Instructions: Solve each of the exercises both by hand (must show all work for credit) and using Mathematica to verify that you get the same answers. Be sure to include the Mathematica code used to verify the solution. For example,

$$\lim_{x\to 0} \frac{\sin(x)}{x} \quad \text{would be} \quad \texttt{Limit[Sin[x] / x, x -> 0]}.$$

1. Compute the following limits.

   (a) $\displaystyle\lim_{h\to 0} \frac{4(x+h-3)^2 - 4(x-3)^2}{h}$

   $\displaystyle= \lim_{h\to 0} \frac{4(x^2+2hx-6x+h^2-6h+9) - 4(x^2-6x+9)}{h}$

   $\displaystyle= \lim_{h\to 0} \frac{4x^2+8hx-24x+4h^2-24h+36-4x^2-24x+36}{h}$

   $\displaystyle= \lim_{h\to 0} \frac{8hx+4h^2-24h}{h}$

   $\displaystyle= \lim_{h\to 0} 8x+4h-24$

   = 8x - 24

   $$Mathematica = Limit[(4(x+h-3)^2 - 4(x-3)^2/h, h- > 0]$$

   (b) $\displaystyle\lim_{x\to\infty} \frac{1}{\sqrt{4x^2-2x-10}+2x}$

   The denominator approaches infinity, so the fraction approaches 0.

   $$Mathematica = Limit[1/(sqrt(4x^2-2x-10)+2x), x- > Infinity]$$

2. Compute the derivatives of the following functions.

   (a) $\displaystyle f(x) = \frac{1}{1-x} = (1-x)^{-1} = -1(1-x)^{-2} * (-1)$

   $$= \frac{1}{(1-x)^2}$$

   Mathematica In[1] := f[x] := 1/(1 - x) In[2] := f'[x]

(b) $f(x) = \displaystyle\sum_{k=1}^{7} k e^{-a_k x^3}$ (the $\{a_k\}$ are constants)

$= \sum_{k=1}^{7} k * \frac{d}{dx}[e^{-a_k x^3}]$

$= 28 * -3x^2 a_k e^{-a_k x^3}$

$= -84x^2 a_k e^{-a_k x^3}$

Mathematica Code: D[Sum[K*E^(−$Subscript[a,k]*x^3$), K, 1, 7], x]

(c) $f(x) = \dfrac{\log\left(\frac{x}{K}\right) + \left(r - q + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}$ ($K > 0$, r, q, $\sigma > 0$, and $T > t$ constant)

$= \frac{log(x) - log(K)}{\sigma\sqrt{T-t}}$

$= \frac{\frac{1}{x}}{\sigma\sqrt{T-t}}$

$= \frac{1}{x\sigma\sqrt{T-t}}$

Mathematica Code: D[((-t + T) (-q + r + $sigma^2$/2) + Log[x/k])/(Sqrt[-t + T] [sigma]), x]

(d) $f(x) = \dfrac{\log\left(\frac{S}{K}\right) + \left(r - q + \frac{x^2}{2}\right)(T-t)}{x\sqrt{T-t}}$ ($S > 0$, $K > 0$, r, q, and $T > t$ constant)

$= \frac{1}{\sqrt{T-t}} * \frac{d}{dx}[\frac{\log\left(\frac{S}{K}\right)+\left(r-q+\frac{x^2}{2}\right)\left(T-t\right)}{x}]$

Quotient Rule

$= \frac{(T-t)*x^2 - log(s/k) - (r-g+\frac{x^2}{2})(T-t)}{x^2}$

$= \frac{(T-t)*x^2 - log(s/k) - (r-g+\frac{x^2}{2})(T-t)}{x^2\sqrt{T-t}}$

Mathematica Code : D[((-t + T) (-q + r + x^2/2) + Log[s/k])/(Sqrt[-t + T] x), x]

(e) $f(x) = \dfrac{\log\left(\frac{S}{K}\right) + \left(x - q + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}$ ($S > 0$, $K > 0$, q, $\sigma > 0$, and $T > t$ constant)

Factor out constants

$= \frac{1}{\sigma\sqrt{T-t}} * \frac{d}{dx}[\log\left(\frac{S}{K}\right) + \left(x - q + \frac{\sigma^2}{2}\right)(T-t)]$

Eliminate constants not multiplied by x

= (T-t)x

$= \frac{d}{dx}[x(T-t)]$

$= T - t$

$= \frac{T-t}{\sigma\sqrt{T-t}}$

$= \frac{T-t}{\sigma\sqrt{T-t}} * \frac{\sqrt{T-t}}{\sqrt{T-t}}$

$= \frac{\sqrt{T-t}}{\sigma}$

Mathematica Code : D[((T - t) (x - q + $sigma^2$/2) + Log[s/k])/(Sqrt[-t + T] sigma), x]

3. Recall that

$$d_+(\cdot) = \frac{\log\left(\frac{S}{K}\right) + \left(r - q + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}$$

(a) Parts (c), (d), and (e) of Problem 2 correspond to partial derivatives of d_+. What partial derivative does each correspond to?

2c corresponds to taking the derivative of d_+ with respect to S.

2d corresponds to taking the derivative of d_+ with respect to σ.

2e corresponds to taking the derivative of d_+ with respect to r.

(b) Compute the partial derivative of d_+ with respect to t.

Quotient Rule: Factor out constant

$= \frac{1}{\sigma} * \frac{d}{dt}\left[\frac{\log\left(\frac{S}{K}\right) + \left(r-q+\frac{\sigma^2}{2}\right)\left(T-t\right)}{\sqrt{T-t}}\right]$

$$= \frac{\sqrt{T-t} - 1 * (r - q + \frac{\sigma^2}{2}) - (log(s/k) + (r - q + \frac{\sigma^2}{2})(T-t))(-\frac{1}{2\sqrt{T-t}})}{T-t}$$

$$= \frac{-\sqrt{T-t}(r - q + \frac{\sigma^2}{2}) + \frac{log(s/k) + (r-q+\frac{\sigma^2}{2})(T-t)}{2\sqrt{T-t}}}{T-t}$$

$$= -\frac{(t-T)(2q - 2r - \sigma^2) - 2log(s/k)}{4(T-t)^{\frac{3}{2}}}$$

Mathematica code : D[((T - t) (r - q + σ^2/2) + Log[s/k])/(Sqrt[-t + T]), t]

4. Compute the following antiderivatives.

(a) $\displaystyle\int x^2 \log(x)\, dx$

Use integration by parts

U =log(x) du = $\frac{1}{x}$

dv = x^2 V = $\frac{x^3}{3}$

UV - $\int V du$

$\frac{x^3}{3} \log x - \int \frac{x^3}{3} * \frac{1}{x}$

$\frac{x^3}{3} \log x - \frac{1}{9} x^3 + C$

Mathematica code Integrate[x^2 Log[x], x]

(b) $\displaystyle\int x^2 e^x\, dx$

Integrate by parts twice
U =x^2 du = 2x
dv = e^x V = e^x

$$= e^x x^2 - \int 2xe^x$$

U =2x du = 2
dv = e^x V = e^x

$$e^x x^2 - (2xe^x - 2e^x) + C$$
$$e^x x^2 - 2xe^x + 2e^x + C$$
$$e^x(x^2 - 2x + 2) + C$$

Mathematica Code : Integrate[$x^2 E^x$, x]

(c) $\displaystyle\int \big[\log(x)\big]^2\, dx$

U = log(x) du = $\frac{1}{x} dx$ e^udu = dx

$$\int u^2 e^u du$$

Same form as 4b.

$$e^u(u^2 - 2u + 2) + C$$

$$e^{log(x)}((log(x))^2 - 2(log(x)) + 2) + C$$
$$x(log^2(x) - 2log(x) + 2) + C$$

Mathematica code : Integrate[(Log[x])^2, x]

5. Evaluate the following definite integrals.

   (a) $\int_4^7 x^2 \log(x)\, dx$

   Same form as 4a.

$$= \frac{x^3}{3}\log x - \frac{1}{9}x^3 \Big|_4^7$$
$$= \frac{7^3}{3}log(7) - \frac{7^3}{9} - \left(\frac{4^3}{3}log(4) - \frac{4^3}{9}\right)$$
$$= \frac{343}{3}log(7) - \frac{64}{3}log(4) - 31$$

   Use Wolfram to simplify further

$$= \frac{343log(343)}{9} - \frac{64log(64)}{9} - 31$$

   Mathematica Code : Integrate[$x^2 Log[x], x, 4, 7$]

   (b) $\int_0^2 \frac{1}{(1+x)^2}\, dx$

   U = 1 + x   du = dx

$$= \int \frac{1}{u^2} du$$
$$= -\frac{1}{u}\Big|_1^3$$
$$= \frac{2}{3}$$

   Mathematica Code : Integrate[1/(1 + x)^2, x, 0, 2]