$ .



*Proof.* Clearly, one always has  $\text{pos } C_N \subset C_\infty$ . Conversely, let  $0 \neq d \in C_\infty$ .

Then there exists  $t_k \rightarrow \infty$ ,  $x_k \in C$  such that

$$d = \lim_{k \rightarrow \infty} \frac{x_k}{t_k} = \lim_{k \rightarrow \infty} \frac{1}{t_k} \cdot \|x_k\| \cdot \frac{x_k}{\|x_k\|}, \text{ with } \|x_k\| \rightarrow \infty.$$

Thus the sequence  $\{t_k^{-1} \|x_k\|\}$  is a nonnegative bounded sequence, and by the Bolzano-Weierstrass theorem, there exists a subsequence  $\{t_k^{-1} \|x_k\|\}_{k \in K}$  with  $K \subset \mathbb{N}$  such that  $\lim_{k \in K} t_k^{-1} \|x_k\| = \lambda \geq 0$ , which means that  $d = \lambda d_N$  with  $d_N \in C_N$ , namely  $d \in \text{pos } C_N$ .  $\square$

#### Proposition 2.1.5

Let  $C$  be a nonempty convex set in  $\mathbb{R}^n$ . Then the asymptotic cone  $C_\infty$  is a closed convex cone. Moreover, define the following sets:

$$\begin{aligned} D(x) &:= \{d \in \mathbb{R}^n \mid x + td \in \text{cl } C, \forall t > 0\} \forall x \in C, \\ E &:= \{d \in \mathbb{R}^n \mid \exists x \in C \text{ s.t. } x + td \in \text{cl } C, \forall t > 0\}, \\ F &:= \{d \in \mathbb{R}^n \mid d + \text{cl } C \subset \text{cl } C\}. \end{aligned}$$

Then  $D(x)$  is in fact independent of  $x$ , which is thus now denoted by  $D$ , and  $C_\infty = D = E = F$ .

*Proof.* We show that

- ( i )  $C_\infty$  is convex,
- ( ii )  $C_\infty \subset D(x)$ ,
- ( iii )  $D(x) \subset E$ ,
- ( iv )  $E \subset C_\infty$ ,
- ( v )  $C_\infty \subset F$ ,
- ( vi )  $C_\infty \supset F$ .

( i ) We'll show that  $C_\infty$  is convex.

It follows that  $C$  is asymptotically regular from Proposition 2.1.3.

For any  $d_1$  and  $d_2$ ,

$$\begin{aligned} &\forall t_k \rightarrow \infty, \exists \{x_k\}_{k=1}^\infty \text{ with } \frac{x_k}{t_k} \rightarrow d_1 \text{ as } k \rightarrow \infty \\ &(\forall \epsilon > 0, \exists k_0 \in \mathbb{N} \text{ s.t. } \forall k \geq k_0, \left\| \frac{x_k}{t_k} - d_1 \right\| < \epsilon), \\ &\forall s_l \rightarrow \infty, \exists \{y_l\}_{l=1}^\infty \text{ with } \frac{y_l}{s_l} \rightarrow d_2 \text{ as } l \rightarrow \infty \\ &(\forall \epsilon > 0, \exists l_0 \in \mathbb{N} \text{ s.t. } \forall l \geq l_0, \left\| \frac{y_l}{s_l} - d_2 \right\| < \epsilon). \end{aligned}$$

Then we'll check the convexity, that is,

$$\lambda d_1 + (1 - \lambda)d_2 \in C_\infty \text{ where } \lambda \in (0, 1).$$

We take a sequence  $\{u_m\}_{m=1}^\infty$  where  $u_m := \max\{t_m, s_m\}$  and  $m_0 := \max\{k_0, l_0\} \in \mathbb{N}$ .

Then you can find  $u_m \rightarrow \infty$ .

Also we define a sequence as  $\{\lambda x_m + (1 - \lambda)y_m\}_{m=1}^\infty$ .

$\forall m \geq m_0$ ,

$$\begin{aligned} \left\| \frac{\lambda x_m + (1 - \lambda)y_m}{u_m} - (\lambda d_1 + (1 - \lambda)d_2) \right\| &\leq \lambda \left\| \frac{x_m}{u_m} - d_1 \right\| + (1 - \lambda) \left\| \frac{y_m}{u_m} - d_2 \right\| \\ &< \lambda \epsilon + (1 - \lambda)\epsilon = \epsilon. \end{aligned}$$

Therefore  $C_\infty$  is convex.

(ii) We now prove that  $C_\infty \subset D(x)$

□