CS 663 Assignment 4

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QUESTION 6

A

$$\begin{aligned}\boldsymbol{y}^\intercal \boldsymbol{P}\boldsymbol{y} &= \boldsymbol{y}^\intercal(\boldsymbol{A}^\intercal \boldsymbol{A})\boldsymbol{y} \\ &= (\boldsymbol{A}\boldsymbol{y})^\intercal(\boldsymbol{A}\boldsymbol{y}) \\ &= ||\boldsymbol{A}\boldsymbol{y}||_2^2 \geq 0 \ \ (\because ||\boldsymbol{v}||_2^2 \geq 0 \, \forall \boldsymbol{v} \in \mathbb{R}^m)\end{aligned}$$

Let $\boldsymbol{u}$ be an eigenvector of $\boldsymbol{P}$ with corresponding eigenvalue of λ,

$$\begin{aligned}&\boldsymbol{P}\boldsymbol{u} = \lambda\boldsymbol{u} \\ \implies &\boldsymbol{u}^\intercal(\boldsymbol{P}\boldsymbol{u}) = \lambda\boldsymbol{u}^\intercal\boldsymbol{u} \\ \implies &\lambda = \frac{\boldsymbol{u}^\intercal\boldsymbol{P}\boldsymbol{u}}{||\boldsymbol{u}||_2^2} \geq 0 \ \ (\because ||\boldsymbol{v}||_2^2 \geq 0 \, \forall \boldsymbol{v} \in \mathbb{R}^n)\end{aligned}$$

$$\begin{aligned}\boldsymbol{z}^\intercal \boldsymbol{Q}\boldsymbol{z} &= \boldsymbol{z}^\intercal(\boldsymbol{A}\boldsymbol{A}^\intercal)\boldsymbol{z} \\ &= (\boldsymbol{A}^\intercal\boldsymbol{z})^\intercal(\boldsymbol{A}^\intercal\boldsymbol{z}) \\ &= ||\boldsymbol{A}^\intercal\boldsymbol{z}||_2^2 \geq 0 \ \ (\because ||\boldsymbol{v}||_2^2 \geq 0 \, \forall \boldsymbol{v} \in \mathbb{R}^n)\end{aligned}$$

Let $\boldsymbol{v}$ be an eigenvector of $\boldsymbol{Q}$ with corresponding eigenvalue of μ,

$$\begin{aligned}&\boldsymbol{Q}\boldsymbol{v} = \mu\boldsymbol{v} \\ \implies &\boldsymbol{v}^\intercal(\boldsymbol{Q}\boldsymbol{v}) = \mu\boldsymbol{v}^\intercal\boldsymbol{v} \\ \implies &\mu = \frac{\boldsymbol{v}^\intercal\boldsymbol{Q}\boldsymbol{v}}{||\boldsymbol{v}||_2^2} \geq 0 \ \ (\because ||\boldsymbol{v}||_2^2 \geq 0 \, \forall \boldsymbol{v} \in \mathbb{R}^m)\end{aligned}$$

Both $\boldsymbol{P}$ and $\boldsymbol{Q}$ are said to be positive semi-definite matrices.

B

$$\begin{aligned}\boldsymbol{Q}(\boldsymbol{A}\boldsymbol{u}) &= (\boldsymbol{A}\boldsymbol{A}^\intercal)(\boldsymbol{A}\boldsymbol{u}) \ \ (\because \boldsymbol{A}\boldsymbol{A}^\intercal = \boldsymbol{Q}) \\ &= \boldsymbol{A}(\boldsymbol{A}^\intercal\boldsymbol{A})\boldsymbol{u} \\ &= \boldsymbol{A}(\boldsymbol{P}\boldsymbol{u}) \ \ (\because \boldsymbol{A}^\intercal\boldsymbol{A} = \boldsymbol{P}) \\ &= \lambda(\boldsymbol{A}\boldsymbol{u}) \ \ (\because \boldsymbol{P}\boldsymbol{u} = \lambda\boldsymbol{u})\end{aligned}$$

$\boldsymbol{u}$ is an eigenvector to a nxn matrix and so has n elements.

$$\begin{aligned}\boldsymbol{P}(\boldsymbol{A}^\intercal\boldsymbol{v}) &= (\boldsymbol{A}^\intercal\boldsymbol{A})(\boldsymbol{A}^\intercal\boldsymbol{v}) \ \ (\because \boldsymbol{A}^\intercal\boldsymbol{A} = \boldsymbol{P}) \\ &= \boldsymbol{A}^\intercal(\boldsymbol{A}\boldsymbol{A}^\intercal)\boldsymbol{v} \\ &= \boldsymbol{A}^\intercal(\boldsymbol{Q}\boldsymbol{v}) \ \ (\because \boldsymbol{A}\boldsymbol{A}^\intercal = \boldsymbol{Q}) \\ &= \mu(\boldsymbol{A}^\intercal\boldsymbol{v}) \ \ (\because \boldsymbol{Q}\boldsymbol{v} = \mu\boldsymbol{v})\end{aligned}$$

$\boldsymbol{v}$ is an eigenvector to a mxm matrix and so has m elements.

C

Let the eigenvalue of $\boldsymbol{Q}$ corresponding to $\boldsymbol{v_i}$ by μ_i

$$\begin{aligned}
\boldsymbol{A}\boldsymbol{u_i} &= \boldsymbol{A}\frac{\boldsymbol{A}^\intercal \boldsymbol{v_i}}{||\boldsymbol{A}^\intercal \boldsymbol{v_i}||_2} \\
&= \frac{\boldsymbol{A}\boldsymbol{A}^\intercal \boldsymbol{v_i}}{||\boldsymbol{A}^\intercal \boldsymbol{v_i}||_2} \\
&= \frac{\boldsymbol{Q}\boldsymbol{v_i}}{||\boldsymbol{A}^\intercal \boldsymbol{v_i}||_2} \\
&= \frac{\mu_i \boldsymbol{v_i}}{||\boldsymbol{A}^\intercal \boldsymbol{v_i}||_2} \\
&= \frac{\mu_i}{||\boldsymbol{A}^\intercal \boldsymbol{v_i}||_2}\boldsymbol{v_i}
\end{aligned}$$

As $\boldsymbol{Q}$ is positive semi-definite, $\mu_i \geq 0$ (equality results in $\boldsymbol{u_i} = \boldsymbol{0}$), also the L2-norm of any non-zero vector is always positive.
So

$$\gamma_i = \frac{\mu_i}{||\boldsymbol{A}^\intercal \boldsymbol{v_i}||_2} \begin{cases} \geq 0, & \text{if } \boldsymbol{A}^\intercal \boldsymbol{v_i} \neq \boldsymbol{0} \\ = 0, & \text{if} \boldsymbol{A}^\intercal \boldsymbol{v_i} = \boldsymbol{0} \end{cases}$$

So, there exists $\gamma_i \geq 0$ such that $\boldsymbol{A}\boldsymbol{u_i} = \gamma_i \boldsymbol{v_i}$

D

From the result obtained in part C, $\boldsymbol{A}\boldsymbol{u}_i = \gamma_i \boldsymbol{v}_i$, $\gamma_i \geq 0 \quad \forall i = 1, 2, \ldots, m$

$$\boldsymbol{U}\boldsymbol{\Gamma} = \begin{bmatrix} \boldsymbol{v}_1 | & \boldsymbol{v}_2 | & \boldsymbol{v}_3 | & \ldots | & \boldsymbol{v}_m \end{bmatrix} \begin{bmatrix} \gamma_1 & 0 & 0 & \ldots & 0 \\ 0 & \gamma_2 & 0 & \ldots & 0 \\ 0 & 0 & \gamma_3 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \gamma_m \end{bmatrix}$$

$$\begin{aligned}
&= \begin{bmatrix} \gamma_1 \boldsymbol{v}_1 | & \gamma_2 \boldsymbol{v}_2 | & \gamma_3 \boldsymbol{v}_3 | & \ldots | & \gamma_m \boldsymbol{v}_m \end{bmatrix} \\
&= \begin{bmatrix} \boldsymbol{A}\boldsymbol{u}_1 | & \boldsymbol{A}\boldsymbol{u}_2 | & \boldsymbol{A}\boldsymbol{u}_3 | & \ldots | & \boldsymbol{A}\boldsymbol{u}_m \end{bmatrix} \\
&= \boldsymbol{A} \begin{bmatrix} \boldsymbol{u}_1 | & \boldsymbol{u}_2 | & \boldsymbol{u}_3 | & \ldots | & \boldsymbol{u}_m \end{bmatrix} \\
&= \boldsymbol{A}\boldsymbol{V}
\end{aligned}$$

$$\boldsymbol{u}_i^\intercal \boldsymbol{u}_j = \frac{\boldsymbol{v}_i^\intercal \boldsymbol{A}\boldsymbol{A}^\intercal \boldsymbol{v_j}}{||\boldsymbol{A}^\intercal \boldsymbol{v_i}||_2 ||\boldsymbol{A}^\intercal \boldsymbol{v_j}||_2} = \begin{cases} 0, & \text{for } i \neq j \\ 1, & \text{for } i = j \end{cases}$$

Thus $\boldsymbol{V}\boldsymbol{V}^\intercal = \boldsymbol{I_n}$ (i.e.$\boldsymbol{V}$ is orthonormal)

$$\begin{aligned}
\therefore \boldsymbol{A}\boldsymbol{V}\boldsymbol{V}^\intercal &= \boldsymbol{U}\boldsymbol{\Gamma}\boldsymbol{V}^\intercal \\
\boldsymbol{A} &= \boldsymbol{U}\boldsymbol{\Gamma}\boldsymbol{V}^\intercal
\end{aligned}$$