

2 Asymptotic Cones and Functions

2.1 Definition of Asymptotic Cones

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April 18, 2023

We use the book; Asymptotic Cones and Functions in Optimization and Variational Inequalities (author: A.AUSLENDER and M.TEBOULLE), pp.25-31.

The set of natural numbers is denoted by \mathbb{N} , so that $k \in \mathbb{N}$ means $k = 1, 2, \dots$. A sequence $\{x_k\}_{k \in \mathbb{N}}$ or simply $\{x_k\}$ in \mathbb{R}^n is said to converge to x if $\|x_k - x\| \rightarrow 0$ as $k \rightarrow \infty$, and this will be indicated by the notation $x_k \rightarrow x$ or $x = \lim_{k \rightarrow \infty} x_k$. We say that x is a cluster point of $\{x_k\}$ if some subsequence converge to x . Recall that every bounded sequence in \mathbb{R}^n converges to x if and only if it is bounded and has x as its unique cluster point.

Let $\{x_k\}$ be a sequence in \mathbb{R}^n . We are interested in knowing how to handle convergence properties, we are led to consider direction $d_k := x_k \|x_k\|^{-1}$ with $x_k \neq 0, k \in \mathbb{N}$. From classical analysis, the Bolzano-Weierstrass theorem implies that we can extract a convergent subsequence $d = \lim_{k \in K} d_k$, $K \subset \mathbb{N}$, with $d \neq 0$. Now suppose that the sequence $\{x_k\} \subset \mathbb{R}^n$ is such that $\|x_k\| \rightarrow +\infty$. Then

$$\exists t_k := \|x_k\|, k \in K \subset \mathbb{N}, \text{ such that } \lim_{k \in K} t_k = +\infty \text{ and } \lim_{k \in K} \frac{x_k}{t_k} = d.$$

This leads us to introduce the following concepts.

Definition 2.1.1

A sequence $\{x_k\} \subset \mathbb{R}^n$ is said to converge to a direction $d \in \mathbb{R}^n$ if

$$\exists \{t_k\}, \text{ with } t_k \rightarrow +\infty \text{ such that } \lim_{k \rightarrow \infty} \frac{x_k}{t_k} = d.$$

Definition 2.1.2

Let C be a nonempty set in \mathbb{R}^n . Then the asymptotic cone of the set C , denoted by C_∞ , is the set of vectors $d \in \mathbb{R}^n$ that are limits in direction of the sequences $\{x_k\} \subset C$, namely

$$C_\infty = \{d \in \mathbb{R}^n \mid \exists t_k \rightarrow +\infty, \exists x_k \in C \text{ with } \lim_{k \rightarrow \infty} \frac{x_k}{t_k} = d\}.$$

From the definition we immediately deduce the following elementary facts.

Proposition 2.1.1

Let $C \subset \mathbb{R}^n$ be nonempty. Then:

- (i) C_∞ is a closed cone.
- (ii) $(\text{cl } C)_\infty = C_\infty$.
- (iii) If C is a cone, then $C_\infty = \text{cl } C$.

Proof. We will prove each part separately.

- (i) C_∞ is a closed cone.

We need to show two propositions: (i-a) C_∞ is a cone and (i-b) C_∞ is a closed set.

(i-a) We show that C_∞ is a cone, that is, $\forall \alpha \geq 0, d \in C_\infty, \alpha d \in C_\infty$.

Since 0 is a element of C_∞ , it is clear in the case of $\alpha = 0$.

(\because Since C is nonempty, we can take a element x_0 from C . In addition we take a sequence $\{t_k\}_{k=1}^\infty$ with $t_k \rightarrow +\infty$ as $k \rightarrow \infty$. Of course this sequence exists, for example $t_k := k$. By using $t_k := k$ and $x_k := x_0$, we can obtain 0 as the limit. Hence 0 is a element of C_∞ .)

Also we consider the other case $\alpha > 0$. To prove that C_∞ is a cone, we take a any direction d from C_∞ . Since d is a element of C_∞ ,

$$\exists t_k \rightarrow +\infty, \exists x_k \in C \text{ with } \lim_{k \rightarrow \infty} \frac{x_k}{t_k} = d.$$

Then we define a sequence $\{t'_k\}_{k=1}^\infty := \frac{t_k}{\alpha}$, exactly whose limit becomes $+\infty$ as $k \rightarrow \infty$. Accordingly there exist $t'_k \rightarrow +\infty$ and $x_k \in C$ with

$$\lim_{k \rightarrow \infty} \frac{x_k}{t'_k} = \lim_{k \rightarrow \infty} \alpha \cdot \frac{x_k}{t_k} = \alpha d.$$

This means $d \in C_\infty$.

By these results, we can get $\forall \alpha \geq 0, d \in C_\infty, \alpha d \in C_\infty$.

Therefore C_∞ is a cone.

(i-b) We show that C_∞ is a closed set. In order to prove closeness, we consider convergency of a sequence of C_∞ . First we take a sequence $\{d_k\}_{k=1}^\infty \subset C_\infty$ with $d_k \rightarrow d$ as $k \rightarrow \infty$ for some d . Then we don't forget that $d \in C_\infty$ is our goal.

For each $k \in \mathbb{N}$,

$$\exists \{x_k^{(n)}\}_{n=1}^\infty \subset C \text{ and } \{t_k^{(n)}\}_{n=1}^\infty \text{ with } t_k^{(n)} \rightarrow \infty \text{ as } n \rightarrow \infty.$$

The below figure represents $x_k^{(n)}$ and $t_k^{(n)}$.

Figure:

$k \setminus n$	1	2	\dots	m	\dots	limit
1	$x_1^{(1)}, t_1^{(1)}$	$x_1^{(2)}, t_1^{(2)}$	\dots	$x_1^{(m)}, t_1^{(m)}$	\dots	d_1
2	$x_2^{(1)}, t_2^{(1)}$	$x_2^{(2)}, t_2^{(2)}$	\dots	$x_2^{(m)}, t_2^{(m)}$	\dots	d_2
\vdots	\vdots	\vdots	\ddots	\vdots	\vdots	\vdots
m	$x_m^{(1)}, t_m^{(1)}$	$x_m^{(2)}, t_m^{(2)}$	\dots	$x_m^{(m)}, t_m^{(m)}$	\dots	d_m
\vdots			\vdots			

Then we define

$$x_m := x_m^{(m)} \text{ and } t_m := t_m^{(m)}.$$

By the definition of convergence of a sequence,

$$\forall \epsilon > 0, \exists \bar{m} \in \mathbb{N} \text{ s.t. } \forall m \geq \bar{m}, \|d_m - d\| < \frac{\epsilon}{2}, \text{ and}$$

$$\forall \epsilon > 0, \exists \hat{m} \in \mathbb{N} \text{ s.t. } \forall m \geq \hat{m}, \left\| \frac{x_m}{t_m} - d_m \right\| < \frac{\epsilon}{2}.$$

Also, we let $\tilde{m} := \max \{\bar{m}, \hat{m}\} \in \mathbb{N}$. By using triangle inequality,

$$\forall \epsilon > 0, \exists \tilde{m} \in \mathbb{N} \text{ s.t. } \forall m \geq \tilde{m}, \left\| \frac{x_m}{t_m} - d \right\| < \epsilon.$$

$$(\because \left\| \frac{x_m}{t_m} - d \right\| \leq \left\| \frac{x_m}{t_m} - d_m \right\| + \|d_m - d\| < \epsilon.)$$

Therefore c_∞ is a closed set.

Then (i)'s proof is completed.

(ii) $(\text{cl } C)_\infty = C_\infty$.

We need to show two relations: (ii-a) $(\text{cl } C)_\infty \supset C_\infty$ (ii-b) $(\text{cl } C)_\infty \subset C_\infty$.

(ii-a) We show that C_∞ is included in $(\text{cl } C)_\infty$. However it is clear from the definition of asymptotic cone.

(ii-b) We show that $(\text{cl } C)_\infty \subset C_\infty$. In order to prove that a element of $(\text{cl } C)_\infty$ satisfies the asymptotic cone's relation, we consider convergency of a sequences of $(\text{cl } C)_\infty$ and $\text{cl } C$. First we take any $d \in (\text{cl } C)_\infty$ which satisfies

$$\exists t_k \rightarrow +\infty, \exists x_k \in \text{cl } C \text{ with } \lim_{k \rightarrow \infty} \frac{x_k}{t_k} = d.$$

For each $k \in \mathbb{N}$,

$$\exists \{y_k^{(n)}\}_{n=1}^\infty \subset C \text{ with } y_k^{(n)} \rightarrow x_k \text{ as } n \rightarrow \infty.$$

The below figure represents $y_k^{(n)}$.

Figure:

$k \setminus n$	1	2	\dots	m	\dots	limit
1	$y_1^{(1)}$	$y_1^{(2)}$	\dots	$y_1^{(m)}$	\dots	x_1
2	$y_2^{(1)}$	$y_2^{(2)}$	\dots	$y_2^{(m)}$	\dots	x_2
\vdots	\vdots	\vdots	\ddots	\vdots	\vdots	\vdots
m	$y_m^{(1)}$	$y_m^{(2)}$	\dots	$y_m^{(m)}$	\dots	x_m
\vdots				\vdots		

Then we define

$$y_m := y_m^{(m)}.$$

By the definition of convergence of a sequence,

$$\begin{aligned} \forall \epsilon > 0, \exists \bar{m} \in \mathbb{N} \text{ s.t. } \forall m \geq \bar{m}, \|d_m - d\| &< \frac{\epsilon}{2}, \\ \forall \epsilon > 0, \exists \hat{m} \in \mathbb{N} \text{ s.t. } \forall m \geq \hat{m}, \|y_m^m - x_m\| &< \frac{\sqrt{\epsilon}}{2}, \text{ and} \\ \forall \epsilon > 0, \exists \tilde{m} \in \mathbb{N} \text{ s.t. } \forall m \geq \tilde{m}, \left| \frac{1}{t_m} \right| &< \sqrt{\epsilon}. \end{aligned}$$

Also, we let $m_0 := \max \{\bar{m}, \hat{m}, \tilde{m}\} \in \mathbb{N}$. By using triangle inequality,

$$\forall \epsilon > 0, \exists m_0 \in \mathbb{N} \text{ s.t. } \forall m \geq \bar{m}, \left\| \frac{y_m}{t_m} - d \right\| < \epsilon.$$

$$(\because \left\| \frac{y_m}{t_m} - d \right\| \leq \frac{1}{|t_m|} \cdot \|y_m - d_m\| + \left\| \frac{y_m}{t_m} - d \right\| < \epsilon.)$$

Therefore $(\text{cl } C)_\infty \subset C_\infty$.

Then (ii)'s proof is also completed.

(iii) If C is a cone, then $C_\infty = \text{cl } C$.

We need to show two relations: (iii-a) $C_\infty \subset \text{cl } C$ and (iii-b) $C_\infty \supset \text{cl } C$.

(iii-a) We take any direction $d \in C_\infty$ which satisfies

$$\exists t_k \rightarrow +\infty, \exists x_k \in C \text{ with } \lim_{k \rightarrow \infty} \frac{x_k}{t_k} = d.$$

Let $d_k := \frac{x_k}{t_k}$ (with $d_k \rightarrow d$ as $k \rightarrow \infty$). Since C is a cone,

$$d_k = \frac{1}{t_k} \cdot x_k \in C.$$

Due to $d_k \in C$, the limit of d_k is a element of $\text{cl } C$, i.e., $d \in \text{cl } C$.

Therefore $C_\infty \subset \text{cl } C$.

(iii-b) We take any $d \in \text{cl } C$ and show $d \in C_\infty$, that is,

$$\exists t_k \rightarrow +\infty, \exists x_k \in C \text{ with } \lim_{k \rightarrow \infty} \frac{x_k}{t_k} = d.$$

By $d \in \text{cl } C$,

$$\exists \{d_k\}_{k=1}^\infty \in C \text{ with } d_k \rightarrow d \text{ as } \infty,$$

in other words,

$$\lim_{k \rightarrow \infty} d_k = d.$$

We define $y_k = k \cdot d_k$ and $s_k = k$ for each k . Since $d_k \in C$ and C is a cone, y_k is also a element of C .

There exist $\{s_k\}_{k=1}^\infty$ with $s_k \rightarrow \infty$ as $k \rightarrow \infty$ and $\{y_k\}_{k=1}^\infty \subset C$ such that

$$\lim_{k \rightarrow \infty} \frac{y_k}{s_k} = \lim_{k \rightarrow \infty} d_k = d.$$

As d is a element of C_∞ , therefore $C_\infty \supset \text{cl } C$.

□

The importance of the asymptotic cone is revealed by the following key property, which is a immediate consequence of its definition.

Proposition 2.1.2

A set $C \subset \mathbb{R}^n$ is bounded if and only if $C_\infty = \{0\}$.

Proof. We show that:

- (i) a set $C \subset \mathbb{R}^n$ is bounded $\Rightarrow C_\infty = \{0\}$, and
- (ii) a set $C \subset \mathbb{R}^n$ is unbounded $\Rightarrow C_\infty \neq \{0\}$.

- (i) By Proposition 2.1.1 (i), $0 \in C_\infty$. Also, by the assumption C is bounded,

$$\exists r > 0, \forall x_k \in C \text{ where } k \in \mathbb{N}, \|x_k\| \leq r.$$

For any sequence $\{t_k\}_{k=1}^{\infty}$ such that $t_k \rightarrow \infty$ as $k \rightarrow \infty$,

$$\lim_{k \rightarrow \infty} \frac{x_k}{t_k} = 0.$$

Thus the limit becomes only 0 for any $\{x_k\}_{k=1}^{\infty} \subset C$ and $\{t_k\}_{k=1}^{\infty}$ with $t_k \rightarrow \infty$ as $k \rightarrow \infty$.

Therefore if C is bounded then $C_{\infty} = \{0\}$.

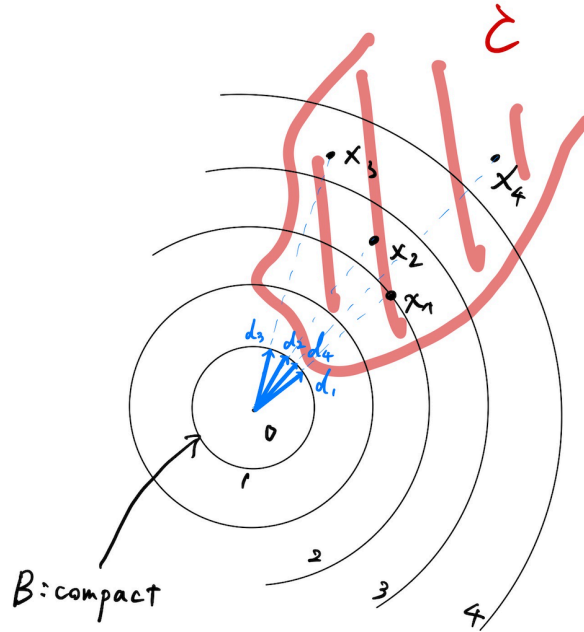
(ii) If C is unbounded, then there exists a sequence $\{x_k\} \subset C$ with $x_k \neq 0, \forall k \in \mathbb{N}$, such that $t_k := \|x_k\| \rightarrow \infty$ and thus the vectors $d_k = t_k^{-1}x_k \in \{d : \|d\| = 1\}$.

By the Bolzano-Weierstrass, we can extract a subsequence of $\{d_k\}$ such that $\lim_{k \in K} d_k = d, K \subset \mathbb{N}$, and with $\|d\| = 1$. This nonzero vector d is an element of C_{∞} by Definition 2.1.2, a contradiction.

□

Figure:

Why do we take a subsequence of $\{d_k\}$?



Associated with the asymptotic cone C_{∞} is the following related concept, which will help us in simplifying the definition of C_{∞} in the particular case where $C \in \mathbb{R}^n$ is assumed convex.

Definition 2.1.3

Let $C \in \mathbb{R}^n$ be nonempty and define

$$C_\infty^1 = \{d \in \mathbb{R}^n \mid \forall t_k \rightarrow +\infty, \exists x_k \in C \text{ with } \lim_{k \rightarrow \infty} \frac{x_k}{t_k} = d\}.$$

We say that C is asymptotically regular if $C_\infty = C_\infty^1$

Remark: A set $D = \{(x, y) \in \mathbb{R}_+ \times \mathbb{R}_+ \mid y = e^x - 1\}$ is not asymptotically regular.

Proposition 2.1.3

Let C be a nonempty convex set in \mathbb{R}^n . Then C is asymptotically regular.

Proof. The inclusion $C_\infty^1 \subset C_\infty$ clearly holds from the definition of C_∞^1 and C_∞ , respectively. Let $d \in C_\infty$. Then $\exists \{x_k\} \in C$, $\exists s_k \rightarrow \infty$ such that $d = \lim_{k \rightarrow \infty} s_k^{-1}x_k$. Let $x \in C$ and define $d_k = s_k^{-1}(x_k - x)$. Then we have

$$d = \lim_{k \rightarrow \infty} d_k, x + s_k d_k \in C.$$

$$\therefore x_k = x + s_k d_k \in C.$$

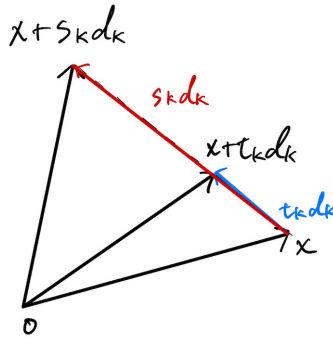
Now note that an arbitrary sequence such that $\lim_{k \rightarrow \infty} t_k = +\infty$. For any fixed $m \in \mathbb{N}$, there exists $k(m)$ with $\lim_{m \rightarrow \infty} k(m) = +\infty$ such that $t_m \leq s_{k(m)}$, and since C is convex, we have $x'_m = x + t_m + t_m d_{m(k)} \in C$. Hence, $d = \lim_{m \rightarrow \infty} t_m^{-1}x'_m$, showing that $d \in C_\infty^1$. \square

Figure:

Why should we consider $k(m)$ with $\lim_{m \rightarrow \infty} k(m) = +\infty$ such that $t_m \leq s_{k(m)}$?

If $\{s_k\} = 1, 2, 3, 7, 8, 9, 13, \dots$ and $\{t_k\} = 1, 3, 4, 6, 8, 10, 11, \dots$, then we can get

$$\{k(m)\} = 1, 3, 4, 5, 6, 6, \dots$$



We note that a set can be nonconvex, yet asymptotically regular. Indeed, consider, for example, sets defined by $C := S + K$, with S compact and K a closed convex cone. Then clearly C is not necessarily convex, but it can be easily seen that $C_\infty = C_\infty^1$.

Remark 2.1.1

Note that the definitions of C_∞ and C_∞^1 are related to the theory of set convergence of Painlevé-Kuratowski. Indeed, for a family $\{C_t\}_{t>0}$ of subsets of \mathbb{R}^n , the outer limit as $t \rightarrow +\infty$ is the set.

$$\limsup_{t \rightarrow +\infty} C_t = \{x \mid \liminf_{t \rightarrow +\infty} d(x, C_t) = 0\},$$

while the inner limit as $t \rightarrow +\infty$ is the set

$$\liminf_{t \rightarrow +\infty} C_t = \{x \mid \limsup_{t \rightarrow +\infty} d(x, C_t) = 0\},$$

It can then be verified that the corresponding asymptotic cones can be written as

$$C_\infty = \limsup_{t \rightarrow +\infty} t^{-1}C, \quad C_\infty^1 = \liminf_{t \rightarrow +\infty} t^{-1}C.$$