CFRM 460: Mathematical Methods for Quantitative Finance Homework 2

Solve the exercises by hand and verify your answers using Mathematica.

1. Let K, T, σ, and r be positive constants and let

$$g(x) = \frac{1}{\sqrt{2\pi}} \int_0^{b(x)} e^{-\frac{y^2}{2}}\, dy$$

where $b(x) = \frac{1}{\sigma\sqrt{T}}\left[\log\left(\frac{x}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)T\right]$. Compute $g'(x)$.

(a) Recall that $\frac{d}{dx}\left[\int_{a(x)}^{b(x)} f(t)dt\right]$ = f(b(x))b'(x) - f(a(x))a'(x)

a(x) = 0 a'(x) = 0

$b(x) = \frac{1}{\sigma\sqrt{T}}\left[\log\left(\frac{x}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)T\right]$
Eliminate constants (anything not x)

$$b'(x) = \frac{1}{\sigma\sqrt{t}} * \log x - \log k$$

$$b'(x) = \frac{1}{\sigma\sqrt{t}} * \frac{d}{dx}\left[\log x\right]$$

$$b'(x) = \frac{1}{x\sigma\sqrt{t}}$$

Plug into a

$$\text{g'(x)} = \frac{1}{\sqrt{2\pi}} e^{-\frac{\left(\frac{1}{\sigma\sqrt{T}}\left[\log\left(\frac{x}{K}\right)+\left(r+\frac{\sigma^2}{2}\right)T\right]\right)^2}{2}} \frac{1}{x\sigma\sqrt{t}}$$

Mathematica Code : D[1/(σ*$\sqrt{T}$)*(Log[x/K] + (r + σ^2/2)*T), x]

2. Let $\phi(u) = \frac{1}{\sqrt{2\pi}}e^{-u^2/2}$ so that $\Phi(x) = \int_{-\infty}^{x} \phi(u)\, du$ (i.e., the $\Phi(x)$ in Black-Scholes).

(a) For $x > 0$, show that $\phi(-x) = \phi(x)$.

$$\phi(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2} = \phi(-x) = \frac{1}{\sqrt{2\pi}}e^{-(-x^2)/2} \implies e^{-x^2/2} == e^{-(-x^2)/2}$$

Example :

$$e^{-2^2/2} = e^{-(-2^2)/2} = e^{-4/2} = e^{-2}$$

$\implies$ that phi(x) = phi(-x) because the only change is u which is squared. The rest of the equation is a constant.

Mathematica Code: $f[u] = \frac{1}{\sqrt{2\Pi}} E^{-}(u^2/2)$
$f[x] == f[-x]$

(b) Given that $\lim_{x\to\infty} \Phi(x) = 1$, use the properties of the integral as well as a substitution to show that $\Phi(-x) = 1 - \Phi(x)$ (again, assuming $x > 0$).
We are going to use the additive property and the observed symmetry for our solution
We know that $\phi(x) = \phi(-x)$ and $\Phi(x) = 1 \implies \int_{-\infty}^{-x} \phi(u)du$

$$\implies \int_{-\infty}^{-x} \phi(u)du - \int_{-x}^{\infty} \phi(u)du$$

$$\implies \int_{-\infty}^{-x} \phi(u)du = 1 - \int_{-x}^{\infty} \phi(u)du \implies \int_{-\infty}^{-x} \phi(u)du = 1 + \int_{-x}^{\infty} \phi(u)du$$

Because of the observed symmetry we know that

$$= \int_{-x}^{\infty} \phi(u)du \implies \int_{-\infty}^{-x} \phi(u)du \implies \int_{-\infty}^{x} \phi(u)du = \Phi(x)$$

Therefore:

$$\Phi(-x) = 1 - \Phi(x)$$

3. (a) Under what condition does the following hold?

$$\iint_D f(x,y)\,dA = \iint_D f(x,y)\,dy\,dx = \iint_D f(x,y)\,dx\,dy$$

Fubini's theorem holds when:

$$\iint_D |f(x,y)|\,dx\,dy < \infty$$

The domain D is bounded and convex for any two points x_1 and x_2 all the points on the segment joining x_1 and x_2 are in D as well. Also assume that there exists two continuous functions such that D can be described as:

$D = \{(x,y) : a \le x \le b,\ \ f_1(x) \le y \le f_2(x)\}$

(1.1)

$$\iint_D f(x,y)\,dA = \iint_D f(x,y)\,dy\,dx = \int_a^b \Big(\int_{f_1(x)}^{f_2(x)} f(x,y)\,dy\Big)\,dx$$

If there exists two continuous functions $g_1(y)$ and $g_2(y)$ such that $D = \{(x, y) : c \leq y \leq d,\ \ g_1(y) \leq x \leq g_2(y)\}$ then by definition,

(1.2)

$$\iint_D f(x,y)\,dA = \iint_D f(x,y)\,dx\,dy = \int_c^d \Big(\int_{g_1(y)}^{g_2(y)} f(x,y)\,dx\Big)\,dy$$

If f(x,y) is continuous then by definition, 1.1 is equal to 1.2 and the double integrals over D are equal to each other therefore the order of integration does not matter and

$$\iint_D f(x,y)\,dA = \iint_D f(x,y)\,dy\,dx = \iint_D f(x,y)\,dx\,dy$$

(b) Evaluate the double integral

$$\iint_D e^{y^2}\,dA$$

where $D = \{(x, y) : 0 \leq y \leq 1,\ \ 0 \leq x \leq y\}$.

$$\Longrightarrow \int_0^1 \int_0^y e^{y^2} dxdy \Longrightarrow \int_0^1 ye^{y^2} dy \Longrightarrow \frac{e}{2} - \frac{1}{2}$$

Mathematica code Integrate[Integrate[E^{y^2}, x, 0, y], y, 0, 1]

4. (a) Transform the double integral

$$\iint_D e^{\frac{x+y}{x-y}}\,dA$$

into an integral of u and v using the change of variables

$$u = x + y \qquad\qquad v = x - y$$

and call the domain in the uv plane S. Solve for x and y in terms of u and v

$$\frac{u+v}{2} = x \qquad \frac{u-v}{2} = y$$

Plug into Jacobian

$$-\frac{1}{2} * \frac{1}{2} - \frac{1}{2} * \frac{1}{2} \quad \Longrightarrow |-\frac{1}{2}| \Longrightarrow \iint_D \frac{1}{2} e^{\frac{u}{v}} dudv$$

Mathematica Code:

$$du/dx = D[(u+v)/2, u] \qquad dv/dx = D[(u+v)/2, v]$$

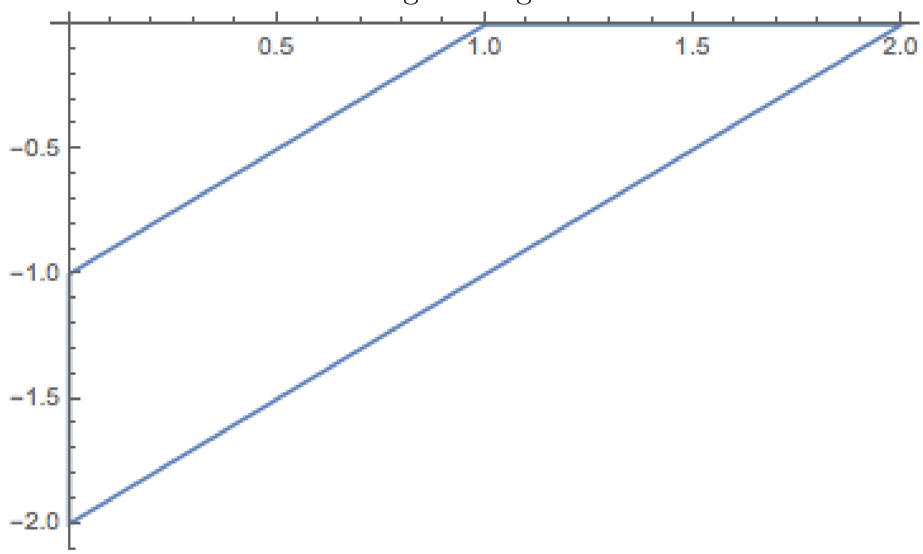
$$du/dy = D[(u-v)/2, u] \qquad dv/dy = D[(u-v)/2, v]$$

(b) Let D be the trapezoidal region with vertices $(1,0)$, $(2,0)$, $(0,-2)$ and $(0,-1)$. Find the corresponding region S in the uv plane by evaluating the transformation at the vertices of D and connecting the dots. Sketch both regions.
Plug (x,y) coordinates into u and v

$$(1,0)_{xy} = (1,1)_{uv} \qquad (2,0)_{xy} = (2,2)_{uv}$$

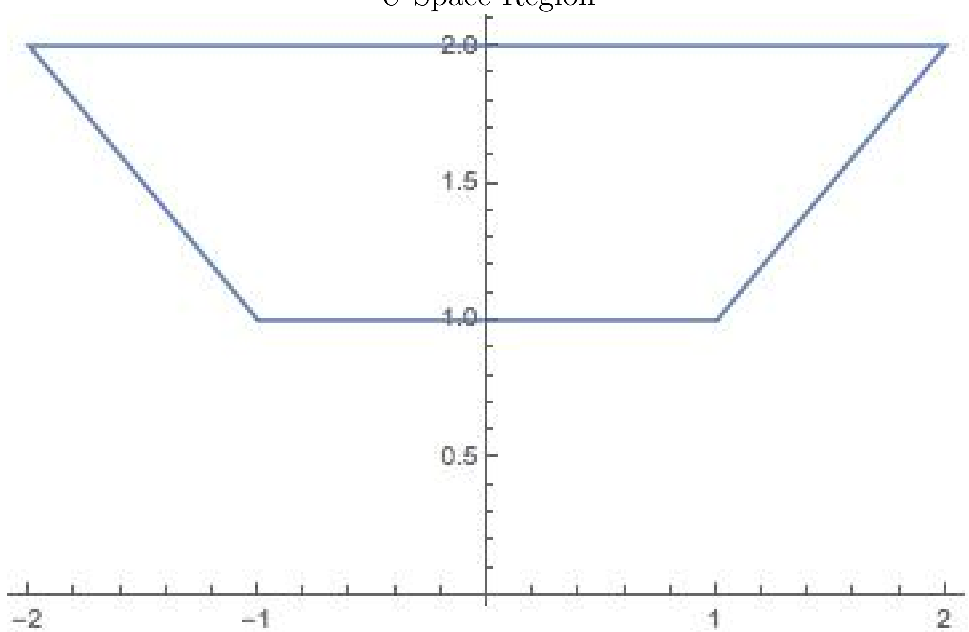
$$(0,-2)_{xy} = (-2,2)_{uv} \qquad (0,-1)_{xy} = (-1,1)_{uv}$$

Original Region



Mathematica Code: ListLinePlot[[1, 0], [2, 0], [0, -2], [-2, 2], [1, 0]]

U-Space Region



Mathematica Code: ListLinePlot[[1, 1], [2, 2], [-2, 2], [-1, 1], [1, 1]]

(c) Compute the integral found in part (a) over the domain S from part (b).

$$\implies \int_1^2 \int_{-v}^{v} \frac{1}{2} e^{\frac{u}{v}} du dv \implies \frac{1}{2} \int_1^2 ve - ve^{-1} \implies \frac{v^2}{4}(e - \frac{1}{e})\bigg|_1^2 \implies \frac{3}{4}\left(e - \frac{1}{e}\right)$$

Mathematica Code: Integrate[Integrate[1/2*$E^{(}u/v)$, u, -v, v], v, 1, 2]

5. (a) Let $D = \{(x, y) : 1 \leq x^2 + y^2 \leq 9,\ y \geq 0\}$. Compute the integral

$$\iint_D \sqrt{x^2 + y^2}\, dx\, dy$$

by changing to polar coordinates. Sketch the domains of integration in both the xy and $r\theta$ (that means r on one axis and θ on the other) planes.

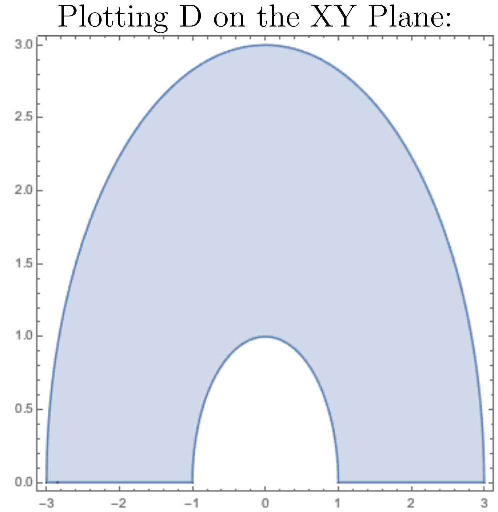
$$x = rcos(\theta) \quad y = rsin(\theta)$$

$$\implies \iint_D \sqrt{r^2cos^2(\theta) + r^2sin^2(\theta)} \implies \iint_D \sqrt{r^2(cos^2(\theta) + sin^2(\theta))} \implies \iint_D \sqrt{r^2} \cdot rdrd\theta$$

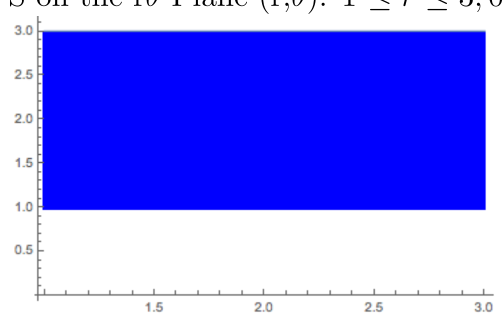
$$\implies \int_0^{\pi} \int_1^3 r^2 drd\theta \implies \int_0^{\pi} \frac{26}{3} d\theta \implies \frac{26\pi}{3}$$

Mathematica Code : Integrate[Integrate[r^2, r, 1, 3], theta, 0, Pi]

Plotting D on the XY Plane:



Plotting S on the rθ Plane (r,θ): $1 \leq r \leq 3, 0 \leq \theta \leq \pi$



(b) Compute the integral

$$\iint_D \sin(\sqrt{x^2+y^2})\,dx\,dy$$

where $D = \{(x,y) : \pi^2 \le x^2 + y^2 \le 4\pi^2\}$.

$$\implies \iint_D sin(\sqrt{r^2cos^2(\theta) + r^2sin^2(\theta)}) \implies \iint_D sin(r)*rdrd\theta \implies \int_\pi^{2\pi}\int_0^{2\pi} rsin(r)d\theta dr$$

$$f = r \qquad f' = dr \qquad g' = sin(r)dr \qquad g = -cos(r)$$

$$\implies \int_\pi^{2\pi} 2\pi rsin(r)dr \implies -2\pi rcos(r)\Big|_\pi^{2\pi} + 2\pi\int_\pi^{2\pi} cos(r)dr$$

$$\implies -2\pi rcos(r)\Big|_\pi^{2\pi} + 2\pi sin(r)\Big|_\pi^{2\pi} \implies -6\pi^2$$

Mathematica Code :

$$Integrate[Integrate[r * Sin[r], theta, 0, 2Pi], r, Pi, 2Pi]$$