2 Asymptotic Cones and Functions

2.1 Definition of Asymptotic Cones

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We use the book; Asymptotic Cones and Functions in Optimization and Variational Inequalities (author: A.AUSLENDER and M.TEBOULLE), pp.25-31.

The set of natural numbers is denoted by $\mathbb{N}$, so that $k \in \mathbb{N}$ means $k = 1, 2, \ldots$. A sequence $\{x_k\}_{k\in\mathbb{N}}$ or simply $\{x_k\}$ in $\mathbb{R}^n$ is said to converge to x if $\|x_k - x\| \to 0$ as $k \to \infty$, and this will be indicated by the notation $x_k \to x$ or $x = \lim_{k\to\infty} x_k$. We say that x is a cluster point of $\{x_k\}$ if some subsequence converge to x. Recall that every bounded sequence in $\mathbb{R}^n$ converges to x if and only if it is bounded and has x as its unique cluster point.

Let $\{x_k\}$ be a sequence in $\mathbb{R}^n$. We are interested in knowing how to handle convergence properties, we are led to consider direction $d_k := x_k \|x_k\|^{-1}$ with $x_k \neq 0$, $k \in \mathbb{N}$. From classical analysis, the Bolzano-Weierstrass theorem implies that we can extract a convergent subsequence $d = \lim_{k\in K} d_k$, $K \subset \mathbb{N}$, with $d \neq 0$. Now suppose that the sequence $\{x_k\} \subset \mathbb{R}^n$ is such that $\|x_k\| \to +\infty$. Then

$$\exists t_k := \|x_k\|, k \in K \subset \mathbb{N}, \text{ such that } \lim_{k\in K} t_k = +\infty \text{ and } \lim_{k\in K} \frac{x_k}{t_k} = d.$$

This leads us to introduce the following concepts.

Definition 2.1.1

A sequence $\{x_k\} \subset \mathbb{R}$ is said to converge to a direction $d \in \mathbb{R}^n$ if

$$\exists\{t_k\},\ with\, t_k \to +\infty \text{ such that } \lim_{k\to\infty} \frac{x_k}{t_k} = d.$$

Definition 2.1.2

Let C be a nonempty set in $\mathbb{R}^n$. Then the asymptotic cone of the set C, denoted by C_∞, is the set of vectors $d \in \mathbb{R}^n$ that are limits in direction of the sequences $\{x_k\} \subset C$, namely

$$C_\infty = \{d \in \mathbb{R}^n \mid \exists t_k \to +\infty, \exists x_k \in C \text{ with } \lim_{k\to\infty} \frac{x_k}{t_k} = d\}.$$

From the definition we immediately deduce the following elementary facts.

Proposition 2.1.1

Let $C \subset \mathbb{R}^n$ be nonempty. Then:

(i) C_∞ is a closed cone.
(ii) $(\operatorname{cl} C)_\infty = C_\infty$.
(iii) If C is a cone, then $C_\infty = \operatorname{cl} C$.

Proof. We will prove each part separately.

(i) C_∞ is a closed cone.

We need to show two propositions: (i-a) C_∞ is a cone and (i-b) C_∞ is a closed set.

(i-a) We show that C_∞ is a cone, that is, $\forall \alpha \geq 0, d \in C_\infty, \alpha d \in C_\infty$.

Since 0 is a element of C_∞, it is clear in the case of $\alpha = 0$.

($\because$ Since C is nonempty, we can take a element x_0 from C. In addition we take a sequence $\{t_k\}_{k=1}^\infty$ with $t_k \to +\infty$ as $k \to \infty$. Of course this sequence exists, for example $t_k := k$. By using $t_k := k$ and $x_k := x_0$, we can obtain 0 as the limit. Hence 0 is a element of C_∞.)

Also we consider the other case $\alpha > 0$. To prove that C_∞ is a cone, we take a any direction d from C_∞. Since d is a element of C_∞,

$$\exists t_k \to +\infty, \exists x_k \in C \text{ with } \lim_{k\to\infty} \frac{x_k}{t_k} = d.$$

Then we define a sequence $\{t'_k\}_{k=1}^\infty := \frac{t_k}{\alpha}$, exactly whose limit becomes $+\infty$ as $k \to \infty$. Accordingly there exist $t'_k \to +\infty$ and $x_k \in C$ with

$$\lim_{k\to\infty} \frac{x_k}{t'_k} = \lim_{k\to\infty} \alpha \cdot \frac{x_k}{t_k} = \alpha d.$$

This means $d \in C_\infty$.

By these results, we can get $\forall \alpha \geq 0, d \in C_\infty, \alpha d \in C_\infty$.

Therefore C_∞ is a cone.

(i-b) We show that C_∞ is a closed set. In order to prove closeness, we consider convergency of a sequence in C_∞. First we take a sequence $\{d_k\}_{k=1}^\infty \subset C_\infty$ with $d_k \to d$ as $k \to \infty$ for some d. To obtain $d \in C_\infty$, we need two sequences like $\{t_n\}_{n=1}^\infty$ with $t_n \to \infty$ as $n \to \infty$ and $\{x_n\}_{n=1}^\infty$ where $\frac{x_n}{t_n} \to d$ as $n \to \infty$.

Since $d_k \to d$ and $t_k^{m-1} \cdot x_k^m \to d_k$ as $m \to \infty$ for each $k \in \mathbb{N}$,

$$\begin{aligned}
&\forall n \in \mathbb{N}, \exists k(n) \in \mathbb{N} \,\text{s.t.}\, \forall j \geq k(n), ||d_j - d|| < \frac{1}{n}, \text{and} \\
&\forall k \in \mathbb{N}(1 \leq k \leq k(n)), \exists m(n,k) \in \mathbb{N} \,\text{s.t.}\, \forall m \geq m(n,k), ||\frac{x_k^m}{t_k^m} - d_k|| < \frac{1}{n}.
\end{aligned}$$

Now we can rearrange

$$\begin{aligned}
k(n) &\coloneqq \max\{k(n-1), k(n)\} + n \text{ and} \\
m(n) &\coloneqq \max_{1 \leq k \leq k(n)} \{m(n,k)\} + n
\end{aligned}$$

as sequences of $n \in \mathbb{N}$. Then it holds that $k(1) \leq k(2) \leq \ldots$ and $m(1) \leq m(2) \leq \ldots$.

Let's define

$$\begin{aligned}
t_n &\coloneqq t_{k(n)}^{m(n)}, \text{ and} \\
x_n &\coloneqq x_{k(n)}^{m(n)}.
\end{aligned}$$

Also we can find that

$$\begin{aligned}
&t_n \to \infty \text{ as } n \to \infty, \\
&x_n \in C, \text{ and} \\
&\frac{x_n}{t_n} = \frac{x_{k(n)}^{m(n)}}{t_{k(n)}^{m(n)}}.
\end{aligned}$$

Hence we get for each $n \in \mathbb{N}$

$$0 \leq \left\| \frac{x_n}{t_n} - d \right\| \leq \left\| \frac{x_{k(n)}^{m(n)}}{t_{k(n)}^{m(n)}} - d_{k(n)} \right\| + \left\| d_{k(n)} - d \right\| < \frac{1}{2n} \to 0$$

as $n \to \infty$.

Thus $d \in C_\infty$, that is, C_∞ is a closed set.

Then (i)'s proof is completed.

(ii) $(\operatorname{cl} C)_\infty = C_\infty$.

We need to show two relations: (ii-a) $(\operatorname{cl} C)_\infty \supset C_\infty$ (ii-b) $(\operatorname{cl} C)_\infty \subset C_\infty$.

(ii-a) We show that C_∞ is included in $(\operatorname{cl} C)_\infty$. However it is clear from the definition of asymptotic cone.

(ii-b) We show that $(\operatorname{cl} C)_\infty \subset C_\infty$. Like (i-b), we'll show that.

Then (ii)'s proof is also completed.

(iii) If C is a cone, then $C_\infty = \operatorname{cl} C$.

We need to show two relations: (iii-a) $C_\infty \subset \operatorname{cl} C$ and (iii-b) $C_\infty \supset \operatorname{cl} C$.

(iii-a) We take any direction $d \in C_\infty$ which satisfies

$$\exists t_k \to +\infty, \exists x_k \in C \ \text{ with } \ \lim_{k\to\infty} \frac{x_k}{t_k} = d.$$

Let $d_k := \frac{x_k}{t_k}$ (with $d_k \to d$ as $k \to \infty$). Since C is a cone,

$$d_k = \frac{1}{t_k} \cdot x_k \in C.$$

Due to $d_k \in C$, the limit of d_k is a element of $\operatorname{cl} C$, i.e., $d \in \operatorname{cl} C$.

Therefore $C_\infty \subset \operatorname{cl} C$.

(iii-b) We take any $d \in \operatorname{cl} C$ and show $d \in C_\infty$, that is,

$$\exists t_k \to +\infty, \exists x_k \in C \ \text{ with } \ \lim_{k\to\infty} \frac{x_k}{t_k} = d.$$

By $d \in \operatorname{cl} C$,

$$\exists \{d_k\}_{k=1}^{\infty} \in C \text{ with } d_k \to d \text{ as } k \to \infty,$$

in other words,

$$\lim_{k\to\infty} d_k = d.$$

We define $y_k = k \cdot d_k$ and $s_k = k$ for each k. Since $d_k \in C$ and C is a cone, y_k is also a element of C.

There exist $\{s_k\}_{k=1}^{\infty}$ with $s_k \to \infty$ as $k \to \infty$ and $\{y_k\}_{k=1}^{\infty} \subset C$ such that

$$\lim_{k\to\infty} \frac{y_k}{s_k} = \lim_{k\to\infty} d_k = d.$$

As d is a element of C_∞, therefore $C_\infty \supset \operatorname{cl} C$.

$\square$

The importance of the asymptotic cone is revealed by the following key property, which is a immediate consequence of its definition.

Proposition 2.1.2

A set $C \subset \mathbb{R}^n$ is bounded if and only if $C_\infty = \{0\}$.

Proof. We show that:

(i) a set $C \subset \mathbb{R}^n$ is bounded $\Rightarrow C_\infty = \{0\}$, and
(ii) a set $C \subset \mathbb{R}^n$ is unbounded $\Rightarrow C_\infty \neq \{0\}$.

(i) By Proposition 2.1.1 (i), $0 \in C_\infty$. Also, by the assumption C is bounded,

$$\exists r > 0, \forall x_k \in C \text{ where } k \in \mathbb{N}, \|x_k\| \leq r.$$

For any sequence $\{t_k\}_{k=1}^{\infty}$ such that $t_k \to \infty$ as $k \to \infty$,

$$\lim_{k \to \infty} \frac{x_k}{t_k} = 0.$$

Thus the limit becomes only 0 for any $\{x_k\}_{k=1}^{\infty} \subset C$ and $\{t_k\}_{k=1}^{\infty}$ with $t_k \to \infty$ as $k \to \infty$.

Therefore if C is bounded then $C_\infty = \{0\}$.

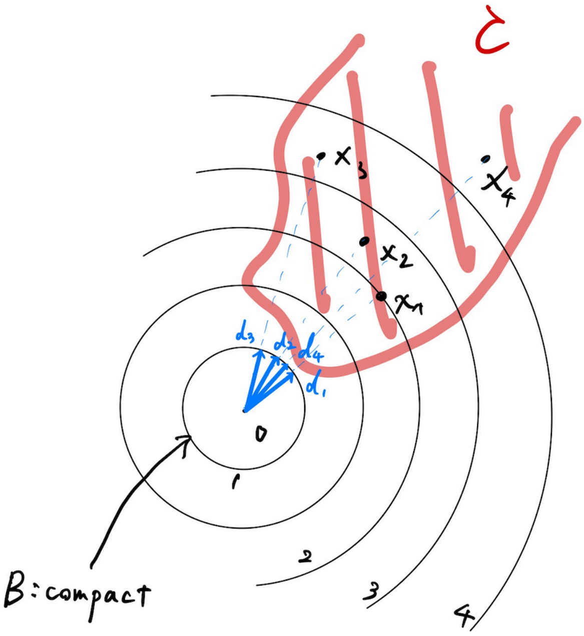
(ii) If C is unbounded, then there exists a sequence $\{x_k\} \subset C$ with $x_k \neq 0$, $\forall k \in \mathbb{N}$, such that $t_k := \|t_k\| \to \infty$ and thus the vectors $d_k = t_k^{-1} x_k \in \{d : \ \|d\| = 1\}$.

By the Bolzano-Weierstrass, we can extract a subsequence of $\{d_k\}$ such that $\lim_{k \in K} d_k = d$, $K \subset \mathbb{N}$, and with $\|d\| = 1$. This nonzero vector d is an element of C_∞ by Definition 2.1.2, a contradiction.

$\square$

Figure:

Why do we take a subsequence of $\{d_k\}$?



Associated with the asymptotic cone C_∞ is the following related concept, which will help us in simplifying the definition of C_∞ in the particular case where $C \subset \mathbb{R}^n$ is assumed convex.

Definition 2.1.3

Let $C \subset \mathbb{R}^n$ be nonempty and define

$$C^1_\infty = \{d \in \mathbb{R}^n \mid \forall t_k \to +\infty, \exists x_k \in C \text{ with } \lim_{k\to\infty} \frac{x_k}{t_k} = d\}.$$

We say that C is asymptotically regular if $C_\infty = C^1_\infty$.

Proposition 2.1.3

Let C be a nonempty convex set in $\mathbb{R}^n$. Then C is asymptotically regular.

Proof. The inclusion $C^1_\infty \subset C_\infty$ clearly holds from the definition of C^1_∞ and C_∞, respectively. Let $d \in C_\infty$. Then $\exists \{x_k\} \in C$, $\exists s_k \to \infty$ such that $d = \lim_{k\to\infty} s_k^{-1} x_k$. Let $x \in C$ and define $d_k = s_k^{-1}(x_k - x)$. Then we have

$$d = \lim_{k\to\infty} d_k, x + s_k d_k \in C.$$

$\because x_k = x + s_k d_k \in C.$

Now note that an arbitrary sequence such that $\lim_{k\to\infty} t_k = +\infty$. For any fixed $m \in \mathbb{N}$, there exists $k(m)$ with $\lim_{m\to\infty} k(m) = +\infty$ such that $t_m \leq s_{k(m)}$, and since C is convex, we have $x'_m := x + t_m d_{m(k)} \in C$. Hence, $d = \lim_{m\to\infty} t_m^{-1} x'_m$, showing that $d \in C^1_\infty$.

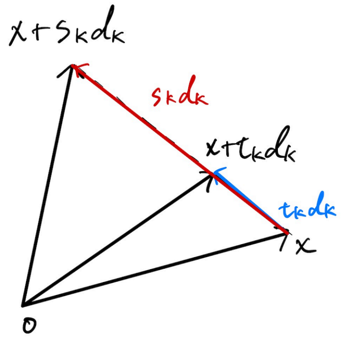
$\square$

Figure:

Why should we consider $k(m)$ with $\lim_{m\to\infty} k(m) = +\infty$ such that $t_m \leq s_{k(m)}$?

If $\{s_k\} = 1, 2, 3, 7, 8, 9, 13, \cdots$ and $\{t_k\} = 1, 3, 4, 6, 8, 10, 11, \cdots$, then we can get

$$\{k(m)\} = 1, 3, 4, 5, 6, 6, \cdots .$$



We note that a set can be nonconvex, yet asymptotically regular Indeed, consider, for example, sets definition by $C := S + K$, with S compact and K a closed convex cone. Then clearly C is not necessarily convex, but it can be easily seen that $C_\infty = C^1_\infty$.

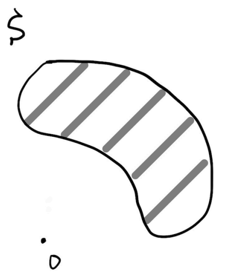


Figure1 S:compact

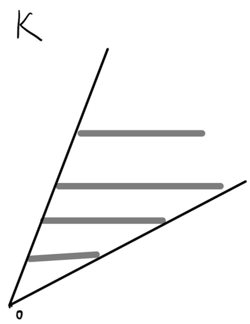


Figure2 K:closed convex cone

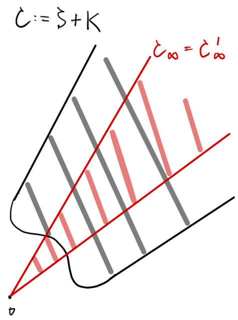


Figure3 $C := S + K$

Proof. We show that for any $d \in C_\infty$, $d \in C^1_\infty$.

By the definition of the asymptotic cone,

$$\exists t_k \to \infty, \exists x_k \in C \text{ with } \lim_{k\to\infty} \frac{x_k}{t_k} = d.$$

As S is compact, S is asymptotically regular, that is,

$$\forall t'_l \to \infty, \exists s_l \in C \text{ with } \lim_{l\to\infty} \frac{x_l}{t'_l} = 0.$$

For each k,

$$\exists s_k \in S, b_k \in K \text{ s.t. } x_k = s_k + b_k.$$

Then we get $d \in k_\infty$ because it holds that

$$\exists t_k \to \infty, \exists b_k \in K \text{ with } \frac{b_k}{t_k} \to d.$$

The convexity of K and Proposition 2.1.3 lead to $d \in K^1_\infty$.
Thus we obtain $d \in C^1_\infty$ because it holds that

$$\forall t'_l \to \infty, \exists x_l \in C \text{ where } x_l \coloneqq b_l + s_l \text{ with } \lim_{l\to\infty} \frac{x_l}{t'_l} = d.$$

Therefore $C_\infty = C^1_\infty$.

□

Remark 2.1.1

Note that the definitions of C_∞ and C^1_∞ are related to the theory of set convergence of Painleve-Kuratowski. Indeed, for a family $\{C_t\}_{t>0}$ of subsets of $\mathbb{R}^n$, the outer limit as $t \to +\infty$ is the set.

$$\limsup_{t\to+\infty} C_t = \{x \mid \liminf_{t\to+\infty} d(x, C_t) = 0\},$$

while the inner limit as $t \to +\infty$ is the set

$$\liminf_{t\to+\infty} C_t = \{x \mid \limsup_{t\to+\infty} d(x, C_t) = 0\},$$

It can then be verified that the corresponding asymptotic cones can be written as

$$C_\infty = \limsup_{t\to+\infty} t^{-1}C, \;\; C^1_\infty = \liminf_{t\to+\infty} t^{-1}C.$$

Proposition

Let $\{C_t\}_{t>0}$, $C \subset \mathbb{R}^n$, and $C \neq \varnothing$. Then,

(i) $C_\infty = \limsup_{t\to+\infty} t^{-1}C$, and
(ii) $C^1_\infty = \liminf_{t\to+\infty} t^{-1}C$.

Proof. First, we show that (i).

(i-a) We prove $C_\infty \subset \limsup_{t\to+\infty} t^{-1}C$.
$\forall d \in C_\infty$,

$$\exists t_k \to \infty, \exists x_k \in C \text{ with } \lim_{k\to\infty} \frac{x_k}{t_k} = d.$$

In other words, it holds that

$$\forall \epsilon > 0, \exists k_0 \in \mathbb{N} \text{ s.t. } \forall k \geq k_0, \left\| \frac{x_k}{t_k} - d \right\| < \epsilon.$$

To obtain $d \in \limsup_{t\to+\infty} t^{-1}C$, we need to show that

$$\forall \epsilon > 0, \exists s_0 \in \mathbb{N} \text{ s.t. } \forall s \geq s_0, \left\| \inf_{u\geq s} \inf_{y\in u^{-1}C} \|d-y\| \right\| < \epsilon.$$

To use the assumption of the asymptotic cone, we define a real value

$$t(k)_m := \max\{k, t_m\} \text{ where } t_m \in \{t_s\}_{s\geq k}^\infty \text{ and } m \in \mathbb{N}.$$

Soon we'll get

$$\begin{aligned} t(k)_m &\geq k, \\ m &\geq k, \text{ and} \\ t(k)_m &\to \infty \text{ as } m \to \infty. \end{aligned}$$

Thus $\forall \epsilon > 0, \exists k_0 \in \mathbb{N}$ s.t. $\forall k \geq k_0$,

$$\left\| \inf_{u\geq k} \inf_{y\in u^{-1}C} \|d-y\| \right\| \leq \inf_{y\in t(k)_m^{-1}C} \|d-y\| \leq \left\| \frac{x_m}{t(k)_m} - d \right\| < \epsilon.$$

Then $C_\infty \subset \limsup_{t\to+\infty} t^{-1}C$.

(i-b) We prove $C_\infty \supset \limsup_{t\to+\infty} t^{-1}C$.
We show that $\forall d \in \limsup_{t\to+\infty} t^{-1}C$,

$$\exists u_m \to \infty, \exists x_m \in C \text{ with } \lim_{t\to\infty} \frac{x_m}{u_m} = d.$$

$\forall d \in \limsup_{t\to+\infty} t^{-1}C$, $\forall \epsilon, \exists k_0 \in \mathbb{N}$ s.t. $\forall k \geq k_0$,

$$\left\| \inf_{u\geq k} \inf_{y\in u^{-1}C} \|d-y\| \right\| < \frac{\epsilon}{3}.$$

we let $\alpha(k) := \left\| \inf_{u\geq k} \inf_{y\in u^{-1}C} \|d-y\| \right\|$. To get $u_m \to \infty$ as $m \to \infty$, we define $u_m := m$ where $m \geq k$.

By the definition of infimum, there exist $u_k, \cdots, u_{m_0}, \cdots$ such that

$$\inf_{y \in u_k^{-1} C} \|d - y\| < \alpha(k) + \frac{\epsilon}{3},$$
$$\vdots$$
$$\inf_{y \in u_{m_0}^{-1} C} \|d - y\| < \alpha(k) + \frac{\epsilon}{3},$$
$$\vdots$$

Also we let $\beta(m) \coloneqq \inf_{y \in u_m^{-1} C} \|d - y\|$ for each $m \geq k$.
By the definition of infimum, there exist $x_k, \cdots, x_{m_0}, \cdots \in C$ such that

$$\left\| \frac{x_k}{u_k} - d \right\| < \frac{\epsilon}{3} + \beta(k) < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \alpha(k) = \epsilon,$$
$$\vdots$$
$$\left\| \frac{x_{m_0}}{u_{m_0}} - d \right\| < \frac{\epsilon}{3} + \beta(m_0) < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \alpha(k) = \epsilon,$$
$$\vdots$$

Thus it holds that

$$\exists u_m \to \infty, \exists x_m \in C \text{ with } \lim_{t \to \infty} \frac{x_m}{u_m} = d.$$

Then $C_\infty \supset \limsup_{t \to +\infty} t^{-1} C$.

Therefore $C_\infty = \limsup_{t \to +\infty} t^{-1} C$.
Second, we show that (ii).

(ii-a) We prove $C_\infty^1 \subset \liminf_{t \to +\infty} t^{-1} C$.
$\forall d \in C_\infty^1$,

$$\forall t_k \to \infty, \exists x_k \in C \text{ with } \lim_{k \to \infty} \frac{x_k}{t_k} = d.$$

Also,

$$\forall \epsilon > 0, \exists k_0 \in \mathbb{N} \text{ s.t. } \forall k \geq k_0, \left\| \frac{x_k}{t_k} - d \right\| < \epsilon.$$

We let $\alpha(s) \coloneqq \sup_{u \geq s} \inf_{y \in u^{-1} C} \|y - d\| \geq 0$.z
For each $s = 1, 2, \cdots$, $\exists t_s \geq s$,

$$-\frac{1}{s} \leq \alpha(s) - \frac{1}{s} < \inf_{y \in t_s^{-1} C} \|y - d\| \leq \left\| \frac{x_s}{t_s} - d \right\|.$$

Now $\{t_k\}_{k \in \mathbb{N}}$ satisfies $t_k \to \infty$.

Since $d \in C^1_\infty$,

$$\exists x_s \in C \text{ s.t. } \lim_{k \to \infty} \frac{x_s}{t_s} = d.$$

Thus $d \in \liminf_{t \to +\infty} t^{-1} C$.

(ii-b) We prove $C^1_\infty \supset \liminf_{t \to +\infty} t^{-1} C$.

$\forall d \in \liminf_{t \to +\infty} t^{-1} C$,

$$\forall \epsilon, \exists s_0 \in \mathbb{N} \text{ s.t. } \forall s \geq s_0, 0 \leq \sup_{u \geq s} \inf_{y \in u^{-1} C} \|y - d\| < \epsilon.$$

$\forall n \geq s$,

$$0 \leq \inf_{y \in n^{-1} C} \|y - d\| \leq \sup_{u \geq s} \inf_{y \in u^{-1} C} \|y - d\| < \epsilon.$$

By the definition of infimum, for any $\{t_k\} \to \infty$,

$$\exists u_0 \geq s \text{ where } t_{u_0} \geq s \text{ s.t. } \forall u \geq u_0, \exists x_u \in C, \left\| \frac{x_u}{t_u} - d \right\| < \epsilon.$$

Thus $d \in C^1_\infty$.

Therefore $C^1_\infty = \liminf_{t \to +\infty} t^{-1} C$. $\square$

Proposition 2.1.4

Let $C \subset \mathbb{R}^n$ be nonempty and define the normalized sets.

$$C_N := \{d \in \mathbb{R}^n \mid \exists \{x_k\} \in C, \|x_k\| \to +\infty \text{ with } d = \lim_{k \to \infty} \frac{x_k}{\|x_k\|}\}.$$

Then $C_\infty = \operatorname{pos} C_N$, where for any set C, $\operatorname{pos} C = \{\lambda x \mid x \in C, \lambda \geq 0\}$.

Proof. Clearly, one always has $\operatorname{pos} C_N \subset C_\infty$. Conversely, let $0 \neq d \in C_\infty$.

Then there exists $t_k \to \infty$, $x_k \in C$ such that

$$d = \lim_{k \to \infty} \frac{x_k}{t_k} = \lim_{k \to \infty} \frac{1}{t_k} \cdot \|x_k\| \cdot \frac{x_k}{\|x_k\|}, \text{ with } \|x_k\| \to \infty.$$

Thus the sequence $\{t_k^{-1} \|x_k\|\}$ is a nonnegative bounded sequence, and by the Bolzano-Weierstrass theorem, there exists a subsequence $\{t_k^{-1} \|x_k\|\}_{k \in K}$ with $K \subset \mathbb{N}$ such that $\lim_{k \in K} t_k^{-1} \|x_k\| = \lambda \geq 0$, which means that $d = \lambda d_N$ with $d_N \in C_N$, namely $d \in \operatorname{pos} C_N$. $\square$

Proposition 2.1.5

Let C be a nonempty convex set in $\mathbb{R}^n$. Then the asymptotic cone C_∞ is a closed convex cone. Moreover, define the following sets:

$$\begin{aligned} D(x) &:= \{d \in \mathbb{R}^n \mid x + td \in \operatorname{cl} C, \forall t > 0\}\ \forall x \in C, \\ E &:= \{d \in \mathbb{R}^n \mid \exists x \in C \text{ s.t. } x + td \in \operatorname{cl} C, \forall t > 0\}, \\ F &:= \{d \in \mathbb{R}^n \mid d + \operatorname{cl} C \subset \operatorname{cl} C\}. \end{aligned}$$

Then $D(x)$ is in fact independent of x, which is thus now denoted by D, and $C_\infty = D = E = F$.

Proof. We show that

(i) C_∞ is convex,

(ii) $C_\infty \subset D(x)$,

(iii) $D(x) \subset E$,

(iv) $E \subset C_\infty$,

(v) $C_\infty \subset F$,

(vi) $C_\infty \supset F$.

(i) We'll show that C_∞ is convex.

It follows that C is asymptotically regular from Proposition 2.1.3.

For any d_1 and d_2,

$$\begin{aligned} &\forall t_k \to \infty, \exists \{x_k\}_{k=1}^\infty \text{ with } \frac{x_k}{t_k} \to d_1 \text{ as } k \to \infty \\ &(\forall \epsilon > 0, \exists k_0 \in \mathbb{N} \text{ s.t. } \forall k \geq k_0, \left\| \frac{x_k}{t_k} - d_1 \right\| < \epsilon), \\ &\forall s_l \to \infty, \exists \{y_l\}_{l=1}^\infty \text{ with } \frac{y_l}{s_l} \to d_2 \text{ as } l \to \infty \\ &(\forall \epsilon > 0, \exists l_0 \in \mathbb{N} \text{ s.t. } \forall l \geq l_0, \left\| \frac{y_l}{s_l} - d_2 \right\| < \epsilon). \end{aligned}$$

Then we'll check the convexity, that is,

$$\lambda d_1 + (1 - \lambda) d_2 \in C_\infty \text{ where } \lambda \in (0, 1).$$

We take a sequence $\{u_m\}_{m=1}^\infty$ where $u_m := \max\{t_m, s_m\}$ and $m_0 := \max\{k_0, l_0\} \in \mathbb{N}$.
Then you can find $u_m \to \infty$.
Also we define a sequence as $\{\lambda x_m + (1 - \lambda) y_m\}_{m=1}^\infty$.
$\forall m \geq m_0$,

$$
\begin{aligned}
\left\| \frac{\lambda x_m + (1-\lambda) y_m}{u_m} - (\lambda d_1 + (1-\lambda) d_2) \right\| &\leq \lambda \left\| \frac{x_m}{u_m} - d_1 \right\| + (1-\lambda) \left\| \frac{y_m}{u_m} - d_2 \right\| \\
&< \lambda \epsilon + (1-\lambda) \epsilon = \epsilon.
\end{aligned}
$$

Therefore C_∞ is convex.

(ii) We now prove that $C_\infty \subset D(x)$

□