1. Let $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 4 & 1 \\ -2 & 1 & 1 \end{bmatrix}$

   (a) Use elimination to turn A into an upper triangular matrix. How many pivots does A have?

   $$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 4 & 1 \\ -2 & 1 & 1 \end{bmatrix} \implies \begin{bmatrix} 1 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 3 & 1 \end{bmatrix} \implies \begin{bmatrix} 1 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

   A contains two pivots

   (b) Let $b = (1, 6, 3)$. Does $Ax = b$ have a solution?
   No it does not have a solution because row 3 contains all zeros and the determinant is zero.

   (c) Let $b = (1, 6, 5)$. Does $Ax = b$ have a solution?
   No it does not have a solution because row 3 contains all zeros and the determinant is zero.

   (d) Can you find multiple solutions in either part (b) or part (c)? If so, find 2.
   No it does not have a solution because row 3 contains all zeros and the determinant is zero.

   (e) Does A have an inverse? Justify your answer using results from this exercise.
   No it does not have a solution because row 3 contains all zeros and the determinant is zero.

2. Suppose $AB = I$ and $CA = I$ where I is the $n \times n$ identity matrix.

   (a) What are the dimensions of the matrices A, B and C?
   The only dimensions that work where I is the $n \times n$ identity matrix is when A,B,C are $n \times n$

   (b) Show that $B = C$.
   (CA)B = C(AB) $\implies$ IB = CI $\implies$ B = C

   (c) Is A invertible?
   The definition of an invertible matrix is when AB = BA = I. In this case B=C and A is a $n \times n$ matrix. $\therefore$ AB = BA = CA = AC = I where A is an invertible matrix.

   [Hint: you can write $B = IB$]

3. Let A be a square matrix with the property that $A^2 = A$. Simplify $(I - A)^2$ and $(I - A)^7$.

$$(I - A)^2 \implies I^2 - IA - AI + A^2 \implies I^2 - 2A + A$$

$$I^2 - A \implies I - A$$

$$(I - A)^7 = (I - A)^2 * (I - A)^2 * (I - A)^2 * (I - A)$$
$$\implies (I - A) * (I - A) * (I - A) * (I - A)$$
$$\implies (I - A)^2 * (I - A)^2$$
$$\implies (I - A) * (I - A)$$
$$\implies (I - A)^2 \implies I - A$$

4. (a) Write the vector $(9, 2, -5)$ as a linear combination of the vectors $(1, 2, 3)$ and $(6, 4, 2)$ or explain why it can't be done.

$$A(1, 2, 3) + B(6, 4, 2) = (9, 2, -5) \implies A = 9 - 6B$$

$$\implies 2(9 - 6B) + 4B = 2 \implies B = 2$$

$$A = 9 - 12 = -3$$

$$\therefore A = -3 \; B = -3$$

(b) How many pivots does a system of equations with coefficient matrix have?

$$A = \begin{bmatrix} 1 & 6 & 9 \\ 2 & 4 & 2 \\ 3 & 2 & -5 \end{bmatrix}$$

Reduce to

$$A = \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

2 pivots

5. Suppose A is a 6×20 matrix and B is a 20×7 matrix.

   (a) What are the dimensions of $C = AB$?
   6×7

   (b) Suppose A, B, and C have been partitioned into block matrices like so:

$$A = \left[\begin{array}{c|c|c} A_{11} & A_{12} & A_{13} \\ \hline A_{21} & A_{22} & A_{23} \end{array}\right], \quad B = \left[\begin{array}{c|c} B_{11} & B_{12} \\ \hline B_{21} & B_{22} \\ \hline B_{31} & B_{32} \end{array}\right], \quad C = \left[\begin{array}{c|c} C_{11} & C_{12} \\ \hline C_{21} & C_{22} \end{array}\right],$$

   Suppose that A_{11} is 2×10, B_{22} is 4×3, and C_{11} is $? \times 4$. What are the dimensions of *each* block of A, B, and C?

$$A = \left[\begin{array}{c|c|c} 2 \times 10 & 2 \times 4 & 2 \times 6 \\ \hline 4 \times 10 & 4 \times 4 & 4 \times 6 \end{array}\right], \quad B = \left[\begin{array}{c|c} 10 \times 4 & 10 \times 3 \\ \hline 4 \times 4 & 4 \times 3 \\ \hline 6 \times 4 & 6 \times 3 \end{array}\right], \quad C = \left[\begin{array}{c|c} 2 \times 4 & 2 \times 3 \\ \hline 4 \times 4 & 4 \times 3 \end{array}\right]$$

   [Hint: Make note of every fact you know, sketch all three matrices, and fill in the unknowns step by step]

   (c) Write each block of C in terms of blocks of A and B.

$$C = \left[\begin{array}{c|c} A_{11} \cdot B_{11} + A_{12} \cdot B_{21} + A_{13} \cdot B_{31} & A_{11} \cdot B_{12} + A_{12} \cdot B_{22} + A_{13} \cdot B_{32} \\ \hline A_{21} \cdot B_{11} + A_{22} \cdot B_{21} + A_{23} \cdot B_{31} & A_{21} \cdot B_{12} + A_{22} \cdot B_{22} + A_{23} \cdot B_{32} \end{array}\right]$$

6. Let A be an $m \times n$ matrix.

   (a) The full $A = QR$ factorization contains more information than necessary to reconstruct A. What are the smallest matrices $\tilde{Q}$ and $\tilde{R}$ such that $\tilde{Q}\tilde{R} = A$?

   Let Q be an orthogonal matrix that is $m \times m$ and R be an upper rectangular matrix $m \times n$. Since R contains m-n rows that are zero the smallest matrix R can be is $n \times n$ and since the lower rows are zero the Q matrix can be cut down to a $m \times n$ matrix.

   (b) Let $\tilde{A}$ be an $m \times n$ matrix ($m > n$) whose columns each sum to zero, and let $\tilde{A} = \tilde{Q}\tilde{R}$ be the reduced QR factorization of $\tilde{A}$. The squared *Mahalanobis* distance to the point $\tilde{x}_i^T$ (the i^{th} row of $\tilde{A}$) is

$$d_i^2 = \tilde{x}_i^T \hat{S}^{-1} \tilde{x}_i$$

   where $\hat{S} = \frac{1}{m-1}\tilde{A}^T\tilde{A}$ is a covariance matrix. Compute d_i^2 without inverting a matrix.

$$\tilde{A}^T\tilde{A} = \tilde{R}^T\tilde{Q}^T\tilde{Q}\tilde{R} \implies \tilde{R}^T\tilde{R}$$
$$\implies S^-1 = (m-1)\tilde{R}^{-1}\tilde{R}^{-T}$$

$\tilde{x}_i^T$ is the i^{th} row of $\tilde{A}$

$$\Longrightarrow\ \tilde{x}_i = \tilde{R}^T \tilde{Q}_i^{\ T}$$

$$\Longrightarrow\ d_i^2 = \tilde{Q}_i \tilde{R}[(m-1)\tilde{R}^{-1}\tilde{R}^{-T}]\tilde{Q}_i^{\ T}\tilde{R}^2$$

$$\Longrightarrow\ (m-1)\tilde{Q}_i \tilde{Q}_i^{\ T}$$

$$d_i^2 = (m-1)\sum_{j=1}^{n} q_{ij}^2$$

where q_{ij} is the (i,j) element of $\tilde{Q}$.