

APS 2023: Session 1

# Modular Combinatorics and Exponentiation in CP

Chahel, Haricharan\_B, Aditya\_Jain\_\_

# QR for Whatsapp Group



# *Topics to be covered*

1. Modulo & its properties
2. Fermat's Little Theorem
3. Modular Multiplicative Inverse i.e. `Mod_Inv`
4. Computing `Mod_Inv`
5. Fast computation of  $a^b$
6. Computing large numbers modulo  $1e9+7$
7. Power Function
8. `Mod_Inv` Function
9. Computing  $n!$  modulo  $p$
10.  $nCr$  Function
11.  $nPr$  Function

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- For the rest of this session, we will only be dealing with  $a \% b$ , where  $a$  is non-negative and  $b$  is positive.



# Properties of Modulo Operator

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Why? Can you prove these 3 properties using  $a = nq + r$  and  $b = ns + t$ , where  $r = a \bmod n$  &  $t = b \bmod n$ ?

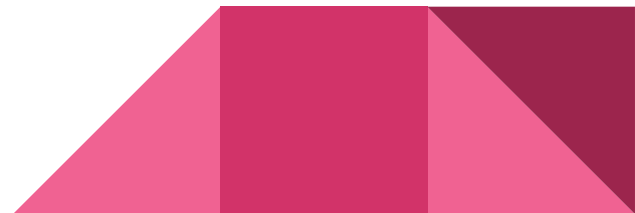


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4. From (3),  $a^b \bmod n = (a \bmod n)^b \bmod n$
- What about  $a/b \bmod n$ ? There is a conditional property to this, but it's not as simple as the previous ones. Before we can understand  $a/b \bmod n$ , there is a prerequisite theorem that should be understood.

# Fermat's Little Theorem

- If  $p$  is a prime number, then for any positive integer  $a$ ,

$$a^p \equiv a \pmod{p}$$

(Means  $a^p \% p = a \% p$ )



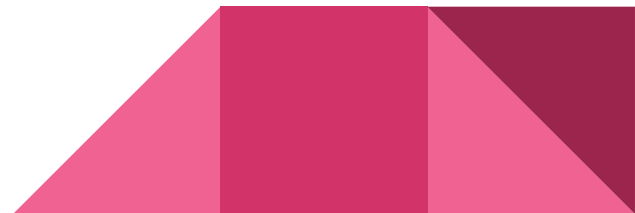
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As LHS is divisible by  $a$ , RHS must also be divisible by  $a$ . As  $p$  is not divisible by  $a$ ,  $q$  must be divisible by  $a$ . Let  $q = ka$ .

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$$\Rightarrow a^p - a = kap$$

$$\Rightarrow a^{p-1} - 1 = kp$$

This becomes:  $a^{p-1} \equiv 1 \pmod{p}$

( $a$  is not divisible by  $p$  &  $p$  is prime)

We will be using this form of the theorem in the upcoming discussions.



# Modular Multiplicative Inverse - Mod\_Inv

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*Yes, under certain terms and conditions!*
- **The modular multiplicative inverse of a number  $a$  (wrt modulo  $n$ ) is an integer  $x$  such that:**

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- Now that we have seen the definition, for what kind of pairs  $(a,n)$  is  $\text{Mod\_Inv}(a,n)$  defined?  
*If and only if  $a$  and  $n$  are coprime i.e.  $\text{GCD}(a,n) = 1$ . Why so?*

# Bezout's Identity

Let  $a$  and  $b$  be integers with greatest common divisor  $d$ . Then there exist integers  $x$  and  $y$  such that  $ax + by = d$ . More generally, the integers of the form  $ax + by$  are exactly the multiples of  $d$ .

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- If  $\text{GCD}(a,n) = g \neq 1$ , then all numbers of the form  $ax - ny$  will be divisible by  $g$ .  
For  $\text{Mod\_Inv}(a,n)$  to exist, we need to have  $ax - 1 = ny$  i.e.  $ax - ny = 1$ .  
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Can the Bezout's Identity be proved by me?

Yes! Can you think of an intuitive proof? I leave it as an exercise to the reader. :)

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You can think about it for some time. :)



# Application of Mod\_Inv

- What exactly is the application of Mod\_Inv for us now?
- Our initial aim was to find  $(a/b) \bmod n$  in terms of  $a$  and  $b$ .

Prerequisite:  $b$  and  $n$  must be coprime.



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If we know  $x$ , we know  $bx$ .

Multiply  $c$  on both sides of the modulo congruence. (Why can we do this?)

If  $a \% n = b \% n$ , then  $ac \% n = bc \% n$ .

(From Property (3))

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But  $cb = a \Rightarrow ax \equiv c \bmod n$ .





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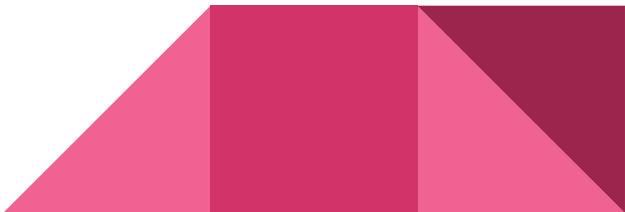
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We now have  $cbx \equiv c \bmod n$ .

But  $cb = a \Rightarrow ax \equiv c \bmod n$ .

Thus,  $(a/b) \% n = [a * \text{Mod\_Inv}(b,n)] \% n$ .

It looks like  $1/b$  got replaced with  $\text{Mod\_Inv}(b,n)$ .



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Fermat's Little Theorem: If  $b$  is not divisible by a prime  $p$ , then  $b^{p-1} \equiv 1 \pmod{p}$ .

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Hence,  $\text{Mod\_Inv}(b,p) = b^{p-2} \% p$ .

[One of the solutions]

Summarizing,  $(a/b) \% p = [a * \text{Mod\_Inv}(b,p)] \% p = [a * (b^{p-2} \% p)] \% p$

$$(a/b) \% p = [(a \% p) * (b^{p-2} \% p)] \% p$$

# How fast can you compute $\text{pow}(a,b)$ i.e. $a^b$ ?

Let us hypothetically say the computer can deal with numbers upto  $10^{(10^{10})}$ .

Also, in this set up, assume addition, subtraction, multiplication & division each take 1 second to do.

Around how many seconds will you take to find  $3^{(10^9)}$ ?

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I give you guarantee that I can do it within 1 minute!

But how can I do it in such short time?





# How am I so fast at multiplication? :)

I will compute  $3^{(2^i)}$  for  $i$  from 1 to  $j$ , where  $j$  is the least number such that  $2^j > 1e9$ .  
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Now I travel through the bits of  $1e9$ . If the  $i^{\text{th}}$  bit of  $1e9$  is 1, I multiply  $3^{(2^i)}$  to result.  
Else I don't.



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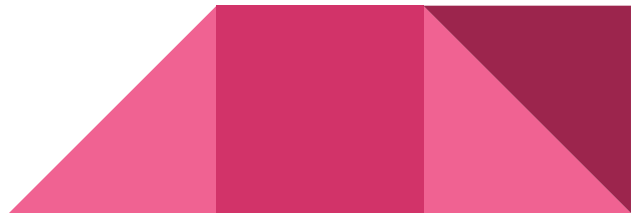
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Now the final result I will have will be  $3^{(\text{summation of } 2^i \text{ such that } i^{\text{th}} \text{ bit of } 1e9 \text{ is } 1)}$   
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Number of multiplications I did  $< 2 * \log_2(1e9) < 60$ .



# Code for the “fast” multiplier

```
typedef long long ll;
ll FastMultiplier(ll a,ll b) // Evaluates a^b
{
    ll result=1, a_pwr=a;
    while(b!=0)
    {
        if(b%2==1)
            result*=a_pwr;
        a_pwr=a_pwr*a_pwr;
        b/=2;
    }
    return result;
}
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Yes, we can.

We exploit property (3) of modulo:  $ab \bmod n = [(a \bmod n)(b \bmod n)] \bmod n$ .



# Power(a,b) - Computes $(a^b)$ modulo $1e9+7$

Time Complexity:  $O(\log b)$ .

```
typedef long long ll;
#define mod 1000000007
// Take a and b as input and returns : The power (a,b) , (a^b) % mod.
// mod need not be a prime OR coprime to b.
ll Power(ll a, ll b)
{
    ll result=1;
    ll a_pwr=a%mod;
    while(b)
    {
        if(b%2==1) result*=a_pwr;
        a_pwr*=a_pwr;
        a_pwr%=mod; // Take modulo everywhere
        result%=mod;
        b/=2;
    }
    return result;
}
```

Note that mod need not be coprime to b or be a prime number.

# Function to compute $\text{Mod\_Inv}(b,p)$ where $p$ is prime

Now that we have developed an efficient function to calculate  $(a^b) \% p$ , we can use this to find  $\text{Mod\_Inv}(b,p)$ .

$$\begin{aligned}\text{Mod\_Inv}(b,p) &= b^{p-2} \% p \\ &= \text{Power}(b,p-2) \text{ where mod}=p.\end{aligned}$$

```
typedef long long ll;
#define p 998244353
// It gives the modulo inverse of a, (1/a)%p.
ll Mod_Inv(ll a)
{
    a%=p;
    ll x= Power(a,p-2);
    return x;
}
```

Time Complexity:  $O(\log p)$

# Computing $n! \% p$ efficiently

How many digits do you think 300C100 has?



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We use the idea of  $\text{factorial}(n) = \text{factorial}(n-1) * n$  & memoization to make our computation process efficient.



# Computing $n! \% p$ efficiently

Time Complexity:  $O(\text{Fact\_Length})$

```
typedef long long ll;
#define mod 1000000007
#define Fact_length 200001

ll Factorial[Fact_length];

ll Make_Factorial()
{
    Factorial[0]=1;
    for(ll i=1;i<Fact_length;i++)
    {
        Factorial[i]=i*Factorial[i-1];
        Factorial[i]%=mod;
    }
    return 0;
}
```



# Computing $nCr \% p$

Given  $n$  and  $r$ , how would you compute  $nCr \% p$ ? (Given  $p$  is a prime)



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$$\text{Hence } nCr \text{ modulo } p = [ (n!,p) * (\text{Mod\_Inv}((n-r)!,p) * (\text{Mod\_Inv}(r!,p) ) \% p$$



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Lemma: If  $a \equiv b \pmod p$  ( $a \% p = b \% p = r$ ) &  $p$  is prime, then  $\text{Mod\_Inv}(a,p) = \text{Mod\_Inv}(b,p)$





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$$\text{Proof: } \text{Mod\_Inv}(a,p) = a^{p-2} \% p = ((a \% p)^{p-2}) \% p = (r^{p-2}) \% p$$

$$\text{Mod\_Inv}(b,p) = b^{p-2} \% p = ((b \% p)^{p-2}) \% p = (r^{p-2}) \% p$$

Hence, both are equal.



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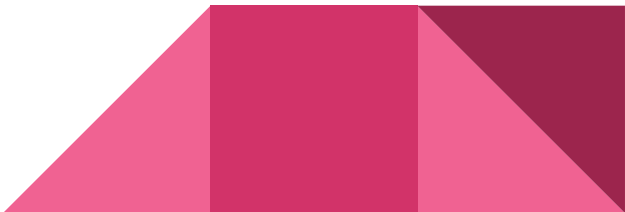
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Hence, both are equal.

Corollary:  $\text{Mod\_Inv}(a,p) = \text{Mod\_Inv}(a\%p,p)$



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Given  $n$  and  $r$ , how would you compute  $nCr \% p$ ? (Given  $p$  is a prime)

$$nCr = n! / (n-r)!r!$$

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Lemma: If  $p$  is prime, then  $\text{Mod\_Inv}(a,p) = \text{Mod\_Inv}(a\%p,p)$

Proof:  $\text{Mod\_Inv}(a,p) = a^{p-2} \% p = ((a\%p)^{p-2}) \% p = \text{Mod\_Inv}(a\%p,p)$

Hence, both are equal.

For this scenario,  $\text{Mod\_Inv}(r!,p) = \text{Mod\_Inv}(r!\%p,p) = \text{Mod\_Inv}(\text{Factorial}[r]);$

This makes it convenient because we can compute  $r! \% p$  easily. In fact, we have precomputed it.

$$\begin{aligned} \text{Hence } nCr \text{ modulo } p &= [ (n!\%p) * (\text{Mod\_Inv}((n-r)!,p) * (\text{Mod\_Inv}(r!,p) ] \% p \\ &= [ (n!\%p) * (\text{Mod\_Inv}((n-r)!\%p,p) * (\text{Mod\_Inv}(r!\%p,p) ] \% p \\ &= [ (\text{Factorial}[n],p) * (\text{Mod\_Inv}(\text{Factorial}[n-r],p)) * (\text{Mod\_Inv}(\text{Factorial}[r],p)) ] \% p \end{aligned}$$

# Computing $nCr \% p$

```
typedef long long ll;
#define mod 1000000007
ll nCr(ll n, ll r)
{
    if(n<r) return 0;
    ll ans=Factorial[n];
    ans%=mod;
    ans*=Mod_Inv(Factorial[r]);
    ans%=mod;
    ans*=Mod_Inv(Factorial[n-r]);
    ans%=mod;
    return ans;
}
```

Time Complexity:  $O(\log \text{mod})$



# Computing $nPr \% p$

$$nPr = n!/(n-r)!$$

$$\begin{aligned} nPr \% p &= [ (n!\%p) * (\text{Mod\_Inv}((n-r)!, p)) ] \% p \\ &= [ (n!\%p) * (\text{Mod\_Inv}((n-r)!\%p, p)) ] \% p \\ &= [ (\text{Factorial}[n], p) * (\text{Mod\_Inv}(\text{Factorial}[n-r], p)) ] \% p \end{aligned}$$

```
typedef long long ll;
#define mod 1000000007
ll nPr(ll n, ll r)
{
    ll ans=Factorial[n];
    ans%=mod;
    ans*=Mod_Inv(Factorial[n-r]);
    ans%=mod;
    return ans;
}
```

Time Complexity:  $O(\log \text{mod})$



## Feedback Form:



# Illustrative Problems

1. [Distributing Apples](#)
2. [Christmas Party](#)
3. [Check out this CF Blog!](#)

