Mathematical Methods for Quantitative Finance Homework 5

1. Give an example of a 2×2 matrix that has no real eigenvectors. Justify your solution with intuition (without solving completely for the eigenvectors and eigenvalues).

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

The reason this wouldn't work is because we know λ^2 would need to be 1 and -1 where $\lambda^2 + 1 = 0$

2. Consider an $n \times p$ matrix A. Show that the number of linear independent rows is the same as the number of linearly independent columns.
Hint: Write $A = CR$ where C is a matrix of the linearly independent columns of A. Why can we write A like this? Then consider the CR product in the "row" interpretation of matrix multiplication.

$$C = \begin{bmatrix} C_1 & C_2 & C_3 & C_n \end{bmatrix}$$

$$R = \begin{bmatrix} R_1 \\ R_2 \\ R_3 \\ R_p \end{bmatrix}$$

If n != p then the matrix A would be overdetermined or undetermined and would then be linearly dependent because it would be equivalent to a homogenous system where Vx = θ has a non-trivial solution. Therefore the number of rows must equal the number of columns.

3. Let A be an $m \times n$ matrix (assume $m > n$). The full singular value factorization $A = U\Sigma V^{\mathrm{T}}$ contains more information than necessary to reconstruct A.

   (a) What are the smallest matrices $\tilde{U}$, $\tilde{\Sigma}$ and $\tilde{V}^{\mathrm{T}}$ such that $\tilde{U}\tilde{\Sigma}\tilde{V}^{\mathrm{T}} = A$?

$$A = m \times n$$
$$\implies U = m \times n$$
$$\implies \Sigma = n \times n$$
$$\implies V = n \times n$$

   This is because m > n, we only need n Eigenvalues

   (b) Let $U = \begin{bmatrix} \tilde{U} & \hat{U} \end{bmatrix}$. That is, think about U from the full singular value factorization as a block matrix consisting of the matrix $\tilde{U}$ found in part (a) and the remaining (unneeded) columns $\hat{U}$.

Find expressions for $\tilde{U}^{\mathrm{T}}\tilde{U}$ and $\tilde{U}\tilde{U}^{\mathrm{T}}$.

$$
\begin{aligned}
\tilde{U}\tilde{U}^{\mathrm{T}} &= \begin{bmatrix} \tilde{U} & \hat{U} \end{bmatrix} \begin{bmatrix} \tilde{U}^T \\ \hat{U}^T \end{bmatrix} \\
\implies UU^{\mathrm{T}} &= \tilde{U}\tilde{U}^T + \hat{U}\hat{U}^T \\
\implies I &= \tilde{U}\tilde{U}^T + \hat{U}\hat{U}^T \\
\implies \tilde{U}\tilde{U}^T &= I - \hat{U}\hat{U}^T
\end{aligned}
$$

$$
\begin{aligned}
\tilde{U}^{\mathrm{T}}\tilde{U} &= \begin{bmatrix} \tilde{U}^T \\ \hat{U}^T \end{bmatrix} \begin{bmatrix} \tilde{U} & \hat{U} \end{bmatrix} \\
\implies &\begin{bmatrix} \tilde{U}^T\tilde{U} & \tilde{U}^T\hat{U} \\ \hat{U}^T\tilde{U} & \hat{U}^T\hat{U} \end{bmatrix} = I
\end{aligned}
$$

(c) Use the *reduced* singular value factorization obtained in part (a) to find an expression for the matrix $H = A(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}$. How many matrices must be inverted (diagonal and orthogonal matrices don't count)?

Assume everything has a tilde

$$
\begin{aligned}
A^TA = (U\Sigma V)^T U\Sigma V^T &\implies V\Sigma^T\Sigma V^T \\
\implies (U\Sigma V)(V\Sigma^T\Sigma V^T)^{-1}&(U\Sigma V^T)^T \\
\implies U\Sigma V^T V\Sigma^{-1}&U^T \\
\implies UU^T &\implies I
\end{aligned}
$$

No matrices need to be inverted.

4. Let x and y be vectors of m elements. The least squares solution for a best-fit line for a plot of y versus x is

$$\hat{\beta} = (X^{\mathrm{T}}X)^{-1}X^{\mathrm{T}}y$$

where

$$X = \begin{bmatrix} | & | \\ 1 & x \\ | & | \end{bmatrix}$$

(a) Suppose you know the **full** singular value factorization $X = U\Sigma V^{\mathrm{T}}$. Find an expression for $\hat{\beta}$ in terms of U, Σ, and V. Hint: Only square matrices can be invertible.

$$
\begin{aligned}
\hat{\beta} &= ((U\Sigma V^T)^T U\Sigma V^T)^{-1}(U\Sigma V^T)^T y \\
&\implies (V\Sigma^{\mathrm{T}}\Sigma V^{\mathrm{T}})^{-1}(U\Sigma V^T)^T y \\
&\implies (V\Sigma^{\mathrm{T}}\Sigma V^{\mathrm{T}})^{-1}(V\Sigma^{\mathrm{T}}U^{\mathrm{T}})y \\
&\implies V\Sigma^{-1}U^T y
\end{aligned}
$$

(b) Repeat part (a) using the reduced singular value factorization $X = \tilde{U}\tilde{\Sigma}\tilde{V}^{\mathrm{T}}$.

$$((\tilde{U}\tilde{\Sigma}\tilde{V}^{\mathrm{T}})^T\tilde{U}\tilde{\Sigma}\tilde{V}^{\mathrm{T}})^{-1}(\tilde{U}\tilde{\Sigma}\tilde{V}^{\mathrm{T}})^T y$$

$$\implies (\tilde{V}\tilde{\Sigma}^{\mathrm{T}}\Sigma\tilde{V}^{\mathrm{T}})^{-1}(\tilde{U}\tilde{\Sigma}\tilde{V}^{T})^T y$$

$$\implies \tilde{V}\tilde{\Sigma}^{-1}\tilde{U}^T y$$

5. Let $\tilde{X}$ be an $m \times n$ matrix $(m > n)$ whose columns have sample mean zero, and let $\tilde{X} = \tilde{U}\tilde{\Sigma}\tilde{V}^{\mathrm{T}}$ be a reduced singular value factorization of $\tilde{X}$. The squared *Mahalanobis* distance to the point $\tilde{x}_i^{\mathrm{T}}$ (the i^{th} row of $\tilde{X}$) is

$$d_i^2 = \tilde{x}_i^{\mathrm{T}}\hat{S}^{-1}\tilde{x}_i$$

where $\hat{S} = \frac{1}{m-1}\tilde{X}^{\mathrm{T}}\tilde{X} = \operatorname{cov}(\tilde{X})$. Explain how to compute d_i^2 without inverting a matrix.

$$\hat{S} = \frac{1}{m-1}(\tilde{U}\tilde{\Sigma}\tilde{V}^{\mathrm{T}})^T\tilde{U}\tilde{\Sigma}\tilde{V}^{\mathrm{T}}$$

$$\implies \tilde{V}\tilde{\Sigma}^T\tilde{U}^T\tilde{U}\tilde{\Sigma}\tilde{V}^T$$

$$\implies \hat{S} = \frac{1}{m-1}\tilde{V}\tilde{\Sigma}^T\tilde{\Sigma}\tilde{V}^T$$

$$\implies \hat{S}^{-1} = (m-1)(\tilde{V}\tilde{\Sigma}^T\tilde{\Sigma}\tilde{V}^T)^{-1}$$

$$\implies \tilde{V}\tilde{\Sigma}^T\tilde{U}^T[(m-1)(\tilde{V}\tilde{\Sigma}^T\tilde{\Sigma}\tilde{V}^T)^{-1}]\tilde{U}\tilde{\Sigma}\tilde{V}^T$$

$$d_i^2 = m-1$$

6. (a) Suppose $A = LU$ where L is lower triangular and U is upper triangular. Explain how you would solve the problem $Ax = b$ using L, U, and the concepts of forward and backward substitution.

Start with the identity matrix then factor A into L and U using elimination matrices. Using L, we can use forward substitution to solve for some vector c. Setting the upper triangular system to c we can solve for b with backward substitution.

(b) Compute the LU factorization of

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & -3 & 2 \\ -2 & 1 & 1 \end{bmatrix}$$

by hand using elimination matrices.

$$L_1 A = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & -1 \\ 3 & -3 & 2 \\ -2 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & -1 \\ 0 & -9 & 5 \\ -2 & 1 & 1 \end{bmatrix}$$

$$L_2 L_1 A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & -1 \\ 0 & -9 & 5 \\ -2 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & -1 \\ 0 & -9 & 5 \\ 0 & 5 & -1 \end{bmatrix}$$

$$L_3 L_2 L_1 A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{5}{9} & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & -1 \\ 0 & -9 & 5 \\ 0 & 5 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & -1 \\ 0 & -9 & 5 \\ 0 & 0 & \frac{16}{9} \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -2 & -\frac{5}{9} & 1 \end{bmatrix}$$

$$U = \begin{bmatrix} 1 & 2 & -1 \\ 0 & -9 & 5 \\ 0 & 0 & \frac{16}{9} \end{bmatrix}$$