1. Consider the constrained optimization problem

$$\begin{aligned} \text{minimize:} \quad & x^2 + 4xy + 3y^2 \\ \text{subject to:} \quad & 2x + 8y = 12 \end{aligned}$$

   (a) Write down the Lagrangian for the problem and compute its gradient.

$$F(x, \lambda) = x^2 + 4xy + 3y^2 + \lambda(2x + 8y - 12)$$
$$\begin{bmatrix} 2x + 4y + 2\lambda \\ 4x + 6y + 8\lambda \\ 2x + 8y \end{bmatrix}$$

   (b) The critical point of the Lagrangian can be found by solving a linear system. Write this system in matrix notation.

$$\begin{bmatrix} 2 & 4 & 2 \\ 4 & 6 & 8 \\ 2 & 8 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ \lambda \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 12 \end{bmatrix}$$

   (c) Use elimination followed by back substitution to compute the critical point. What is the value of the objective at the critical point?

$$\implies \begin{bmatrix} 2 & 4 & 2 \\ 0 & -2 & 4 \\ 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \\ \lambda \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 12 \end{bmatrix}$$

   Z = 100 + 4(-10)(4) + 3(16) $\implies -12$

   (d) Use the constraint to reduce the objective to a function of 1 variable. Find the critical point of this new 1 variable objective and verify that it is a minimum.

$$x = 6 - 4y$$
$$(6 - 4y)^2 + 4y(6 - 4y) + 3y^2$$
$$\implies 3y^2 - 24y + 36$$
$$\implies \frac{df}{dy} = 6y - 24$$
$$y = 6$$
$$\frac{d^2f}{dy^2} = 6 > 0 \therefore min$$

2. Consider using Lagrange's method to solve the following optimization problem.

$$\begin{aligned} \text{minimize:} \quad & 3x_1 - 4x_2 + x_3 - 2x_4 \\ \text{subject to:} \quad & -x_2^2 + x_3^2 + x_4^2 = 1 \\ & 3x_1^2 + x_3^2 + 2x_4^2 = 6 \end{aligned}$$

Write down the Lagrangian and its gradient and explain why it is difficult to find a critical point.

$$F(x,\lambda) = 3x_1 - 4x_2 + x_3 - 2x_4 + \lambda_1(-x_2^2 + x_3^2 + x_4^2 - 1) + \lambda_2(3x_1^2 + x_3^2 + 2x_4^2 - 6)$$

$$DF(x,\lambda) = \begin{bmatrix} 3 + 6\lambda_2 x_1 \\ -4 - 2\lambda_1 x_2 \\ 1 + 2\lambda_1 x_3 + 2\lambda_2 x_3 \\ -2 + 2\lambda_1 x_4 + 4\lambda_2 x_4 \\ -x_2^2 + x_3^2 + x_4^2 - 1 \\ 3x_1^2 + x_3^2 + 2x_4^2 - 6 \end{bmatrix}$$

There are lots of variables and no lambda is given

3. Compute the gradient of the Lagrangian for the maximum expected return portfolio subject to risk = $(20\%)^2$.

$$\begin{aligned} \text{maximize:} \quad & \mu^{\mathrm{T}} w \\ \text{subject to:} \quad & e^{\mathrm{T}} w - 1 = 0 \\ & w^{\mathrm{T}} \Sigma w - 0.04 = 0 \end{aligned}$$

$$F(w,\lambda) = \mu^{\mathrm{T}} w + \lambda_1(e^{\mathrm{T}} w - 1) + \lambda_2(w^{\mathrm{T}} \Sigma w - 0.04)$$

$$DF(w,\lambda) = \begin{bmatrix} \mu^{\mathrm{T}} + e^{\mathrm{T}}\lambda_1 + 2\lambda_2 \Sigma w \\ e^{\mathrm{T}} w - 1 \\ w^{\mathrm{T}} \Sigma w - 0.04 \end{bmatrix}$$