# 2 Asymptotic Cones and Functions

## 2.1 Definition of Asymptotic Cones

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We use the book; Asymptotic Cones and Functions in Optimization and Variational Inequalities (author: A.AUSLENDER and M.TEBOULLE), pp.25-31.

The set of natural numbers is denoted by $\mathbb{N}$, so that $k \in \mathbb{N}$ means $k = 1, 2, \ldots$. A sequence $\{x_k\}_{k\in\mathbb{N}}$ or simply $\{x_k\}$ in $\mathbb{R}^n$ is said to converge to $x$ if $\|x_k - x\| \to 0$ as $k \to \infty$, and this will be indicated by the notation $x_k \to x$ or $x = \lim_{k\to\infty} x_k$. We say that $x$ is a cluster point of $\{x_k\}$ if some subsequence converge to $x$. Recall that every bounded sequence in $\mathbb{R}^n$ converges to $x$ if and only if it is bounded and has $x$ as its unique cluster point.

Let $\{x_k\}$ be a sequence in $\mathbb{R}^n$. We are interested in knowing how to handle convergence properties, we are led to consider direction $d_k := x_k \|x_k\|^{-1}$ with $x_k \neq 0$, $k \in \mathbb{N}$. From classical analysis, the Bolzano-Weierstrass theorem implies that we can extract a convergent subsequence $d = \lim_{k\in K} d_k$, $K \subset \mathbb{N}$, with $d \neq 0$. Now suppose that the sequence $\{x_k\} \subset \mathbb{R}^n$ is such that $\|x_k\| \to +\infty$. Then

$$\exists t_k := \|x_k\|, k \in K \subset \mathbb{N}, \text{ such that } \lim_{k\in K} t_k = +\infty \text{ and } \lim_{k\in K} \frac{x_k}{t_k} = d.$$

This leads us to introduce the following concepts.

**Definition 2.1.1**

A sequence $\{x_k\} \subset \mathbb{R}$ is said to converge to a direction $d \in \mathbb{R}^n$ if

$$\exists \{t_k\},\ with\, t_k \to +\infty \text{ such that } \lim_{k\to\infty} \frac{x_k}{t_k} = d.$$

Definition 2.1.2

Let $C$ be a nonempty set in $\mathbb{R}^n$. Then the asymptotic cone of the set $C$, denoted by $C_\infty$, is the set of vectors $d \in \mathbb{R}^n$ that are limits in direction of the sequences $\{x_k\} \subset C$, namely

$$C_\infty = \{d \in \mathbb{R}^n \mid \exists t_k \to +\infty, \exists x_k \in C \text{ with } \lim_{k\to\infty} \frac{x_k}{t_k} = d\}.$$

From the definition we immediately deduce the following elementary facts.

Proposition 2.1.1

Let $C \subset \mathbb{R}^n$ be nonempty. Then:

( i ) $C_\infty$ is a closed cone.

( ii ) $(\operatorname{cl} C)_\infty = C_\infty$.

(iii) If $C$ is a cone, then $C_\infty = \operatorname{cl} C$.

*Proof.* We will prove each part separately.

( i ) $C_\infty$ is a closed cone.

We need to show two propositions: (i-a) $C_\infty$ is a cone and (i-b) $C_\infty$ is a closed set.

(i-a) We show that $C_\infty$ is a cone, that is, $\forall \alpha \geq 0, d \in C_\infty, \alpha d \in C_\infty$.

Since 0 is a element of $C_\infty$, it is clear in the case of $\alpha = 0$.

($\because$ Since $C$ is nonempty, we can take a element $x_0$ from $C$. In addition we take a sequence $\{t_k\}_{k=1}^\infty$ with $t_k \to +\infty$ as $k \to \infty$. Of course this sequence exists, for example $t_k \coloneqq k$. By using $t_k \coloneqq k$ and $x_k \coloneqq x_0$, we can obtain 0 as the limit. Hence 0 is a element of $C_\infty$.)

Also we consider the other case $\alpha > 0$. To prove that $C_\infty$ is a cone, we take a any direction $d$ from $C_\infty$. Since d is a element of $C_\infty$,

$$\exists t_k \to +\infty, \exists x_k \in C \text{ with } \lim_{k\to\infty} \frac{x_k}{t_k} = d.$$

Then we define a sequence $\{t'_k\}_{k=1}^\infty \coloneqq \frac{t_k}{\alpha}$, exactly whose limit becomes $+\infty$ as $k \to \infty$. Accordingly there exist $t'_k \to +\infty$ and $x_k \in C$ with

$$\lim_{k\to\infty} \frac{x_k}{t'_k} = \lim_{k\to\infty} \alpha \cdot \frac{x_k}{t_k} = \alpha d.$$

This means $d \in C_\infty$.

By these results, we can get

$$\forall \alpha \geq 0, d \in C_\infty, \alpha d \in C_\infty$$

. Therefore $C_\infty$ is a cone.

(i-b) We show that $C_\infty$ is a closed set. In order to prove closeness, we consider convergency of a sequence of $C_\infty$. First we take a sequence $\{d_k\}_{k=1}^\infty \subset C_\infty$ with $d_k \to d$ as $k \to \infty$ for some $d$. Then we don't forget that $d \in C_\infty$ is our goal.

For each $d_k$,

( ii ) $(\mathrm{cl}\, C)_\infty = C_\infty$.

(iii) If $C$ is a cone, then $C_\infty = \mathrm{cl}\, C$.

□

The importance of the asymptotic cone is revealed by the following key property, which is a immediate consequence of its definition.

**Proposition 2.1.2**

A set $C \subset \mathbb{R}^n$ is bounded if and only if $C_\infty = \{0\}$.