The GMWM: A New Framework for Inertial Sensor Calibration

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joint work with
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Introduction

General Framework:

- We propose **a new framework for inertial sensor calibration.**
- It is based on the **Generalized Method of Wavelet Moments (GMWM)** of *Guerrier et al, JASA, 2013* which is a new statistical approach to estimate the parameters of (complex) time series models.
- The GMWM is able to estimate efficiently time series models which are commonly used to describe the errors of inertial sensors.
- This calibration approach provides considerable improvements (terms of navigation performance) compared to existing methods.
- This methodology is **robust** (potentially applicable for FDI purposes) and is able to **automatically select a suitable model**.

IMU Calibration

Errors in Inertial Sensors

- Possible causes:
  - Non-orthogonalities of the sensor axes
  - Environmental conditions (e.g. temperature)
  - Electronics
  - Dynamics
  - ...
- Error types:
  - Deterministic (calibration models, physical models, ...)
  - **Random error components** (typically latent time series models,...)

Correct stochastic sensor error modeling implies:

- Correct stochastic assumptions for inference
- **Better navigation or post-processing performance**

Effect on position of error model

Emulation setting:

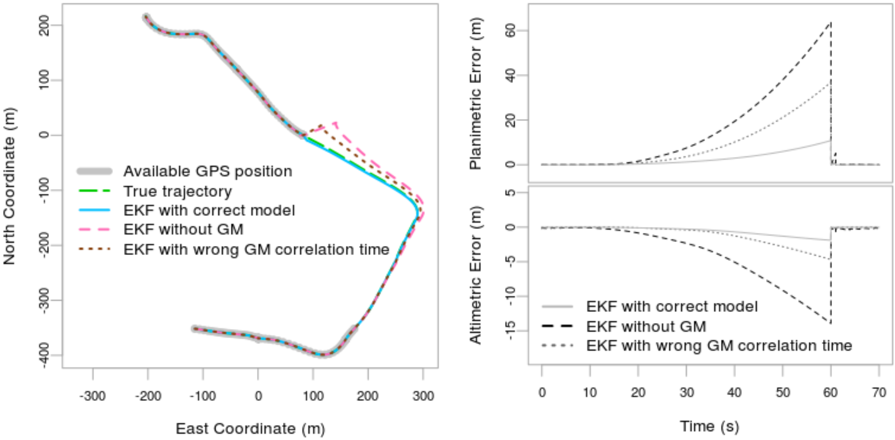
- Suppose the following model for inertial sensors:

$$Y_t = \exp(-\beta\Delta t)Y_{t-1} + \varepsilon_t,\ \varepsilon_t \overset{iid}{\sim} \mathcal{N}(0, \sigma^2_{GM}(1 - \exp(-2\beta\Delta t)))$$
$$X_t \overset{iid}{\sim} \mathcal{N}(0, \sigma^2),\ Z_t = X_t + Y_t.$$

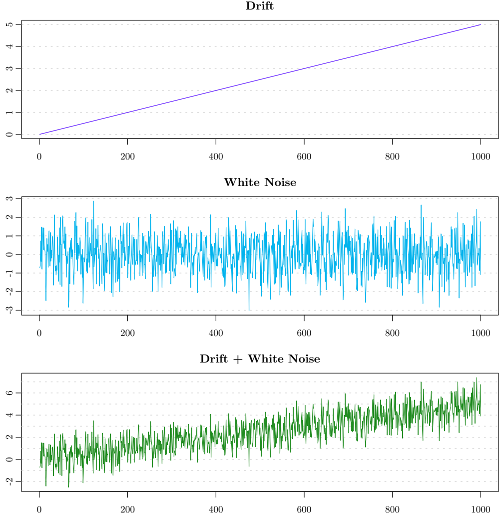
- Consider the following three models:

Sensor	Scenario	β	σ_{GM}	σ_{WN}
Acc.	Correct model	10^{-4}	50.0	70.0
	Wrong β	10^{-2}	50.0	70.0
	Without GM	–	–	70.0
Gyro.	Correct model	10^{-4}	10.0	30.0
	Wrong β	10^{-2}	10.0	30.0
	Without GM	–	–	30.0

Effect on position of error model



An easy latent time series model



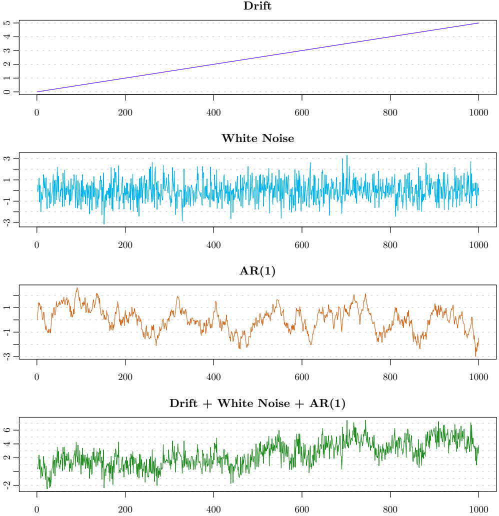
Remarks:

- Simple linear regression model:

$$y_t = \omega t + \varepsilon_t$$
$$\varepsilon_t \overset{iid}{\sim} \mathcal{N}\left(0, \sigma^2\right)$$

- MLE is perfectly fine.
- **What if we add an AR(1) process?**

Adding an autoregressive process



Remarks:

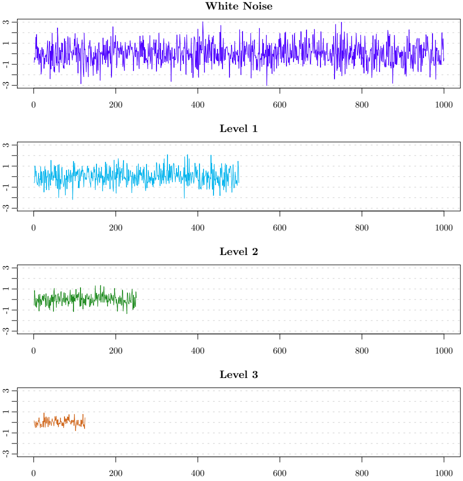
- Not a linear regression model but a **state space model**.
- Computing the likelihood is not an easy task (Kalman filter).
- **MLE (in fact EM-KF) fails**.

Estimation of Latent Time Series Models

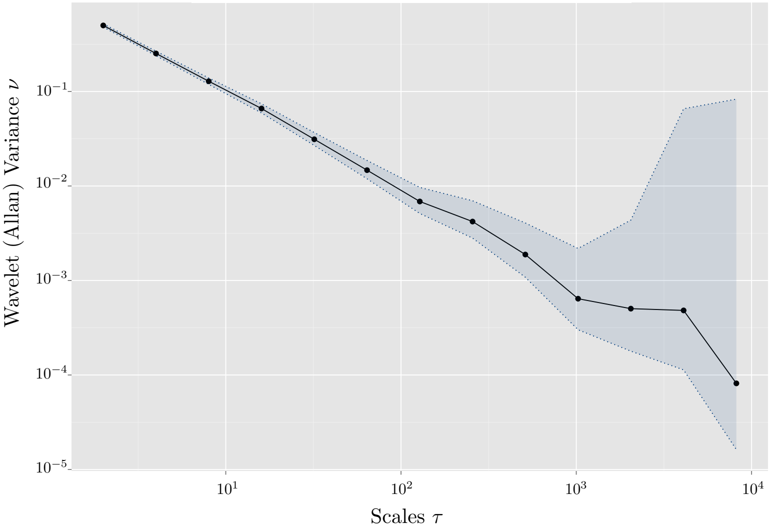
Existing methods:

- Transforming into a "non-latent" model (e.g. ARMA)
  - Does not work in general.
  - Tends to diverge when one latent time series is "close" to unit root.
  - Difficult to "inverse".
- MLE of an associated state-space (possibly using EM algorithm)
  - Computationally intensive.
  - Systematically diverges with "complex" models.
  - A lot of work is needed for a new model (see *Stebler et al, Meas Sci Tech, 2012*).
- "Graphical" method
  - Limited to a few possible models.
  - Not consistent in general (see *Guerrier et al, 2013b*).
  - "Inefficient" (see *Guerrier et al, 2013b*).

Looking differently at a time series using the Wavelet Variance



The Wavelet Variance



Initial idea:

Match the WV:

- Exploit the relationship that exists between the model $F_{\boldsymbol{\theta}}$ and the WV $\boldsymbol{\nu}(\boldsymbol{\theta})$ (i.e. **mapping $\boldsymbol{\theta} \mapsto \boldsymbol{\nu}(\boldsymbol{\theta})$**).
- "Inverse" this mapping by minimizing some discrepancy between empirical WV ($\hat{\boldsymbol{\nu}}$) and the theoretical WV $\boldsymbol{\nu}(\boldsymbol{\theta})$.
- This should provide an approximation of the point $\boldsymbol{\theta}(\hat{\boldsymbol{\nu}})$.

Wavelet Variance

Empirical WV:

- The WV (ν_{τ_j}) is the **variance of wavelet coefficients** for the scale τ_j.
- Wavelet coefficients ($\overline{W}_{j,t}$) are weighted averages computed on the series Y_t.
- The weights are called wavelet filters h_j: e.g. the Haar wavelet filter.
- The wavelet filters give non-zero weights to observations at a given lag (windows size of length L_j). Hence, there are as many WV as there are scales.
- The wavelet filters can be computed on consecutive windows, or on overlapping windows (to get $\widetilde{W}_{j,t}$ using $\tilde{h}_j$). Overlapping windows lead to more efficient estimators (such as the MODWT).

Wavelet Variance

Definition:

The Wavelet Variance (WV) is the variance of the wavelet coefficients, i.e.

$$\nu_{\tau_j} = \mathrm{var}\left(\widetilde{W}_{j,t}\right), \quad \text{where} \quad \widetilde{W}_{j,t} = \sum_{l=0}^{L_j-1} \tilde{h}_{j,l} y_{t-l}, \; t \in \mathbb{Z}$$

and where $\tilde{h}_{j,l}$ are wavelet filters based on MODWT.

Remark:

The Allan variance is a special case of the WV, in fact $\sigma_{\bar{y}}(\tau) = 2\nu_\tau$ where ν_τ is based on Haar wavelet.

Estimation of the WV

MODWT estimator:

A consistent estimator for ν_{τ_j} is given by the MODWT estimator defined in *Percival, Biometrika, 1995*

$$\hat{\nu}(\tau_j) = \frac{1}{M(T_j)} \sum_{t \in T_j} \widetilde{W}_{j,t}^2$$

Theorem: Asymptotic Normality

Serroukh et al, JASA, 2000 show that under suitable conditions

$$\sqrt{M(T_j)}\left(\hat{\nu}(\tau_j) - \nu_{\tau_j}\right) \xmapsto[T \to \infty]{\mathcal{D}} \mathcal{N}\left(0, S_{W_j}(0)\right)$$

A more general theorem

Theorem: Multivariate Extension

We extended this result to the multivariate case and demonstrated that under some regularity conditions

$$\sqrt{T}\left(\hat{\boldsymbol{\nu}} - \mathbb{E}[\hat{\boldsymbol{\nu}}]\right) \xmapsto[T\to\infty]{\mathcal{D}} \mathcal{N}\left(0, \mathbf{V}\right)$$

where $\mathbf{V} = [\sigma^2_{kl}]_{k,l=1,\dots,J}$.

Remark:

- This theorem generalizes *Serroukh et al, JASA, 2000* result and enables to compute the (asymptotic) covariance between the WV (or the AV) at two different scales.
- We proposed an estimator for σ^2_{kl}.

Theoretical WV

WV implied by $F_{\boldsymbol{\theta}}$:

Given a model $F_{\boldsymbol{\theta}}$ one can compute the theoretical WV as:

$$\nu_{\tau_j} = f\left(\boldsymbol{\theta}\right) = \int_{-1/2}^{1/2} |\widetilde{H}_j(f)|^2 S_{F_{\boldsymbol{\theta}}}(f) df$$

Example:

Consider an AR(1):

$$Y_t = \phi_1 Y_{t-1} + \varepsilon_t, \;\; \varepsilon_t \stackrel{iid}{\sim} \mathcal{N}\left(0, \sigma_\varepsilon^2\right), \;\; t = 1, ..., T$$

The theoretical (haar) WV of such process is given by

$$\nu_{\tau_j} = \frac{\left(\frac{\tau_j}{2} - 3\phi_1 - \frac{\tau_j \phi_1^2}{2} + 4\phi_1^{\frac{\tau_j}{2}+1} - \phi_1^{\tau_j+1}\right)\sigma_\varepsilon^2}{\frac{\tau_j^2}{2}\left(1-\phi_1\right)^2\left(1-\phi_1^2\right)}$$

WV of latent time series models

A very useful property:

Suppose we have

$$Y_t = X_t^{(1)} + ... + X_t^{(k)}$$

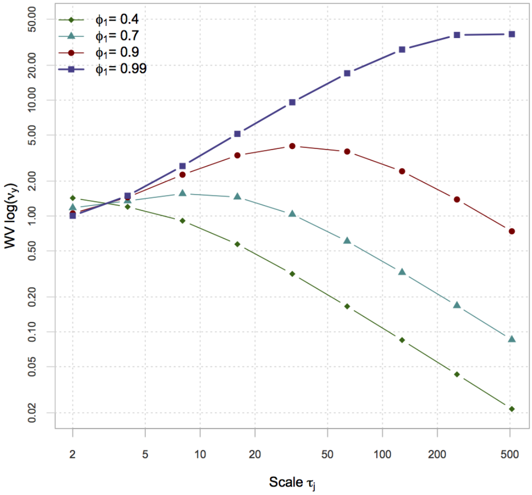
then the PSD of Y_t is

$$S_{Y_t} = S_{X_t^{(1)}} + ... + S_{X_t^{(k)}}$$

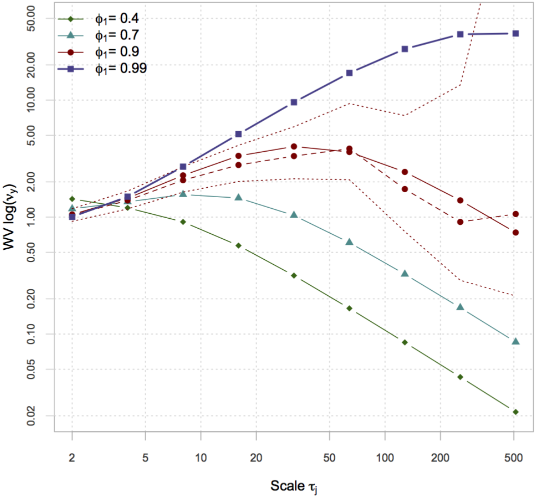
so the WV of Y_t is given by

$$\nu_{Y_t,\tau_j} = \int_{-1/2}^{1/2} |\widetilde{H}_j(f)|^2 \left(\sum_{i=1}^{k} S_{X_t^{(i)}} \right) df = \sum_{i=1}^{k} \nu_{X_t^{(i)},\tau_j}$$

Principle of the GMWM



Principle of the GMWM



The GMWM estimator

Definition:

The GMWM estimator is the solution of the following optimization problem

$$\hat{\boldsymbol{\theta}} = \underset{\boldsymbol{\theta} \in \boldsymbol{\Theta}}{\text{argmin}} \; \left(\hat{\boldsymbol{\nu}} - \boldsymbol{\nu}(\boldsymbol{\theta})\right)^T \boldsymbol{\Omega} \left(\hat{\boldsymbol{\nu}} - \boldsymbol{\nu}(\boldsymbol{\theta})\right)$$

in which $\boldsymbol{\Omega}$, a positive definite weighting matrix, is chosen in a suitable manner such that the above quadratic form is convex.

Theorem: Consistency

$\hat{\boldsymbol{\theta}}$ is a consistent estimator of $\boldsymbol{\theta}$ (under some regularity conditions) for a large class of (latent) models.

Asymptotic distribution of $\hat{\boldsymbol{\theta}}$

Theorem: Asymptotic Normality

We showed that (under some regularity conditions)

$$\sqrt{T}\left(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}_0\right) \xrightarrow[T\to\infty]{\mathcal{D}} \mathcal{N}\left(0, \boldsymbol{\Sigma}\right)$$

where $\boldsymbol{\Sigma} = \mathbf{B}\mathbf{V}\mathbf{B}^T$, $\mathbf{B} = \left(\mathbf{D}^T\boldsymbol{\Omega}\mathbf{D}\right)^{-1}\mathbf{D}^T\boldsymbol{\Omega}$, $\mathbf{D} = \partial\boldsymbol{\nu}(\boldsymbol{\theta})/\partial\boldsymbol{\theta}^T$ and $\mathbf{V}$ is the asymptotic covariance matrix of $\hat{\boldsymbol{\nu}}$.

Choosing $\boldsymbol{\Omega}$:

The most efficient estimator is (asymptotically) obtained by choosing $\boldsymbol{\Omega} = \mathbf{V}^{-1}$, leading then to $\boldsymbol{\Sigma} = (\mathbf{D}^T\mathbf{V}^{-1}\mathbf{D})^{-1}$.

A small Example...

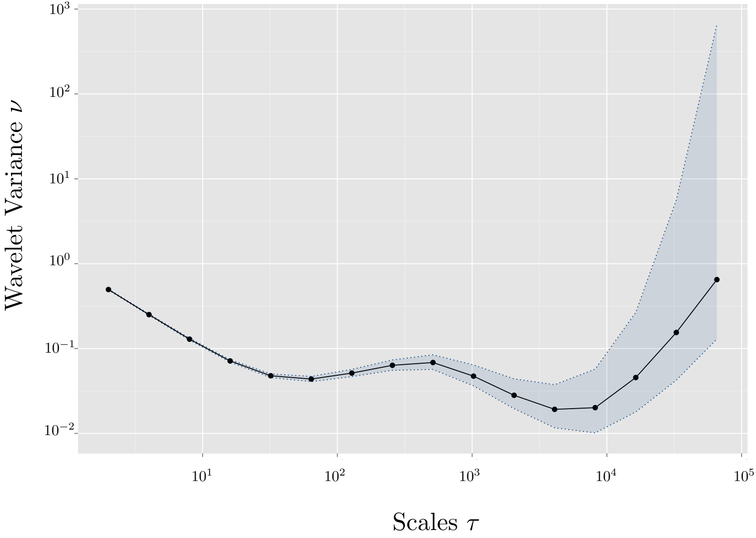
A simulated example:

Let $(y_t) \ : t = 1, ..., 10^5$ be a simulated signal composed of a:

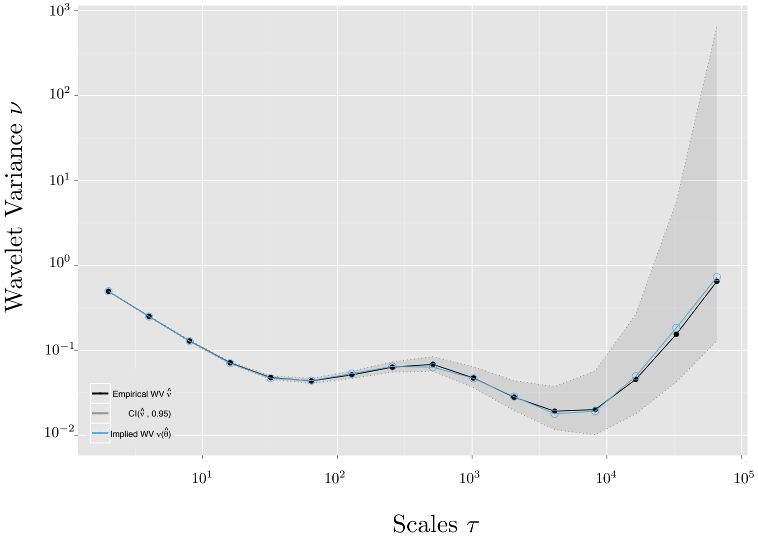
- First-order Gauss-Markov:
  $Y_t = \exp(-\beta \Delta t) Y_{t-1} + \varepsilon_t, \ \ \varepsilon_t \overset{iid}{\sim} \mathcal{N}\left(0, \sigma^2_{GM}\left(1 - \exp(-2\beta \Delta t)\right)\right)$
- Gaussian White Noise: $X_t \overset{iid}{\sim} \mathcal{N}\left(0, \sigma^2_{WN}\right)$
- Rate ramp (drift): $R_t = \omega t$

The observed process is therefore $Z_t = Y_t + X_t + R_t$ and we have that $Z_t \sim F_{\boldsymbol{\theta}}$ where $\boldsymbol{\theta} = (\sigma_{WN}, \sigma_{GM}, \beta, \omega)$

A small Example...



GMWM estimation results



GMWM estimation results

Estimated parameters:

	θ_0	$\hat{\theta}$	$\text{IC}(\theta_0, 0.95)$
σ^2_{WN}	1.00	1.00	$(0.99; 1.01)$
σ^2_{GM}	0.60	0.58	$(0.55; 0.61)$
β	10^{-2}	$1.07 \cdot 10^{-2}$	$(0.99 \cdot 10^{-2}; 1.12 \cdot 10^{-2})$
ω	$5 \cdot 10^{-5}$	$4.87 \cdot 10^{-5}$	$(4.67 \cdot 10^{-5}; 5.07 \cdot 10^{-5})$

Selecting the best model(s)

Goodness-of-Fit (GoF) test

The GMWM properties allow to perform a test for over-identifying restrictions (also known as "J-test" or "Goodness-of-Fit" (GoF)) based on the objective function $g(\boldsymbol{\theta}) \equiv (\hat{\boldsymbol{\nu}} - \boldsymbol{\nu}(\boldsymbol{\theta}))^T \boldsymbol{\Omega} (\hat{\boldsymbol{\nu}} - \boldsymbol{\nu}(\boldsymbol{\theta}))$.

The hypotheses tested are

$$H_0 : g(\boldsymbol{\theta}) = 0$$
$$H_1 : g(\boldsymbol{\theta}) \neq 0$$

Distribution of $g(\hat{\boldsymbol{\theta}})$

Under H_0, $g(\hat{\boldsymbol{\theta}}) \sim \chi^2_{J-p}$

Selecting the best model(s)

Wavelet Information Criterion (WIC)

To jointly evaluate the descriptive and predictive capacity of a model, *Guerrier et al, 2015* proposed the Wavelet Information Criterion (WIC):

$$WIC = \left(\hat{\boldsymbol{\nu}} - \boldsymbol{\nu}(\hat{\boldsymbol{\theta}})\right)^T \boldsymbol{\Omega} \left(\hat{\boldsymbol{\nu}} - \boldsymbol{\nu}(\hat{\boldsymbol{\theta}})\right) + 2\,\mathrm{tr}\left[\widehat{\mathrm{cov}}\left[\hat{\boldsymbol{\nu}}, \boldsymbol{\Omega}\boldsymbol{\nu}(\hat{\boldsymbol{\theta}})\right]^T\right].$$

The model(s) which minimize(s) this criterion can be considered as the best.

Extensions and Developments

The GMWM has many extensions:

- A **robust version** of the GMWM has been developed to reduce the impact of outliers on the estimation procedure and provides a *basis for Fault Detection and Isolation*.
- A **multivariate version** of the GMWM is being developed to take into account the *dependence between gyros and accelerometers* from the same IMU to improve estimation and hence navigation precision.
- A **non-stationary version** of the GMWM is being developed to address the problems linked to the estimation of IMU error signals in dynamic conditions.

Software

R package

- Supports the Classical and Robust forms of GMWM.
- Ability to **automatically select models**.
- **Computationally efficient.** Indeed, it has the ability to process large time series ($n < 10$ million) in under 2 minutes.
- Computational backend written using the **Armadillo C++ Linear Algebra Library**.
- The computational code is platform independent (e.g. compatible with MATLAB's C++ API).

Conclusions

The GMWM approach...

- is a statistically rigorous approach for sensor calibration
- is able to model signals of complex spectral structure (where classical methods fail!)
- is consistent (unlike AV) and asymptotically *normally distributed*
- can be computed (in some situations) using a computationally efficient algorithm (unlike EM) even with very large sample sizes (e.g. few hours of static IMU data)
- can be used to model any type stochastic process that can be simulated (and fulfill some regularity conditions)
- **considerably increases navigation accuracy**.

Any questions before the “fun” part?

Main References on the GMWM

Methodology-related:

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