Set-Valued Fan-Takahashi Inequalities Via Scalarization

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Introduction

Introduction

Let us consider a common scalar optimization problem

$$\min g(x) \quad \text{s.t.} \quad x \in C \tag{1}$$

where C is a given nonempty set in a space X and $g\colon C \to \mathbb{R}$ a given function. Let $x_0 \in C$ be a solution of the problem (1), which implies

$$g(x_0) \leq g(y) \quad \forall y \in C$$

Setting $f(x, y) := g(x) - g(y)$ for $x, y \in C$, x_0 also solves

$$\text{find} \quad x_0 \in C \quad \text{such that} \quad f(x_0, y) \leq 0 \quad \forall y \in C. \tag{2}$$

Theorem (Takahashi [6] in 1976)
Let X be a nonempty compact convex subset of a Hausdorff topological vector space and $f : X \times X \to \mathbb{R}$. If f satisfies the following conditions:

1. for each fixed $y \in X$, $f(\cdot, y)$ is lower semicontinuous,
2. for each fixed $x \in X$, $f(x, \cdot)$ is quasi concave,
3. $f(x, x) \leq 0$ for all $x \in X$,

then there exists $\bar{x} \in X$ such that $f(\bar{x}, y) \leq 0$ for all $y \in X$.

Theorem (Fan [2] in 1972)
Let X be a nonempty compact convex set in a Hausdorff topological vector space and $f : X \times X \to \mathbb{R}$. If f satisfies the following conditions:

1. for each fixed $y \in X$, $f(\cdot, y)$ is lower semicontinuous,
2. for each fixed $x \in X$, $f(x, \cdot)$ is quasi concave,

then the minimax inequality

$$\min_{x \in X} \sup_{y \in X} f(x, y) \leq \sup_{x \in X} f(x, x)$$

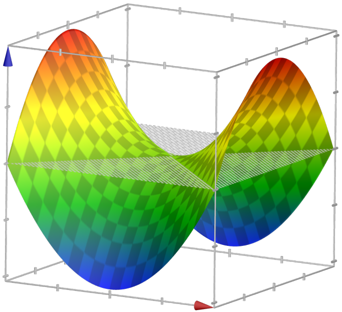
holds.

Motivation

Obseve Fan-Takahashi minimax inequality with an example.

Consider $\min\{g(x) := x^2 \mid x \in [-2, 2]\}$. Letting $f(x, y) := x^2 - y^2$, we have the result of Fan-Takahashi minimax inequality. Checking the function satisfying the assumptions,

- for each fixed $y \in [-2, 2]$, $f(\cdot, y)$ is continuous,
- for each fixed $x \in [-2, 2]$, $f(x, \cdot)$ is concave, and
- $f(x, x) \leq 0$ for all $x \in [-2, 2]$.



In this case, 0 is a solution of the minimization.

Background

Background

- Georgiev and Tanaka [3] extended the minimax inequality to the form of set-valued maps.

- Kuwano, Tanaka, and Yamada [5] constructed the result of four types set-valued minimax inequalities with set relations.

- Our goal is to generalize the result of four types set-valued minimax inequalities without set-relations and the scalarization functions.

Theorem [5]
Let X be a nonempty compact convex subset of a Hausdorff topological vector space, Y a real topological vector space, C a proper closed convex cone in Y with $\mathrm{int}\, C \neq \emptyset$ and $F\colon X \times X \to \mathcal{P}(Y)\backslash\{\emptyset\}$ If F satisfies the following conditions:

1. F is C-bounded on $X \times X$,
2. for each fixed $y \in X$, $F(\cdot, y)$ is C-lower continuous,
3. for each fixed $x \in X$, $f(x, \cdot)$ is type (5) properly C-quasi concave,
4. for all $x \in X$, $F(x, x) \subset -C$,

then there exists $\bar{x} \in X$ such that $F(\bar{x}, y) \subset -C$ for all $y \in X$.

Preliminaries

Let X be a topological space, Y a real topological vector space, and θ_Y be a zero vector in Y. Define that $\mathcal{P}(Y)$ is the set of all nonempty subsets of Y. The sets of neighborhoods of $x \in X$ and $y \in Y$ is denoted by $\mathcal{N}_X(x)$ and $\mathcal{N}_Y(y)$, respectively.

Definition [1]

Let $F\colon X \to \mathcal{P}(Y)$, $x_0 \in X$, $\preccurlyeq$ a binary relation on $\mathcal{P}(Y)$ and $C \subset Y$ a convex cone. We say that F is $(\preccurlyeq, C)$-continuous at x_0 if

$$\forall W \subset Y, W \text{ open}, W \preccurlyeq F(x_0), \exists V \in \mathcal{N}_X(x_0) \text{ s.t. } W + C \preccurlyeq F(x), \forall x \in V.$$

Preliminaries (Lower semicontinuity)

Definition [1]
Let $\varphi \colon \mathcal{P}(Y) \to \mathbb{R} \cup \{\pm\infty\}$, $A_0 \in \mathcal{P}(Y)$, $\preccurlyeq$ a binary relation on $\mathcal{P}(Y)$, and C a convex cone in Y with $C \neq Y$. Then, we say that φ is $(\preccurlyeq, C)$-lower semicontinuous at A_0 if

$$\forall r < \varphi(A_0), \exists W \in \mathcal{P}(Y), W \text{ open, s.t. } W \preccurlyeq A_0 \text{ and } r > \varphi(A), \forall A \in U(W + C, \preccurlyeq);$$

where $U(V, \preccurlyeq) := \{A \in \mathcal{P}(Y) \mid V \preccurlyeq A\}$.

Theorem [1]
Let $F \colon X \to \mathcal{P}(Y)$, $\varphi \colon \mathcal{P}(Y) \to \mathbb{R} \cup \{\pm\infty\}$, $x_0 \in X$, $\preccurlyeq$ a binary relation on $\mathcal{P}(Y)$, and C a convex cone. If F is $(\preccurlyeq, C)$-continuous at x_0 and φ is $(\preccurlyeq, C)$-lower semicontinuous at $F(x_0)$, then $(\varphi \circ F)$ is lower semicontinuous at x_0.

Preliminaries (Convexity)

Definition [4]
Let $\mathcal{A} \subset \mathcal{P}(Y) \backslash \{\emptyset\}$. $\mathcal{A}$ is said to be convex if for each $A_1, A_2 \in \mathcal{A}$ and $\lambda \in (0, 1)$,

$$\lambda A_1 + (1 - \lambda) A_2 \in \mathcal{A}.$$

Definition [4]
Let $\varphi \colon \mathcal{P}(Y) \to \mathbb{R} \cup \{\pm\infty\}$. Then,

1. φ is quasi convex if for any $\alpha \in \mathbb{R}$, $\mathrm{lev}\,(\varphi, \leq, \alpha) := \{A \in \mathcal{P}(Y) \backslash \{\emptyset\} \mid \varphi(A) \leq \alpha\}$ is convex.
2. φ is quasi concave if for any $\alpha \in \mathbb{R}$, $\mathrm{lev}\,(\varphi, \geq, \alpha) := \{A \in \mathcal{P}(Y) \backslash \{\emptyset\} \mid \varphi(A) \geq \alpha\}$ is convex.

Preliminaries (Convexity)

Definition

Let X be a nonempty set, Y a real topological vector space, C a convex cone in Y, and $F\colon X \to 2^Y \backslash \{\emptyset\}$ a set-valued map.

1. F is called ($\preccurlyeq$)-naturally quasi convex if for each $x, y \in X$ and $\lambda \in (0,1)$, there exists $\mu \in [0,1]$ such that

$$F(\lambda x + (1-\lambda)y) \preccurlyeq \mu F(x) + (1-\mu)F(y).$$

2. F is called ($\preccurlyeq$)-naturally quasi concave if for each $x, y \in X$ and $\lambda \in (0,1)$, there exists $\mu \in [0,1]$ such that

$$\mu F(x) + (1-\mu)F(y) \preccurlyeq F(\lambda x + (1-\lambda)y).$$

Main results

Specific scalarization function

To extend Ky Fan inequality for set-valued maps with a binary relation, consider assumptions of scalarization fucntions. To begin with, introduce four properties;

1. φ is $(\preccurlyeq, C)$-lower semicontinuous,
2. φ is quasi concave,
3. φ is $(\preccurlyeq)$-monotone,
4. $\varphi(\{\theta\}) = 0$,

and define the set of functions satisfying these properties as $\Phi(\preccurlyeq, C)$. In addition, establish three vaital properties for Ky Fan inequality;

$$\varphi(A) \leq 0 \Rightarrow A \preccurlyeq \{\theta\}. \tag{A1}$$

Main results

Theorem

Let X be a nonempty compact convex subset of a topological vector space, Y a real topological vector space, $\preccurlyeq$ a binary relation on $\mathcal{P}(Y)$, C a convex cone in Y, $\varphi\colon \mathcal{P}(Y) \to \mathbb{R} \cup \{\pm\infty\}$, and $F\colon X \times X \to \mathcal{P}(Y) \setminus \{\emptyset\}$ a set-valued map. For the scaralization function $\varphi \in \Phi(\preccurlyeq, C)$ satisfying Assumption (A1), if F satisfies the following conditions:

1. $(\varphi \circ F)(x, y) \in \mathbb{R}$ for all $x, y \in X$,
2. for each fixed $y \in X$, $F(\cdot, y)$ is $(\preccurlyeq, C)$-continuous,
3. for each fixed $x \in X$, $F(x, \cdot)$ is $(\preccurlyeq)$-naturally quasi concave,
4. for all $x \in X$, $F(x, x) \preccurlyeq \{\theta\}$,

then there exists $\bar{x} \in X$ such that $F(\bar{x}, y) \preccurlyeq \{\theta\}$ for all $y \in X$.

Conclusion

Conclusion

- Fan-Takahashi minimax inequalitiy and beckgrounds of my consequence were introduced with an example.
- We gave a new result of set-valued Fan-Takahashi inequalities via scalarization .
- One of the next steps is to find a scalarization function satisfying the properties $(A1)$ in $\Phi(\preccurlyeq, C)$.

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