Set-Valued Fan-Takahashi Inequalities with Set-Relations Based on Scalarization Methods

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Introduction

- Georgiev and Tanaka [2] extended Fan-Takahashi minimax inequality to the form of set-valued maps.

- Kuwano, Tanaka, and Yamada [4] constructed the result of four types of set-valued minimax inequalities with set relations.

- Our goal is to generalize the result of four types of set-valued minimax inequalities which is not related to the specific set-relations and scalarization functions.

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[5] Issei Kuwano, Tamaki Tanaka, and Syuuji Yamada. "Unified scalarization for sets and set-valued Ky Fan minimax inequality". In: J. Nonlinear Convex Anal. 11.3 (2010), pp. 513–525.

Background

- Georgiev and Tanaka [2] extended Fan-Takahashi minimax inequality to the form of set-valued maps.

- Kuwano, Tanaka, and Yamada [4] constructed the result of four types of set-valued minimax inequalities with set relations.

- Our goal is to generalize the result of four types of set-valued minimax inequalities which is not related to the specific set-relations and scalarization functions.

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Fan-Takahashi Minimax Inequality

Theorem (Fan-Takahashi [5])
Let X be a nonempty compact convex subset of a Hausdorff topological vector space and $f : X \times X \to \mathbb{R}$. If f satisfies the following conditions:

1. for each fixed $y \in X$, $f(\cdot, y)$ is lower semicontinuous,
2. for each fixed $x \in X$, $f(x, \cdot)$ is quasi concave,
3. $f(x, x) \leq 0$ for all $x \in X$,

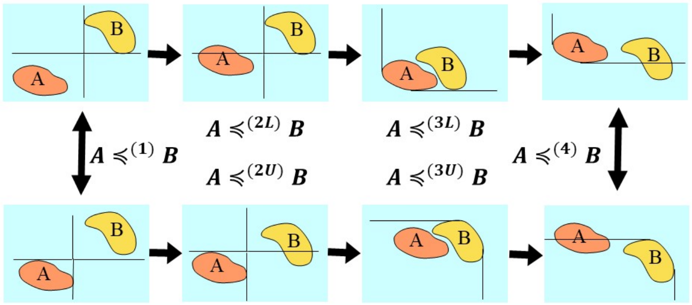
then there exists $\bar{x} \in X$ such that $f(\bar{x}, y) \leq 0$ for all $y \in X$.

Ordering and Set-relations

Let $(Y, \preccurlyeq)$ be an ordered space, generally.
$A, B \subset Y$: nonempty sets. $A \preccurlyeq^{(j)} B$ $(j = 1, 2L, 2U, 3L, 3U, 4)$ is defined below.

(1) $\forall a \in A, \forall b \in B, a \preccurlyeq b$

(2L) $\exists a \in A$ s.t. $\forall b \in B, a \preccurlyeq b$ **(3L)** $\forall b \in B, \exists a \in A$ s.t. $a \preccurlyeq b$

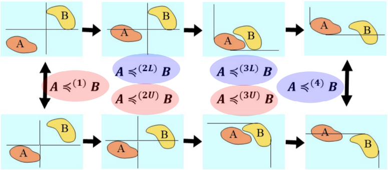


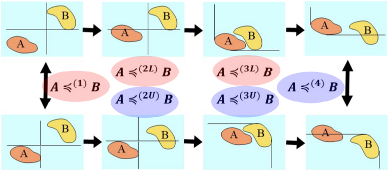
(2U) $\exists b \in B$ s.t. $\forall a \in A, a \preccurlyeq b$ **(3U)** $\forall a \in A, \exists b \in B$ s.t. $a \preccurlyeq b$
(4) $\exists a \in A, \exists b \in B$ s.t. $a \preccurlyeq b$

Lemma

Let Y be a real topological vector space, C a convex cone with $\operatorname{int} C \neq \emptyset$, and $A, \{\theta_Y\} \subset Y$. Then

1. $A \preccurlyeq^{(1)}_C \{\theta_Y\} \Leftrightarrow A \preccurlyeq^{(2U)}_C \{\theta_Y\} \Leftrightarrow A \preccurlyeq^{(3U)}_C \{\theta_Y\} \Leftrightarrow A \subset -C$,
2. $A \preccurlyeq^{(2L)}_C \{\theta_Y\} \Leftrightarrow A \preccurlyeq^{(3L)}_C \{\theta_Y\} \Leftrightarrow A \preccurlyeq^{(4)}_C \{\theta_Y\} \Leftrightarrow A \cap (-C) \neq \emptyset$,
3. $\{\theta_Y\} \not\preccurlyeq^{(1)}_{\operatorname{int} C} A \Leftrightarrow \{\theta_Y\} \not\preccurlyeq^{(2L)}_{\operatorname{int} C} A \Leftrightarrow \{\theta_Y\} \not\preccurlyeq^{(3L)}_{\operatorname{int} C} A \Leftrightarrow A \cap \operatorname{int} C = \emptyset$,
4. $\{\theta_Y\} \not\preccurlyeq^{(2U)}_{\operatorname{int} C} A \Leftrightarrow \{\theta_Y\} \not\preccurlyeq^{(3U)}_{\operatorname{int} C} A \Leftrightarrow \{\theta_Y\} \not\preccurlyeq^{(4)}_{\operatorname{int} C} A \Leftrightarrow A \not\subset \operatorname{int} C$.





Four types of Set-Valued Minimax Inequalities with Set-relations

Let $f : X \to \mathbb{R}$ and $F : X \to \mathcal{P}_0(Y)$.

$$f(\bar{x}, y) \leq 0 \quad \Longleftrightarrow \quad 0 \not< f(\bar{x}, y)$$

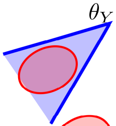
$$\Downarrow \qquad\qquad\qquad\qquad \Downarrow$$

$$\boxed{F(\bar{x}, y) \preccurlyeq_C^{(j)} \{\theta_Y\}} \qquad \boxed{\{\theta_Y\} \not\preccurlyeq_{\mathrm{int}\,C}^{(j)} F(\bar{x}, y)}$$

Case A1

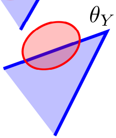
$F(\bar{x}, y) \subset -C$

$\Downarrow$



Case A2

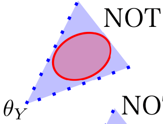
$F(\bar{x}, y) \cap (-C) \neq \emptyset$



Case B1

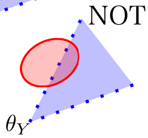
$F(\bar{x}, y) \not\subset \mathrm{int} C$

$\Uparrow$



Case B2

$F(\bar{x}, y) \cap \mathrm{int} C = \emptyset$



Four types of Set-Valued Minimax Inequalities with Set-relations

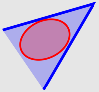
Assumption: F is (j) naturally quasi C-concave or (j) properly quasi C-concave

or $I^{(j)}_{k,\{\theta_Y\}}$

$S^{(j)}_{k,\{\theta_Y\}}$

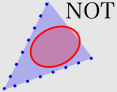
Case A1

$F(\bar{x}, y) \subset -C$



Case B1

$F(\bar{x}, y) \not\subset \mathrm{int}C$



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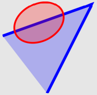

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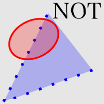

proved by Kuwano, Tanaka, Yamada.

$\boxed{\text{Fan-Takahashi minimax inequality}}$

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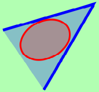
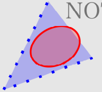


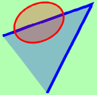
Case B1

$F(\bar{x}, y) \not\subset \mathrm{int} C$



Case A2

$F(\bar{x}, y) \cap (-C) \neq \emptyset$



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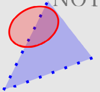

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Four types of Set-Valued Minimax Inequalities with Set-relations

Assumption: F is (j) naturally quasi C-concave

or

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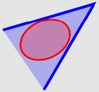
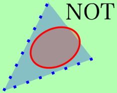
Case A1

$F(\bar{x},y) \subset -C$



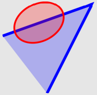
Case B1

$F(\bar{x},y) \not\subset \mathrm{int}C$



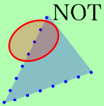
Case A2

$F(\bar{x},y) \cap (-C) \neq \emptyset$



Case B2

$F(\bar{x},y) \cap \mathrm{int}C = \emptyset$



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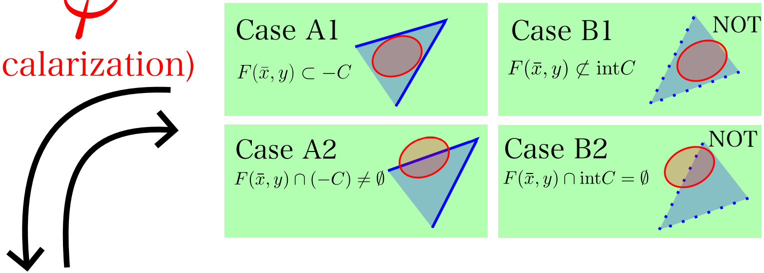
$\boxed{\text{Fan-Takahashi minimax inequality}}$

Assumption: F is (j) naturally quasi C-concave

φ

(scalarization)



$\boxed{\text{Fan-Takahashi minimax inequality}}$

Preliminaries

Let X be a topological space, Y a real topological vector space, and θ_Y be a zero vector in Y. Define that $\mathcal{P}_0(Y)$ is the set of all nonempty subsets of Y. The sets of neighborhoods of $x \in X$ and $y \in Y$ is denoted by $\mathcal{N}_X(x)$ and $\mathcal{N}_Y(y)$, respectively.

Definition

For $A, B \in \mathcal{P}_0(Y)$, we define two binary relations on $\mathcal{P}_0(Y)$:

$$A \preccurlyeq_1 B \overset{\text{def}}{\Longleftrightarrow} A \cap B \neq \emptyset \quad \text{and} \quad A \preccurlyeq_2 B \overset{\text{def}}{\Longleftrightarrow} B \subset A.$$

Definition

A is said to be C-bounded if for each neighborhood U of θ_Y there exists $t > 0$ such that $A \subset tU + C$.

Definition

Let $f : Y \to \mathbb{R} \cup \{\pm\infty\}$ and $y_0 \in Y$. We say that f is lower semicontinuous (l.s.c. shortly) at y_0 if

$$\forall r < f(y_0), \exists V \in \mathcal{N}_Y(y_0) \text{ s.t. } r < f(y), \forall y \in V;$$

Definition [1]

Let $F\colon X \to \mathcal{P}_0(Y)$, $x_0 \in X$, $\preccurlyeq$ a binary relation on $\mathcal{P}_0(Y)$ and $C \subset Y$ a convex cone. We say that F is $(\preccurlyeq, C)$-continuous at x_0 if

$$\forall W \subset Y, W \text{ open}, W \preccurlyeq F(x_0), \exists V \in \mathcal{N}_X(x_0) \text{ s.t. } W + C \preccurlyeq F(x), \forall x \in V.$$

Remark

As special cases, $(\preccurlyeq_1, C)$-continuity and $(\preccurlyeq_2, C)$-continuity coincide with “ C-lower continuity ” and “ C-upper continuity ” for set-valued maps, respectively.

Definition [1]
Let $\varphi\colon \mathcal{P}_0(Y) \to \mathbb{R} \cup \{\pm\infty\}$, $A_0 \in \mathcal{P}_0(Y)$, $\preccurlyeq$ a binary relation on $\mathcal{P}_0(Y)$, and C a convex cone in Y with $C \neq Y$. Then, we say that φ is $(\preccurlyeq, C)$-lower semicontinuous at A_0 if

$$\forall r < \varphi(A_0), \exists W \in \mathcal{P}_0(Y), W \text{ open, s.t. } W \preccurlyeq A_0 \text{ and } r > \varphi(A), \forall A \in U(W + C, \preccurlyeq);$$

where $U(V, \preccurlyeq) := \{A \in \mathcal{P}_0(Y) \mid V \preccurlyeq A\}$.

Theorem [1]
Let $F\colon X \to \mathcal{P}_0(Y)$, $\varphi\colon \mathcal{P}_0(Y) \to \mathbb{R} \cup \{\pm\infty\}$, $x_0 \in X$, $\preccurlyeq$ a binary relation on $\mathcal{P}_0(Y)$, and C a convex cone. If F is $(\preccurlyeq, C)$-continuous at x_0 and φ is $(\preccurlyeq, C)$-lower semicontinuous at $F(x_0)$, then $(\varphi \circ F)$ is lower semicontinuous at x_0.

Definition

Let X be a nonempty set, Y a real topological vector space, C a convex cone in Y, and $F\colon X \to \mathcal{P}_0(Y)$ a set-valued map.

1. F is called type (j) properly quasi C-concave if for each $x, y \in X$ and $\lambda \in (0,1)$,

$$F(x) \preccurlyeq_C^{(j)} F(\lambda x + (1-\lambda)y) \quad \text{or} \quad F(y) \preccurlyeq_C^{(j)} F(\lambda x + (1-\lambda)y)$$

2. F is called type (j) naturally quasi C-concave if for each $x, y \in X$ and $\lambda \in (0,1)$, there exists $\mu \in [0,1]$ such that

$$\mu F(x) + (1-\mu)F(y) \preccurlyeq_C^{(j)} F(\lambda x + (1-\lambda)y).$$

Remark

If F is type (j) properly quasi C-concave, then F is type (j) naturally quasi C-concave.

Definition [3]

Let $\mathcal{A} \subset \mathcal{P}_0(Y)$. $\mathcal{A}$ is said to be convex if for each $A_1, A_2 \in \mathcal{A}$ and $\lambda \in (0,1)$,

$$\lambda A_1 + (1-\lambda)A_2 \in \mathcal{A}.$$

Definition [3]

Let $\varphi \colon \mathcal{P}_0(Y) \to \mathbb{R} \cup \{\pm\infty\}$. Then,

1. φ is concave if for $A, B \in \mathcal{P}_0(Y)$, $\varphi(\lambda A + (1-\lambda)B) \geq \lambda\varphi(A) + (1-\lambda)\varphi(B)$,
2. φ is quasi concave if for any $\alpha \in \mathbb{R}$, $\mathrm{lev}\,(\varphi, \geq, \alpha) \coloneqq \{A \in \mathcal{P}_0(Y) \mid \varphi(A) \geq \alpha\}$ is convex.

Remark

If φ is concave, then φ is quasi concave.

Preliminaries (Monotonicity)

Definition

Let C be a convex cone in Y. A scalarization function φ is $(\preccurlyeq_C^{(j)})$-monotone if for any $A, B \in \mathcal{P}_0(Y)$ with $A \preccurlyeq_C^{(j)} B$, $\varphi(A) \leq \varphi(B)$.

Definition

Let C be a convex cone in Y. A scalarization function φ is $(\preccurlyeq_{\mathrm{int}\, C}^{(j)})$-monotone if for any $A, B \in \mathcal{P}_0(Y)$ with $A \preccurlyeq_{\mathrm{int}\, C}^{(j)} B$, $\varphi(A) < \varphi(B)$.

Proposition

Let φ be $(\preccurlyeq_C^{(j)})$-monotone and quasi concave. If F is type (j) naturally quasi C-concave, then $(\varphi \circ F)$ is quasi concave.

Proposition

Let φ be $(\preccurlyeq_{\mathrm{int}\, C}^{(j)})$-monotone and quasi concave. If F is type (j) naturally quasi C-concave, then $(\varphi \circ F)$ is quasi concave.

Main results

Specific Scalarization Function

Let $\varphi : \mathcal{P}_0(Y) \to \mathbb{R} \cup \{\pm\infty\}$, $\preccurlyeq$ a binary relation on $\mathcal{P}_0(Y)$, and $C' \subset Y$ a convex cone. In order to generalize four types of set-valued minimax inequalities [4], we provide a new class of scalarization functions which satisfy;

1. φ is $(\preccurlyeq, C')$-lower semicontinuous,
2. φ is quasi concave,
3. $\varphi(\{\theta_Y\}) = 0$,

In addition, we define conditions between inequalities and set-relations as follows;

(A1) φ is $(\preccurlyeq^{(j)}_{\mathrm{int}\, C})$-monotone,
(A2) $\varphi(A) > 0 \Rightarrow \{\theta_Y\} \preccurlyeq^{(j)}_{\mathrm{int}\, C} A$ for any $A \in \mathcal{P}_0(Y)$.

If φ satisfies conditions (i)–(iii), (A1), and (A2), we write the notation as $\varphi \in \Phi(\preccurlyeq^{(j)}_{\mathrm{int}\, C}, \preccurlyeq, C')$.

Main Result

Theorem

Let X be a nonempty compact convex subset of a Hausdorff topological vector space, Y a real topological vector space, $\preccurlyeq$ a binary relation on $\mathcal{P}(Y)$, C a convex cone in Y, C' a convex cone in Y, $\varphi\colon \mathcal{P}_0(Y) \to \mathbb{R} \cup \{\pm\infty\}$, and $F\colon X \times X \to \mathcal{P}_0(Y)$. For the scaralization function $\varphi \in \Phi(\preccurlyeq^{(j)}_{\mathrm{int}\, C}, \preccurlyeq, C')$, if F satisfies the following conditions:

1. $(\varphi \circ F)(x, y) \in \mathbb{R}$ for all $x, y \in X$,
2. for each fixed $y \in X$, $F(\cdot, y)$ is $(\preccurlyeq, C')$-continuous,
3. for each fixed $x \in X$, $F(x, \cdot)$ is j-naturally quasi C-concave,
4. for all $x \in X$, $\{\theta_Y\} \not\preccurlyeq^{(j)}_{\mathrm{int}\, C} F(x, x)$,

then there exists $\bar{x} \in X$ such that $\{\theta_Y\} \not\preccurlyeq^{(j)}_{\mathrm{int}\, C} F(\bar{x}, y)$ for all $y \in X$.

Conclusion

Conclusion

- We introduce the background and the basic notion.
- We gave a new result of set-valued Fan-Takahashi inequalities via scalarization methods.
- Next step is to check other scalarization functions to satisfy new assumption.

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