Advanced Bayesian Learning
Approximate Methods
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Topic overview

- Variational Bayes (VB)
- Approximate Bayesian Computations (ABC)

Variational Bayes

- Let $\theta = (\theta_1, ..., \theta_M)$. Approximate the posterior $p(\theta|y)$ with a (simpler) distribution $q(\theta)$.
  - **Nonparametric/Factorization/Mean field approximation**

$$q(\theta) = \prod_{i=1}^{M} q_i(\theta_i)$$

  - **Parametric**, where $q_\lambda(\theta)$ is a parametric family with parameters λ.
- Find the $q(\theta)$ that **minimizes the Kullback-Leibler distance**:

$$KL(p, q) = \int p(\theta|y) \ln \frac{p(\theta|y)}{q(\theta)} d\theta = E_p \left[\ln \frac{p(\theta|y)}{q(\theta)} \right].$$

- Computing the expectation wrt $p(\theta|y)$ is often hard.
- **Reverse KL** problem is often simpler (but somewhat unnatural):

$$KL(q, p) = \int q(\theta) \ln \frac{q(\theta)}{p(\theta|y)} d\theta = E_q \left[\ln \frac{q(\theta)}{p(\theta|y)} \right].$$

VB gives a lower bound on $p(y)$

- Using that $p(\theta|\mathbf{y}) = p(\mathbf{y}, \theta)/p(\mathbf{y})$ we have

$$
\begin{aligned}
KL(q,p) &= \int q(\theta) \ln \frac{q(\theta)}{p(\theta|\mathbf{y})} d\theta \\
&= \int q(\theta) \ln \left(\frac{q(\theta)}{p(\mathbf{y},\theta)} \right) d\theta + \int q(\theta) \ln p(\mathbf{y}) d\theta \\
&= \int q(\theta) \ln \frac{q(\theta)}{p(\mathbf{y},\theta)} d\theta + \ln p(\mathbf{y})
\end{aligned}
$$

- Since $KL(q,p) \geq 0$, we have the following **lower bound** for $\ln p(\mathbf{y})$

$$
\ln p(\mathbf{y}) \geq - \int q(\theta) \ln \frac{q(\theta)}{p(\mathbf{y},\theta)} d\theta = \int q(\theta) \ln \frac{p(\mathbf{y},\theta)}{q(\theta)} d\theta \overset{def}{=} \ln \underline{p}(\mathbf{y};q),
$$

  where $p(\mathbf{y},\theta) = p(\mathbf{y}|\theta)p(\theta)$ is the unnormalized posterior.
- Minimizing $KL(q,p)$ is the same as maximizing $\ln \underline{p}(\mathbf{y};q)$.

Mean field approximation

- Factorization

$$q(\theta) = \prod_{i=1}^{M} q_i(\theta_i)$$

- **No functional forms are assumed** for the $q_i(\theta)$. Nonparametric.
- **Optimal densities** can be shown to satisfy:

$$q_i(\theta) \propto \exp\left(E_{-\theta_i} \ln p(\mathbf{y}, \theta)\right)$$

  where $E_{-\theta_i}(\cdot)$ is the expectation with respect to $\prod_{i \neq j} q_j(\theta_j)$.
- Alternative formulation that **connects to Gibbs sampling**

$$q_i(\theta) \propto \exp\left(E_{-\theta_i} \ln p(\theta_i|\mathrm{rest})\right)$$

  where $p(\theta_i|\mathrm{rest})$ is the full conditional posterior of θ_i.
- **Structured mean field approximation**. Group subset of parameters in tractable blocks.

Mean field approximation - algorithm

- Initialize: $q_2^*(\theta_2), \ldots, q_M^*(\theta_M)$
- Repeat until increase in $\ln \underline{p}(\mathbf{y}; q)$ is negligible:
  - $q_1^*(\theta_1) \leftarrow \frac{\exp\left[E_{-\theta_1} \ln p(\mathbf{y},\theta)\right]}{\int \exp\left[E_{-\theta_1} \ln p(\mathbf{y},\theta)\right] d\theta_1}$
  - $\vdots$
  - $q_M^*(\theta_M) \leftarrow \frac{\exp\left[E_{-\theta_M} \ln p(\mathbf{y},\theta)\right]}{\int \exp\left[E_{-\theta_M} \ln p(\mathbf{y},\theta)\right] d\theta_M}$
- Note: we make no assumptions about parametric form of the $q_i(\theta)$, but the optimal $q_i(\theta)$ often turn out to be parametric (normal, gamma etc).
- The updates above then boil down to just updating of hyperparameters in the optimal densities.

Mean field approximation - Normal model

- **Model**: $X_i|\mu,\sigma^2 \overset{iid}{\sim} N(\mu,\sigma^2)$.
- **Prior**: $\mu \sim N(\mu_\mu, \sigma_\mu^2)$ **independent** of $\sigma^2 \sim IG(A,B)$.
- Note: this is NOT the conjugate prior.
- **Variational approximation**: $q(\mu,\sigma^2) = q_\mu(\mu)\cdot q_{\sigma^2}(\sigma^2)$.
- Optimal densities

$$q_\mu^*(\mu) \propto \exp\left[E_{q(\sigma^2)} \ln p(\mu|\sigma^2,\mathbf{x})\right]$$
$$q_{\sigma^2}^*(\sigma^2) \propto \exp\left[E_{q(\mu)} \ln p(\sigma^2|\mu,\mathbf{x})\right]$$

- Full conditional posteriors

$$\mu|\sigma^2,\mathbf{x} \sim N\left(\frac{n\bar{x}/\sigma^2 + \mu_\mu/\sigma_\mu^2}{n/\sigma^2 + 1/\sigma_\mu^2}, \frac{1}{n/\sigma^2 + 1/\sigma_\mu^2}\right)$$
$$\sigma^2|\mu,\mathbf{x} \sim IG\left(A + \frac{n}{2}, B + \frac{1}{2}(\mathbf{x}-\mu)'(\mathbf{x}-\mu)\right)$$

Normal model example - Updating $q^*_{\sigma^2}(\sigma^2)$

- Full conditional posterior of σ^2

$$\sigma^2|\mu,\mathbf{x} \sim IG\left(A+\frac{n}{2}, B+\frac{1}{2}(\mathbf{x}-\mu)'(\mathbf{x}-\mu)\right)$$

- So, $E_{q(\mu)} \ln p(\sigma^2|\mu,\mathbf{x})$ is proportional to

$$E_{q(\mu)}\left[-\left(A+\frac{n}{2}+1\right)\ln\sigma^2 - \left(B+\frac{1}{2}(\mathbf{x}-\mu)'(\mathbf{x}-\mu)\right)/\sigma^2\right]$$

  and therefore

$$q^*_{\sigma^2}(\sigma^2) \propto \left(\sigma^2\right)^{-(A+n/2+1)}\exp\left(-\left(B+\frac{1}{2}E_{q(\mu)}(\mathbf{x}-\mu)'(\mathbf{x}-\mu)/\sigma^2\right)\right)$$

  which shows that $q^*_{\sigma^2}(\sigma^2)$ is

$$IG\left(A+\frac{n}{2}, B+\frac{1}{2}E_{q(\mu)}(\mathbf{x}-\mu)'(\mathbf{x}-\mu)\right)$$

Normal model example - Updating $q^*_{\sigma^2}(\sigma^2)$

- So $q^*_{\sigma^2}(\sigma^2)$ is

$$IG\left(A + \frac{n}{2}, B + \frac{1}{2}E_{q(\mu)}(\mathbf{x}-\mu)'(\mathbf{x}-\mu)\right)$$

  and

$$E_{q(\mu)}(\mathbf{x}-\mu)'(\mathbf{x}-\mu) =$$

$$E_{q(\mu)}\left[\left(\mathbf{x}-E_{q(\mu)}(\mu)\right) + \left(E_{q(\mu)}(\mu)-\mu\right)\mathbf{1}_n\right]'\left[\left(\mathbf{x}-E_{q(\mu)}(\mu)\right) + \left(E_{q(\mu}\right.\right.$$

$$= E_{q(\mu)}\left[\left(\mathbf{x}-E_{q(\mu)}(\mu)\right)'\left(\mathbf{x}-E_{q(\mu)}(\mu)\right) + \left(E_{q(\mu)}(\mu)-\mu\right)^2 n\right]$$

$$= \left(\mathbf{x}-E_{q(\mu)}(\mu)\right)'\left(\mathbf{x}-E_{q(\mu)}(\mu)\right) + n \cdot Var_{q(\mu)}(\mu)$$

  because $E_{q(\mu)}\left(E_{q(\mu)}(\mu)-\mu\right) = 0$.
- Important: $E_{q(\mu)}(\mu)$ and $Var_{q(\mu)}(\mu)$ is the mean and variance of the q_μ distribution.

Normal model example - Updating $q^*_\mu(\mu)$

- Easier to go back to the form $q_\mu(\mu) \propto \exp\left(E_{q(\sigma^2)} \ln p(\mathbf{y}, \mu, \sigma^2)\right)$

$$\ln p(\mathbf{y}, \mu, \sigma^2) = \ln p(\mathbf{y}|\mu, \sigma^2) + \ln p(\mu) + \ln p(\sigma^2)$$
$$= -\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i - \mu)^2 - \frac{1}{2\sigma^2_\mu}(\mu - \mu_\mu)^2 + const$$

$$q_\mu(\mu) \propto \exp\left(-E_{q(\sigma^2)}\left(\frac{1}{\sigma^2}\right)\frac{1}{2}\sum_{i=1}^{n}(x_i - \mu)^2 - \frac{1}{2\sigma^2_\mu}(\mu - \mu_\mu)^2\right)$$

- Completing the square shows

$$q_\mu(\mu) = N\left(E_{q(\mu)}(\mu), Var_{q(\mu)}(\mu)\right)$$

where

$$E_\mu(\mu) = \frac{n\bar{x}E_{q(\sigma^2)}\left(\frac{1}{\sigma^2}\right) + \mu_\mu/\sigma^2_\mu}{nE_{q(\sigma^2)}\left(\frac{1}{\sigma^2}\right) + 1/\sigma^2_\mu}$$

$$Var_\mu(\mu) = \frac{1}{nE_{q(\sigma^2)}\left(\frac{1}{\sigma^2}\right) + 1/\sigma^2_\mu}$$

Normal model example - summary

- Variational density for σ^2

$$q_{\sigma^2}(\sigma^2) = IG\left(A_q, B_q\right)$$

where $A_q = A + n/2$ and $B_q = B + \frac{1}{2}\left(\|\mathbf{x} - \mu_q \cdot \mathbf{1}_n\|^2 + n \cdot \sigma_q^2\right)$

- Variational density for μ

$$q_\mu(\mu) = N\left(\mu_q, \sigma_q^2\right)$$

where

$$\sigma_q^2 = \frac{1}{n\frac{A_q}{B_q} + 1/\sigma_\mu^2}$$

$$\mu_q = \left(n\bar{x}\frac{A_q}{B_q} + \mu_\mu/\sigma_\mu^2\right)\sigma_q^2$$

Normal model example - algorithm

- Set $A_q = A + n/2$.
- Initialize $E_\mu(\mu) = \bar{x}$ and $Var_\mu(\mu) = s^2/n$.
- Repeat
  - $$B_q \leftarrow B + \frac{1}{2}\left(\|\mathbf{x} - \mu_q \cdot \mathbf{1}_n\|^2 + n \cdot \sigma_q^2\right)$$
  - $$\sigma_q^2 \leftarrow \frac{1}{n\frac{A_q}{B_q} + 1/\sigma_\mu^2}$$
  - $$\mu_q \leftarrow \left(n\bar{x}\frac{A_q}{B_q} + \mu_\mu/\sigma_\mu^2\right)\sigma_q^2$$
- Until the change in

$$\ln \underline{p}(\mathbf{x}; q) = \frac{1}{2} - \frac{n}{2}\log(2\pi) + \frac{1}{2}\ln\left(\sigma_q^2/\sigma_\mu^2\right) - \frac{(\mu_q - \mu_\mu)^2 + \sigma_q^2}{2\sigma_\mu^2}$$

is negligible.

Probit regression

- **Model**:
$$Y_i | x_i \overset{ind.}{\sim} Bern\left[\Phi(x_i'\beta)\right]$$
- **Prior**: $\beta \sim N(\mu_\beta, \Sigma_\beta)$
- **Latent variable formulation** with $\mathbf{a} = (a_1, ..., a_n)'$
$$\mathbf{a}|\beta \sim N(X\beta, 1)$$
and
$$p(y_i|a_i) = I(a_i \geq 0)^{y_i} I(a_i < 0)^{1-y_i}$$
- Factorized **variational approximation**
$$q(\mathbf{a}, \beta) = q_{\mathbf{a}}(\mathbf{a}) q_\beta(\beta)$$

Probit regression - updating A

- **Log joint distribution**

$$\ln p(y,\beta,\mathbf{a}) = \ln p(y|\beta,\mathbf{a}) + \ln p(\mathbf{a}|\beta) + \ln p(\beta)$$
$$\propto \sum_{i=1}^{n} \left(y_i \ln I(a_i \geq 0) + (1-y_i)\ln I(a_i < 0) \right)$$
$$- \frac{1}{2}(\mathbf{a}-\mathbf{X}\beta)'(\mathbf{a}-\mathbf{X}\beta) - \frac{1}{2}(\beta-\mu_\beta)'\Sigma_\beta(\beta-\mu_\beta)$$

- Updating **a**

$$\ln q_{\mathbf{a}}(\mathbf{a}) \propto E_{q(\beta)} \ln p(y,\beta,\mathbf{a}) \propto \sum_{i=1}^{n} \left(y_i \ln I(a_i \geq 0) + (1-y_i)\ln I(a_i < 0) \right)$$
$$- \frac{1}{2} E_{q(\beta)}(\mathbf{a}-\mathbf{X}\beta)'(\mathbf{a}-\mathbf{X}\beta) + const$$

Note that

$$E_{q(\beta)}(\mathbf{a}-\mathbf{X}\beta)'(\mathbf{a}-\mathbf{X}\beta) = \mathbf{a}'\mathbf{a} - 2\mathbf{a}'\mathbf{X}E_{q(\beta)}(\beta) + const$$
$$= (\mathbf{a}-\mathbf{X}E_{q(\beta)}(\beta))'(\mathbf{a}-\mathbf{X}E_{q(\beta)}(\beta)) + const$$

PROBIT REGRESSION - UPDATING A

- So

$$q_{\mathbf{a}}(\mathbf{a}) \propto \prod_{i=1}^{n} I(a_i \geq 0)^{y_i} I(a_i < 0)^{1-y_i}$$
$$\times \exp\left(-\frac{1}{2}(\mathbf{a} - \mathbf{X}\mu_{q(\beta)})'(\mathbf{a} - \mathbf{X}\mu_{q(\beta)})\right)$$

- Normalizing gives

$$q_{\mathbf{a}}(\mathbf{a}) = \prod_{i=1}^{n} \left[\frac{I(a_i \geq 0)}{\Phi\left((\mathbf{X}\mu_{q(\beta)})_i\right)}\right]^{y_i} \left[\frac{I(a_i < 0)}{1 - \Phi\left((\mathbf{X}\mu_{q(\beta)})_i\right)}\right]^{1-y_i}$$
$$\times (2\pi)^{-n/2} \exp\left(-\frac{1}{2}\left\|\mathbf{a} - \mathbf{X}\mu_{q(\beta)}\right\|^2\right)$$

which is a product of n truncated normals.

Probit regression - updating β

- Updating β

$$\begin{aligned}\ln q_\beta(\beta) &\propto E_{q(\mathbf{a})} \ln p(y, \beta, \mathbf{a}) \\ &\propto -\frac{1}{2} E_{q(\mathbf{a})} (\mathbf{a} - \mathbf{X}\beta)'(\mathbf{a} - \mathbf{X}\beta) - \frac{1}{2}(\beta - \mu_\beta)'\Sigma_\beta(\beta - \mu_\beta)\end{aligned}$$

- For any random vector $\mathbf{y}$ with mean μ and covariance matrix Ω

$$E(\mathbf{y} - \mathbf{m})'(\mathbf{y} - \mathbf{m}) = trace(\Omega) + (\mu - \mathbf{m})'(\mu - \mathbf{m})$$

- So

$$\begin{aligned}\ln q_\beta(\beta) \propto -\frac{1}{2} trace(\Sigma_{\mathbf{a}}) - \frac{1}{2}(\mu_{q(\mathbf{a})} - \mathbf{X}\beta)'(\mu_{q(\mathbf{a})} - \mathbf{X}\beta) \\ - \frac{1}{2}(\beta - \mu_\beta)'\Sigma_\beta(\beta - \mu_\beta)\end{aligned}$$

- The variational approximation of β is like the posterior from regressing $\mu_{q(\mathbf{a})}$ on $\mathbf{X}$ with prior $\beta \sim N(\mu_\beta, \Sigma_\beta)$.

Probit regression - updating β

- We therefore have

$$q_\beta(\beta) = N\left(\mu_{q(\beta)}, \left(\mathbf{X}'\mathbf{X} + \Sigma_\beta^{-1}\right)^{-1}\right)$$

and

$$\mu_{q(\beta)} = \left(\mathbf{X}'\mathbf{X} + \Sigma_\beta^{-1}\right)^{-1}\left(\mathbf{X}'\mu_{q(\mathbf{a})} + \Sigma_\beta^{-1}\mu_\beta\right)$$

where

$$\mu_{q(\mathbf{a})} = X\mu_{q(\beta)} + \frac{\phi\left(X\mu_{q(\beta)}\right)}{\Phi\left(X\mu_{q(\beta)}\right)^{\mathbf{y}}\left[\Phi\left(X\mu_{q(\beta)}\right) - \mathbf{1}_n\right]^{\mathbf{1_n}-\mathbf{y}}}$$

which follows from the expected value formula for a truncated distribution.

- The lower bound $\ln \underline{p}(\mathbf{y}; q)$ is given in Ormerod and Wand (2010) where also the complete algorithm in given as Algorithm 4.

Poisson regression

- **Model**:

$$Y_i | x_i, \beta \overset{ind.}{\sim} Poisson\left(x_i'\beta\right)$$

- **Prior**: $\beta \sim N(\mu_\beta, \Sigma_\beta)$
- Posterior is intractable.
- Assume **variational approximation** to be Gaussian

$$q_\beta(\beta) = N(\mu_q, \Sigma_q)$$

- Use Newton-Raphson to find μ_q, Σ_q that maximizes the (tractable) lower bound $\ln \underline{p}(\mathbf{y}; \mu_q, \Sigma_q)$. See Ormerod and Wand (2010).

Variational Bayes EM (VBEM)

- Latent variable models:
  - Model parameters, $\theta = (\theta_1, ..., \theta_p)$
  - Latent variables, $\mathbf{z} = (z_1, ..., z_n)$
- Examples:
  - Mixture models: $z_i \in \{1, ..., K\}$ is the mixture allocation for the ith observation.
  - Missing data: $\mathbf{z}$ contains the missing values.
- EM:
  - Get rid of $\mathbf{z}$ by computing the expected log-likelihood $E_{\mathbf{z}|\theta} \ln L(\theta, \mathbf{z})$ (E-step)
  - Maximize $E_{\mathbf{z}|\theta} \ln L(\theta, \mathbf{z})$ wrt to θ (M-step)
- VBEM approximation of posterior

$$p(\theta, \mathbf{z}|\mathbf{y}) \approx q(\theta)q(\mathbf{z}) = q(\theta)\prod_{i=1}^{n} q(z_i)$$

- Improves on EM by modelling the uncertainty in θ.

Indirect use of VB

- VB can play a role as **initial values for MCMC**.
- **Variational MCMC**. Use VB to construct Metropolis-Hastings proposal. Need to combine it with Metropolis random walk moves since VB typically underestimates the posterior variance.
- VB is fast and accurate for computing **log predictive scores (LPS)**

$$\sum_{t=T+1}^{T^*} \ln p(y_t|y_{t-1}^H) = \sum_{t=T+1}^{T^*} \int \ln p(y_t|y_{t-1}^H, \theta) p(\theta|y_{t-1}^H) d\theta$$

- VB is **fast** in this setting since it can approximate each sequential posterior $p(\theta|y_{t-1}^H)$ using the mode of $\hat{p}(\theta|y_{t-2}^H)$ as excellent initial values.
- VB seems **accurate** for approximating LPS, at least when the prediction uncertainty is mainly dominated by the future error uncertainty and not by parameter uncertainty.

Approximate Bayesian Computations (ABC)

- Suitable when the likelihood is very costly or even infeasible to compute, but simulation from the model is cheap.
- Examples:
  - Likelihood is given by an **intractable high-dimensional integral**

$$\ell(\theta|\mathbf{y}) = \int \ell^*(\theta|\mathbf{y}, \mathbf{u}) d\mathbf{u}.$$

  - **Normalizing constant Z_θ is costly** or intractable

$$\ell(\theta|\mathbf{y}) = \ell_1(\theta|\mathbf{y}) / Z_\theta$$

  - Likelihood is unavailable because the **PDF does not exist in closed form**. α-stable distributions.
- ABC is often very crude.
- Arbitrary (creative) choices needed when implementing it.
- Early days, likely to improve.

Likelihood-free rejection sampler 1

- Idea: θ's with large posterior should generate data $\mathbf{z}$ that look like the actual data $\mathbf{y}$.
- Assume the data $\mathbf{y}$ takes values in a finite or countable set $\mathcal{D}$.
- **for** $i = 1$ to N do
  - **repeat**
    - Generate θ' from the prior distribution $\pi(\cdot)$
    - Generate $\mathbf{z}$ from the data distribution $f(\cdot|\theta')$
  - **until** $\mathbf{z} = \mathbf{y}$
  - set $\theta_i = \theta'$
- **end for**
- Algorithm 1 produces a sample $\theta_1, ..., \theta_N$ from the posterior $\pi(\theta|\mathbf{y})$:

$$f(\theta_i) \propto \sum_{\mathbf{z}\in\mathcal{D}} \pi(\theta_i) f(\mathbf{z}|\theta_i) \mathbb{I}_{\mathbf{y}}(\mathbf{z}) = \pi(\theta_i) f(\mathbf{y}|\theta_i) \propto \pi(\theta_i|\mathbf{y}).$$

Likelihood-free rejection sampler 2

- Extension to continuous sample spaces where $Pr(\mathbf{z} = \mathbf{y}|\theta) = 0$.
- Define summary statistics $\eta(\mathbf{z})$ and a distance function $\rho\left[\eta(\mathbf{z}), \eta(\mathbf{y})\right]$.
- **for** $i = 1$ to N do
  - **repeat**
    - Generate θ' from the prior distribution $\pi(\cdot)$
    - Generate $\mathbf{z}$ from the data distribution $f(\cdot|\theta')$
  - **until** $\rho\left[\eta(\mathbf{z}), \eta(\mathbf{y})\right] \leq \varepsilon$
  - set $\theta_i = \theta'$
- **end for**
- **Algorithmic choices**:
  - η - a function on $\mathcal{D}$ defining a summary statistic (close to sufficient)
  - $\rho > 0$, a distance on $\eta(\mathcal{D})$
  - $\varepsilon > 0$, a tolerance level.

Likelihood-free rejection sampler 2, cont.

- The algorithm samples from the joint distribution

$$\pi_\varepsilon(\theta, \mathbf{z}|\mathbf{y}) = \frac{\pi(\theta) f(\mathbf{z}|\theta) \mathbb{I}_{A_{\varepsilon,\mathbf{y}}}(\mathbf{z})}{\int_{A_{\varepsilon,\mathbf{y}} \times \theta} \pi(\theta) f(\mathbf{z}|\theta) d\mathbf{z} d\theta}$$

  where

$$A_{\varepsilon,\mathbf{y}} = \{\mathbf{z} \in \mathcal{D} | \rho\left[\eta(\mathbf{z}), \eta(\mathbf{y})\right] \leq \varepsilon\}$$

- The hope is that

$$\pi(\theta|\mathbf{y}) \approx \pi_\varepsilon(\theta|\mathbf{y}) = \int \pi_\varepsilon(\theta, \mathbf{z}|\mathbf{y}) d\mathbf{z}$$

  is a good approximation.
- "*The basic idea behind ABC is that using a representative (enough) summary statistic η coupled with a small (enough) tolerance ε should produce a good (enough) approximation to the posterior distribution*" (Marin et al, 2012).

ABC - AN EXAMPLE

- $MA(q)$ model. Fairly complicated likelihood. Easy to simulate time series from $MA(q)$.

- Summary statistics:
  - Raw distance between time series: $\rho\left[(z_1, ..., z_n), (y_1, ..., y_n)\right] = \sqrt{\sum_{i=1}^{n}(y_i - z_i)^2}$
  - Distance between estimated autocorrelation functions: $\sum_{j=1}^{K}(\tau_{y,j} - \tau_{z,j})^2$.

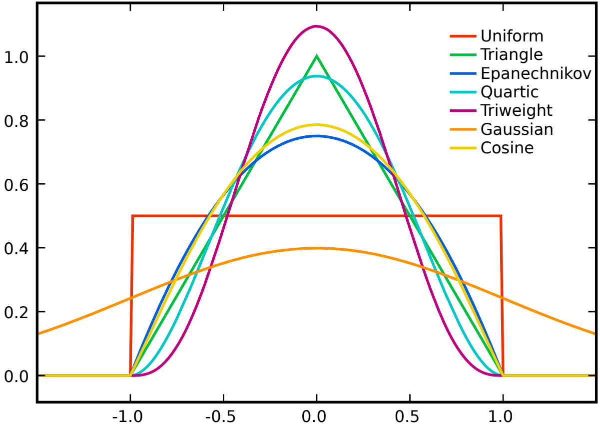
MCMC - ABC

- Likelihood-free rejection sampler 2 is inefficient since it proposes θ's from the prior $\pi(\theta)$, which is often far from the posterior (with informative data).
- Initialize $(\theta^{(0)}, \mathbf{z}^{(0)})$
- **for** $i = 1$ to N do
  - Propose θ' from the Markov kernel $q\left(\cdot|\theta^{(t-1)}\right)$
  - Generate $\mathbf{z}'$ from the data distribution $f(\cdot|\theta')$
  - Generate u from $\mathcal{U}_{[0,1]}$
  - **if** $u \leq \frac{\pi(\theta')q\left(\theta^{(t-1)}|\theta'\right)}{\pi\left(\theta^{(t-1)}\right)q\left(\theta'|\theta^{(t-1)}\right)}$ and $\rho\left[\eta(\mathbf{z}'), \eta(\mathbf{y})\right] \leq \varepsilon$ **then**
    - set $(\theta^{(t)}, \mathbf{z}^{(t)}) = (\theta', \mathbf{z}')$
  - **else**
    - set $(\theta^{(t)}, \mathbf{z}^{(t)}) = (\theta^{(t-1)}, \mathbf{z}^{(t-1)})$
  - **end if**
- **end for**

Noisy ABC

- Replacing the crude $\rho\left[\eta(\mathbf{z}),\eta(\mathbf{y})\right]\leq\varepsilon$ rejection rule with a smoother version
$$\pi_\varepsilon(\theta,\mathbf{z}|\mathbf{y})=\frac{\pi(\theta)f(\mathbf{z}|\theta)K_\varepsilon(\mathbf{y}-\mathbf{z})}{\int\pi(\theta)f(\mathbf{z}|\theta)K_\varepsilon(\mathbf{y}-\mathbf{z})dzd\theta}$$
- $K_\varepsilon(\cdot)$ is a kernel (think normal density) parametrized by the bandwidth $\varepsilon>0$ (think variance).
- The Bayes estimator from $\pi_\varepsilon(\theta|\mathbf{y})$ is converging to the true value when $n\to\infty$ and $\varepsilon\to 0$.
- See Wilkinson (2008) for details about the algorithm.

Some common kernels



Choosing the algoritmic settings in ABC

- Summary statistics $\eta(\cdot)$ should be nearly sufficient. Which ones? Creativity ...
- Choice of $\eta(\cdot)$ is crucial.
- Choice of ε is less important. Smaller ε gives better approximation at higher computational cost.
- Common choice of ε: small (0.1% or so) percentile of simulated distances $\rho\left[\eta(\mathbf{z}'),\eta(\mathbf{y})\right]$.

Post-processing of ABC output

- Same algorithms, but allowing for larger ε by post-processing the ABC output.
- Keep **all** θ draws regardless of how far $\mathbf{z}$ is from the actual data $\mathbf{y}$, but shrink the θ draws using

$$\theta* = \theta - (\eta(\mathbf{z}) - \eta(\mathbf{y}))' \hat{\beta}$$

- $\hat{\beta}$ is obtained from a local kernel regression of θ on $\rho\left[\eta(\mathbf{z}), \eta(\mathbf{y})\right]$ with weights given by the kernel

$$K_\delta \left\{\rho\left[\eta(\mathbf{z}), \eta(\mathbf{y})\right]\right\}.$$

- Kernel bandwidth δ can for example be set equal to ABC tolerance ε.
- Alternative: heteroscedastic nonlinear regression.

ABC for model choice

- You are entertaining a **set of M different** (competing, possibly non-nested) **models**.
- Let $\mathcal{M}$ denote the unknown true model, and let $\pi(\mathcal{M} = m)$ denote the **prior distribution** over the **model space**.
- The **Bayesian solution for model inference** is the posterior distribution: $\pi(\mathcal{M} = m|\mathbf{y})$.
- **ABC solution**: include $\mathcal{M}$ in the set of parameters.
- Let $\eta(\mathbf{z}) = (\eta_1(\mathbf{z}), ..., \eta_M(\mathbf{z}))$ be the concatenation of the **summary statistics** used for all models.

ABC algorithm for model choice

- **for** $i = 1$ to N **do**
  - **repeat**
    - Generate m from the prior $\pi(\mathcal{M} = m)$
    - Generate θ_m from the prior $\pi_m(\theta_m)$
    - Generate $\mathbf{z}$ from the data distribution $f_m(\cdot|\theta_m)$
  - **until** $\rho\left[\eta(\mathbf{z}), \eta(\mathbf{y})\right] \leq \varepsilon$
  - Set $m^{(i)} = m$ and $\theta^{(i)} = \theta_m$
- **end for**

- **ABC estimate**

$$\pi(\mathcal{M} = m|\mathbf{y}) \approx \frac{1}{N}\sum_{i=1}^{N} \mathbb{I}_{m^{(i)}=m}$$

- Example - comparing:
  - AR-processes (homoscedastic variance)
  - GARCH models (volatility clustering)