1. The Black-Scholes price for a European put option is

$$P(S,t,K,T,r,q,\sigma) = Ke^{-r(T-t)}\Phi(-d_-) - Se^{-q(T-t)}\Phi(-d_+) \tag{1}$$

where

$$d_+ = \frac{\log\left(\frac{S}{K}\right) + \left(r - q + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}$$

and

$$d_- = d_+ - \sigma\sqrt{T-t} = \frac{\log\left(\frac{S}{K}\right) + \left(r - q - \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}$$

Compute each of

(a) $\Delta(P) = \dfrac{\partial P}{\partial S}$

$$\Delta(P) = \frac{\partial P}{\partial S} = \frac{\partial}{\partial S}\left[Ke^{-r(T-t)}\Phi(-d_-) - Se^{-q(T-t)}\Phi(-d_+)\right]$$

$$\implies Ke^{-r(T-t)}\phi(-d_-)\frac{\partial}{\partial S}[-d_-] - [e^{-q(T-t)}\Phi(-d_+) + Se^{-q(T-t)}\frac{\partial}{\partial S}\phi(-d_+)]$$

$$\implies -e^{-q(T-t)}\Phi(-d_+) + [Ke^{-r(T-t)}\phi(-d_-)\frac{\partial}{\partial S}(-d_-) - Se^{-q(T-t)}\frac{\partial}{\partial S}\phi(-d_+)]$$

Lemma 3.15 states $Ke^{-r(T-t)}\phi(d_-) = Se^{-q(T-t)}\phi(d_+)$

Let's prove that so we can state $Ke^{-r(T-t)}\phi(-d_-)\frac{\partial}{\partial S}(-d_-) - Se^{-q(T-t)}\frac{\partial}{\partial S}\phi(-d_+) = 0$

$$\phi(d_-) = \frac{1}{\sqrt{2\pi}}exp(-\frac{d_-^2}{2})$$

Using the fact that $d_- = d_+ - \sigma\sqrt{T-t}$

$$\implies \frac{1}{\sqrt{2\pi}}exp\left(-\frac{(d_+ - \sigma\sqrt{T-t})^2}{2}\right) \implies \phi(d_+)exp(d_+\sigma\sqrt{T-t}))exp(-\sigma^2(T-t)/2)$$

$$\implies \phi(d_+)exp\left[\log\left(\frac{S}{K}\right) + \left(r - q - \frac{\sigma^2}{2}\right)(T-t)\right]exp\left[-\frac{\sigma^2(T-t)}{2}\right]$$

$$\implies \phi(d_+)\frac{S}{K}exp\Bigg[(r-q+\sigma^2/2)(T-t)-\frac{\sigma^2}{2}(T-t)\Bigg]$$

$$\implies \phi(d_+)\frac{S}{K}e^{r(T-t)}e^{-q(T-t)}$$

Going back to

$$Ke^{-r(T-t)}\phi(-d_-)-Se^{-q(T-t)}\phi(-d_+)=0$$

Plug in $\phi(d_-)=\phi(d_+)\frac{S}{K}e^{r(T-t)}e^{-q(T-t)}$

$$=Ke^{-r(T-t)}[\phi(d_+)\frac{S}{K}e^{r(T-t)}e^{-q(T-t)}]-Se^{-q(T-t)}\phi(d_+)$$

$$\implies Se^{-q(T-t)}\phi(d_+)-Se^{-q(T-t)}\phi(d_+)=0$$

$$\therefore \Delta(P)=\frac{\partial}{\partial S}=-e^{-q(T-t)}\Phi(-d_+)$$

Verify using put-call parity:

$$P=C-Se^{-q(T-t)}+Ke^{-r(T-t)}$$

$$\text{Delta}(P)=\frac{\partial P}{\partial S}=\frac{\partial}{\partial S}\big[C-Se^{-q(T-t)}+Ke^{-r(T-t)}\big]$$

$$=\frac{\partial C}{\partial S}-e^{-q(T-t)}$$

From this we can conclude:

$$\frac{\partial}{\partial S}[P(\cdot)]+e^{-q(T-t)}=\frac{\partial}{\partial S}[C(\cdot)]$$

(b) $\Gamma(P)=\dfrac{\partial^2 P}{\partial S^2}$

$$\Gamma(P)=\frac{\partial^2 P}{\partial S^2}=\frac{\partial}{\partial S^2}[-e^{-q(T-t)}\phi(-d_+)]$$

$$\implies -e^{-q(T-t)}\frac{\partial}{\partial S}[\phi(-d_+)] \implies -e^{-q(T-t)}\frac{\partial}{\partial d_+}\phi(-d_+)\frac{\partial d_+}{\partial S}$$

$$\implies \frac{1}{s\sigma\sqrt{T-t}}\cdot e^{-q(T-t)}\phi(d_+)$$

Verify using put call parity:

$$\frac{\partial^2}{\partial S^2}[P(\cdot)]+\frac{\partial^2}{\partial S^2}[e^{-q(T-t)}]=\frac{\partial^2}{\partial S^2}[C(\cdot)]$$

$$\frac{\partial^2}{\partial S^2}[P(\cdot)]=\frac{\partial^2}{\partial S^2}[C(\cdot)]$$

(c) $\theta(P) = \dfrac{\partial P}{\partial t}$

$$\theta(P) = \frac{\partial P}{\partial t} = \frac{\partial}{\partial t}\left[Ke^{-r(T-t)}\Phi(-d_-) - Se^{-q(T-t)}\Phi(-d_+)\right]$$

$$\implies rKe^{-r(T-t)}\phi(-d_-)+Ke^{-r(T-t)}\phi(-d_-)\frac{\partial}{\partial t}(-d_-)-[qSe^{-q(T-t)}\phi(-d_+)+Se^{-q(T-t)}\phi(-d_+\frac{\partial}{\partial t}(-d_+)]$$

$$\implies rKe^{-r(T-t)}\phi(-d_-)-qSe^{-q(T-t)}\phi(-d_+)+[Ke^{-r(T-t)}\phi(-d_-)\frac{\partial}{\partial t}(-d_-)-Se^{-q(T-t)}\phi(-d_-)\frac{\partial}{\partial t}(-d_+)]$$

$$Ke^{-r(T-t)}\phi(-d_-)\frac{\partial}{\partial t}(-d_-) - Se^{-q(T-t)}\phi(-d_+)\frac{\partial}{\partial t}(-d_+)$$

$$\implies Se^{-q(T-t)}\phi(-d_+)\frac{\partial}{\partial t}(-d_-) - Se^{-q(T-t)}\phi(-d_+)\frac{\partial}{\partial t}(-d_+)$$

$$\implies Se^{-q(T-t)}\phi(-d_+)\left[\frac{\partial}{\partial t}(-d_-) - \frac{\partial}{\partial t}(-d_+)\right]$$

$$\implies Se^{-q(T-t)}\phi(-d_+)\left[\frac{\partial}{\partial t}(d_+) - \frac{\partial}{\partial t}(d_-)\right]$$

$$\frac{\partial}{\partial t}(d_+) - \frac{\partial}{\partial t}(d_-)$$

$$d_+ - d_- = \sigma\sqrt{T-t} \implies \frac{\partial}{\partial t}\sigma\sqrt{T-t} \implies -\frac{\sigma}{2\sqrt{T-t}}$$

$$\implies Se^{-q(T-t)}\phi(-d_+)\cdot -\frac{\sigma}{2\sqrt{T-t}}$$

$$\implies rKe^{-r(T-t)}\phi(-d_-) - qSe^{-q(T-t)}\phi(-d_+) - \frac{Se^{-q(T-t)}\sigma}{2\sqrt{T-t}}\phi(-d_+)$$

Verify using put call parity:

$$P = C - Se^{-q(T-t)} + Ke^{-r(T-t)}$$

$$\begin{aligned}\text{Theta}(P) = \frac{\partial P}{\partial t} &= \frac{\partial}{\partial t}\left[C - Se^{-q(T-t)} + Ke^{-r(T-t)}\right] \\ &= \frac{\partial C}{\partial t} - Sqe^{-q(T-t)} + rKe^{-r(T-t)}\end{aligned}$$

(d) $\rho(P) = \dfrac{\partial P}{\partial r}$

$$\rho(P) = \frac{\partial P}{\partial r} = \frac{\partial}{\partial r}\left[Ke^{-r(T-t)}\Phi(-d_-) - Se^{-q(T-t)}\Phi(-d_+)\right]$$

$$\implies -K(T-t)e^{-r(T-t)}\Phi(-d_-)+[Ke^{-r(T-t)}\phi(-d_-)\frac{\partial}{\partial r}\phi(-d_-)-Se^{-q(T-t)}\phi(-d_+)\frac{\partial}{\partial r}\phi(-d_+)]$$

$$\implies -K(T-t)e^{-r(T-t)}\Phi(-d_-)+[Se^{-q(T-t)}\phi(-d_+)\frac{\partial}{\partial r}\phi(-d_-)-Se^{-q(T-t)}\phi(-d_+)\frac{\partial}{\partial r}\phi(-d_+)]$$

$$\implies -K(T-t)e^{-r(T-t)}\Phi(-d_-)+Se^{-q(T-t)}\phi(-d_+)\left[\frac{\partial}{\partial r}\phi(-d_-)-\frac{\partial}{\partial r}\phi(-d_+)\right]$$

$$\frac{\partial}{\partial r}\left[d_+ - d_- = \sigma\sqrt{T-t}\right] = 0$$

$$\implies -K(T-t)e^{-r(T-t)}\Phi(-d_-)$$

Verify using put call parity:

$$\begin{aligned} P &= C - Se^{-q(T-t)} + Ke^{-r(T-t)} \\ \text{Rho}(P) = \frac{\partial P}{\partial r} &= \frac{\partial}{\partial r}\left[C - Se^{-q(T-t)} + Ke^{-r(T-t)}\right] \\ &= \frac{\partial C}{\partial r} - K(T-t)e^{-r(T-t)} \end{aligned}$$

Example

Compute the vega of a European put option.

$$
\begin{aligned}
\text{vega}(P) = \frac{\partial P}{\partial \sigma} &= \frac{\partial}{\partial \sigma}\left[Ke^{-r(T-t)}\Phi(-d_-) - Se^{-q(T-t)}\Phi(-d_+)\right] \\
&= Ke^{-r(T-t)}\phi(-d_-)\frac{\partial}{\partial \sigma}(-d_-) - Se^{-q(T-t)}\phi(-d_+)\frac{\partial}{\partial \sigma}(-d_+) \\
&= Ke^{-r(T-t)}\phi(d_-)\frac{\partial}{\partial \sigma}(-d_-) - Se^{-q(T-t)}\phi(d_+)\frac{\partial}{\partial \sigma}(-d_+)
\end{aligned}
$$

Lemma 3.15 states that $Ke^{-r(T-t)}\phi(d_-) = Se^{-q(T-t)}\phi(d_+)$, thus

$$
\begin{aligned}
\text{vega}(P) = \frac{\partial P}{\partial \sigma} &= Se^{-q(T-t)}\phi(d_+)\frac{\partial}{\partial \sigma}(-d_-) - Se^{-q(T-t)}\phi(d_+)\frac{\partial}{\partial \sigma}(-d_+) \\
&= Se^{-q(T-t)}\phi(d_+)\left[\frac{\partial}{\partial \sigma}(-d_-) - \frac{\partial}{\partial \sigma}(-d_+)\right] \\
&= Se^{-q(T-t)}\phi(d_+)\left[\frac{\partial}{\partial \sigma}(d_+) - \frac{\partial}{\partial \sigma}(d_-)\right] \\
&= Se^{-q(T-t)}\phi(d_+)\ \frac{\partial}{\partial \sigma}\left[d_+ - d_-\right]
\end{aligned}
$$

But $d_- = d_+ - \sigma\sqrt{T-t} \implies d_+ - d_- = \sigma\sqrt{T-t}$, thus

$$
\begin{aligned}
\text{vega}(P) = \frac{\partial P}{\partial \sigma} &= Se^{-q(T-t)}\phi(d_+)\ \frac{\partial}{\partial \sigma}\left[\sigma\sqrt{T-t}\right] \\
\text{vega}(P) = \frac{\partial P}{\partial \sigma} &= Se^{-q(T-t)}\phi(d_+)\ \sqrt{T-t}
\end{aligned}
$$

Check the result using put-call parity:

$$
\begin{aligned}
P &= C - Se^{-q(T-t)} + Ke^{-r(T-t)} \\
\text{vega}(P) = \frac{\partial P}{\partial \sigma} &= \frac{\partial}{\partial \sigma}\left[C - Se^{-q(T-t)} + Ke^{-r(T-t)}\right] \\
&= \frac{\partial C}{\partial \sigma}
\end{aligned}
$$

The vega of a European put option is the same as the vega for a European call option.